The Isoperimetric Inequality for Rectifiable Simple Closed Curves

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Abstract

This entry formalises the planar isoperimetric inequality for rectifiable simple closed curves. If a curve has length L, the area of its inside is at most $L^2/(4\pi)$; equality implies that its image is a circle. The development proves a Wirtinger inequality, a special area formula for convex curves, and a convexification argument. It builds on the `Fourier` AFP entry and analysis material in Isabelle2026.

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1 Proof outline and provenance

The proof first establishes the inequality for convex curves. Arc-length reparametrisation and the Wirtinger inequality supply its analytic core; a special case of Green's area formula relates the resulting integral to the enclosed area. Convexification then reduces a general rectifiable simple closed curve to the convex case and accounts for equality.

This development was ported from John Harrison's HOL Light formalisation `100/isoperimetric.ml`.[1] The arc-length material also follows `Multivariate/integration.ml`.[2] Claude assisted with the translation; the Isabelle proofs required substantial manual revision. There were no additional textual sources. The Wirtinger proof uses the AFP entry *Fourier Series*.[3]

2 Part 1: The Wirtinger inequality

theory *Wirtinger*
imports *HOL−Analysis.Rectifiable-Path Fourier.Square-Integrable*

begin

Convex curve lemmas: Switching between views of a convex simple closed curve.

lemma *convex-hull-eq-closure-inside*:
fixes *g* :: *real* ⇒ *complex*
assumes *g*: *simple-path g pathfinish g* = *pathstart g*
and *conv*: *convex* (*inside* (*path-image g*))
shows *convex hull* (*path-image g*) = *closure* (*inside* (*path-image g*))
⟨*proof*⟩

lemma *frontier-convex-hull-eq-path-image*:
fixes *g* :: *real* ⇒ *complex*
assumes *simple-path g pathfinish g* = *pathstart g convex* (*inside* (*path-image g*))
shows *frontier* (*convex hull* (*path-image g*)) = *path-image g*
⟨*proof*⟩

lemma *frontier-convex-hull-subset-path-image*:
fixes *g* :: *real* ⇒ *complex*
assumes *simple-path g pathfinish g* = *pathstart g*
path-image g ⊆ *frontier* (*convex hull* (*path-image g*))
shows *frontier* (*convex hull path-image g*) ⊆ *path-image g*
⟨*proof*⟩

locale *W* =
fixes *f f′* :: *real* ⇒ *real* **and** *a*::*real*
assumes *f′hsd*: ⋀*x*. *x* ∈ {*0*..*2*∗*pi*} ⟹ (*f′ has-integral* (*f x* − *f 0*)) {*0*..*x*}
and *feq*: *f* (*2*∗*pi*) = *f 0*
and *f0*: (*f has-integral 0*) {*0*..*2*∗*pi*}
and *f′2*: (λ*x*. (*f′ x*)²) *integrable-on* {*0*..*2*∗*pi*}
and *a*: *0* ≤ *a a* < *pi f* (*a* + *pi*) = *f a*

[1] https://github.com/jrh13/hol-light/blob/master/100/isoperimetric.ml
[2] https://github.com/jrh13/hol-light/blob/master/Multivariate/integration.ml
[3] https://isa-afp.org/entries/Fourier.html

begin

definition g **where** $g \equiv \lambda x.\ (f\ x - f\ a)^2\ /\ tan\ (x - a)$
definition g' **where** $g' \equiv \lambda x.\ (f'\ x)^2 - (f\ x - f\ a)^2 - (f'\ x - (f\ x - f\ a)\ /\ tan\ (x - a))^2$

lemma *f'*: ‹f' *integrable-on* $\{0..2*pi\}$›
⟨*proof*⟩

lemma *f'abs*: ‹f' *absolutely-integrable-on* $\{0..2*pi\}$›
⟨*proof*⟩

lemma *contf*: ‹*continuous-on* $\{0..2*pi\}$ f›
⟨*proof*⟩

The integral over completely trouble-free intervals.

lemma *trouble-free*: (g' *has-integral* $g\ d - g\ c$) $\{c..d\}$
if $c \le d$
and *sub-cd*: $\{c..d\} \subseteq \{0..2*pi\}$
and *sin-nz*: $\bigwedge x.\ x \in \{c..d\} \Longrightarrow sin\ (x - a) \neq 0$
for $c\ d$
⟨*proof*⟩

Continuity of g.

lemma *g-cont*: *continuous-on* $\{0..2*pi\}$ g
⟨*proof*⟩

The integral over mainly trouble-free intervals: we only need $sin(x - a) \neq 0$ on the open interior, allowing zeros at the endpoints.

lemma *mainly-trouble-free*:
assumes $c \le d$ **and** *cd-sub*: $\{c..d\} \subseteq \{0..2*pi\}$ **and** *sin-nz*: $\bigwedge x.\ x \in \{c<..<d\} \Longrightarrow sin\ (x - a) \neq 0$
shows (g' *has-integral* $g\ d - g\ c$) $\{c..d\}$
⟨*proof*⟩

end

theorem *Wirtinger-inequality*:
fixes $f\ f' :: real \Rightarrow real$
assumes *f'hsd*: $\bigwedge x.\ x \in \{0..2*pi\} \Longrightarrow (f'$ *has-integral* $(f\ x - f\ 0))\ \{0..x\}$
and *feq*: $f\ (2*pi) = f\ 0$
and *f0*: (f *has-integral* 0) $\{0..2*pi\}$
and *f'2*: $(\lambda x.\ (f'\ x)^2)$ *integrable-on* $\{0..2*pi\}$
shows $(\lambda x.\ (f\ x)^2)$ *integrable-on* $\{0..2*pi\}$
and *integral* $\{0..2*pi\}$ $(\lambda x.\ (f\ x)^2) \le$ *integral* $\{0..2*pi\}$ $(\lambda x.\ (f'\ x)^2)$
and *integral* $\{0..2*pi\}$ $(\lambda x.\ (f\ x)^2) =$ *integral* $\{0..2*pi\}$ $(\lambda x.\ (f'\ x)^2) \Longrightarrow$
$\exists\, c\ a.\ \forall\, x \in \{0..2*pi\}.\ f\ x = c * sin\ (x - a)$
⟨*proof*⟩

theorem *scaled-Wirtinger-inequality*:
fixes $f\ f' :: real \Rightarrow real$
assumes f': $\bigwedge x.\ x \in \{0..1\} \Longrightarrow (f'\ has\text{-}integral\ (f\ x - f\ 0))\ \{0..x\}$
and $f\ 1 = f\ 0$
and *f-int*: $(f\ has\text{-}integral\ 0)\ \{0..1\}$
and f'*-int*: $(\lambda x.\ (f'\ x)^2)\ integrable\text{-}on\ \{0..1\}$
shows $(\lambda x.\ (f\ x)^2)\ integrable\text{-}on\ \{0..1\}$
and $integral\ \{0..1\}\ (\lambda x.\ (2{*}pi * f\ x)^2) \leq integral\ \{0..1\}\ (\lambda x.\ (f'\ x)^2)$
and $integral\ \{0..1\}\ (\lambda x.\ (2{*}pi * f\ x)^2) = integral\ \{0..1\}\ (\lambda x.\ (f'\ x)^2) \Longrightarrow$
$\exists c\ a.\ \forall x \in \{0..1\}.\ f\ x = c * sin\ (2{*}pi{*}x - a)$
⟨*proof*⟩

end

3 A very special case of Green's theorem for a convex area

theory *Green-Variant*
imports *Wirtinger*

begin

3.1 Area under an arc.

locale *Area* =
fixes $g :: real \Rightarrow complex$ **and** $g' :: real \Rightarrow complex$ **and** $u\ v\ S$
assumes *uv*: $u \leq v$
and *Re-g-le*: $Re\ (g\ u) \leq Re\ (g\ v)$
and *acont-g*: $absolutely\text{-}continuous\text{-}on\ \{u..v\}\ g$
and *gim*: $g\ `\ \{u..v\} \subseteq \{z.\ Im\ z \geq 0\}$
and *inj-g*: $inj\text{-}on\ g\ \{u..v\}$
and *inj-Re*: $inj\text{-}on\ Re\ (g\ `\ \{u..v\})$
and *negS*: $negligible\ S$
and *gder*: $\bigwedge t.\ t \in \{u..v\} - S \Longrightarrow (g\ has\text{-}vector\text{-}derivative\ g'\ t)\ (at\ t)$

begin

lemma *below-arclet*:
shows $(\lambda t.\ Re\ (g'\ t) * Im\ (g\ t))\ absolutely\text{-}integrable\text{-}on\ \{u..v\}$ (**is** *?th1*)
and $integral\ \{u..v\}\ (\lambda t.\ Re\ (g'\ t) * Im\ (g\ t)) =$
$measure\ lebesgue\ \{z.\ \exists w \in g\ `\ \{u..v\}.\ Re\ w = Re\ z \wedge 0 \leq Im\ z \wedge Im\ z \leq Im\ w\}$ (**is** *?th2*)
⟨*proof*⟩

end

3.2 Area above an arc.

lemma *area-above-arclet*:
fixes $g :: real \Rightarrow complex$ **and** $g' :: real \Rightarrow complex$
assumes $u \leq v$
and *Re-g-ge*: $Re\ (g\ v) \leq Re\ (g\ u)$
and *ac-g*: *absolutely-continuous-on* $\{u..v\}\ g$
and *gim*: $g\ `\ \{u..v\} \subseteq \{z.\ Im\ z \leq 0\}$
and *injg*: *inj-on* $g\ \{u..v\}$
and *injRe*: *inj-on* $Re\ (g\ `\ \{u..v\})$
and *negligible* S
and *vder-g*: $\bigwedge t.\ t \in \{u..v\} - S \Longrightarrow (g\ has\text{-}vector\text{-}derivative\ g'\ t)\ (at\ t)$
shows $(\lambda t.\ Re\ (g'\ t) * Im\ (g\ t))$ *absolutely-integrable-on* $\{u..v\}$ (**is** *?th1*)
and *integral* $\{u..v\}\ (\lambda t.\ Re\ (g'\ t) * Im\ (g\ t)) =$
measure lebesgue $\{z.\ \exists w \in g\ `\ \{u..v\}.\ Re\ w = Re\ z \wedge Im\ w \leq Im\ z \wedge Im\ z \leq 0\}$ (**is** *?th2*)
⟨*proof*⟩

3.3 Lemmas for Green's theorem

definition *Green-concl* :: $(real \Rightarrow complex) \Rightarrow (real \Rightarrow complex) \Rightarrow bool$ **where**
Green-concl $g\ g' \equiv (\lambda t.\ Re\ (g'\ t) * Im\ (g\ t))$ *absolutely-integrable-on* $\{0..1\}$
$\wedge\ |integral\ \{0..1\}\ (\lambda t.\ Re\ (g'\ t) * Im\ (g\ t))| = measure\ lebesgue\ (inside\ (path\text{-}image\ g))$

At most two points on the frontier of a 2-dimensional convex body can share the same inner product with a non-zero vector. Consequence: if three distinct points on the frontier have the same *Re* (or *Im*), the body must lie on one side.

lemma *frontier-vertical-at-most-two*:
fixes $S :: complex\ set$ **and** $c :: real$
assumes *convex* S *compact* S *interior* $S \neq \{\}$
and *sides*: $\exists p \in S.\ Re\ p < c\ \exists q \in S.\ c < Re\ q$
and *xyz*: $x \in frontier\ S\ y \in frontier\ S\ z \in frontier\ S$
$Re\ x = c\ Re\ y = c\ Re\ z = c$
shows $\neg\ (x \neq y \wedge x \neq z \wedge y \neq z)$
⟨*proof*⟩

locale *Green* =
fixes $g :: real \Rightarrow complex$ **and** $g' :: real \Rightarrow complex$
and $U :: real\ set$
and $a\ b :: complex$
assumes *g*: *simple-path* g *pathstart* $g = a$ *pathfinish* $g = a$
and *b*: $b \in path\text{-}image\ g\ Re\ a < Re\ b\ Im\ a = Im\ b$
and *dab*: $dist\ a\ b = diameter\ (path\text{-}image\ g)$
and *conv*: $convex\ (inside\ (path\text{-}image\ g))$
and *cont*: *absolutely-continuous-on* $\{0..1\}\ g$
and *U*: *negligible* U
and *vder*: $\bigwedge t.\ t \in \{0..1\} - U \Longrightarrow (g\ has\text{-}vector\text{-}derivative\ g'\ t)\ (at\ t)$

begin

definition $gop \equiv cnj \circ reversepath\ g$
definition $gop' \equiv uminus \circ cnj \circ reversepath\ g'$

lemma *cnj-rev*: $Green\ gop\ gop'\ ((\lambda t.\ 1-t)\ `\ U)\ (cnj\ a)\ (cnj\ b)$
⟨*proof*⟩

lemma *rev*: $Green\ (reversepath\ g)\ (uminus \circ reversepath\ g')\ ((\lambda t.\ 1-t)\ `\ U)\ a\ b$
⟨*proof*⟩

lemma *f-abs-int*: $(\lambda s.\ Re\ (g'\ s) * Im\ (g\ s))\ absolutely\text{-}integrable\text{-}on\ \{0..1\}$
⟨*proof*⟩

lemma *arc-inj-on*:
assumes *huv*: $0 \le u\ v \le 1\ u < v$ **and** *hne*: $u > 0 \vee v < 1$
shows *inj-on g* $\{u..v\}$
⟨*proof*⟩

lemma *arc-Re-inj-on*:
assumes *inj-on g* $\{u..v\}$
and $\bigwedge s1\ s2.\ [\![s1 \in \{u..v\};\ s2 \in \{u..v\};\ Re\ (g\ s1) = Re\ (g\ s2);\ s1 \neq s2]\!] \Longrightarrow Re\ (g\ u) = Re\ (g\ v)$
and $Re\ (g\ u) \neq Re\ (g\ v)$
shows *inj-on Re* $(g\ `\ \{u..v\})$
⟨*proof*⟩

lemma *Re-inj-upper-gen*:
assumes *s1t*: $s1 \in \{0..t\}$ **and** *s2t*: $s2 \in \{0..t\}$
and *Re-eq*: $Re\ (g\ s1) = Re\ (g\ s2)$ **and** *neq*: $s1 \neq s2$
and *geq0*: $g\ 0 = 0\ g\ 1 = 0$
and *ht*: $0 < t\ t < 1\ g\ t = b$
shows $(s1 = 0 \wedge s2 = t) \vee (s1 = t \wedge s2 = 0)$
⟨*proof*⟩

lemma *not-all-above*:
assumes *Reb*: $Re\ b > 0$
assumes $a = 0$
assumes *real-on-curve*: $\bigwedge z.\ z \in path\text{-}image\ g \Longrightarrow Im\ z = 0 \Longrightarrow z = 0 \vee z = b$
shows $\neg\ (path\text{-}image\ g \subseteq \{z.\ 0 \le Im\ z\})$
⟨*proof*⟩

interpretation *CR*: $Green\ gop\ gop'\ (\lambda t.\ 1-t)\ `\ U\ cnj\ a\ cnj\ b$
⟨*proof*⟩

lemma *Green-area-zero-A*:
assumes $a = 0$

and t: $0 < t\ t < 1$
and *hgt*: $g\ t = b$
and *above*: $g\ `\ \{0..t\} \subseteq \{z.\ 0 \leq Im\ z\}$
and *below*: $g\ `\ \{t..1\} \subseteq \{z.\ Im\ z \leq 0\}$
shows *Green-concl g g'*
⟨*proof*⟩

3.4 Green's theorem special case at zero

interpretation *R*: *Green reversepath g uminus ∘ reversepath g' (λt. 1−t) ` U a b*
⟨*proof*⟩

lemma *Green-area-zero*:
assumes $a = 0$
shows *Green-concl g g'*
⟨*proof*⟩

3.5 Conclusion of Green's theorem and the signed area formula for a convex closed curve.

lemma *Green-invariant*:
assumes $\bigwedge g\ g'\ U\ b.$ *Green g g' U 0 b* $\Longrightarrow$ *Green-concl g g'*
shows *Green-concl g g'*
⟨*proof*⟩

theorem *area-theorem*:
obtains $(\lambda t.\ Re\ (g'\ t) * Im\ (g\ t))$ *absolutely-integrable-on* $\{0..1\}$
and $|integral\ \{0..1\}\ (\lambda t.\ Re\ (g'\ t) * Im\ (g\ t))| =$
measure lebesgue (*inside* (*path-image g*))
⟨*proof*⟩

end

end

4 Isoperimetric theorem for convex curves

theory *Isoperimetric*
imports *Green-Variant*

begin

Formalisation of the isoperimetric inequality, following John Harrison's HOL Light proof in *100/isoperimetric.ml*.

The proof has six parts:

1. Convex curve lemmas (switching between views of a convex simple closed curve)

2. The Wirtinger inequality
3. A special case of Green's theorem for convex area
4. The isoperimetric theorem for convex curves
5. Convexification of an arbitrary rectifiable simple closed curve
6. The full isoperimetric theorem

Infrastructure is provided by the prerequisite theories:

- *Bounded-Variation*: bounded variation and vector variation
- *Absolute-Continuity*: absolute continuity and FTC
- *Rectifiable-Path*: rectifiable paths and path length
- *Green-Variant*: a form of the Green Area Theorem

AFP dependencies:

- *Fourier*: trigonometric orthonormal system, Bessel inequality, L2 Fourier convergence (useful for Wirtinger inequality)
- *Lp* (via Fourier): Hölder inequality, Minkowski inequality

The kernel lemma: the isoperimetric inequality for a convex curve that has been normalized to arc-length parametrization with zero-mean imaginary part and diameter along the real axis starting at a point with $Re = 0$. This is where the Wirtinger inequality is applied.

lemma *isoperimetric-kernel*:
fixes *g* :: *real* ⇒ *complex* **and** *L* :: *real* **and** *a b* :: *complex*
assumes *0* < *L*
and *conv-in*: *convex* (*inside* (*path-image g*))
and *ab*: *a* ∈ *path-image g b* ∈ *path-image g*
and *dist-ab*: *dist a b* = *diameter* (*path-image g*)
and *bma*: *b* − *a* = *of-real* (*dist a b*)
and *ga*: *pathstart g* = *a pathfinish g* = *a*
and *g*: *rectifiable-path g simple-path g*
and *L*: *path-length g* = *L*
and *arc-length*: ⋀*t*. *t* ∈ {*0..1*} ⟹ *path-length* (*subpath 0 t g*) = *L* ∗ *t*
and *lipschitz*: ⋀*x y*. *x* ∈ {*0..1*} ⟹ *y* ∈ {*0..1*} ⟹ *dist* (*g x*) (*g y*) ≤ *L* ∗ *dist x y*
and *Re a* = *0*
and (*Im* ∘ *g has-integral 0*) {*0..1*}
shows *measure lebesgue* (*inside* (*path-image g*)) ≤ *L*^{2} / (*4* ∗ *pi*)
and *measure lebesgue* (*inside* (*path-image g*)) = *L*^{2} / (*4* ∗ *pi*) ⟹
∃ *c r*. *path-image g* = *sphere c r*
⟨*proof*⟩

Reduction lemmas for the reparametrization steps.

lemma *isoperimetric-reduce-shift*:
fixes $g :: real \Rightarrow complex$
assumes *rectifiable-path g simple-path g*
pathfinish g = pathstart g
convex (inside (path-image g))
path-length g = L
$a \in$ *path-image g*
obtains h **where** *rectifiable-path h simple-path h*
pathfinish h = pathstart h pathstart h = a
convex (inside (path-image h))
path-length h = L
path-image h = path-image g
⟨*proof*⟩

lemma *isoperimetric-reduce-rotate-translate*:
fixes $g :: real \Rightarrow complex$ **and** $a\ b :: complex$
assumes *rectifiable-path g simple-path g*
pathfinish g = pathstart g pathstart g = a
convex (inside (path-image g))
path-length g = L
$b \in$ *path-image g dist a b = diameter (path-image g)*
$a \neq b$
obtains $h\ a'\ b'$ **where** *rectifiable-path h simple-path h*
pathfinish h = pathstart h pathstart h = a'
convex (inside (path-image h))
path-length h = L
$b' \in$ *path-image h dist a' b' = diameter (path-image h)*
$b' - a' =$ *of-real (dist a' b')*
Re a' = 0
measure lebesgue (inside (path-image h)) = measure lebesgue (inside (path-image g))
$\bigwedge c\ r.$ *path-image h = sphere c r* $\Longrightarrow \exists\, c'\ r'.$ *path-image g = sphere c' r'*
⟨*proof*⟩

lemma *isoperimetric-reduce-arc-length*:
fixes $g :: real \Rightarrow complex$
assumes *rectifiable-path g simple-path g*
pathfinish g = pathstart g
convex (inside (path-image g))
path-length g = L $0 < L$
obtains h **where** *rectifiable-path h simple-path h*
pathfinish h = pathstart h pathstart h = pathstart g
convex (inside (path-image h))
path-length h = L
path-image h = path-image g
$\bigwedge t.\ t \in \{0..1\} \Longrightarrow$ *path-length (subpath 0 t h) = L * t*
$\bigwedge x\ y.\ x \in \{0..1\} \Longrightarrow y \in \{0..1\} \Longrightarrow$ *dist (h x) (h y)* $\leq L *$ *dist x y*

⟨*proof*⟩

lemma *isoperimetric-reduce-zero-mean*:
fixes g :: *real* ⇒ *complex* **and** b :: *complex*
assumes *rectifiable-path g simple-path g*
pathfinish g = *pathstart g*
convex (*inside* (*path-image g*))
path-length g = L
$b \in$ *path-image g*
dist (*pathstart g*) b = *diameter* (*path-image g*)
b − *pathstart g* = *of-real* (*dist* (*pathstart g*) b)
Re (*pathstart g*) = 0
and *subp*: $\bigwedge t.\ t \in \{0..1\} \Longrightarrow$ *path-length* (*subpath 0 t g*) = $L * t$
and *dist*: $\bigwedge x\ y.\ x \in \{0..1\} \Longrightarrow y \in \{0..1\} \Longrightarrow$ *dist* (*g x*) (*g y*) ≤ $L *$ *dist x y*
obtains $h\ a'\ b'$ **where** *rectifiable-path h simple-path h*
pathfinish h = *pathstart h*
convex (*inside* (*path-image h*))
path-length h = L
$a' \in$ *path-image h* $b' \in$ *path-image h*
dist $a'\ b'$ = *diameter* (*path-image h*)
$b' - a'$ = *of-real* (*dist* $a'\ b'$)
pathstart h = a' *pathfinish h* = a'
Re a' = 0
$\bigwedge t.\ t \in \{0..1\} \Longrightarrow$ *path-length* (*subpath 0 t h*) = $L * t$
$\bigwedge x\ y.\ x \in \{0..1\} \Longrightarrow y \in \{0..1\} \Longrightarrow$ *dist* (*h x*) (*h y*) ≤ $L *$ *dist x y*
(*Im* ∘ *h has-integral 0*) $\{0..1\}$
measure lebesgue (*inside* (*path-image h*)) = *measure lebesgue* (*inside* (*path-image g*))
$\bigwedge c\ r.$ *path-image h* = *sphere c r* $\Longrightarrow \exists\, c'\ r'.$ *path-image g* = *sphere* $c'\ r'$
⟨*proof*⟩

theorem *isoperimetric-theorem-convex*:
fixes g :: *real* ⇒ *complex*
assumes *rectifiable-path g simple-path g*
pathfinish g = *pathstart g*
convex (*inside* (*path-image g*))
path-length g = L
shows *measure lebesgue* (*inside* (*path-image g*)) ≤ L^2 / (4 * *pi*)
and *measure lebesgue* (*inside* (*path-image g*)) = L^2 / (4 * *pi*) ⟹
$\exists\, a\ r.$ *path-image g* = *sphere a r*
⟨*proof*⟩

4.1 Convexification

A connected subset of a segment that contains both endpoints is the whole segment. (HOL Light: CONNECTED_SUBSET_SEGMENT.)

lemma *connected-subset-segment*:
fixes $a\ b$ :: *'a*::*euclidean-space*
assumes *connected* S $S \subseteq$ *closed-segment* $a\ b$ $a \in S$ $b \in S$

shows $S = \textit{closed-segment}\ a\ b$
⟨*proof*⟩

A connected subset of two arcs joined only at their common endpoints, containing both endpoints, must contain one of the two arcs in full. (HOL Light: CONNECTED_SUBSET_ARC_PAIR.)

lemma *connected-subset-arc-pair*:
fixes $g\ h :: \textit{real} \Rightarrow 'a{::}\textit{euclidean-space}$
assumes *arc g arc h*
pathstart g = pathstart h pathfinish g = pathfinish h
and *pai-eq*: *path-image g ∩ path-image h = {pathstart g, pathfinish g}*
connected S **and** *Ssub*: $S \subseteq \textit{path-image}\ g \cup \textit{path-image}\ h$
and *pathstart g ∈ S pathfinish g ∈ S*
shows *path-image g ⊆ S ∨ path-image h ⊆ S*
⟨*proof*⟩

If two frontier points of a bounded convex 2-dimensional set are joined by a frontier arc whose interior (the arc minus the two points) stays connected and whose convex hull is the whole set, then the straight segment between the two points also lies on the frontier. The connectedness of the arc-minus-endpoints is what rules out the spurious "chord through the interior" case.

lemma *seg-frontier-aux*:
fixes $S :: \textit{complex set}$
assumes *cvx*: *convex S* **and** *cpt*: *compact S* **and** *intne*: *interior S ≠ {}*
and *ga-fr*: *ga ∈ frontier S* **and** *gb-fr*: *gb ∈ frontier S* **and** *ne*: *ga ≠ gb*
and *D1fr*: *path-image D1 ⊆ frontier S*
and *D1con*: *connected (path-image D1 − {ga, gb})*
and *gaD1*: *ga ∈ path-image D1* **and** *gbD1*: *gb ∈ path-image D1*
and *hullD1*: *convex hull (path-image D1) = S*
shows *closed-segment ga gb ⊆ frontier S*
⟨*proof*⟩

The step lemma: replacing an arc that deviates from the convex hull frontier with a straight segment shortens the path while preserving the convex hull.

lemma *step-lemma*:
fixes $g :: \textit{real} \Rightarrow \textit{complex}$
assumes *simple-path g pathfinish g = pathstart g*
and *cont*: $\bigwedge x\ y.\ x \in \{0..1\} \Longrightarrow y \in \{0..1\} \Longrightarrow \textit{dist}\ (g\ x)\ (g\ y) \leq L * \textit{dist}\ x\ y$
and $a < b$
and *ab01*: $a \in \{0..1\}\ b \in \{0..1\}$
and *ga*: *g a ∈ frontier (convex hull (path-image g))*
and *gb*: *g b ∈ frontier (convex hull (path-image g))*
and *disj*: *g ' {a<..<b} ∩ frontier (convex hull (path-image g)) = {}*
obtains *h* **where** *simple-path h*
and *pathstart h = pathstart g* **and** *pathfinish h = pathstart g*
and $\bigwedge x\ y.\ x \in \{0..1\} \Longrightarrow y \in \{0..1\} \Longrightarrow \textit{dist}\ (h\ x)\ (h\ y) \leq L * \textit{dist}\ x\ y$
and *path-length h < path-length g*

and *convex hull* (*path-image h*) = *convex hull* (*path-image g*)
and $\bigwedge x.\ x \notin \{a<..<b\} \Longrightarrow h\ x = g\ x$
and *h* ' $\{a..b\} \subseteq$ *frontier* (*convex hull* (*path-image g*))
⟨*proof*⟩

A nonempty bounded open connected subset of the reals is an open interval.

lemma *open-bounded-connected-real-is-interval*:
fixes *c* :: *real set*
assumes *open c connected c c* ≠ {} *bounded c*
shows *c* = {*Inf c*<..<*Sup c*}
⟨*proof*⟩

The reduced convexification subgoal: a unit-speed (Lipschitz) simple closed rectifiable loop starting on the frontier of its convex hull can be replaced by a no-longer loop with the same convex hull whose image IS that frontier. The components of the part of the parameter interval whose image avoids the frontier are countably many open intervals.

proposition *convexification-unit-speed*:
fixes γ :: *real* ⇒ *complex*
assumes *rect*: *rectifiable-path* γ **and** *simp*: *simple-path* γ **and** *loop*: *pathfinish* γ = *pathstart* γ
and *frstart*: *pathstart* $\gamma \in$ *frontier* (*convex hull* (*path-image* γ))
and *lip*: $\bigwedge x\ y.\ x \in \{0..1\} \Longrightarrow y \in \{0..1\} \Longrightarrow$ *dist* (γ *x*) (γ *y*) ≤ *path-length* γ ∗ *dist x y*
shows ∃ *h*. *rectifiable-path h* ∧ *simple-path h* ∧ *pathfinish h* = *pathstart h* ∧ *path-length h* ≤ *path-length* γ ∧ *convex hull* (*path-image h*) = *convex hull* (*path-image* γ) ∧ *path-image h* = *frontier* (*convex hull* (*path-image* γ))
⟨*proof*⟩

theorem *isoperimetric-convexification*:
fixes *g* :: *real* ⇒ *complex*
assumes *rectifiable-path g simple-path g*
pathfinish g = *pathstart g*
obtains *h* **where** *rectifiable-path h* **and** *simple-path h*
and *pathfinish h* = *pathstart h*
and *path-length h* ≤ *path-length g*
and *convex hull* (*path-image h*) = *convex hull* (*path-image g*)
and *path-image h* = *frontier* (*convex hull* (*path-image g*))
⟨*proof*⟩

theorem *isoperimetric-convexification-strict*:
fixes *g* :: *real* ⇒ *complex*
assumes *rectifiable-path g simple-path g*
pathfinish g = *pathstart g*
¬ *convex* (*inside* (*path-image g*))
obtains *h* **where** *rectifiable-path h* **and** *simple-path h*
and *pathfinish h* = *pathstart h*
and *path-length h* ≤ *path-length g*

and *convex hull* (*path-image h*) = *convex hull* (*path-image g*)
and *path-image h* = *frontier* (*convex hull* (*path-image g*))
and *measure lebesgue* (*inside* (*path-image g*)) < *measure lebesgue* (*inside* (*path-image h*))
⟨*proof*⟩

4.2 The isoperimetric theorem

theorem *isoperimetric-theorem*:
fixes *g* :: *real* ⇒ *complex*
assumes *rectifiable-path g simple-path g*
pathfinish g = *pathstart g*
path-length g = *L*
shows *measure lebesgue* (*inside* (*path-image g*)) ≤ L^2 / (4 ∗ *pi*) ∧
(*measure lebesgue* (*inside* (*path-image g*)) = L^2 / (4 ∗ *pi*) ⟶ (∃ *a r*. *path-image g* = *sphere a r*))
⟨*proof*⟩

end