

Kneser's Theorem and the Cauchy–Davenport Theorem

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Abstract

We formalise Kneser's Theorem in combinatorics [2, 3] following the proof from the 2014 paper "A short proof of Kneser's addition theorem for abelian groups" by Matt DeVos [1]. We also show a strict version of Kneser's Theorem as well as the Cauchy–Davenport Theorem as a corollary of Kneser's Theorem.

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1 Preliminaries

theory *Kneser-Cauchy-Davenport-preliminaries*

imports

Complex-Main

Pluenecke-Ruzsa-Inequality.Pluenecke-Ruzsa-Inequality

HOL-Number-Theory.Prime-Powers

begin

context *subgroup-of-group*

begin

interpretation *left: left-translations-of-group ..*

interpretation *right: right-translations-of-group ..*

interpretation *transformation-group left.translation ' H G ..*

lemma *Right-Coset-eq-iff:*

assumes $x \in G$ **and** $y \in G$

shows $H \cdot x = (H \cdot y) \iff H \cdot x \cap (H \cdot y) \neq \{\}$

using *assms Right-Coset-is-orbit*

by (*metis Int-absorb orbit.disjoint orbit.natural.map-closed orbit.non-vacuous*)

end

context *additive-abelian-group*

begin

1.1 Elementary lemmas on sumsets

lemma *sumset-translate-eq-right:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$

shows $(\text{sumset } A \{x\} = \text{sumset } B \{x\}) \iff A = B$ **using** *assms*

by (*smt (verit, best) Diff-Int-distrib2 Diff-eq-empty-iff*

Int-Un-eq(1) Int-absorb2 Un-Diff-cancel2 Un-commute insert-disjoint(2)

subset-refl sumset-is-empty-iff sumsetdiff-sing)

lemma *sumset-translate-eq-left:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$

shows $(\text{sumset } \{x\} A = \text{sumset } \{x\} B) \iff A = B$ **using** *assms*

by (*simp add: sumset-commute sumset-translate-eq-right*)

lemma *differenceset-translate-eq-right:*

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$

shows $(\text{differenceset } A \{x\} = \text{differenceset } B \{x\}) \iff A = B$ **using** *assms*

by (*metis Int-absorb2 differenceset-commute minus-minusset minusset-subset-carrier*
sumset-translate-eq-left)

lemma *differenceset-translate-eq-left*:
assumes $A \subseteq G$ **and** $B \subseteq G$ **and** $x \in G$
shows $(\text{differenceset } \{x\} A = \text{differenceset } \{x\} B) \longleftrightarrow A = B$ **using** *assms*
by (*metis differenceset-commute differenceset-translate-eq-right*)

lemma *sumset-inter-union-subset*:
 $\text{sumset } (A \cap B) (A \cup B) \subseteq \text{sumset } A B$
by (*metis Int-Diff-Un Int-Un-eq(2) Un-subset-iff sumset-commute sumset-subset-Un(2)*
sumset-subset-Un2)

lemma *differenceset-group-eq*:
 $G = \text{differenceset } G G$
using *equalityE minusset-eq minusset-triv subset-antisym sumset-D(1) sumset-subset-carrier*
sumset-mono image-mono Int-absorb **by** *metis*

lemma *card-sumset-singleton-subset-eq*:
assumes $a \in G$ **and** $A \subseteq G$
shows $\text{card } (\text{sumset } \{a\} A) = \text{card } A$
using *assms card-sumset-singleton-eq card.infinite card-sumset-0-iff' le-iff-inf*
sumset-commute
by *metis*

lemma *card-differenceset-singleton-mem-eq*:
assumes $a \in G$ **and** $A \subseteq G$
shows $\text{card } A = \text{card } (\text{differenceset } A \{a\})$
using *assms* **by** (*metis card-minusset' card-sumset-singleton-subset-eq difference-*
set-commute
minusset-subset-carrier)

lemma *card-singleton-differenceset-eq*:
assumes $a \in G$ **and** $A \subseteq G$
shows $\text{card } A = \text{card } (\text{differenceset } \{a\} A)$
using *assms* **by** (*metis card-minusset' card-sumset-singleton-subset-eq minus-*
set-subset-carrier)

lemma *sumset-eq-Union-left*:
assumes $A \subseteq G$
shows $\text{sumset } A B = (\bigcup a \in A. \text{sumset } \{a\} B)$
proof
show $\text{sumset } A B \subseteq (\bigcup a \in A. \text{sumset } \{a\} B)$
using *assms sumset.cases Int-absorb2 Int-iff UN-iff singletonI sumset.sumsetI*
by (*smt (verit, del-insts) subsetI*)
next

show $(\bigcup a \in A. \text{sumset } \{a\} B) \subseteq \text{sumset } A B$
using *sumset* **by** *auto*
qed

lemma *sumset-eq-Union-right*:
assumes $B \subseteq G$
shows $\text{sumset } A B = (\bigcup b \in B. \text{sumset } A \{b\})$
using *assms sumset-commute sumset-eq-Union-left* **by** (*metis (no-types, lifting) Sup.SUP-cong*)

lemma *sumset-singletons-eq*:
assumes $a \in G$ **and** $b \in G$
shows $\text{sumset } \{a\} \{b\} = \{a \oplus b\}$
using *assms sumset.simps subset-antisym* **by** *auto*

lemma *sumset-eq-subset-differenceset*:
assumes $K \subseteq G$ **and** $K \neq \{\}$ **and** $A \subseteq G$ **and** $\text{sumset } A K = \text{sumset } B K$
shows $A \subseteq \text{differenceset } (\text{sumset } B K) K$
proof
fix a **assume** $ha: a \in A$
obtain k **where** $hk: k \in K$ **using** *assms(2)* **by** *blast*
then have $a \oplus k \in \text{sumset } B K$ **using** *assms sumset.sumsetI ha* **by** *blast*
then have $a \oplus (k \ominus k) \in \text{differenceset } (\text{sumset } B K) K$ **using** hk *assms ha minusset.minussetI*
subset-iff sumset.sumsetI **by** (*smt (verit) associative composition-closed inverse-closed*)
then show $a \in \text{differenceset } (\text{sumset } B K) K$ **using** hk ha *subsetD assms right-unit*
by (*metis invertible invertible-right-inverse*)
qed

end

locale *subgroup-of-additive-abelian-group* =
subgroup-of-abelian-group $H G (\oplus) \mathbf{0} +$ *additive-abelian-group* $G (\oplus) \mathbf{0}$
for $H G$ **and** *addition* (*infixl* $\langle \oplus \rangle$ 65) **and** *zero* ($\langle \mathbf{0} \rangle$)

begin

notation *Left-Coset* (*infixl* $\langle \cdot | \rangle$ 70)

lemma *Left-Coset-eq-sumset*:
assumes $x \in G$
shows $\text{sumset } \{x\} H = x \cdot | H$
using *assms Left-Coset-memI sumset.simps* **by** *fastforce*

lemma *sumset-subgroup-eq-iff*:
assumes $a \in G$ **and** $b \in G$

shows $\text{sumset } \{a\} H = \text{sumset } \{b\} H \longleftrightarrow$
 $(\text{sumset } \{a\} H) \cap (\text{sumset } \{b\} H) \neq \{\}$
using *Right-Coset-eq-iff* *assms* *Left-Coset-eq-sumset* *Left-equals-Right-coset* **by**
presburger

lemma *card-divide-sumset*:
assumes $A \subseteq G$
shows $\text{card } H \text{ dvd } \text{card } (\text{sumset } A H)$
proof(*cases* $\text{finite } H \wedge \text{finite } A$)
case *hfin*: *True*
then have *hfinsum*: $\bigwedge X. X \in ((\lambda a. \text{sumset } \{a\} H) ' A) \implies \text{finite } X$
using *finite-sumset* **by** *force*
moreover have *pairwise disjoint* $((\lambda a. \text{sumset } \{a\} H) ' A)$
using *pairwise-imageI disjoint-def* *sumset-subgroup-eq-iff* *subset-eq* *assms* **by** (*smt*
(verit, best))
moreover have $\text{card } H \text{ dvd } \text{sum } \text{card } ((\lambda a. \text{sumset } \{a\} H) ' A)$
proof(*intro* *dvd-sum*)
fix X **assume** $X \in (\lambda a. \text{sumset } \{a\} H) ' A$
then show $\text{card } H \text{ dvd } \text{card } X$ **using** *dvd-refl*
using *Left-equals-Right-coset* *Right-Coset-cardinality* *assms* *Left-Coset-eq-sumset*
by *auto*
qed
ultimately show *?thesis* **using** *assms* *sumset-eq-Union-left* *card-Union-disjoint*
by *metis*
next
case *False*
then show *?thesis* **using** *assms* *card-sumset-0-iff* **by** (*metis* *card-eq-0-iff* *dvd-0-right*
sub *subsetI*)
qed

lemma *sumset-subgroup-eq-Class-Union*:
assumes $A \subseteq G$
shows $\text{sumset } A H = (\bigcup (\text{Class } ' A))$
proof
show $\text{sumset } A H \subseteq \bigcup (\text{Class } ' A)$
proof
fix x **assume** $x \in \text{sumset } A H$
then obtain $a b$ **where** $ha: a \in A$ **and** $b \in H$ **and** $x = a \oplus b$
using *sumset.cases* **by** *blast*
then have $x \in \text{Class } a$ **using** *Left-Coset-Class-unit* *Left-Coset-eq-sumset* *assms*
by *blast*
thus $x \in \bigcup (\text{Class } ' A)$ **using** *ha* **by** *blast*
qed
next
show $\bigcup (\text{Class } ' A) \subseteq \text{sumset } A H$
proof(*intro* *Union-least*)
fix X **assume** $X \in \text{Class } ' A$
then obtain a **where** $a \in A$ **and** $X = \text{Class } a$ **by** *blast*
moreover hence $\{a\} \subseteq A$ **by** *auto*

ultimately show $X \subseteq \text{sumset } A \ H$ **using** *Left-Coset-Class-unit*
Left-Coset-eq-sumset assms sumset-mono subset-refl in-mono **by** *metis*
qed
qed

lemma *Class-image-sumset-subgroup-eq*:
assumes $A \subseteq G$
shows $\text{Class } 'A (\text{sumset } A \ H) = \text{Class } 'A$
proof
show $\text{Class } 'A \ \text{sumset } A \ H \subseteq \text{Class } 'A$
proof
fix x **assume** $x \in \text{Class } 'A \ \text{sumset } A \ H$
then obtain c **where** $hc: c \in \text{sumset } A \ H$ **and** $x = \text{Class } c$ **by** *blast*
moreover obtain $a \ b$ **where** $ha: a \in A$ **and** $b \in H$ **and** $c = a \oplus b$ **using** hc
sumset.cases
by *blast*
ultimately show $x \in \text{Class } 'A$ **using** ha *Class-eq CongruenceI assms composition-closed*
sumset.cases Partition-def commutative image-eqI left-unit sub unit-closed
by (*smt (verit, ccfv-threshold) Block-self Class-cong Normal-def*)
qed
next
show $\text{Class } 'A \subseteq \text{Class } 'A \ \text{sumset } A \ H$ **using** *assms right-unit subsetD subsetI sumset.sumsetI*
unit-closed **by** (*smt (verit, del-insts) image-subset-iff sub-unit-closed*)
qed

lemma *Class-cover-imp-subset-or-disj*:
assumes $A = (\bigcup (\text{Class } 'C))$ **and** $x \in G$ **and** $C \subseteq G$
shows $\text{Class } x \subseteq A \vee \text{Class } x \cap A = \{\}$
proof(*intro disjCI*)
assume $\text{Class } x \cap A \neq \{\}$
then obtain c **where** $c \in C$ **and** $\text{Class } x \cap \text{Class } c \neq \{\}$ **using** *assms* **by** *blast*
then show $\text{Class } x \subseteq A$ **using** *assms not-disjoint-implies-equal Sup-upper imageI subset-iff*
by *blast*
qed

end

context *additive-abelian-group*

begin

1.2 Stabilizer and basic properties

We define the stabilizer or group of periods of a nonempty subset of an abelian group.

definition *stabilizer::'a set \Rightarrow 'a set* **where**

```

stabilizer S ≡ {x ∈ G. sumset {x} (S ∩ G) = S ∩ G}

lemma stabilizer-is-subgroup: fixes S :: 'a set
  shows subgroup (stabilizer S) G (⊕) (0)
proof (intro subgroupI)
  show stabilizer S ⊆ G using stabilizer-def by auto
next
  show 0 ∈ stabilizer S using stabilizer-def by (simp add: Int-absorb1 Int-commute)
next
  fix a b assume haS: a ∈ stabilizer S and hbS: b ∈ stabilizer S
  then have haG: a ∈ G and hbG: b ∈ G using stabilizer-def by auto
  have sumset {a ⊕ b} (S ∩ G) = sumset {a} (sumset {b} (S ∩ G))
  proof
    show sumset {a ⊕ b} (S ∩ G) ⊆ sumset {a} (sumset {b} (S ∩ G)) using haG
    hbG
      empty-subsetI insert-subset subsetI sumset.simps sumset-assoc sumset-mono
      by metis
    show sumset {a} (sumset {b} (S ∩ G)) ⊆ sumset {a ⊕ b} (S ∩ G)
      using empty-iff insert-iff sumset.simps sumset-assoc by (smt (verit, best)
subsetI)
  qed
  then show a ⊕ b ∈ stabilizer S using haS hbS stabilizer-def by auto
next
  fix g assume g ∈ stabilizer S
  thus invertible g using stabilizer-def by auto
next
  fix g assume hgS: g ∈ stabilizer S
  then have hinvsom : inverse g ⊕ g = 0 using stabilizer-def by simp
  have sumset {inverse g} (sumset {g} (S ∩ G)) = (S ∩ G)
  proof
    show sumset {inverse g} (sumset {g} (S ∩ G)) ⊆ (S ∩ G) using
      empty-iff insert-iff sumset.simps sumset-assoc subsetI left-unit hinvsom
      by (smt (verit, ccfv-threshold))
    show (S ∩ G) ⊆ sumset {inverse g} (sumset {g} (S ∩ G))
    proof
      fix s assume hs: s ∈ (S ∩ G)
      then have inverse g ⊕ g ⊕ s ∈ sumset {inverse g} (sumset {g} (S ∩ G))
    using
      hgS stabilizer-def additive-abelian-group.sumset.sumsetI
      additive-abelian-group-axioms associative in-mono inverse-closed mem-Collect-eq
      singletonI
      by (smt (z3) IntD2)
    thus s ∈ sumset {inverse g} (sumset {g} (S ∩ G)) using hinvsom hs
      by simp
    qed
  qed
  thus inverse g ∈ stabilizer S using hgS stabilizer-def by auto
qed

```

interpretation *subgroup-of-additive-abelian-group stabilizer A G* (\oplus) **0**
using *stabilizer-is-subgroup subgroup-of-abelian-group-def*
by (*metis abelian-group-axioms additive-abelian-group-axioms group-axioms*
subgroup-of-additive-abelian-group-def subgroup-of-group-def)

lemma *zero-mem-stabilizer*: $\mathbf{0} \in \text{stabilizer } A$..

lemma *stabilizer-is-nonempty*:
shows *stabilizer* $S \neq \{\}$
using *sub-unit-closed* **by** *blast*

lemma *Left-Coset-eq-sumset-stabilizer*:
assumes $x \in G$
shows *sumset* $\{x\} (\text{stabilizer } B) = x \cdot | (\text{stabilizer } B)$
by (*simp add: Left-Coset-eq-sumset assms*)

lemma *stabilizer-subset-difference-singleton*:
assumes $S \subseteq G$ **and** $s \in S$
shows *stabilizer* $S \subseteq \text{differenceset } S \{s\}$

proof

fix x **assume** hx : $x \in \text{stabilizer } S$
then obtain t **where** ht : $t \in S$ **and** $x \oplus s = t$ **using** *assms stabilizer-def* **by**
blast
then have $x = t \ominus s$ **using** hx *stabilizer-def assms associative*
by (*metis (no-types, lifting) in-mono inverse-closed invertible invertible-right-inverse*
sub
sub.right-unit)
thus $x \in \text{differenceset } S \{s\}$ **using** ht *assms*
by (*simp add: minusset.minussetI subsetD sumset.sumsetI*)

qed

lemma *stabilizer-subset-singleton-difference*:
assumes $S \subseteq G$ **and** $s \in S$
shows *stabilizer* $S \subseteq \text{differenceset } \{s\} S$

proof–

have *stabilizer* $S \subseteq \text{minusset } (\text{stabilizer } S)$ **using** *assms Int-absorb2 minusset-eq*

subgroup.image-of-inverse submonoid.sub subset-eq

by (*smt (verit) stabilizer-is-subgroup stabilizer-subset-difference-singleton*
sumset-subset-carrier)

moreover have *minusset* $(\text{stabilizer } S) \subseteq \text{minusset } (\text{differenceset } S \{s\})$

proof

fix x **assume** $hx1$: $x \in \text{minusset } (\text{stabilizer } S)$

then have hx : *inverse* $x \in \text{stabilizer } S$

by (*metis invertible invertible-inverse-inverse minusset.cases*)

then obtain t **where** ht : $t \in S$ **and** *inverse* $x \oplus s = t$ **using** *assms stabilizer-def*

by *blast*

then have $hx2$: *inverse* $x = t \ominus s$ **using** hx *stabilizer-def assms*

by (*smt (verit, ccfv-threshold) commutative in-mono inverse-closed invertible*)

invertible-left-inverse2 sub
thus $x \in \text{minusset } (\text{differenceset } S \{s\})$ **using** *ht assms*
by (*smt (verit, best) hx1 additive-abelian-group.sumset.sumsetI additive-abelian-group-axioms*
inverse-closed invertible invertible-inverse-inverse minusset.cases minus-
set.minussetI
singletonI subset-iff)
qed
ultimately show *?thesis* **using** *differenceset-commute assms* **by** *blast*
qed

lemma *stabilizer-subset-nempty:*
assumes $S \neq \{\}$ **and** $S \subseteq G$
shows *stabilizer* $S \subseteq \text{differenceset } S S$
proof
fix x **assume** *hx: x ∈ stabilizer S*
then obtain s **where** *hs: s ∈ S ∩ G* **using** *assms* **by** *blast*
then have $x \oplus s \in S \cap G$ **using** *stabilizer-def assms hx mem-Collect-eq singletonI*
sumset.sumsetI sumset-Int-carrier **by** *blast*
then obtain t **where** *ht: t ∈ S* **and** $x \oplus s = t$ **by** *blast*
then have $x = t \ominus s$ **using** *hx stabilizer-def assms(2) hs ht associative*
by (*metis IntD2 inverse-closed invertible invertible-right-inverse sub sub.right-unit*)
thus $x \in \text{differenceset } S S$ **using** *ht hs* **using** *assms(2)* **by** *blast*
qed

lemma *stabilizer-coset-subset:*
assumes $A \subseteq G$ **and** $x \in A$
shows *sumset* $\{x\} (\text{stabilizer } A) \subseteq A$
proof
fix y **assume** *y ∈ sumset {x} (stabilizer A)*
moreover hence *stabilizer A ⊆ differenceset A {x}* **using** *assms*
stabilizer-subset-difference-singleton **by** *auto*
moreover have *sumset {x} (differenceset A {x}) ⊆ A*
proof
fix z **assume** *z ∈ sumset {x} (differenceset A {x})*
then obtain a **where** $a \in A$ **and** $z = x \oplus (a \ominus x)$
using *additive-abelian-group.sumset.cases additive-abelian-group-axioms singletonD*
minusset.cases assms subsetD **by** (*smt (verit, ccfv-SIG)*)
thus $z \in A$ **using** *assms*
by (*metis additive-abelian-group.inverse-closed additive-abelian-group-axioms commutative in-mono invertible invertible-right-inverse2*)
qed
ultimately show $y \in A$ **by** (*meson in-mono subset-singleton-iff sumset-mono*)
qed

lemma *stabilizer-subset-stabilizer-dvd:*

```

assumes stabilizer  $A \subseteq$  stabilizer  $B$ 
shows card (stabilizer  $A$ ) dvd card (stabilizer  $B$ )
proof(cases finite (stabilizer  $B$ ))
  case  $hB$ : True
    interpret  $H$ : subgroup-of-group stabilizer  $A$  stabilizer  $B$  ( $\oplus$ )  $\mathbf{0}$ 
    proof(unfold-locales)
      show stabilizer  $A \subseteq$  stabilizer  $B$  using assms by blast
    next
      show  $\bigwedge a b. a \in$  stabilizer  $A \implies b \in$  stabilizer  $A \implies a \oplus b \in$  stabilizer  $A$ 
        using stabilizer-is-subgroup group-axioms by simp
    next
      show  $\mathbf{0} \in$  stabilizer  $A$  using sub-unit-closed by blast
    qed
    show ?thesis using  $H.lagrange$   $hB$  by auto
  next
    case False
      then show ?thesis by simp
qed

```

```

lemma stabilizer-coset-Un:
  assumes  $A \subseteq G$ 
  shows  $(\bigcup x \in A. \text{sumset } \{x\} (\text{stabilizer } A)) = A$ 
proof
  show  $(\bigcup x \in A. \text{sumset } \{x\} (\text{stabilizer } A)) \subseteq A$ 
    using stabilizer-coset-subset assms by blast
  next
    show  $A \subseteq (\bigcup x \in A. \text{sumset } \{x\} (\text{stabilizer } A))$ 
    proof
      fix  $x$  assume  $hx: x \in A$ 
      then have  $x \in \text{sumset } \{x\} (\text{stabilizer } A)$  using sub-unit-closed assms
        by (metis in-mono right-unit singletonI sumset.sumsetI unit-closed)
      thus  $x \in (\bigcup x \in A. \text{sumset } \{x\} (\text{stabilizer } A))$  using  $hx$  by blast
    qed
qed

```

```

lemma stabilizer-empty: stabilizer  $\{\}$  =  $G$ 
  using sumset-empty Int-empty-left stabilizer-def subset-antisym
  by (smt (verit, best) mem-Collect-eq subsetI sumset-Int-carrier-eq(1) sumset-commute)

```

```

lemma stabilizer-finite:
  assumes  $S \subseteq G$  and  $S \neq \{\}$  and finite  $S$ 
  shows finite (stabilizer  $S$ )
  using stabilizer-subset-nempty assms
  by (meson finite-minusset finite-sumset rev-finite-subset)

```

```

lemma stabilizer-subset-group:
  shows stabilizer  $S \subseteq G$  using stabilizer-def by blast

```

```

lemma sumset-stabilizer-eq-iff:

```

assumes $a \in G$ **and** $b \in G$
shows $\text{sumset } \{a\} (\text{stabilizer } A) = \text{sumset } \{b\} (\text{stabilizer } A) \iff$
 $(\text{sumset } \{a\} (\text{stabilizer } A)) \cap (\text{sumset } \{b\} (\text{stabilizer } A)) \neq \{\}$
by (*simp add: assms sumset-subgroup-eq-iff*)

lemma *sumset-stabilizer-eq-Class-Union*:
assumes $A \subseteq G$
shows $\text{sumset } A (\text{stabilizer } B) = (\bigcup (\text{Class } B \text{ ' } A))$
by (*simp add: assms sumset-subgroup-eq-Class-Union*)

lemma *card-stabilizer-divide-sumset*:
assumes $A \subseteq G$
shows $\text{card } (\text{stabilizer } B) \text{ dvd } \text{card } (\text{sumset } A (\text{stabilizer } B))$
by (*simp add: assms card-divide-sumset*)

lemma *Class-image-sumset-stabilizer-eq*:
assumes $A \subseteq G$
shows $\text{Class } B \text{ ' } (\text{sumset } A (\text{stabilizer } B)) = \text{Class } B \text{ ' } A$
by (*simp add: Class-image-sumset-subgroup-eq assms*)

lemma *Class-cover-imp-subset-or-disj*:
assumes $A = (\bigcup (\text{Class } B \text{ ' } C))$ **and** $x \in G$ **and** $C \subseteq G$
shows $\text{Class } B \text{ } x \subseteq A \vee \text{Class } B \text{ } x \cap A = \{\}$
by (*simp add: Class-cover-imp-subset-or-disj assms*)

lemma *stabilizer-sumset-disjoint*:
fixes $S1 \ S2 :: 'a \text{ set}$
assumes $\text{stabilizer } S1 \cap \text{stabilizer } S2 = \{0\}$ **and** $S1 \subseteq G$ **and** $S2 \subseteq G$
and *finite* $S1$ **and** *finite* $S2$ **and** $S1 \neq \{\}$ **and** $S2 \neq \{\}$
shows $\text{card } (\text{sumset } (\text{stabilizer } S1) (\text{stabilizer } S2)) =$
 $\text{card } (\text{stabilizer } S1) * \text{card } (\text{stabilizer } S2)$

proof–

have *inj-on* : *inj-on* $(\lambda (a, b). a \oplus b)$ $(\text{stabilizer } S1 \times \text{stabilizer } S2)$

proof(*intro inj-onI*)

fix $x \ y$ **assume** $x \in \text{stabilizer } S1 \times \text{stabilizer } S2$ **and** $y \in \text{stabilizer } S1 \times \text{stabilizer } S2$ **and**

$(\text{case } x \text{ of } (a, b) \Rightarrow a \oplus b) = (\text{case } y \text{ of } (a, b) \Rightarrow a \oplus b)$

then obtain $a \ b \ c \ d$ **where** $hx: x = (a, b)$ **and** $hy: y = (c, d)$ **and** $ha: a \in \text{stabilizer } S1$ **and**

$hb: b \in \text{stabilizer } S2$ **and** $hc: c \in \text{stabilizer } S1$ **and** $hd: d \in \text{stabilizer } S2$ **and**
 $habcd: a \oplus b = c \oplus d$ **by** *auto*

then have $haG: a \in G$ **using** *stabilizer-def* **by** *blast*

have $hbG: b \in G$ **using** *hb stabilizer-def* **by** *blast*

have $hcG: c \in G$ **using** *hc stabilizer-def* **by** *blast*

have $hdG: d \in G$ **using** *hd stabilizer-def* **by** *blast*

then have $a \oplus c = d \oplus b$ **using** *habcd haG hbG hcG hdG*

by (*metis (full-types) associative commutative composition-closed inverse-equality invertible*

invertible-def invertible-left-inverse2)

moreover have $a \oplus c \in \text{stabilizer } S1$ **using** *ha hc stabilizer-is-subgroup
subgroup.axioms(1) submonoid.sub-composition-closed*
by *(metis group.invertible group-axioms hcG subgroup.subgroup-inverse-iff)*
moreover have $d \oplus b \in \text{stabilizer } S2$ **using** *hd hb stabilizer-is-subgroup
subgroup.axioms(1) submonoid.sub-composition-closed*
by *(metis group.invertible group-axioms hbG subgroup.subgroup-inverse-iff)*
ultimately have $a \oplus c = \mathbf{0}$ **and** $d \oplus b = \mathbf{0}$ **using** *assms(1)* **by** *auto*
thus $x = y$ **using** *hx hy haG hbG hcG hdG*
by *(metis inverse-closed invertible invertible-right-cancel invertible-right-inverse)*

qed

moreover have $\text{himage} : (\lambda (a, b). a \oplus b) ' (\text{stabilizer } S1 \times \text{stabilizer } S2) =$
 $\text{sumset } (\text{stabilizer } S1) (\text{stabilizer } S2)$

proof

show $(\lambda (a, b). a \oplus b) ' (\text{stabilizer } S1 \times \text{stabilizer } S2) \subseteq \text{sumset } (\text{stabilizer } S1)$
 $(\text{stabilizer } S2)$

using *stabilizer-subset-group* **by** *force*

next

show $\text{sumset } (\text{stabilizer } S1) (\text{stabilizer } S2) \subseteq (\lambda (a, b). a \oplus b) ' (\text{stabilizer } S1$
 $\times \text{stabilizer } S2)$

proof

fix x **assume** $x \in \text{sumset } (\text{stabilizer } S1) (\text{stabilizer } S2)$

then obtain $s1$ $s2$ **where** $hs1: s1 \in \text{stabilizer } S1$ **and** $hs2: s2 \in \text{stabilizer } S2$ **and**

$x = s1 \oplus s2$ **by** *(meson sumset.cases)*

thus $x \in (\lambda (a, b). a \oplus b) ' (\text{stabilizer } S1 \times \text{stabilizer } S2)$ **using** $hs1$ $hs2$ **by**
auto

qed

qed

ultimately show *?thesis* **using** *card-image card-cartesian-product* **by** *fastforce*

qed

lemma *stabilizer-sub-sumset-left:*

$\text{stabilizer } A \subseteq \text{stabilizer } (\text{sumset } A B)$

proof

fix x **assume** $hx: x \in \text{stabilizer } A$

then have $\text{sumset } \{x\} (\text{sumset } A B) = \text{sumset } A B$ **using** *stabilizer-def sum-*
set-assoc

mem-Collect-eq **by** *(smt (verit, del-insts) sumset-Int-carrier-eq(1) sumset-commute)*

thus $x \in \text{stabilizer } (\text{sumset } A B)$ **using** hx *stabilizer-def*

by *(metis (mono-tags, lifting) mem-Collect-eq sumset-Int-carrier)*

qed

lemma *stabilizer-sub-sumset-right:*

$\text{stabilizer } B \subseteq \text{stabilizer } (\text{sumset } A B)$

using *stabilizer-sub-sumset-left sumset-commute* **by** *fastforce*

lemma *not-mem-stabilizer-obtain:*

assumes $A \neq \{\}$ **and** $x \notin \text{stabilizer } A$ **and** $x \in G$ **and** $A \subseteq G$ **and** *finite* A
obtains a **where** $a \in A$ **and** $x \oplus a \notin A$
proof –
have $\text{sumset } \{x\} A \neq A$ **using** *assms stabilizer-def*
by (*metis (mono-tags, lifting) inf.orderE mem-Collect-eq*)
moreover **have** $\text{card } (\text{sumset } \{x\} A) = \text{card } A$ **using** *assms*
by (*metis card-sumset-singleton-eq inf.orderE sumset-commute*)
ultimately obtain y **where** $y \in \text{sumset } \{x\} A$ **and** $y \notin A$ **using** *assms*
by (*meson card-subset-eq subsetI*)
then obtain a **where** $a \in A$ **and** $x \oplus a \notin A$ **using** *assms*
by (*metis singletonD sumset.cases*)
thus *?thesis* **using** *that* **by** *blast*
qed

lemma *sumset-eq-sub-stabilizer*:

assumes $A \subseteq G$ **and** $B \subseteq G$ **and** *finite* B
shows $\text{sumset } A B = B \implies A \subseteq \text{stabilizer } B$
proof
fix x **assume** $hsum: \text{sumset } A B = B$ **and** $hx: x \in A$
have $\text{sumset } \{x\} B = B$
proof –
have $\text{sumset } \{x\} B \subseteq B$ **using** $hsum$ hx
by (*metis empty-subsetI equalityE insert-subset sumset-mono*)
moreover **have** $\text{card } (\text{sumset } \{x\} B) = \text{card } B$ **using** *assms*
by (*metis IntD1 Int-absorb1 card-sumset-singleton-eq hx inf-commute sum-set-commute*)
ultimately show *?thesis* **using** *card-subset-eq assms(3)* **by** *auto*
qed
thus $x \in \text{stabilizer } B$ **using** hx *assms(1)* *stabilizer-def*
by (*metis (mono-tags, lifting) assms(2) inf.orderE mem-Collect-eq subsetD*)
qed

lemma *sumset-stabilizer-eq*:

shows $\text{sumset } (\text{stabilizer } A) (\text{stabilizer } A) = \text{stabilizer } A$
proof
show $\text{sumset } (\text{stabilizer } A) (\text{stabilizer } A) \subseteq \text{stabilizer } A$
using *stabilizer-is-subgroup subgroup.axioms(1) subsetI*
by (*metis (mono-tags, lifting) additive-abelian-group.sumset.simps additive-abelian-group-axioms*
submonoid.sub-composition-closed)
next
show $\text{stabilizer } A \subseteq \text{sumset } (\text{stabilizer } A) (\text{stabilizer } A)$
using *Left-Coset-eq-sumset stabilizer-is-nonempty*
stabilizer-subset-group sub-unit-closed additive-abelian-group-axioms right-unit
subset-iff sumsetI **by** (*smt (verit, best)*)
qed

```

lemma differenceset-stabilizer-eq:
  shows differenceset (stabilizer A) (stabilizer A) = stabilizer A
proof
  show differenceset (stabilizer A) (stabilizer A) ⊆ stabilizer A
  proof
    fix x assume x ∈ differenceset (stabilizer A) (stabilizer A)
    then obtain a b where a ∈ stabilizer A and b ∈ stabilizer A and x = a ⊖ b
      by (metis minusset.cases sumset.cases)
    thus x ∈ stabilizer A using stabilizer-is-subgroup subgroup.axioms(1)
      by (smt (verit, ccfv-threshold) in-mono invertible stabilizer-subset-group
        subgroup-inverse-iff sub-composition-closed)
  qed
next
  show stabilizer A ⊆ differenceset (stabilizer A) (stabilizer A)
  proof
    fix x assume hx: x ∈ stabilizer A
    then have x ⊖ 0 ∈ differenceset (stabilizer A) (stabilizer A) by blast
    then show x ∈ differenceset (stabilizer A) (stabilizer A) using hx by simp
  qed
qed

lemma stabilizer2-sub-stabilizer:
  shows stabilizer(stabilizer A) ⊆ stabilizer A
proof(cases A ≠ {})
  case True
    then have stabilizer(stabilizer A) ⊆ differenceset (stabilizer A) (stabilizer A)
      by (simp add: stabilizer-is-nonempty stabilizer-subset-group stabilizer-subset-nempty)
    thus ?thesis using differenceset-stabilizer-eq by blast
  next
  case False
    then show ?thesis by (simp add: stabilizer-empty stabilizer-subset-group)
qed

lemma stabilizer-left-sumset-invariant:
  assumes a ∈ G and A ⊆ G
  shows stabilizer (sumset {a} A) = stabilizer A

proof
  show stabilizer (sumset {a} A) ⊆ stabilizer A
  proof
    fix x assume hx: x ∈ stabilizer (sumset {a} A)
    then have hxG: x ∈ G using stabilizer-def by blast
    have sumset {x} (sumset {a} A) = sumset {a} A using stabilizer-def hx
      by (metis (mono-tags, lifting) mem-Collect-eq sumset-Int-carrier)
    then have sumset {x} A = A using assms
      by (metis (full-types) sumset-assoc sumset-commute sumset-subset-carrier
        sumset-translate-eq-right)
    thus x ∈ stabilizer A using hxG stabilizer-def
  
```

by (metis (mono-tags, lifting) mem-Collect-eq sumset-Int-carrier)
 qed
 next
 show stabilizer $A \subseteq$ stabilizer (sumset $\{a\}$ A) using stabilizer-def
 using stabilizer-sub-sumset-right by meson
 qed

lemma stabilizer-right-sumset-invariant:
 assumes $a \in G$ and $A \subseteq G$
 shows stabilizer (sumset $A \{a\}$) = stabilizer A
 using sumset-commute stabilizer-left-sumset-invariant assms by simp

lemma stabilizer-right-differenceset-invariant:
 assumes $b \in G$ and $A \subseteq G$
 shows stabilizer (differenceset $A \{b\}$) = stabilizer A
 using assms minuset-eq stabilizer-right-sumset-invariant by auto

lemma stabilizer-unchanged:
 assumes $a \in G$ and $b \in G$
 shows stabilizer (sumset $A B$) = stabilizer (sumset A (sumset (differenceset $B \{b\}$) $\{a\}$))

proof –
 have sumset A (sumset (differenceset $B \{b\}$) $\{a\}$) = sumset (differenceset (sumset $A B$) $\{b\}$) $\{a\}$
 by (simp add: sumset-assoc)
 thus ?thesis using stabilizer-right-sumset-invariant
 stabilizer-right-differenceset-invariant assms sumset-subset-carrier by simp
 qed

lemma subset-stabilizer-of-subset-sumset:
 assumes $A \subseteq$ sumset $\{x\}$ (stabilizer B) and $x \in G$ and $A \neq \{\}$ and $A \subseteq G$
 shows stabilizer $A \subseteq$ stabilizer B

proof –
 obtain a where $ha: a \in A$ using assms by blast
 moreover then obtain b where $hb: b \in$ stabilizer B and $haxb: a = x \oplus b$
 using sumset.cases assms by blast
 ultimately have stabilizer $A \subseteq$ differenceset $A \{a\}$ using assms sumset-subset-carrier

stabilizer-subset-difference-singleton by (meson subset-trans)
 also have ... = sumset $\{inverse\ a\}$ A using sumset-commute ha assms(4) inverse-closed

subsetD minuset-eq by auto
 also have ... \subseteq sumset $\{inverse\ x \oplus inverse\ b\}$ (sumset $\{x\}$ (stabilizer B))
 using assms sumset-mono $haxb$ inverse-closed hb stabilizer-subset-group subsetD

commutative inverse-composition-commute by (metis invertible subset-singleton-iff)
 also have ... = sumset $\{inverse\ b\}$ (stabilizer B) using sumset-singletons-eq

commutative
assms sumset-assoc hb stabilizer-subset-group inverse-closed invertible
by (*metis composition-closed invertible-right-inverse2 sub*)
also have $\dots = \text{stabilizer } B$ **using** *hb Left-Coset-eq-sumset sub-unit-closed sub subset-iff*
additive-abelian-group-axioms calculation disjoint-iff-not-equal factor-unit inverse-closed
sumset-subgroup-eq-iff **by** (*smt (verit, del-insts)*)
finally show *?thesis* .
qed

lemma *sumset-stabilizer-eq-self*:
assumes $A \subseteq G$
shows *sumset (stabilizer A) A = A*
using *assms sumset-eq-Union-left[OF stabilizer-subset-group]*
Int-absorb2 stabilizer-coset-Un sumset-commute sumset-eq-Union-left **by** *presburger*

lemma *stabilizer-neq-subset-sumset*:
assumes $A \subseteq \text{sumset } \{x\}$ (*stabilizer B*) **and** $x \in A$ **and** $\neg \text{sumset } \{x\}$ (*stabilizer B*) $\subseteq C$ **and**
 $A \subseteq C$ **and** $C \subseteq G$
shows *stabilizer A \neq stabilizer B*
proof
assume *heq: stabilizer A = stabilizer B*
obtain *a* **where** $a \in \text{sumset } \{x\}$ (*stabilizer B*) **and**
 $a \notin C$ **using** *assms* **by** *blast*
moreover then obtain *b* **where** $b \in \text{stabilizer B}$ **and** $a = x \oplus b$ **using** *sumset.cases* **by** *blast*
ultimately have $b \oplus x \notin A$ **using** *commutative stabilizer-subset-group assms in-mono* **by** *metis*
thus *False* **using** *assms heq stabilizer-coset-subset subset-trans* **by** *metis*
qed

lemma *subset-stabilizer-Un*:
shows *stabilizer A \cap stabilizer B \subseteq stabilizer (A \cup B)*
proof
fix *x* **assume** *hx: x \in stabilizer A \cap stabilizer B*
then have *sumset {x} (A \cap G) = A \cap G* **using** *stabilizer-def* **by** *blast*
moreover have *sumset {x} (B \cap G) = (B \cap G)* **using** *stabilizer-def hx* **by** *blast*
ultimately have *sumset {x} ((A \cup B) \cap G) = (A \cup B) \cap G* **using** *sumset-subset-Un2*
boolean-algebra.conj-disj-distrib2 **by** *auto*
then show $x \in \text{stabilizer (A \cup B)}$ **using** *hx stabilizer-subset-group stabilizer-def*
by *blast*
qed

lemma *mem-stabilizer-Un-and-left-imp-right*:
assumes *finite B* **and** $x \in \text{stabilizer (A \cup B)}$ **and** $x \in \text{stabilizer A}$ **and** *disjnt A*

B
shows $x \in \text{stabilizer } B$
proof –
have $(A \cap G) \cup \text{sumset } \{x\} (B \cap G) = (A \cap G) \cup (B \cap G)$
using *assms(2) sumset-subset-Un2[of {x} A ∩ G B ∩ G] stabilizer-def[of A ∪ B]*
Int-Un-distrib2[of A B G] assms(3) stabilizer-def
by (*metis (mono-tags, lifting) mem-Collect-eq*)
then have $B \cap G \subseteq \text{sumset } \{x\} (B \cap G)$ **using** *assms(4) disjnt-def Int-Un-distrib2 Int-commute*
sumset-subset-Un1 **by** (*smt (verit, del-insts) Int-assoc Un-Int-eq(2) inf.orderI insert-is-Un*
sumset-empty(2))
then show $x \in \text{stabilizer } B$ **using** *stabilizer-def[of B] assms(1) assms(3) card-subset-eq*

card-sumset-singleton-subset-eq finite.emptyI finite.insertI finite-Int finite-sumset

inf.cobounded2 stabilizer-subset-group subsetD **by** (*smt (verit) mem-Collect-eq*)
qed

lemma *mem-stabilizer-Un-and-right-imp-left:*
assumes *finite A* **and** $x \in \text{stabilizer } (A \cup B)$ **and** $x \in \text{stabilizer } B$ **and** *disjnt A B*
shows $x \in \text{stabilizer } A$
using *mem-stabilizer-Un-and-left-imp-right Un-commute assms disjnt-sym* **by** *metis*

lemma *Union-stabilizer-Class-eq:*
assumes $A \subseteq G$
shows $A = (\bigcup (\text{Class } A ' A))$ **using** *assms sumset-commute sumset-subgroup-eq-Class-Union*

sumset-stabilizer-eq-self **by** *presburger*

lemma *card-stabilizer-sumset-divide-sumset:*
card (stabilizer (sumset A B)) dvd card (sumset A B) **using** *card-divide-sumset sumset-commute sumset-stabilizer-eq-self sumset-subset-carrier* **by** *metis*

lemma *card-stabilizer-le:*
assumes $A \subseteq G$ **and** *finite A* **and** $A \neq \{\}$
shows $\text{card } (\text{stabilizer } A) \leq \text{card } A$ **using** *assms*
by (*metis card-le-sumset finite.cases insertCI insert-subset stabilizer-finite stabilizer-subset-group sumset-commute sumset-stabilizer-eq-self*)

lemma *sumset-Inter-subset-sumset:*
assumes $a \in G$ **and** $b \in G$
shows $\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C))$
 \subseteq
 $\text{sumset } \{a \oplus b\} (\text{stabilizer } C)$ (**is** $\text{sumset } ?A ?B \subseteq -$)
proof

fix x **assume** $x \in \text{sumset } ?A \ ?B$
then obtain $d1 \ d2$ **where** $d1 \in \text{sumset } \{a\}$ (*stabilizer* C) **and**
 $d2 \in \text{sumset } \{b\}$ (*stabilizer* C) **and** $x = d1 \oplus d2$ **by** (*meson IntD2 sumset.cases*)
then obtain $c1 \ c2$ **where** $hc1: c1 \in \text{stabilizer } C$ **and** $hc2: c2 \in \text{stabilizer } C$
and
 $x = (a \oplus c1) \oplus (b \oplus c2)$ **using** *sumset.simps* **by** *auto*
then have $x = (a \oplus b) \oplus (c1 \oplus c2)$ **using** $hc1 \ hc2$ *assms associative commutative*

stabilizer-subset-group **by** *simp*
thus $x \in \text{sumset } \{a \oplus b\}$ (*stabilizer* C) **using** *stabilizer-is-subgroup hc1 hc2*
stabilizer-subset-group sumset.simps sumset-stabilizer-eq assms **by** *blast*
qed

1.3 Convergent

definition *convergent* $:: 'a \text{ set} \Rightarrow 'a \text{ set} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$ **where**
convergent $C \ A \ B \equiv C \subseteq \text{sumset } A \ B \wedge C \neq \{\}$ \wedge
 $\text{card } C + \text{card } (\text{stabilizer } C) \geq \text{card } (A \cap B) + \text{card } (\text{sumset } (A \cup B) \ (\text{stabilizer } C))$

definition *convergent-set* $:: 'a \text{ set} \Rightarrow 'a \text{ set} \Rightarrow 'a \text{ set set}$ **where**
convergent-set $A \ B = \text{Collect } (\lambda \ C. \text{convergent } C \ A \ B)$

lemma *convergent-set-sub-powerset*:
convergent-set $A \ B \subseteq \text{Pow } (\text{sumset } A \ B)$ **using** *convergent-set-def convergent-def*
by *blast*

lemma *finite-convergent-set*:
assumes *finite* A **and** *finite* B
shows *finite* (*convergent-set* $A \ B$)
using *convergent-set-sub-powerset finite-Pow-iff finite-sumset assms finite-subset*
by *metis*

1.4 Technical lemmas from DeVos's proof of Kneser's Theorem

The following lemmas correspond to intermediate arguments in the proof of Kneser's Theorem by DeVos that we will be following [1].

lemma *stabilizer-sumset-psubset-stabilizer*:
assumes $a \in G$ **and** $b \in G$ **and** $A \cap \text{sumset } \{a\}$ (*stabilizer* C) $\neq \{\}$ **and**
 $B \cap \text{sumset } \{b\}$ (*stabilizer* C) $\neq \{\}$ **and** $\text{hnotsub: } \neg \text{sumset } \{a \oplus b\}$ (*stabilizer* C) $\subseteq \text{sumset } A \ B$
shows *stabilizer* (*sumset* ($A \cap \text{sumset } \{a\}$ (*stabilizer* C)) ($B \cap \text{sumset } \{b\}$ (*stabilizer* C))) \subset
stabilizer C (**is** $?H \subset -$)
proof
have *sumset* ($A \cap \text{sumset } \{a\}$ (*stabilizer* C)) ($B \cap \text{sumset } \{b\}$ (*stabilizer* C)) $\neq \{\}$
using *assms* **by** (*simp add: inf-assoc*)

then show $?H \subseteq \text{stabilizer } C$
by (*meson* *assms*(1) *assms*(2) *composition-closed subset-stabilizer-of-subset-sumset* *sumset-Inter-subset-sumset* *sumset-subset-carrier*)
next
obtain $c1\ c2$ **where** $a \oplus c1 \in A$ **and** $b \oplus c2 \in B$ **and** $hc1: c1 \in \text{stabilizer } C$
and $hc2: c2 \in \text{stabilizer } C$
using *assms*(1, 2, 3, 4) *Left-Coset-eq-sumset-stabilizer* **by** *fastforce*
then have $hc1mem: (a \oplus c1) \in A \cap \text{sumset } \{a\}$ (*stabilizer } C*) **and** $hc1G: a \oplus c1 \in G$ **and**
 $hc2mem: (b \oplus c2) \in B \cap \text{sumset } \{b\}$ (*stabilizer } C*) **and** $hc2G: b \oplus c2 \in G$
by (*auto simp add: assms*(1, 2) *sumset.sumsetI*)
have $(a \oplus c1) \oplus (b \oplus c2) \in \text{sumset } \{a \oplus b\}$ (*stabilizer } C*) **using** *assms* $hc1\ hc2$
by (*smt* (*verit*) *associative commutative composition-closed insertI1* *sub sub-composition-closed* *sumset.sumsetI*)
then have $\text{sumset } \{a \oplus b\}$ (*stabilizer } C*) \cap $\text{sumset } \{(a \oplus c1) \oplus (b \oplus c2)\}$
(*stabilizer } C*) $\neq \{\}$
using *zero-mem-stabilizer* **by** (*smt* (*verit*, *ccfv-threshold*) *composition-closed* *disjoint-iff-not-equal* $hc1G\ hc2G$ *insertCI* *right-unit* *sumset.sumsetI* *unit-closed*)
then have $hsumeq: \text{sumset } \{a \oplus b\}$ (*stabilizer } C*) $= \text{sumset } \{(a \oplus c1) \oplus (b \oplus c2)\}$
(*stabilizer } C*)
using *sumset-stabilizer-eq-iff* *assms* $hc1G\ hc2G$ *composition-closed* **by** *presburger*
have $(\text{sumset } (A \cap \text{sumset } \{a\}$ (*stabilizer } C*)) $(B \cap \text{sumset } \{b\}$ (*stabilizer } C*)))
 $\subseteq \text{sumset } \{a \oplus b\}$ (*stabilizer } C*)
by (*simp add: assms*(1, 2) *sumset-Inter-subset-sumset*)
have $hsummem: (a \oplus c1) \oplus (b \oplus c2) \in \text{sumset } (A \cap \text{sumset } \{a\}$ (*stabilizer } C*))
 $(B \cap \text{sumset } \{b\}$ (*stabilizer } C*))
using $hc1mem\ hc2mem\ hc1G\ hc2G\ \text{sumset.sumsetI}$ **by** *blast*
show $?H \neq \text{stabilizer } C$
using *stabilizer-neq-subset-sumset*[*OF - hsummem*] *hnotsub* $hsumeq$ *sumset-Inter-subset-sumset* *assms*
sumset-subset-carrier *composition-closed* *sumset-mono* *sumset.sumsetI* *zero-mem-stabilizer*

inf.cobounded1 *right-unit* *unit-closed* **by** *metis*
qed

lemma *stabilizer-eq-stabilizer-union*:

assumes $a \in G$ **and** $b \in G$ **and** $A \cap \text{sumset } \{a\}$ (*stabilizer } C*) $\neq \{\}$ **and**
 $B \cap \text{sumset } \{b\}$ (*stabilizer } C*) $\neq \{\}$ **and** *hnotsub*: $\neg \text{sumset } \{a \oplus b\}$ (*stabilizer } C*)
 $\subseteq \text{sumset } A\ B$ **and**
 $C \subseteq \text{sumset } A\ B$ **and** *finite } C* **and**
 $C \cap \text{sumset } (A \cap \text{sumset } \{a\}$ (*stabilizer } C*)) $(B \cap \text{sumset } \{b\}$ (*stabilizer } C*)) $=$
 $\{\}$ **and** $C \neq \{\}$ **and**
finite } A **and** *finite } B*
shows *stabilizer* ($\text{sumset } (A \cap \text{sumset } \{a\}$ (*stabilizer } C*)) $(B \cap \text{sumset } \{b\}$
(*stabilizer } C*))) $=$
stabilizer ($C \cup \text{sumset } (A \cap \text{sumset } \{a\}$ (*stabilizer } C*)) $(B \cap \text{sumset } \{b\}$
(*stabilizer } C*))) **(is** *stabilizer } ?H = stabilizer } ?K*)
proof

```

show stabilizer ?H  $\subseteq$  stabilizer ?K using subset-stabilizer-Un Int-absorb1
  stabilizer-sumset-psubset-stabilizer psubset-imp-subset assms by metis
next
  have hCG : C  $\subseteq$  G using assms(6) sumset-subset-carrier by force
  show stabilizer ?K  $\subseteq$  stabilizer ?H
  proof
    fix x assume hxC1: x  $\in$  stabilizer ?K
    moreover have x  $\in$  stabilizer C
    proof–
      have hC-Un: C = ( $\bigcup$  (Class C ‘ C)) using Union-stabilizer-Class-eq hCG
    by simp
      have hCsumx: sumset {x} C = ( $\bigcup$  y  $\in$  Class C ‘ C. sumset {x} y)
      proof
        show sumset {x} C  $\subseteq$   $\bigcup$  (sumset {x} ‘ Class C ‘ C)
        proof
          fix y assume hy: y  $\in$  sumset {x} C
          then obtain c where hc: c  $\in$  C and hyc: y = x  $\oplus$  c using sumset.cases
        by blast
          then obtain K where hK: K  $\in$  Class C ‘ C and c  $\in$  K using hC-Un
        by blast
          then have y  $\in$  sumset {x} K using hyc hc by (metis sumset.cases
            sumset.sumsetI hCG hy singletonD subset-iff)
          then show y  $\in$   $\bigcup$  (sumset {x} ‘ Class C ‘ C) using hK by auto
        qed
      next
        show  $\bigcup$  (sumset {x} ‘ Class C ‘ C)  $\subseteq$  sumset {x} C
        proof(intro Union-least)
          fix X assume X  $\in$  sumset {x} ‘ Class C ‘ C
          then obtain K where K  $\in$  Class C ‘ C and X = sumset {x} K by blast
          then show X  $\subseteq$  sumset {x} C using sumset-mono[of {x} {x} K C]
            hC-Un subset-refl by blast
          qed
        qed
        have x  $\notin$  stabilizer C  $\implies$  False
        proof–
          assume hxC: x  $\notin$  stabilizer C
          then have hxG: x  $\in$  G using hxC1 stabilizer-subset-group by blast
          then have hxCne: sumset {x} C  $\neq$  C using stabilizer-def[of C] hCG
            Int-absorb2
          hxC by (metis (mono-tags, lifting) mem-Collect-eq)
          moreover have hxsplit: sumset {x} C  $\cup$  sumset {x} ?H = C  $\cup$  ?H
          using hxC1 stabilizer-def[of ?K] sumset-subset-carrier assms(6) sum-
            set-subset-Un2 by force
          have sumset {x} C  $\cap$  ?H  $\neq$  {}
          proof
            assume sumset {x} C  $\cap$  ?H = {}
            then have sumset {x} C  $\subset$  C using hxsplit hxCne by blast
            thus False using hCG assms(6) assms(7) hxC1 stabilizer-subset-group
              psubset-card-mono

```

by (*metis card-sumset-singleton-eq sumset-Int-carrier sumset-commute
 sumset-stabilizer-eq-self hxG less-irrefl-nat*)
 qed
 then obtain c where $hc: c \in C$ and
 $hxcne: \text{sumset } \{x\} (Class\ C\ c) \cap ?H \neq \{\}$ using $hCsumx$ by *blast*
 then have $hxc: \text{sumset } \{x\} (Class\ C\ c) = Class\ C\ (x \oplus c)$
 using hxG *assms(6) Left-Coset-Class-unit Left-Coset-eq-sumset-stabilizer
 sumset-assoc
 sumset-singletons-eq composition-closed sumset.cases sumset-stabilizer-eq-self
 hCG* by (*smt (verit)*)
 have $hClassEmpty: Class\ C\ (x \oplus c) \cap C = \{\}$
 proof–
 have $\neg Class\ C\ (x \oplus c) \subseteq C$ using $hxc\ hxcne\ assms(8)$ by *blast*
 then show *?thesis* using *Class-cover-imp-subset-or-disj[OF hC-Un - hCG]*
 by (*meson composition-closed hCG hc hxG subsetD*)
 qed
 have $Class\ C\ (x \oplus c) \subseteq \text{sumset } \{x\}\ C$ using $hCsumx\ hc\ hxc$ by *blast*
 then have $Class\ C\ (x \oplus c) \subseteq ?H$ using $hClassEmpty\ hxsplit$ by *auto*
 moreover have $\text{card } (Class\ C\ (x \oplus c)) = \text{card } (\text{stabilizer } C)$ using $hxG\ hc$
*hCG
 composition-closed Right-Coset-Class-unit Right-Coset-cardinality sum-
 set-Int-carrier
 Class-cover-imp-subset-or-disj assms* by *auto*
 ultimately have $\text{card } (\text{stabilizer } C) \leq \text{card } ?H$ using *card-mono finite-sumset
 assms(10, 11) finite-Int* by *metis*
 moreover have $\text{card } ?H < \text{card } (\text{sumset } \{a \oplus b\} (\text{stabilizer } C))$
 proof (*intro psubset-card-mono psubsetI sumset-Inter-subset-sumset assms(1)
 assms(2)*)
 show *finite (sumset {a ⊕ b} (stabilizer C))*
 using *stabilizer-finite assms finite-sumset* by (*simp add: hCG*)
 next
 show $?H \neq \text{sumset } \{a \oplus b\} (\text{stabilizer } C)$
 using *hnotsub sumset-mono* by (*metis Int-lower1*)
 qed
 ultimately show *False*
 using *assms(1, 2) stabilizer-subset-group* by (*simp add: card-sumset-singleton-subset-eq*)

 qed
 then show *?thesis* by *auto*
 qed
 moreover have *finite ?H* using *finite-sumset assms(10, 11) finite-Int* by *simp*
 ultimately show $x \in \text{stabilizer } ?H$ using *mem-stabilizer-Un-and-right-imp-left[of
 ?H x C]*
disjnt-def assms Un-commute by (*metis disjoint-iff-not-equal*)
 qed
 qed

 lemma *sumset-inter-ineq*:
 assumes $B \cap \text{sumset } \{a\} (\text{stabilizer } C) = \{\}$ and $\text{stabilizer } (\text{sumset } (A \cap \text{sumset }$

$\{a\} (\text{stabilizer } C) (B \cap \text{sumset } \{b\} (\text{stabilizer } C)) \subseteq \text{stabilizer } C$ **and**
 $a \in A$ **and** $a \in G$ **and** *finite* A **and** *finite* B **and** $A \neq \{\}$ **and** $B \neq \{\}$ **and** *finite*
 $(\text{stabilizer } C)$
shows $\text{int} (\text{card} (\text{sumset } (A \cup B) (\text{stabilizer } C))) - \text{card} (\text{sumset } (A \cup B)$
 $(\text{stabilizer } (\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C)))) \geq$
 $\text{int} (\text{card} (\text{stabilizer } C)) - \text{card} (\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C))$
 $(\text{stabilizer } (\text{sumset } (A \cap \text{sumset } \{a\} (\text{stabilizer } C)) (B \cap \text{sumset } \{b\} (\text{stabilizer } C))))$
(is $\text{int} (\text{card} (\text{sumset } (A \cup B) (\text{stabilizer } C))) - \text{card} (\text{sumset } (A \cup B) ?H1) \geq$
 $\text{int} (\text{card} (\text{stabilizer } C)) - \text{card} (\text{sumset } ?A1 ?H1)$

proof –

have $\text{hfinsumH1} : \text{finite} (\text{sumset } (A \cup B) ?H1)$
using *finite-sumset* *assms* **by** (*meson* *finite-Un* *psubsetE* *rev-finite-subset*)
have $\text{hsubsumH1} : \text{sumset } (A \cup B) ?H1 \subseteq \text{sumset } (A \cup B) (\text{stabilizer } C)$
using *sumset.cases* *assms* **by** (*meson* *psubsetE* *subset-refl* *sumset-mono*)
have $\text{hsumH1card-le} : \text{card} (\text{sumset } (A \cup B) ?H1) \leq \text{card} (\text{sumset } (A \cup B)$
 $(\text{stabilizer } C))$
using *card-mono* *finite-sumset* *stabilizer-finite* *assms*
by (*metis* *equalityE* *finite-UnI* *psubset-imp-subset* *sumset-mono*)
have $\text{hsub} : \text{sumset } ?A1 ?H1 \subseteq \text{sumset } \{a\} (\text{stabilizer } C)$

proof

fix x **assume** $x \in \text{sumset } ?A1 ?H1$
then obtain $h1$ f **where** $h1 \in ?A1$ **and** $hf : f \in ?H1$ **and**
 $hx : x = h1 \oplus f$ **by** (*meson* *sumset.cases*)
then obtain c **where** $hc : c \in \text{stabilizer } C$ **and** $hac : h1 = a \oplus c$
by (*metis* *Int-iff* *empty-iff* *insert-iff* *sumset.cases*)
then have $hcf : c \oplus f \in \text{stabilizer } C$ **using** hf *assms*(2) *stabilizer-is-subgroup*
subgroup-def *monoid-axioms* *Group-Theory.group.axioms*(1)
Group-Theory.monoid-def *subset-iff* *psubset-imp-subset*
by (*smt* (*verit*) *stabilizer-subset-group* *sumset.sumsetI* *sumset-stabilizer-eq*)
have $hcG : c \in G$ **using** hc *stabilizer-subset-group* **by** *auto*
have $hfG : f \in G$ **using** hf *stabilizer-subset-group* **by** *auto*
show $x \in \text{sumset } \{a\} (\text{stabilizer } C)$ **using** hx hac *assms* *stabilizer-subset-group*
 hcf
using *Left-Coset-eq-sumset-stabilizer* *Left-Coset-memI* *associative* hcG hfG

by *presburger*

qed

moreover have *finite* $(\text{sumset } ?A1 ?H1)$ **using** *finite-sumset* *assms* *stabilizer-finite*
finite-subset
by (*metis* *finite.simps* $hsub$)
ultimately have $\text{card} (\text{sumset } \{a\} (\text{stabilizer } C)) - \text{card} (\text{sumset } ?A1 ?H1) =$
 $\text{card} (\text{sumset } \{a\} (\text{stabilizer } C)) - \text{sumset } ?A1 ?H1)$
using *card-Diff-subset* **by** *metis*
moreover have $\text{card} (\text{sumset } ?A1 ?H1) \leq \text{card} (\text{sumset } \{a\} (\text{stabilizer } C))$
using *card-mono* $hsub$ *finite-sumset* *assms* **by** (*metis* *finite.simps*)
ultimately have $\text{int} (\text{card} (\text{sumset } \{a\} (\text{stabilizer } C))) - \text{card} (\text{sumset } ?A1$
 $?H1) =$
 $\text{card} (\text{sumset } \{a\} (\text{stabilizer } C)) - \text{sumset } ?A1 ?H1)$ **by** *linarith*

also have $\dots \leq \text{card} ((\text{sumset } (A \cup B) (\text{stabilizer } C)) - (\text{sumset } (A \cup B) ?H1))$
proof–
have $\text{sumset } \{a\} (\text{stabilizer } C) - \text{sumset } ?A1 ?H1 \subseteq \text{sumset } (A \cup B) (\text{stabilizer } C) - \text{sumset } (A \cup B) ?H1$
proof
fix x **assume** $hx: x \in \text{sumset } \{a\} (\text{stabilizer } C) - \text{sumset } ?A1 ?H1$
then obtain c **where** $hxac: x = a \oplus c$ **and** $hc: c \in \text{stabilizer } C$ **and** $hcG: c \in G$
using *sumset.cases* **by** *blast*
then have $x \in \text{sumset } (A \cup B) (\text{stabilizer } C)$ **using** *assms sumset.cases* **by** *blast*
moreover have $x \notin \text{sumset } (A \cup B) ?H1$
proof
assume $x \in \text{sumset } (A \cup B) ?H1$
then obtain y $h1$ **where** $hy: y \in A \cup B$ **and** $hyG: y \in G$ **and** $hh1G: h1 \in G$
and $hh1: h1 \in ?H1$ **and** $hxy: x = y \oplus h1$ **by** (*meson sumset.cases*)
then have $y = a \oplus (c \oplus \text{inverse } h1)$ **using** $hxac$ hxy *assms associative commutative composition-closed inverse-closed invertible invertible-left-inverse2* **by** (*metis hcG*)
moreover have $h1 \in \text{stabilizer } C$ **using** $hh1$ *assms* **by** *auto*
moreover hence $c \oplus \text{inverse } h1 \in \text{stabilizer } C$ **using** hc *stabilizer-is-subgroup subgroup-def group-axioms invertible subgroup.subgroup-inverse-iff submonoid.sub-composition-closed*
 $hh1G$ **by** *metis*
ultimately have $y \in \text{sumset } \{a\} (\text{stabilizer } C)$
using *assms hcG hh1G* **by** *blast*
moreover hence $y \in A$ **using** *assms(1) hy* **by** *auto*
ultimately have $x \in \text{sumset } ?A1 ?H1$ **using** hxy $hh1$
by (*simp add: hyG hh1G sumset.sumsetI*)
thus *False* **using** hx **by** *auto*
qed
ultimately show $x \in \text{sumset } (A \cup B) (\text{stabilizer } C) - \text{sumset } (A \cup B) ?H1$
by *simp*
qed
thus *?thesis* **using** *card-mono finite-Diff finite-sumset assms*
by (*metis finite-UnI nat-int-comparison(3)*)
qed
also have $\dots = \text{int} (\text{card} (\text{sumset } (A \cup B) (\text{stabilizer } C))) - \text{card} (\text{sumset } (A \cup B) ?H1)$
using *card-Diff-subset[OF hfinsumH1 hsubsumH1] hsumH1card-le* **by** *linarith*
finally show $\text{int} (\text{card} (\text{sumset } (A \cup B) (\text{stabilizer } C))) - \text{card} (\text{sumset } (A \cup B) ?H1) \geq \text{int} (\text{card} (\text{stabilizer } C)) - \text{card} (\text{sumset } ?A1 ?H1)$
using *assms* **by** (*metis card-sumset-singleton-subset-eq stabilizer-subset-group*)
qed

lemma *exists-convergent-min-stabilizer:*

assumes *hind*: $\forall m < n. \forall C D. C \subseteq G \longrightarrow D \subseteq G \longrightarrow \text{finite } C \longrightarrow \text{finite } D \longrightarrow$

$C \neq \{\}$ \longrightarrow
 $D \neq \{\}$ \longrightarrow $\text{card}(\text{sumset } C D) + \text{card } C = m \longrightarrow$
 $\text{card}(\text{sumset } C (\text{stabilizer}(\text{sumset } C D))) + \text{card}(\text{sumset } D (\text{stabilizer}(\text{sumset } C D))) -$
 $\text{card}((\text{stabilizer}(\text{sumset } C D)))$
 $\leq \text{card}(\text{sumset } C D)$ **and** $hAG: A \subseteq G$ **and** $hBG: B \subseteq G$ **and** $hA: \text{finite } A$
and
 $hB: \text{finite } B$ **and** $hAne: A \neq \{\}$ **and** $A \cap B \neq \{\}$ **and**
 $hcardsum: \text{card}(\text{sumset } A B) + \text{card } A = n$ **and** $hintercardA: \text{card}(A \cap B) <$
 $\text{card } A$
obtains X **where** $\text{convergent } X A B$ **and** $\bigwedge Y. Y \in \text{convergent-set } A B \implies$
 $\text{card}(\text{stabilizer } Y) \geq \text{card}(\text{stabilizer } X)$
proof –
let $?C0 = \text{sumset}(A \cap B)(A \cup B)$
have $hC0ne: ?C0 \neq \{\}$ **using** assms **by** fast
moreover **have** $\text{finite } ?C0$ **using** $\text{sumset-inter-union-subset finite-sumset assms}$
by auto
ultimately **have** $\text{finite}(\text{stabilizer } ?C0)$ **using** stabilizer-finite
using $\text{sumset-subset-carrier}$ **by** presburger
then **have** $hcard\text{-sumset-le}: \text{card}(A \cap B) \leq \text{card}(\text{sumset}(A \cap B)(\text{stabilizer } ?C0))$
using $\text{card-le-sumset sumset-commute sub-unit-closed assms}$
by $(\text{metis Int-Un-eq(3) Un-subset-iff finite-Int unit-closed})$
have $\text{card } ?C0 \leq \text{card}(\text{sumset } A B)$
using $\text{card-mono sumset-inter-union-subset finite-sumset assms}$
by $(\text{simp add: card-mono finite-sumset hA hB sumset-inter-union-subset})$
then **have** $\text{card } ?C0 + \text{card}(A \cap B) < \text{card}(\text{sumset } A B) + \text{card } A$
using $hintercardA$ **by** auto
then **obtain** m **where** $m < n$ **and** $\text{card } ?C0 + \text{card}(A \cap B) = m$ **using**
 $hcardsum$ **by** auto
then **have** $\text{card}(\text{sumset}(A \cap B)(\text{stabilizer } ?C0)) +$
 $\text{card}(\text{sumset}(A \cup B)(\text{stabilizer } ?C0)) - \text{card}(\text{stabilizer } ?C0) \leq \text{card } ?C0$
using $\text{assms finite-Un finite-Int}$
by $(\text{metis Int-Un-eq(4) Un-empty Un-subset-iff})$
then **have** $\text{card } ?C0 + \text{card}(\text{stabilizer } ?C0) \geq$
 $\text{card}(A \cap B) + \text{card}(\text{sumset}(A \cup B)(\text{stabilizer } ?C0))$ **using** $hcard\text{-sumset-le}$
by auto
then **have** $?C0 \in \text{convergent-set } A B$ **using** $\text{convergent-set-def convergent-def}$
 $\text{sumset-inter-union-subset hC0ne}$ **by** auto
then **have** $hconvergent\text{-ne}: \text{convergent-set } A B \neq \{\}$ **by** auto
define KS **where** $KS \equiv (\lambda X. \text{card}(\text{stabilizer } X))$ ‘ $\text{convergent-set } A B$
define K **where** $K \equiv \text{Min } KS$
define C **where** $C \equiv @C. C \in \text{convergent-set } A B \wedge K = \text{card}(\text{stabilizer } C)$
obtain $KS: \text{finite } KS$ $KS \neq \{\}$
using $hconvergent\text{-ne finite-convergent-set assms KS-def}$ **by** auto
then **have** $K \in KS$ **using** $K\text{-def Min-in}$ **by** blast
then **have** $\exists X. X \in \text{convergent-set } A B \wedge K = \text{card}(\text{stabilizer } X)$
using $KS\text{-def}$ **by** auto
then **obtain** $C \in \text{convergent-set } A B$ **and** $\text{Keq}: K = \text{card}(\text{stabilizer } C)$

by (*metis* (*mono-tags*, *lifting*) *C-def someI-ex*)
then have $hC: C \subseteq \text{sumset } A B$ **and** $hCne: C \neq \{\}$ **and**
 $hCcard: \text{card } C + \text{card } (\text{stabilizer } C) \geq$
 $\text{card } (A \cap B) + \text{card } (\text{sumset } (A \cup B) (\text{stabilizer } C))$
using *convergent-set-def convergent-def* **by** *auto*
have $hCmin: \bigwedge Y. Y \in \text{convergent-set } A B \implies$
 $\text{card } (\text{stabilizer } Y) \geq \text{card } (\text{stabilizer } C)$
using *K-def KS-def Keq Min-le KS(1)* **by** *auto*
show *?thesis* **using** $hCmin$ hC $hCcard$ $hCne$ *local.convergent-def* **that** **by** *pres-*
burger
qed

end

context *normal-subgroup*
begin

1.5 A function that picks coset representatives randomly

definition $\varphi :: 'a \text{ set} \Rightarrow 'a$ **where**

$\varphi = (\lambda x. \text{if } x \in G // K \text{ then } (\text{SOME } a. a \in G \wedge x = a \cdot | K) \text{ else undefined})$

definition *quot-comp-alt* $:: 'a \Rightarrow 'a \Rightarrow 'a$ **where** *quot-comp-alt* $a b = \varphi ((a \cdot b) \cdot | K)$

lemma *phi-eq-coset*:

assumes $\varphi x = a$ **and** $a \in G$ **and** $x \in G // K$

shows $x = a \cdot | K$

proof –

have $(\text{SOME } a. a \in G \wedge x = a \cdot | K) = a$ **using** *φ-def assms* **by** *simp*

then show *?thesis* **using** *some-eq-ex representant-exists Left-Coset-Class-unit assms*

by (*metis* (*mono-tags*, *lifting*))

qed

lemma *phi-coset-mem*:

assumes $a \in G$

shows $\varphi (a \cdot | K) \in a \cdot | K$

proof –

obtain x **where** $hx: x = \varphi (a \cdot | K)$ **by** *auto*

then have $x = (\text{SOME } x. x \in G \wedge a \cdot | K = x \cdot | K)$ **using** *φ-def assms*

Class-in-Partition Left-Coset-Class-unit **by** *presburger*

then show *?thesis* **using** *φ-def Class-self Left-Coset-Class-unit hx assms*

by (*smt* (*verit*, *ccfv-SIG*) *tfl-some*)

qed

lemma *phi-coset-eq*:

assumes $a \in G$ **and** $\varphi x = a$ **and** $x \in G // K$

shows $\varphi (a \cdot | K) = a$ **using** *phi-eq-coset assms* **by** *metis*

lemma *phi-inverse-right*:
assumes $g \in G$
shows $\text{quot-comp-alt } g (\varphi (\text{inverse } g \cdot | K)) = \varphi K$
proof –
have $g \cdot (\varphi (\text{inverse } g \cdot | K)) \in (g \cdot (\text{inverse } g) \cdot | K)$
using *phi-coset-mem assms by (smt (z3) Left-Coset-memE factor-unit invertible invertible-right-inverse invertible-inverse-closed invertible-inverse-inverse sub invertible-left-inverse2)*
then have $g \cdot (\varphi (\text{inverse } g \cdot | K)) \cdot | K = K$
using *Block-self Left-Coset-Class-unit Normal-def quotient.unit-closed sub*
by (*metis assms composition-closed invertible invertible-inverse-closed invertible-right-inverse*)
then show *?thesis using quot-comp-alt-def by auto*
qed

lemma *phi-inverse-left*:
assumes $g \in G$
shows $\text{quot-comp-alt } (\varphi (\text{inverse } g \cdot | K)) g = \varphi K$
proof –
have $(\varphi (\text{inverse } g \cdot | K)) \cdot g \in ((\text{inverse } g) \cdot g) \cdot | K$ **using** *phi-coset-mem assms*
by (*metis Left-Coset-memE factor-unit invertible invertible-inverse-closed invertible-left-inverse normal*)
then have $(\varphi (\text{inverse } g \cdot | K)) \cdot g \cdot | K = K$ **using** *Block-self Left-Coset-Class-unit Normal-def*
quotient.unit-closed sub **by** (*smt (verit, best) assms composition-closed invertible invertible-inverse-closed invertible-left-inverse*)
then show *?thesis using quot-comp-alt-def by auto*
qed

lemma *phi-mem-coset-eq*:
assumes $a \in G // K$ **and** $b \in G$
shows $\varphi a \in b \cdot | K \implies a = (b \cdot | K)$
proof –
assume $\varphi a \in b \cdot | K$
then have $a \cap (b \cdot | K) \neq \{\}$
by (*metis Class-closed Class-is-Left-Coset Int-iff assms empty-iff phi-coset-mem phi-eq-coset*)
then show $a = b \cdot | K$ **by** (*metis Class-in-Partition Class-is-Left-Coset assms disjoint*)
qed

lemma *forall-unique-repr*:
 $\forall x \in G // K. \exists! k \in \varphi '(G // K). x = k \cdot | K$
proof
fix x **assume** $hx: x \in G // K$
then have $\varphi x \cdot | K = x$

by (*metis Class-is-Left-Coset block-closed phi-coset-mem phi-eq-coset representant-exists*)
then have $hex: \exists k \in \varphi '(G // K). x = k \cdot | K$ **using** hx **by** *blast*
moreover have $\bigwedge a b. a \in \varphi '(G // K) \implies x = a \cdot | K \implies b \in \varphi '(G // K)$
 $\implies x = b \cdot | K \implies$
 $a = b$
proof–
fix $a b$ **assume** $a \in \varphi '(G // K)$ **and** $hxa: x = a \cdot | K$ **and** $b \in \varphi '(G // K)$
and
 $hxb: x = b \cdot | K$
then obtain $z w$ **where** $a = \varphi (z \cdot | K)$ **and** $b = \varphi (w \cdot | K)$ **and** $z \in G$ **and**
 $w \in G$
using *representant-exists Left-Coset-Class-unit* **by** *force*
then show $a = b$ **using** $hxa hxb$
by (*metis Class-in-Partition Class-is-Left-Coset block-closed phi-coset-mem phi-eq-coset*)
qed
ultimately show $\exists! k \in \varphi '(G // K). x = k \cdot | K$ **by** *blast*
qed

lemma *phi-inj-on*:
shows *inj-on* $\varphi (G // K)$
proof(*intro inj-onI*)
fix $x y$ **assume** $x \in G // K$ **and** $hy: y \in G // K$ **and** $hxy: \varphi x = \varphi y$
then obtain $a b$ **where** $x = a \cdot | K$ **and** $y = b \cdot | K$ **and** $a \in G$ **and** $b \in G$
using *representant-exists Left-Coset-Class-unit* **by** *metis*
then show $x = y$ **using** $hxy hy$ **by** (*metis phi-coset-mem phi-mem-coset-eq*)
qed

lemma *phi-coset-eq-self*:
assumes $a \in G // K$
shows $\varphi a \cdot | K = a$
by (*metis Class-closed Class-is-Left-Coset assms phi-coset-mem phi-eq-coset representant-exists*)

lemma *phi-coset-comp-eq*:
assumes $a \in G // K$ **and** $b \in G // K$
shows $\varphi a \cdot \varphi b \cdot | K = a \cdot | b$ **using** *assms phi-coset-eq-self*
by (*metis Class-is-Left-Coset block-closed factor-composition phi-coset-mem representant-exists*)

lemma *phi-comp-eq*:
assumes $a \in G // K$ **and** $b \in G // K$
shows $\varphi (a \cdot | b) = \text{quot-comp-alt } (\varphi a) (\varphi b)$
using *phi-coset-comp-eq quot-comp-alt-def assms* **by** *auto*

lemma *phi-image-subset*:
 $\varphi '(G // K) \subseteq G$
proof(*intro image-subsetI, simp add: phi-def*)

```

fix  $x$  assume  $x \in G // K$ 
then show (SOME  $a. a \in G \wedge x = a \cdot | K \in G$ )
using Left-Coset-Class-unit representant-exists someI-ex by (metis (mono-tags, lifting))
qed

lemma phi-image-group:
  Group-Theory.group ( $\varphi \text{ ' } (G // K)$ ) quot-comp-alt ( $\varphi K$ )
proof –
  have hmonoid: Group-Theory.monoid ( $\varphi \text{ ' } (G // K)$ ) quot-comp-alt ( $\varphi K$ )
  proof
    show  $\bigwedge a b. a \in \varphi \text{ ' } (G // K) \implies b \in \varphi \text{ ' } (G // K) \implies$ 
      quot-comp-alt  $a b \in \varphi \text{ ' } (G // K)$  using quot-comp-alt-def imageI phi-image-subset
      by (metis Class-in-Partition Left-Coset-Class-unit composition-closed subset-iff)
    next
      show ( $\varphi K \in \varphi \text{ ' } (G // K)$ ) using  $\varphi$ -def Left-Coset-Class-unit imageI Normal-def by blast
    next
      show  $\bigwedge a b c. a \in \varphi \text{ ' } \textit{Partition} \implies b \in \varphi \text{ ' } \textit{Partition} \implies c \in \varphi \text{ ' } \textit{Partition}$ 
       $\implies$ 
        quot-comp-alt (quot-comp-alt  $a b$ )  $c = \textit{quot-comp-alt}$   $a$  (quot-comp-alt  $b c$ )
      proof –
        fix  $a b c$  assume  $ha: a \in \varphi \text{ ' } (G // K)$  and  $hb: b \in \varphi \text{ ' } (G // K)$  and  $hc: c \in \varphi \text{ ' } (G // K)$ 
        have  $habc: a \cdot b \cdot c \in G$  using  $ha hb hc$  composition-closed phi-image-subset by (meson subsetD)
        have  $hab: \textit{quot-comp-alt}$   $a b \in (a \cdot b) \cdot | K$  using phi-image-subset quot-comp-alt-def  $ha hb$ 
        by (metis composition-closed phi-coset-mem subsetD)
        then have quot-comp-alt (quot-comp-alt  $a b$ )  $c \in (a \cdot b \cdot c) \cdot | K$  using quot-comp-alt-def phi-image-subset  $ha hb hc$ 
        by (smt (z3) Block-self Class-closed Class-in-Partition Left-Coset-Class-unit composition-closed
          natural.commutates-with-composition phi-coset-mem subset-iff)
        moreover have  $hbc: \textit{quot-comp-alt}$   $b c \in (b \cdot c) \cdot | K$  using  $hb hc$  phi-image-subset quot-comp-alt-def
        by (metis composition-closed phi-coset-mem subset-iff)
        moreover hence quot-comp-alt  $a$  (quot-comp-alt  $b c$ )  $\in (a \cdot b \cdot c) \cdot | K$ 
using quot-comp-alt-def phi-image-subset  $ha hb hc$ 
        by (smt (verit, del-Insts) Block-self Class-closed Class-in-Partition Left-Coset-Class-unit
          associative composition-closed natural.commutates-with-composition phi-coset-mem subset-iff)
        moreover have  $a \cdot (\textit{quot-comp-alt}$   $b c) \cdot | K \in G // K$  using  $ha hb hc$  phi-image-subset
        by (metis Class-closed Class-in-Partition Class-is-Left-Coset hbc composition-closed in-mono subset-eq)
        moreover have (quot-comp-alt  $a b$ )  $\cdot c \cdot | K \in G // K$  using  $ha hb hc$ 

```

phi-image-subset
by (*metis Class-closed Class-in-Partition Left-Coset-Class-unit hab com-position-closed in-mono*)
ultimately show *quot-comp-alt (quot-comp-alt a b) c = quot-comp-alt a (quot-comp-alt b c)*
using *phi-mem-coset-eq[OF - habc] quot-comp-alt-def* **by** *metis*
qed
next
show $\bigwedge a. a \in \varphi \text{ ' Partition} \implies \text{quot-comp-alt } (\varphi K) a = a$ **using** *quot-comp-alt-def*
phi-def
phi-image-subset image-iff phi-coset-eq subsetD **by** (*smt (z3) Normal-def*
Partition-def
natural.image.sub-unit-closed phi-comp-eq quotient.left-unit)
next
show $\bigwedge a. a \in \varphi \text{ ' Partition} \implies \text{quot-comp-alt } a (\varphi K) = a$ **using** *quot-comp-alt-def*
phi-def
phi-image-subset image-iff phi-coset-eq subsetD **by** (*smt (verit) Normal-def*
factor-composition factor-unit normal-subgroup.phi-coset-eq-self normal-subgroup-axioms
quotient.unit-closed right-unit unit-closed)
qed
moreover show *Group-Theory.group* $(\varphi \text{ ' } (G // K))$ *quot-comp-alt* (φK)
proof (*simp add: group-def group-axioms-def hmonoid*)
show $\forall u. u \in \varphi \text{ ' Partition} \longrightarrow \text{monoid.invertible } (\varphi \text{ ' Partition}) \text{ quot-comp-alt}$
 $(\varphi K) u$
proof (*intro allI impI*)
fix *g* **assume** *hg: g ∈ φ ' (G // K)*
then have *quot-comp-alt g (φ ((inverse g) · | K)) = (φ K)*
and *quot-comp-alt (φ ((inverse g) · | K)) g = (φ K)*
using *phi-image-subset phi-inverse-right phi-inverse-left* **by** *auto*
moreover have $\varphi ((\text{inverse } g) \cdot | K) \in \varphi \text{ ' } (G // K)$ **using** *imageI hg*
phi-image-subset
by (*metis (no-types, opaque-lifting) Class-in-Partition Left-Coset-Class-unit*
in-mono
invertible invertible-inverse-closed)
ultimately show *monoid.invertible* $(\varphi \text{ ' Partition}) \text{ quot-comp-alt } (\varphi K) g$
using *monoid.invertibleI[OF hmonoid] hg* **by** *presburger*
qed
qed
qed

lemma *phi-map: Set-Theory.map* $\varphi \text{ Partition } (\varphi \text{ ' Partition})$
by (*auto simp add: Set-Theory.map-def phi-def*)

lemma *phi-image-isomorphic:*
group-isomorphism $\varphi (G // K) ([\cdot]) (\text{Class } \mathbf{1}) (\varphi \text{ ' } (G // K)) \text{ quot-comp-alt } (\varphi K)$
proof –
have *bijjective-map* $\varphi \text{ Partition } (\varphi \text{ ' Partition})$

```

    using bijective-map-def bijective-def bij-betw-def phi-inj-on phi-map by blast
    moreover have Group-Theory.monoid ( $\varphi$  ' Partition) quot-comp-alt ( $\varphi$  K)
    using phi-image-group group-def by metis
    moreover have  $\varphi$  (Class 1) =  $\varphi$  K using Left-Coset-Class-unit Normal-def by
    auto
    ultimately show ?thesis
    by (auto simp add: group-isomorphism-def group-homomorphism-def monoid-homomorphism-def

    phi-image-group quotient.monoid-axioms quotient.group-axioms monoid-homomorphism-axioms-def

    phi-comp-eq phi-map)
qed

end

context subgroup-of-additive-abelian-group

begin

lemma Union-Coset-card-eq:
  assumes hSG:  $S \subseteq G$  and hSU:  $(\bigcup (Class ' S)) = S$ 
  shows card S = card H * card (Class ' S)
proof (cases finite H)
  case hH: True
  have hfin:  $\bigwedge A. A \in Class ' S \implies finite A$  using hSG Right-Coset-Class-unit
    Right-Coset-cardinality hH card-eq-0-iff empty-iff sub-unit-closed subsetD
  by (smt (verit, del-Insts) imageE)
  have card S = card H * card (Class ' S) when hS: finite S
  proof -
    have hdisj: pairwise ( $\lambda s t. disjoint s t$ ) (Class ' S)
    proof (intro pairwiseI)
      fix x y assume  $x \in Class ' S$  and  $y \in Class ' S$  and hxy:  $x \neq y$ 
      then obtain a b where  $x \in Class a$  and  $y \in Class b$  and
         $a \in S$  and  $b \in S$  by blast
      then show disjoint x y using disjoint-def hxy
      by (smt (verit, ccfv-threshold) not-disjoint-implies-equal hSG subsetD)
    qed
    then have card  $(\bigcup (Class ' S)) = sum card (Class ' S)$  using card-Union-disjoint
    hfin by blast
    moreover have finite (Class ' S) using hS by blast
    ultimately have card  $(\bigcup (Class ' S)) = (\sum a \in Class ' S. card a)$ 
      using sum-card-image hdisj by blast
    moreover have  $\bigwedge a. a \in Class ' S \implies card a = card H$ 
      using hSG Right-Coset-Class-unit Right-Coset-cardinality by auto
    ultimately show card S = card H * card (Class ' S)
      using hSU by simp
  qed
  moreover have card S = card H * card (Class ' S) when hS:  $\neg finite S$ 
  using finite-Union hfin hS hSU by (metis card-eq-0-iff mult-0-right)

```

```

ultimately show ?thesis by blast
next
case hH: False
have card S = card H * card (Class ' S) when S = {}
  by (simp add: that)
then have hinf:  $\bigwedge A. A \in \text{Class ' } S \implies \text{infinite } A$  using hSG Right-Coset-Class-unit

  Right-Coset-cardinality hH card-eq-0-iff empty-iff sub-unit-closed subsetD
  by (smt (verit) Class-self imageE)
moreover have card S = card H * card (Class ' S) when S  $\neq$  {} using hSU
by (metis Class-closed2
  Normal-def card.infinite card-sumset-0-iff hH hSG mult-is-0 sumset-subgroup-eq-Class-Union
  unit-closed)
ultimately show ?thesis by fastforce
qed

end

context subgroup-of-abelian-group
begin

interpretation GH: additive-abelian-group G // H ([·]) Class 1
proof
  fix x y assume x  $\in$  G // H and y  $\in$  G // H
  then show x [·] y = y [·] x using Class-commutes-with-composition commutative
  representant-exists
  by metis
qed

interpretation GH-repr: additive-abelian-group  $\varphi$  ' (G // H) quot-comp-alt  $\varphi$  H
proof (simp add: additive-abelian-group-def abelian-group-def phi-image-group
  commutative-monoid-def commutative-monoid-axioms-def, intro conjI allI impI)
  show Group-Theory.monoid ( $\varphi$  ' Partition) quot-comp-alt ( $\varphi$  H)
  using phi-image-group group-def by metis
next
  show  $\bigwedge x y. x \in \varphi$  ' Partition  $\implies y \in \varphi$  ' Partition  $\implies$  quot-comp-alt x y =
  quot-comp-alt y x
  by (auto) (metis GH.commutative phi-comp-eq)
qed

lemma phi-image-sumset-eq:
  assumes A  $\subseteq$  G // H and B  $\subseteq$  G // H
  shows  $\varphi$  ' (GH.sumset A B) = GH-repr.sumset ( $\varphi$  ' A) ( $\varphi$  ' B)
proof (intro subset-antisym image-subsetI subsetI)
  fix x assume x  $\in$  GH.sumset A B
  then obtain c d where x = quotient-composition c d and hc: c  $\in$  A and hd:
  d  $\in$  B
  using GH.sumset.cases by blast
  then have  $\varphi$  x = quot-comp-alt ( $\varphi$  c) ( $\varphi$  d)

```

```

    using phi-comp-eq assms subsetD by blast
  then show  $\varphi x \in GH\text{-repr.sumset } (\varphi \text{ ' } A) (\varphi \text{ ' } B)$ 
    using hc hd assms subsetD GH-repr.sumsetI imageI by auto
next
  fix x assume  $x \in GH\text{-repr.sumset } (\varphi \text{ ' } A) (\varphi \text{ ' } B)$ 
  then obtain a b where  $x = \text{quot-comp-alt } a \ b$  and ha:  $a \in \varphi \text{ ' } A$  and hb:  $b \in \varphi \text{ ' } B$ 
    using GH-repr.sumset.cases by metis
  moreover obtain c d where  $a = \varphi \ c$  and  $b = \varphi \ d$  and  $c \in A$  and  $d \in B$ 
    using ha hb by blast
  ultimately show  $x \in \varphi \text{ ' } GH.\text{sumset } A \ B$  using phi-comp-eq assms imageI GH.sumsetI
    by (smt (verit, del-insts) subsetD)
qed

```

lemma *phi-image-stabilizer-eq*:

```

  assumes  $A \subseteq G // H$ 
  shows  $\varphi \text{ ' } (GH.\text{stabilizer } A) = GH\text{-repr.stabilizer } (\varphi \text{ ' } A)$ 
proof(intro subset-antisym image-subsetI subsetI)
  fix x assume  $x \in GH.\text{stabilizer } A$ 
  then have  $GH.\text{sumset } \{x\} A = A$  and hx:  $x \in G // H$  using GH.stabilizer-def
  assms by auto
  then have  $GH\text{-repr.sumset } (\varphi \text{ ' } \{x\}) (\varphi \text{ ' } A) = \varphi \text{ ' } A$  using assms phi-image-sumset-eq
    by (metis empty-subsetI insert-subset)
  then show  $\varphi x \in GH\text{-repr.stabilizer } (\varphi \text{ ' } A)$  using GH-repr.stabilizer-def assms
    by (smt (z3) GH-repr.sumset-Int-carrier hx image-empty image-eqI image-insert
  mem-Collect-eq)
next
  fix x assume  $x \in GH\text{-repr.stabilizer } (\varphi \text{ ' } A)$ 
  then have hstab:  $GH\text{-repr.sumset } \{x\} (\varphi \text{ ' } A) = (\varphi \text{ ' } A)$  and hx:  $x \in \varphi \text{ ' } (G // H)$ 
    using GH-repr.stabilizer-def assms phi-image-subset by auto
  then obtain B where hB:  $B \in G // H$  and hBx:  $\varphi B = x$  by blast
  then have  $GH\text{-repr.sumset } (\varphi \text{ ' } \{B\}) (\varphi \text{ ' } A) = \varphi \text{ ' } A$  using hstab by auto
  then have  $GH.\text{sumset } \{B\} A = A$  using phi-image-sumset-eq phi-inj-on assms
  hB
  GH.sumset-subset-carrier by (smt (z3) GH.sumset-singletons-eq inj-on-image-eq-iff

  quotient.right-unit quotient.unit-closed)
  then show  $x \in \varphi \text{ ' } (GH.\text{stabilizer } A)$  using assms hBx GH.stabilizer-def
    by (smt (z3) GH.sumset-Int-carrier hB image-iff mem-Collect-eq)
qed

```

end

1.6 Useful group-theoretic results

lemma *residue-group: abelian-group* $\{0..(m :: nat)-1\} (\lambda x y. ((x + y) \text{ mod } m))$
 $(0 :: int)$

```

proof(cases m > 1)
  case hm: True
    then have hmonoid: Group-Theory.monoid {0..m-1} (λ x y. ((x + y) mod m))
    (0 :: int)
    by (unfold-locales, auto simp add: of-nat-diff, presburger)
    moreover have monoid.invertible {0..int (m - 1)} (λ x y. (x + y) mod int m)
    0 u if u ∈ {0..int (m - 1)} for u
    proof(cases u = 0)
      case True
        then show ?thesis using monoid.invertible-def[OF hmonoid that] monoid.unit-invertible[OF
hmonoid] by simp
      next
        case hx: False
          then have ((m - u) + u) mod m = 0 and (u + (m - u)) mod m = 0 and
m - u ∈ {0..int(m-1)}
          using atLeastAtMost-iff hx that by auto
          then show ?thesis using monoid.invertible-def[OF hmonoid that] by metis
        qed
        moreover have commutative-monoid {0..m-1} (λ x y. ((x + y) mod m)) (0 ::
int)
        using hmonoid commutative-monoid-def commutative-monoid-axioms-def by
(smt (verit))
        ultimately show ?thesis by (simp add: abelian-group-def group-def group-axioms-def
hmonoid)
      next
        case hm: False
          moreover have hmonoid: Group-Theory.monoid {0} (λ x y. ((x + y) mod m))
(0 :: int)
          by (unfold-locales, auto)
          moreover have monoid.invertible {0} (λ x y. (x + y) mod int m) 0 0 using
monoid.invertible-def[OF hmonoid]
          monoid.unit-invertible[OF hmonoid] hm by simp
          ultimately show ?thesis by (unfold-locales, auto)
        qed

lemma (in subgroup-of-group) prime-order-simple:
  assumes prime (card G)
  shows H = {1} ∨ H = G
proof -
  have card H dvd card G using lagrange assms card.infinite dvdI not-prime-0 by
fastforce
  then have card H = 1 ∨ card H = card G using assms prime-nat-iff by blast
  then show ?thesis using card-1-singletonE sub-unit-closed card.infinite card-subset-eq
sub
  assms not-prime-0 subsetI insertE empty-iff by metis
qed

lemma residue-group-simple:

```

```

assumes prime p and subgroup H {0..(p :: nat)-1} (λ x y. ((x + y) mod p))
(0 :: int)
shows  $H = \{0\} \vee H = \{0..int(p-1)\}$ 
proof –
  have hprime: prime (card {0..int(p-1)}) using card-atLeastAtMost-int assms
int-ops by auto
  moreover have hsub:subgroup-of-group H {0..(p :: nat)-1} (λ x y. ((x + y)
mod p)) (0 :: int)
  using subgroup-of-group-def assms abelian-group-def residue-group by fast
  ultimately show ?thesis using assms subgroup-of-group.prime-order-simple[OF
hsub hprime] by blast
qed

end

```

2 Kneser’s Theorem and the Cauchy–Davenport Theorem: main proofs

```

theory Kneser-Cauchy-Davenport-main-proofs
imports
  Kneser-Cauchy-Davenport-preliminaries

```

```

begin

```

```

context additive-abelian-group

```

```

begin

```

2.1 Proof of Kneser’s Theorem

The proof we formalise follows the paper [1]. This version of Kneser’s Theorem corresponds to Theorem 3.2 in [3], or to Theorem 4.3 in [2].

```

theorem Kneser:

```

```

assumes  $A \subseteq G$  and  $B \subseteq G$  and finite A and finite B and hAne: A ≠ {} and
hBne: B ≠ {}

```

```

shows  $\text{card}(\text{sumset } A \ B) \geq \text{card}(\text{sumset } A \ (\text{stabilizer}(\text{sumset } A \ B))) +$ 
 $\text{card}(\text{sumset } B \ (\text{stabilizer}(\text{sumset } A \ B))) - \text{card}(\text{stabilizer}(\text{sumset } A \ B))$ 

```

```

proof –

```

```

have  $\bigwedge n \ A \ B. \text{additive-abelian-group } G \ (\oplus) \ \mathbf{0} \implies A \subseteq G \implies B \subseteq G \implies$ 
 $\text{finite } A \implies \text{finite } B \implies A \neq \{\} \implies B \neq \{\} \implies \text{card}(\text{sumset } A \ B) + \text{card}$ 
 $A = n \implies$ 

```

```

 $\text{card}(\text{sumset } A \ B) \geq \text{card}(\text{sumset } A \ (\text{stabilizer}(\text{sumset } A \ B))) +$ 
 $\text{card}(\text{sumset } B \ (\text{stabilizer}(\text{sumset } A \ B))) - \text{card}((\text{stabilizer}(\text{sumset } A \ B)))$ 

```

```

proof –

```

```

fix n

```

```

show  $\bigwedge A \ B. \text{additive-abelian-group } G \ (\oplus) \ \mathbf{0} \implies A \subseteq G \implies B \subseteq G \implies$ 
 $\text{finite } A \implies \text{finite } B \implies A \neq \{\} \implies B \neq \{\} \implies \text{card}(\text{sumset } A \ B) + \text{card}$ 
 $A = n \implies$ 

```

```

card (sumset A B) ≥ card (sumset A (stabilizer (sumset A B))) +
card (sumset B (stabilizer (sumset A B))) - card ((stabilizer (sumset A B)))
proof(induction n arbitrary : G (⊕) 0 rule: nat-less-induct)
  fix n
  fix A B G :: 'a set
  fix add (infixl ⟨[⊕]⟩ 65)
  fix zero (⟨[0]⟩)
  assume hind: ∀ m < n. ∀ x xa xb :: 'a set. ∀ xc xd.
    additive-abelian-group xb xc xd → x ⊆ xb →
    xa ⊆ xb → finite x → finite xa → x ≠ {} → xa ≠ {} →
    card (additive-abelian-group.sumset xb xc x xa) + card x = m →
    card (additive-abelian-group.sumset xb xc x (additive-abelian-group.stabilizer
xb xc
    (additive-abelian-group.sumset xb xc x xa))) + card (additive-abelian-group.sumset
xb xc xa
    (additive-abelian-group.stabilizer xb xc (additive-abelian-group.sumset xb xc
x xa))) -
    card (additive-abelian-group.stabilizer xb xc (additive-abelian-group.sumset
xb xc x xa))
    ≤ card (additive-abelian-group.sumset xb xc x xa) and
    hGroupG: additive-abelian-group G ([⊕]) [0] and hAG: A ⊆ G and hBG: B
⊆ G and
    hA: finite A and hB: finite B and hAne: A ≠ {} and hBne: B ≠ {} and
    hcardsum: card (additive-abelian-group.sumset G ([⊕]) A B) + card A = n
  interpret G: additive-abelian-group G ([⊕]) [0] using hGroupG by simp
  have hindG: ∀ m < n. ∀ C D. C ⊆ G →
    D ⊆ G → finite C → finite D → C ≠ {} → D ≠ {} →
    card (G.sumset C D) + card C = m →
    card (G.sumset C (G.stabilizer
(G.sumset C D))) + card (G.sumset D
(G.stabilizer (G.sumset C D))) -
    card (G.stabilizer (G.sumset C D))
    ≤ card (G.sumset C D) using hind hGroupG by blast
  show card (G.sumset A (G.stabilizer (G.sumset A B))) +
    card (G.sumset B (G.stabilizer (G.sumset A B))) -
    card (G.stabilizer (G.sumset A B)) ≤ card (G.sumset A B)
  proof(cases G.stabilizer (G.sumset A B) = {[0]})
    case hstab0: True
      show ?thesis
      proof (cases card A = 1)
        case True
          then obtain a where A = {a} and a ∈ G
            by (metis hAG card-1-singletonE insert-subset)
          then show ?thesis using G.card-sumset-singleton-subset-eq
            G.stabilizer-left-sumset-invariant G.stabilizer-subset-group G.sumset-commute

            G.sumset-stabilizer-eq-self hBG by (metis diff-add-inverse eq-imp-le)
        next
          case False

```

then have $\text{card } A \geq 2$ **using** *Suc-1 order-antisym-conv*
by (*metis Suc-eq-plus1 bot.extremum card-seteq hA hAne le-add2 not-less-eq-eq*)
then obtain $a \ a'$ **where** $\text{haA}: \{a', a\} \subseteq A$ **and** $\text{hanot}: a' \neq a$ **and** $\text{ha1G}: a \in G$ **and**
 $\text{ha2G}: a' \in G$
by (*smt (verit, ccfv-threshold) card-2-iff hAG insert-subset obtain-subset-with-card-n subset-trans*)
then have $(a' [\oplus] (G.\text{inverse } a)) \notin G.\text{stabilizer } B$ **using** *G.stabilizer-sub-sumset-right*

 $\text{hstab0 subset-singletonD}$ **by** (*metis G.commutative G.inverse-closed G.invertible G.invertible-right-inverse2 G.right-unit empty-iff insert-iff*)
then obtain b **where** $\text{hb}: b \in B$ **and** $(a' [\oplus] (G.\text{inverse } a)) [\oplus] b \notin B$
using *G.stabilizer-def G.not-mem-stabilizer-obtain hBG hB hBne ha1G ha2G*
 $G.\text{composition-closed } G.\text{inverse-closed}$ **by** (*metis (no-types, lifting)*)
then have $\text{habG}: a' [\oplus] b [\oplus] G.\text{inverse } a \notin B$
using $\text{hb } hBG \text{ ha1G } ha2G$ **by** (*metis G.associative G.commutative G.inverse-closed subset-iff*)
have $\text{hbG}: b \in G$ **using** $\text{hb } hBG$ **by** *auto*
let $?B' = G.\text{sumset } (G.\text{differenceset } B \ \{b\}) \ \{a\}$
have $\text{hB'}: \text{finite } ?B'$ **using** hB
by (*simp add: G.finite-minusset G.finite-sumset*)
have $\text{hB'G}: ?B' \subseteq G$ **using** *G.sumset-subset-carrier* **by** *blast*
have $\text{hB'ne}: ?B' \neq \{\}$ **using** $\text{hBne } hbG \text{ ha1G } \text{sumset-is-empty-iff } hBG$ **by**
auto
have $\text{hstabeq}: G.\text{stabilizer } (G.\text{sumset } A \ B) = G.\text{stabilizer } (G.\text{sumset } A \ ?B')$
using $\text{hbG } ha1G \text{ hAG } hBG$ *G.stabilizer-unchanged* **by** *blast*
have $\text{hstab0'}: G.\text{stabilizer } (G.\text{sumset } A \ ?B') = \{[0]\}$ **using** $\text{hstab0 } \text{hstabeq}$
by *blast*
have $\text{ha1B'}: a \in ?B'$
proof–
have $(b [\oplus] G.\text{inverse } b) [\oplus] a \in ?B'$ **using** $\text{hBG } hbG \text{ ha1G } hb$
G.minusset.minussetI **by** *blast*
thus $a \in ?B'$ **by** (*simp add: hbG ha1G*)
qed
then have $\text{hinter-nempty}: A \cap ?B' \neq \{\}$ **using** $\text{ha1B'} \text{ haA}$ **by** *blast*
have $\text{ha2B'}: a' \notin ?B'$
proof–
have $\text{h1}: (a' [\oplus] b [\oplus] G.\text{inverse } a) [\oplus] G.\text{inverse } b \notin G.\text{differenceset } B$
 $\{b\}$
proof
assume $(a' [\oplus] b [\oplus] G.\text{inverse } a) [\oplus] G.\text{inverse } b \in G.\text{differenceset } B$
 $\{b\}$
then obtain b' **where** $(a' [\oplus] b [\oplus] G.\text{inverse } a) [\oplus] G.\text{inverse } b = b'$
 $[\oplus] G.\text{inverse } b$
and $b' \in B$ **using** *hbG G.minusset.simps G.sumset.cases* **by** *force*
then have $(a' [\oplus] b [\oplus] G.\text{inverse } a) \in B$ **using** hbG

```

      by (smt (verit) G.composition-closed hBG ha1G ha2G G.inverse-closed
G.invertible
      G.invertible-right-cancel subsetD)
      thus False using habG by auto
    qed
    have ((a' [⊕] b [⊕] G.inverse a) [⊕] G.inverse b) [⊕] a ∉ ?B'
    proof
      assume ((a' [⊕] b [⊕] G.inverse a) [⊕] G.inverse b) [⊕] a ∈ ?B'
      then obtain b' where ((a' [⊕] b [⊕] G.inverse a) [⊕] G.inverse b) [⊕]
a = b' [⊕] a
      and b' ∈ G.differenceset B {b} using ha1G G.sumset.simps by auto
      then have ((a' [⊕] b [⊕] G.inverse a) [⊕] G.inverse b) ∈ G.differenceset
B {b}
      using ha1G by (smt (z3) G.sumset.simps G.additive-abelian-group-axioms

      G.composition-closed ha2G hbG G.inverse-closed G.invertible
G.invertible-right-cancel)
      thus False using h1 by auto
    qed
    thus a' ∉ ?B' using ha1G hbG
  by (smt (verit, del-Insts) G.associative G.commutative G.composition-closed
ha2G
      G.inverse-closed G.invertible G.invertible-left-inverse G.right-unit)
  qed
  have hinterA: A ∩ ?B' ≠ A using haA ha2B' by auto
  have hintercard0: 0 < card (A ∩ ?B')
  using hA hB hinter-nempty card-gt-0-iff by blast
  have hintercardA: card (A ∩ ?B') < card A using hA hB hinterA card-mono
  by (simp add: psubsetI psubset-card-mono)
  have inj: inj-on (λ x. x [⊕] G.inverse b [⊕] a) G using inj-onI ha1G hb
G.invertible
      G.inverse-closed G.composition-closed by (smt (verit) G.invertible-right-cancel
hbG)
  have 1: card (G.sumset A B) = card (G.sumset A ?B')
  using G.card-differenceset-singleton-mem-eq G.card-sumset-singleton-subset-eq
hAG hB'G
      ha1G hbG G.sumset-assoc G.sumset-commute G.sumset-subset-carrier
  by (smt (verit, del-Insts))
  obtain C where hCconv: G.convergent C A ?B' and hCmin: ⋀ Y. Y ∈
G.convergent-set A ?B'
  ⇒ card (G.stabilizer Y) ≥ card (G.stabilizer C)
  using G.exists-convergent-min-stabilizer[of n A ?B']
      hindG hA hB' hAG hB'G hinter-nempty hAne hcardsum hintercardA 1
hGroupG by auto
  have hC: C ⊆ G.sumset A ?B' and hCne: C ≠ {} and
  hCcard: card C + card (G.stabilizer C) ≥
  card (A ∩ ?B') + card (G.sumset (A ∪ ?B') (G.stabilizer C))
  using G.convergent-def hCconv by auto
  then have hCfinite: finite C using hC G.finite-sumset hA hB'

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    by (meson finite-subset)
    have htranslate: card (G.sumset A {[0]}) + card (G.sumset ?B' {[0]}) -
card {[0]} ≤
    card (G.sumset A ?B')
  proof(cases G.stabilizer C = {[0]})
    case hCstab0: True
      have card (G.sumset A {[0]}) + card (G.sumset ?B' {[0]}) - card {[0]}
= card A + card ?B' -
      card {[0]} using hAG hB'G G.card-minusset' by fastforce
      also have ... = card (A ∩ ?B') + card (A ∪ ?B') - card {[0]}
      using hA hB' card-Un-Int by fastforce
      also have ... = card (A ∩ ?B') + card (G.sumset (A ∪ ?B') {[0]}) -
card {[0]}
      by (simp add: Int-absorb1 Int-commute hAG G.sumset-subset-carrier)
      also have ... ≤ card C using hCcard hCstab0 by auto
      also have ... ≤ card (G.sumset A ?B')
      using hC card-mono G.finite-sumset hA hB' by metis
      finally show ?thesis by simp
    next
      case hCstab-ne0: False
        have hCG: C ⊆ G using hC by (meson subset-trans G.sumset-subset-carrier)
        then have hstabC: finite (G.stabilizer C) using G.stabilizer-finite hCne
hC
          G.finite-sumset hA hB' by (metis Nat.add-0-right add-leE card.infinite
hCcard
          hintercard0 le-0-eq not-gr0)
        then have hcardstabC-gt-1: card (G.stabilizer C) > 1 using G.zero-mem-stabilizer
hCstab-ne0 hCG by (metis card-1-singletonE card-gt-0-iff diffs0-imp-equal
empty-iff
          gr-zeroI insertE less-one zero-less-diff)
        have G.sumset (G.stabilizer C) (G.sumset A ?B') ≠ G.sumset A ?B'
        using G.finite-sumset G.stabilizer-is-nonempty G.stabilizer-subset-group
          G.sumset-eq-sub-stabilizer G.sumset-subset-carrier hA hB' hCstab-ne0
hstab0'
          subset-singletonD by metis
        moreover have card (G.sumset (G.stabilizer C) (G.sumset A ?B')) ≥
card (G.sumset A ?B')
        using G.card-le-sumset G.finite-sumset hA hB' hstabC
        by (meson hCG G.sumset-subset-carrier G.unit-closed G.zero-mem-stabilizer)
        ultimately have ¬ G.sumset (G.stabilizer C) (G.sumset A ?B') ⊆
G.sumset A ?B'
        using G.finite-sumset hA hB' card-seteq by metis
        then obtain x where hx1: x ∈ G.sumset (G.stabilizer C) (G.sumset A
?B') and
          hx2: x ∉ G.sumset A ?B' by auto
        then obtain a1 b1 c where x = c [⊕] (a1 [⊕] b1) and c ∈ G.stabilizer
C and
          ha1A: a1 ∈ A and hb1B': b1 ∈ ?B' and ha1memG: a1 ∈ G and
hb1memG: b1 ∈ G

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    by (metis (no-types, lifting) G.sumset.cases)
  then have hx:  $x \in G.\text{sumset } (G.\text{stabilizer } C) \{a1 \oplus b1\}$ 
    using hx1 by (meson G.composition-closed hAG hB'G insertI1
G.stabilizer-subset-group
  subsetD G.sumset.sumsetI)
  then have hnotsubAB:  $\neg G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer } C) \subseteq$ 
G.sumset A ?B'
    using hx2 G.sumset-commute by auto
  let ?A1 =  $A \cap (G.\text{sumset } \{a1\} (G.\text{stabilizer } C))$ 
  let ?A2 =  $A \cap (G.\text{sumset } \{b1\} (G.\text{stabilizer } C))$ 
  let ?B1 =  $?B' \cap (G.\text{sumset } \{b1\} (G.\text{stabilizer } C))$ 
  let ?B2 =  $?B' \cap (G.\text{sumset } \{a1\} (G.\text{stabilizer } C))$ 
  let ?C1 =  $C \cup (G.\text{sumset } ?A1 ?B1)$ 
  let ?C2 =  $C \cup (G.\text{sumset } ?A2 ?B2)$ 
  let ?H1 =  $G.\text{stabilizer } (G.\text{sumset } ?A1 ?B1)$ 
  let ?H2 =  $G.\text{stabilizer } (G.\text{sumset } ?A2 ?B2)$ 
  have hA1ne:  $?A1 \neq \{\}$  using ha1A G.zero-mem-stabilizer hCG
    by (metis (full-types) disjoint-iff-not-equal hAG insertCI G.right-unit
subset-eq
  G.sumset.sumsetI G.unit-closed)
  have hB1ne:  $?B1 \neq \{\}$  using hb1B' G.zero-mem-stabilizer hCG
    by (metis G.composition-closed disjoint-iff-not-equal insertCI G.left-unit

  G.sumset.cases G.sumset.sumsetI G.sumset-commute G.unit-closed)
  have hnotsubC:  $\neg G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer } C) \subseteq C$  using
hnotsubAB hC by blast
  have habstabempty:  $G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer } C) \cap C = \{\}$ 
  proof(rule ccontr)
    assume  $G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer } C) \cap C \neq \{\}$ 
    then obtain z where
      hz:  $G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer } C) \cap G.\text{sumset } \{z\} (G.\text{stabilizer }
C) \neq \{\}$  and
      hzC:  $z \in C$  using G.stabilizer-coset-Un hCG by blast
    then have  $G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer } C) = G.\text{sumset } \{z\}$ 
(G.stabilizer C) using hz
      G.sumset-stabilizer-eq-iff hCG G.sumset.simps hx by auto
    then have  $G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer } C) \subseteq C$  using hzC
      by (simp add: hCG G.stabilizer-coset-subset)
    thus False using hnotsubC by simp
  qed
  have hA1B1sub:  $G.\text{sumset } ?A1 ?B1 \subseteq G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer }
C)$  and
    hB2A2sub:  $G.\text{sumset } ?B2 ?A2 \subseteq G.\text{sumset } \{a1 \oplus b1\} (G.\text{stabilizer }
C)$ 
    using G.sumset-Inter-subset-sumset ha1memG hb1memG by auto
  then have hA1B1Cempty:  $G.\text{sumset } ?A1 ?B1 \cap C = \{\}$  using
habstabempty by blast
  then have heardC1:  $\text{card } ?C1 = \text{card } C + \text{card } (G.\text{sumset } ?A1 ?B1)$ 
using card-Un-disjoint

```

$G.\text{finite-sumset } hA \ hB' \ \text{finite-Int } hC \ \text{finite-subset Int-commute}$ **by** *metis*
have $hA1B1\text{cardless: card } (G.\text{sumset } ?A1 \ ?B1) < \text{card } (G.\text{sumset } A \ B)$
proof-
have $G.\text{sumset } ?A1 \ ?B1 \subseteq G.\text{sumset } A \ ?B'$ **using** $G.\text{sumset-mono}$ **by**
auto
moreover have $G.\text{sumset } ?A1 \ ?B1 \neq G.\text{sumset } A \ ?B'$
using $hA1B1\text{Cempty } hC \ hCne \ hA1B1\text{sub}$ **by** *auto*
ultimately show $\text{card } (G.\text{sumset } ?A1 \ ?B1) < \text{card } (G.\text{sumset } A \ B)$
using $G.\text{finite-sumset } hA \ hB' \ \text{psubset-card-mono psubset-eq 1}$ **by**
metis
qed
have $hB2A2\text{Cempty: } G.\text{sumset } ?B2 \ ?A2 \cap C = \{\}$ **using** $h\text{abstabempty}$
 $hB2A2\text{sub}$ **by** *blast*
then have $h\text{cardC2: card } ?C2 = \text{card } C + \text{card } (G.\text{sumset } ?B2 \ ?A2)$
using card-Un-disjoint
 $G.\text{finite-sumset } hA \ hB' \ \text{finite-Int } hC \ \text{finite-subset Int-commute}$
 $G.\text{sumset-commute}$ **by** *metis*
have $hA2B2\text{cardless: card } (G.\text{sumset } ?A2 \ ?B2) < \text{card } (G.\text{sumset } A \ B)$
proof-
have $G.\text{sumset } ?A2 \ ?B2 \subset G.\text{sumset } A \ ?B'$
using $G.\text{sumset-mono } hB2A2\text{Cempty } hC \ hCne \ hB2A2\text{sub } G.\text{sumset-commute}$
 psubset-eq
by (*metis Int-absorb1 Int-lower1*)
then show $?thesis$ **by** (*simp add: 1 G.finite-sumset hA hB' psubset-card-mono*)
qed
have $\text{card } ?A1 \leq \text{card } A$ **using** $\text{card-mono } hA$ **by** (*metis Int-lower1*)
then have $\text{card } (G.\text{sumset } ?A1 \ ?B1) + \text{card } ?A1 < \text{card } (G.\text{sumset } A \ B) + \text{card } A$
using $hA1B1\text{cardless}$ **by** *linarith*
then obtain l **where** $hln: l < n$ **and** $hind1: \text{card } (G.\text{sumset } ?A1 \ ?B1) + \text{card } ?A1 = l$
using $h\text{cardsum}$ **by** *auto*
have $\text{card } ?A2 \leq \text{card } A$ **using** $\text{card-mono } hA \ \text{Int-lower1}$ **by** *metis*
then have $\text{card } (G.\text{sumset } ?A2 \ ?B2) + \text{card } ?A2 < \text{card } (G.\text{sumset } A \ B) + \text{card } A$
using $hA2B2\text{cardless}$ **by** *linarith*
then obtain k **where** $hkn: k < n$ **and** $hind2: \text{card } (G.\text{sumset } ?A2 \ ?B2) + \text{card } ?A2 = k$
using $h\text{cardsum}$ **by** *auto*
have $hH1\text{stabC: } ?H1 \subset G.\text{stabilizer } C$ **using** $G.\text{stabilizer-sumset-psubset-stabilizer}$
 $hA1ne \ hB1ne \ ha1\text{memG} \ hb1\text{memG} \ h\text{notsubAB}$ **by** *presburger*
then have $\text{card } ?H1 < \text{card } (G.\text{stabilizer } C)$ **using** psubset-card-mono
 $h\text{stabC}$ **by** *auto*
moreover have $hC1H1: ?H1 = G.\text{stabilizer } ?C1$ **using** $G.\text{stabilizer-eq-stabilizer-union}$
by (*metis Int-commute hA hA1B1Cempty hA1ne hB' hB1ne hC hCfinite hCne ha1memG hb1memG hnotsubAB*)
ultimately have $hC1\text{notconv: } \neg G.\text{convergent } ?C1 \ A \ ?B'$ **using** $hC\text{min}$

G.convergent-set-def
le-antisym less-imp-le-nat less-not-refl2 **by** *fastforce*
have *hC1ne: ?C1 ≠ {}* **and** *hC2ne: ?C2 ≠ {}* **using** *hCne* **by** *auto*
have *hC1AB': ?C1 ⊆ G.sumset A ?B'* **and** *hC2AB': ?C2 ⊆ G.sumset*
A ?B' **using** *hC*
by (*auto simp add: G.sumset-mono*)
have *hA1G: ?A1 ⊆ G* **and** *hA1 : finite ?A1* **and** *hB1G: ?B1 ⊆ G* **and**
hB1: finite ?B1
using *hAG hB'G hA hB' finite-Int* **by** *auto*
then have *hindA1B1: card (G.sumset ?A1 ?H1) + card (G.sumset ?B1*
?H1) - card ?H1 ≤
card (G.sumset ?A1 ?B1) **using** *hindG hGroupG hA1ne hB1ne hind1*
hln hAG hB'G
hA hB' **by** *metis*
have *hC1notconv-ineq:*
(int (card ?C1) + card ?H1 - card (A ∩ ?B')) < int (card (G.sumset
(A ∪ ?B') ?H1))
using *hC1notconv hC1ne hC1AB' hC1H1 G.convergent-def* **by** *auto*
have *int (card (G.sumset (A ∪ ?B') (G.stabilizer C))) - card (G.sumset*
(A ∪ ?B') ?H1)
≤ (int (card C) + card (G.stabilizer C) - card (A ∩ ?B')) - card
(G.sumset (A ∪ ?B') ?H1)
using *hCcard* **by** *linarith*
then have *int (card (G.sumset (A ∪ ?B') (G.stabilizer C))) - card*
(G.sumset (A ∪ ?B') ?H1) <
(int (card C) + card (G.stabilizer C) - card (A ∩ ?B')) -
(int (card ?C1) + card ?H1 - card (A ∩ ?B')) **using** *hC1notconv-ineq*
by *linarith*
also have *... = int (card (G.stabilizer C)) - card ?H1 - card (G.sumset*
?A1 ?B1)
using *hcardC1* **by** *presburger*
also have *... ≤ int (card (G.stabilizer C)) - card (G.sumset ?A1 ?H1)*
- card (G.sumset ?B1 ?H1)
using *hindA1B1* **by** *linarith*

Finally, we deduce the inequality that is referred to as inequality (1) in [1] for $A \cap G.\text{sumset } \{a1\} (G.\text{stabilizer } C)$ and $G.\text{sumset } (G.\text{differenceset } B \{b\}) \{a\} \cap G.\text{sumset } \{b1\} (G.\text{stabilizer } C)$.

finally have *hA1B1ineq: int (card (G.sumset (A ∪ ?B') (G.stabilizer*
C))) -
card (G.sumset (A ∪ ?B') ?H1) < int (card (G.stabilizer C)) -
card (G.sumset ?A1 ?H1) - card (G.sumset ?B1 ?H1) **by** *simp*
have *hB2ne: ?B2 ≠ {}* **using** *G.sumset-inter-ineq hA1B1ineq ha1A ha1G*
hA hB' hAne hB'ne
hstabC hH1stabC ha1memG of-nat-0-le-iff **by** (*smt (verit, del-insts)*)
have *hA2ne: ?A2 ≠ {}* **using** *G.sumset-inter-ineq[of A b1 C ?B' a1]*
hA1B1ineq
hb1B' hb1memG hA hB' hAne hB'ne hstabC hH1stabC of-nat-0-le-iff
G.sumset-commute Un-commute **by** (*smt (verit, ccfv-SIG)*)

have $hH2stabC$: $?H2 \subseteq G.stabilizer\ C$ **using** $G.stabilizer-sumset-psubset-stabilizer$
 $G.commutative\ hA2ne\ hB2ne\ ha1memG\ hb1memG\ hnotsubAB$ **by**
presburger
then have $card\ ?H2 < card\ (G.stabilizer\ C)$ **using** $psubset-card-mono$
 $hstabC$ **by** *auto*
moreover have $hC2H2$: $?H2 = G.stabilizer\ ?C2$ **using** $G.stabilizer-eq-stabilizer-union$
by $(smt\ (verit,\ ccfv-threshold)\ G.sumset-commute\ Int-commute\ hA$
 $hA2ne\ hB'\ hB2A2Cempty$
 $hB2ne\ hC\ hCfinite\ hCne\ ha1memG\ hb1memG\ hnotsubAB)$
ultimately have $hC2notconv$: $\neg G.convergent\ ?C2\ A\ ?B'$
using $hCmin\ G.convergent-set-def\ le-antisym\ less-imp-le-nat\ less-not-refl2$

by *fastforce*
have $hA2G$: $?A2 \subseteq G$ **and** $hA2$: $finite\ ?A2$ **and** $hB2G$: $?B2 \subseteq G$ **and**
 $hB2$: $finite\ ?B2$
using $hAG\ hB'G\ hA\ hB'\ finite-Int$ **by** *auto*
then have $hindA2B2$: $card\ (G.sumset\ ?A2\ ?H2) + card\ (G.sumset\ ?B2$
 $?H2) - card\ ?H2 \leq$
 $card\ (G.sumset\ ?A2\ ?B2)$
using $hindG\ hGroupG\ hA2ne\ hB2ne\ hind2\ hkn\ hAG\ hB'G\ hA\ hB'$ **by**
metis
have $hC2notconv-ineq$:
 $(int\ (card\ ?C2) + card\ ?H2 - card\ (A \cap ?B')) < int\ (card\ (G.sumset$
 $(A \cup ?B')\ ?H2))$
using $hC2notconv\ hC2ne\ hC2AB'\ hC2H2\ G.convergent-def$ **by** *auto*
have $int\ (card\ (G.sumset\ (A \cup ?B')\ (G.stabilizer\ C))) - card\ (G.sumset$
 $(A \cup ?B')\ ?H2)$
 $\leq (int\ (card\ C) + card\ (G.stabilizer\ C) - card\ (A \cap ?B')) - card$
 $(G.sumset\ (A \cup ?B')\ ?H2)$
using $hCcard$ **by** *linarith*
then have $int\ (card\ (G.sumset\ (A \cup ?B')\ (G.stabilizer\ C))) - card$
 $(G.sumset\ (A \cup ?B')\ ?H2) <$
 $(int\ (card\ C) + card\ (G.stabilizer\ C) - card\ (A \cap ?B')) -$
 $(int\ (card\ ?C2) + card\ ?H2 - card\ (A \cap ?B'))$ **using** $hC2notconv-ineq$
by *linarith*
also have $\dots = int\ (card\ (G.stabilizer\ C)) - card\ ?H2 - card\ (G.sumset$
 $?A2\ ?B2)$
using $hcardC2\ G.sumset-commute$ **by** *simp*
also have $\dots \leq int\ (card\ (G.stabilizer\ C)) - card\ (G.sumset\ ?A2\ ?H2)$
 $- card\ (G.sumset\ ?B2\ ?H2)$
using $hindA2B2$ **by** *linarith*

Analogously, we deduce the inequality that is referred to as inequality (1) in [1] for $A \cap G.sumset\ \{b1\}\ (G.stabilizer\ C)$ and $G.sumset\ (G.differenceset\ B\ \{b\})\ \{a\} \cap G.sumset\ \{a1\}\ (G.stabilizer\ C)$.

finally have $hA2B2ineq$: $int\ (card\ (G.sumset\ (A \cup ?B')\ (G.stabilizer$
 $C))) -$
 $card\ (G.sumset\ (A \cup ?B')\ ?H2) < int\ (card\ (G.stabilizer\ C)) -$

$\text{card } (G.\text{sumset } ?A2 \ ?H2) - \text{card } (G.\text{sumset } ?B2 \ ?H2)$ **by simp**
have $\text{hfinsumH2} : \text{finite } (G.\text{sumset } (A \cup ?B') \ ?H2)$
using $G.\text{finite-sumset } hA \ hB' \ \text{finite-UnI } \text{hstabC } \text{hH2stabC } \text{finite-subset}$
psubset-imp-subset
by metis
have $\text{hsubsumH2} : G.\text{sumset } (A \cup ?B') \ ?H2 \subseteq G.\text{sumset } (A \cup ?B')$
 $(G.\text{stabilizer } C)$
using $G.\text{sumset.cases } hAG \ hB'G \ G.\text{stabilizer-subset-group } \text{hH2stabC}$
psubset-imp-subset
by (*smt* (*verit*, *best*) *subset-Un-eq* $G.\text{sumset-commute } G.\text{sumset-subset-UnI}$)
then have $\text{hsumH2card-le} : \text{card } (G.\text{sumset } (A \cup ?B') \ ?H2) \leq$
 $\text{card } (G.\text{sumset } (A \cup ?B') \ (G.\text{stabilizer } C))$
using $\text{card-mono } G.\text{finite-sumset } G.\text{stabilizer-finite } hC \ hCne \ hCG \ hA$
 hB'
by (*metis* *finite-UnI* *hstabC*)
have $\text{hfinsumH1} : \text{finite } (G.\text{sumset } (A \cup ?B') \ ?H1)$
using $\text{finite-sumset } \text{finite-Un } \text{psubsetE}$ **by** (*metis* $G.\text{finite-sumset}$
 $G.\text{stabilizer-finite}$
 $G.\text{sumset-subset-carrier } \text{Int-Un-eq}(4) \ hA \ hA1 \ hB' \ hB1 \ hC1H1 \ hH1stabC$
 habstabempty)
have $\text{hsubsumH1} : G.\text{sumset } (A \cup ?B') \ ?H1 \subseteq G.\text{sumset } (A \cup ?B')$
 $(G.\text{stabilizer } C)$
using $G.\text{sumset.cases } \text{psubsetE } \text{subset-refl } G.\text{sumset-mono } \text{hH1stabC}$
by simp
have $\text{hsumH1card-le} : \text{card } (G.\text{sumset } (A \cup ?B') \ ?H1) \leq \text{card } (G.\text{sumset}$
 $(A \cup ?B') \ (G.\text{stabilizer } C))$
using $\text{card-mono } G.\text{finite-sumset } G.\text{stabilizer-finite}$ **by** (*metis* *finite-UnI*
 $hA \ hB' \ \text{hstabC } \text{hsubsumH1}$)
have $\text{ha1b1stabCne} : G.\text{sumset } \{a1\} \ (G.\text{stabilizer } C) \neq G.\text{sumset } \{b1\}$
 $(G.\text{stabilizer } C)$
proof
assume $\text{ha1b1} : G.\text{sumset } \{a1\} \ (G.\text{stabilizer } C) = G.\text{sumset } \{b1\}$
 $(G.\text{stabilizer } C)$
have $\text{hfin} : \text{finite } (G.\text{sumset } ?A1 \ ?H1 \cup G.\text{sumset } ?B1 \ ?H1)$
using $\text{finite-UnI } G.\text{finite-sumset } hA1 \ hB1 \ \text{hH1stabC } \text{hstabC}$ *psub-*
set-imp-subset
by (*metis* *finite-subset*)
have $\text{int } (\text{card } (G.\text{sumset } \{a1\} \ (G.\text{stabilizer } C))) - \text{card } (G.\text{sumset}$
 $?A1 \ ?H1) -$
 $\text{card } (G.\text{sumset } ?B1 \ ?H1) \leq \text{int } (\text{card } (G.\text{sumset } \{a1\} \ (G.\text{stabilizer}$
 $C))) -$
 $\text{card } (G.\text{sumset } ?A1 \ ?H1 \cup G.\text{sumset } ?B1 \ ?H1)$
using card-Un-le [of $G.\text{sumset } ?A1 \ ?H1 \ G.\text{sumset } ?B1 \ ?H1$] **by** *linarith*
also have $\dots \leq \text{card } (G.\text{sumset } \{a1\} \ (G.\text{stabilizer } C) - (G.\text{sumset}$
 $?A1 \ ?H1 \cup G.\text{sumset } ?B1 \ ?H1))$
using $\text{diff-card-le-card-Diff}$ [of $G.\text{sumset } ?A1 \ ?H1 \cup G.\text{sumset } ?B1$
 $?H1$
 $G.\text{sumset } \{a1\} \ (G.\text{stabilizer } C)$] *hfin* **by** *linarith*
also have $\dots \leq \text{card } (G.\text{sumset } \{a1\} \ (G.\text{stabilizer } C) - G.\text{sumset } (?A1$

$\cup ?B1) ?H1)$
using $G.sumset-subset-Un1$ **by** *auto*
also have $\dots \leq \text{card } (G.sumset (A \cup ?B') (G.stabilizer C) - G.sumset (A \cup ?B') ?H1)$
proof–
have $hsub: G.sumset \{a1\} (G.stabilizer C) - G.sumset (?A1 \cup ?B1)$
 $?H1 \subseteq$
 $G.sumset (A \cup ?B') (G.stabilizer C) - G.sumset (A \cup ?B') ?H1$
proof
fix x **assume** $hx: x \in G.sumset \{a1\} (G.stabilizer C) - G.sumset (?A1 \cup ?B1) ?H1$
then obtain c **where** $hxac: x = a1 [\oplus] c$ **and** $hc: c \in G.stabilizer C$
using $G.sumset.cases$ **by** *blast*
then have $x \in G.sumset (A \cup ?B') (G.stabilizer C)$ **using** $ha1A$
 hAG
 $G.stabilizer-subset-group$ **by** (*simp add: subset-iff G.sumset.sumsetI*)
moreover have $x \notin G.sumset (A \cup ?B') ?H1$
proof
assume $hx1: x \in G.sumset (A \cup ?B') ?H1$
then obtain y $h1$ **where** $hxy: x = y [\oplus] h1$ **and** $hy: y \in A \cup ?B'$
and
 $hh1: h1 \in ?H1$ **using** $G.sumset.cases$ **by** *blast*
then have $hyG: y \in G$ **and** $hcG: c \in G$ **and** $hh1G: h1 \in G$
using $hAG hB'G G.stabilizer-subset-group hc$ **by** *auto*
then have $y = a1 [\oplus] (c [\oplus] G.inverse h1)$ **using** $hxac hxy ha1A$
 hAG
by (*metis G.associative G.commutative G.composition-closed*)
in-mono
 $G.inverse-closed G.invertible G.invertible-left-inverse2)$
moreover have $h1 \in G.stabilizer C$ **using** $hh1 hH1stabC$ **by** *auto*
moreover hence $c [\oplus] G.inverse h1 \in G.stabilizer C$ **using** hc
 $G.stabilizer-is-subgroup subgroup-def G.group-axioms$
 $group.invertible subgroup.subgroup-inverse-iff$
 $submonoid.sub-composition-closed hh1G$ **by** *metis*
ultimately have $y \in G.sumset \{a1\} (G.stabilizer C)$ **using** $ha1G$
 $hcG hAG ha1A hh1G$
by *blast*
then have $y \in ?A1 \cup ?B1$ **using** hy **by** (*simp add: ha1b1*)
thus *False* **using** $hx hxy hh1 hh1G hyG$ **by** *auto*
qed
ultimately show $x \in G.sumset (A \cup ?B') (G.stabilizer C) - G.sumset (A \cup ?B') ?H1$ **by** *simp*
qed
moreover have $finite (G.sumset \{a1\} (G.stabilizer C) - G.sumset (?A1 \cup ?B1) ?H1)$
using $finite-subset G.finite-sumset hstabC$ **by** *simp*
moreover hence $finite (G.sumset (A \cup ?B') (G.stabilizer C) - G.sumset (A \cup ?B') ?H1)$

using *finite-subset G.finite-sumset hstabC hA hB' finite-UnI* **by**
simp
moreover have $\text{card } (G.\text{sumset } \{a1\} (G.\text{stabilizer } C)) - G.\text{sumset}$
 $(?A1 \cup ?B1) ?H1) \leq$
 $\text{card } (G.\text{sumset } (A \cup ?B') (G.\text{stabilizer } C)) - G.\text{sumset } (A \cup ?B')$
 $?H1)$
using *card-mono hsub calculation(3)* **by** *auto*
ultimately show *?thesis using card-Diff-subset by linarith*
qed
also have $\dots = \text{int } (\text{card } (G.\text{sumset } (A \cup ?B') (G.\text{stabilizer } C))) -$
 $\text{card } (G.\text{sumset } (A \cup ?B') ?H1)$
using *card-Diff-subset[OF hfinsumH1 hsubsumH1] hsumH1card-le* **by**
linarith
finally have $\text{int } (\text{card } (G.\text{stabilizer } C)) - \text{card } (G.\text{sumset } ?A1 ?H1)$
 $- \text{card } (G.\text{sumset } ?B1 ?H1)$
 $\leq \text{int } (\text{card } (G.\text{sumset } (A \cup ?B') (G.\text{stabilizer } C))) - \text{card } (G.\text{sumset}$
 $(A \cup ?B') ?H1)$
using *hCG subset-iff G.card-sumset-singleton-subset-eq G.stabilizer-subset-group*

hAG ha1A **by** *auto*
thus *False using hA1B1ineq* **by** *linarith*
qed
have $\text{int } (\text{card } (G.\text{sumset } (A \cup ?B') (G.\text{stabilizer } C))) -$
 $\text{card } (G.\text{sumset } (A \cup ?B') ?H1) \geq 0$ **by** (*simp add: hsumH1card-le*)
then have $\text{int } (\text{card } (G.\text{stabilizer } C)) -$
 $\text{card } (G.\text{sumset } ?A1 ?H1) - \text{card } (G.\text{sumset } ?B1 ?H1) > 0$ **using**
hA1B1ineq **by** *linarith*
moreover have $\text{int } (\text{card } ?H1) \text{ dvd } \text{int } (\text{card } (G.\text{stabilizer } C)) -$
 $\text{card } (G.\text{sumset } ?A1 ?H1) - \text{card } (G.\text{sumset } ?B1 ?H1)$ **using**
G.stabilizer-subset-stabilizer-dvd hH1stabC psubset-imp-subset int-dvd-int-iff
dvd-diff
G.card-stabilizer-divide-sumset[OF hA1G] G.card-stabilizer-divide-sumset[OF
hB1G]
by *fastforce*
ultimately have $\text{int } (\text{card } (G.\text{stabilizer } C)) -$
 $\text{card } (G.\text{sumset } ?A1 ?H1) - \text{card } (G.\text{sumset } ?B1 ?H1) \geq \text{int } (\text{card}$
 $?H1)$
using *zdvd-imp-le* **by** *blast*
moreover have *hA1-le-sum: card ?A1 ≤ card (G.sumset ?A1 ?H1)*
using *G.sumset-commute G.card-le-sumset G.zero-mem-stabilizer hA1G*
hA1 hH1stabC hstabC
by (*metis finite-subset psubset-imp-subset G.unit-closed*)
moreover have *hB1-le-sum: card ?B1 ≤ card (G.sumset ?B1 ?H1)*
using *G.sumset-commute G.card-le-sumset G.zero-mem-stabilizer hB1G*
hB1 hH1stabC hstabC
by (*metis finite-subset psubset-imp-subset G.unit-closed*)

The above facts combined allow us to deduce the inequality that is referred to as inequality (2) in [1] for $A \cap G.\text{sumset } \{a1\} (G.\text{stabilizer } C)$, $G.\text{sumset } (G.\text{differenceset } B \{b\}) \{a\} \cap G.\text{sumset } \{b1\} (G.\text{stabilizer } C)$ and

$G.stabilizer (G.sumset (A \cap G.sumset \{a1\} (G.stabilizer C)) (G.sumset (G.differenceset B \{b\}) \{a\} \cap G.sumset \{b1\} (G.stabilizer C)))$.

ultimately have $21: int (card (G.stabilizer C)) \geq int (card ?H1) + card ?A1 + card ?B1$
by *linarith*
have $int (card (G.sumset (A \cup ?B') (G.stabilizer C))) - card (G.sumset (A \cup ?B') ?H2) \geq 0$ **by** (*simp add: hsumH2card-le*)
then have $int (card (G.stabilizer C)) - card (G.sumset ?A2 ?H2) - card (G.sumset ?B2 ?H2) > 0$ **using** *hA2B2ineq* **by** *linarith*
moreover have $int (card ?H2) dvd int (card (G.stabilizer C)) - card (G.sumset ?A2 ?H2) - card (G.sumset ?B2 ?H2)$ **using** *psubset-imp-subset*
 $G.stabilizer-subset-stabilizer-dvd$ *hH2stabC int-dvd-int-iff dvd-diff*
 $G.card-stabilizer-divide-sumset[OF hA2G]$ $G.card-stabilizer-divide-sumset[OF hB2G]$
by *fastforce*
ultimately have $int (card (G.stabilizer C)) - card (G.sumset ?A2 ?H2) - card (G.sumset ?B2 ?H2) \geq int (card ?H2)$
using *zdvd-imp-le* **by** *blast*
moreover have $hA2-le-sum: card ?A2 \leq card (G.sumset ?A2 ?H2)$
using $G.sumset-commute$ $G.card-le-sumset$ $G.zero-mem-stabilizer$ $G.stabilizer-subset-group$
 $hA2G$ $hA2$ $hH2stabC$ $hstabC$ *psubset-imp-subset* **by** (*metis finite-subset G.unit-closed*)
moreover have $hB2-le-sum: card ?B2 \leq card (G.sumset ?B2 ?H2)$
using $G.sumset-commute$ $G.card-le-sumset$ $G.zero-mem-stabilizer$ $G.stabilizer-subset-group$
 $hB2G$ $hB2$ $hH2stabC$ $hstabC$ *psubset-imp-subset* **by** (*metis finite-subset G.unit-closed*)

Analogously, the above facts combined allow us to deduce the inequality that is referred to as inequality (2) in [1] for $A \cap G.sumset \{b1\} (G.stabilizer C)$, $G.sumset (G.differenceset B \{b\}) \{a\} \cap G.sumset \{a1\} (G.stabilizer C)$ and $G.stabilizer (G.sumset (A \cap G.sumset \{b1\} (G.stabilizer C)) (G.sumset (G.differenceset B \{b\}) \{a\} \cap G.sumset \{a1\} (G.stabilizer C)))$.

ultimately have $22: int (card (G.stabilizer C)) \geq int (card ?H2) + card ?A2 + card ?B2$
by *linarith*
let $?S = G.sumset \{a1\} (G.stabilizer C) - (?A1 \cup ?B2)$
let $?T = G.sumset \{b1\} (G.stabilizer C) - (?A2 \cup ?B1)$
have $hS : finite ?S$ **and** $hT : finite ?T$ **using** $G.finite-sumset$ $hstabC$ **by** *auto*
have $hST: ?S \cap ?T = \{\}$ **using** $ha1b1stabCne$ $Diff-Int2$ $Diff-Int-distrib2$ $Int-Diff$ $Int-Un-eq(4)$
 $Int-absorb$ $Int-commute$ $Int-empty-right$ $Int-insert-right$ $Un-empty$ $empty-Diff$ $hA1ne$

$hA2ne$ $G.sumset-commute$ $G.sumset-is-empty-iff$ $G.sumset-stabilizer-eq-iff$
by (smt ($verit$, $ccfv-threshold$) $G.sumset-assoc$)
have $hSTcard-le$: $card ?S + card ?T + card (G.sumset (A \cup ?B') \{[0]\})$
 \leq
 $card (G.sumset (A \cup ?B') (G.stabilizer C))$
proof-
have $G.sumset \{a1\} (G.stabilizer C) \subseteq G.sumset (A \cup ?B') (G.stabilizer$
 $C)$ **and**
 $G.sumset \{b1\} (G.stabilizer C) \subseteq G.sumset (A \cup ?B') (G.stabilizer$
 $C)$
using $G.sumset-mono$ $ha1A$ $hb1B'$ $subset-refl$ $empty-subsetI$ $insert-subset$
by $auto$
moreover **have** $(G.sumset (A \cup ?B') \{[0]\}) \subseteq G.sumset (A \cup ?B')$
 $(G.stabilizer C)$
using $G.sumset-mono$ $subset-refl$ $empty-subsetI$ $insert-subset$
 $G.zero-mem-stabilizer$
by $metis$
ultimately **have** $hsub$: $?S \cup ?T \cup (G.sumset (A \cup ?B') \{[0]\}) \subseteq$
 $G.sumset (A \cup ?B') (G.stabilizer C)$ **by** $blast$
have $?S \cap G.sumset (A \cup ?B') \{[0]\} = \{\}$ **by** $auto$
moreover **have** $(?S \cup ?T) \cap G.sumset (A \cup ?B') \{[0]\} = \{\}$ **by** $auto$
ultimately **have** $card ?S + card ?T + card (G.sumset (A \cup ?B')$
 $\{[0]\}) =$
 $card (?S \cup ?T \cup (G.sumset (A \cup ?B') \{[0]\}))$ **using** $card-Un-disjoint$
 hS hT
 $G.finite-sumset$ $finite-UnI$ hA hB' hST **by** ($metis$ $finite.emptyI$
 $finite.insertI$)
also **have** $\dots \leq card (G.sumset (A \cup ?B') (G.stabilizer C))$ **using**
 $card-mono$
 $G.finite-sumset$ $finite-UnI$ hA hB' $hstabC$ $hsub$ **by** $metis$
finally **show** $?thesis$ **by** $simp$
qed
have $hAB-not-conv$: $\neg G.convergent (G.sumset A ?B') A ?B'$ **using**
 $hCmin$ $hstab0$
 $G.convergent-set-def$ $hcardstabC-gt-1$ $hstab0'$ **by** $fastforce$
then **have** $card (G.sumset A ?B') + card \{[0]\} < card (A \cap ?B') +$
 $card (G.sumset (A \cup ?B') \{[0]\})$ **using** $G.convergent-def$ $hAne$ $hB'ne$
 hAG $hB'G$ $hstab0'$
 $subset-refl$ **by** $auto$
then **have** $hAB'sum$: $int (card (G.sumset (A \cup ?B') \{[0]\})) + card (A$
 $\cap ?B') -$
 $card (G.sumset A ?B') > 1$ **by** $simp$
moreover **have** $int (card ?A1) + card ?B1 \leq int (card (G.sumset ?A1$
 $?H1)) +$
 $card (G.sumset ?B1 ?H1)$ **using** $hA1-le-sum$ $hB1-le-sum$ **by** $linarith$
moreover **hence** $int (card ?A1) + card ?B1 - card ?H1 \leq card$
 $(G.sumset ?A1 ?B1)$
using $hindA1B1$ **by** $linarith$
ultimately **have** $int (card ?S) + card ?T + card ?A1 + card ?B1 -$

$\text{card } ?H1 <$
 $\text{int } (\text{card } ?S) + \text{card } ?T + \text{card } (G.\text{sumset } (A \cup ?B') \{[0]\}) + \text{card } (A \cap ?B') -$
 $\text{card } (G.\text{sumset } A ?B') + \text{card } (G.\text{sumset } ?A1 ?B1)$ **by** *linarith*
also have $\dots \leq \text{int } (\text{card } (G.\text{sumset } (A \cup ?B') (G.\text{stabilizer } C))) + \text{card } (A \cap ?B') - \text{card } C$
proof–
have $G.\text{sumset } ?A1 ?B1 \cup C \subseteq G.\text{sumset } A ?B'$ **using** hC *G.sumset-mono*
by *auto*
then have $\text{card } (G.\text{sumset } ?A1 ?B1) + \text{card } C \leq$
 $\text{card } (G.\text{sumset } A ?B')$ **using** *card-Un-disjoint hCfinite G.finite-sumset hA1 hB1 hA1B1Cempty*
card-mono hA hB' **by** *metis*
then show *?thesis* **using** *hSTcard-le* **by** *linarith*
qed

From this, inequality (3) [1] follows for $A \cap G.\text{sumset } \{a1\} (G.\text{stabilizer } C)$, $G.\text{sumset } (G.\text{differenceset } B \{b\}) \{a\} \cap G.\text{sumset } \{b1\} (G.\text{stabilizer } C)$ and $G.\text{stabilizer } (G.\text{sumset } (A \cap G.\text{sumset } \{a1\} (G.\text{stabilizer } C)) (G.\text{sumset } (G.\text{differenceset } B \{b\}) \{a\} \cap G.\text{sumset } \{b1\} (G.\text{stabilizer } C)))$.

finally have $31: \text{int } (\text{card } ?S) + \text{card } ?T + \text{card } ?A1 + \text{card } ?B1 -$
 $\text{card } ?H1 <$
 $\text{card } (G.\text{stabilizer } C)$ **using** $hC\text{card}$ **by** *linarith*
have $\text{int } (\text{card } ?A2) + \text{card } ?B2 \leq \text{int } (\text{card } (G.\text{sumset } ?A2 ?H2)) +$
 $\text{card } (G.\text{sumset } ?B2 ?H2)$ **using** *hA2-le-sum hB2-le-sum* **by** *linarith*
moreover hence $\text{int } (\text{card } ?A2) + \text{card } ?B2 - \text{card } ?H2 \leq \text{card } (G.\text{sumset } ?A2 ?B2)$
using *hindA2B2* **by** *linarith*
ultimately have $\text{int } (\text{card } ?S) + \text{card } ?T + \text{card } ?A2 + \text{card } ?B2 -$
 $\text{card } ?H2 <$
 $\text{int } (\text{card } ?S) + \text{card } ?T + \text{card } (G.\text{sumset } (A \cup ?B') \{[0]\}) + \text{card } (A \cap ?B') -$
 $\text{card } (G.\text{sumset } A ?B') + \text{card } (G.\text{sumset } ?A2 ?B2)$ **using** *hAB'sum*
by *linarith*
also have $\dots \leq \text{int } (\text{card } (G.\text{sumset } (A \cup ?B') (G.\text{stabilizer } C))) + \text{card } (A \cap ?B') - \text{card } C$
proof–
have $G.\text{sumset } ?A2 ?B2 \cup C \subseteq G.\text{sumset } A ?B'$ **using** hC *G.sumset-mono*
by *auto*
then have $\text{card } (G.\text{sumset } ?A2 ?B2) + \text{card } C \leq$
 $\text{card } (G.\text{sumset } A ?B')$ **using** *card-Un-disjoint hCfinite G.finite-sumset hA2 hB2 hA1B1Cempty*
card-mono hA hB' hB2A2Cempty G.sumset-commute **by** *metis*
then show *?thesis* **using** *hSTcard-le* **by** *linarith*
qed

From this, inequality (3) [1] follows for $A \cap G.\text{sumset } \{b1\} (G.\text{stabilizer } C)$, $G.\text{sumset } (G.\text{differenceset } B \{b\}) \{a\} \cap G.\text{sumset } \{a1\} (G.\text{stabilizer } C)$ and $G.\text{stabilizer } (G.\text{sumset } (A \cap G.\text{sumset } \{b1\} (G.\text{stabilizer } C)) (G.\text{sumset } (G.\text{differenceset } B \{b\}) \{a\} \cap G.\text{sumset } \{a1\} (G.\text{stabilizer } C)))$.

$(G.differenceset\ B\ \{b\})\ \{a\} \cap G.sumset\ \{a1\}\ (G.stabilizer\ C))$.

finally have $32: int\ (card\ ?S) + card\ ?T + card\ ?A2 + card\ ?B2 - card\ ?H2 < card\ (G.stabilizer\ C)$ **using** $hCcard$ **by** $linarith$

Adding together the four inequalities obtained by versions of inequalities (2) and (3) and dividing by 2 gives the following inequality:

have $4: 2 * int\ (card\ (G.stabilizer\ C)) > int\ (card\ ?A1) + card\ ?B2 + card\ ?S + card\ ?T + card\ ?B1 + card\ ?A2$
using $21\ 22\ 31\ 32$ **by** $linarith$

have $G.sumset\ \{a1\}\ (G.stabilizer\ C) = (?S \cup ?A1) \cup ?B2$ **and**
 $hb1T: G.sumset\ \{b1\}\ (G.stabilizer\ C) = (?T \cup ?A2) \cup ?B1$ **by** $auto$
then have $card\ (G.stabilizer\ C) \leq card\ ?S + card\ ?A1 + card\ ?B2$
using $card-Un-le[of\ ?S\ ?A1]$ $card-Un-le[of\ ?S \cup ?A1\ ?B2]$
 $G.card-sumset-singleton-subset-eq\ G.stabilizer-subset-group\ ha1A\ hAG$

by $auto$

moreover have $card\ (G.stabilizer\ C) \leq card\ ?T + card\ ?A2 + card\ ?B1$

using $hb1T\ card-Un-le[of\ ?T\ ?A2]$ $card-Un-le[of\ ?T \cup ?A2\ ?B1]$
 $G.card-sumset-singleton-subset-eq\ G.stabilizer-subset-group\ hb1B'\ hb'G$

by $auto$

Combining the inequality labelled $4::'b$ above with the above facts, the claim follows:

ultimately show $?thesis$ **using** 4 **by** $linarith$
qed

It remains to transfer the statement of inequality labelled 1 into an analogous one, which replaces $G.sumset\ (G.differenceset\ B\ \{b\})\ \{a\}$ with B .

have $2: card\ (G.sumset\ A\ (G.stabilizer\ (G.sumset\ A\ B))) = card\ (G.sumset\ A\ (G.stabilizer\ (G.sumset\ A\ ?B')))$
using $hstabeq$ **by** $auto$

have $3: card\ (G.sumset\ B\ (G.stabilizer\ (G.sumset\ A\ B))) = card\ (G.sumset\ ?B'\ (G.stabilizer\ (G.sumset\ A\ ?B')))$ **using** $hstabeq$
 $hstab0'\ G.sumset-commute$

by $(metis\ G.card-differenceset-singleton-mem-eq\ G.card-sumset-singleton-subset-eq$

$G.sumset-D(1)\ G.sumset-commute\ G.sumset-subset-carrier\ Int-absorb1\ Int-commute\ hBG\ ha1G\ hbG)$

then show $?thesis$ **using** $1\ 2\ 3\ hstabeq\ hstab0'\ htranslate$ **by** $auto$
qed

next

case $hstabne0: False$

let $?K = G.stabilizer\ (G.sumset\ A\ B)$

have $hcardK-gt-1: card\ ?K > 1$ **using** $G.stabilizer-finite\ G.sumset-subset-carrier\ G.finite-sumset$

$hA\ hB\ hstabne0\ G.zero-mem-stabilizer$ **by** $(metis\ card-0-eq\ card-1-singletonE\ G.card-sumset-0-iff$

```

empty-iff hAG hAne hBG hBne insertE less-one linorder-neqE-nat)
interpret K: subgroup-of-additive-abelian-group ?K G ([⊕]) [0]
using G.stabilizer-is-subgroup subgroup-of-additive-abelian-group-def
by (metis G.abelian-group-axioms G.group-axioms hGroupG subgroup-of-abelian-group-def

subgroup-of-group.intro)
let ?φ = K.Class
have hφ:  $\bigwedge a. a \in G \implies ?\phi a = (\lambda x. G.sumset \{x\} ?K) a$ 
using K.Left-Coset-Class-unit G.Left-Coset-eq-sumset-stabilizer by simp
interpret GK: additive-abelian-group G.Factor-Group G ?K K.quotient-composition
K.Class [0]
proof
fix x y assume x ∈ K.Partition and y ∈ K.Partition
then obtain a b where x = K.Class a and y = K.Class b and a ∈ G
and b ∈ G
by (meson K.representant-exists)
then show K.quotient-composition x y = K.quotient-composition y x
using K.Class-commutes-with-composition G.commutative by presburger
qed
have hGroupGK: additive-abelian-group (G.Factor-Group G ?K) K.quotient-composition
(K.Class [0]) ..

```

Here, we specialize the induction hypothesis to the factor group.

```

let ?K-repr = K.φ ‘ K.Partition
have  $\bigwedge x y. x \in ?K\text{-repr} \implies y \in ?K\text{-repr} \implies K.\text{quot-comp-alt } x y =$ 
K.quot-comp-alt y x
using K.quot-comp-alt-def G.commutative K.phi-image-subset subsetD by
(metis (full-types))
then interpret K-repr: additive-abelian-group ?K-repr K.quot-comp-alt K.φ
?K
using Group-Theory.group.axioms(1)[OF K.phi-image-group]
by (auto simp add: additive-abelian-group-def abelian-group-def K.phi-image-group
commutative-monoid-axioms-def commutative-monoid-def)
have hindrepr:  $\bigwedge m C D. m < n \longrightarrow C \subseteq ?K\text{-repr} \longrightarrow D \subseteq ?K\text{-repr} \longrightarrow$ 
finite C  $\longrightarrow$ 
finite D  $\longrightarrow C \neq \{\}$   $\longrightarrow D \neq \{\}$   $\longrightarrow \text{card } (K\text{-repr.sumset } C D) + \text{card } C$ 
= m  $\longrightarrow$ 
card (K-repr.sumset C (K-repr.stabilizer (K-repr.sumset C D))) +
card (K-repr.sumset D (K-repr.stabilizer (K-repr.sumset C D))) - card
(K-repr.stabilizer (K-repr.sumset C D)) ≤
card (K-repr.sumset C D) using hind K-repr.additive-abelian-group-axioms
by blast
have hindfactor:  $\bigwedge m C D. m < n \longrightarrow C \subseteq K.Partition \longrightarrow D \subseteq$ 
K.Partition  $\longrightarrow$  finite C  $\longrightarrow$ 
finite D  $\longrightarrow C \neq \{\}$   $\longrightarrow D \neq \{\}$   $\longrightarrow \text{card } (GK.sumset C D) + \text{card } C =$ 
m  $\longrightarrow$ 
card (GK.sumset C (GK.stabilizer (GK.sumset C D))) +
card (GK.sumset D (GK.stabilizer (GK.sumset C D))) - card (GK.stabilizer
(GK.sumset C D)) ≤

```

$card (GK.sumset C D)$
proof(*intro impI*)
fix $m C D$ **assume** $hmn: m < n$ **and** $hCK: C \subseteq K.Partition$ **and** $hDK: D \subseteq K.Partition$ **and**
 $hC: finite C$ **and** $hD: finite D$ **and** $hCne: C \neq \{\}$ **and** $hDne: D \neq \{\}$ **and**
 $hCDcard: card (GK.sumset C D) + card C = m$
let $?C = K.\varphi ' C$ **and** $?D = K.\varphi ' D$
have $hCrepr: ?C \subseteq ?K-repr$ **and** $hDrepr: ?D \subseteq ?K-repr$ **using** $hCK hDK$
by *auto*
have $hCfin: finite ?C$ **and** $hDfin: finite ?D$ **and** $hCne-1: ?C \neq \{\}$ **and**
 $hDne-1: ?D \neq \{\}$ **using** $hC hD hCne hDne$ **by** *auto*
have $hcardC: card ?C = card C$ **using** $K.phi-inj-on hC card-image$
inj-on-subset hCK **by** *metis*
have $card (GK.sumset C D) = card (K-repr.sumset ?C ?D)$
using $card-image K.phi-inj-on inj-on-subset K.phi-image-sumset-eq$
 $GK.sumset-subset-carrier hCK hDK$ **by** (*smt (verit, best)*)
then **have** $card (K-repr.sumset ?C ?D) + card ?C = m$ **using** $hCDcard$
 $hcardC$ **by** *presburger*
then **have** $card (K-repr.sumset ?C (K-repr.stabilizer (K-repr.sumset ?C$
 $?D))) +$
 $card (K-repr.sumset ?D (K-repr.stabilizer (K-repr.sumset ?C ?D))) - card$
 $(K-repr.stabilizer (K-repr.sumset ?C ?D)) \leq$
 $card (K-repr.sumset ?C ?D)$
using $hindrepr hCfin hDfin hCne-1 hDne-1 hCrepr hDrepr hmn$ **by** *blast*
then **show** $card (GK.sumset C (GK.stabilizer (GK.sumset C D))) +$
 $card (GK.sumset D (GK.stabilizer (GK.sumset C D))) - card (GK.stabilizer$
 $(GK.sumset C D)) \leq$
 $card (GK.sumset C D)$ **using** $K.phi-image-sumset-eq K.phi-image-stabilizer-eq$
 $K.phi-inj-on inj-on-subset hCK hDK card-image$
by (*smt (z3) GK.stabilizer-subset-group GK.sumset-subset-carrier*)
qed
have $hstab0: GK.stabilizer (?\varphi '(G.sumset A B)) = \{K.Class [0]\}$
proof
show $GK.stabilizer (?\varphi ' G.sumset A B) \subseteq \{K.Class [0]\}$
proof
fix x **assume** $hx: x \in GK.stabilizer (?\varphi ' G.sumset A B)$
moreover **have** $?\varphi ' G.sumset A B \subseteq K.Partition$
using $K.natural.map-closed G.sumset-subset-carrier$ **by** *blast*
ultimately **have** $hsum: GK.sumset \{x\} (?\varphi ' G.sumset A B) = ?\varphi ' G.sumset A B$
using $GK.stabilizer-def$ **by** *auto*
obtain x' **where** $hx\varphi: x = ?\varphi x'$ **and** $hx'G: x' \in G$
using $hx GK.stabilizer-subset-group K.representant-exists$ **by** *force*
have $hsumset: GK.sumset \{x\} (?\varphi ' G.sumset A B) = (\lambda a. ?\varphi (x' [\oplus$
 $a])) ' G.sumset A B$
proof
show $GK.sumset \{x\} (?\varphi ' G.sumset A B) \subseteq (\lambda a. ?\varphi (x' [\oplus a])) ' G.sumset A B$

```

proof
  fix  $y$  assume  $y \in GK.sumset \{x\}$  ( $?φ \text{ ' } G.sumset A B$ )
  then obtain  $z$  where  $z \in G.sumset A B$  and  $y = K.quotient-composition$ 
( $?φ x'$ ) ( $?φ z$ )
    using  $GK.sumset.cases \ hxφ$  by blast
    then show  $y \in (\lambda a. ?φ (x' [\oplus] a)) \text{ ' } G.sumset A B$ 
      using  $K.Class-commutes-with-composition \ G.composition-closed$ 
 $hx'G \ G.sumset.cases$ 
       $imageI$  by metis
    qed
  next
    show  $(\lambda a. ?φ (x' [\oplus] a)) \text{ ' } G.sumset A B \subseteq GK.sumset \{x\}$  ( $?φ \text{ ' }$ 
 $G.sumset A B$ )
    proof
      fix  $y$  assume  $y \in (\lambda a. ?φ (x' [\oplus] a)) \text{ ' } G.sumset A B$ 
      then obtain  $z$  where  $hz: z \in G.sumset A B$  and  $y = ?φ (x' [\oplus] z)$ 
by blast
        then have  $y = K.quotient-composition$  ( $?φ x'$ ) ( $?φ z$ ) using
 $K.Class-commutes-with-composition \ G.composition-closed \ hx'G$ 
 $G.sumset.cases$  by metis
        then show  $y \in GK.sumset \{x\}$  ( $?φ \text{ ' } G.sumset A B$ )
          using  $hxφ \ hz \ imageI \ hx \ GK.sumset.sumsetI \ K.natural.map-closed$ 
          by (metis  $G.composition-closed \ hx'G \ insertCI \ G.sumset.cases$ )
        qed
      qed
    have  $G.sumset \{x'\} (G.sumset A B) \subseteq G.sumset (G.sumset A B) \ ?K$ 
    proof
      fix  $y$  assume  $y \in G.sumset \{x'\} (G.sumset A B)$ 
      then obtain  $z$  where  $hz: z \in G.sumset A B$  and  $hy: y = x' [\oplus] z$ 
using  $G.sumset.cases$  by blast
      then have  $?φ (x' [\oplus] z) \in ?φ \text{ ' } G.sumset A B$  using  $hsum \ hsumset$ 
by blast
      then obtain  $w$  where  $hw: \{w\} \subseteq G.sumset A B$  and  $w \in G.sumset$ 
 $A B$ 
        and  $?φ (x' [\oplus] z) = ?φ w$  by auto
        then have  $(x' [\oplus] z) \in G.differenceset (G.sumset \{w\} \ ?K) \ ?K$ 
using  $hφ \ G.sumset-subset-carrier \ hx'G \ hz \ G.sumset-eq-subset-differenceset$ 
 $G.composition-closed \ G.stabilizer-is-nonempty \ G.stabilizer-subset-group$ 
 $G.sumset.cases$ 
         $K.Class-self \ G.differenceset-stabilizer-eq \ G.sumset-assoc$  by metis
        moreover have  $G.differenceset (G.sumset \{w\} \ ?K) \ ?K \subseteq G.sumset$ 
 $\{w\} \ ?K$ 
          using  $hw$  by (simp add: G.differenceset-stabilizer-eq \ G.sumset-assoc)
          ultimately show  $y \in G.sumset (G.sumset A B) \ ?K$  using  $hy \ hw$ 
 $G.sumset-mono \ subsetD$ 
           $subset-refl$  by blast
        qed
      moreover have  $G.sumset (G.sumset A B) \ ?K = G.sumset A B$ 

```

```

    using G.sumset-commute G.sumset-stabilizer-eq-self G.sumset-subset-carrier
  by auto
    ultimately have G.sumset {x'} (G.sumset A B) = G.sumset A B
      by (metis G.finite-sumset G.sumset-subset-carrier card-subset-eq
        G.card-sumset-singleton-subset-eq hA hB hx'G)
    then have x' ∈ ?K using hx'G by (meson empty-subsetI G.finite-sumset
hA hB insert-subset
      G.sumset-eq-sub-stabilizer G.sumset-subset-carrier)
    then show x ∈ {K.Class [0]} using hxφ
      by (metis K.Block-self K.Normal-def K.quotient.unit-closed insertCI)
    qed
  next
  show {K.Class [0]} ⊆ GK.stabilizer (K.Class ' G.sumset A B)
    using GK.zero-mem-stabilizer by auto
  qed
  interpret group-epimorphism ?φ G ([⊕]) [0] G.Factor-Group G ?K
    K.quotient-composition K.Class [0] ..
  interpret GKN: normal-subgroup-in-kernel K.Class G ([⊕]) [0] G.Factor-Group
G ?K
    K.quotient-composition K.Class [0] ?K
  proof
    show ?K ⊆ Ker using K.Block-self K.Normal-def K.quotient.unit-closed
  by blast
  qed
  have hsumK: card (G.sumset A B) = card ?K * card (?φ ' (G.sumset A
B))
  using G.finite-sumset hA hB G.sumset-subset-carrier G.Union-stabilizer-Class-eq

    G.sumset-subset-carrier K.Union-Coset-card-eq by simp
  have hGKsumset: GK.sumset (?φ ' A) (?φ ' B) = ?φ ' (G.sumset A B)
  proof
    show GK.sumset (?φ ' A) (?φ ' B) ⊆ ?φ ' G.sumset A B
  proof
    fix x assume x ∈ GK.sumset (?φ ' A) (?φ ' B)
    then obtain a b where ha: a ∈ A and hb: b ∈ B and
      x = K.quotient-composition (?φ a) (?φ b) using GK.sumset.cases by
blast
    then have x = ?φ (a [⊕] b) by (meson K.Class-commutes-with-composition
hAG hBG in-mono)
    then show x ∈ ?φ ' G.sumset A B using ha hb hAG hBG by blast
  qed
  next
  show ?φ ' G.sumset A B ⊆ GK.sumset (?φ ' A) (?φ ' B)
  proof
    fix x assume x ∈ ?φ ' G.sumset A B
    then obtain c where c ∈ G.sumset A B and x = ?φ c by blast
    then obtain a b where a ∈ A and b ∈ B and x = ?φ (a [⊕] b)
      using G.sumset.cases by metis
    then show x ∈ GK.sumset (?φ ' A) (?φ ' B) using GK.sumset.cases

```

```

      K.Class-commutes-with-composition hAG hBG in-mono
    by (smt (verit, best) GK.sumset.simps K.natural.map-closed imageI)
  qed
  have hAK: card (G.sumset A ?K) = card ?K * card (?φ ' A) using hAG
K.Union-Coset-card-eq hA
      G.sumset-stabilizer-eq-Class-Union G.Class-image-sumset-stabilizer-eq
    by (smt (verit, ccfv-threshold) card-0-eq G.card-sumset-0-iff G.finite-sumset
hAne hB hBG hBne
      G.stabilizer-finite G.sumset-subset-carrier)
    have hBK: card (G.sumset B ?K) = card ?K * card (?φ ' B) using hBG
K.Union-Coset-card-eq hB
      G.sumset-stabilizer-eq-Class-Union G.Class-image-sumset-stabilizer-eq
    by (smt (verit, ccfv-SIG) card-0-eq G.card-sumset-0-iff G.finite-sumset hA
hAG hAne hBne
      G.stabilizer-finite G.sumset-subset-carrier)
    have card (?φ ' A) ≤ card A by (simp add: card-image-le hA)
    moreover have card (?φ ' (G.sumset A B)) < card (G.sumset A B) using
hsumK hcardK-gt-1
      G.card-sumset-0-iff hA hB hAne hBne by (metis card-eq-0-iff card-image-le
hAG hBG
      le-neq-implies-less less-not-refl3 mult-cancel2 nat-mult-1)
    ultimately have card (GK.sumset (?φ ' A) (?φ ' B)) + card (?φ ' A) <
card (G.sumset A B) + card A
      using hGKsumset by auto
    then obtain m where m < n and card (GK.sumset (?φ ' A) (?φ ' B)) +
card (?φ ' A) = m
      using hcardsum by blast
    moreover have hφAsub: ?φ ' A ⊆ G.Factor-Group G ?K
    proof
      fix x assume x ∈ ?φ ' A then obtain a where a ∈ G and ?φ a = x
using hAG by blast
      then show x ∈ G.Factor-Group G ?K by blast
    qed
    moreover have hφBsub: ?φ ' B ⊆ G.Factor-Group G ?K
    proof
      fix x assume x ∈ ?φ ' B then obtain b where b ∈ G and ?φ b = x
using hBG by blast
      then show x ∈ G.Factor-Group G ?K by blast
    qed
    moreover have hφA: finite (?φ ' A) and hφB: finite (?φ ' B) and hφAne:
?φ ' A ≠ {} and
      hφBne: ?φ ' B ≠ {} using hA hB hAne hBne by auto
    moreover have additive-abelian-group (G.Factor-Group G ?K) K.quotient-composition
      (K.Class [0]) ..
    moreover have GK.stabilizer (GK.sumset (?φ ' A) (?φ ' B)) = {K.Class
[0]}
      using hstab0 hGKsumset by auto

```

ultimately have $\text{hind}\varphi: \text{card} (GK.\text{sumset} (? \varphi ' A) (? \varphi ' B)) \geq$
 $\text{card} (GK.\text{sumset} (? \varphi ' A) \{K.\text{Class} [0]\}) + \text{card} (GK.\text{sumset} (? \varphi ' B)$
 $\{K.\text{Class} [0]\}) - 1$
using $\text{hindfactor}[of m ? \varphi ' A ? \varphi ' B]$ **by** simp
have $h\varphi\text{sum}A: GK.\text{sumset} (? \varphi ' A) \{K.\text{Class} [0]\} = ? \varphi ' A$
by $(\text{simp add: Int-absorb1 Int-commute } h\varphi\text{Asub})$
have $h\varphi\text{sum}B: GK.\text{sumset} (? \varphi ' B) \{K.\text{Class} [0]\} = ? \varphi ' B$
by $(\text{simp add: Int-absorb1 Int-commute } h\varphi\text{Bsub})$
have $\text{card} (G.\text{sumset} A ?K) + \text{card} (G.\text{sumset} B ?K) - \text{card} ?K =$
 $\text{card} ?K * (\text{card} (? \varphi ' A) + \text{card} (? \varphi ' B) - 1)$
using $hAK hBK \text{add-mult-distrib2 diff-mult-distrib2 nat-mult-1-right}$ **by**
 presburger
also have $\dots \leq \text{card} ?K * \text{card} (GK.\text{sumset} (? \varphi ' A) (? \varphi ' B))$
using $\text{hind}\varphi h\varphi\text{sum}A h\varphi\text{sum}B$ **by** simp
finally show $?thesis$ **by** $(\text{simp add: hGKsumset hsumK})$
qed
qed
qed
thus $?thesis$ **using** $\text{assms } hAne hBne \text{additive-abelian-group-axioms}$ **by** blast
qed

2.2 Strict version of Kneser's Theorem

We show a strict version of Kneser's Theorem as presented in Theorem 3.2 of [3].

theorem Kneser-strict-aux : **fixes** A and B **assumes** $hAG: A \subseteq G$ and $hBG: B \subseteq G$ and hA : $\text{finite } A$
and hB : $\text{finite } B$ **and** $hAne: A \neq \{\}$ **and** $hBne: B \neq \{\}$ **and**
 $hineq: \text{card} (\text{sumset } A B) > \text{card} (\text{sumset } A (\text{stabilizer} (\text{sumset } A B))) +$
 $\text{card} (\text{sumset } B (\text{stabilizer} (\text{sumset } A B))) - \text{card} (\text{stabilizer} (\text{sumset } A B))$
shows $\text{card} (\text{sumset } A B) \geq \text{card } A + \text{card } B$

proof –

let $?H = \text{stabilizer} (\text{sumset } A B)$
have $hfin$: $\text{finite } ?H$ **using** $\text{stabilizer-subset-group stabilizer-finite sumset-subset-carrier}$
 $\text{finite-sumset assms sumset-is-empty-iff sumset-stabilizer-eq-self}$ **by** metis
have $\text{card } ?H \text{ dvd } \text{card} (\text{sumset } A B)$ **and** $\text{card } ?H \text{ dvd} (\text{card} (\text{sumset } A (\text{stabilizer}$
 $(\text{sumset } A B))) +$
 $\text{card} (\text{sumset } B (\text{stabilizer} (\text{sumset } A B))) - \text{card } ?H$
using $\text{card-stabilizer-divide-sumset } hAG hBG \text{card-stabilizer-sumset-divide-sumset}$
by auto
then have $\text{card} (\text{sumset } A B) \geq \text{card} (\text{sumset } A ?H) + \text{card} (\text{sumset } B ?H)$
using $hineq$
by $(\text{metis } \text{diff-le-mono2 dvd-add-right-iff dvd-imp-le le-diff-conv less-imp-add-positive})$
moreover have $\text{card } A + \text{card } B \leq \text{card} (\text{sumset } A ?H) + \text{card} (\text{sumset } B ?H)$
using $\text{card-le-sumset sumset-commute assms stabilizer-subset-group stabilizer-is-nonempty}$
 $\text{Int-emptyI inf.orderE add-mono hfin}$ **by** metis

ultimately show *?thesis* by *linarith*
qed

theorem *Kneser-strict*: fixes A and B assumes $A \subseteq G$ and $B \subseteq G$ and *finite* A and *finite* B
and *stabilizer* (*sumset* A B) = H and $A \neq \{\}$ and $B \neq \{\}$ and *card* (*sumset* A B) < *card* A + *card* B
shows *card* (*sumset* A B) = *card* (*sumset* A (*stabilizer* (*sumset* A B))) +
card (*sumset* B (*stabilizer* (*sumset* A B))) - *card* (*stabilizer* (*sumset* A B))
using *Kneser* *Kneser-strict-ax* *assms* *le-antisym* *nat-less-le* by *metis*

2.3 The Cauchy–Davenport Theorem

We show the Cauchy–Davenport Theorem as a corollary of Kneser’s Theorem, following a comment on Theorem 3.2 in [3].

interpretation *Z-p*: *additive-abelian-group* $\{0..int ((p :: nat)-1)\}$ ($\lambda x y. ((x + y) \bmod int p)$) $0::int$
using *additive-abelian-group-def* *residue-group*[*of p*] by *fastforce*

theorem *Cauchy-Davenport*:

fixes $p :: nat$
assumes *prime* p and $A \neq \{\}$ and $B \neq \{\}$ and *finite* A and *finite* B and
 $A \subseteq \{0..p-1\}$ and $B \subseteq \{0..p-1\}$
shows *card* (*Z-p.sumset* p A B) \geq *Min* $\{p, \text{card } A + \text{card } B - 1\}$

proof(*cases* *Z-p.stabilizer* p (*Z-p.sumset* p A B) = $\{0\}$)

case *True*

moreover have *Z-p.sumset* p A $\{0\}$ = A and *Z-p.sumset* p B $\{0\}$ = B

using *assms* *Z-p.sumset-D(1)* by *auto*

ultimately show *?thesis* using *Z-p.Kneser*[*of A p B*] *assms* by *simp*

next

case *hne*: *False*

let $?H = \text{Z-p.stabilizer } p (\text{Z-p.sumset } p A B)$

have $?H = \{0..int(p-1)\}$ using *hne* *Z-p.stabilizer-is-subgroup*[*of p Z-p.sumset p A B*]

residue-group-simple[*OF assms(1)*] by *blast*

moreover have $p \geq 2$ using *assms(1)* by (*simp add: prime-ge-2-nat*)

ultimately have *card* $?H = p$ using *card-atLeastAtMost* by (*simp add: of-nat-diff*)

then have $p \leq \text{card } (\text{Z-p.sumset } p A B)$

using *Z-p.card-stabilizer-le card-0-eq* *assms* *Z-p.card-sumset-0-iff* *Z-p.sumset.cases*

Z-p.sumset-subset-carrier *Z-p.finite-sumset* by *metis*

then show *?thesis* by *auto*

qed

end

end

References

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- [2] M. B. Nathanson. *Additive Number Theory: Inverse Problems and the Geometry of Sumsets*, volume 165 of *Graduate Texts in Mathematics*. Springer-Verlag, 1996.
- [3] I. Z. Ruzsa. Sumsets and structure, 2008. Course notes, available on <https://www.math.cmu.edu/users/af1p/Teaching/AdditiveCombinatorics/Additive-Combinatorics.pdf>.