IsaGeoCoq: Partial porting of GeoCoq 2.4.0. Case studies: Tarski's postulate of parallels implies the 5th postulate of Euclid, the postulate of Playfair and the original postulate of Euclid.

Roland Coghetto

February 6, 2026

Abstract

The GeoCoq library contains a formalization of geometry using the Coq proof assistant. It contains both proofs about the foundations of geometry [20, 15, 6, 16] and high-level proofs in the same style as in high-school. [1](Code Repository https://github.com/GeoCoq/GeoCoq).

Some theorems also inspired by [20] are also formalized with others ITP(Metamath, Mizar) or ATP [24, 25, 3, 23, 4, 2, 17, 5, 11, 19, 8, 9, 10].

We port a part of the GeoCoq 2.4.0 library within the Isabelle/Hol proof assistant: more precisely, the files Chap02.v to Chap13_3.v, suma.v as well as the associated definitions and some useful files for the demonstration of certain parallel postulates.

While the demonstrations in Coq are written in procedural language [26], the transcript is done in declarative language Isar[18].

The synthetic approach of the demonstrations are directly inspired by those contained in GeoCoq. Some demonstrations are credited to G.E Martin(«lemma bet_le_lt:» in Ch11_angles.thy, proved by Martin as Theorem 18.17 in [14]) or Gupta H.N (Krippen Lemma, proved by Gupta in its PhD in 1965 as Theorem 3.45). (See [12]).

In this work, the proofs are not contructive. The sledeghammer tool being used to find some demonstrations.

The names of the lemmas and theorems used are kept as far as possible as well as the definitions. A different translation has been proposed when the name was already used in Isabel/Hol ("Len" is translated as "TarskiLen") or that characters were not allowed in Isabel/Hol ("anga'" in Ch13_angles.v is translated as "angaP"). For some definitions the highlighting of a variable has changed the order or the position of the variables (Midpoint, Out, Inter,...).

All the lemmas are valid in absolute/neutral space defined with Tarski's axioms.

It should be noted that T.J.M. Makarios [13] has begun some demonstrations of certain proposals mainly those corresponding to SST chapters 2 and 3. It uses a definition that does not quite coincide with the definition used in Geocoq and here. As an example, Makarios introduces the axiom A11 (Axiom of continuity) in the definition of the locale "Tarski_absolute_space".

Furthermore, the definition of the locale "TarskiAbsolute" [22, 21] is not not identical to the one defined in the "Tarski_neutral_dimensionless" class of GeoCoq. Indeed this one does not contain the axiom "upper_dimension". In some cases particular, it is nevertheless to use the axiom "upper_dimension". The addition of the word "_2D" in the file indicates its presence.

In the last part, it is formalized that, in the neutral/absolute space, the axiom of the parallels of the system of Tarski implies the Playfair axiom, the 5th postulate of euclide and the postulate original from Euclid. These proofs, which are not constructive, are directly inspired by [12, 7].

Contents

theory *Tarski-Neutral*

imports
Main

begin

1 Neutral geometry dimensionless

1.1 Tarski's axiom system for neutral geometry: dimensionless

locale *Tarski-neutral-dimensionless* =
fixes *Bet* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ ((- − - − -))
and *Cong* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
and *TPA TPB TPC* :: $'p$
assumes *cong-pseudo-reflexivity*: ∀ *a b*.

Cong a b b a

and *cong-inner-transitivity*: ∀ *a b p q r s*.

Cong a b p q ∧
Cong a b r s
⟶
Cong p q r s

and *cong-identity*: ∀ *a b c*.

Cong a b c c
⟶
$a = b$

and *segment-construction*: ∀ *a b c q*.

∃ *x*. (*Bet q a x* ∧ *Cong a x b c*)

and *five-segment*: ∀ $a\ b\ c\ d\ a'\ b'\ c'\ d'$.

$a \neq b$ ∧
Bet a b c ∧
Bet $a'\ b'\ c'$∧
Cong $a\ b\ a'\ b'$ ∧
Cong $b\ c\ b'\ c'$ ∧
Cong $a\ d\ a'\ d'$ ∧
Cong $b\ d\ b'\ d'$
⟶
Cong $c\ d\ c'\ d'$

and *between-identity*: ∀ *a b*.

Bet a b a
⟶
$a = b$

and *inner-pasch*: ∀ *a b c p q*.

Bet a p c ∧
Bet b q c
⟶
(∃ *x*. *Bet p x b* ∧ *Bet q x a*)

and *lower-dim*: ¬ *Bet TPA TPB TPC* ∧ ¬ *Bet TPB TPC TPA* ∧ ¬ *Bet TPC TPA TPB*

context *Tarski-neutral-dimensionless*

begin

1.2 Definitions

definition *OFSC* ::
  $['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
  (- - - - *OFSC* - - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
  where
    $A\ B\ C\ D\ OFSC\ A'\ B'\ C'\ D'$
$\equiv$
$Bet\ A\ B\ C\ \wedge$
$Bet\ A'\ B'\ C'\ \wedge$
$Cong\ A\ B\ A'\ B'\ \wedge$
$Cong\ B\ C\ B'\ C'\ \wedge$
$Cong\ A\ D\ A'\ D'\ \wedge$
$Cong\ B\ D\ B'\ D'$

definition *Cong3* ::
  $['p,'p,'p,'p,'p,'p] \Rightarrow bool$
  (- - - *Cong3* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
  where
    $A\ B\ C\ Cong3\ A'\ B'\ C'$
$\equiv$
$Cong\ A\ B\ A'\ B'\ \wedge$
$Cong\ A\ C\ A'\ C'\ \wedge$
$Cong\ B\ C\ B'\ C'$

definition *Col* ::
  $['p,'p,'p] \Rightarrow bool$
  (*Col* - - - [*99*,*99*,*99*] *50*)
  where
    Col A B C
$\equiv$
$Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B$

definition *Bet4* ::
  $['p,'p,'p,'p] \Rightarrow bool$
  (*Bet4* - - - - [*99*,*99*,*99*,*99*] *50*)
  where
    Bet4 A1 A2 A3 A4
$\equiv$
$Bet\ A1\ A2\ A3\ \wedge$
$Bet\ A2\ A3\ A4\ \wedge$
$Bet\ A1\ A3\ A4\ \wedge$
$Bet\ A1\ A2\ A4$

definition *BetS* ::
  $['p,'p,'p] \Rightarrow bool$ (*BetS* - - - [*99*,*99*,*99*] *50*)
  where
    BetS A B C
$\equiv$
$Bet\ A\ B\ C\ \wedge$
$A \neq B\ \wedge$
$B \neq C$

definition *SumS* ::
  $['p,'p,'p,'p,'p,'p] \Rightarrow bool$
  (- - - - *SumS* - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
  where
    A B C D SumS E F
$\equiv$
$\exists\ P\ Q\ R.\ Bet\ P\ Q\ R \wedge Cong\ P\ Q\ A\ B \wedge Cong\ Q\ R\ C\ D \wedge Cong\ P\ R\ E\ F$

definition *FSC* ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
$(\text{- - - - } FSC \text{ - - - - } [99,99,99,99,99,99,99,99]\ 50)$
where
$A\ B\ C\ D\ FSC\ A'\ B'\ C'\ D'$
$\equiv$
$Col\ A\ B\ C\ \wedge$
$A\ B\ C\ Cong3\ A'\ B'\ C'\ \wedge$
$Cong\ A\ D\ A'\ D'\ \wedge$
$Cong\ B\ D\ B'\ D'$

definition *IFSC* ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
$(\text{- - - - } IFSC \text{ - - - - } [99,99,99,99,99,99,99,99]\ 50)$
where
$A\ B\ C\ D\ IFSC\ A'\ B'\ C'\ D'$
$\equiv$
$Bet\ A\ B\ C\ \wedge$
$Bet\ A'\ B'\ C'\ \wedge$
$Cong\ A\ C\ A'\ C'\ \wedge$
$Cong\ B\ C\ B'\ C'\ \wedge$
$Cong\ A\ D\ A'\ D'\ \wedge$
$Cong\ C\ D\ C'\ D'$

definition *Le* ::
$['p,'p,'p,'p] \Rightarrow bool$
$(\text{- - } Le \text{ - - } [99,99,99,99]\ 50)$
where
$A\ B\ Le\ C\ D$
$\equiv$
$\exists\ E.\ (Bet\ C\ E\ D \wedge Cong\ A\ B\ C\ E)$

definition *Lt* ::
$['p,'p,'p,'p] \Rightarrow bool$
$(\text{- - } Lt \text{ - - } [99,99,99,99]\ 50)$
where
$A\ B\ Lt\ C\ D$
$\equiv$
$A\ B\ Le\ C\ D \wedge \neg\ Cong\ A\ B\ C\ D$

definition *Ge* ::
$['p,'p,'p,'p] \Rightarrow bool$
$(\text{- -}Ge \text{ - - } [99,99,99,99]\ 50)$
where
$A\ B\ Ge\ C\ D$
$\equiv$
$C\ D\ Le\ A\ B$

definition *Gt* ::
$['p,'p,'p,'p] \Rightarrow bool$
$(\text{- - } Gt \text{ - - } [99,99,99,99]\ 50)$
where
$A\ B\ Gt\ C\ D$
$\equiv$
$C\ D\ Lt\ A\ B$

definition *Out* ::
$['p,'p,'p] \Rightarrow bool$
$(\text{- } Out \text{ - - } [99,99,99]\ 50)$
where
$P\ Out\ A\ B$
$\equiv$
$A \neq P\ \wedge$
$B \neq P\ \wedge$
$(Bet\ P\ A\ B \vee Bet\ P\ B\ A)$

definition *Midpoint* ::
$[{}'p,{}'p,{}'p] \Rightarrow bool$
$(\text{-}\ Midpoint\ \text{-}\ \text{-}\ [99,99,99]\ 50)$
where
$M\ Midpoint\ A\ B$
$\equiv$
$Bet\ A\ M\ B\ \wedge$
$Cong\ A\ M\ M\ B$

definition *Per* ::
$[{}'p,{}'p,{}'p] \Rightarrow bool$
$(Per\ \text{-}\ \text{-}\ \text{-}\ [99,99,99]\ 50)$
where
$Per\ A\ B\ C$
$\equiv$
$\exists\ C'.\ (B\ Midpoint\ C\ C' \wedge Cong\ A\ C\ A\ C')$

definition *PerpAt* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(\text{-}\ PerpAt\ \text{-}\ \text{-}\ \text{-}\ \text{-}\ [99,99,99,99,99]\ 50)$
where
$X\ PerpAt\ A\ B\ C\ D$
$\equiv$
$A \neq B\ \wedge$
$C \neq D\ \wedge$
$Col\ X\ A\ B\ \wedge$
$Col\ X\ C\ D\ \wedge$
$(\forall\ U\ V.\ ((Col\ U\ A\ B \wedge Col\ V\ C\ D) \longrightarrow Per\ U\ X\ V))$

definition *Perp* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(\text{-}\ \text{-}\ Perp\ \text{-}\ \text{-}\ [99,99,99,99]\ 50)$
where
$A\ B\ Perp\ C\ D$
$\equiv$
$\exists\ X{::}{}'p.\ X\ PerpAt\ A\ B\ C\ D$

definition *Coplanar* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(Coplanar\ \text{-}\ \text{-}\ \text{-}\ \text{-}\ [99,99,99,99]\ 50)$
where
$Coplanar\ A\ B\ C\ D$
$\equiv$
$\exists\ X.\ (Col\ A\ B\ X \wedge Col\ C\ D\ X) \vee$
$(Col\ A\ C\ X \wedge Col\ B\ D\ X) \vee$
$(Col\ A\ D\ X \wedge Col\ B\ C\ X)$

definition *TS* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(\text{-}\ \text{-}\ TS\ \text{-}\ \text{-}\ [99,99,99,99]\ 50)$
where
$A\ B\ TS\ P\ Q$
$\equiv$
$\neg\ Col\ P\ A\ B \wedge \neg\ Col\ Q\ A\ B \wedge (\exists\ T{::}{}'p.\ Col\ T\ A\ B \wedge Bet\ P\ T\ Q)$

definition *ReflectL* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(\text{-}\ \text{-}\ ReflectL\ \text{-}\ \text{-}\ [99,99,99,99]\ 50)$
where
$P'\ P\ ReflectL\ A\ B$
$\equiv$
$(\exists\ X.\ X\ Midpoint\ P\ P' \wedge Col\ A\ B\ X) \wedge (A\ B\ Perp\ P\ P' \vee P = P')$

definition *Reflect* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(\text{-}\ \text{-}\ Reflect\ \text{-}\ \text{-}\ [99,99,99,99]\ 50)$

where
P' P *Reflect* A B
$\equiv$
$(A \neq B \wedge P'$ P *ReflectL* A $B) \vee (A = B \wedge A$ *Midpoint* P $P')$

definition *InAngle* ::
$['p,'p,'p,'p] \Rightarrow bool$
(- *InAngle* - - - $[99,99,99,99]$ 50)
where
P *InAngle* A B C
$\equiv$
$A \neq B \wedge C \neq B \wedge P \neq B \wedge$
$(\exists$ X. *Bet* A X $C \wedge (X = B \vee B$ *Out* X $P))$

definition *ParStrict*::
$['p,'p,'p,'p] \Rightarrow bool$
(- - *ParStrict* - - $[99,99,99,99]$ 50)
where
A B *ParStrict* C D
$\equiv$
Coplanar A B C D $\wedge$
$\neg$ $(\exists$ X. *Col* X A B $\wedge$ *Col* X C $D)$

definition *Par*::
$['p,'p,'p,'p] \Rightarrow bool$
(- - *Par* - - $[99,99,99,99]$ 50)
where
A B *Par* C D
$\equiv$
A B *ParStrict* C $D \vee (A \neq B \wedge C \neq D \wedge$ *Col* A C $D \wedge$ *Col* B C $D)$

definition *Plg*::
$['p,'p,'p,'p] \Rightarrow bool$
(*Plg* - - - - $[99,99,99,99]$ 50)
where
Plg A B C D
$\equiv$
$(A \neq C \vee B \neq D) \wedge (\exists$ M. M *Midpoint* A $C \wedge M$ *Midpoint* B $D)$

definition *ParallelogramStrict*::
$['p,'p,'p,'p] \Rightarrow bool$
(*ParallelogramStrict* - - - - $[99,99,99,99]$ 50)
where
ParallelogramStrict A B A' B'
$\equiv$
A A' *TS* B B' $\wedge$
A B *Par* A' B' $\wedge$
Cong A B A' B'

definition *ParallelogramFlat*::
$['p,'p,'p,'p] \Rightarrow bool$
(*ParallelogramFlat* - - - - $[99,99,99,99]$ 50)
where
ParallelogramFlat A B A' B'
$\equiv$
Col A B A' $\wedge$
Col A B B' $\wedge$
Cong A B A' B' $\wedge$
Cong A B' A' B $\wedge$
$(A \neq A' \vee B \neq B')$

definition *Parallelogram*::
$['p,'p,'p,'p] \Rightarrow bool$
(*Parallelogram* - - - - $[99,99,99,99]$ 50)
where
Parallelogram A B A' B'

≡
ParallelogramStrict A B A′ B′ ∨ *ParallelogramFlat A B A′ B′*

definition *Rhombus*::
  [′p,′p,′p,′p] ⇒ *bool*
  (*Rhombus* - - - - [*99*,*99*,*99*,*99*] *50*)
  where
    Rhombus A B C D
≡
Plg A B C D ∧ *Cong A B B C*

definition *Rectangle*::
  [′p,′p,′p,′p] ⇒ *bool*
  (*Rectangle* - - - - [*99*,*99*,*99*,*99*] *50*)
  where
    Rectangle A B C D
≡
Plg A B C D ∧ *Cong A C B D*

definition *Square*::
  [′p,′p,′p,′p] ⇒ *bool*
  (*Square* - - - - [*99*,*99*,*99*,*99*] *50*)
  where
    Square A B C D
≡
Rectangle A B C D ∧ *Cong A B B C*

definition *Kite*::
  [′p,′p,′p,′p] ⇒ *bool*
  (*Kite* - - - - [*99*,*99*,*99*,*99*] *50*)
  where
    Kite A B C D
≡
Cong B C C D ∧ *Cong D A A B*

definition *Lambert*::
  [′p,′p,′p,′p] ⇒ *bool*
  (*Lambert* - - - - [*99*,*99*,*99*,*99*] *50*)
  where
    Lambert A B C D
≡
$A \neq B \wedge B \neq C \wedge C \neq D \wedge A \neq D \wedge$
Per B A D ∧
Per A D C ∧
Per A B C ∧
Coplanar A B C D

definition *OS* ::
  [′p,′p,′p,′p] ⇒ *bool*
  (- - *OS* - - [*99*,*99*,*99*,*99*] *50*)
  where
    A B OS P Q
≡
∃ *R*::′*p*. *A B TS P R* ∧ *A B TS Q R*

definition *TSP* ::
  [′p,′p,′p,′p,′p] ⇒ *bool*
  (- - - *TSP* - - [*99*,*99*,*99*,*99*,*99*] *50*)
  where
    A B C TSP P Q
≡
(¬ *Coplanar A B C P*) ∧ (¬ *Coplanar A B C Q*) ∧
(∃ *T*. *Coplanar A B C T* ∧ *Bet P T Q*)

definition *OSP* ::
  [′p,′p,′p,′p,′p] ⇒ *bool*

(- - - *OSP* - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
A B C OSP P Q
≡
∃ *R*. ((*A B C TSP P R*) ∧ (*A B C TSP Q R*))

definition *Saccheri*::
[′*p*,′*p*,′*p*,′*p*] ⇒ *bool*
(*Saccheri* - - - - [*99*,*99*,*99*,*99*] *50*)
where
Saccheri A B C D
≡
Per B A D ∧
Per A D C ∧
Cong A B C D ∧ *A D OS B C*

definition *ReflectLAt* ::
[′*p*,′*p*,′*p*,′*p*,′*p*] ⇒ *bool*
(- *ReflectLAt* - - - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
M ReflectLAt P′ P A B
≡
(*M Midpoint P P′* ∧ *Col A B M*) ∧ (*A B Perp P P′* ∨ *P* = *P′*)

definition *ReflectAt* ::
[′*p*,′*p*,′*p*,′*p*,′*p*] ⇒ *bool*
(- *ReflectAt* - - - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
M ReflectAt P′ P A B
≡
(*A* ≠ *B* ∧ *M ReflectLAt P′ P A B*) ∨ (*A* = *B* ∧ *A* = *M* ∧ *M Midpoint P P′*)

definition *upper-dim-axiom* ::
bool
(*UpperDimAxiom* [] *50*)
where
upper-dim-axiom
≡
∀ *A B C P Q*.
P ≠ *Q* ∧
Cong A P A Q ∧
Cong B P B Q ∧
Cong C P C Q
⟶
(*Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*)

definition *all-coplanar-axiom* ::
bool
(*AllCoplanarAxiom* [] *50*)
where
AllCoplanarAxiom
≡
∀ *A B C P Q*.
P ≠ *Q* ∧
Cong A P A Q ∧
Cong B P B Q ∧
Cong C P C Q
⟶
(*Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*)

definition *upper-dim-3-axiom* ::
bool
where
upper-dim-3-axiom

$\equiv$
$\forall\ A\ B\ C\ P\ Q\ R.\ P \neq Q \wedge Q \neq R \wedge P \neq R \wedge$
$\quad Cong\ A\ P\ A\ Q \wedge Cong\ B\ P\ B\ Q \wedge Cong\ C\ P\ C\ Q \wedge$
$\quad Cong\ A\ P\ A\ R \wedge Cong\ B\ P\ B\ R \wedge Cong\ C\ P\ C\ R \longrightarrow$
$\quad (Bet\ A\ B\ C \vee Bet\ B\ C\ A \vee Bet\ C\ A\ B)$

definition *median-planes-axiom* ::
bool
where
median-planes-axiom
$\equiv$
$\forall\ A\ B\ C\ D\ P\ Q.\ P \neq Q \wedge$
$\quad Cong\ A\ P\ A\ Q \wedge Cong\ B\ P\ B\ Q \wedge Cong\ C\ P\ C\ Q \wedge Cong\ D\ P\ D\ Q \longrightarrow$
$\quad Coplanar\ A\ B\ C\ D$

definition *plane-intersection-axiom* ::
bool
where
plane-intersection-axiom
$\equiv$
$\forall\ A\ B\ C\ D\ E\ F\ P.$
$\quad Coplanar\ A\ B\ C\ P \wedge Coplanar\ D\ E\ F\ P \longrightarrow$
$(\exists\ Q.\ Coplanar\ A\ B\ C\ Q \wedge Coplanar\ D\ E\ F\ Q \wedge P \neq Q)$

definition *space-separation-axiom* ::
bool
where
space-separation-axiom
$\equiv$
$\forall\ A\ B\ C\ P\ Q.$
$\quad \neg\ Coplanar\ A\ B\ C\ P \wedge \neg\ Coplanar\ A\ B\ C\ Q \longrightarrow$
$\quad (A\ B\ C\ TSP\ P\ Q \vee A\ B\ C\ OSP\ P\ Q)$

definition *orthonormal-family-axiom* ::
bool
where
orthonormal-family-axiom
$\equiv$
$\forall\ S\ U1'\ U1\ U2\ U3\ U4.$
$\quad \neg\ (S \neq U1' \wedge Bet\ U1\ S\ U1' \wedge$
$\quad\quad Cong\ S\ U1\ S\ U1' \wedge Cong\ S\ U2\ S\ U1' \wedge Cong\ S\ U3\ S\ U1' \wedge Cong\ S\ U4\ S\ U1' \wedge$
$\quad\quad Cong\ U1\ U2\ U1'\ U2 \wedge Cong\ U1\ U3\ U1'\ U2 \wedge Cong\ U1\ U4\ U1'\ U2 \wedge$
$\quad\quad Cong\ U2\ U3\ U1'\ U2 \wedge Cong\ U2\ U4\ U1'\ U2 \wedge Cong\ U3\ U4\ U1'\ U2)$

definition *CongA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - - *CongA* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ CongA\ D\ E\ F$
$\equiv$
$A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E \wedge$
$(\exists\ A'\ C'\ D'\ F'.\ Bet\ B\ A\ A' \wedge Cong\ A\ A'\ E\ D \wedge Bet\ B\ C\ C' \wedge Cong\ C\ C'\ E\ F \wedge$
$Bet\ E\ D\ D' \wedge Cong\ D\ D'\ B\ A \wedge Bet\ E\ F\ F' \wedge Cong\ F\ F'\ B\ C \wedge$
$Cong\ A'\ C'\ D'\ F')$

definition *LeA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - - *LeA* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)

where
$A\ B\ C\ LeA\ D\ E\ F$
$\equiv$
$\exists\ P.\ (P\ InAngle\ D\ E\ F \wedge A\ B\ C\ CongA\ D\ E\ P)$

definition *LtA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - - *LtA* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ LtA\ D\ E\ F$
$\equiv$
$A\ B\ C\ LeA\ D\ E\ F \wedge \neg\ A\ B\ C\ CongA\ D\ E\ F$

definition *GtA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - - *GtA* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ GtA\ D\ E\ F$
$\equiv$
$D\ E\ F\ LtA\ A\ B\ C$

definition *Acute* ::
$['p,'p,'p] \Rightarrow bool$
(*Acute* - - - [*99*,*99*,*99*] *50*)
where
$Acute\ A\ B\ C$
$\equiv$
$\exists\ A'\ B'\ C'.\ (Per\ A'\ B'\ C' \wedge A\ B\ C\ LtA\ A'\ B'\ C')$

definition *Obtuse* ::
$['p,'p,'p] \Rightarrow bool$
(*Obtuse* - - - [*99*,*99*,*99*] *50*)
where
$Obtuse\ A\ B\ C$
$\equiv$
$\exists\ A'\ B'\ C'.\ (Per\ A'\ B'\ C' \wedge A'\ B'\ C'\ LtA\ A\ B\ C)$

definition *OrthAt* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- *OrthAt* - - - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$X\ OrthAt\ A\ B\ C\ U\ V$
$\equiv$
$\neg\ Col\ A\ B\ C \wedge U \neq V \wedge Coplanar\ A\ B\ C\ X \wedge Col\ U\ V\ X \wedge$
$(\forall\ P\ Q.\ (Coplanar\ A\ B\ C\ P \wedge Col\ U\ V\ Q) \longrightarrow Per\ P\ X\ Q)$

definition *Orth* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$
(- - - *Orth* - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ Orth\ U\ V$
$\equiv$
$\exists\ X.\ X\ OrthAt\ A\ B\ C\ U\ V$

definition *SuppA* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - - *SuppA* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
$A\ B\ C\ SuppA\ D\ E\ F$
$\equiv$
$A \neq B \wedge (\exists\ A'.\ Bet\ A\ B\ A' \wedge D\ E\ F\ CongA\ C\ B\ A')$

definition *SumA* ::
$['p,'p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - - - - - *SumA* - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where

A B C D E F SumA G H I
≡
∃ *J*. (*C B J CongA D E F* ∧ ¬ *B C OS A J* ∧ *Coplanar A B C J* ∧ *A B J CongA G H I*)

definition *TriSumA* ::
['p,'p,'p,'p,'p,'p] ⇒ *bool*
(- - - *TriSumA* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
A B C TriSumA D E F
≡
∃ *G H I*. (*A B C B C A SumA G H I* ∧ *G H I C A B SumA D E F*)

definition *SAMS* ::
['p,'p,'p,'p,'p,'p] ⇒ *bool*
(*SAMS* - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
SAMS A B C D E F
≡
(*A* ≠ *B* ∧
(*E Out D F* ∨ ¬ *Bet A B C*)) ∧
(∃ *J*. (*C B J CongA D E F* ∧ ¬ (*B C OS A J*) ∧ ¬ (*A B TS C J*) ∧ *Coplanar A B C J*))

definition *Inter* ::
['p,'p,'p,'p,'p] ⇒ *bool*
(- *Inter* - - - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
X Inter A1 A2 B1 B2
≡
B1 ≠ *B2* ∧
(∃ *P*::'p. (*Col P B1 B2* ∧ ¬ *Col P A1 A2*)) ∧
Col A1 A2 X ∧ *Col B1 B2 X*

definition *Perp2* ::
['p,'p,'p,'p,'p] ⇒ *bool*
(- *Perp2* - - - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
P Perp2 A B C D
≡
∃ *X Y*. (*Col P X Y* ∧ *X Y Perp A B* ∧ *X Y Perp C D*)

definition *Perp-bisect* ::
['p,'p,'p,'p] ⇒ *bool*
(- - *PerpBisect* - - [*99*,*99*,*99*,*99*] *50*)
where
P Q PerpBisect A B
≡
A B ReflectL P Q ∧ *A* ≠ *B*

definition *Perp-bisect-bis* ::
['p,'p,'p,'p] ⇒ *bool*
(- - *PerpBisectBis* - - [*99*,*99*,*99*,*99*] *50*)
where
P Q PerpBisectBis A B
≡
∃ *I*. *I PerpAt P Q A B* ∧ *I Midpoint A B*

definition *Is-on-perp-bisect* ::
['p,'p,'p] ⇒ *bool*
(- *IsOnPerpBisect* - - [*99*,*99*,*99*] *50*)
where
P IsOnPerpBisect A B
≡
Cong A P P B

definition *isosceles*::
['p,'p,'p] ⇒ *bool*

(- - - *isosceles* [*99*,*99*,*99*] *50*)
where
A B C isosceles
≡
Cong A B B C

definition *equilateral*::
['*p*,'*p*,'*p*] ⇒ *bool*
(- - - *equilateral* [*99*,*99*,*99*] *50*)
where
A B C equilateral
≡
Cong A B B C ∧ *Cong B C C A*

definition *equilateralStrict*::
['*p*,'*p*,'*p*] ⇒ *bool*
(- - - *equilateralStrict* [*99*,*99*,*99*] *50*)
where
A B C equilateralStrict
≡
A B C equilateral ∧ *A* ≠ *B*

definition *QCong*::
(['*p*,'*p*] ⇒ *bool*) ⇒ *bool*
(*QCong* - [*99*] *50*)
where
QCong l
≡
∃ *A B*. (∀ *X Y*. (*Cong A B X Y* ⟷ *l X Y*))

definition *TarskiLen*::
['*p*,'*p*,(['*p*,'*p*] ⇒ *bool*)] ⇒ *bool*
(*TarskiLen* - - - [*99*,*99*,*99*] *50*)
where
TarskiLen A B l
≡
QCong l ∧ *l A B*

definition *QCongNull* ::
(['*p*,'*p*] ⇒ *bool*) ⇒ *bool*
(*QCongNull* - [*99*] *50*)
where
QCongNull l
≡
QCong l ∧ (∃ *A*. *l A A*)

definition *QCongA* ::
(['*p*, '*p*, '*p*] ⇒ *bool*) ⇒ *bool*
(*QCongA* - [*99*] *50*)
where
QCongA a
≡
∃ *A B C*. (*A* ≠ *B* ∧ *C* ≠ *B* ∧ (∀ *X Y Z*. *A B C CongA X Y Z* ⟷ *a X Y Z*))

definition *Ang* ::
['*p*,'*p*,'*p*, (['*p*, '*p*, '*p*] ⇒ *bool*)] ⇒ *bool*
(- - - *Ang* - [*99*,*99*,*99*,*99*] *50*)
where
A B C Ang a
≡
QCongA a ∧
a A B C

definition *QCongAAcute* ::
(['*p*, '*p*, '*p*] ⇒ *bool*) ⇒ *bool*
(*QCongAACute* - [*99*] *50*)

where
  QCongAAcute a
≡
$\exists$ *A B C*. (*Acute A B C* $\wedge$ ($\forall$ *X Y Z*. (*A B C CongA X Y Z* $\longleftrightarrow$ *a X Y Z*)))

definition *AngAcute* ::
  ['*p*,'*p*,'*p*, (['*p*,'*p*,'*p*] $\Rightarrow$ *bool*)] $\Rightarrow$ *bool*
  (- - - *AngAcute* - [*99*,*99*,*99*,*99*] *50*)
  where
  A B C AngAcute a
≡
((*QCongAAcute a*) $\wedge$ (*a A B C*))

definition *QCongANullAcute* ::
  (['*p*,'*p*,'*p*] $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
  (*QCongANullAcute* - [*99*] *50*)
  where
  QCongANullAcute a
≡
QCongAAcute a $\wedge$
($\forall$ *A B C*. (*a A B C* $\longrightarrow$ *B Out A C*))

definition *QCongAnNull* ::
  (['*p*,'*p*,'*p*] $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
  (*QCongAnNull* - [*99*] *50*)
  where
  QCongAnNull a
≡
QCongA a $\wedge$
($\forall$ *A B C*. (*a A B C* $\longrightarrow$ $\neg$ *B Out A C*))

definition *QCongAnFlat* ::
  (['*p*,'*p*,'*p*] $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
  (*QCongAnFlat* - [*99*] *50*)
  where
  QCongAnFlat a
≡
QCongA a $\wedge$
($\forall$ *A B C*. (*a A B C* $\longrightarrow$ $\neg$ *Bet A B C*))

definition *IsNullAngaP* ::
  (['*p*,'*p*,'*p*] $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
  (*IsNullAngaP* - [*99*] *50*)
  where
  IsNullAngaP a
≡
QCongAAcute a $\wedge$
($\exists$ *A B C*. (*a A B C* $\wedge$ *B Out A C*))

definition *QCongANull* ::
  (['*p*,'*p*,'*p*] $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
  (*QCongANull* - [*99*] *50*)
  where
  QCongANull a
≡
QCongA a $\wedge$
($\forall$ *A B C*. (*a A B C* $\longrightarrow$ *B Out A C*))

definition *AngFlat* ::
  (['*p*, '*p*, '*p*] $\Rightarrow$ *bool*) $\Rightarrow$ *bool*
  (*AngFlat* - [*99*] *50*)
  where
  AngFlat a
≡
QCongA a $\wedge$
($\forall$ *A B C*. (*a A B C* $\longrightarrow$ *Bet A B C*))

definition *EqLTarski* ::
$(['p,\ 'p] \Rightarrow bool) \Rightarrow (['p,\ 'p] \Rightarrow bool) \Rightarrow bool$
(- *EqLTarski* - [*99*,*99*] *50*)
where
l1 EqLTarski l2
$\equiv$
$\forall\ A\ B.\ l1\ A\ B \longleftrightarrow l2\ A\ B$

definition *EqA* ::
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow (['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow bool$
(- *EqA* - [*99*,*99*] *50*)
where
a1 EqA a2
$\equiv$
$\forall\ A\ B\ C.\ a1\ A\ B\ C \longleftrightarrow a2\ A\ B\ C$

definition *hypothesis-of-right-saccheri-quadrilaterals* ::
bool
(*HypothesisRightSaccheriQuadrilaterals*)
where
hypothesis-of-right-saccheri-quadrilaterals
$\equiv$
$\forall\ A\ B\ C\ D.\ Saccheri\ A\ B\ C\ D \longrightarrow Per\ A\ B\ C$

definition *hypothesis-of-acute-saccheri-quadrilaterals* ::
bool
(*HypothesisAcuteSaccheriQuadrilaterals*)
where
hypothesis-of-acute-saccheri-quadrilaterals
$\equiv$
$\forall\ A\ B\ C\ D.\ Saccheri\ A\ B\ C\ D \longrightarrow Acute\ A\ B\ C$

definition *hypothesis-of-obtuse-saccheri-quadrilaterals* ::
bool
(*HypothesisObtuseSaccheriQuadrilaterals*)
where
hypothesis-of-obtuse-saccheri-quadrilaterals
$\equiv$
$\forall\ A\ B\ C\ D.\ Saccheri\ A\ B\ C\ D \longrightarrow Obtuse\ A\ B\ C$

definition *Defect* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Defect* - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
Defect A B C D E F
$\equiv$
$(\exists\ G\ H\ I.\ (A\ B\ C\ TriSumA\ G\ H\ I \land G\ H\ I\ SuppA\ D\ E\ F))$

definition *Ar1* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$
(*Ar1* - - - - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
Ar1 PO E A B C
$\equiv$
$PO \neq E \land Col\ PO\ E\ A \land Col\ PO\ E\ B \land Col\ PO\ E\ C$

definition *Ar2* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Ar2* - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
Ar2 PO E E' A B C
$\equiv$
$\neg\ Col\ PO\ E\ E' \land Col\ PO\ E\ A \land Col\ PO\ E\ B \land Col\ PO\ E\ C$

definition *Pj* ::
$['p,'p,'p,'p] \Rightarrow bool$
(- - *Pj* - - [99,99,99,99] 50)
where
$A\ B\ Pj\ C\ D$
$\equiv$
$A\ B\ Par\ C\ D \lor C = D$

definition *Sum* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Sum* - - - - - - [99,99,99,99,99,99] 50)
where
$Sum\ PO\ E\ E'\ A\ B\ C$
$\equiv$
$Ar2\ PO\ E\ E'\ A\ B\ C \land$
$(\exists\ A'\ C'.\ E\ E'\ Pj\ A\ \ A' \land Col\ PO\ E'\ A' \land PO\ E\ Pj\ A'\ C' \land$
$PO\ E'\ Pj\ B\ \ C' \land E'\ E\ Pj\ C'\ C)$

definition *Proj* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(- - *Proj* - - - - [99,99,99,99,99,99] 50)
where
$P\ Q\ Proj\ A\ B\ X\ Y$
$\equiv$
$A \neq B \land X \neq Y \land \neg\ A\ B\ Par\ X\ Y \land Col\ A\ B\ Q \land (P\ Q\ Par\ X\ Y \lor P = Q)$

definition *Sump* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Sump* - - - - - - [99,99,99,99,99,99] 50)
where
$Sump\ PO\ E\ E'\ A\ B\ C$
$\equiv$
$Col\ PO\ E\ A \land Col\ PO\ E\ B \land$
$(\exists\ A'\ C'\ P'.\ A\ A'\ Proj\ PO\ E'\ E\ E' \land PO\ E\ Par\ A'\ P' \land$
$B\ C'\ Proj\ A'\ P'\ PO\ E' \land C'\ C\ Proj\ PO\ E\ E\ E')$

definition *Prod* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Prod* - - - - - - [99,99,99,99,99,99] 50)
where
$Prod\ PO\ E\ E'\ A\ B\ C$
$\equiv$
$Ar2\ PO\ E\ E'\ A\ B\ C \land$
$(\exists\ B'.\ E\ E'\ Pj\ B\ B' \land Col\ PO\ E'\ B' \land E'\ A\ Pj\ B'\ C)$

definition *Prodp* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Prodp* - - - - - - [99,99,99,99,99,99] 50)
where
$Prodp\ PO\ E\ E'\ A\ B\ C$
$\equiv$
$Col\ PO\ E\ A \land Col\ PO\ E\ B \land$
$(\exists\ B'.\ B\ B'\ Proj\ PO\ E'\ E\ E' \land B'\ C\ Proj\ PO\ E\ A\ E')$

definition *Opp* ::
$['p,'p,'p,'p,'p] \Rightarrow bool$
(*Opp* - - - - - [99,99,99,99,99] 50)

where
Opp PO E E' A B
≡
Sum PO E E' B A PO

definition *Diff* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Diff* - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
Diff PO E E' A B C
≡
$\exists$ *B'*. *Opp PO E E' B B'* $\wedge$ *Sum PO E E' A B' C*

definition *sum3* ::
$['p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*sum3* - - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
sum3 PO E E' A B C S
≡
$\exists$ *AB*. *Sum PO E E' A B AB* $\wedge$ *Sum PO E E' AB C S*

definition *sum4* ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Sum4* - - - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
Sum4 PO E E' A B C D S
≡
$\exists$ *ABC*. *sum3 PO E E' A B C ABC* $\wedge$ *Sum PO E E' ABC D S*

definition *sum22* ::
$['p,'p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*sum22* - - - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
sum22 PO E E' A B C D S
≡
$\exists$ *AB CD*. *Sum PO E E' A B AB* $\wedge$ *Sum PO E E' C D CD* $\wedge$ *Sum PO E E' AB CD S*

definition *Ar2p4* ::
$['p,'p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*Ar2p4* - - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
Ar2p4 PO E E' A B C D
≡
$\neg$ *Col PO E E'* $\wedge$ *Col PO E A* $\wedge$ *Col PO E B* $\wedge$ *Col PO E C* $\wedge$ *Col PO E D*

definition *Ps* ::
$['p,'p,'p] \Rightarrow bool$
(*Ps* - - - [*99*,*99*,*99*] *50*)
where
Ps X E A
≡
X Out A E

definition *Ng* ::
$['p,'p,'p] \Rightarrow bool$
(*Ng* - - - [*99*,*99*,*99*] *50*)
where
Ng X E A
≡
$A \neq X \wedge E \neq X \wedge$ *Bet A X E*

definition *LtP* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(LtP$ - - - - - $[99,99,99,99,99]$ $50)$
where
$LtP\ X\ E\ E'\ A\ B$
$\equiv$
$\exists\ D.\ Diff\ X\ E\ E'\ B\ A\ D \wedge Ps\ X\ E\ D$

definition *LeP* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(LeP$ - - - - - $[99,99,99,99,99]$ $50)$
where
$LeP\ X\ E\ E'\ A\ B$
$\equiv$
$LtP\ X\ E\ E'\ A\ B \vee A = B$

definition *Length* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(Length$ - - - - - - $[99,99,99,99,99,99]$ $50)$
where
$Length\ X\ E\ E'\ A\ B\ L$
$\equiv$
$X \neq E \wedge Col\ X\ E\ L \wedge LeP\ X\ E\ E'\ X\ L \wedge Cong\ X\ L\ A\ B$

definition *IsLength* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(IsLength$ - - - - - - $[99,99,99,99,99,99]$ $50)$
where
$IsLength\ X\ E\ E'\ A\ B\ L$
$\equiv$
$Length\ X\ E\ E'\ A\ B\ L \vee (X = E \wedge X = L)$

definition *Sumg* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(Sumg$ - - - - - - $[99,99,99,99,99,99]$ $50)$
where
$Sumg\ X\ E\ E'\ A\ B\ C$
$\equiv$
$Sum\ X\ E\ E'\ A\ B\ C \vee (\neg\ Ar2\ X\ E\ E'\ A\ B\ B \wedge C = X)$

definition *Prodg* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(Prodg$ - - - - - - $[99,99,99,99,99,99]$ $50)$
where
$Prodg\ X\ E\ E'\ A\ B\ C$
$\equiv$
$Prod\ X\ E\ E'\ A\ B\ C \vee (\neg\ Ar2\ X\ E\ E'\ A\ B\ B \wedge C = X)$

definition *PythRel* ::
$[{}'p,{}'p,{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(PythRel$ - - - - - - $[99,99,99,99,99,99]$ $50)$
where
$PythRel\ X\ E\ E'\ A\ B\ C$
$\equiv$
$Ar2\ X\ E\ E'\ A\ B\ C \wedge$
$((X = B \wedge (A = C \vee Opp\ X\ E\ E'\ A\ C)) \vee (\exists\ B'.\ X\ B'\ Perp\ X\ B \wedge Cong\ X\ B'\ X\ B \wedge Cong\ X\ C\ A\ B'))$

definition *SignEq* ::
$[{}'p,{}'p,{}'p,{}'p] \Rightarrow bool$
$(SignEq$ - - - - $[99,99,99,99]$ $50)$
where
$SignEq\ X\ E\ A\ B$
$\equiv$

Ps X E A ∧ *Ps X E B* ∨ *Ng X E A* ∧ *Ng X E B*

definition *LtPs* ::
 [′p,′p,′p,′p,′p] ⇒ *bool*
 (*LtPs* - - - - - [*99*,*99*,*99*,*99*,*99*] *50*)
 where
 LtPs X E E′ A B
≡
∃ *D*. *Ps X E D* ∧ *Sum X E E′ A D B*

definition *IsOrthocenter* ::
 [′p,′p,′p,′p] ⇒ *bool*
 (- *IsOrthocenter* - - - [*99*,*99*,*99*,*99*] *50*)
 where
 H IsOrthocenter A B C ≡ ¬ *Col A B C* ∧
 A H Perp B C ∧
 B H Perp A C ∧
 C H Perp A B

definition *IsCircumcenter* ::
 [′p,′p,′p,′p] ⇒ *bool*
 (- *IsCircumcenter* - - - [*99*,*99*,*99*,*99*] *50*)
 where
 G IsCircumcenter A B C ≡
Cong A G B G ∧
Cong B G C G ∧
Coplanar G A B C

definition *IsGravityCenter* ::
 [′p,′p,′p,′p] ⇒ *bool*
 (- *IsGravityCenter* - - - [*99*,*99*,*99*,*99*] *50*)
 where
 G IsGravityCenter A B C ≡ ¬ *Col A B C* ∧
 (∃ *I J*. *I Midpoint B C* ∧
 J Midpoint A C ∧
 Col G A I ∧
 Col G B J)

definition *Lcos* :: ([′p,′p] ⇒ *bool*) ⇒
([′p,′p] ⇒ *bool*) ⇒
([′p, ′p, ′p] ⇒ *bool*) ⇒
bool
 where
 Lcos lb lc a ≡
 QCong lb ∧ *QCong lc* ∧ *QCongAAcute a* ∧
 (∃ *A B C*. (*Per C B A* ∧ *lb A B* ∧ *lc A C* ∧ *a B A C*))

definition *EqLcos* :: ([′p,′p] ⇒ *bool*) ⇒
([′p, ′p, ′p] ⇒ *bool*) ⇒
([′p,′p] ⇒ *bool*) ⇒
([′p, ′p, ′p] ⇒ *bool*) ⇒
bool
 where
 EqLcos la a lb b ≡ (∃ *lp*. *Lcos lp la a* ∧ *Lcos lp lb b*)

definition *Lcos2* :: ([′p,′p] ⇒ *bool*) ⇒
([′p,′p] ⇒ *bool*) ⇒
([′p, ′p, ′p] ⇒ *bool*) ⇒
([′p, ′p, ′p] ⇒ *bool*) ⇒
bool
 where
 Lcos2 lp l a b ≡ ∃ *la*. *Lcos la l a* ∧ *Lcos lp la b*

definition *EqLcos2* :: ([′p,′p] ⇒ *bool*) ⇒

$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
bool
where
EqLcos2 l1 a b l2 c d $\equiv$ ($\exists$ *lp*. *Lcos2 lp l1 a b* $\wedge$ *Lcos2 lp l2 c d*)

definition *Lcos3* :: $(['p,'p] \Rightarrow bool) \Rightarrow$
$(['p,'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
bool
where
Lcos3 lp l a b c $\equiv \exists$ *la lab*. *Lcos la l a* $\wedge$
Lcos lab la b $\wedge$ *Lcos lp lab c*

definition *EqLcos3* :: $(['p,'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
$(['p,\ 'p,\ 'p] \Rightarrow bool) \Rightarrow$
bool
where
EqLcos3 l1 a b c l2 d e f $\equiv$ ($\exists$ *lp*. *Lcos3 lp l1 a b c* $\wedge$ *Lcos3 lp l2 d e f*)

definition *EqV* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- - *EqV* - - [*99*,*99*,*99*,*99*] *50*)
where
A B EqV C D $\equiv$ *Parallelogram A B D C* $\vee$ ($A = B \wedge C = D$)

definition *SumV* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- - - - *SumV* - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
A B C D SumV E F $\equiv \forall$ D'. *C D EqV B* $D' \longrightarrow$ *A* D' *EqV E F*

definition *SumVExists* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- - - - *SumVExists* - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
A B C D SumVExists E F $\equiv$ ($\exists$ D'. *B* D' *EqV C D* $\wedge$ *A* D' *EqV E F*)

definition *SameDir* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- - *SameDir* - - [*99*,*99*,*99*,*99*] *50*)
where
A B SameDir C D $\equiv$
($A = B \wedge C = D$) $\vee$ ($\exists$ D'. *C Out D* D' $\wedge$ *A B EqV C* D')

definition *OppDir* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- - *OppDir* - - [*99*,*99*,*99*,*99*] *50*)
where
A B OppDir C D $\equiv$ *A B SameDir D C*

definition *CongA3* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- - - *CongA3* - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
A B C CongA3 A' B' C' $\equiv$
A B C CongA A' B' C' $\wedge$ *B C A CongA* B' C' A' $\wedge$ *C A B CongA* C' A' B'

definition *Projp* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- - *Projp* - - [*99*,*99*,*99*,*99*] *50*)
where
$P\ Q\ Projp\ A\ B \equiv$
$A \neq B \wedge ((Col\ A\ B\ Q \wedge A\ B\ Perp\ P\ Q) \vee (Col\ A\ B\ P \wedge P = Q))$

1.3 Propositions

lemma *cong-reflexivity*:
shows *Cong A B A B*
⟨*proof*⟩

lemma *cong-symmetry*:
assumes *Cong A B C D*
shows *Cong C D A B*
⟨*proof*⟩

lemma *cong-transitivity*:
assumes *Cong A B C D* **and** *Cong C D E F*
shows *Cong A B E F*
⟨*proof*⟩

lemma *cong-left-commutativity*:
assumes *Cong A B C D*
shows *Cong B A C D*
⟨*proof*⟩

lemma *cong-right-commutativity*:
assumes *Cong A B C D*
shows *Cong A B D C*
⟨*proof*⟩

lemma *cong-3421*:
assumes *Cong A B C D*
shows *Cong C D B A*
⟨*proof*⟩

lemma *cong-4312*:
assumes *Cong A B C D*
shows *Cong D C A B*
⟨*proof*⟩

lemma *cong-4321*:
assumes *Cong A B C D*
shows *Cong D C B A*
⟨*proof*⟩

lemma *cong-trivial-identity*:
shows *Cong A A B B*
⟨*proof*⟩

lemma *cong-reverse-identity*:
assumes *Cong A A C D*
shows $C = D$
⟨*proof*⟩

lemma *cong-commutativity*:
assumes *Cong A B C D*
shows *Cong B A D C*
⟨*proof*⟩

lemma *not-cong-2134*:
assumes ¬ *Cong A B C D*
shows ¬ *Cong B A C D*

⟨*proof*⟩

lemma *not-cong-1243*:
assumes $\neg$ *Cong* A B C D
shows $\neg$ *Cong* A B D C
⟨*proof*⟩

lemma *not-cong-2143*:
assumes $\neg$ *Cong* A B C D
shows $\neg$ *Cong* B A D C
⟨*proof*⟩

lemma *not-cong-3412*:
assumes $\neg$ *Cong* A B C D
shows $\neg$ *Cong* C D A B
⟨*proof*⟩

lemma *not-cong-4312*:
assumes $\neg$ *Cong* A B C D
shows $\neg$ *Cong* D C A B
⟨*proof*⟩

lemma *not-cong-3421*:
assumes $\neg$ *Cong* A B C D
shows $\neg$ *Cong* C D B A
⟨*proof*⟩

lemma *not-cong-4321*:
assumes $\neg$ *Cong* A B C D
shows $\neg$ *Cong* D C B A
⟨*proof*⟩

lemma *five-segment-with-def*:
assumes A B C D *OFSC* A' B' C' D' **and** $A \neq B$
shows *Cong* C D C' D'
⟨*proof*⟩

lemma *cong-diff*:
assumes $A \neq B$ **and** *Cong* A B C D
shows $C \neq D$
⟨*proof*⟩

lemma *cong-diff-2*:
assumes $B \neq A$ **and** *Cong* A B C D
shows $C \neq D$
⟨*proof*⟩

lemma *cong-diff-3*:
assumes $C \neq D$ **and** *Cong* A B C D
shows $A \neq B$
⟨*proof*⟩

lemma *cong-diff-4*:
assumes $D \neq C$ **and** *Cong* A B C D
shows $A \neq B$
⟨*proof*⟩

lemma *cong-3-sym*:
assumes A B C *Cong3* A' B' C'
shows A' B' C' *Cong3* A B C
⟨*proof*⟩

lemma *cong-3-swap*:
assumes A B C *Cong3* A' B' C'
shows B A C *Cong3* B' A' C'
⟨*proof*⟩

lemma *cong-3-swap-2*:
  assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$
  shows $A\ C\ B\ Cong3\ A'\ C'\ B'$
  ⟨*proof*⟩

lemma *cong3-transitivity*:
  assumes $A0\ B0\ C0\ Cong3\ A1\ B1\ C1$ **and**
    $A1\ B1\ C1\ Cong3\ A2\ B2\ C2$
  shows $A0\ B0\ C0\ Cong3\ A2\ B2\ C2$
  ⟨*proof*⟩

lemma *eq-dec-points*:
  shows $A = B \lor \neg\ A = B$
  ⟨*proof*⟩

lemma *distinct*:
  assumes $P \neq Q$
  shows $R \neq P \lor R \neq Q$
  ⟨*proof*⟩

lemma *l2-11*:
  assumes $Bet\ A\ B\ C$ **and**
    $Bet\ A'\ B'\ C'$ **and**
    $Cong\ A\ B\ A'\ B'$ **and**
    $Cong\ B\ C\ B'\ C'$
  shows $Cong\ A\ C\ A'\ C'$
⟨*proof*⟩

lemma *bet-cong3*:
  assumes $Bet\ A\ B\ C$ **and**
    $Cong\ A\ B\ A'\ B'$
  shows $\exists\ C'.\ A\ B\ C\ Cong3\ A'\ B'\ C'$
  ⟨*proof*⟩

lemma *construction-uniqueness*:
  assumes $Q \neq A$ **and**
    $Bet\ Q\ A\ X$ **and**
    $Cong\ A\ X\ B\ C$ **and**
    $Bet\ Q\ A\ Y$ **and**
    $Cong\ A\ Y\ B\ C$
  shows $X = Y$
  ⟨*proof*⟩

lemma *Cong-cases*:
  assumes $Cong\ A\ B\ C\ D \lor Cong\ A\ B\ D\ C \lor Cong\ B\ A\ C\ D \lor Cong\ B\ A\ D\ C \lor Cong\ C\ D\ A\ B \lor$
$Cong\ C\ D\ B\ A \lor Cong\ D\ C\ A\ B \lor Cong\ D\ C\ B\ A$
  shows $Cong\ A\ B\ C\ D$
  ⟨*proof*⟩

lemma *Cong-perm* :
  assumes $Cong\ A\ B\ C\ D$
  shows $Cong\ A\ B\ C\ D \land Cong\ A\ B\ D\ C \land Cong\ B\ A\ C\ D \land Cong\ B\ A\ D\ C \land Cong\ C\ D\ A\ B \land$
$Cong\ C\ D\ B\ A \land Cong\ D\ C\ A\ B \land Cong\ D\ C\ B\ A$
  ⟨*proof*⟩

lemma *bet-col*:
  assumes $Bet\ A\ B\ C$
  shows $Col\ A\ B\ C$
  ⟨*proof*⟩

lemma *between-trivial*:
  shows $Bet\ A\ B\ B$
  ⟨*proof*⟩

lemma *between-symmetry*:

assumes *Bet A B C*
shows *Bet C B A*
⟨*proof*⟩

lemma *Bet-cases*:
assumes *Bet A B C* ∨ *Bet C B A*
shows *Bet A B C*
⟨*proof*⟩

lemma *Bet-perm*:
assumes *Bet A B C*
shows *Bet A B C* ∧ *Bet C B A*
⟨*proof*⟩

lemma *between-trivial2*:
shows *Bet A A B*
⟨*proof*⟩

lemma *between-equality*:
assumes *Bet A B C* **and** *Bet B A C*
shows $A = B$
⟨*proof*⟩

lemma *between-equality-2*:
assumes *Bet A B C* **and**
Bet A C B
shows $B = C$
⟨*proof*⟩

lemma *between-exchange3*:
assumes *Bet A B C* **and**
Bet A C D
shows *Bet B C D*
⟨*proof*⟩

lemma *bet-neq12--neq*:
assumes *Bet A B C* **and**
$A \neq B$
shows $A \neq C$
⟨*proof*⟩

lemma *bet-neq21--neq*:
assumes *Bet A B C* **and**
$B \neq A$
shows $A \neq C$
⟨*proof*⟩

lemma *bet-neq23--neq*:
assumes *Bet A B C* **and**
$B \neq C$
shows $A \neq C$
⟨*proof*⟩

lemma *bet-neq32--neq*:
assumes *Bet A B C* **and**
$C \neq B$
shows $A \neq C$
⟨*proof*⟩

lemma *not-bet-distincts*:
assumes ¬ *Bet A B C*
shows $A \neq B \wedge B \neq C$
⟨*proof*⟩

lemma *between-inner-transitivity*:
assumes *Bet A B D* **and**

Bet B C D
shows *Bet A B C*
⟨*proof*⟩

lemma *outer-transitivity-between2*:
assumes *Bet A B C* **and**
Bet B C D **and**
$B \neq C$
shows *Bet A C D*
⟨*proof*⟩

lemma *between-exchange2*:
assumes *Bet A B D* **and**
Bet B C D
shows *Bet A C D*
⟨*proof*⟩

lemma *outer-transitivity-between*:
assumes *Bet A B C* **and**
Bet B C D **and**
$B \neq C$
shows *Bet A B D*
⟨*proof*⟩

lemma *between-exchange4*:
assumes *Bet A B C* **and**
Bet A C D
shows *Bet A B D*
⟨*proof*⟩

lemma *l3-9-4*:
assumes *Bet4 A1 A2 A3 A4*
shows *Bet4 A4 A3 A2 A1*
⟨*proof*⟩

lemma *l3-17*:
assumes *Bet A B C* **and**
Bet A′ B′ C **and**
Bet A P A′
shows ∃ *Q*. *Bet P Q C* ∧ *Bet B Q B′*
⟨*proof*⟩

lemma *lower-dim-ex*:
∃ *A B C*. ¬ (*Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*)
⟨*proof*⟩

lemma *two-distinct-points*:
∃ *X*::′*p*. ∃ *Y*::′*p*. $X \neq Y$
⟨*proof*⟩

lemma *point-construction-different*:
∃ *C*. *Bet A B C* ∧ $B \neq C$
⟨*proof*⟩

lemma *another-point*:
∃ *B*::′*p*. $A \neq B$
⟨*proof*⟩

lemma *Cong-stability*:
assumes ¬ ¬ *Cong A B C D*
shows *Cong A B C D*
⟨*proof*⟩

lemma *l2-11-b*:
assumes *Bet A B C* **and**
Bet A′ B′ C′ **and**

$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ B\ C\ B'\ C'$
shows $Cong\ A\ C\ A'\ C'$
⟨*proof*⟩

lemma *cong-dec-eq-dec-b*:
assumes $\neg\ A \neq B$
shows $A = B$
⟨*proof*⟩

lemma *BetSEq*:
assumes $BetS\ A\ B\ C$
shows $Bet\ A\ B\ C \wedge A \neq B \wedge A \neq C \wedge B \neq C$
⟨*proof*⟩

lemma *l4-2*:
assumes $A\ B\ C\ D\ IFSC\ A'\ B'\ C'\ D'$
shows $Cong\ B\ D\ B'\ D'$
⟨*proof*⟩

lemma *l4-3*:
assumes $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$Cong\ A\ C\ A'\ C'$
and $Cong\ B\ C\ B'\ C'$
shows $Cong\ A\ B\ A'\ B'$
⟨*proof*⟩

lemma *l4-3-1*:
assumes $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ A\ C\ A'\ C'$
shows $Cong\ B\ C\ B'\ C'$
⟨*proof*⟩

lemma *l4-5*:
assumes $Bet\ A\ B\ C$ **and**
$Cong\ A\ C\ A'\ C'$
shows $\exists\ B'.\ (Bet\ A'\ B'\ C' \wedge A\ B\ C\ Cong3\ A'\ B'\ C')$
⟨*proof*⟩

lemma *l4-6*:
assumes $Bet\ A\ B\ C$ **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $Bet\ A'\ B'\ C'$
⟨*proof*⟩

lemma *cong3-bet-eq*:
assumes $Bet\ A\ B\ C$ **and**
$A\ B\ C\ Cong3\ A\ X\ C$
shows $X = B$
⟨*proof*⟩

lemma *col-permutation-1*:
assumes $Col\ A\ B\ C$
shows $Col\ B\ C\ A$
⟨*proof*⟩

lemma *col-permutation-2*:
assumes $Col\ A\ B\ C$
shows $Col\ C\ A\ B$
⟨*proof*⟩

lemma *col-permutation-3*:

assumes *Col A B C*
shows *Col C B A*
⟨*proof*⟩

lemma *col-permutation-4*:
assumes *Col A B C*
shows *Col B A C*
⟨*proof*⟩

lemma *col-permutation-5*:
assumes *Col A B C*
shows *Col A C B*
⟨*proof*⟩

lemma *not-col-permutation-1*:
assumes ¬ *Col A B C*
shows ¬ *Col B C A*
⟨*proof*⟩

lemma *not-col-permutation-2*:
assumes ¬ *Col A B C*
shows ¬ *Col C A B*
⟨*proof*⟩

lemma *not-col-permutation-3*:
assumes ¬ *Col A B C*
shows ¬ *Col C B A*
⟨*proof*⟩

lemma *not-col-permutation-4*:
assumes ¬ *Col A B C*
shows ¬ *Col B A C*
⟨*proof*⟩

lemma *not-col-permutation-5*:
assumes ¬ *Col A B C*
shows ¬ *Col A C B*
⟨*proof*⟩

lemma *Col-cases*:
assumes *Col A B C* ∨ *Col A C B* ∨ *Col B A C* ∨ *Col B C A* ∨ *Col C A B* ∨ *Col C B A*
shows *Col A B C*
⟨*proof*⟩

lemma *Col-perm*:
assumes *Col A B C*
shows *Col A B C* ∧ *Col A C B* ∧ *Col B A C* ∧ *Col B C A* ∧ *Col C A B* ∧ *Col C B A*
⟨*proof*⟩

lemma *col-trivial-1*:
Col A A B
⟨*proof*⟩

lemma *col-trivial-2*:
Col A B B
⟨*proof*⟩

lemma *col-trivial-3*:
Col A B A
⟨*proof*⟩

lemma *l4-13*:
assumes *Col A B C* **and**
A B C Cong3 A′ B′ C′
shows *Col A′ B′ C′*
⟨*proof*⟩

lemma *l4-14R1*:
  assumes *Bet A B C* **and**
    Cong A B A′ B′
  shows ∃ *C′*. *A B C Cong3 A′ B′ C′*
  ⟨*proof*⟩

lemma *l4-14R2*:
  assumes *Bet B C A* **and**
    Cong A B A′ B′
  shows ∃ *C′*. *A B C Cong3 A′ B′ C′*
  ⟨*proof*⟩

lemma *l4-14R3*:
  assumes *Bet C A B* **and**
    Cong A B A′ B′
  shows ∃ *C′*. *A B C Cong3 A′ B′ C′*
  ⟨*proof*⟩

lemma *l4-14*:
  assumes *Col A B C* **and**
    Cong A B A′ B′
  shows ∃ *C′*. *A B C Cong3 A′ B′ C′*
  ⟨*proof*⟩

lemma *l4-16R1*:
  assumes *A B C D FSC A′ B′ C′ D′* **and**
    $A \neq B$ **and**
    Bet A B C
  shows *Cong C D C′ D′*
⟨*proof*⟩

lemma *l4-16R2*:
  assumes *A B C D FSC A′ B′ C′ D′*
    and *Bet B C A*
  shows *Cong C D C′ D′*
⟨*proof*⟩

lemma *l4-16R3*:
  assumes *A B C D FSC A′ B′ C′ D′* **and**
    $A \neq B$ **and**
    Bet C A B
  shows *Cong C D C′ D′*
⟨*proof*⟩

lemma *l4-16*:
  assumes *A B C D FSC A′ B′ C′ D′* **and**
    $A \neq B$
  shows *Cong C D C′ D′*
  ⟨*proof*⟩

lemma *l4-17*:
  assumes $A \neq B$ **and**
    Col A B C **and**
    Cong A P A Q **and**
    Cong B P B Q
  shows *Cong C P C Q*
⟨*proof*⟩

lemma *l4-18*:
  assumes $A \neq B$ **and**
    Col A B C **and**
    Cong A C A C′ **and**
    Cong B C B C′
  shows $C = C'$
  ⟨*proof*⟩

lemma *l4-19*:
assumes *Bet A C B* **and**
Cong A C A C′ **and**
Cong B C B C′
shows $C = C'$
⟨*proof*⟩

lemma *not-col-distincts*:
assumes ¬ *Col A B C*
shows ¬ *Col A B C* ∧ $A \neq B \wedge B \neq C \wedge A \neq C$
⟨*proof*⟩

lemma *NCol-cases*:
assumes ¬ *Col A B C* ∨ ¬ *Col A C B* ∨ ¬ *Col B A C* ∨ ¬ *Col B C A* ∨ ¬ *Col C A B* ∨ ¬ *Col C B A*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *NCol-perm*:
assumes ¬ *Col A B C*
shows ¬ *Col A B C* ∧ ¬ *Col A C B* ∧ ¬ *Col B A C* ∧ ¬ *Col B C A* ∧ ¬ *Col C A B* ∧ ¬ *Col C B A*
⟨*proof*⟩

lemma *col-cong-3-cong-3-eq*:
assumes $A \neq B$
and *Col A B C*
and *A B C Cong3 A′ B′ C1*
and *A B C Cong3 A′ B′ C2*
shows *C1* = *C2*
⟨*proof*⟩

lemma *l5-1*:
assumes $A \neq B$ **and**
Bet A B C **and**
Bet A B D
shows *Bet A C D* ∨ *Bet A D C*
⟨*proof*⟩

lemma *l5-2*:
assumes $A \neq B$ **and**
Bet A B C **and**
Bet A B D
shows *Bet B C D* ∨ *Bet B D C*
⟨*proof*⟩

lemma *segment-construction-2*:
assumes $A \neq Q$
shows ∃ *X*. ((*Bet Q A X* ∨ *Bet Q X A*) ∧ *Cong Q X B C*)
⟨*proof*⟩

lemma *l5-3*:
assumes *Bet A B D* **and**
Bet A C D
shows *Bet A B C* ∨ *Bet A C B*
⟨*proof*⟩

lemma *bet3--bet*:
assumes *Bet A B E* **and**
Bet A D E **and**
Bet B C D
shows *Bet A C E*
⟨*proof*⟩

lemma *le-bet*:
assumes *C D Le A B*
shows ∃ *X*. (*Bet A X B* ∧ *Cong A X C D*)

⟨*proof*⟩

lemma *l5-5-1*:
assumes $A\ B\ Le\ C\ D$
shows $\exists\ X.\ (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
⟨*proof*⟩

lemma *l5-5-2*:
assumes $\exists\ X.\ (Bet\ A\ B\ X \wedge Cong\ A\ X\ C\ D)$
shows $A\ B\ Le\ C\ D$
⟨*proof*⟩

lemma *l5-6*:
assumes $A\ B\ Le\ C\ D$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ C\ D\ C'\ D'$
shows $A'\ B'\ Le\ C'\ D'$
⟨*proof*⟩

lemma *le-reflexivity*:
shows $A\ B\ Le\ A\ B$
⟨*proof*⟩

lemma *le-transitivity*:
assumes $A\ B\ Le\ C\ D$ **and**
$C\ D\ Le\ E\ F$
shows $A\ B\ Le\ E\ F$
⟨*proof*⟩

lemma *between-cong*:
assumes $Bet\ A\ C\ B$ **and**
$Cong\ A\ C\ A\ B$
shows $C = B$
⟨*proof*⟩

lemma *cong3-symmetry*:
assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $A'\ B'\ C'\ Cong3\ A\ B\ C$
⟨*proof*⟩

lemma *between-cong-2*:
assumes $Bet\ A\ D\ B$ **and**
$Bet\ A\ E\ B$ **and**
$Cong\ A\ D\ A\ E$
shows $D = E$
⟨*proof*⟩

lemma *between-cong-3*:
assumes $A \neq B$
and $Bet\ A\ B\ D$
and $Bet\ A\ B\ E$
and $Cong\ B\ D\ B\ E$
shows $D = E$
⟨*proof*⟩

lemma *le-anti-symmetry*:
assumes $A\ B\ Le\ C\ D$ **and**
$C\ D\ Le\ A\ B$
shows $Cong\ A\ B\ C\ D$
⟨*proof*⟩

lemma *cong-dec*:
shows $Cong\ A\ B\ C\ D \vee \neg\ Cong\ A\ B\ C\ D$
⟨*proof*⟩

lemma *bet-dec*:

shows *Bet A B C* ∨ ¬ *Bet A B C*
⟨*proof*⟩

lemma *col-dec*:
shows *Col A B C* ∨ ¬ *Col A B C*
⟨*proof*⟩

lemma *le-trivial*:
shows *A A Le C D*
⟨*proof*⟩

lemma *le-cases*:
shows *A B Le C D* ∨ *C D Le A B*
⟨*proof*⟩

lemma *le-zero*:
assumes *A B Le C C*
shows *A* = *B*
⟨*proof*⟩

lemma *le-diff*:
assumes *A* ≠ *B* **and** *A B Le C D*
shows *C* ≠ *D*
⟨*proof*⟩

lemma *lt-diff*:
assumes *A B Lt C D*
shows *C* ≠ *D*
⟨*proof*⟩

lemma *bet-cong-eq*:
assumes *Bet A B C* **and**
Bet A C D **and**
Cong B C A D
shows *C* = *D* ∧ *A* = *B*
⟨*proof*⟩

lemma *cong--le*:
assumes *Cong A B C D*
shows *A B Le C D*
⟨*proof*⟩

lemma *cong--le3412*:
assumes *Cong A B C D*
shows *C D Le A B*
⟨*proof*⟩

lemma *le1221*:
shows *A B Le B A*
⟨*proof*⟩

lemma *le-left-comm*:
assumes *A B Le C D*
shows *B A Le C D*
⟨*proof*⟩

lemma *le-right-comm*:
assumes *A B Le C D*
shows *A B Le D C*
⟨*proof*⟩

lemma *le-comm*:
assumes *A B Le C D*
shows *B A Le D C*
⟨*proof*⟩

lemma *ge-left-comm*:
assumes *A B Ge C D*
shows *B A Ge C D*
⟨*proof*⟩

lemma *ge-right-comm*:
assumes *A B Ge C D*
shows *A B Ge D C*
⟨*proof*⟩

lemma *ge-comm0*:
assumes *A B Ge C D*
shows *B A Ge D C*
⟨*proof*⟩

lemma *lt-right-comm*:
assumes *A B Lt C D*
shows *A B Lt D C*
⟨*proof*⟩

lemma *lt-left-comm*:
assumes *A B Lt C D*
shows *B A Lt C D*
⟨*proof*⟩

lemma *lt-comm*:
assumes *A B Lt C D*
shows *B A Lt D C*
⟨*proof*⟩

lemma *gt-left-comm0*:
assumes *A B Gt C D*
shows *B A Gt C D*
⟨*proof*⟩

lemma *gt-right-comm*:
assumes *A B Gt C D*
shows *A B Gt D C*
⟨*proof*⟩

lemma *gt-comm*:
assumes *A B Gt C D*
shows *B A Gt D C*
⟨*proof*⟩

lemma *cong2-lt--lt*:
assumes *A B Lt C D* **and**
Cong A B A′ B′ **and**
Cong C D C′ D′
shows *A′ B′ Lt C′ D′*
⟨*proof*⟩

lemma *fourth-point*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Col A B P **and**
Bet A B C
shows *Bet P A B* ∨ *Bet A P B* ∨ *Bet B P C* ∨ *Bet B C P*
⟨*proof*⟩

lemma *third-point*:
assumes *Col A B P*
shows *Bet P A B* ∨ *Bet A P B* ∨ *Bet A B P*
⟨*proof*⟩

lemma *l5-12-a*:

assumes *Bet A B C*
shows *A B Le A C* ∧ *B C Le A C*
⟨*proof*⟩

lemma *bet--le1213*:
assumes *Bet A B C*
shows *A B Le A C*
⟨*proof*⟩

lemma *bet--le2313*:
assumes *Bet A B C*
shows *B C Le A C*
⟨*proof*⟩

lemma *bet--lt1213*:
assumes *B* ≠ *C* **and**
Bet A B C
shows *A B Lt A C*
⟨*proof*⟩

lemma *bet--lt2313*:
assumes *A* ≠ *B* **and**
Bet A B C
shows *B C Lt A C*
⟨*proof*⟩

lemma *l5-12-b*:
assumes *Col A B C* **and**
A B Le A C **and**
B C Le A C
shows *Bet A B C*
⟨*proof*⟩

lemma *bet-le-eq*:
assumes *Bet A B C*
and *A C Le B C*
shows *A* = *B*
⟨*proof*⟩

lemma *or-lt-cong-gt*:
A B Lt C D ∨ *A B Gt C D* ∨ *Cong A B C D*
⟨*proof*⟩

lemma *lt--le*:
assumes *A B Lt C D*
shows *A B Le C D*
⟨*proof*⟩

lemma *le1234-lt--lt*:
assumes *A B Le C D* **and**
C D Lt E F
shows *A B Lt E F*
⟨*proof*⟩

lemma *le3456-lt--lt*:
assumes *A B Lt C D* **and**
C D Le E F
shows *A B Lt E F*
⟨*proof*⟩

lemma *lt-transitivity*:
assumes *A B Lt C D* **and**
C D Lt E F
shows *A B Lt E F*
⟨*proof*⟩

lemma *not-and-lt*:
¬ (*A B Lt C D* ∧ *C D Lt A B*)
⟨*proof*⟩

lemma *nlt*:
¬ *A B Lt A B*
⟨*proof*⟩

lemma *le--nlt*:
assumes *A B Le C D*
shows ¬ *C D Lt A B*
⟨*proof*⟩

lemma *cong--nlt*:
assumes *Cong A B C D*
shows ¬ *A B Lt C D*
⟨*proof*⟩

lemma *nlt--le*:
assumes ¬ *A B Lt C D*
shows *C D Le A B*
⟨*proof*⟩

lemma *lt--nle*:
assumes *A B Lt C D*
shows ¬ *C D Le A B*
⟨*proof*⟩

lemma *nle--lt*:
assumes ¬ *A B Le C D*
shows *C D Lt A B*
⟨*proof*⟩

lemma *lt1123*:
assumes $B \neq C$
shows *A A Lt B C*
⟨*proof*⟩

lemma *bet2-le2--le-R1*:
assumes *Bet a P b* **and**
Bet A Q B **and**
P a Le Q A **and**
P b Le Q B **and**
$B = Q$
shows *a b Le A B*
⟨*proof*⟩

lemma *bet2-le2--le-R2*:
assumes *Bet a Po b* **and**
Bet A PO B **and**
Po a Le PO A **and**
Po b Le PO B **and**
$A \neq PO$ **and**
$B \neq PO$
shows *a b Le A B*
⟨*proof*⟩

lemma *bet2-le2--le*:
assumes *Bet a P b* **and**
Bet A Q B **and**
P a Le Q A **and**
P b Le Q B
shows *a b Le A B*
⟨*proof*⟩

lemma *Le-cases*:

assumes $A\ B\ Le\ C\ D \lor B\ A\ Le\ C\ D \lor A\ B\ Le\ D\ C \lor B\ A\ Le\ D\ C$
shows $A\ B\ Le\ C\ D$
⟨*proof*⟩

lemma *Lt-cases*:
assumes $A\ B\ Lt\ C\ D \lor B\ A\ Lt\ C\ D \lor A\ B\ Lt\ D\ C \lor B\ A\ Lt\ D\ C$
shows $A\ B\ Lt\ C\ D$
⟨*proof*⟩

lemma *bet-out*:
assumes $B \neq A$ **and**
Bet A B C
shows *A Out B C*
⟨*proof*⟩

lemma *bet-out-1*:
assumes $B \neq A$ **and**
Bet C B A
shows *A Out B C*
⟨*proof*⟩

lemma *out-dec*:
shows $P\ Out\ A\ B \lor \neg\ P\ Out\ A\ B$
⟨*proof*⟩

lemma *out-diff1*:
assumes *A Out B C*
shows $B \neq A$
⟨*proof*⟩

lemma *out-diff2*:
assumes *A Out B C*
shows $C \neq A$
⟨*proof*⟩

lemma *out-distinct*:
assumes *A Out B C*
shows $B \neq A \land C \neq A$
⟨*proof*⟩

lemma *out-col*:
assumes *A Out B C*
shows *Col A B C*
⟨*proof*⟩

lemma *l6-2*:
assumes $A \neq P$ **and**
$B \neq P$ **and**
$C \neq P$ **and**
Bet A P C
shows $Bet\ B\ P\ C \longleftrightarrow P\ Out\ A\ B$
⟨*proof*⟩

lemma *bet-out--bet*:
assumes *Bet A P C* **and**
P Out A B
shows *Bet B P C*
⟨*proof*⟩

lemma *l6-3-1*:
assumes *P Out A B*
shows $A \neq P \land B \neq P \land (\exists\ C.\ (C \neq P \land Bet\ A\ P\ C \land Bet\ B\ P\ C))$
⟨*proof*⟩

lemma *l6-3-2*:
assumes $A \neq P$ **and**

$B \neq P$ **and**
$\exists\ C.\ (C \neq P \wedge Bet\ A\ P\ C \wedge Bet\ B\ P\ C)$
shows $P\ Out\ A\ B$
⟨*proof*⟩

lemma *l6-4-1*:
assumes $P\ Out\ A\ B$ **and**
$Col\ A\ P\ B$
shows $\neg\ Bet\ A\ P\ B$
⟨*proof*⟩

lemma *l6-4-2*:
assumes $Col\ A\ P\ B$
and $\neg\ Bet\ A\ P\ B$
shows $P\ Out\ A\ B$
⟨*proof*⟩

lemma *out-trivial*:
assumes $A \neq P$
shows $P\ Out\ A\ A$
⟨*proof*⟩

lemma *l6-6*:
assumes $P\ Out\ A\ B$
shows $P\ Out\ B\ A$
⟨*proof*⟩

lemma *l6-7*:
assumes $P\ Out\ A\ B$ **and**
$P\ Out\ B\ C$
shows $P\ Out\ A\ C$
⟨*proof*⟩

lemma *bet-out-out-bet*:
assumes $Bet\ A\ B\ C$ **and**
$B\ Out\ A\ A'$ **and**
$B\ Out\ C\ C'$
shows $Bet\ A'\ B\ C'$
⟨*proof*⟩

lemma *out2-bet-out*:
assumes $B\ Out\ A\ C$ **and**
$B\ Out\ X\ P$ **and**
$Bet\ A\ X\ C$
shows $B\ Out\ A\ P \wedge B\ Out\ C\ P$
⟨*proof*⟩

lemma *l6-11-uniqueness*:
assumes $A\ Out\ X\ R$ **and**
$Cong\ A\ X\ B\ C$ **and**
$A\ Out\ Y\ R$ **and**
$Cong\ A\ Y\ B\ C$
shows $X = Y$
⟨*proof*⟩

lemma *l6-11-existence*:
assumes $R \neq A$ **and**
$B \neq C$
shows $\exists\ X.\ (A\ Out\ X\ R \wedge Cong\ A\ X\ B\ C)$
⟨*proof*⟩

lemma *segment-construction-3*:
assumes $A \neq B$ **and**
$X \neq Y$
shows $\exists\ C.\ (A\ Out\ B\ C \wedge Cong\ A\ C\ X\ Y)$

⟨*proof*⟩

lemma *l6-13-1*:
assumes *P Out A B* **and**
P A Le P B
shows *Bet P A B*
⟨*proof*⟩

lemma *l6-13-2*:
assumes *P Out A B* **and**
Bet P A B
shows *P A Le P B*
⟨*proof*⟩

lemma *l6-16-1*:
assumes $P \neq Q$ **and**
Col S P Q **and**
Col X P Q
shows *Col X P S*
⟨*proof*⟩

lemma *col-transitivity-1*:
assumes $P \neq Q$ **and**
Col P Q A **and**
Col P Q B
shows *Col P A B*
⟨*proof*⟩

lemma *col-transitivity-2*:
assumes $P \neq Q$ **and**
Col P Q A **and**
Col P Q B
shows *Col Q A B*
⟨*proof*⟩

lemma *l6-21*:
assumes ¬ *Col A B C* **and**
$C \neq D$ **and**
Col A B P **and**
Col A B Q **and**
Col C D P **and**
Col C D Q
shows $P = Q$
⟨*proof*⟩

lemma *col2--eq*:
assumes *Col A X Y* **and**
Col B X Y **and**
¬ *Col A X B*
shows $X = Y$
⟨*proof*⟩

lemma *not-col-exists*:
assumes $A \neq B$
shows ∃ *C*. ¬ *Col A B C*
⟨*proof*⟩

lemma *col3*:
assumes $X \neq Y$ **and**
Col X Y A **and**
Col X Y B **and**
Col X Y C
shows *Col A B C*
⟨*proof*⟩

lemma *colx*:

assumes $A \neq B$ **and**
Col X Y A **and**
Col X Y B **and**
Col A B C
shows *Col X Y C*
⟨*proof*⟩

lemma *out2--bet*:
assumes *A Out B C* **and**
C Out A B
shows *Bet A B C*
⟨*proof*⟩

lemma *bet2-le2--le1346*:
assumes *Bet A B C* **and**
Bet A′ B′ C′ **and**
A B Le A′ B′ **and**
B C Le B′ C′
shows *A C Le A′ C′*
⟨*proof*⟩

lemma *bet2-le2--le2356-R1*:
assumes *Bet A A C* **and**
Bet A′ B′ C′ **and**
A A Le A′ B′ **and**
A′ C′ Le A C
shows *B′ C′ Le A C*
⟨*proof*⟩

lemma *bet2-le2--le2356-R2*:
assumes $A \neq B$ **and**
Bet A B C **and**
Bet A′ B′ C′ **and**
A B Le A′ B′ **and**
A′ C′ Le A C
shows *B′ C′ Le B C*
⟨*proof*⟩

lemma *bet2-le2--le2356*:
assumes *Bet A B C* **and**
Bet A′ B′ C′ **and**
A B Le A′ B′ **and**
A′ C′ Le A C
shows *B′ C′ Le B C*
⟨*proof*⟩

lemma *bet2-le2--le1245*:
assumes *Bet A B C* **and**
Bet A′ B′ C′ **and**
B C Le B′ C′ **and**
A′ C′ Le A C
shows *A′ B′ Le A B*
⟨*proof*⟩

lemma *cong-preserves-bet*:
assumes *Bet B A′ A0* **and**
Cong B A′ E D′ **and**
Cong B A0 E D0 **and**
E Out D′ D0
shows *Bet E D′ D0*
⟨*proof*⟩

lemma *out-cong-cong*:
assumes *B Out A A0* **and**
E Out D D0 **and**
Cong B A E D **and**

Cong B A0 E D0
shows *Cong A A0 D D0*
⟨*proof*⟩

lemma *not-out-bet*:
assumes *Col A B C* **and**
¬ *B Out A C*
shows *Bet A B C*
⟨*proof*⟩

lemma *or-bet-out*:
shows *Bet A B C* ∨ *B Out A C* ∨ ¬ *Col A B C*
⟨*proof*⟩

lemma *not-bet-out*:
assumes *Col A B C* **and**
¬ *Bet A B C*
shows *B Out A C*
⟨*proof*⟩

lemma *not-bet-and-out*:
shows ¬ (*Bet A B C* ∧ *B Out A C*)
⟨*proof*⟩

lemma *out-to-bet*:
assumes *Col A′ B′ C′* **and**
B Out A C ⟷ *B′ Out A′ C′* **and**
Bet A B C
shows *Bet A′ B′ C′*
⟨*proof*⟩

lemma *col-out2-col*:
assumes *Col A B C* **and**
B Out A AA **and**
B Out C CC
shows *Col AA B CC*
⟨*proof*⟩

lemma *bet2-out-out*:
assumes *B* ≠ *A* **and**
B′ ≠ *A* **and**
A Out C C′ **and**
Bet A B C **and**
Bet A B′ C′
shows *A Out B B′*
⟨*proof*⟩

lemma *bet2--out*:
assumes *A* ≠ *B* **and**
A ≠ *B′* **and**
Bet A B C
and *Bet A B′ C*
shows *A Out B B′*
⟨*proof*⟩

lemma *out-bet-out-1*:
assumes *P Out A C* **and**
Bet A B C
shows *P Out A B*
⟨*proof*⟩

lemma *out-bet-out-2*:
assumes *P Out A C* **and**
Bet A B C
shows *P Out B C*
⟨*proof*⟩

lemma *out-bet--out*:
  assumes *Bet P Q A* **and**
    Q Out A B
  shows *P Out A B*
  ⟨*proof*⟩

lemma *segment-reverse*:
  assumes *Bet A B C*
  shows $\exists\ B'. \ Bet\ A\ B'\ C \land Cong\ C\ B'\ A\ B$
  ⟨*proof*⟩

lemma *diff-col-ex*:
  shows $\exists\ C.\ A \neq C \land B \neq C \land Col\ A\ B\ C$
  ⟨*proof*⟩

lemma *diff-bet-ex3*:
  assumes *Bet A B C*
  shows $\exists\ D.\ A \neq D \land B \neq D \land C \neq D \land Col\ A\ B\ D$
  ⟨*proof*⟩

lemma *diff-col-ex3*:
  assumes *Col A B C*
  shows $\exists\ D.\ A \neq D \land B \neq D \land C \neq D \land Col\ A\ B\ D$
  ⟨*proof*⟩

lemma *Out-cases*:
  assumes $A\ Out\ B\ C \lor A\ Out\ C\ B$
  shows *A Out B C*
  ⟨*proof*⟩

lemma *ex-sums*:
  shows $\exists\ E\ F.\ A\ B\ C\ D\ SumS\ E\ F$
⟨*proof*⟩

lemma *sums-sym*:
  assumes *A B C D SumS E F*
  shows *C D A B SumS E F*
⟨*proof*⟩

lemma *sums2--cong56*:
  assumes *A B C D SumS E F* **and**
    $A\ B\ C\ D\ SumS\ E'\ F'$
  shows $Cong\ E\ F\ E'\ F'$
⟨*proof*⟩

lemma *sums2--cong12*:
  assumes *A B C D SumS E F*
    and $A'\ B'\ C\ D\ SumS\ E\ F$
  shows $Cong\ A\ B\ A'\ B'$
⟨*proof*⟩

lemma *sums2--cong34*:
  assumes *A B C D SumS E F* **and**
    $A\ B\ C'\ D'\ SumS\ E\ F$
  shows $Cong\ C\ D\ C'\ D'$
  ⟨*proof*⟩

lemma *cong3-sums--sums*:
assumes *Cong A B A′ B′* **and**
Cong C D C′ D′ **and**
Cong E F E′ F′ **and**
A B C D SumS E F
shows *A′ B′ C′ D′ SumS E′ F′*
⟨*proof*⟩

lemma *sums123312*:
shows *A B C C SumS A B*
⟨*proof*⟩

lemma *sums--cong1245*:
assumes *A B C C SumS D E*
shows *Cong A B D E*
⟨*proof*⟩

lemma *sums--eq34*:
assumes *A B C D SumS A B*
shows $C = D$
⟨*proof*⟩

lemma *sums112323*:
shows *A A B C SumS B C*
⟨*proof*⟩

lemma *sums--cong2345*:
assumes *A A B C SumS D E*
shows *Cong B C D E*
⟨*proof*⟩

lemma *sums--eq12*:
assumes *A B C D SumS C D*
shows $A = B$
⟨*proof*⟩

lemma *sums-left-comm*:
assumes *A B C D SumS E F*
shows *B A C D SumS E F*
⟨*proof*⟩

lemma *sums-middle-comm*:
assumes *A B C D SumS E F*
shows *A B D C SumS E F*
⟨*proof*⟩

lemma *sums-right-comm*:
assumes *A B C D SumS E F*
shows *A B C D SumS F E*
⟨*proof*⟩

lemma *sums-comm*:
assumes *A B C D SumS E F*
shows *B A D C SumS F E*
⟨*proof*⟩

lemma *bet--sums*:
assumes *Bet A B C*

shows *A B B C SumS A C*
⟨*proof*⟩

lemma *sums-assoc-1*:
assumes *A B C D SumS G H* **and**
C D E F SumS I J **and**
G H E F SumS K L
shows *A B I J SumS K L*
⟨*proof*⟩

lemma *sums-assoc-2*:
assumes *A B C D SumS G H* **and**
C D E F SumS I J **and**
A B I J SumS K L
shows *G H E F SumS K L*
⟨*proof*⟩

lemma *sums-assoc*:
assumes *A B C D SumS G H* **and**
C D E F SumS I J
shows *G H E F SumS K L* $\longleftrightarrow$ *A B I J SumS K L*
⟨*proof*⟩

lemma *sums--le1256*:
assumes *A B C D SumS E F*
shows *A B Le E F*
⟨*proof*⟩

lemma *sums--le3456*:
assumes *A B C D SumS E F*
shows *C D Le E F*
⟨*proof*⟩

lemma *eq-sums--eq*:
assumes *A B C D SumS E E*
shows $A = B \wedge C = D$
⟨*proof*⟩

lemma *sums-diff-1*:
assumes $A \neq B$ **and**
A B C D SumS E F
shows $E \neq F$
⟨*proof*⟩

lemma *sums-diff-2*:
assumes $C \neq D$ **and**
A B C D SumS E F
shows $E \neq F$
⟨*proof*⟩

lemma *le2-sums2--le*:
assumes *A B Le* A' B' **and**
C D Le C' D' **and**
A B C D SumS E F **and**
A' B' C' D' *SumS* E' F'
shows *E F Le* E' F'

⟨*proof*⟩

lemma *le2-sums2--cong12*:
assumes *A B Le A′ B′* **and**
C D Le C′ D′ **and**
A B C D SumS E F **and**
A′ B′ C′ D′ SumS E F
shows *Cong A B A′ B′*
⟨*proof*⟩

lemma *le2-sums2--cong34*:
assumes *A B Le A′ B′* **and**
C D Le C′ D′ **and**
A B C D SumS E F **and**
A′ B′ C′ D′ SumS E F
shows *Cong C D C′ D′*
⟨*proof*⟩

lemma *le-lt12-sums2--lt*:
assumes *A B Lt A′ B′* **and**
C D Le C′ D′ **and**
A B C D SumS E F **and**
A′ B′ C′ D′ SumS E′ F′
shows *E F Lt E′ F′*
⟨*proof*⟩

lemma *le-lt34-sums2--lt*:
assumes *A B Le A′ B′* **and**
C D Lt C′ D′ **and**
A B C D SumS E F **and**
A′ B′ C′ D′ SumS E′ F′
shows *E F Lt E′ F′*
⟨*proof*⟩

lemma *lt2-sums2--lt*:
assumes *A B Lt A′ B′* **and**
C D Lt C′ D′ **and**
A B C D SumS E F **and**
A′ B′ C′ D′ SumS E′ F′
shows *E F Lt E′ F′*
⟨*proof*⟩

lemma *le2-sums2--le12*:
assumes *C′ D′ Le C D* **and**
E F Le E′ F′ **and**
A B C D SumS E F **and**
A′ B′ C′ D′ SumS E′ F′
shows *A B Le A′ B′*
⟨*proof*⟩

lemma *le2-sums2--le34*:
assumes *A′ B′ Le A B* **and**
E F Le E′ F′ **and**
A B C D SumS E F **and**
A′ B′ C′ D′ SumS E′ F′
shows *C D Le C′ D′*
⟨*proof*⟩

lemma *le-lt34-sums2--lt12*:
assumes C' D' *Lt* C D **and**
E F *Le* E' F' **and**
A B C D *SumS* E F **and**
A' B' C' D' *SumS* E' F'
shows A B *Lt* A' B'
⟨*proof*⟩

lemma *le-lt12-sums2--lt34*:
assumes A' B' *Lt* A B **and**
E F *Le* E' F' **and**
A B C D *SumS* E F **and**
A' B' C' D' *SumS* E' F'
shows C D *Lt* C' D'
⟨*proof*⟩

lemma *le-lt56-sums2--lt12*:
assumes C' D' *Le* C D **and**
E F *Lt* E' F' **and**
A B C D *SumS* E F **and**
A' B' C' D' *SumS* E' F'
shows A B *Lt* A' B'
⟨*proof*⟩

lemma *le-lt56-sums2--lt34*:
assumes A' B' *Le* A B **and**
E F *Lt* E' F' **and**
A B C D *SumS* E F **and**
A' B' C' D' *SumS* E' F'
shows C D *Lt* C' D'
⟨*proof*⟩

lemma *lt2-sums2--lt12*:
assumes C' D' *Lt* C D **and**
E F *Lt* E' F' **and**
A B C D *SumS* E F **and**
A' B' C' D' *SumS* E' F'
shows A B *Lt* A' B'
⟨*proof*⟩

lemma *lt2-sums2--lt34*:
assumes A' B' *Lt* A B **and**
E F *Lt* E' F' **and**
A B C D *SumS* E F **and**
A' B' C' D' *SumS* E' F'
shows C D *Lt* C' D'
⟨*proof*⟩

lemma *midpoint-dec*:
I *Midpoint* A B $\vee$ $\neg$ I *Midpoint* A B
⟨*proof*⟩

lemma *is-midpoint-id*:
assumes A *Midpoint* A B
shows $A = B$
⟨*proof*⟩

lemma *is-midpoint-id-2*:
assumes A *Midpoint* B A
shows $A = B$
⟨*proof*⟩

lemma *l7-2*:
assumes M *Midpoint* A B

shows *M Midpoint B A*
⟨*proof*⟩

lemma *l7-3*:
assumes *M Midpoint A A*
shows *M* = *A*
⟨*proof*⟩

lemma *l7-3-2*:
A Midpoint A A
⟨*proof*⟩

lemma *symmetric-point-construction*:
∃ *P*′. *A Midpoint P P*′
⟨*proof*⟩

lemma *symmetric-point-uniqueness*:
assumes *P Midpoint A P1* **and**
P Midpoint A P2
shows *P1* = *P2*
⟨*proof*⟩

lemma *l7-9*:
assumes *A Midpoint P X* **and**
A Midpoint Q X
shows *P* = *Q*
⟨*proof*⟩

lemma *l7-9-bis*:
assumes *A Midpoint P X* **and**
A Midpoint X Q
shows *P* = *Q*
⟨*proof*⟩

lemma *l7-13-R1*:
assumes *A* ≠ *P* **and**
A Midpoint P′ *P* **and**
A Midpoint Q′ *Q*
shows *Cong P Q P*′ *Q*′
⟨*proof*⟩

lemma *l7-13*:
assumes *A Midpoint P*′ *P* **and**
A Midpoint Q′ *Q*
shows *Cong P Q P*′ *Q*′
⟨*proof*⟩

lemma *l7-15*:
assumes *A Midpoint P P*′ **and**
A Midpoint Q Q′ **and**
A Midpoint R R′ **and**
Bet P Q R
shows *Bet P*′ *Q*′ *R*′
⟨*proof*⟩

lemma *l7-16*:
assumes *A Midpoint P P*′ **and**
A Midpoint Q Q′ **and**
A Midpoint R R′ **and**
A Midpoint S S′ **and**
Cong P Q R S
shows *Cong P*′ *Q*′ *R*′ *S*′
⟨*proof*⟩

lemma *symmetry-preserves-midpoint*:
assumes *Z Midpoint A D* **and**

Z Midpoint B E **and**
Z Midpoint C F **and**
B Midpoint A C
shows *E Midpoint D F*
⟨*proof*⟩

lemma *Mid-cases*:
assumes *A Midpoint B C* ∨ *A Midpoint C B*
shows *A Midpoint B C*
⟨*proof*⟩

lemma *Mid-perm*:
assumes *A Midpoint B C*
shows *A Midpoint B C* ∧ *A Midpoint C B*
⟨*proof*⟩

lemma *l7-17*:
assumes *A Midpoint P P′* **and**
B Midpoint P P′
shows $A = B$
⟨*proof*⟩

lemma *l7-17-bis*:
assumes *A Midpoint P P′* **and**
B Midpoint P′ P
shows $A = B$
⟨*proof*⟩

lemma *l7-20*:
assumes *Col A M B* **and**
Cong M A M B
shows $A = B$ ∨ *M Midpoint A B*
⟨*proof*⟩

lemma *l7-20-bis*:
assumes $A \neq B$ **and**
Col A M B **and**
Cong M A M B
shows *M Midpoint A B*
⟨*proof*⟩

lemma *cong-col-mid*:
assumes $A \neq C$ **and**
Col A B C **and**
Cong A B B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *l7-21-R1*:
assumes ¬ *Col A B C* **and**
$B \neq D$ **and**
Cong A B C D **and**
Cong B C D A **and**
Col A P C **and**
Col B P D
shows *P Midpoint A C*
⟨*proof*⟩

lemma *l7-21*:
assumes ¬ *Col A B C* **and**
$B \neq D$ **and**
Cong A B C D **and**
Cong B C D A **and**
Col A P C **and**
Col B P D
shows *P Midpoint A C* ∧ *P Midpoint B D*

⟨*proof*⟩

lemma *l7-22-aux-R1*:
assumes *Bet A1 C C* **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C C C B2 **and**
M1 Midpoint A1 B1 **and**
*M2 Midpoint A2 B2***and**
C A1 Le C C
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *l7-22-aux-R2*:
assumes $A2 \neq C$ **and**
Bet A1 C A2 **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C A2 C B2 **and**
M1 Midpoint A1 B1 **and**
M2 Midpoint A2 B2 **and**
C A1 Le C A2
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *l7-22-aux*:
assumes *Bet A1 C A2* **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C A2 C B2 **and**
M1 Midpoint A1 B1 **and**
M2 Midpoint A2 B2 **and**
C A1 Le C A2
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *l7-22*:
assumes *Bet A1 C A2* **and**
Bet B1 C B2 **and**
Cong C A1 C B1 **and**
Cong C A2 C B2 **and**
M1 Midpoint A1 B1 **and**
M2 Midpoint A2 B2
shows *Bet M1 C M2*
⟨*proof*⟩

lemma *bet-col1*:
assumes *Bet A B D* **and**
Bet A C D
shows *Col A B C*
⟨*proof*⟩

lemma *l7-25-R1*:
assumes *Cong C A C B* **and**
Col A B C
shows ∃ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *l7-25-R2*:
assumes *Cong C A C B* **and**
¬ *Col A B C*
shows ∃ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *l7-25*:
assumes *Cong C A C B*

shows $\exists$ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *midpoint-distinct-1*:
assumes $A \neq B$ **and**
I Midpoint A B
shows $I \neq A \wedge I \neq B$
⟨*proof*⟩

lemma *midpoint-distinct-2*:
assumes $I \neq A$ **and**
I Midpoint A B
shows $A \neq B \wedge I \neq B$
⟨*proof*⟩

lemma *midpoint-distinct-3*:
assumes $I \neq B$ **and**
I Midpoint A B
shows $A \neq B \wedge I \neq A$
⟨*proof*⟩

lemma *midpoint-def*:
assumes *Bet A B C* **and**
Cong A B B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *midpoint-bet*:
assumes *B Midpoint A C*
shows *Bet A B C*
⟨*proof*⟩

lemma *midpoint-col*:
assumes *M Midpoint A B*
shows *Col M A B*
⟨*proof*⟩

lemma *midpoint-cong*:
assumes *B Midpoint A C*
shows *Cong A B B C*
⟨*proof*⟩

lemma *midpoint-out*:
assumes $A \neq C$ **and**
B Midpoint A C
shows *A Out B C*
⟨*proof*⟩

lemma *midpoint-out-1*:
assumes $A \neq C$ **and**
B Midpoint A C
shows *C Out A B*
⟨*proof*⟩

lemma *midpoint-not-midpoint*:
assumes $A \neq B$ **and**
I Midpoint A B
shows $\neg$ *B Midpoint A I*
⟨*proof*⟩

lemma *swap-diff*:
assumes $A \neq B$
shows $B \neq A$
⟨*proof*⟩

lemma *cong-cong-half-1*:

assumes *M Midpoint A B* **and**
M′ Midpoint A′ B′ **and**
Cong A B A′ B′
shows *Cong A M A′ M′*
⟨*proof*⟩

lemma *cong-cong-half-2*:
assumes *M Midpoint A B* **and**
M′ Midpoint A′ B′ **and**
Cong A B A′ B′
shows *Cong B M B′ M′*
⟨*proof*⟩

lemma *cong-mid2--cong*:
assumes *M Midpoint A B* **and**
M′ Midpoint A′ B′ **and**
Cong A M A′ M′
shows *Cong A B A′ B′*
⟨*proof*⟩

lemma *mid--lt*:
assumes $A \neq B$ **and**
M Midpoint A B
shows *A M Lt A B*
⟨*proof*⟩

lemma *le-mid2--le13*:
assumes *M Midpoint A B* **and**
M′ Midpoint A′ B′ **and**
A M Le A′ M′
shows *A B Le A′ B′*
⟨*proof*⟩

lemma *le-mid2--le12*:
assumes *M Midpoint A B* **and**
M′ Midpoint A′ B′
and *A B Le A′ B′*
shows *A M Le A′ M′*
⟨*proof*⟩

lemma *lt-mid2--lt13*:
assumes *M Midpoint A B* **and**
M′ Midpoint A′ B′ **and**
A M Lt A′ M′
shows *A B Lt A′ B′*
⟨*proof*⟩

lemma *lt-mid2--lt12*:
assumes *M Midpoint A B* **and**
M′ Midpoint A′ B′ **and**
A B Lt A′ B′
shows *A M Lt A′ M′*
⟨*proof*⟩

lemma *midpoint-preserves-out*:
assumes *A Out B C* **and**
M Midpoint A A′ **and**
M Midpoint B B′ **and**
M Midpoint C C′
shows *A′ Out B′ C′*
⟨*proof*⟩

lemma *col-cong-bet*:
assumes *Col A B D* **and**
Cong A B C D **and**
Bet A C B

shows *Bet C A D* ∨ *Bet C B D*
⟨*proof*⟩

lemma *col-cong2-bet1*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Cong A C B D
shows *Bet C B D*
⟨*proof*⟩

lemma *col-cong2-bet2*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Cong A D B C
shows *Bet C A D*
⟨*proof*⟩

lemma *col-cong2-bet3*:
assumes *Col A B D* **and**
Bet A B C **and**
Cong A B C D **and**
Cong A C B D
shows *Bet B C D*
⟨*proof*⟩

lemma *col-cong2-bet4*:
assumes *Col A B C* **and**
Bet A B D **and**
Cong A B C D **and**
Cong A D B C
shows *Bet B D C*
⟨*proof*⟩

lemma *col-bet2-cong1*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Bet C B D
shows *Cong A C D B*
⟨*proof*⟩

lemma *col-bet2-cong2*:
assumes *Col A B D* **and**
Bet A C B **and**
Cong A B C D **and**
Bet C A D
shows *Cong D A B C*
⟨*proof*⟩

lemma *bet2-lt2--lt*:
assumes *Bet a Po b* **and**
Bet A PO B **and**
Po a Lt PO A **and**
Po b Lt PO B
shows *a b Lt A B*
⟨*proof*⟩

lemma *bet2-lt-le--lt*:
assumes *Bet a Po b* **and**
Bet A PO B **and**
Cong Po a PO A **and**
Po b Lt PO B
shows *a b Lt A B*
⟨*proof*⟩

lemma *per-dec*:
Per A B C ∨ ¬ *Per A B C*
⟨*proof*⟩

lemma *l8-2*:
assumes *Per A B C*
shows *Per C B A*
⟨*proof*⟩

lemma *Per-cases*:
assumes *Per A B C* ∨ *Per C B A*
shows *Per A B C*
⟨*proof*⟩

lemma *Per-perm* :
assumes *Per A B C*
shows *Per A B C* ∧ *Per C B A*
⟨*proof*⟩

lemma *l8-3* :
assumes *Per A B C* **and**
$A \neq B$ **and**
Col B A A′
shows *Per A′ B C*
⟨*proof*⟩

lemma *l8-4*:
assumes *Per A B C* **and**
B Midpoint C C′
shows *Per A B C′*
⟨*proof*⟩

lemma *l8-5*:
shows *Per A B B*
⟨*proof*⟩

lemma *l8-6*:
assumes *Per A B C* **and**
Per A′ B C **and**
Bet A C A′
shows $B = C$
⟨*proof*⟩

lemma *l8-7*:
assumes *Per A B C* **and**
Per A C B
shows $B = C$
⟨*proof*⟩

lemma *l8-8*:
assumes *Per A B A*
shows $A = B$
⟨*proof*⟩

lemma *per-distinct*:
assumes *Per A B C* **and**
$A \neq B$
shows $A \neq C$
⟨*proof*⟩

lemma *per-distinct-1*:
assumes *Per A B C* **and**
$B \neq C$
shows $A \neq C$
⟨*proof*⟩

lemma *l8-9*:
  assumes *Per A B C* **and**
    Col A B C
  shows $A = B \vee C = B$
  ⟨*proof*⟩

lemma *l8-10*:
  assumes *Per A B C* **and**
    A B C Cong3 A′ B′ C′
  shows *Per A′ B′ C′*
⟨*proof*⟩

lemma *col-col-per-per*:
  assumes $A \neq X$ **and**
    $C \neq X$ **and**
    Col U A X **and**
    Col V C X **and**
    Per A X C
  shows *Per U X V*
  ⟨*proof*⟩

lemma *perp-in-dec*:
  $X\ PerpAt\ A\ B\ C\ D \vee \neg\ X\ PerpAt\ A\ B\ C\ D$
  ⟨*proof*⟩

lemma *perp-distinct*:
  assumes *A B Perp C D*
  shows $A \neq B \wedge C \neq D$
  ⟨*proof*⟩

lemma *l8-12*:
  assumes *X PerpAt A B C D*
  shows *X PerpAt C D A B*
  ⟨*proof*⟩

lemma *per-col*:
  assumes $B \neq C$ **and**
    Per A B C **and**
    Col B C D
  shows *Per A B D*
  ⟨*proof*⟩

lemma *l8-13-2*:
  assumes $A \neq B$ **and**
    $C \neq D$ **and**
    Col X A B **and**
    Col X C D **and**
    $\exists\ U.\ \exists\ V.\ Col\ U\ A\ B \wedge Col\ V\ C\ D \wedge U \neq X \wedge V \neq X \wedge Per\ U\ X\ V$
  shows *X PerpAt A B C D*
⟨*proof*⟩

lemma *l8-14-1*:
  $\neg\ A\ B\ Perp\ A\ B$
  ⟨*proof*⟩

lemma *l8-14-2-1a*:
  assumes *X PerpAt A B C D*
  shows *A B Perp C D*
  ⟨*proof*⟩

lemma *perp-in-distinct*:
  assumes *X PerpAt A B C D*
  shows $A \neq B \wedge C \neq D$
  ⟨*proof*⟩

lemma *l8-14-2-1b*:
assumes *X PerpAt A B C D* **and**
Col Y A B **and**
Col Y C D
shows $X = Y$
⟨*proof*⟩

lemma *l8-14-2-1b-bis*:
assumes *A B Perp C D* **and**
Col X A B **and**
Col X C D
shows *X PerpAt A B C D*
⟨*proof*⟩

lemma *l8-14-2-2*:
assumes *A B Perp C D* **and**
$\forall\ Y.\ (Col\ Y\ A\ B \wedge Col\ Y\ C\ D) \longrightarrow X = Y$
shows *X PerpAt A B C D*
⟨*proof*⟩

lemma *l8-14-3*:
assumes *X PerpAt A B C D* **and**
Y PerpAt A B C D
shows $X = Y$
⟨*proof*⟩

lemma *l8-15-1*:
assumes *Col A B X* **and**
A B Perp C X
shows *X PerpAt A B C X*
⟨*proof*⟩

lemma *l8-15-2*:
assumes *Col A B X* **and**
X PerpAt A B C X
shows *A B Perp C X*
⟨*proof*⟩

lemma *perp-in-per*:
assumes *B PerpAt A B B C*
shows *Per A B C*
⟨*proof*⟩

lemma *perp-sym*:
assumes *A B Perp A B*
shows *C D Perp C D*
⟨*proof*⟩

lemma *perp-col0*:
assumes *A B Perp C D* **and**
$X \neq Y$ **and**
Col A B X **and**
Col A B Y
shows *C D Perp X Y*
⟨*proof*⟩

lemma *per-perp-in*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C
shows *B PerpAt A B B C*
⟨*proof*⟩

lemma *per-perp*:
assumes $A \neq B$ **and**
$B \neq C$ **and**

Per A B C
shows *A B Perp B C*
⟨*proof*⟩

lemma *perp-left-comm*:
assumes *A B Perp C D*
shows *B A Perp C D*
⟨*proof*⟩

lemma *perp-right-comm*:
assumes *A B Perp C D*
shows *A B Perp D C*
⟨*proof*⟩

lemma *perp-comm*:
assumes *A B Perp C D*
shows *B A Perp D C*
⟨*proof*⟩

lemma *perp-in-sym*:
assumes *X PerpAt A B C D*
shows *X PerpAt C D A B*
⟨*proof*⟩

lemma *perp-in-left-comm*:
assumes *X PerpAt A B C D*
shows *X PerpAt B A C D*
⟨*proof*⟩

lemma *perp-in-right-comm*:
assumes *X PerpAt A B C D*
shows *X PerpAt A B D C*
⟨*proof*⟩

lemma *perp-in-comm*:
assumes *X PerpAt A B C D*
shows *X PerpAt B A D C*
⟨*proof*⟩

lemma *Perp-cases*:
assumes *A B Perp C D* ∨ *B A Perp C D* ∨ *A B Perp D C* ∨ *B A Perp D C* ∨ *C D Perp A B* ∨ *C D Perp B A* ∨ *D C Perp A B* ∨ *D C Perp B A*
shows *A B Perp C D*
⟨*proof*⟩

lemma *Perp-perm* :
assumes *A B Perp C D*
shows *A B Perp C D* ∧ *B A Perp C D* ∧ *A B Perp D C* ∧ *B A Perp D C* ∧ *C D Perp A B* ∧ *C D Perp B A* ∧ *D C Perp A B* ∧ *D C Perp B A*
⟨*proof*⟩

lemma *Perp-in-cases*:
assumes *X PerpAt A B C D* ∨ *X PerpAt B A C D* ∨ *X PerpAt A B D C* ∨ *X PerpAt B A D C* ∨ *X PerpAt C D A B* ∨ *X PerpAt C D B A* ∨ *X PerpAt D C A B* ∨ *X PerpAt D C B A*
shows *X PerpAt A B C D*
⟨*proof*⟩

lemma *Perp-in-perm*:
assumes *X PerpAt A B C D*
shows *X PerpAt A B C D* ∧ *X PerpAt B A C D* ∧ *X PerpAt A B D C* ∧ *X PerpAt B A D C* ∧ *X PerpAt C D A B* ∧ *X PerpAt C D B A* ∧ *X PerpAt D C A B* ∧ *X PerpAt D C B A*
⟨*proof*⟩

lemma *perp-in-col*:
assumes *X PerpAt A B C D*
shows *Col A B X* ∧ *Col C D X*

⟨*proof*⟩

lemma *perp-perp-in*:
assumes *A B Perp C A*
shows *A PerpAt A B C A*
⟨*proof*⟩

lemma *perp-per-1*:
assumes *A B Perp C A*
shows *Per B A C*
⟨*proof*⟩

lemma *perp-per-2*:
assumes *A B Perp A C*
shows *Per B A C*
⟨*proof*⟩

lemma *perp-col*:
assumes $A \neq E$ **and**
A B Perp C D **and**
Col A B E
shows *A E Perp C D*
⟨*proof*⟩

lemma *perp-col2*:
assumes *A B Perp X Y* **and**
$C \neq D$ **and**
Col A B C **and**
Col A B D
shows *C D Perp X Y*
⟨*proof*⟩

lemma *perp-col4*:
assumes $P \neq Q$ **and**
$R \neq S$ **and**
Col A B P **and**
Col A B Q **and**
Col C D R **and**
Col C D S **and**
A B Perp C D
shows *P Q Perp R S*
⟨*proof*⟩

lemma *perp-not-eq-1*:
assumes *A B Perp C D*
shows $A \neq B$
⟨*proof*⟩

lemma *perp-not-eq-2*:
assumes *A B Perp C D*
shows $C \neq D$
⟨*proof*⟩

lemma *diff-per-diff*:
assumes $A \neq B$ **and**
Cong A P B R **and**
Per B A P
and *Per A B R*
shows $P \neq R$
⟨*proof*⟩

lemma *per-not-colp*:
assumes $A \neq B$ **and**
$A \neq P$ **and**
$B \neq R$ **and**
Per B A P

and *Per A B R*
shows ¬ *Col P A R*
⟨*proof*⟩

lemma *per-not-col*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *perp-not-col2*:
assumes *A B Perp C D*
shows ¬ *Col A B C* ∨ ¬ *Col A B D*
⟨*proof*⟩

lemma *perp-not-col*:
assumes *A B Perp P A*
shows ¬ *Col A B P*
⟨*proof*⟩

lemma *perp-in-col-perp-in*:
assumes $C \neq E$ **and**
Col C D E **and**
P PerpAt A B C D
shows *P PerpAt A B C E*
⟨*proof*⟩

lemma *perp-col2-bis*:
assumes *A B Perp C D* **and**
Col C D P **and**
Col C D Q **and**
$P \neq Q$
shows *A B Perp P Q*
⟨*proof*⟩

lemma *perp-in-perp-bis-R1*:
assumes $X \neq A$ **and**
X PerpAt A B C D
shows *X B Perp C D* ∨ *A X Perp C D*
⟨*proof*⟩

lemma *perp-in-perp-bis*:
assumes *X PerpAt A B C D*
shows *X B Perp C D* ∨ *A X Perp C D*
⟨*proof*⟩

lemma *col-per-perp*:
assumes $A \neq B$ **and**
$B \neq C$ **and**

$D \neq C$ **and**
Col B C D **and**
Per A B C
shows *C D Perp A B*
⟨*proof*⟩

lemma *per-cong-mid-R1*:
assumes $B = H$ **and**

Bet A B C **and**
Cong A H C H **and**
Per H B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *per-cong-mid-R2*:
assumes
$B \neq C$ **and**
Bet A B C **and**
Cong A H C H **and**
Per H B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *per-cong-mid*:
assumes $B \neq C$ **and**
Bet A B C **and**
Cong A H C H **and**
Per H B C
shows *B Midpoint A C*
⟨*proof*⟩

lemma *per-double-cong*:
assumes *Per A B C* **and**
B Midpoint C C′
shows *Cong A C A C′*
⟨*proof*⟩

lemma *cong-perp-or-mid-R1*:
assumes *Col A B X* **and**
$A \neq B$ **and**
M Midpoint A B **and**
Cong A X B X
shows $X = M \lor \neg$ *Col A B X* $\land$ *M PerpAt X M A B*
⟨*proof*⟩

lemma *cong-perp-or-mid-R2*:
assumes $\neg$ *Col A B X* **and**
$A \neq B$ **and**
M Midpoint A B **and**
Cong A X B X
shows $X = M \lor \neg$ *Col A B X* $\land$ *M PerpAt X M A B*
⟨*proof*⟩

lemma *cong-perp-or-mid*:
assumes $A \neq B$ **and**
M Midpoint A B **and**
Cong A X B X
shows $X = M \lor \neg$ *Col A B X* $\land$ *M PerpAt X M A B*
⟨*proof*⟩

lemma *col-per2-cases*:
assumes $B \neq C$ **and**
$B' \neq C$ **and**
$C \neq D$ **and**
Col B C D **and**
Per A B C **and**
Per A B′ C
shows $B = B' \lor \neg$ *Col B′ C D*
⟨*proof*⟩

lemma *l8-16-1*:
assumes *Col A B X* **and**
Col A B U **and**
A B Perp C X
shows $\neg$ *Col A B C* $\land$ *Per C X U*
⟨*proof*⟩

lemma *l8-16-2*:
assumes *Col A B X* **and**
Col A B U **and**

U ≠ *X* **and**
¬ *Col A B C* **and**
Per C X U
shows *A B Perp C X*
⟨*proof*⟩

lemma *l8-18-uniqueness*:
assumes
Col A B X **and**
A B Perp C X **and**
Col A B Y **and**
A B Perp C Y
shows *X* = *Y*
⟨*proof*⟩

lemma *midpoint-distinct*:
assumes ¬ *Col A B C* **and**
Col A B X **and**
X Midpoint C C'
shows *C* ≠ *C'*
⟨*proof*⟩

lemma *l8-20-1-R1*:
assumes *A* = *B*
shows *Per B A P*
⟨*proof*⟩

lemma *l8-20-1-R2*:
assumes *A* ≠ *B* **and**
Per A B C **and**
P Midpoint C' D **and**
A Midpoint C' C **and**
B Midpoint D C
shows *Per B A P*
⟨*proof*⟩

lemma *l8-20-1*:
assumes *Per A B C* **and**
P Midpoint C' D **and**
A Midpoint C' C **and**
B Midpoint D C
shows *Per B A P*
⟨*proof*⟩

lemma *l8-20-2*:
assumes *P Midpoint C' D* **and**
A Midpoint C' C **and**
B Midpoint D C **and**
B ≠ *C*
shows *A* ≠ *P*
⟨*proof*⟩

lemma *perp-col1*:
assumes *C* ≠ *X* **and**
A B Perp C D **and**
Col C D X
shows *A B Perp C X*
⟨*proof*⟩

lemma *l8-18-existence*:
assumes ¬ *Col A B C*
shows ∃ *X*. *Col A B X* ∧ *A B Perp C X*
⟨*proof*⟩

lemma *l8-21-aux*:
assumes ¬ *Col A B C*

shows $\exists$ *P*. $\exists$ *T*. (*A B Perp P A* $\wedge$ *Col A B T* $\wedge$ *Bet C T P*)
⟨*proof*⟩

lemma *l8-21*:
assumes $A \neq B$
shows $\exists$ *P T*. *A B Perp P A* $\wedge$ *Col A B T* $\wedge$ *Bet C T P*
⟨*proof*⟩

lemma *per-cong*:
assumes $A \neq B$ **and**
$A \neq P$ **and**
Per B A P **and**
Per A B R **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R
shows *Cong A R P B*
⟨*proof*⟩

lemma *perp-cong*:
assumes $A \neq B$ **and**
$A \neq P$ **and**
A B Perp P A **and**
A B Perp R B **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R
shows *Cong A R P B*
⟨*proof*⟩

lemma *perp-exists*:
assumes $A \neq B$
shows $\exists$ *X*. *PO X Perp A B*
⟨*proof*⟩

lemma *perp-vector*:
assumes $A \neq B$
shows $\exists$ *X Y*. *A B Perp X Y*
⟨*proof*⟩

lemma *midpoint-existence-aux*:
assumes $A \neq B$ **and**
A B Perp Q B **and**
A B Perp P A **and**
Col A B T **and**
Bet Q T P **and**
A P Le B Q
shows $\exists$ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *midpoint-existence*:
$\exists$ *X*. *X Midpoint A B*
⟨*proof*⟩

lemma *MidR-uniq-aux*:
shows $\exists!x$. *x Midpoint A B*
⟨*proof*⟩

lemma *SymR-uniq-aux*:
assumes *B Midpoint A x* **and**
B Midpoint A y
shows $x = y$
⟨*proof*⟩

lemma *perp-in-id*:
assumes *X PerpAt A B C A*

shows *X* = *A*
⟨*proof*⟩

lemma *l8-22*:
assumes *A* ≠ *B* **and**
A ≠ *P* **and**
Per B A P **and**
Per A B R **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R **and**
Cong A R P B
shows *X Midpoint A B* ∧ *X Midpoint P R*
⟨*proof*⟩

lemma *l8-22-bis*:
assumes *A* ≠ *B* **and**
A ≠ *P* **and**
A B Perp P A **and**
A B Perp R B **and**
Cong A P B R **and**
Col A B X **and**
Bet P X R
shows *Cong A R P B* ∧ *X Midpoint A B* ∧ *X Midpoint P R*
⟨*proof*⟩

lemma *perp-in-perp*:
assumes *X PerpAt A B C D*
shows *A B Perp C D*
⟨*proof*⟩

lemma *perp-proj*:
assumes *A B Perp C D* **and**
¬ *Col A C D*
shows ∃ *X*. *Col A B X* ∧ *A X Perp C D*
⟨*proof*⟩

lemma *l8-24* :
assumes *P A Perp A B* **and**
Q B Perp A B **and**
Col A B T **and**
Bet P T Q **and**
Bet B R Q **and**
Cong A P B R
shows ∃ *X*. *X Midpoint A B* ∧ *X Midpoint P R*
⟨*proof*⟩

lemma *col-per2--per*:
assumes *A* ≠ *B* **and**
Col A B C **and**
Per A X P **and**
Per B X P
shows *Per C X P*
⟨*proof*⟩

lemma *perp-in-per-1*:
assumes *X PerpAt A B C D*
shows *Per A X C*
⟨*proof*⟩

lemma *perp-in-per-2*:
assumes *X PerpAt A B C D*
shows *Per A X D*
⟨*proof*⟩

lemma *perp-in-per-3*:

assumes *X PerpAt A B C D*
shows *Per B X C*
⟨*proof*⟩

lemma *perp-in-per-4*:
assumes *X PerpAt A B C D*
shows *Per B X D*
⟨*proof*⟩

lemma *coplanar-perm-1*:
assumes *Coplanar A B C D*
shows *Coplanar A B D C*
⟨*proof*⟩

lemma *coplanar-perm-2*:
assumes *Coplanar A B C D*
shows *Coplanar A C B D*
⟨*proof*⟩

lemma *coplanar-perm-3*:
assumes *Coplanar A B C D*
shows *Coplanar A C D B*
⟨*proof*⟩

lemma *coplanar-perm-4*:
assumes *Coplanar A B C D*
shows *Coplanar A D B C*
⟨*proof*⟩

lemma *coplanar-perm-5*:
assumes *Coplanar A B C D*
shows *Coplanar A D C B*
⟨*proof*⟩

lemma *coplanar-perm-6*:
assumes *Coplanar A B C D*
shows *Coplanar B A C D*
⟨*proof*⟩

lemma *coplanar-perm-7*:
assumes *Coplanar A B C D*
shows *Coplanar B A D C*
⟨*proof*⟩

lemma *coplanar-perm-8*:
assumes *Coplanar A B C D*
shows *Coplanar B C A D*
⟨*proof*⟩

lemma *coplanar-perm-9*:
assumes *Coplanar A B C D*
shows *Coplanar B C D A*
⟨*proof*⟩

lemma *coplanar-perm-10*:
assumes *Coplanar A B C D*
shows *Coplanar B D A C*
⟨*proof*⟩

lemma *coplanar-perm-11*:
assumes *Coplanar A B C D*
shows *Coplanar B D C A*
⟨*proof*⟩

lemma *coplanar-perm-12*:
assumes *Coplanar A B C D*

shows *Coplanar C A B D*
⟨*proof*⟩

lemma *coplanar-perm-13*:
assumes *Coplanar A B C D*
shows *Coplanar C A D B*
⟨*proof*⟩

lemma *coplanar-perm-14*:
assumes *Coplanar A B C D*
shows *Coplanar C B A D*
⟨*proof*⟩

lemma *coplanar-perm-15*:
assumes *Coplanar A B C D*
shows *Coplanar C B D A*
⟨*proof*⟩

lemma *coplanar-perm-16*:
assumes *Coplanar A B C D*
shows *Coplanar C D A B*
⟨*proof*⟩

lemma *coplanar-perm-17*:
assumes *Coplanar A B C D*
shows *Coplanar C D B A*
⟨*proof*⟩

lemma *coplanar-perm-18*:
assumes *Coplanar A B C D*
shows *Coplanar D A B C*
⟨*proof*⟩

lemma *coplanar-perm-19*:
assumes *Coplanar A B C D*
shows *Coplanar D A C B*
⟨*proof*⟩

lemma *coplanar-perm-20*:
assumes *Coplanar A B C D*
shows *Coplanar D B A C*
⟨*proof*⟩

lemma *coplanar-perm-21*:
assumes *Coplanar A B C D*
shows *Coplanar D B C A*
⟨*proof*⟩

lemma *coplanar-perm-22*:
assumes *Coplanar A B C D*
shows *Coplanar D C A B*
⟨*proof*⟩

lemma *coplanar-perm-23*:
assumes *Coplanar A B C D*
shows *Coplanar D C B A*
⟨*proof*⟩

lemma *ncoplanar-perm-1*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A B D C*
⟨*proof*⟩

lemma *ncoplanar-perm-2*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A C B D*

⟨*proof*⟩

lemma *ncoplanar-perm-3*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A C D B*
⟨*proof*⟩

lemma *ncoplanar-perm-4*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A D B C*
⟨*proof*⟩

lemma *ncoplanar-perm-5*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar A D C B*
⟨*proof*⟩

lemma *ncoplanar-perm-6*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B A C D*
⟨*proof*⟩

lemma *ncoplanar-perm-7*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B A D C*
⟨*proof*⟩

lemma *ncoplanar-perm-8*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B C A D*
⟨*proof*⟩

lemma *ncoplanar-perm-9*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B C D A*
⟨*proof*⟩

lemma *ncoplanar-perm-10*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B D A C*
⟨*proof*⟩

lemma *ncoplanar-perm-11*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar B D C A*
⟨*proof*⟩

lemma *ncoplanar-perm-12*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C A B D*
⟨*proof*⟩

lemma *ncoplanar-perm-13*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C A D B*
⟨*proof*⟩

lemma *ncoplanar-perm-14*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C B A D*
⟨*proof*⟩

lemma *ncoplanar-perm-15*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C B D A*
⟨*proof*⟩

lemma *ncoplanar-perm-16*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C D A B*
⟨*proof*⟩

lemma *ncoplanar-perm-17*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar C D B A*
⟨*proof*⟩

lemma *ncoplanar-perm-18*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D A B C*
⟨*proof*⟩

lemma *ncoplanar-perm-19*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D A C B*
⟨*proof*⟩

lemma *ncoplanar-perm-20*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D B A C*
⟨*proof*⟩

lemma *ncoplanar-perm-21*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D B C A*
⟨*proof*⟩

lemma *ncoplanar-perm-22*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D C A B*
⟨*proof*⟩

lemma *ncoplanar-perm-23*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Coplanar D C B A*
⟨*proof*⟩

lemma *coplanar-trivial*:
shows *Coplanar A A B C*
⟨*proof*⟩

lemma *col--coplanar*:
assumes *Col A B C*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *ncop--ncol*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *ncop--ncols*:
assumes ¬ *Coplanar A B C D*
shows ¬ *Col A B C* ∧ ¬ *Col A B D* ∧ ¬ *Col A C D* ∧ ¬ *Col B C D*
⟨*proof*⟩

lemma *bet--coplanar*:
assumes *Bet A B C*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *out--coplanar*:

assumes *A Out B C*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *midpoint--coplanar*:
assumes *A Midpoint B C*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *perp--coplanar*:
assumes *A B Perp C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *ts--coplanar*:
assumes *A B TS C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *reflectl--coplanar*:
assumes *A B ReflectL C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *reflect--coplanar*:
assumes *A B Reflect C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *inangle--coplanar*:
assumes *A InAngle B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *pars--coplanar*:
assumes *A B ParStrict C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *par--coplanar*:
assumes *A B Par C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *plg--coplanar*:
assumes *Plg A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *plgs--coplanar*:
assumes *ParallelogramStrict A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *plgf--coplanar*:
assumes *ParallelogramFlat A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *parallelogram--coplanar*:
assumes *Parallelogram A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *rhombus--coplanar*:
assumes *Rhombus A B C D*

shows *Coplanar A B C D*
⟨*proof*⟩

lemma *rectangle--coplanar*:
assumes *Rectangle A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *square--coplanar*:
assumes *Square A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *lambert--coplanar*:
assumes *Lambert A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *ts-distincts*:
assumes *A B TS P Q*
shows $A \neq B \wedge A \neq P \wedge A \neq Q \wedge B \neq P \wedge B \neq Q \wedge P \neq Q$
⟨*proof*⟩

lemma *l9-2*:
assumes *A B TS P Q*
shows *A B TS Q P*
⟨*proof*⟩

lemma *invert-two-sides*:
assumes *A B TS P Q*
shows *B A TS P Q*
⟨*proof*⟩

lemma *l9-3*:
assumes *P Q TS A C* **and**
Col M P Q **and**
M Midpoint A C **and**
Col R P Q **and**
R Out A B
shows *P Q TS B C*
⟨*proof*⟩

lemma *mid-preserves-col*:
assumes *Col A B C* **and**
M Midpoint A A′ **and**
M Midpoint B B′ **and**
M Midpoint C C′
shows *Col A′ B′ C′*
⟨*proof*⟩

lemma *per-mid-per*:
assumes
Per X A B **and**
M Midpoint A B **and**
M Midpoint X Y
shows *Cong A X B Y* ∧ *Per Y B A*
⟨*proof*⟩

lemma *sym-preserve-diff*:
assumes $A \neq B$ **and**
M Midpoint A A′ **and**
M Midpoint B B′
shows $A' \neq B'$
⟨*proof*⟩

lemma *l9-4-1-aux-R1*:

assumes $R = S$ **and**
$S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

lemma *l9-4-1-aux-R21*:
assumes $R \neq S$ **and**
$S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

lemma *l9-4-1-aux*:
assumes $S\ C\ Le\ R\ A$ **and**
$P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ (M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C'))$
⟨*proof*⟩

lemma *per-col-eq*:
assumes $Per\ A\ B\ C$ **and**
$Col\ A\ B\ C$ **and**
$B \neq C$
shows $A = B$
⟨*proof*⟩

lemma *l9-4-1*:
assumes $P\ Q\ TS\ A\ C$ **and**
$Col\ R\ P\ Q$ **and**
$P\ Q\ Perp\ A\ R$ **and**
$Col\ S\ P\ Q$ **and**
$P\ Q\ Perp\ C\ S$ **and**
$M\ Midpoint\ R\ S$
shows $\forall\ U\ C'.\ M\ Midpoint\ U\ C' \longrightarrow (R\ Out\ U\ A \longleftrightarrow S\ Out\ C\ C')$
⟨*proof*⟩

lemma *mid-two-sides*:
assumes $M\ Midpoint\ A\ B$ **and**
$\neg\ Col\ A\ B\ X$ **and**
$M\ Midpoint\ X\ Y$
shows $A\ B\ TS\ X\ Y$
⟨*proof*⟩

lemma *col-preserves-two-sides*:
assumes $C \neq D$ **and**
$Col\ A\ B\ C$ **and**
$Col\ A\ B\ D$ **and**
$A\ B\ TS\ X\ Y$
shows $C\ D\ TS\ X\ Y$
⟨*proof*⟩

lemma *out-out-two-sides*:
  assumes $A \neq B$ **and**
    A B TS X Y **and**
    Col I A B **and**
    Col I X Y **and**
    I Out X U **and**
    I Out Y V
  shows *A B TS U V*
⟨*proof*⟩

lemma *l9-4-2-aux-R1*:
  assumes $R = S$ **and**
    S C Le R A **and**
    P Q TS A C **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
⟨*proof*⟩

lemma *l9-4-2-aux-R2*:
  assumes $R \neq S$ **and**
    S C Le R A **and**
    P Q TS A C **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
⟨*proof*⟩

lemma *l9-4-2-aux*:
  assumes *S C Le R A* **and**
    P Q TS A C **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
  ⟨*proof*⟩

lemma *l9-4-2*:
  assumes *P Q TS A C* **and**
    Col R P Q **and**
    P Q Perp A R **and**
    Col S P Q **and**
    P Q Perp C S **and**
    R Out U A **and**
    S Out V C
  shows *P Q TS U V*
⟨*proof*⟩

lemma *l9-5*:
  assumes *P Q TS A C* **and**
    Col R P Q **and**
    R Out A B
  shows *P Q TS B C*
⟨*proof*⟩

lemma *outer-pasch-R1*:
assumes *Col P Q C* **and**
Bet A C P **and**
Bet B Q C
shows $\exists$ *X*. *Bet A X B* $\wedge$ *Bet P Q X*
⟨*proof*⟩

lemma *outer-pasch-R2*:
assumes $\neg$ *Col P Q C* **and**
Bet A C P **and**
Bet B Q C
shows $\exists$ *X*. *Bet A X B* $\wedge$ *Bet P Q X*
⟨*proof*⟩

lemma *outer-pasch*:
assumes *Bet A C P* **and**
Bet B Q C
shows $\exists$ *X*. *Bet A X B* $\wedge$ *Bet P Q X*
⟨*proof*⟩

lemma *os-distincts*:
assumes *A B OS X Y*
shows $A \neq B \wedge A \neq X \wedge A \neq Y \wedge B \neq X \wedge B \neq Y$
⟨*proof*⟩

lemma *invert-one-side*:
assumes *A B OS P Q*
shows *B A OS P Q*
⟨*proof*⟩

lemma *l9-8-1*:
assumes *P Q TS A C* **and**
P Q TS B C
shows *P Q OS A B*
⟨*proof*⟩

lemma *not-two-sides-id*:
shows $\neg$ *P Q TS A A*
⟨*proof*⟩

lemma *l9-8-2*:
assumes *P Q TS A C* **and**
P Q OS A B
shows *P Q TS B C*
⟨*proof*⟩

lemma *l9-9*:
assumes *P Q TS A B*
shows $\neg$ *P Q OS A B*
⟨*proof*⟩

lemma *l9-9-bis*:
assumes *P Q OS A B*
shows $\neg$ *P Q TS A B*
⟨*proof*⟩

lemma *one-side-chara*:
assumes *P Q OS A B*
shows $\forall$ *X*. *Col X P Q* $\longrightarrow$ $\neg$ *Bet A X B*
⟨*proof*⟩

lemma *l9-10*:

assumes ¬ *Col A P Q*
shows ∃ *C*. *P Q TS A C*
⟨*proof*⟩

lemma *one-side-reflexivity*:
assumes ¬ *Col A P Q*
shows *P Q OS A A*
⟨*proof*⟩

lemma *one-side-symmetry*:
assumes *P Q OS A B*
shows *P Q OS B A*
⟨*proof*⟩

lemma *one-side-transitivity*:
assumes *P Q OS A B* **and**
P Q OS B C
shows *P Q OS A C*
⟨*proof*⟩

lemma *l9-17*:
assumes *P Q OS A C* **and**
Bet A B C
shows *P Q OS A B*
⟨*proof*⟩

lemma *l9-18-R1*:
assumes *Col X Y P* **and**
Col A B P
and *X Y TS A B*
shows *Bet A P B* ∧ ¬ *Col X Y A* ∧ ¬ *Col X Y B*
⟨*proof*⟩

lemma *l9-18-R2*:
assumes *Col X Y P* **and**
Col A B P **and**
Bet A P B **and**
¬ *Col X Y A* **and**
¬ *Col X Y B*
shows *X Y TS A B*
⟨*proof*⟩

lemma *l9-18*:
assumes *Col X Y P* **and**
Col A B P
shows *X Y TS A B* ⟷ (*Bet A P B* ∧ ¬ *Col X Y A* ∧ ¬ *Col X Y B*)
⟨*proof*⟩

lemma *l9-19-R1*:
assumes *Col X Y P* **and**
Col A B P **and**
X Y OS A B
shows *P Out A B* ∧ ¬ *Col X Y A*
⟨*proof*⟩

lemma *l9-19-R2*:
assumes *Col X Y P* **and**

P Out A B **and**
¬ *Col X Y A*
shows *X Y OS A B*
⟨*proof*⟩

lemma *l9-19*:
assumes *Col X Y P* **and**
Col A B P

shows *X Y OS A B* $\longleftrightarrow$ (*P Out A B* $\wedge$ $\neg$ *Col X Y A*)
⟨*proof*⟩

lemma *one-side-not-col123*:
assumes *A B OS X Y*
shows $\neg$ *Col A B X*
⟨*proof*⟩

lemma *one-side-not-col124*:
assumes *A B OS X Y*
shows $\neg$ *Col A B Y*
⟨*proof*⟩

lemma *col-two-sides*:
assumes *Col A B C* **and**
A $\neq$ *C* **and**
A B TS P Q
shows *A C TS P Q*
⟨*proof*⟩

lemma *col-one-side*:
assumes *Col A B C* **and**
A $\neq$ *C* **and**
A B OS P Q
shows *A C OS P Q*
⟨*proof*⟩

lemma *out-out-one-side*:
assumes *A B OS X Y* **and**
A Out Y Z
shows *A B OS X Z*
⟨*proof*⟩

lemma *out-one-side*:
assumes $\neg$ *Col A B X* $\vee$ $\neg$ *Col A B Y* **and**
A Out X Y
shows *A B OS X Y*
⟨*proof*⟩

lemma *bet--ts*:
assumes *A* $\neq$ *Y* **and**
$\neg$ *Col A B X* **and**
Bet X A Y
shows *A B TS X Y*
⟨*proof*⟩

lemma *bet-ts--ts*:
assumes *A B TS X Y* **and**
Bet X Y Z
shows *A B TS X Z*
⟨*proof*⟩

lemma *bet-ts--os*:
assumes *A B TS X Y* **and**
Bet X Y Z
shows *A B OS Y Z*
⟨*proof*⟩

lemma *l9-31* :
assumes *A X OS Y Z* **and**
A Z OS Y X
shows *A Y TS X Z*
⟨*proof*⟩

lemma *col123--nos*:
assumes *Col P Q A*

shows ¬ *P Q OS A B*
⟨*proof*⟩

lemma *col124--nos*:
assumes *Col P Q B*
shows ¬ *P Q OS A B*
⟨*proof*⟩

lemma *col2-os--os*:
assumes $C \neq D$ **and**
Col A B C **and**
Col A B D **and**
A B OS X Y
shows *C D OS X Y*
⟨*proof*⟩

lemma *os-out-os*:
assumes *Col A B P* **and**
A B OS C D **and**
P Out C C'
shows *A B OS C' D*
⟨*proof*⟩

lemma *ts-ts-os*:
assumes *A B TS C D* **and**
C D TS A B
shows *A C OS B D*
⟨*proof*⟩

lemma *col-one-side-out*:
assumes *Col A X Y* **and**
A B OS X Y
shows *A Out X Y*
⟨*proof*⟩

lemma *col-two-sides-bet*:
assumes *Col A X Y* **and**
A B TS X Y
shows *Bet X A Y*
⟨*proof*⟩

lemma *os-ts1324--os*:
assumes *A X OS Y Z* **and**
A Y TS X Z
shows *A Z OS X Y*
⟨*proof*⟩

lemma *ts2--ex-bet2*:
assumes *A C TS B D* **and**
B D TS A C
shows ∃ *X*. *Bet A X C* ∧ *Bet B X D*
⟨*proof*⟩

lemma *out-one-side-1*:
assumes ¬ *Col A B C* **and**
Col A B X **and**
X Out C D
shows *A B OS C D*
⟨*proof*⟩

lemma *out-two-sides-two-sides*:
assumes
Col A B PX **and**
PX Out X P **and**
A B TS P Y
shows *A B TS X Y*

⟨*proof*⟩

lemma *l8-21-bis*:
assumes $X \neq Y$ **and**
¬ *Col C A B*
shows ∃ *P*. *Cong A P X Y* ∧ *A B Perp P A* ∧ *A B TS C P*
⟨*proof*⟩

lemma *ts--ncol*:
assumes *A B TS X Y*
shows ¬ *Col A X Y* ∨ ¬ *Col B X Y*
⟨*proof*⟩

lemma *one-or-two-sides-aux*:
assumes ¬ *Col C A B* **and**
¬ *Col D A B* **and**
Col A C X
and *Col B D X*
shows *A B TS C D* ∨ *A B OS C D*
⟨*proof*⟩

lemma *cop--one-or-two-sides*:
assumes *Coplanar A B C D* **and**
¬ *Col C A B* **and**
¬ *Col D A B*
shows *A B TS C D* ∨ *A B OS C D*
⟨*proof*⟩

lemma *os--coplanar*:
assumes *A B OS C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *coplanar-trans-1*:
assumes ¬ *Col P Q R* **and**
Coplanar P Q R A **and**
Coplanar P Q R B
shows *Coplanar Q R A B*
⟨*proof*⟩

lemma *col-cop--cop*:
assumes *Coplanar A B C D* **and**
$C \neq D$ **and**
Col C D E
shows *Coplanar A B C E*
⟨*proof*⟩

lemma *bet-cop--cop*:
assumes *Coplanar A B C E* **and**
Bet C D E
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *col2-cop--cop*:
assumes *Coplanar A B C D* **and**
$C \neq D$ **and**
Col C D E **and**
Col C D F
shows *Coplanar A B E F*
⟨*proof*⟩

lemma *col-cop2--cop*:
assumes $U \neq V$ **and**
Coplanar A B C U **and**
Coplanar A B C V **and**
Col U V P

shows *Coplanar A B C P*
⟨*proof*⟩

lemma *bet-cop2--cop*:
assumes *Coplanar A B C U* **and**
Coplanar A B C W **and**
Bet U V W
shows *Coplanar A B C V*
⟨*proof*⟩

lemma *coplanar-pseudo-trans-lem1*:
assumes ¬ *Col P Q R* **and**
Coplanar P Q R A **and**
Coplanar P Q R B **and**
Coplanar P Q R C
shows *Coplanar A B C R*
⟨*proof*⟩

lemma *coplanar-pseudo-trans*:
assumes ¬ *Col P Q R* **and**
Coplanar P Q R A **and**
Coplanar P Q R B **and**
Coplanar P Q R C **and**
Coplanar P Q R D
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *l9-30*:
assumes ¬ *Coplanar A B C P* **and**
¬ *Col D E F* **and**
Coplanar D E F P **and**
Coplanar A B C X **and**
Coplanar A B C Y **and**
Coplanar A B C Z **and**
Coplanar D E F X **and**
Coplanar D E F Y **and**
Coplanar D E F Z
shows *Col X Y Z*
⟨*proof*⟩

lemma *cop-per2--col*:
assumes *Coplanar A X Y Z* **and**
$A \neq Z$ **and**
Per X Z A **and**
Per Y Z A
shows *Col X Y Z*
⟨*proof*⟩

lemma *cop-perp2--col*:
assumes *Coplanar A B Y Z* **and**
X Y Perp A B **and**
X Z Perp A B
shows *Col X Y Z*
⟨*proof*⟩

lemma *two-sides-dec*:
shows *A B TS C D* ∨ ¬ *A B TS C D*
⟨*proof*⟩

lemma *cop-nts--os*:
assumes *Coplanar A B C D* **and**
¬ *Col C A B* **and**
¬ *Col D A B* **and**
¬ *A B TS C D*
shows *A B OS C D*
⟨*proof*⟩

lemma *cop-nos--ts*:
assumes *Coplanar A B C D* **and**
¬ *Col C A B* **and**
¬ *Col D A B* **and**
¬ *A B OS C D*
shows *A B TS C D*
⟨*proof*⟩

lemma *one-side-dec*:
A B OS C D ∨ ¬ *A B OS C D*
⟨*proof*⟩

lemma *cop-dec*:
Coplanar A B C D ∨ ¬ *Coplanar A B C D*
⟨*proof*⟩

lemma *ex-diff-cop*:
∃ *E*. *Coplanar A B C E* ∧ *D* ≠ *E*
⟨*proof*⟩

lemma *ex-ncol-cop*:
assumes *D* ≠ *E*
shows ∃ *F*. *Coplanar A B C F* ∧ ¬ *Col D E F*
⟨*proof*⟩

lemma *ex-ncol-cop2*:
∃ *E F*. (*Coplanar A B C E* ∧ *Coplanar A B C F* ∧ ¬ *Col D E F*)
⟨*proof*⟩

lemma *col2-cop2--eq*:
assumes ¬ *Coplanar A B C U* **and**
U ≠ *V* **and**
Coplanar A B C P **and**
Coplanar A B C Q **and**
Col U V P **and**
Col U V Q
shows *P* = *Q*
⟨*proof*⟩

lemma *cong3-cop2--col*:
assumes *Coplanar A B C P* **and**
Coplanar A B C Q **and**
P ≠ *Q* **and**
Cong A P A Q **and**
Cong B P B Q **and**
Cong C P C Q
shows *Col A B C*
⟨*proof*⟩

lemma *l9-38*:
assumes *A B C TSP P Q*
shows *A B C TSP Q P*
⟨*proof*⟩

lemma *l9-39*:
assumes *A B C TSP P R* **and**
Coplanar A B C D **and**
D Out P Q
shows *A B C TSP Q R*
⟨*proof*⟩

lemma *l9-41-1*:
assumes *A B C TSP P R* **and**
A B C TSP Q R
shows *A B C OSP P Q*

⟨*proof*⟩

lemma *l9-41-2*:
assumes *A B C TSP P R* **and**
A B C OSP P Q
shows *A B C TSP Q R*
⟨*proof*⟩

lemma *tsp-exists*:
assumes ¬ *Coplanar A B C P*
shows ∃ *Q*. *A B C TSP P Q*
⟨*proof*⟩

lemma *osp-reflexivity*:
assumes ¬ *Coplanar A B C P*
shows *A B C OSP P P*
⟨*proof*⟩

lemma *osp-symmetry*:
assumes *A B C OSP P Q*
shows *A B C OSP Q P*
⟨*proof*⟩

lemma *osp-transitivity*:
assumes *A B C OSP P Q* **and**
A B C OSP Q R
shows *A B C OSP P R*
⟨*proof*⟩

lemma *cop3-tsp--tsp*:
assumes ¬ *Col D E F* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar A B C F **and**
A B C TSP P Q
shows *D E F TSP P Q*
⟨*proof*⟩

lemma *cop3-osp--osp*:
assumes ¬ *Col D E F* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar A B C F **and**
A B C OSP P Q
shows *D E F OSP P Q*
⟨*proof*⟩

lemma *ncop-distincts*:
assumes ¬ *Coplanar A B C D*
shows $A \neq B \wedge A \neq C \wedge A \neq D \wedge B \neq C \wedge B \neq D \wedge C \neq D$
⟨*proof*⟩

lemma *tsp-distincts*:
assumes *A B C TSP P Q*
shows $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q \wedge P \neq Q$
⟨*proof*⟩

lemma *osp-distincts*:
assumes *A B C OSP P Q*
shows $A \neq B \wedge A \neq C \wedge B \neq C \wedge A \neq P \wedge B \neq P \wedge C \neq P \wedge A \neq Q \wedge B \neq Q \wedge C \neq Q$
⟨*proof*⟩

lemma *tsp--ncop1*:
assumes *A B C TSP P Q*
shows ¬ *Coplanar A B C P*
⟨*proof*⟩

lemma *tsp--ncop2*:
assumes *A B C TSP P Q*
shows ¬ *Coplanar A B C Q*
⟨*proof*⟩

lemma *osp--ncop1*:
assumes *A B C OSP P Q*
shows ¬ *Coplanar A B C P*
⟨*proof*⟩

lemma *osp--ncop2*:
assumes *A B C OSP P Q*
shows ¬ *Coplanar A B C Q*
⟨*proof*⟩

lemma *tsp--nosp*:
assumes *A B C TSP P Q*
shows ¬ *A B C OSP P Q*
⟨*proof*⟩

lemma *osp--ntsp*:
assumes *A B C OSP P Q*
shows ¬ *A B C TSP P Q*
⟨*proof*⟩

lemma *osp-bet--osp*:
assumes *A B C OSP P R* **and**
Bet P Q R
shows *A B C OSP P Q*
⟨*proof*⟩

lemma *l9-18-3*:
assumes *Coplanar A B C P* **and**
Col X Y P
shows *A B C TSP X Y* ⟷ (*Bet X P Y* ∧ ¬ *Coplanar A B C X* ∧ ¬ *Coplanar A B C Y*)
⟨*proof*⟩

lemma *bet-cop--tsp*:
assumes ¬ *Coplanar A B C X* **and**
P ≠ *Y* **and**
Coplanar A B C P **and**
Bet X P Y
shows *A B C TSP X Y*
⟨*proof*⟩

lemma *cop-out--osp*:
assumes ¬ *Coplanar A B C X* **and**
Coplanar A B C P **and**
P Out X Y
shows *A B C OSP X Y*
⟨*proof*⟩

lemma *l9-19-3*:
assumes *Coplanar A B C P* **and**
Col X Y P
shows *A B C OSP X Y* ⟷ (*P Out X Y* ∧ ¬ *Coplanar A B C X*)
⟨*proof*⟩

lemma *cop2-ts--tsp*:
assumes ¬ *Coplanar A B C X* **and** *Coplanar A B C D* **and**
Coplanar A B C E **and** *D E TS X Y*
shows *A B C TSP X Y*
⟨*proof*⟩

lemma *cop2-os--osp*:

assumes $\neg$ *Coplanar A B C X* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
D E OS X Y
shows *A B C OSP X Y*
⟨*proof*⟩

lemma *cop3-tsp--ts*:
assumes $D \neq E$ **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar D E X Y **and**
A B C TSP X Y
shows *D E TS X Y*
⟨*proof*⟩

lemma *cop3-osp--os*:
assumes $D \neq E$ **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar D E X Y **and**
A B C OSP X Y
shows *D E OS X Y*
⟨*proof*⟩

lemma *cop-tsp--ex-cop2*:
assumes
A B C TSP D E
shows $\exists$ *Q*. (*Coplanar A B C Q* $\wedge$ *Coplanar D E P Q* $\wedge$ $P \neq Q$)
⟨*proof*⟩

lemma *cop-osp--ex-cop2*:
assumes *Coplanar A B C P* **and**
A B C OSP D E
shows $\exists$ *Q*. *Coplanar A B C Q* $\wedge$ *Coplanar D E P Q* $\wedge$ $P \neq Q$
⟨*proof*⟩

lemma *sac--coplanar*:
assumes *Saccheri A B C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *ex-sym*:
$\exists$ *Y*. (*A B Perp X Y* $\vee$ $X = Y$) $\wedge$ ($\exists$ *M*. *Col A B M* $\wedge$ *M Midpoint X Y*)
⟨*proof*⟩

lemma *is-image-is-image-spec*:
assumes $A \neq B$
shows P' *P Reflect A B* $\longleftrightarrow$ P' *P ReflectL A B*
⟨*proof*⟩

lemma *ex-sym1*:
assumes $A \neq B$
shows $\exists$ *Y*. (*A B Perp X Y* $\vee$ $X = Y$) $\wedge$ ($\exists$ *M*. *Col A B M* $\wedge$ *M Midpoint X Y* $\wedge$ *X Y Reflect A B*)
⟨*proof*⟩

lemma *l10-2-uniqueness*:
assumes *P1 P Reflect A B* **and**
P2 P Reflect A B
shows $P1 = P2$
⟨*proof*⟩

lemma *l10-2-uniqueness-spec*:
assumes *P1 P ReflectL A B* **and**
P2 P ReflectL A B
shows $P1 = P2$

⟨*proof*⟩

lemma *l10-2-existence-spec*:
  $\exists$ *P'*. *P' P ReflectL A B*
⟨*proof*⟩

lemma *l10-2-existence*:
  $\exists$ *P'*. *P' P Reflect A B*
  ⟨*proof*⟩

lemma *l10-4-spec*:
  assumes *P P' ReflectL A B*
  shows *P' P ReflectL A B*
⟨*proof*⟩

lemma *l10-4*:
  assumes *P P' Reflect A B*
  shows *P' P Reflect A B*
  ⟨*proof*⟩

lemma *l10-5*:
  assumes *P' P Reflect A B* **and**
    P'' P' Reflect A B
  shows *P = P''*
  ⟨*proof*⟩

lemma *l10-6-uniqueness*:
  assumes *P P1 Reflect A B* **and**
    P P2 Reflect A B
  shows *P1 = P2*
  ⟨*proof*⟩

lemma *l10-6-uniqueness-spec*:
  assumes *P P1 ReflectL A B* **and**
    P P2 ReflectL A B
  shows *P1 = P2*
  ⟨*proof*⟩

lemma *l10-6-existence-spec*:
  assumes $A \neq B$
  shows $\exists$ *P*. *P' P ReflectL A B*
  ⟨*proof*⟩

lemma *l10-6-existence*:
  $\exists$ *P*. *P' P Reflect A B*
  ⟨*proof*⟩

lemma *l10-7*:
  assumes *P' P Reflect A B* **and**
    Q' Q Reflect A B **and**
    P' = Q'
  shows *P = Q*
  ⟨*proof*⟩

lemma *l10-8*:
  assumes *P P Reflect A B*
  shows *Col P A B*
  ⟨*proof*⟩

lemma *col--refl*:
  assumes *Col P A B*
  shows *P P ReflectL A B*
  ⟨*proof*⟩

lemma *is-image-col-cong*:
  assumes $A \neq B$ **and**

P P' *Reflect* A B **and**
Col A B X
shows *Cong* P X P' X
⟨*proof*⟩

lemma *is-image-spec-col-cong*:
assumes P P' *ReflectL* A B **and**
Col A B X
shows *Cong* P X P' X
⟨*proof*⟩

lemma *image-id*:
assumes $A \neq B$ **and**
Col A B T **and**
T T' *Reflect* A B
shows $T = T'$
⟨*proof*⟩

lemma *osym-not-col*:
assumes P P' *Reflect* A B **and**
$\neg$ *Col* A B P
shows $\neg$ *Col* A B P'
⟨*proof*⟩

lemma *midpoint-preserves-image*:
assumes $A \neq B$ **and**
Col A B M **and**
P P' *Reflect* A B **and**
M *Midpoint* P Q **and**
M *Midpoint* P' Q'
shows Q Q' *Reflect* A B
⟨*proof*⟩

lemma *image-in-is-image-spec*:
assumes M *ReflectLAt* P P' A B
shows P P' *ReflectL* A B
⟨*proof*⟩

lemma *image-in-gen-is-image*:
assumes M *ReflectAt* P P' A B
shows P P' *Reflect* A B
⟨*proof*⟩

lemma *image-image-in*:
assumes $P \neq P'$ **and**
P P' *ReflectL* A B **and**
Col A B M **and**
Col P M P'
shows M *ReflectLAt* P P' A B
⟨*proof*⟩

lemma *image-in-col*:
assumes Y *ReflectLAt* P P' A B
shows *Col* P P' Y
⟨*proof*⟩

lemma *is-image-spec-rev*:
assumes P P' *ReflectL* A B
shows P P' *ReflectL* B A
⟨*proof*⟩

lemma *is-image-rev*:
assumes P P' *Reflect* A B
shows P P' *Reflect* B A
⟨*proof*⟩

lemma *midpoint-preserves-per*:
assumes *Per A B C* **and**
M Midpoint A A1 **and**
M Midpoint B B1 **and**
M Midpoint C C1
shows *Per A1 B1 C1*
⟨*proof*⟩

lemma *col--image-spec*:
assumes *Col A B X*
shows *X X ReflectL A B*
⟨*proof*⟩

lemma *image-triv*:
A A Reflect A B
⟨*proof*⟩

lemma *cong-midpoint--image*:
assumes *Cong A X A Y* **and**
B Midpoint X Y
shows *Y X Reflect A B*
⟨*proof*⟩

lemma *col-image-spec--eq*:
assumes *Col A B P* **and**
P P′ ReflectL A B
shows $P = P'$
⟨*proof*⟩

lemma *image-spec-triv*:
A A ReflectL B B
⟨*proof*⟩

lemma *image-spec--eq*:
assumes *P P′ ReflectL A A*
shows $P = P'$
⟨*proof*⟩

lemma *image--midpoint*:
assumes *P P′ Reflect A A*
shows *A Midpoint P′ P*
⟨*proof*⟩

lemma *is-image-spec-dec*:
A B ReflectL C D ∨ ¬ *A B ReflectL C D*
⟨*proof*⟩

lemma *l10-14*:
assumes $P \neq P'$ **and**
$A \neq B$ **and**
P P′ Reflect A B
shows *A B TS P P′*
⟨*proof*⟩

lemma *l10-15*:
assumes *Col A B C* **and**
¬ *Col A B P*
shows ∃ *Q. A B Perp Q C* ∧ *A B OS P Q*
⟨*proof*⟩

lemma *ex-per-cong*:
assumes $A \neq B$ **and**
$X \neq Y$ **and**
Col A B C **and**
¬ *Col A B D*

shows $\exists$ *P*. *Per P C A* $\wedge$ *Cong P C X Y* $\wedge$ *A B OS P D*
⟨*proof*⟩

lemma *exists-cong-per*:
$\exists$ *C*. *Per A B C* $\wedge$ *Cong B C X Y*
⟨*proof*⟩

lemma *upper-dim-implies-per2--col*:
assumes *upper-dim-axiom*
shows $\forall$ *A B C X*. (*Per A X C* $\wedge$ *X* $\neq$ *C* $\wedge$ *Per B X C*) $\longrightarrow$ *Col A B X*
⟨*proof*⟩

lemma *upper-dim-implies-col-perp2--col*:
assumes *upper-dim-axiom*
shows $\forall$ *A B X Y P*. (*Col A B P* $\wedge$ *A B Perp X P* $\wedge$ *P A Perp Y P*) $\longrightarrow$ *Col Y X P*
⟨*proof*⟩

lemma *upper-dim-implies-perp2--col*:
assumes *upper-dim-axiom*
shows $\forall$ *X Y Z A B*. (*X Y Perp A B* $\wedge$ *X Z Perp A B*) $\longrightarrow$ *Col X Y Z*
⟨*proof*⟩

lemma *upper-dim-implies-not-two-sides-one-side-aux*:
assumes *upper-dim-axiom*
shows $\forall$ *A B X Y PX*. (*A* $\neq$ *B* $\wedge$ *PX* $\neq$ *A* $\wedge$ *A B Perp X PX* $\wedge$ *Col A B PX* $\wedge$
$\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B* $\wedge$ $\neg$ *A B TS X Y*) $\longrightarrow$ *A B OS X Y*
⟨*proof*⟩

lemma *upper-dim-implies-not-two-sides-one-side*:
assumes *upper-dim-axiom*
shows $\forall$ *A B X Y*. ($\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B* $\wedge$ $\neg$ *A B TS X Y*) $\longrightarrow$ *A B OS X Y*
⟨*proof*⟩

lemma *upper-dim-implies-not-one-side-two-sides*:
assumes *upper-dim-axiom*
shows $\forall$ *A B X Y*. ($\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B* $\wedge$ $\neg$ *A B OS X Y*) $\longrightarrow$ *A B TS X Y*
⟨*proof*⟩

lemma *upper-dim-implies-one-or-two-sides*:
assumes *upper-dim-axiom*
shows $\forall$ *A B X Y*. ($\neg$ *Col X A B* $\wedge$ $\neg$ *Col Y A B*) $\longrightarrow$ (*A B TS X Y* $\vee$ *A B OS X Y*)
⟨*proof*⟩

lemma *upper-dim-implies-all-coplanar*:
assumes *upper-dim-axiom*
shows *all-coplanar-axiom*
⟨*proof*⟩

lemma *all-coplanar-implies-upper-dim*:
assumes *all-coplanar-axiom*
shows *upper-dim-axiom*
⟨*proof*⟩

lemma *all-coplanar-upper-dim*:
shows *all-coplanar-axiom* $\longleftrightarrow$ *upper-dim-axiom*
⟨*proof*⟩

lemma *upper-dim-stab*:
shows $\neg$ $\neg$ *upper-dim-axiom* $\longrightarrow$ *upper-dim-axiom*
⟨*proof*⟩

lemma *cop--cong-on-bissect*:
assumes *Coplanar A B X P* **and**
M Midpoint A B **and**
M PerpAt A B P M **and**
Cong X A X B

shows *Col M P X*
⟨*proof*⟩

lemma *cong-cop-mid-perp--col*:
assumes *Coplanar A B X P* **and**
Cong A X B X **and**
M Midpoint A B **and**
A B Perp P M
shows *Col M P X*
⟨*proof*⟩

lemma *cop-image-in2--col*:
assumes *Coplanar A B P Q* **and**
M ReflectLAt P P′ A B **and**
M ReflectLAt Q Q′ A B
shows *Col M P Q*
⟨*proof*⟩

lemma *l10-10-spec*:
assumes *P′ P ReflectL A B* **and**
Q′ Q ReflectL A B
shows *Cong P Q P′ Q′*
⟨*proof*⟩

lemma *l10-10*:
assumes *P′ P Reflect A B* **and**
Q′ Q Reflect A B
shows *Cong P Q P′ Q′*
⟨*proof*⟩

lemma *image-preserves-bet*:
assumes *A A′ ReflectL X Y* **and**
B B′ ReflectL X Y **and**
C C′ ReflectL X Y **and**
Bet A B C
shows *Bet A′ B′ C′*
⟨*proof*⟩

lemma *image-gen-preserves-bet*:
assumes *A A′ Reflect X Y* **and**
B B′ Reflect X Y **and**
C C′ Reflect X Y **and**
Bet A B C
shows *Bet A′ B′ C′*
⟨*proof*⟩

lemma *image-preserves-col*:
assumes *A A′ ReflectL X Y* **and**
B B′ ReflectL X Y **and**
C C′ ReflectL X Y **and**
Col A B C
shows *Col A′ B′ C′* ⟨*proof*⟩

lemma *image-gen-preserves-col*:
assumes *A A′ Reflect X Y* **and**
B B′ Reflect X Y **and**
C C′ Reflect X Y **and**
Col A B C
shows *Col A′ B′ C′*
⟨*proof*⟩

lemma *image-gen-preserves-ncol*:
assumes *A A′ Reflect X Y* **and**
B B′ Reflect X Y **and**
C C′ Reflect X Y **and**
¬ *Col A B C*

shows ¬ *Col* A' B' C'
⟨*proof*⟩

lemma *image-gen-preserves-inter*:
assumes *A* A' *Reflect X Y* **and**
B B' *Reflect X Y* **and**
C C' *Reflect X Y* **and**
D D' *Reflect X Y* **and**
¬ *Col A B C* **and**
$C \neq D$ **and**
Col A B I **and**
Col C D I **and**
Col A' B' I' **and**
Col C' D' I'
shows *I* I' *Reflect X Y*
⟨*proof*⟩

lemma *intersection-with-image-gen*:
assumes *A* A' *Reflect X Y* **and**
B B' *Reflect X Y* **and**
¬ *Col A B* A' **and**
Col A B C **and**
Col A' B' *C*
shows *Col C X Y*
⟨*proof*⟩

lemma *image-preserves-midpoint* :
assumes *A* A' *ReflectL X Y* **and**
B B' *ReflectL X Y* **and**
C C' *ReflectL X Y* **and**
A Midpoint B C
shows A' *Midpoint* B' C'
⟨*proof*⟩

lemma *image-spec-preserves-per*:
assumes *A* A' *ReflectL X Y* **and**
B B' *ReflectL X Y* **and**
C C' *ReflectL X Y* **and**
Per A B C
shows *Per* A' B' C'
⟨*proof*⟩

lemma *image-preserves-per*:
assumes *A* A' *Reflect X Y* **and**
B B' *Reflect X Y***and**
C C' *Reflect X Y* **and**
Per A B C
shows *Per* A' B' C'
⟨*proof*⟩

lemma *l10-12*:
assumes *Per A B C* **and**
Per A' B' C' **and**
Cong A B A' B' **and**
Cong B C B' C'
shows *Cong A C* A' C'
⟨*proof*⟩

lemma *l10-16*:
assumes ¬ *Col A B C* **and**
¬ *Col* A' B' *P* **and**
Cong A B A' B'
shows ∃ C'. *A B C Cong3* A' B' C' ∧ A' B' *OS P* C'
⟨*proof*⟩

lemma *cong-cop-image--col*:

assumes $P \neq P'$ **and**
$P\ P'$ *Reflect* $A\ B$ **and**
Cong $P\ X\ P'\ X$ **and**
Coplanar $A\ B\ P\ X$
shows *Col* $A\ B\ X$
⟨*proof*⟩

lemma *cong-cop-per2-1*:
assumes $A \neq B$ **and**
Per $A\ B\ X$ **and**
Per $A\ B\ Y$ **and**
Cong $B\ X\ B\ Y$ **and**
Coplanar $A\ B\ X\ Y$
shows $X = Y \vee B$ *Midpoint* $X\ Y$
⟨*proof*⟩

lemma *cong-cop-per2*:
assumes $A \neq B$ **and**
Per $A\ B\ X$ **and**
Per $A\ B\ Y$ **and**
Cong $B\ X\ B\ Y$ **and**
Coplanar $A\ B\ X\ Y$
shows $X = Y \vee X\ Y$ *ReflectL* $A\ B$
⟨*proof*⟩

lemma *cong-cop-per2-gen*:
assumes $A \neq B$ **and**
Per $A\ B\ X$ **and**
Per $A\ B\ Y$ **and**
Cong $B\ X\ B\ Y$ **and**
Coplanar $A\ B\ X\ Y$
shows $X = Y \vee X\ Y$ *Reflect* $A\ B$
⟨*proof*⟩

lemma *ex-perp-cop*:
assumes $A \neq B$
shows $\exists\ Q.\ A\ B$ *Perp* $Q\ C \wedge$ *Coplanar* $A\ B\ P\ Q$
⟨*proof*⟩

lemma *hilbert-s-version-of-pasch-aux*:
assumes *Coplanar* $A\ B\ C\ P$ **and**
$\neg$ *Col* $A\ I\ P$ **and**
$\neg$ *Col* $B\ C\ P$ **and**
Bet $B\ I\ C$ **and**
$B \neq I$ **and**
$I \neq C$ **and**
$B \neq C$
shows $\exists\ X.$ *Col* $I\ P\ X \wedge$
$((Bet\ A\ X\ B \wedge A \neq X \wedge X \neq B \wedge A \neq B) \vee (Bet\ A\ X\ C \wedge A \neq X \wedge X \neq C \wedge A \neq C))$
⟨*proof*⟩

lemma *hilbert-s-version-of-pasch*:
assumes *Coplanar* $A\ B\ C\ P$ **and**
$\neg$ *Col* $C\ Q\ P$ **and**
$\neg$ *Col* $A\ B\ P$ **and**
BetS $A\ Q\ B$
shows $\exists\ X.$ *Col* $P\ Q\ X \wedge$ (*BetS* $A\ X\ C \vee$ *BetS* $B\ X\ C$)
⟨*proof*⟩

lemma *two-sides-cases*:
assumes $\neg$ *Col* *PO* $A\ B$ **and**
PO P *OS* $A\ B$
shows *PO* A *TS* $P\ B \vee$ *PO* B *TS* $P\ A$
⟨*proof*⟩

lemma *not-par-two-sides*:

assumes $C \neq D$ **and**
Col A B I **and**
Col C D I **and**
$\neg$ *Col A B C*
shows $\exists$ *X Y*. *Col C D X* $\wedge$ *Col C D Y* $\wedge$ *A B TS X Y*
⟨*proof*⟩

lemma *cop-not-par-other-side*:
assumes $C \neq D$ **and**
Col A B I **and**
Col C D I **and**
$\neg$ *Col A B C* **and**
$\neg$ *Col A B P* **and**
Coplanar A B C P
shows $\exists$ *Q*. *Col C D Q* $\wedge$ *A B TS P Q*
⟨*proof*⟩

lemma *cop-not-par-same-side*:
assumes $C \neq D$ **and**
Col A B I **and**
Col C D I **and**
$\neg$ *Col A B C* **and**
$\neg$ *Col A B P* **and**
Coplanar A B C P
shows $\exists$ *Q*. *Col C D Q* $\wedge$ *A B OS P Q*
⟨*proof*⟩

lemma *perp-bisect-equiv-defA*:
assumes *P Q PerpBisect A B*
shows *P Q PerpBisectBis A B*
⟨*proof*⟩

lemma *perp-bisect-equiv-defB*:
assumes *P Q PerpBisectBis A B*
shows *P Q PerpBisect A B*
⟨*proof*⟩

lemma *perp-bisect-equiv-def*:
shows *P Q PerpBisect A B* $\longleftrightarrow$ *P Q PerpBisectBis A B*
⟨*proof*⟩

lemma *perp-bisect-sym-1*:
assumes *P Q PerpBisect A B*
shows *Q P PerpBisect A B*
⟨*proof*⟩

lemma *perp-bisect-sym-2*:
assumes *P Q PerpBisect A B*
shows *P Q PerpBisect B A*
⟨*proof*⟩

lemma *perp-bisect-sym-3*:
assumes *A B PerpBisect C D*
shows *B A PerpBisect D C*
⟨*proof*⟩

lemma *perp-bisect-perp*:
assumes *P Q PerpBisect A B*
shows *P Q Perp A B*
⟨*proof*⟩

lemma *perp-bisect-cong-1*:
assumes *P Q PerpBisect A B*
shows *Cong A P B P*
⟨*proof*⟩

lemma *perp-bisect-cong-2*:
assumes *P Q PerpBisect A B*
shows *Cong A Q B Q*
⟨*proof*⟩

lemma *perp-bisect-cong2*:
assumes *P Q PerpBisect A B*
shows *Cong A P B P* ∧ *Cong A Q B Q*
⟨*proof*⟩

lemma *perp-bisect-cong*:
assumes
Cong A pO B pO **and**
Cong B pO C pO
shows *Cong A pO C pO*
⟨*proof*⟩

lemma *cong-cop-perp-bisect*:
assumes $P \neq Q$ **and**
$A \neq B$ **and**
Coplanar P Q A B **and**
Cong A P B P **and**
Cong A Q B Q
shows *P Q PerpBisect A B*
⟨*proof*⟩

lemma *cong-mid-perp-bisect*:
assumes $P \neq Q$ **and**
$A \neq B$ **and**
Cong A P B P **and**
Q Midpoint A B
shows *P Q PerpBisect A B*
⟨*proof*⟩

lemma *perp-bisect-is-on-perp-bisect*:
assumes *P IsOnPerpBisect A B* **and**
P IsOnPerpBisect B C
shows *P IsOnPerpBisect A C*
⟨*proof*⟩

lemma *perp-mid-perp-bisect*:
assumes *C Midpoint A B* **and**
C D Perp A B
shows *C D PerpBisect A B*
⟨*proof*⟩

lemma *cong-cop2-perp-bisect-col*:
assumes *Coplanar A C D E* **and**
Coplanar B C D E **and**
Cong C D C E **and**
A B PerpBisect D E
shows *Col A B C*
⟨*proof*⟩

lemma *perp-bisect-cop2-existence*:
assumes $A \neq B$
shows ∃ *P Q*. *P Q PerpBisect A B* ∧ *Coplanar A B C P* ∧ *Coplanar A B C Q*
⟨*proof*⟩

lemma *perp-bisect-existence*:
assumes $A \neq B$
shows ∃ *P Q*. *P Q PerpBisect A B*
⟨*proof*⟩

lemma *perp-bisect-existence-cop*:
assumes $A \neq B$

shows $\exists$ *P Q*. *P Q PerpBisect A B* $\wedge$ *Coplanar A B C P* $\wedge$ *Coplanar A B C Q*
⟨*proof*⟩

lemma *l11-3*:
assumes *A B C CongA D E F*
shows $\exists$ A' C' D' F'. *B Out* A' *A* $\wedge$ *B Out C* C' $\wedge$
E Out D' *D* $\wedge$ *E Out F* F' $\wedge$
A' *B* C' *Cong3* D' *E* F'
⟨*proof*⟩

lemma *l11-aux*:
assumes *B Out A* A' **and**
E Out D D' **and**
Cong B A' *E* D' **and**
Bet B A A0 **and**
Bet E D D0 **and**
Cong A A0 E D **and**
Cong D D0 B A
shows *Cong B A0 E D0* $\wedge$ *Cong* A' *A0* D' *D0*
⟨*proof*⟩

lemma *l11-3-bis*:
assumes $\exists$ A' C' D' F'.
(*B Out* A' *A* $\wedge$ *B Out* C' *C* $\wedge$ *E Out* D' *D* $\wedge$ *E Out* F' *F* $\wedge$ A' *B* C' *Cong3* D' *E* F')
shows *A B C CongA D E F*
⟨*proof*⟩

lemma *l11-4-1*:
assumes *A B C CongA D E F* **and**

B Out A' *A* **and**
B Out C' *C* **and**
E Out D' *D* **and**
E Out F' *F* **and**
Cong B A' *E* D' **and**
Cong B C' *E* F'
shows *Cong* A' C' D' F'
⟨*proof*⟩

lemma *l11-4-2*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$D \neq E$ **and**
$F \neq E$ **and**
$\forall$ A' C' D' F'. (*B Out* A' *A* $\wedge$ *B Out* C' *C* $\wedge$ *E Out* D' *D* $\wedge$ *E Out* F' *F* $\wedge$ *Cong B* A' *E* D' $\wedge$
Cong B C' *E* F' $\longrightarrow$ *Cong* A' C' D' F')
shows *A B C CongA D E F*
⟨*proof*⟩

lemma *conga-refl*:
assumes $A \neq B$ **and**
$C \neq B$
shows *A B C CongA A B C*
⟨*proof*⟩

lemma *conga-sym*:
assumes *A B C CongA* A' B' C'
shows A' B' C' *CongA A B C*
⟨*proof*⟩

lemma *l11-10*:
assumes *A B C CongA D E F* **and**
B Out A' *A* **and**
B Out C' *C* **and**
E Out D' *D* **and**
E Out F' *F*

shows A' B C' *CongA* D' E F'
⟨*proof*⟩

lemma *out2--conga*:
assumes B *Out* A' A **and**
B *Out* C' C
shows A B C *CongA* A' B C'
⟨*proof*⟩

lemma *cong3-diff*:
assumes $A \neq B$ **and**
A B C *Cong3* A' B' C'
shows $A' \neq B'$
⟨*proof*⟩

lemma *cong3-diff2*:
assumes $B \neq C$ **and**
A B C *Cong3* A' B' C'
shows $B' \neq C'$
⟨*proof*⟩

lemma *cong3-conga*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
A B C *Cong3* A' B' C'
shows A B C *CongA* A' B' C'
⟨*proof*⟩

lemma *cong3-conga2*:
assumes A B C *Cong3* A' B' C' **and**
A B C *CongA* A'' B'' C''
shows A' B' C' *CongA* A'' B'' C''
⟨*proof*⟩

lemma *conga-diff1*:
assumes A B C *CongA* A' B' C'
shows $A \neq B$
⟨*proof*⟩

lemma *conga-diff2*:
assumes A B C *CongA* A' B' C'
shows $C \neq B$
⟨*proof*⟩

lemma *conga-diff45*:
assumes A B C *CongA* A' B' C'
shows $A' \neq B'$
⟨*proof*⟩

lemma *conga-diff56*:
assumes A B C *CongA* A' B' C'
shows $C' \neq B'$
⟨*proof*⟩

lemma *conga-trans*:
assumes A B C *CongA* A' B' C' **and**
A' B' C' *CongA* A'' B'' C''
shows A B C *CongA* A'' B'' C''
⟨*proof*⟩

lemma *conga-pseudo-refl*:
assumes $A \neq B$ **and**
$C \neq B$
shows A B C *CongA* C B A
⟨*proof*⟩

lemma *conga-trivial-1*:
assumes $A \neq B$ **and**
$C \neq D$
shows $A\ B\ A\ CongA\ C\ D\ C$
⟨*proof*⟩

lemma *l11-13*:
assumes $A\ B\ C\ CongA\ D\ E\ F$ **and**
$Bet\ A\ B\ A'$ **and**
$A' \neq B$ **and**
$Bet\ D\ E\ D'$ **and**
$D' \neq E$
shows $A'\ B\ C\ CongA\ D'\ E\ F$
⟨*proof*⟩

lemma *conga-right-comm*:
assumes $A\ B\ C\ CongA\ D\ E\ F$
shows $A\ B\ C\ CongA\ F\ E\ D$
⟨*proof*⟩

lemma *conga-left-comm*:
assumes $A\ B\ C\ CongA\ D\ E\ F$
shows $C\ B\ A\ CongA\ D\ E\ F$
⟨*proof*⟩

lemma *conga-comm*:
assumes $A\ B\ C\ CongA\ D\ E\ F$
shows $C\ B\ A\ CongA\ F\ E\ D$
⟨*proof*⟩

lemma *conga-line*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A' \neq B'$ **and**
$B' \neq C'$
and $Bet\ A\ B\ C$ **and**
$Bet\ A'\ B'\ C'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l11-14*:
assumes $Bet\ A\ B\ A'$ **and**
$A \neq B$ **and**
$A' \neq B$ **and**
$Bet\ C\ B\ C'$ **and**
$B \neq C$ **and**
$B \neq C'$
shows $A\ B\ C\ CongA\ A'\ B\ C'$
⟨*proof*⟩

lemma *l11-16*:
assumes $Per\ A\ B\ C$ **and**
$A \neq B$ **and**
$C \neq B$ **and**
$Per\ A'\ B'\ C'$ **and**
$A' \neq B'$ **and**
$C' \neq B'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l11-17*:
assumes $Per\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$
shows $Per\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l11-18-1*:
assumes *Bet C B D* **and**
$B \neq C$ **and**
$B \neq D$ **and**
$A \neq B$ **and**
Per A B C
shows *A B C CongA A B D*
⟨*proof*⟩

lemma *l11-18-2*:
assumes *Bet C B D* **and**
A B C CongA A B D
shows *Per A B C*
⟨*proof*⟩

lemma *cong3-preserves-out*:
assumes *A Out B C* **and**
$A\ B\ C\ Cong3\ A'\ B'\ C'$
shows $A'\ Out\ B'\ C'$
⟨*proof*⟩

lemma *l11-21-a*:
assumes *B Out A C* **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$
shows $B'\ Out\ A'\ C'$
⟨*proof*⟩

lemma *l11-21-b*:
assumes *B Out A C* **and**
$B'\ Out\ A'\ C'$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *conga-cop--or-out-ts*:
assumes $Coplanar\ A\ B\ C\ C'$ **and**
$A\ B\ C\ CongA\ A\ B\ C'$
shows $B\ Out\ C\ C' \vee A\ B\ TS\ C\ C'$
⟨*proof*⟩

lemma *conga-os--out*:
assumes $A\ B\ C\ CongA\ A\ B\ C'$ **and**
$A\ B\ OS\ C\ C'$
shows $B\ Out\ C\ C'$
⟨*proof*⟩

lemma *cong2-conga-cong*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ B\ C\ B'\ C'$
shows $Cong\ A\ C\ A'\ C'$
⟨*proof*⟩

lemma *angle-construction-1*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ A'\ B'\ P$
shows $\exists\ C'.\ (A\ B\ C\ CongA\ A'\ B'\ C' \wedge A'\ B'\ OS\ C'\ P)$
⟨*proof*⟩

lemma *angle-construction-2*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$\neg\ Col\ A'\ B'\ P$
shows $\exists\ C'.\ (A\ B\ C\ CongA\ A'\ B'\ C' \wedge (A'\ B'\ OS\ C'\ P \vee Col\ A'\ B'\ C'))$
⟨*proof*⟩

lemma *ex-conga-ts*:

assumes ¬ *Col A B C* **and**
¬ *Col A′ B′ P*
shows ∃ *C′*. *A B C CongA A′ B′ C′* ∧ *A′ B′ TS C′ P*
⟨*proof*⟩

lemma *l11-15*:
assumes ¬ *Col A B C* **and**
¬ *Col D E P*
shows
∃ *F*. (*A B C CongA D E F* ∧ *E D OS F P*) ∧
(∀ *F1 F2*. ((*A B C CongA D E F1* ∧ *E D OS F1 P*) ∧
(*A B C CongA D E F2* ∧ *E D OS F2 P*))
⟶ *E Out F1 F2*)
⟨*proof*⟩

lemma *l11-19*:
assumes *Per A B P1* **and**
Per A B P2 **and**
A B OS P1 P2
shows *B Out P1 P2*
⟨*proof*⟩

lemma *l11-22-bet*:
assumes *Bet A B C* **and**
P′ B′ TS A′ C′ **and**
A B P CongA A′ B′ P′ **and**
P B C CongA P′ B′ C′
shows *Bet A′ B′ C′*
⟨*proof*⟩

lemma *l11-22a*:
assumes *B P TS A C* **and**
B′ P′ TS A′ C′ **and**
A B P CongA A′ B′ P′ **and**
P B C CongA P′ B′ C′
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *l11-22b*:
assumes *B P OS A C* **and**
B′ P′ OS A′ C′ **and**
A B P CongA A′ B′ P′ **and**
P B C CongA P′ B′ C′
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *l11-22*:
assumes ((*B P TS A C* ∧ *B′ P′ TS A′ C′*)∨(*B P OS A C* ∧ *B′ P′ OS A′ C′*)) **and**
A B P CongA A′ B′ P′ **and**
P B C CongA P′ B′ C′
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *l11-24*:
assumes *P InAngle A B C*
shows *P InAngle C B A*
⟨*proof*⟩

lemma *col-in-angle*:
assumes *A* ≠ *B* **and**
C ≠ *B* **and**
P ≠ *B* **and**
B Out A P ∨ *B Out C P*
shows *P InAngle A B C*
⟨*proof*⟩

lemma *out321--inangle*:
assumes $C \neq B$ **and**
B *Out* A P
shows P *InAngle* A B C
⟨*proof*⟩

lemma *inangle1123*:
assumes $A \neq B$ **and**
$C \neq B$
shows A *InAngle* A B C
⟨*proof*⟩

lemma *out341--inangle*:
assumes $A \neq B$ **and**
B *Out* C P
shows P *InAngle* A B C
⟨*proof*⟩

lemma *inangle3123*:
assumes $A \neq B$ **and**
$C \neq B$
shows C *InAngle* A B C
⟨*proof*⟩

lemma *in-angle-two-sides*:
assumes ¬ *Col* B A P **and**
¬ *Col* B C P **and**
P *InAngle* A B C
shows P B *TS* A C
⟨*proof*⟩

lemma *in-angle-out*:
assumes B *Out* A C **and**
P *InAngle* A B C
shows B *Out* A P
⟨*proof*⟩

lemma *col-in-angle-out*:
assumes *Col* B A P **and**
¬ *Bet* A B C **and**
P *InAngle* A B C
shows B *Out* A P
⟨*proof*⟩

lemma *l11-25-aux*:
assumes P *InAngle* A B C **and**
¬ *Bet* A B C **and**
B *Out* A' A
shows P *InAngle* A' B C
⟨*proof*⟩

lemma *l11-25*:
assumes P *InAngle* A B C **and**
B *Out* A' A **and**
B *Out* C' C **and**
B *Out* P' P
shows P' *InAngle* A' B C'
⟨*proof*⟩

lemma *inangle-distincts*:
assumes P *InAngle* A B C
shows $A \neq B \wedge C \neq B \wedge P \neq B$
⟨*proof*⟩

lemma *segment-construction-0*:
shows $\exists$ B'. *Cong* A' B' A B

⟨*proof*⟩

lemma *angle-construction-3*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$A' \neq B'$
shows $\exists\ C'.\ A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *l11-28*:
assumes $A\ B\ C\ Cong3\ A'\ B'\ C'$ **and**
$Col\ A\ C\ D$
shows $\exists\ D'.\ (Cong\ A\ D\ A'\ D' \wedge Cong\ B\ D\ B'\ D' \wedge Cong\ C\ D\ C'\ D')$
⟨*proof*⟩

lemma *bet-conga--bet*:
assumes $Bet\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$
shows $Bet\ A'\ B'\ C'$
⟨*proof*⟩

lemma *in-angle-one-side*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ B\ A\ P$ **and**
$P\ InAngle\ A\ B\ C$
shows $A\ B\ OS\ P\ C$
⟨*proof*⟩

lemma *inangle-one-side*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ A\ B\ P$ **and**
$\neg\ Col\ A\ B\ Q$ **and**
$P\ InAngle\ A\ B\ C$ **and**
$Q\ InAngle\ A\ B\ C$
shows $A\ B\ OS\ P\ Q$
⟨*proof*⟩

lemma *inangle-one-side2*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$\neg\ Col\ A\ B\ P$ **and**
$\neg\ Col\ A\ B\ Q$ **and**
$\neg\ Col\ C\ B\ P$ **and**
$\neg\ Col\ C\ B\ Q$ **and**
$P\ InAngle\ A\ B\ C$ **and**
$Q\ InAngle\ A\ B\ C$
shows $A\ B\ OS\ P\ Q \wedge C\ B\ OS\ P\ Q$
⟨*proof*⟩

lemma *col-conga-col*:
assumes $Col\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ D\ E\ F$
shows $Col\ D\ E\ F$
⟨*proof*⟩

lemma *ncol-conga-ncol*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ D\ E\ F$
shows $\neg\ Col\ D\ E\ F$
⟨*proof*⟩

lemma *angle-construction-4*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$A' \neq B'$
shows $\exists C'.\ (A\ B\ C\ CongA\ A'\ B'\ C' \wedge Coplanar\ A'\ B'\ C'\ P)$
⟨*proof*⟩

lemma *lea-distincts*:
assumes *A B C LeA D E F*
shows $A{\neq}B \wedge C{\neq}B \wedge D{\neq}E \wedge F{\neq}E$
⟨*proof*⟩

lemma *l11-29-a*:
assumes *A B C LeA D E F*
shows ∃ *Q*. (*C InAngle A B Q* ∧ *A B Q CongA D E F*)
⟨*proof*⟩

lemma *in-angle-line*:
assumes $P \neq B$ **and**
$A \neq B$ **and**
$C \neq B$ **and**
Bet A B C
shows *P InAngle A B C*
⟨*proof*⟩

lemma *l11-29-b*:
assumes ∃ *Q*. (*C InAngle A B Q* ∧ *A B Q CongA D E F*)
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *bet-in-angle-bet*:
assumes *Bet A B P* **and**
P InAngle A B C
shows *Bet A B C*
⟨*proof*⟩

lemma *lea-line*:
assumes *Bet A B P* **and**
A B P LeA A B C
shows *Bet A B C*
⟨*proof*⟩

lemma *eq-conga-out*:
assumes *A B A CongA D E F*
shows *E Out D F*
⟨*proof*⟩

lemma *out-conga-out*:
assumes *B Out A C* **and**
A B C CongA D E F
shows *E Out D F*
⟨*proof*⟩

lemma *conga-ex-cong3*:
assumes *A B C CongA* A' B' C'
shows ∃ *AA CC*. ((*B Out A AA* ∧ *B Out C CC*) ⟶ *AA B CC Cong3* A' B' C')
⟨*proof*⟩

lemma *conga-preserves-in-angle*:
assumes *A B C CongA* A' B' C' **and**
A B I CongA A' B' I' **and**
I InAngle A B C **and** A' B' *OS* I' C'
shows I' *InAngle* A' B' C'
⟨*proof*⟩

lemma *l11-30*:
assumes *A B C LeA D E F* **and**
A B C CongA A' B' C' **and**
D E F CongA D' E' F'
shows A' B' C' *LeA* D' E' F'
⟨*proof*⟩

lemma *l11-31-1*:
assumes B *Out* A C **and**
$D \neq E$ **and**
$F \neq E$
shows A B C *LeA* D E F
⟨*proof*⟩

lemma *l11-31-2*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
$D \neq E$ **and**
$F \neq E$ **and**
Bet D E F
shows A B C *LeA* D E F
⟨*proof*⟩

lemma *lea-refl*:
assumes $A \neq B$ **and**
$C \neq B$
shows A B C *LeA* A B C
⟨*proof*⟩

lemma *conga--lea*:
assumes A B C *CongA* D E F
shows A B C *LeA* D E F
⟨*proof*⟩

lemma *conga--lea456123*:
assumes A B C *CongA* D E F
shows D E F *LeA* A B C
⟨*proof*⟩

lemma *lea-left-comm*:
assumes A B C *LeA* D E F
shows C B A *LeA* D E F
⟨*proof*⟩

lemma *lea-right-comm*:
assumes A B C *LeA* D E F
shows A B C *LeA* F E D
⟨*proof*⟩

lemma *lea-comm*:
assumes$A$ B C *LeA* D E F
shows C B A *LeA* F E D
⟨*proof*⟩

lemma *lta-left-comm*:
assumes A B C *LtA* D E F
shows C B A *LtA* D E F
⟨*proof*⟩

lemma *lta-right-comm*:
assumes A B C *LtA* D E F
shows A B C *LtA* F E D
⟨*proof*⟩

lemma *lta-comm*:
assumes A B C *LtA* D E F
shows C B A *LtA* F E D
⟨*proof*⟩

lemma *lea-out4--lea*:
assumes A B C *LeA* D E F **and**
B *Out* A A' **and**
B *Out* C C' **and**

E Out D D' **and**
E Out F F'
shows *A' B C' LeA D' E F'*
⟨*proof*⟩

lemma *lea121345*:
assumes $A \neq B$ **and**
$C \neq D$ **and**
$D \neq E$
shows *A B A LeA C D E*
⟨*proof*⟩

lemma *inangle--lea*:
assumes *P InAngle A B C*
shows *A B P LeA A B C*
⟨*proof*⟩

lemma *inangle--lea-1*:
assumes *P InAngle A B C*
shows *P B C LeA A B C*
⟨*proof*⟩

lemma *inangle--lta*:
assumes ¬ *Col P B C* **and**
P InAngle A B C
shows *A B P LtA A B C*
⟨*proof*⟩

lemma *in-angle-trans*:
assumes *C InAngle A B D* **and**
D InAngle A B E
shows *C InAngle A B E*
⟨*proof*⟩

lemma *lea-trans*:
assumes *A B C LeA A1 B1 C1* **and**
A1 B1 C1 LeA A2 B2 C2
shows *A B C LeA A2 B2 C2*
⟨*proof*⟩

lemma *in-angle-asym*:
assumes *D InAngle A B C* **and**
C InAngle A B D
shows *A B C CongA A B D*
⟨*proof*⟩

lemma *lea-asym*:
assumes *A B C LeA D E F* **and**
D E F LeA A B C
shows *A B C CongA D E F*
⟨*proof*⟩

lemma *col-lta--bet*:
assumes *Col X Y Z* **and**
A B C LtA X Y Z
shows *Bet X Y Z*
⟨*proof*⟩

lemma *col-lta--out*:
assumes *Col A B C* **and**
A B C LtA X Y Z
shows *B Out A C*
⟨*proof*⟩

lemma *lta-distincts*:

assumes *A B C LtA D E F*
shows $A{\neq}B \wedge C{\neq}B \wedge D{\neq}E \wedge F{\neq}E \wedge D \neq F$
⟨*proof*⟩

lemma *gta-distincts*:
assumes *A B C GtA D E F*
shows $A{\neq}B \wedge C{\neq}B \wedge D{\neq}E \wedge F{\neq}E \wedge A \neq C$
⟨*proof*⟩

lemma *acute-distincts*:
assumes *Acute A B C*
shows $A{\neq}B \wedge C{\neq}B$
⟨*proof*⟩

lemma *obtuse-distincts*:
assumes *Obtuse A B C*
shows $A{\neq}B \wedge C{\neq}B \wedge A \neq C$
⟨*proof*⟩

lemma *two-sides-in-angle*:
assumes $B \neq P'$ **and**
B P TS A C **and**
Bet P B P'
shows *P InAngle A B C* $\vee$ P' *InAngle A B C*
⟨*proof*⟩

lemma *in-angle-reverse*:
assumes $A' \neq B$ **and**
Bet A B A' **and**
C InAngle A B D
shows *D InAngle* A' *B C*
⟨*proof*⟩

lemma *in-angle-trans2*:
assumes *C InAngle A B D* **and**
D InAngle A B E
shows *D InAngle C B E*
⟨*proof*⟩

lemma *l11-36-aux1*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
Bet A B A' **and**
Bet D E D' **and**
A B C LeA D E F
shows D' *E F LeA* A' *B C*
⟨*proof*⟩

lemma *l11-36-aux2*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**
Bet A B A' **and**
Bet D E D' **and**
D' *E F LeA* A' *B C*
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *l11-36*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
$D \neq E$ **and**
$D' \neq E$ **and**

$Bet\ A\ B\ A'$ **and**
$Bet\ D\ E\ D'$
shows $A\ B\ C\ LeA\ D\ E\ F \longleftrightarrow D'\ E\ F\ LeA\ A'\ B\ C$
⟨*proof*⟩

lemma *l11-41-aux*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$Bet\ B\ A\ D$ **and**
$A \neq D$
shows $A\ C\ B\ LtA\ C\ A\ D$
⟨*proof*⟩

lemma *l11-41*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$Bet\ B\ A\ D$ **and**
$A \neq D$
shows $A\ C\ B\ LtA\ C\ A\ D \wedge A\ B\ C\ LtA\ C\ A\ D$
⟨*proof*⟩

lemma *not-conga*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$\neg\ A\ B\ C\ CongA\ D\ E\ F$
shows $\neg\ A'\ B'\ C'\ CongA\ D\ E\ F$
⟨*proof*⟩

lemma *not-conga-sym*:
assumes $\neg\ A\ B\ C\ CongA\ D\ E\ F$
shows $\neg\ D\ E\ F\ CongA\ A\ B\ C$
⟨*proof*⟩

lemma *not-and-lta*:
shows $\neg\ (A\ B\ C\ LtA\ D\ E\ F \wedge D\ E\ F\ LtA\ A\ B\ C)$
⟨*proof*⟩

lemma *conga-preserves-lta*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$D\ E\ F\ CongA\ D'\ E'\ F'$ **and**
$A\ B\ C\ LtA\ D\ E\ F$
shows $A'\ B'\ C'\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *lta-trans*:
assumes $A\ B\ C\ LtA\ A1\ B1\ C1$ **and**
$A1\ B1\ C1\ LtA\ A2\ B2\ C2$
shows $A\ B\ C\ LtA\ A2\ B2\ C2$
⟨*proof*⟩

lemma *obtuse-sym*:
assumes $Obtuse\ A\ B\ C$
shows $Obtuse\ C\ B\ A$
⟨*proof*⟩

lemma *acute-sym*:
assumes $Acute\ A\ B\ C$
shows $Acute\ C\ B\ A$
⟨*proof*⟩

lemma *acute-col--out*:
assumes $Col\ A\ B\ C$ **and**
$Acute\ A\ B\ C$
shows $B\ Out\ A\ C$
⟨*proof*⟩

lemma *col-obtuse--bet*:
assumes $Col\ A\ B\ C$ **and**
$Obtuse\ A\ B\ C$

shows *Bet A B C*
⟨*proof*⟩

lemma *out--acute*:
assumes *B Out A C*
shows *Acute A B C*
⟨*proof*⟩

lemma *bet--obtuse*:
assumes *Bet A B C* **and**
$A \neq B$ **and** $B \neq C$
shows *Obtuse A B C*
⟨*proof*⟩

lemma *l11-43-aux*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Per B A C $\vee$ *Obtuse B A C*
shows *Acute A B C*
⟨*proof*⟩

lemma *l11-43*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Per B A C $\vee$ *Obtuse B A C*
shows *Acute A B C* $\wedge$ *Acute A C B*
⟨*proof*⟩

lemma *acute-lea-acute*:
assumes *Acute D E F* **and**
A B C LeA D E F
shows *Acute A B C*
⟨*proof*⟩

lemma *lea-obtuse-obtuse*:
assumes *Obtuse D E F* **and**
D E F LeA A B C
shows *Obtuse A B C*
⟨*proof*⟩

lemma *l11-44-1-a*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Cong B A B C
shows *B A C CongA B C A*
⟨*proof*⟩

lemma *l11-44-2-a*:
assumes $\neg$ *Col A B C* **and**
B A Lt B C
shows *B C A LtA B A C*
⟨*proof*⟩

lemma *not-lta-and-conga*:
$\neg$ (*A B C LtA D E F* $\wedge$ *A B C CongA D E F*)
⟨*proof*⟩

lemma *conga-sym-equiv*:
A B C CongA A' B' C' $\longleftrightarrow$ *A' B' C' CongA A B C*
⟨*proof*⟩

lemma *conga-dec*:
A B C CongA D E F $\vee$ $\neg$ *A B C CongA D E F*
⟨*proof*⟩

lemma *lta-not-conga*:

assumes *A B C LtA D E F*
shows ¬ *A B C CongA D E F*
⟨*proof*⟩

lemma *lta--lea*:
assumes *A B C LtA D E F*
shows *A B C LeA D E F*
⟨*proof*⟩

lemma *nlta*:
¬ *A B C LtA A B C*
⟨*proof*⟩

lemma *lea--nlta*:
assumes *A B C LeA D E F*
shows ¬ *D E F LtA A B C*
⟨*proof*⟩

lemma *lta--nlea*:
assumes *A B C LtA D E F*
shows ¬ *D E F LeA A B C*
⟨*proof*⟩

lemma *l11-44-1-b*:
assumes ¬ *Col A B C* **and**
B A C CongA B C A
shows *Cong B A B C*
⟨*proof*⟩

lemma *l11-44-2-b*:
assumes *B A C LtA B C A*
shows *B C Lt B A*
⟨*proof*⟩

lemma *l11-44-1*:
assumes ¬ *Col A B C*
shows *B A C CongA B C A* ⟷ *Cong B A B C*
⟨*proof*⟩

lemma *l11-44-2*:
assumes ¬ *Col A B C*
shows *B A C LtA B C A* ⟷ *B C Lt B A*
⟨*proof*⟩

lemma *l11-44-2bis*:
assumes ¬ *Col A B C*
shows *B A C LeA B C A* ⟷ *B C Le B A*
⟨*proof*⟩

lemma *l11-46*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C ∨ *Obtuse A B C*
shows *B A Lt A C* ∧ *B C Lt A C*
⟨*proof*⟩

lemma *l11-47*:
assumes *Per A C B* **and**
H PerpAt C H A B
shows *Bet A H B* ∧ $A \neq H$ ∧ $B \neq H$
⟨*proof*⟩

lemma *l11-49*:
assumes *A B C CongA* A' B' C' **and**
Cong B A B' A' **and**
Cong B C B' C'

shows $Cong\ A\ C\ A'\ C' \wedge (A \neq C \longrightarrow (B\ A\ C\ CongA\ B'\ A'\ C' \wedge B\ C\ A\ CongA\ B'\ C'\ A'))$
⟨*proof*⟩

lemma *l11-50-1*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$B\ A\ C\ CongA\ B'\ A'\ C'$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$
shows $Cong\ A\ C\ A'\ C' \wedge Cong\ B\ C\ B'\ C' \wedge A\ C\ B\ CongA\ A'\ C'\ B'$
⟨*proof*⟩

lemma *l11-50-2*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$B\ C\ A\ CongA\ B'\ C'\ A'$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$Cong\ A\ B\ A'\ B'$
shows $Cong\ A\ C\ A'\ C' \wedge Cong\ B\ C\ B'\ C' \wedge C\ A\ B\ CongA\ C'\ A'\ B'$
⟨*proof*⟩

lemma *l11-51*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$Cong\ A\ C\ A'\ C'$ **and**
$Cong\ B\ C\ B'\ C'$
shows
$B\ A\ C\ CongA\ B'\ A'\ C' \wedge A\ B\ C\ CongA\ A'\ B'\ C' \wedge B\ C\ A\ CongA\ B'\ C'\ A'$
⟨*proof*⟩

lemma *conga-distinct*:
assumes $A\ B\ C\ CongA\ D\ E\ F$
shows $A \neq B \wedge C \neq B \wedge D \neq E \wedge F \neq E$
⟨*proof*⟩

lemma *l11-52*:
assumes $A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$Cong\ A\ C\ A'\ C'$ **and**
$Cong\ B\ C\ B'\ C'$ **and**
$B\ C\ Le\ A\ C$
shows $Cong\ B\ A\ B'\ A' \wedge B\ A\ C\ CongA\ B'\ A'\ C' \wedge B\ C\ A\ CongA\ B'\ C'\ A'$
⟨*proof*⟩

lemma *l11-53*:
assumes $Per\ D\ C\ B$ **and**
$C \neq D$ **and**
$A \neq B$ **and**
$B \neq C$ **and**
$Bet\ A\ B\ C$
shows $C\ A\ D\ LtA\ C\ B\ D \wedge B\ D\ Lt\ A\ D$
⟨*proof*⟩

lemma *cong2-conga-obtuse--cong-conga2*:
assumes $Obtuse\ A\ B\ C$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$Cong\ A\ C\ A'\ C'$ **and**
$Cong\ B\ C\ B'\ C'$
shows $Cong\ B\ A\ B'\ A' \wedge B\ A\ C\ CongA\ B'\ A'\ C' \wedge B\ C\ A\ CongA\ B'\ C'\ A'$
⟨*proof*⟩

lemma *cong2-per2--cong-conga2*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$Per\ A\ B\ C$ **and**
$Per\ A'\ B'\ C'$ **and**
$Cong\ A\ C\ A'\ C'$ **and**

Cong B C B′ C′
shows *Cong B A B′ A′* ∧ *B A C CongA B′ A′ C′* ∧ *B C A CongA B′ C′ A′*
⟨*proof*⟩

lemma *cong2-per2--cong*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
Cong A C A′ C′ **and**
Cong B C B′ C′
shows *Cong B A B′ A′*
⟨*proof*⟩

lemma *cong2-per2--cong-3*:
assumes *Per A B C*
Per A′ B′ C′ **and**
Cong A C A′ C′ **and**
Cong B C B′ C′
shows *A B C Cong3 A′ B′ C′*
⟨*proof*⟩

lemma *cong-lt-per2--lt*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
Cong A B A′ B′ **and**
B C Lt B′ C′
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *cong-le-per2--le*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
Cong A B A′ B′ **and**
B C Le B′ C′
shows *A C Le A′ C′*
⟨*proof*⟩

lemma *lt2-per2--lt*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Lt A′ B′ **and**
B C Lt B′ C′
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *le-lt-per2--lt*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Le A′ B′ **and**
B C Lt B′ C′
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *le2-per2--le*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Le A′ B′ **and**
B C Le B′ C′
shows *A C Le A′ C′*
⟨*proof*⟩

lemma *cong-lt-per2--lt-1*:
assumes *Per A B C* **and**
Per A′ B′ C′ **and**
A B Lt A′ B′ **and**
Cong A C A′ C′
shows *B′ C′ Lt B C*

⟨*proof*⟩

lemma *symmetry-preserves-conga*:
assumes $A \neq B$ **and** $C \neq B$ **and**
M Midpoint A A' **and**
M Midpoint B B' **and**
M Midpoint C C'
shows *A B C CongA A' B' C'*
⟨*proof*⟩

lemma *l11-57*:
assumes *A A' OS B B'* **and**
Per B A A' **and**
Per B' A' A **and**
A A' OS C C' **and**
Per C A A' **and**
Per C' A' A
shows *B A C CongA B' A' C'*
⟨*proof*⟩

lemma *cop3-orth-at--orth-at*:
assumes ¬ *Col D E F* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar A B C F **and**
X OrthAt A B C U V
shows *X OrthAt D E F U V*
⟨*proof*⟩

lemma *col2-orth-at--orth-at*:
assumes $U \neq V$ **and**
Col P Q U **and**
Col P Q V **and**
X OrthAt A B C P Q
shows *X OrthAt A B C U V*
⟨*proof*⟩

lemma *col-orth-at--orth-at*:
assumes $U \neq W$ **and**
Col U V W **and**
X OrthAt A B C U V
shows *X OrthAt A B C U W*
⟨*proof*⟩

lemma *orth-at-symmetry*:
assumes *X OrthAt A B C U V*
shows *X OrthAt A B C V U*
⟨*proof*⟩

lemma *orth-at-distincts*:
assumes *X OrthAt A B C U V*
shows $A \neq B \wedge B \neq C \wedge A \neq C \wedge U \neq V$
⟨*proof*⟩

lemma *orth-at-chara*:
X OrthAt A B C X P ⟷
(¬ *Col A B C* ∧ $X \neq P$ ∧ *Coplanar A B C X* ∧ (∀ *D*.(*Coplanar A B C D* ⟶ *Per D X P*)))
⟨*proof*⟩

lemma *cop3-orth--orth*:
assumes ¬ *Col D E F* **and**
Coplanar A B C D **and**
Coplanar A B C E **and**
Coplanar A B C F **and**
A B C Orth U V
shows *D E F Orth U V*

⟨*proof*⟩

lemma *col2-orth--orth*:
assumes $U \neq V$ **and**
Col P Q U **and**
Col P Q V **and**
A B C Orth P Q
shows *A B C Orth U V*
⟨*proof*⟩

lemma *col-orth--orth*:
assumes $U \neq W$ **and**
Col U V W **and**
A B C Orth U V
shows *A B C Orth U W*
⟨*proof*⟩

lemma *orth-symmetry*:
assumes *A B C Orth U V*
shows *A B C Orth V U*
⟨*proof*⟩

lemma *orth-distincts*:
assumes *A B C Orth U V*
shows $A \neq B \land B \neq C \land A \neq C \land U \neq V$
⟨*proof*⟩

lemma *col-cop-orth--orth-at*:
assumes *A B C Orth U V* **and**
Coplanar A B C X **and**
Col U V X
shows *X OrthAt A B C U V*
⟨*proof*⟩

lemma *l11-60-aux*:
assumes ¬ *Col A B C* **and**
Cong A P A Q **and**
Cong B P B Q **and**
Cong C P C Q **and**
Coplanar A B C D
shows *Cong D P D Q*
⟨*proof*⟩

lemma *l11-60*:
assumes ¬ *Col A B C* **and**
Per A D P **and**
Per B D P **and**
Per C D P **and**
Coplanar A B C E
shows *Per E D P*
⟨*proof*⟩

lemma *l11-60-bis*:
assumes ¬ *Col A B C* **and**
$D \neq P$ **and**
Coplanar A B C D **and**
Per A D P **and**
Per B D P **and**
Per C D P
shows *D OrthAt A B C D P*
⟨*proof*⟩

lemma *l11-61*:
assumes $A \neq A'$ **and**
$A \neq B$ **and**
$A \neq C$ **and**

Coplanar A A′ B B′ **and**
Per B A A′ **and**
Per B′ A′ A **and**
Coplanar A A′ C C′ **and**
Per C A A′ **and**
Per B A C
shows *Per B′ A′ C′*
⟨*proof*⟩

lemma *l11-61-bis*:
assumes *D OrthAt A B C D P* **and**
D E Perp E Q **and**
Coplanar A B C E **and**
Coplanar D E P Q
shows *E OrthAt A B C E Q*
⟨*proof*⟩

lemma *l11-62-unicity*:
assumes *Coplanar A B C D* **and**
Coplanar A B C D′ **and**
$\forall$ *E. Coplanar A B C E* $\longrightarrow$ *Per E D P* **and**
$\forall$ *E. Coplanar A B C E* $\longrightarrow$ *Per E D′ P*
shows $D = D'$
⟨*proof*⟩

lemma *l11-62-unicity-bis*:
assumes *X OrthAt A B C X U* **and**
Y OrthAt A B C Y U
shows $X = Y$
⟨*proof*⟩

lemma *orth-at2--eq*:
assumes *X OrthAt A B C U V* **and**
Y OrthAt A B C U V
shows $X = Y$
⟨*proof*⟩

lemma *col-cop-orth-at--eq*:
assumes *X OrthAt A B C U V* **and**
Coplanar A B C Y **and**
Col U V Y
shows $X = Y$
⟨*proof*⟩

lemma *orth-at--ncop1*:
assumes $U \neq X$ **and**
X OrthAt A B C U V
shows ¬ *Coplanar A B C U*
⟨*proof*⟩

lemma *orth-at--ncop2*:
assumes $V \neq X$ **and**
X OrthAt A B C U V
shows ¬ *Coplanar A B C V*
⟨*proof*⟩

lemma *orth-at--ncop*:
assumes *X OrthAt A B C X P*
shows ¬ *Coplanar A B C P*
⟨*proof*⟩

lemma *l11-62-existence*:
$\exists$ *D*. (*Coplanar A B C D* $\wedge$ ($\forall$ *E*. (*Coplanar A B C E* $\longrightarrow$ *Per E D P*)))
⟨*proof*⟩

lemma *l11-62-existence-bis*:

assumes ¬ *Coplanar A B C P*
shows ∃ *X*. *X OrthAt A B C X P*
⟨*proof*⟩

lemma *l11-63-aux*:
assumes *Coplanar A B C D* **and**
$D \neq E$ **and**
E OrthAt A B C E P
shows ∃ *Q*. (*D E OS P Q* ∧ *A B C Orth D Q*)
⟨*proof*⟩

lemma *l11-63-existence*:
assumes *Coplanar A B C D* **and**
¬ *Coplanar A B C P*
shows ∃ *Q*. *A B C Orth D Q*
⟨*proof*⟩

lemma *l8-21-3*:
assumes *Coplanar A B C D* **and**
¬ *Coplanar A B C X*
shows
∃ *P T*. (*A B C Orth D P* ∧ *Coplanar A B C T* ∧ *Bet X T P*)
⟨*proof*⟩

lemma *mid2-orth-at2--cong*:
assumes *X OrthAt A B C X P* **and**
Y OrthAt A B C Y Q **and**
X Midpoint P P' **and**
Y Midpoint Q Q'
shows *Cong P Q P' Q'*
⟨*proof*⟩

lemma *orth-at2-tsp--ts*:
assumes $P \neq Q$ **and**
P OrthAt A B C P X **and**
Q OrthAt A B C Q Y **and**
A B C TSP X Y
shows *P Q TS X Y*
⟨*proof*⟩

lemma *orth-dec*:
shows *A B C Orth U V* ∨ ¬ *A B C Orth U V*
⟨*proof*⟩

lemma *orth-at-dec*:
shows *X OrthAt A B C U V* ∨ ¬ *X OrthAt A B C U V*
⟨*proof*⟩

lemma *tsp-dec*:
shows *A B C TSP X Y* ∨ ¬ *A B C TSP X Y*
⟨*proof*⟩

lemma *osp-dec*:
shows *A B C OSP X Y* ∨ ¬ *A B C OSP X Y*
⟨*proof*⟩

lemma *ts2--inangle*:
assumes *A C TS B P* **and**
B P TS A C
shows *P InAngle A B C*
⟨*proof*⟩

lemma *os-ts--inangle*:
assumes *B P TS A C* **and**
B A OS C P
shows *P InAngle A B C*

⟨*proof*⟩

lemma *os2--inangle*:
assumes *B A OS C P* **and**
B C OS A P
shows *P InAngle A B C*
⟨*proof*⟩

lemma *acute-conga--acute*:
assumes *Acute A B C* **and**
A B C CongA D E F
shows *Acute D E F*
⟨*proof*⟩

lemma *acute-out2--acute*:
assumes *B Out A′ A* **and**
B Out C′ C **and**
Acute A B C
shows *Acute A′ B C′*
⟨*proof*⟩

lemma *conga-obtuse--obtuse*:
assumes *Obtuse A B C* **and**
A B C CongA D E F
shows *Obtuse D E F*
⟨*proof*⟩

lemma *obtuse-out2--obtuse*:
assumes *B Out A′ A* **and**
B Out C′ C **and**
Obtuse A B C
shows *Obtuse A′ B C′*
⟨*proof*⟩

lemma *bet-lea--bet*:
assumes *Bet A B C* **and**
A B C LeA D E F
shows *Bet D E F*
⟨*proof*⟩

lemma *out-lea--out*:
assumes *E Out D F* **and**
A B C LeA D E F
shows *B Out A C*
⟨*proof*⟩

lemma *bet2-lta--lta*:
assumes *A B C LtA D E F* **and**
Bet A B A′ **and**
A′ ≠ *B* **and**
Bet D E D′ **and**
D′ ≠ *E*
shows *D′ E F LtA A′ B C*
⟨*proof*⟩

lemma *lea123456-lta--lta*:
assumes *A B C LeA D E F* **and**
D E F LtA G H I
shows *A B C LtA G H I*
⟨*proof*⟩

lemma *lea456789-lta--lta*:
assumes *A B C LtA D E F* **and**
D E F LeA G H I
shows *A B C LtA G H I*
⟨*proof*⟩

lemma *acute-per--lta*:
  assumes $Acute\ A\ B\ C$ **and**
    $D \neq E$ **and**
    $E \neq F$ **and**
    $Per\ D\ E\ F$
  shows $A\ B\ C\ LtA\ D\ E\ F$
⟨*proof*⟩

lemma *obtuse-per--lta*:
  assumes $Obtuse\ A\ B\ C$ **and**
    $D \neq E$ **and**
    $E \neq F$ **and**
    $Per\ D\ E\ F$
  shows $D\ E\ F\ LtA\ A\ B\ C$
⟨*proof*⟩

lemma *acute-obtuse--lta*:
  assumes $Acute\ A\ B\ C$ **and**
    $Obtuse\ D\ E\ F$
  shows $A\ B\ C\ LtA\ D\ E\ F$
  ⟨*proof*⟩

lemma *lea-in-angle*:
  assumes $A\ B\ P\ LeA\ A\ B\ C$ **and**
    $A\ B\ OS\ C\ P$
  shows $P\ InAngle\ A\ B\ C$
⟨*proof*⟩

lemma *acute-bet--obtuse*:
  assumes $Bet\ A\ B\ A'$ **and**
    $A' \neq B$ **and**
    $Acute\ A\ B\ C$
  shows $Obtuse\ A'\ B\ C$
⟨*proof*⟩

lemma *bet-obtuse--acute*:
  assumes $Bet\ A\ B\ A'$ **and**
    $A' \neq B$ **and**
    $Obtuse\ A\ B\ C$
  shows $Acute\ A'\ B\ C$
⟨*proof*⟩

lemma *inangle-dec*:
  $P\ InAngle\ A\ B\ C \vee \neg\ P\ InAngle\ A\ B\ C$
  ⟨*proof*⟩

lemma *lea-dec*:
  $A\ B\ C\ LeA\ D\ E\ F \vee \neg\ A\ B\ C\ LeA\ D\ E\ F$
  ⟨*proof*⟩

lemma *lta-dec*:
  $A\ B\ C\ LtA\ D\ E\ F \vee \neg\ A\ B\ C\ LtA\ D\ E\ F$
  ⟨*proof*⟩

lemma *lea-total*:
  assumes $A \neq B$ **and**
    $B \neq C$ **and**
    $D \neq E$ **and**
    $E \neq F$
  shows $A\ B\ C\ LeA\ D\ E\ F \vee D\ E\ F\ LeA\ A\ B\ C$
⟨*proof*⟩

lemma *or-lta2-conga*:
  assumes $A \neq B$ **and**
    $C \neq B$ **and**

$D \neq E$ **and**
$F \neq E$
shows $A\ B\ C\ LtA\ D\ E\ F \vee D\ E\ F\ LtA\ A\ B\ C \vee A\ B\ C\ CongA\ D\ E\ F$
⟨*proof*⟩

lemma *angle-partition*:
assumes $A \neq B$ **and**
$B \neq C$
shows $Acute\ A\ B\ C \vee Per\ A\ B\ C \vee Obtuse\ A\ B\ C$
⟨*proof*⟩

lemma *acute-chara-1*:
assumes $Bet\ A\ B\ A'$ **and**
$B \neq A'$ **and**
$Acute\ A\ B\ C$
shows $A\ B\ C\ LtA\ A'\ B\ C$
⟨*proof*⟩

lemma *acute-chara-2*:
assumes $Bet\ A\ B\ A'$ **and**
$A\ B\ C\ LtA\ A'\ B\ C$
shows $Acute\ A\ B\ C$
⟨*proof*⟩

lemma *acute-chara*:
assumes $Bet\ A\ B\ A'$ **and**
$B \neq A'$
shows $Acute\ A\ B\ C \longleftrightarrow A\ B\ C\ LtA\ A'\ B\ C$
⟨*proof*⟩

lemma *obtuse-chara*:
assumes $Bet\ A\ B\ A'$ **and**
$B \neq A'$
shows $Obtuse\ A\ B\ C \longleftrightarrow A'\ B\ C\ LtA\ A\ B\ C$
⟨*proof*⟩

lemma *conga--acute*:
assumes $A\ B\ C\ CongA\ A\ C\ B$
shows $Acute\ A\ B\ C$
⟨*proof*⟩

lemma *cong--acute*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$Cong\ A\ B\ A\ C$
shows $Acute\ A\ B\ C$
⟨*proof*⟩

lemma *nlta--lea*:
assumes $\neg\ A\ B\ C\ LtA\ D\ E\ F$ **and**
$A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
shows $D\ E\ F\ LeA\ A\ B\ C$
⟨*proof*⟩

lemma *nlea--lta*:
assumes $\neg\ A\ B\ C\ LeA\ D\ E\ F$ **and**
$A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
shows $D\ E\ F\ LtA\ A\ B\ C$
⟨*proof*⟩

lemma *triangle-strict-inequality*:
assumes *Bet A B D* **and**
Cong B C B D **and**
¬ *Bet A B C*
shows *A C Lt A D*
⟨*proof*⟩

lemma *triangle-inequality*:
assumes *Bet A B D* **and**
Cong B C B D
shows *A C Le A D*
⟨*proof*⟩

lemma *triangle-strict-inequality-2*:
assumes *Bet A′ B′ C′* **and**
Cong A B A′ B′ **and**
Cong B C B′ C′ **and**
¬ *Bet A B C*
shows *A C Lt A′ C′*
⟨*proof*⟩

lemma *triangle-inequality-2*:
assumes *Bet A′ B′ C′* **and**
Cong A B A′ B′ **and**
Cong B C B′ C′
shows *A C Le A′ C′*
⟨*proof*⟩

lemma *triangle-strict-reverse-inequality*:
assumes *A Out B D* **and**
Cong A C A D **and**
¬ *A Out B C*
shows *B D Lt B C*
⟨*proof*⟩

lemma *triangle-reverse-inequality*:
assumes *A Out B D* **and**
Cong A C A D
shows *B D Le B C*
⟨*proof*⟩

lemma *os3--lta*:
assumes *A B OS C D* **and**
B C OS A D **and**
A C OS B D
shows *B A C LtA B D C*
⟨*proof*⟩

lemma *bet-le--lt*:
assumes *Bet A D B* **and**
A ≠ *D* **and**
D ≠ *B* **and**
A C Le B C
shows *D C Lt B C*
⟨*proof*⟩

lemma *cong2--ncol*:
assumes *A* ≠ *B* **and**
B ≠ *C* **and**
A ≠ *C* **and**
Cong A P B P **and**
Cong A P C P
shows ¬ *Col A B C*

⟨proof⟩

lemma *cong4-cop2--eq*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
Cong A P B P **and**
Cong A P C P **and**
Coplanar A B C P **and**
Cong A Q B Q **and**
Cong A Q C Q **and**
Coplanar A B C Q
shows $P = Q$
⟨proof⟩

lemma *t18-18-aux*:
assumes *Cong* A B D E **and**
Cong A C D F **and**
F D E *LtA* C A B **and**
¬ *Col* A B C **and**
¬ *Col* D E F **and**
D F *Le* D E
shows E F *Lt* B C
⟨proof⟩

lemma *t18-18*:
assumes *Cong* A B D E **and**
Cong A C D F **and**
F D E *LtA* C A B
shows E F *Lt* B C
⟨proof⟩

lemma *t18-19*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Cong A B D E **and**
Cong A C D F **and**
E F *Lt* B C
shows F D E *LtA* C A B
⟨proof⟩

lemma *acute-trivial*:
assumes $A \neq B$
shows *Acute* A B A
⟨proof⟩

lemma *acute-not-per*:
assumes *Acute* A B C
shows ¬ *Per* A B C
⟨proof⟩

lemma *angle-bisector*:
assumes $A \neq B$ **and**
$C \neq B$
shows $\exists$ P. (P *InAngle* A B C $\wedge$ P B A *CongA* P B C)
⟨proof⟩

lemma *reflectl--conga*:
assumes $A \neq B$ **and**
$B \neq P$ **and**
P P' *ReflectL* A B
shows A B P *CongA* A B P'
⟨proof⟩

lemma *conga-cop-out-reflectl--out*:
assumes ¬ B *Out* A C **and**

Coplanar A B C P **and**
P B A CongA P B C **and**
B Out A T **and**
T T′ ReflectL B P
shows *B Out C T′*
⟨*proof*⟩

lemma *col-conga-cop-reflectl--col*:
assumes ¬ *B Out A C* **and**
Coplanar A B C P **and**
P B A CongA P B C **and**
Col B A T **and**
T T′ ReflectL B P
shows *Col B C T′*
⟨*proof*⟩

lemma *conga2-cop2--col*:
assumes ¬ *B Out A C* **and**
P B A CongA P B C **and**
P′ B A CongA P′ B C **and**
Coplanar A B P P′ **and**
Coplanar B C P P′
shows *Col B P P′*
⟨*proof*⟩

lemma *conga2-cop2--col-1*:
assumes ¬ *Col A B C* **and**
P B A CongA P B C **and**
P′ B A CongA P′ B C **and**
Coplanar A B C P **and**
Coplanar A B C P′
shows *Col B P P′*
⟨*proof*⟩

lemma *col-conga--conga*:
assumes *P B A CongA P B C* **and**
Col B P P′ **and**
$B \neq P'$
shows *P′ B A CongA P′ B C*
⟨*proof*⟩

lemma *cop-inangle--ex-col-inangle*:
assumes ¬ *B Out A C* **and**
P InAngle A B C **and**
Coplanar A B C Q
shows ∃ *R*. (*R InAngle A B C* ∧ $P \neq R$ ∧ *Col P Q R*)
⟨*proof*⟩

lemma *col-inangle2--out*:
assumes ¬ *Bet A B C* **and**
P InAngle A B C **and**
Q InAngle A B C **and**
Col B P Q
shows *B Out P Q*
⟨*proof*⟩

lemma *inangle2--lea*:
assumes *P InAngle A B C* **and**
Q InAngle A B C
shows *P B Q LeA A B C*
⟨*proof*⟩

lemma *conga-inangle-per--acute*:
assumes *Per A B C* **and**
P InAngle A B C **and**
P B A CongA P B C

shows *Acute A B P*
⟨*proof*⟩

lemma *conga-inangle2-per--acute*:
assumes *Per A B C* **and**
P InAngle A B C **and**
P B A CongA P B C **and**
Q InAngle A B C
shows *Acute P B Q*
⟨*proof*⟩

lemma *lta-os--ts*:
assumes
A O1 P LtA A O1 B **and**
O1 A OS B P
shows *O1 P TS A B*
⟨*proof*⟩

lemma *bet--suppa*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$B \neq A'$ **and**
Bet A B A'
shows *A B C SuppA C B* A'
⟨*proof*⟩

lemma *ex-suppa*:
assumes $A \neq B$ **and**
$B \neq C$
shows $\exists$ *D E F. A B C SuppA D E F*
⟨*proof*⟩

lemma *suppa-distincts*:
assumes *A B C SuppA D E F*
shows $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$
⟨*proof*⟩

lemma *suppa-right-comm*:
assumes *A B C SuppA D E F*
shows *A B C SuppA F E D*
⟨*proof*⟩

lemma *suppa-left-comm*:
assumes *A B C SuppA D E F*
shows *C B A SuppA D E F*
⟨*proof*⟩

lemma *suppa-comm*:
assumes *A B C SuppA D E F*
shows *C B A SuppA F E D*
⟨*proof*⟩

lemma *suppa-sym*:
assumes *A B C SuppA D E F*
shows *D E F SuppA A B C*
⟨*proof*⟩

lemma *conga2-suppa--suppa*:
assumes *A B C CongA* A' B' C' **and**
D E F CongA D' E' F' **and**
A B C SuppA D E F
shows A' B' C' *SuppA* D' E' F'
⟨*proof*⟩

lemma *suppa2--conga456*:
assumes *A B C SuppA D E F* **and**

$A\ B\ C\ SuppA\ D'\ E'\ F'$
shows $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *suppa2--conga123*:
assumes $A\ B\ C\ SuppA\ D\ E\ F$ **and**
$A'\ B'\ C'\ SuppA\ D\ E\ F$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *bet-out--suppa*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$Bet\ A\ B\ C$ **and**
$E\ Out\ D\ F$
shows $A\ B\ C\ SuppA\ D\ E\ F$
⟨*proof*⟩

lemma *bet-suppa--out*:
assumes $Bet\ A\ B\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
shows $E\ Out\ D\ F$
⟨*proof*⟩

lemma *out-suppa--bet*:
assumes $B\ Out\ A\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
shows $Bet\ D\ E\ F$
⟨*proof*⟩

lemma *per-suppa--per*:
assumes $Per\ A\ B\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
shows $Per\ D\ E\ F$
⟨*proof*⟩

lemma *per2--suppa*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
$Per\ A\ B\ C$ **and**
$Per\ D\ E\ F$
shows $A\ B\ C\ SuppA\ D\ E\ F$
⟨*proof*⟩

lemma *suppa--per*:
assumes $A\ B\ C\ SuppA\ A\ B\ C$
shows $Per\ A\ B\ C$
⟨*proof*⟩

lemma *acute-suppa--obtuse*:
assumes $Acute\ A\ B\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
shows $Obtuse\ D\ E\ F$
⟨*proof*⟩

lemma *obtuse-suppa--acute*:
assumes $Obtuse\ A\ B\ C$ **and**
$A\ B\ C\ SuppA\ D\ E\ F$
shows $Acute\ D\ E\ F$
⟨*proof*⟩

lemma *lea-suppa2--lea*:
assumes $A\ B\ C\ SuppA\ A'\ B'\ C'$ **and**
$D\ E\ F\ SuppA\ D'\ E'\ F'$

$A\ B\ C\ LeA\ D\ E\ F$
shows $D'\ E'\ F'\ LeA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *lta-suppa2--lta*:
assumes $A\ B\ C\ SuppA\ A'\ B'\ C'$
and $D\ E\ F\ SuppA\ D'\ E'\ F'$
and $A\ B\ C\ LtA\ D\ E\ F$
shows $D'\ E'\ F'\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *suppa-dec*:
$A\ B\ C\ SuppA\ D\ E\ F \lor \neg\ A\ B\ C\ SuppA\ D\ E\ F$
⟨*proof*⟩

lemma *acute-one-side-aux*:
assumes $C\ A\ OS\ P\ B$ **and**
$Acute\ A\ C\ P$ **and**
$C\ A\ Perp\ B\ C$
shows $C\ B\ OS\ A\ P$
⟨*proof*⟩

lemma *acute-one-side-aux0*:
assumes $Col\ A\ C\ P$ **and**
$Acute\ A\ C\ P$ **and**
$C\ A\ Perp\ B\ C$
shows $C\ B\ OS\ A\ P$
⟨*proof*⟩

lemma *acute-cop-perp--one-side*:
assumes $Acute\ A\ C\ P$ **and**
$C\ A\ Perp\ B\ C$ **and**
$Coplanar\ A\ B\ C\ P$
shows $C\ B\ OS\ A\ P$
⟨*proof*⟩

lemma *acute--not-obtuse*:
assumes $Acute\ A\ B\ C$
shows $\neg\ Obtuse\ A\ B\ C$
⟨*proof*⟩

lemma *suma-distincts*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
shows $A \neq B \land B \neq C \land D \neq E \land E \neq F \land G \neq H \land H \neq I$
⟨*proof*⟩

lemma *trisuma-distincts*:
assumes $A\ B\ C\ TriSumA\ D\ E\ F$
shows $A \neq B \land B \neq C \land A \neq C \land D \neq E \land E \neq F$
⟨*proof*⟩

lemma *ex-suma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$
shows $\exists\ G\ H\ I.\ A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
⟨*proof*⟩

lemma *suma2--conga*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ CongA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *suma-sym*:

assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
shows $D\ E\ F\ A\ B\ C\ SumA\ G\ H\ I$
⟨*proof*⟩

lemma *conga3-suma--suma*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$D\ E\ F\ CongA\ D'\ E'\ F'$ **and**
$G\ H\ I\ CongA\ G'\ H'\ I'$
shows $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *out6-suma--suma*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$B\ Out\ A\ A'$ **and**
$B\ Out\ C\ C'$ **and**
$E\ Out\ D\ D'$ **and**
$E\ Out\ F\ F'$ **and**
$H\ Out\ G\ G'$ **and**
$H\ Out\ I\ I'$
shows $A'\ B\ C'\ D'\ E\ F'\ SumA\ G'\ H\ I'$
⟨*proof*⟩

lemma *out546-suma--conga*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$E\ Out\ D\ F$
shows $A\ B\ C\ CongA\ G\ H\ I$
⟨*proof*⟩

lemma *out546--suma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$E\ Out\ D\ F$
shows $A\ B\ C\ D\ E\ F\ SumA\ A\ B\ C$
⟨*proof*⟩

lemma *out213-suma--conga*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$B\ Out\ A\ C$
shows $D\ E\ F\ CongA\ G\ H\ I$
⟨*proof*⟩

lemma *out213--suma*:
assumes $D \neq E$ **and**
$E \neq F$ **and**
$B\ Out\ A\ C$
shows $A\ B\ C\ D\ E\ F\ SumA\ D\ E\ F$
⟨*proof*⟩

lemma *suma-left-comm*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
shows $C\ B\ A\ D\ E\ F\ SumA\ G\ H\ I$
⟨*proof*⟩

lemma *suma-middle-comm*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
shows $A\ B\ C\ F\ E\ D\ SumA\ G\ H\ I$
⟨*proof*⟩

lemma *suma-right-comm*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
shows $A\ B\ C\ D\ E\ F\ SumA\ I\ H\ G$
⟨*proof*⟩

lemma *suma-comm*:
assumes $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$

shows *C B A F E D SumA I H G*
⟨*proof*⟩

lemma *ts--suma*:
assumes *A B TS C D*
shows *C B A A B D SumA C B D*
⟨*proof*⟩

lemma *ts--suma-1*:
assumes *A B TS C D*
shows *C A B B A D SumA C A D*
⟨*proof*⟩

lemma *inangle--suma*:
assumes *P InAngle A B C*
shows *A B P P B C SumA A B C*
⟨*proof*⟩

lemma *bet--suma*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$P \neq B$ **and** *Bet A B C*
shows *A B P P B C SumA A B C*
⟨*proof*⟩

lemma *sams-chara*:
assumes $A \neq B$ **and**
$A' \neq B$ **and**
Bet A B A'
shows *SAMS A B C D E F* $\longleftrightarrow$ *D E F LeA C B A'*
⟨*proof*⟩

lemma *sams-distincts*:
assumes *SAMS A B C D E F*
shows $A \neq B \wedge B \neq C \wedge D \neq E \wedge E \neq F$
⟨*proof*⟩

lemma *sams-sym*:
assumes *SAMS A B C D E F*
shows *SAMS D E F A B C*
⟨*proof*⟩

lemma *sams-right-comm*:
assumes *SAMS A B C D E F*
shows *SAMS A B C F E D*
⟨*proof*⟩

lemma *sams-left-comm*:
assumes *SAMS A B C D E F*
shows *SAMS C B A D E F*
⟨*proof*⟩

lemma *sams-comm*:
assumes *SAMS A B C D E F*
shows *SAMS C B A F E D*
⟨*proof*⟩

lemma *conga2-sams--sams*:
assumes *A B C CongA A' B' C'* **and**
D E F CongA D' E' F' **and**
SAMS A B C D E F
shows *SAMS A' B' C' D' E' F'*
⟨*proof*⟩

lemma *out546--sams*:
assumes $A \neq B$ **and**

$B \neq C$ **and**
E *Out* D F
shows *SAMS* A B C D E F
⟨*proof*⟩

lemma *out213--sams*:
assumes $D \neq E$ **and**
$E \neq F$ **and**
B *Out* A C
shows *SAMS* A B C D E F
⟨*proof*⟩

lemma *bet-suma--sams*:
assumes A B C D E F *SumA* G H I **and**
Bet G H I
shows *SAMS* A B C D E F
⟨*proof*⟩

lemma *bet--sams*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$P \neq B$ **and**
Bet A B C
shows *SAMS* A B P P B C
⟨*proof*⟩

lemma *suppa--sams*:
assumes A B C *SuppA* D E F
shows *SAMS* A B C D E F
⟨*proof*⟩

lemma *os-ts--sams*:
assumes B P *TS* A C **and**
A B *OS* P C
shows *SAMS* A B P P B C
⟨*proof*⟩

lemma *os2--sams*:
assumes A B *OS* P C **and**
C B *OS* P A
shows *SAMS* A B P P B C
⟨*proof*⟩

lemma *inangle--sams*:
assumes P *InAngle* A B C
shows *SAMS* A B P P B C
⟨*proof*⟩

lemma *sams-suma--lea123789*:
assumes A B C D E F *SumA* G H I **and**
SAMS A B C D E F
shows A B C *LeA* G H I
⟨*proof*⟩

lemma *sams-suma--lea456789*:
assumes A B C D E F *SumA* G H I **and**
SAMS A B C D E F
shows D E F *LeA* G H I
⟨*proof*⟩

lemma *sams-lea2--sams*:
assumes *SAMS* A' B' C' D' E' F' **and**
A B C *LeA* A' B' C' **and**
D E F *LeA* D' E' F'
shows *SAMS* A B C D E F
⟨*proof*⟩

lemma *sams-lea456-suma2--lea*:
assumes $D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A\ B\ C\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A\ B\ C\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ LeA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams-lea123-suma2--lea*:
assumes $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$SAMS\ A'\ B'\ C'\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D\ E\ F\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ LeA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams-lea2-suma2--lea*:
assumes $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
$D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
$SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $G\ H\ I\ LeA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams2-suma2--conga456*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ A\ B\ C\ D'\ E'\ F'$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A\ B\ C\ D'\ E'\ F'\ SumA\ G\ H\ I$
shows $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams2-suma2--conga123*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ A'\ B'\ C'\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D\ E\ F\ SumA\ G\ H\ I$
shows $A\ B\ C\ CongA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *suma-assoc-1*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M$
shows $A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
⟨*proof*⟩

lemma *suma-assoc-2*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$ **and**
$A\ B\ C\ D'\ E'\ F'\ SumA\ K\ L\ M$
shows $A'\ B'\ C'\ G\ H\ I\ SumA\ K\ L\ M$
⟨*proof*⟩

lemma *suma-assoc*:
assumes $SAMS\ A\ B\ C\ D\ E\ F$ **and**
$SAMS\ D\ E\ F\ G\ H\ I$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ A'\ B'\ C'$ **and**
$D\ E\ F\ G\ H\ I\ SumA\ D'\ E'\ F'$
shows

A' B' C' G H I *SumA* K L M $\longleftrightarrow$ A B C D' E' F' *SumA* K L M
⟨*proof*⟩

lemma *sams-assoc-1*:
assumes *SAMS* A B C D E F **and**
SAMS D E F G H I **and**
A B C D E F *SumA* A' B' C' **and**
D E F G H I *SumA* D' E' F' **and**
SAMS A' B' C' G H I
shows *SAMS* A B C D' E' F'
⟨*proof*⟩

lemma *sams-assoc-2*:
assumes *SAMS* A B C D E F **and**
SAMS D E F G H I **and**
A B C D E F *SumA* A' B' C' **and**
D E F G H I *SumA* D' E' F' **and**
SAMS A B C D' E' F'
shows *SAMS* A' B' C' G H I
⟨*proof*⟩

lemma *sams-assoc*:
assumes *SAMS* A B C D E F **and**
SAMS D E F G H I **and**
A B C D E F *SumA* A' B' C' **and**
D E F G H I *SumA* D' E' F'
shows (*SAMS* A' B' C' G H I) $\longleftrightarrow$ (*SAMS* A B C D' E' F')
⟨*proof*⟩

lemma *sams-nos--nts*:
assumes *SAMS* A B C C B J **and**
$\neg$ B C *OS* A J
shows $\neg$ A B *TS* C J
⟨*proof*⟩

lemma *conga-sams-nos--nts*:
assumes *SAMS* A B C D E F **and**
C B J *CongA* D E F **and**
$\neg$ B C *OS* A J
shows $\neg$ A B *TS* C J
⟨*proof*⟩

lemma *sams-lea2-suma2--conga123*:
assumes A B C *LeA* A' B' C' **and**
D E F *LeA* D' E' F' **and**
SAMS A' B' C' D' E' F' **and**
A B C D E F *SumA* G H I **and**
A' B' C' D' E' F' *SumA* G H I
shows A B C *CongA* A' B' C'
⟨*proof*⟩

lemma *sams-lea2-suma2--conga456*:
assumes A B C *LeA* A' B' C' **and**
D E F *LeA* D' E' F' **and**
SAMS A' B' C' D' E' F' **and**
A B C D E F *SumA* G H I **and**
A' B' C' D' E' F' *SumA* G H I
shows D E F *CongA* D' E' F'
⟨*proof*⟩

lemma *sams-suma--out213*:
assumes A B C D E F *SumA* D E F **and**
SAMS A B C D E F
shows B *Out* A C
⟨*proof*⟩

lemma *sams-suma--out546*:
  assumes $A\ B\ C\ D\ E\ F\ SumA\ A\ B\ C$ **and**
    $SAMS\ A\ B\ C\ D\ E\ F$
  shows $E\ Out\ D\ F$
⟨*proof*⟩

lemma *sams-lea-lta123-suma2--lta*:
  assumes $A\ B\ C\ LtA\ A'\ B'\ C'$ **and**
    $D\ E\ F\ LeA\ D'\ E'\ F'$ **and**
    $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
    $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
    $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
  shows $G\ H\ I\ LtA\ G'\ H'\ I'$
⟨*proof*⟩

lemma *sams-lea-lta456-suma2--lta*:
  assumes $A\ B\ C\ LeA\ A'\ B'\ C'$ **and**
    $D\ E\ F\ LtA\ D'\ E'\ F'$ **and**
    $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
    $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
    $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
  shows $G\ H\ I\ LtA\ G'\ H'\ I'$
  ⟨*proof*⟩

lemma *sams-lta2-suma2--lta*:
  assumes $A\ B\ C\ LtA\ A'\ B'\ C'$ **and**
    $D\ E\ F\ LtA\ D'\ E'\ F'$ **and**
    $SAMS\ A'\ B'\ C'\ D'\ E'\ F'$ **and**
    $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
    $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
  shows $G\ H\ I\ LtA\ G'\ H'\ I'$
  ⟨*proof*⟩

lemma *sams-lea2-suma2--lea123*:
  assumes $D'\ E'\ F'\ LeA\ D\ E\ F$ **and**
    $G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
    $SAMS\ A\ B\ C\ D\ E\ F$ **and**
    $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
    $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
  shows $A\ B\ C\ LeA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lea2-suma2--lea456*:
  assumes $A'\ B'\ C'\ LeA\ A\ B\ C$ **and**
    $G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
    $SAMS\ A\ B\ C\ D\ E\ F$ **and**
    $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
    $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
  shows $D\ E\ F\ LeA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-lea-lta456-suma2--lta123*:
  assumes $D'\ E'\ F'\ LtA\ D\ E\ F$ **and**
    $G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
    $SAMS\ A\ B\ C\ D\ E\ F$ **and**
    $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
    $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
  shows $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lea-lta123-suma2--lta456*:
  assumes $A'\ B'\ C'\ LtA\ A\ B\ C$ **and**
    $G\ H\ I\ LeA\ G'\ H'\ I'$ **and**
    $SAMS\ A\ B\ C\ D\ E\ F$ **and**
    $A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
    $A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$

shows $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-lea-lta789-suma2--lta123*:
assumes $D'\ E'\ F'\ LeA\ D\ E\ F$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lea-lta789-suma2--lta456*:
assumes $A'\ B'\ C'\ LeA\ A\ B\ C$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams-lta2-suma2--lta123*:
assumes $D'\ E'\ F'\ LtA\ D\ E\ F$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $A\ B\ C\ LtA\ A'\ B'\ C'$
⟨*proof*⟩

lemma *sams-lta2-suma2--lta456*:
assumes $A'\ B'\ C'\ LtA\ A\ B\ C$ **and**
$G\ H\ I\ LtA\ G'\ H'\ I'$ **and**
$SAMS\ A\ B\ C\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$ **and**
$A'\ B'\ C'\ D'\ E'\ F'\ SumA\ G'\ H'\ I'$
shows $D\ E\ F\ LtA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *sams123231*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$
shows $SAMS\ A\ B\ C\ B\ C\ A$
⟨*proof*⟩

lemma *col-suma--col*:
assumes $Col\ D\ E\ F$ **and**
$A\ B\ C\ B\ C\ A\ SumA\ D\ E\ F$
shows $Col\ A\ B\ C$
⟨*proof*⟩

lemma *ncol-suma--ncol*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$A\ B\ C\ B\ C\ A\ SumA\ D\ E\ F$
shows $\neg\ Col\ D\ E\ F$
⟨*proof*⟩

lemma *per2-suma--bet*:
assumes $Per\ A\ B\ C$ **and**
$Per\ D\ E\ F$ **and**
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
shows $Bet\ G\ H\ I$
⟨*proof*⟩

lemma *bet-per2--suma*:

assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
$G \neq H$ **and**
$H \neq I$ **and**
Per A B C **and**
Per D E F **and**
Bet G H I
shows *A B C D E F SumA G H I*
⟨*proof*⟩

lemma *per2--sams*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$D \neq E$ **and**
$E \neq F$ **and**
Per A B C **and**
Per D E F
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *bet-per-suma--per456*:
assumes *Per A B C* **and**
Bet G H I **and**
A B C D E F SumA G H I
shows *Per D E F*
⟨*proof*⟩

lemma *bet-per-suma--per123*:
assumes *Per D E F* **and**
Bet G H I **and**
A B C D E F SumA G H I
shows *Per A B C*
⟨*proof*⟩

lemma *bet-suma--per*:
assumes *Bet D E F* **and**
A B C A B C SumA D E F
shows *Per A B C*
⟨*proof*⟩

lemma *acute--sams*:
assumes *Acute A B C*
shows *SAMS A B C A B C*
⟨*proof*⟩

lemma *acute-suma--nbet*:
assumes *Acute A B C* **and**
A B C A B C SumA D E F
shows ¬ *Bet D E F*
⟨*proof*⟩

lemma *acute2--sams*:
assumes *Acute A B C* **and**
Acute D E F
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *acute2-suma--nbet-a*:
assumes *Acute A B C* **and**
D E F LeA A B C **and**
A B C D E F SumA G H I
shows ¬ *Bet G H I*
⟨*proof*⟩

lemma *acute2-suma--nbet*:
assumes *Acute A B C* **and**
Acute D E F **and**
A B C D E F SumA G H I
shows ¬ *Bet G H I*
⟨*proof*⟩

lemma *acute-per--sams*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C **and**
Acute D E F
shows *SAMS A B C D E F*
⟨*proof*⟩

lemma *acute-per-suma--nbet*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C **and**
Acute D E F **and**
A B C D E F SumA G H I
shows ¬ *Bet G H I*
⟨*proof*⟩

lemma *obtuse--nsams*:
assumes *Obtuse A B C*
shows ¬ *SAMS A B C A B C*
⟨*proof*⟩

lemma *nbet-sams-suma--acute*:
assumes ¬ *Bet D E F* **and**
SAMS A B C A B C **and**
A B C A B C SumA D E F
shows *Acute A B C*
⟨*proof*⟩

lemma *nsams--obtuse*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
¬ *SAMS A B C A B C*
shows *Obtuse A B C*
⟨*proof*⟩

lemma *sams2-suma2--conga*:
assumes *SAMS A B C A B C* **and**
A B C A B C SumA D E F **and**
SAMS A' B' C' A' B' C' **and**
A' B' C' A' B' C' *SumA D E F*
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *acute2-suma2--conga*:
assumes *Acute A B C* **and**
A B C A B C SumA D E F **and**
Acute A' B' C' **and**
A' B' C' A' B' C' *SumA D E F*
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *bet2-suma--out*:
assumes *Bet A B C* **and**
Bet D E F **and**
A B C D E F SumA G H I
shows *H Out G I*
⟨*proof*⟩

lemma *col2-suma--col*:
assumes *Col A B C* **and**
Col D E F **and**
A B C D E F SumA G H I
shows *Col G H I*
⟨*proof*⟩

lemma *suma-suppa--bet*:
assumes *A B C SuppA D E F* **and**
A B C D E F SumA G H I
shows *Bet G H I*
⟨*proof*⟩

lemma *bet-suppa--suma*:
assumes $G \neq H$ **and**
$H \neq I$ **and**
A B C SuppA D E F **and**
Bet G H I
shows *A B C D E F SumA G H I*
⟨*proof*⟩

lemma *bet-suma--suppa*:
assumes *A B C D E F SumA G H I* **and**
Bet G H I
shows *A B C SuppA D E F*
⟨*proof*⟩

lemma *bet2-suma--suma*:
assumes $A' \neq B$ **and**
$D' \neq E$ **and**
Bet A B A' **and**
Bet D E D' **and**
A B C D E F SumA G H I
shows A' *B C* D' *E F SumA G H I*
⟨*proof*⟩

lemma *suma-suppa2--suma*:
assumes *A B C SuppA* A' B' C' **and**
D E F SuppA D' E' F' **and**
A B C D E F SumA G H I
shows A' B' C' D' E' F' *SumA G H I*
⟨*proof*⟩

lemma *suma2-obtuse2--conga*:
assumes *Obtuse A B C* **and**
A B C A B C SumA D E F **and**
Obtuse A' B' C' **and**
A' B' C' A' B' C' *SumA D E F*
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *bet-suma2--or-conga*:
assumes $A0 \neq B$ **and**
Bet A B A0 **and**
A B C A B C SumA D E F **and**
A' B' C' A' B' C' *SumA D E F*
shows *A B C CongA* A' B' C' $\vee$ *A0 B C CongA* A' B' C'
⟨*proof*⟩

lemma *suma2--or-conga-suppa*:
assumes *A B C A B C SumA D E F* **and**
A' B' C' A' B' C' *SumA D E F*
shows *A B C CongA* A' B' C' $\vee$ *A B C SuppA* A' B' C'
⟨*proof*⟩

lemma *ex-trisuma*:

assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$
shows $\exists\ D\ E\ F.\ A\ B\ C\ TriSumA\ D\ E\ F$
⟨*proof*⟩

lemma *trisuma-perm-231*:
assumes *A B C TriSumA D E F*
shows *B C A TriSumA D E F*
⟨*proof*⟩

lemma *trisuma-perm-312*:
assumes *A B C TriSumA D E F*
shows *C A B TriSumA D E F*
⟨*proof*⟩

lemma *trisuma-perm-321*:
assumes *A B C TriSumA D E F*
shows *C B A TriSumA D E F*
⟨*proof*⟩

lemma *trisuma-perm-213*:
assumes *A B C TriSumA D E F*
shows *B A C TriSumA D E F*
⟨*proof*⟩

lemma *trisuma-perm-132*:
assumes *A B C TriSumA D E F*
shows *A C B TriSumA D E F*
⟨*proof*⟩

lemma *conga-trisuma--trisuma*:
assumes *A B C TriSumA D E F* **and**
$D\ E\ F\ CongA\ D'\ E'\ F'$
shows $A\ B\ C\ TriSumA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *trisuma2--conga*:
assumes *A B C TriSumA D E F* **and**
$A\ B\ C\ TriSumA\ D'\ E'\ F'$
shows $D\ E\ F\ CongA\ D'\ E'\ F'$
⟨*proof*⟩

lemma *conga3-trisuma--trisuma*:
assumes *A B C TriSumA D E F* **and**
$A\ B\ C\ CongA\ A'\ B'\ C'$ **and**
$B\ C\ A\ CongA\ B'\ C'\ A'$ **and**
$C\ A\ B\ CongA\ C'\ A'\ B'$
shows $A'\ B'\ C'\ TriSumA\ D\ E\ F$
⟨*proof*⟩

lemma *col-trisuma--bet*:
assumes *Col A B C* **and**
A B C TriSumA P Q R
shows *Bet P Q R*
⟨*proof*⟩

lemma *suma-dec*:
$A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I \lor \neg\ A\ B\ C\ D\ E\ F\ SumA\ G\ H\ I$
⟨*proof*⟩

lemma *sams-dec*:
$SAMS\ A\ B\ C\ D\ E\ F \lor \neg\ SAMS\ A\ B\ C\ D\ E\ F$
⟨*proof*⟩

lemma *trisuma-dec*:

A B C TriSumA P Q R ∨ ¬ *A B C TriSumA P Q R*
⟨*proof*⟩

lemma *acute-not-bet*:
assumes *Acute A B C*
shows ¬ *Bet A B C*
⟨*proof*⟩

lemma *upper-dim-3-stab*:
assumes ¬ ¬ *upper-dim-3-axiom*
shows *upper-dim-3-axiom*
⟨*proof*⟩

lemma *median-planes-implies-upper-dim*:
assumes *median-planes-axiom*
shows *upper-dim-3-axiom*
⟨*proof*⟩

lemma *median-planes-aux*:
assumes ∀ *A B C P Q M*. *P* ≠ *Q* ∧ *Cong A P A Q* ∧ *Cong B P B Q* ∧
Cong C P C Q ∧ *M Midpoint P Q* ⟶ *Coplanar M A B C*
shows *median-planes-axiom*
⟨*proof*⟩

lemma *orthonormal-family-aux-1*:
assumes *orthonormal-family-axiom*
shows ∀ *A B X P Q*. ¬ *Col P Q X* ∧ *Per A X P* ∧ *Per A X Q* ∧ *Per B X P* ∧ *Per B X Q* ⟶ *Col A B X*
⟨*proof*⟩

lemma *orthonormal-family-aux-2*:
assumes ∀ *A B X P Q*. ¬ *Col P Q X* ∧ *Per A X P* ∧ *Per A X Q* ∧
Per B X P ∧ *Per B X Q* ⟶ *Col A B X*
shows *orthonormal-family-axiom*
⟨*proof*⟩

lemma *orthonormal-family-aux*:
shows *orthonormal-family-axiom* ⟷
(∀ *A B X P Q*. ¬ *Col P Q X* ∧ *Per A X P* ∧ *Per A X Q* ∧ *Per B X P* ∧ *Per B X Q* ⟶ *Col A B X*)
⟨*proof*⟩

lemma *upper-dim-implies-orthonormal-family-axiom*:
assumes *upper-dim-3-axiom*
shows *orthonormal-family-axiom*
⟨*proof*⟩

lemma *orthonormal-family-axiom-implies-orth-at2--col*:
assumes *orthonormal-family-axiom*
shows ∀ *A B C P Q X*. *X OrthAt A B C X P* ∧ *X OrthAt A B C X Q* ⟶ *Col P Q X*
⟨*proof*⟩

lemma *orthonormal-family-axiom-implies-not-two-sides-one-side*:
assumes *orthonormal-family-axiom*
shows ∀ *A B C X Y*. ¬ *Coplanar A B C X* ∧ ¬ *Coplanar A B C Y* ∧
¬ *A B C TSP X Y* ⟶ *A B C OSP X Y*
⟨*proof*⟩

lemma *orthonormal-family-axiom-implies-space-separation*:
assumes *orthonormal-family-axiom*
shows *space-separation-axiom*
⟨*proof*⟩

lemma *space-separation-implies-plane-intersection*:
assumes *space-separation-axiom*
shows *plane-intersection-axiom*
⟨*proof*⟩

lemma *plane-intersection-implies-space-separation*:
  assumes *plane-intersection-axiom*
  shows *space-separation-axiom*
⟨*proof*⟩

lemma *space-separation-implies-median-planes*:
  assumes *space-separation-axiom*
  shows *median-planes-axiom*
⟨*proof*⟩

theorem *upper-dim-3-equivalent-axioms*:
  shows (*upper-dim-3-axiom* ⟷ *orthonormal-family-axiom*) ∧
    (*orthonormal-family-axiom* ⟷ *space-separation-axiom*) ∧
    (*space-separation-axiom* ⟷ *plane-intersection-axiom*) ∧
    (*plane-intersection-axiom* ⟷ *median-planes-axiom*)
  ⟨*proof*⟩

lemma *par-reflexivity*:
  assumes $A \neq B$
  shows *A B Par A B*
  ⟨*proof*⟩

lemma *par-strict-irreflexivity*:
  ¬ *A B ParStrict A B*
  ⟨*proof*⟩

lemma *not-par-strict-id*:
  ¬ *A B ParStrict A C*
  ⟨*proof*⟩

lemma *par-id*:
  assumes *A B Par A C*
  shows *Col A B C*
  ⟨*proof*⟩

lemma *par-strict-not-col-1*:
  assumes *A B ParStrict C D*
  shows ¬ *Col A B C*
  ⟨*proof*⟩

lemma *par-strict-not-col-2*:
  assumes *A B ParStrict C D*
  shows ¬ *Col B C D*
  ⟨*proof*⟩

lemma *par-strict-not-col-3*:
  assumes *A B ParStrict C D*
  shows ¬ *Col C D A*
  ⟨*proof*⟩

lemma *par-strict-not-col-4*:
  assumes *A B ParStrict C D*
  shows ¬ *Col A B D*
  ⟨*proof*⟩

lemma *par-id-1*:
  assumes *A B Par A C*
  shows *Col B A C*
  ⟨*proof*⟩

lemma *par-id-2*:
  assumes *A B Par A C*
  shows *Col B C A*
  ⟨*proof*⟩

lemma *par-id-3*:

assumes *A B Par A C*
shows *Col A C B*
⟨*proof*⟩

lemma *par-id-4*:
assumes *A B Par A C*
shows *Col C B A*
⟨*proof*⟩

lemma *par-id-5*:
assumes *A B Par A C*
shows *Col C A B*
⟨*proof*⟩

lemma *par-strict-symmetry*:
assumes *A B ParStrict C D*
shows *C D ParStrict A B*
⟨*proof*⟩

lemma *par-symmetry*:
assumes *A B Par C D*
shows *C D Par A B*
⟨*proof*⟩

lemma *par-left-comm*:
assumes *A B Par C D*
shows *B A Par C D*
⟨*proof*⟩

lemma *par-right-comm*:
assumes *A B Par C D*
shows *A B Par D C*
⟨*proof*⟩

lemma *par-comm*:
assumes *A B Par C D*
shows *B A Par D C*
⟨*proof*⟩

lemma *par-strict-left-comm*:
assumes *A B ParStrict C D*
shows *B A ParStrict C D*
⟨*proof*⟩

lemma *par-strict-right-comm*:
assumes *A B ParStrict C D*
shows *A B ParStrict D C*
⟨*proof*⟩

lemma *par-strict-comm*:
assumes *A B ParStrict C D*
shows *B A ParStrict D C*
⟨*proof*⟩

lemma *par-strict-neq1*:
assumes *A B ParStrict C D*
shows $A \neq B$
⟨*proof*⟩

lemma *par-strict-neq2*:
assumes *A B ParStrict C D*
shows $C \neq D$
⟨*proof*⟩

lemma *par-neq1*:
assumes *A B Par C D*

shows *A* ≠ *B*
⟨*proof*⟩

lemma *par-neq2*:
assumes *A B Par C D*
shows *C* ≠ *D*
⟨*proof*⟩

lemma *Par-cases*:
assumes *A B Par C D* ∨ *B A Par C D* ∨ *A B Par D C* ∨ *B A Par D C* ∨
C D Par A B ∨ *C D Par B A* ∨ *D C Par A B* ∨ *D C Par B A*
shows *A B Par C D*
⟨*proof*⟩

lemma *Par-perm*:
assumes *A B Par C D*
shows *A B Par C D* ∧ *B A Par C D* ∧ *A B Par D C* ∧ *B A Par D C* ∧
C D Par A B ∧ *C D Par B A* ∧ *D C Par A B* ∧ *D C Par B A*
⟨*proof*⟩

lemma *Par-strict-cases*:
assumes *A B ParStrict C D* ∨ *B A ParStrict C D* ∨ *A B ParStrict D C* ∨
B A ParStrict D C ∨ *C D ParStrict A B* ∨ *C D ParStrict B A* ∨
D C ParStrict A B ∨ *D C ParStrict B A*
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *Par-strict-perm*:
assumes *A B ParStrict C D*
shows *A B ParStrict C D* ∧ *B A ParStrict C D* ∧ *A B ParStrict D C* ∧
B A ParStrict D C ∧ *C D ParStrict A B* ∧ *C D ParStrict B A* ∧
D C ParStrict A B ∧ *D C ParStrict B A*
⟨*proof*⟩

lemma *l12-6*:
assumes *A B ParStrict C D*
shows *A B OS C D*
⟨*proof*⟩

lemma *pars--os3412*:
assumes *A B ParStrict C D*
shows *C D OS A B*
⟨*proof*⟩

lemma *perp-dec*:
A B Perp C D ∨ ¬ *A B Perp C D*
⟨*proof*⟩

lemma *col-cop2-perp2--col*:
assumes *X1 X2 Perp A B* **and**
Y1 Y2 Perp A B **and**
Col X1 Y1 Y2 **and**
Coplanar A B X2 Y1 **and**
Coplanar A B X2 Y2
shows *Col X2 Y1 Y2*
⟨*proof*⟩

lemma *col-perp2-ncol-col*:
assumes *X1 X2 Perp A B* **and**
Y1 Y2 Perp A B **and**
Col X1 Y1 Y2 **and**
¬ *Col X1 A B*
shows *Col X2 Y1 Y2*
⟨*proof*⟩

lemma *l12-9*:

assumes
Coplanar C1 C2 A1 B1 **and**
Coplanar C1 C2 A1 B2 **and**
Coplanar C1 C2 A2 B1 **and**
Coplanar C1 C2 A2 B2 **and**
A1 A2 Perp C1 C2 **and**
B1 B2 Perp C1 C2
shows *A1 A2 Par B1 B2*
⟨*proof*⟩

lemma *parallel-existence*:
assumes $A \neq B$
shows $\exists$ *C D*. $C \neq D \wedge$ *A B Par C D* $\wedge$ *Col P C D*
⟨*proof*⟩

lemma *par-col-par*:
assumes $C \neq D'$ **and**
A B Par C D **and**
Col C D D'
shows *A B Par C* D'
⟨*proof*⟩

lemma *parallel-existence1*:
assumes $A \neq B$
shows $\exists$ *Q*. *A B Par P Q*
⟨*proof*⟩

lemma *par-not-col*:
assumes *A B ParStrict C D* **and**
Col X A B
shows ¬ *Col X C D*
⟨*proof*⟩

lemma *not-strict-par1*:
assumes *A B Par C D* **and**
Col A B X **and**
Col C D X
shows *Col A B C*
⟨*proof*⟩

lemma *not-strict-par2*:
assumes *A B Par C D* **and**
Col A B X **and**
Col C D X
shows *Col A B D*
⟨*proof*⟩

lemma *not-strict-par*:
assumes *A B Par C D* **and**
Col A B X **and**
Col C D X
shows *Col A B C* $\wedge$ *Col A B D*
⟨*proof*⟩

lemma *not-par-not-col*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
¬ *A B Par A C*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *not-par-inter-uniqueness*:
assumes $A \neq B$ **and**
$C \neq D$ **and**
¬ *A B Par C D* **and**
Col A B X **and**

Col C D X **and**
Col A B Y **and**
Col C D Y
shows *X = Y*
⟨*proof*⟩

lemma *inter-uniqueness-not-par*:
assumes ¬ *Col A B C* **and**
Col A B P **and**
Col C D P
shows ¬ *A B Par C D*
⟨*proof*⟩

lemma *col-not-col-not-par*:
assumes ∃ *P*. *Col A B P* ∧ *Col C D P* **and**
∃ *Q*. *Col C D Q* ∧ ¬*Col A B Q*
shows ¬ *A B Par C D*
⟨*proof*⟩

lemma *par-distincts*:
assumes *A B Par C D*
shows *A B Par C D* ∧ *A* ≠ *B* ∧ *C* ≠ *D*
⟨*proof*⟩

lemma *par-not-col-strict*:
assumes *A B Par C D* **and**
Col C D P **and**
¬ *Col A B P*
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *col-cop-perp2--pars*:
assumes ¬ *Col A B P* **and**
Col C D P **and**
Coplanar A B C D **and**
A B Perp P Q **and**
C D Perp P Q
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *all-one-side-par-strict*:
assumes *C* ≠ *D* **and**
∀ *P*. *Col C D P* ⟶ *A B OS C P*
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *par-col-par-2*:
assumes *A* ≠ *P* **and**
Col A B P **and**
A B Par C D
shows *A P Par C D*
⟨*proof*⟩

lemma *par-col2-par*:
assumes *E* ≠ *F* **and**
A B Par C D **and**
Col C D E **and**
Col C D F
shows *A B Par E F*
⟨*proof*⟩

lemma *par-col2-par-bis*:
assumes *E* ≠ *F* **and**
A B Par C D **and**
Col E F C **and**
Col E F D

shows *A B Par E F*
⟨*proof*⟩

lemma *par-strict-col-par-strict*:
assumes $C \neq E$ **and**
A B ParStrict C D **and**
Col C D E
shows *A B ParStrict C E*
⟨*proof*⟩

lemma *par-strict-col2-par-strict*:
assumes $E \neq F$ **and**
A B ParStrict C D **and**
Col C D E **and**
Col C D F
shows *A B ParStrict E F*
⟨*proof*⟩

lemma *line-dec*:
(*Col C1 B1 B2* ∧ *Col C2 B1 B2*) ∨ ¬ (*Col C1 B1 B2* ∧ *Col C2 B1 B2*)
⟨*proof*⟩

lemma *par-distinct*:
assumes *A B Par C D*
shows $A \neq B \land C \neq D$
⟨*proof*⟩

lemma *par-col4--par*:
assumes $E \neq F$ **and**
$G \neq H$ **and**
A B Par C D **and**
Col A B E **and**
Col A B F **and**
Col C D G **and**
Col C D H
shows *E F Par G H*
⟨*proof*⟩

lemma *par-strict-col4--par-strict*:
assumes $E \neq F$ **and**
$G \neq H$ **and**
A B ParStrict C D **and**
Col A B E **and**
Col A B F **and**
Col C D G **and**
Col C D H
shows *E F ParStrict G H*
⟨*proof*⟩

lemma *par-strict-one-side*:
assumes *A B ParStrict C D* **and**
Col C D P
shows *A B OS C P*
⟨*proof*⟩

lemma *par-strict-all-one-side*:
assumes *A B ParStrict C D*
shows ∀ *P*. *Col C D P* ⟶ *A B OS C P*
⟨*proof*⟩

lemma *inter-trivial*:
assumes ¬ *Col A B X*
shows *X Inter A X B X*
⟨*proof*⟩

lemma *inter-sym*:

assumes *X Inter A B C D*
shows *X Inter C D A B*
⟨*proof*⟩

lemma *inter-left-comm*:
assumes *X Inter A B C D*
shows *X Inter B A C D*
⟨*proof*⟩

lemma *inter-right-comm*:
assumes *X Inter A B C D*
shows *X Inter A B D C*
⟨*proof*⟩

lemma *inter-comm*:
assumes *X Inter A B C D*
shows *X Inter B A D C*
⟨*proof*⟩

lemma *l12-17*:
assumes $A \neq B$ **and**
P Midpoint A C **and**
P Midpoint B D
shows *A B Par C D*
⟨*proof*⟩

lemma *l12-18-a*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *A B Par C D*
⟨*proof*⟩

lemma *l12-18-b*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *B C Par D A*
⟨*proof*⟩

lemma *l12-18-c*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *B D TS A C*
⟨*proof*⟩

lemma *l12-18-d*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows *A C TS B D*

⟨*proof*⟩

lemma *l12-18*:
assumes *Cong A B C D* **and**
Cong B C D A **and**
¬ *Col A B C* **and**
$B \neq D$ **and**
Col A P C **and**
Col B P D
shows $A\ B\ Par\ C\ D \wedge B\ C\ Par\ D\ A \wedge B\ D\ TS\ A\ C \wedge A\ C\ TS\ B\ D$
⟨*proof*⟩

lemma *par-two-sides-two-sides*:
assumes *A B Par C D* **and**
B D TS A C
shows *A C TS B D*
⟨*proof*⟩

lemma *par-one-or-two-sides*:
assumes *A B ParStrict C D*
shows $(A\ C\ TS\ B\ D \wedge B\ D\ TS\ A\ C) \vee (A\ C\ OS\ B\ D \wedge B\ D\ OS\ A\ C)$
⟨*proof*⟩

lemma *l12-21-b*:
assumes *A C TS B D* **and**
B A C CongA D C A
shows *A B Par C D*
⟨*proof*⟩

lemma *l12-22-aux*:
assumes $P \neq A$ **and**
$A \neq C$ **and**
Bet P A C **and**
P A OS B D **and**
B A P CongA D C P
shows *A B Par C D*
⟨*proof*⟩

lemma *l12-22-b*:
assumes *P Out A C* **and**
P A OS B D **and**
B A P CongA D C P
shows *A B Par C D*
⟨*proof*⟩

lemma *par-strict-par*:
assumes *A B ParStrict C D*
shows *A B Par C D*
⟨*proof*⟩

lemma *par-strict-distinct*:
assumes *A B ParStrict C D*
shows $A \neq B \wedge C \neq D$
⟨*proof*⟩

lemma *col-par*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Col A B C
shows *A B Par B C*
⟨*proof*⟩

lemma *acute-col-perp--out*:
assumes *Acute A B C* **and**
Col B C A' **and**
B C Perp A A'

shows *B Out A′ C*
⟨*proof*⟩

lemma *acute-col-perp--out-1*:
assumes *Acute A B C* **and**
Col B C A′ **and**
B A Perp A A′
shows *B Out A′ C*
⟨*proof*⟩

lemma *conga-inangle-per2--inangle*:
assumes *Per A B C* **and**
T InAngle A B C **and**
P B A CongA P B C **and**
Per B P T **and**
Coplanar A B C P
shows *P InAngle A B C*
⟨*proof*⟩

lemma *perp-not-par*:
assumes *A B Perp X Y*
shows ¬ *A B Par X Y*
⟨*proof*⟩

lemma *cong-conga-perp*:
assumes *B P TS A C* **and**
Cong A B C B **and**
A B P CongA C B P
shows *A C Perp B P*
⟨*proof*⟩

lemma *perp-inter-exists*:
assumes *A B Perp C D*
shows ∃ *P*. *Col A B P* ∧ *Col C D P*
⟨*proof*⟩

lemma *perp-inter-perp-in*:
assumes *A B Perp C D*
shows ∃ *P*. *Col A B P* ∧ *Col C D P* ∧ *P PerpAt A B C D*
⟨*proof*⟩

lemma *cop-npars--inter-exists*:
assumes *Coplanar A1 B1 A2 B2* **and**
¬ *A1 B1 ParStrict A2 B2*
shows ∃ *X*. *Col X A1 B1* ∧ *Col X A2 B2*
⟨*proof*⟩

lemma *cop-npar--inter-exists*:
assumes *Coplanar A1 B1 A2 B2* **and**
¬ *A1 B1 Par A2 B2*
shows ∃ *X*. *Col X A1 B1* ∧ *Col X A2 B2*
⟨*proof*⟩

lemma *cop-npar--inter*:
assumes *A1* ≠ *B1* **and**
A2 ≠ *B2* **and**
Coplanar A1 B1 A2 B2 **and**
¬ *A1 B1 Par A2 B2*
shows ∃ *X*. *X Inter A1 B1 A2 B2*
⟨*proof*⟩

lemma *inter--npar*:
assumes *X Inter A1 A2 B1 B2*
shows ¬ *A1 A2 Par B1 B2*
⟨*proof*⟩

lemma *cong-identity-inv*:
assumes $A \neq B$
shows $\neg$ *Cong* $A\ B\ C\ C$
⟨*proof*⟩

lemma *midpoint-midpoint-col*:
assumes $A \neq B$ **and**
M Midpoint $A\ A'$ **and**
M Midpoint $B\ B'$ **and**
Col $A\ B\ B'$
shows $A' \neq B' \wedge$ *Col* $A\ A'\ B' \wedge$ *Col* $B\ A'\ B'$
⟨*proof*⟩

lemma *midpoint-par-strict*:
assumes $\neg$ *Col* $A\ B\ B'$ **and**
M Midpoint $A\ A'$ **and**
M Midpoint $B\ B'$
shows $A\ B$ *ParStrict* $A'\ B'$
⟨*proof*⟩

lemma *bet3-cong3-bet*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$A \neq D$ **and**
Bet $D\ A\ C$ **and**
Bet $A\ C\ B$ **and**
Bet $D\ C\ D'$ **and**
Cong $A\ B\ C\ D$ **and**
Cong $A\ D\ B\ C$ **and**
Cong $D\ C\ C\ D'$
shows *Bet* $C\ B\ D'$
⟨*proof*⟩

lemma *bet-double-bet*:
assumes B' *Midpoint* $A\ B$ **and**
C' *Midpoint* $A\ C$ **and**
Bet $A\ B'\ C'$
shows *Bet* $A\ B\ C$
⟨*proof*⟩

lemma *bet-half-bet*:
assumes *Bet* $A\ B\ C$ **and**
B' *Midpoint* $A\ B$ **and**
C' *Midpoint* $A\ C$
shows *Bet* $A\ B'\ C'$
⟨*proof*⟩

lemma *midpoint-preserves-bet*:
assumes B' *Midpoint* $A\ B$ **and**
C' *Midpoint* $A\ C$
shows *Bet* $A\ B\ C \longleftrightarrow$ *Bet* $A\ B'\ C'$
⟨*proof*⟩

lemma *symmetry-preseves-bet1*:
assumes *M Midpoint* $A\ A'$ **and**
M Midpoint $B\ B'$ **and**
Bet $M\ A\ B$
shows *Bet* $M\ A'\ B'$
⟨*proof*⟩

lemma *symmetry-preseves-bet2*:
assumes *M Midpoint* $A\ A'$ **and**
M Midpoint $B\ B'$ **and**
Bet $M\ A'\ B'$
shows *Bet* $M\ A\ B$
⟨*proof*⟩

lemma *symmetry-preserves-bet*:
  assumes M *Midpoint* $A\ A'$ **and**
    M *Midpoint* $B\ B'$
  shows *Bet* $M\ A'\ B' \longleftrightarrow$ *Bet* $M\ A\ B$
  ⟨*proof*⟩

lemma *bet-cong-bet*:
  assumes $A \neq B$ **and**
    Bet $A\ B\ C$ **and**
    Bet $A\ B\ D$ **and**
    Cong $A\ C\ B\ D$
  shows *Bet* $B\ C\ D$
  ⟨*proof*⟩

lemma *col-cong-mid* :
  assumes $A\ B$ *Par* $A'\ B'$ **and**
    $\neg$ $A\ B$ *ParStrict* $A'\ B'$ **and**
    Cong $A\ B\ A'\ B'$
  shows $\exists\ M.$ ((M *Midpoint* $A\ A' \wedge M$ *Midpoint* $B\ B'$) $\vee$
(M *Midpoint* $A\ B' \wedge M$ *Midpoint* $B\ A'$))
⟨*proof*⟩

lemma *mid-par-cong1*:
  assumes $A \neq B$ **and**
    M *Midpoint* $A\ A'$ **and**
    M *Midpoint* $B\ B'$
  shows *Cong* $A\ B\ A'\ B' \wedge A\ B$ *Par* $A'\ B'$
  ⟨*proof*⟩

lemma *mid-par-cong2*:
  assumes $A \neq B'$ **and**
    M *Midpoint* $A\ A'$ **and**
    M *Midpoint* $B\ B'$
  shows *Cong* $A\ B'\ A'\ B \wedge A\ B'$ *Par* $A'\ B$
  ⟨*proof*⟩

lemma *mid-par-cong*:
  assumes $A \neq B$ **and**
    $A \neq B'$ **and**
    M *Midpoint* $A\ A'$ **and**
    M *Midpoint* $B\ B'$
  shows *Cong* $A\ B\ A'\ B' \wedge$ *Cong* $A\ B'\ A'\ B \wedge A\ B$ *Par* $A'\ B' \wedge A\ B'$ *Par* $A'\ B$
  ⟨*proof*⟩

lemma *Parallelogram-strict-Parallelogram*:
  assumes *ParallelogramStrict* $A\ B\ C\ D$
  shows *Parallelogram* $A\ B\ C\ D$
  ⟨*proof*⟩

lemma *plgf-permut*:
  assumes *ParallelogramFlat* $A\ B\ C\ D$
  shows *ParallelogramFlat* $B\ C\ D\ A$
⟨*proof*⟩

lemma *plgf-sym*:
  assumes *ParallelogramFlat* $A\ B\ C\ D$
  shows *ParallelogramFlat* $C\ D\ A\ B$
  ⟨*proof*⟩

lemma *plgf-irreflexive*:
  shows $\neg$ *ParallelogramFlat* $A\ B\ A\ B$
  ⟨*proof*⟩

lemma *plgs-irreflexive*:
  shows $\neg$ *ParallelogramStrict* $A\ B\ A\ B$

⟨*proof*⟩

lemma *plg-irreflexive*:
  shows ¬ *Parallelogram A B A B*
⟨*proof*⟩

lemma *plgf-mid*:
  assumes *ParallelogramFlat A B C D*
  shows ∃ *M*. *M Midpoint A C* ∧ *M Midpoint B D*
⟨*proof*⟩

lemma *mid-plgs*:
  assumes ¬ *Col A B C* **and**
    M Midpoint A C **and**
    M Midpoint B D
  shows *ParallelogramStrict A B C D*
⟨*proof*⟩

lemma *mid-plgf-aux*:
  assumes *A* ≠ *C* **and**
    Col A B C **and**
    M Midpoint A C **and**
    M Midpoint B D
  shows *ParallelogramFlat A B C D*
⟨*proof*⟩

lemma *mid-plgf-1*:
  assumes *A* ≠ *C*

    Col A B C **and**
    M Midpoint A C **and**
    M Midpoint B D
  shows *ParallelogramFlat A B C D*
  ⟨*proof*⟩

lemma *mid-plgf-2*:
  assumes *B* ≠ *D*
    Col A B C **and**
    M Midpoint A C **and**
    M Midpoint B D
  shows *ParallelogramFlat A B C D*
  ⟨*proof*⟩

lemma *mid-plgf*:
  assumes *A* ≠ *C* ∨ *B* ≠ *D*
    Col A B C **and**
    M Midpoint A C **and**
    M Midpoint B D
  shows *ParallelogramFlat A B C D*
  ⟨*proof*⟩

lemma *mid-plg*:
  assumes *A* ≠ *C* ∨ *B* ≠ *D* **and**
    M Midpoint A C **and**
    M Midpoint B D
  shows *Parallelogram A B C D*
  ⟨*proof*⟩

lemma *midpoint-cong-uniqueness*:
  assumes *Col A B C* **and**
    M Midpoint A B **and**
    M Midpoint C D **and**
    Cong A B C D
  shows *A* = *C* ∧ *B* = *D* ∨ *A* = *D* ∧ *B* = *C*
⟨*proof*⟩

lemma *plgf-not-comm1*:
assumes $A \neq B$ **and**
ParallelogramFlat A B C D
shows ¬ *ParallelogramFlat A B D C*
⟨*proof*⟩

lemma *plgf-not-comm2*:
assumes $A \neq B$ **and**
ParallelogramFlat A B C D
shows ¬ *ParallelogramFlat B A C D*
⟨*proof*⟩

lemma *plgf-not-comm*:
assumes $A \neq B$ **and**
ParallelogramFlat A B C D
shows ¬ *ParallelogramFlat A B D C* ∧ ¬ *ParallelogramFlat B A C D*
⟨*proof*⟩

lemma *plgf-cong*:
assumes *ParallelogramFlat A B C D*
shows *Cong A B C D* ∧ *Cong A D B C*
⟨*proof*⟩

lemma *plg-to-parallelogram*:
assumes *Plg A B C D*
shows *Parallelogram A B C D*
⟨*proof*⟩

lemma *plgs-one-side*:
assumes *ParallelogramStrict A B C D*
shows *A B OS C D* ∧ *C D OS A B*
⟨*proof*⟩

lemma *parallelogram-strict-not-col*:
assumes *ParallelogramStrict A B C D*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *parallelogram-strict-not-col-2*:
assumes *ParallelogramStrict A B C D*
shows ¬ *Col B C D*
⟨*proof*⟩

lemma *parallelogram-strict-not-col-3*:
assumes *ParallelogramStrict A B C D*
shows ¬ *Col C D A*
⟨*proof*⟩

lemma *parallelogram-strict-not-col-4*:
assumes *ParallelogramStrict A B C D*
shows ¬ *Col A B D*
⟨*proof*⟩

lemma *plgs--pars*:
assumes *ParallelogramStrict A B C D*
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *plgs-sym*:
assumes *ParallelogramStrict A B C D*
shows *ParallelogramStrict C D A B*
⟨*proof*⟩

lemma *plg-sym*:
assumes *Parallelogram A B C D*
shows *Parallelogram C D A B*

⟨proof⟩

lemma *Rhombus-Plg*:
assumes *Rhombus A B C D*
shows *Plg A B C D*
⟨proof⟩

lemma *Rectangle-Plg*:
assumes *Rectangle A B C D*
shows *Plg A B C D*
⟨proof⟩

lemma *Rectangle-Parallelogram*:
assumes *Rectangle A B C D*
shows *Parallelogram A B C D*
⟨proof⟩

lemma *plg-cong-rectangle*:
assumes *Plg A B C D* **and**
Cong A C B D
shows *Rectangle A B C D*
⟨proof⟩

lemma *plg-trivial*:
assumes $A \neq B$
shows *Parallelogram A B B A*
⟨proof⟩

lemma *plg-trivial1*:
assumes $A \neq B$
shows *Parallelogram A A B B*
⟨proof⟩

lemma *col-not-plgs*:
assumes *Col A B C*
shows ¬*ParallelogramStrict A B C D*
⟨proof⟩

lemma *plg-col-plgf*:
assumes *Col A B C* **and**
Parallelogram A B C D
shows *ParallelogramFlat A B C D*
⟨proof⟩

lemma *plg-bet1*:
assumes *Parallelogram A B C D* **and**
Bet A C B
shows *Bet D A C*
⟨proof⟩

lemma *plgf-trivial1*:
assumes $A \neq B$
shows *ParallelogramFlat A B B A*
⟨proof⟩

lemma *plgf-trivial2*:
assumes $A \neq B$
shows *ParallelogramFlat A A B B*
⟨proof⟩

lemma *plgf-not-point*:
assumes *ParallelogramFlat A A B B*
shows $A \neq B$
⟨proof⟩

lemma *plgf-trivial-neq*:

assumes *ParallelogramFlat A A C D*
shows $C = D \wedge A \neq C$
⟨*proof*⟩

lemma *plgf-trivial-trans*:
assumes *ParallelogramFlat A A B B* **and**
ParallelogramFlat B B C C
shows *ParallelogramFlat A A C C* $\vee A = C$
⟨*proof*⟩

lemma *plgf-trivial*:
assumes $A \neq B$
shows *ParallelogramFlat A B B A*
⟨*proof*⟩

lemma *plgf3-mid*:
assumes *ParallelogramFlat A B A C*
shows *A Midpoint B C*
⟨*proof*⟩

lemma *cong3-id*:
assumes $A \neq B$ **and**
Col A B C **and**
Col A B D **and**
Cong A B C D **and**
Cong A D B C **and**
Cong A C B D
shows $A = D \wedge B = C \vee A = C \wedge B = D$
⟨*proof*⟩

lemma *col-cong-mid1*:
assumes $A \neq D$ **and**
Col A B C **and**
Col A B D **and**
Cong A B C D **and**
Cong A C B D
shows $\exists$ *M. M Midpoint A D* $\wedge$ *M Midpoint B C*
⟨*proof*⟩

lemma *col-cong-mid2*:
assumes $A \neq C$ **and**
Col A B C **and**
Col A B D **and**
Cong A B C D **and**
Cong A D B C
shows $\exists$ *M. M Midpoint A C* $\wedge$ *M Midpoint B D*
⟨*proof*⟩

lemma *plgs-not-col*:
assumes *ParallelogramStrict A B C D*
shows $\neg$ *Col A B C* $\wedge \neg$ *Col B C D* $\wedge \neg$ *Col C D A* $\wedge \neg$ *Col A B D*
⟨*proof*⟩

lemma *not-col-sym-not-col*:
assumes $\neg$ *Col A B C* **and**
A Midpoint B B'
shows $\neg$ *Col A B' C*
⟨*proof*⟩

lemma *plg-existence*:
assumes $A \neq B$
shows $\exists$ *D. Parallelogram A B C D*
⟨*proof*⟩

lemma *plgs-diff*:
assumes *ParallelogramStrict A B C D*

shows $ParallelogramStrict\ A\ B\ C\ D \wedge A \neq B \wedge B \neq C \wedge C \neq D \wedge D \neq A \wedge A \neq C \wedge B \neq D$
⟨*proof*⟩

lemma *sym-par*:
assumes $A \neq B$ **and**
$M\ Midpoint\ A\ A'$ **and**
$M\ Midpoint\ B\ B'$
shows $A\ B\ Par\ A'\ B'$
⟨*proof*⟩

lemma *symmetry-preserves-two-sides*:
assumes $Col\ X\ Y\ M$ **and**
$X\ Y\ TS\ A\ B$ **and**
$M\ Midpoint\ A\ A'$ **and**
$M\ Midpoint\ B\ B'$
shows $X\ Y\ TS\ A'\ B'$
⟨*proof*⟩

lemma *symmetry-preserves-one-side*:
assumes $Col\ X\ Y\ M$ **and**
$X\ Y\ OS\ A\ B$ **and**
$M\ Midpoint\ A\ A'$ **and**
$M\ Midpoint\ B\ B'$
shows $X\ Y\ OS\ A'\ B'$
⟨*proof*⟩

lemma *plgf-bet*:
assumes $ParallelogramFlat\ A\ B\ B'\ A'$
shows $Bet\ A'\ B'\ A \wedge Bet\ B'\ A\ B$
$\vee\ Bet\ A'\ A\ B' \wedge Bet\ A\ B'\ B$
$\vee\ Bet\ A\ A'\ B \wedge Bet\ A'\ B\ B'$
$\vee\ Bet\ A\ B\ A' \wedge Bet\ B\ A'\ B'$
⟨*proof*⟩

lemma *plgs-existence*:
assumes $A \neq B$
shows $\exists\ C.\ \exists\ D.\ ParallelogramStrict\ A\ B\ C\ D$
⟨*proof*⟩

lemma *Rectangle-not-triv*:
shows $\neg\ Rectangle\ A\ A\ A\ A$
⟨*proof*⟩

lemma *Plg-triv*:
assumes $A \neq B$
shows $Plg\ A\ A\ B\ B$
⟨*proof*⟩

lemma *Rectangle-triv*:
assumes $A \neq B$
shows $Rectangle\ A\ A\ B\ B$
⟨*proof*⟩

lemma *Rectangle-not-triv-2*:
shows $\neg\ Rectangle\ A\ B\ A\ B$
⟨*proof*⟩

lemma *Square-not-triv*:
shows $\neg\ Square\ A\ A\ A\ A$
⟨*proof*⟩

lemma *Square-not-triv-2*:
shows $\neg\ Square\ A\ A\ B\ B$
⟨*proof*⟩

lemma *Square-not-triv-3*:

shows ¬ *Square A B A B*
⟨*proof*⟩

lemma *Square-Rectangle*:
assumes *Square A B C D*
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *Square-Parallelogram*:
assumes *Square A B C D*
shows *Parallelogram A B C D*
⟨*proof*⟩

lemma *Rhombus-Rectangle-Square*:
assumes *Rhombus A B C D* **and**
Rectangle A B C D
shows *Square A B C D*
⟨*proof*⟩

lemma *rhombus-cong-square*:
assumes *Rhombus A B C D* **and**
Cong A C B D
shows *Square A B C D*
⟨*proof*⟩

lemma *Kite-comm*:
assumes *Kite A B C D*
shows *Kite C D A B*
⟨*proof*⟩

lemma *per2-col-eq*:
assumes $A \neq P$ **and**
$A \neq P'$ **and**
Per A P B **and**
Per A P' B **and**
Col P A P'
shows $P = P'$
⟨*proof*⟩

lemma *per2-preserves-diff*:
assumes $PO \neq A'$ **and**
$PO \neq B'$ **and**
Col PO A' B' **and**
Per PO A' A **and**
Per PO B' B **and**
$A' \neq B'$
shows $A \neq B$
⟨*proof*⟩

lemma *per23-preserves-bet*:
assumes *Bet A B C* **and**
$A \neq B'$ **and** $A \neq C'$ **and**
Col A B' C' **and**
Per A B' B **and**
Per A C' C
shows *Bet A B' C'*
⟨*proof*⟩

lemma *per23-preserves-bet-inv*:
assumes *Bet A B' C'* **and**
$A \neq B'$ **and**
Col A B C **and**
Per A B' B **and**
Per A C' C
shows *Bet A B C*
⟨*proof*⟩

lemma *per13-preserves-bet*:
  assumes *Bet A B C* **and**
    $B \neq A'$ **and**
    $B \neq C'$ **and**
    Col A' *B* C' **and**
    Per B A' *A* **and**
    Per B C' *C*
  shows *Bet* A' *B* C'
  ⟨*proof*⟩

lemma *per13-preserves-bet-inv*:
  assumes *Bet* A' *B* C' **and**
    $B \neq A'$ **and**
    $B \neq C'$ **and**
    Col A B C **and**
    Per B A' *A* **and**
    Per B C' *C*
  shows *Bet A B C*
  ⟨*proof*⟩

lemma *per3-preserves-bet1*:
  assumes *Col PO A B* **and**
    Bet A B C **and**
    $PO \neq A'$ **and**
    $PO \neq B'$ **and**
    $PO \neq C'$ **and**
    Per PO A' *A* **and**
    Per PO B' *B* **and**
    Per PO C' *C* **and**
    Col A' B' C' **and**
    Col PO A' B'
  shows *Bet* A' B' C'
⟨*proof*⟩

lemma *per3-preserves-bet2-aux*:
  assumes *Col PO A C* **and**
    $A \neq C'$ **and**
    Bet A B' C' **and**
    $PO \neq A$ **and**
    $PO \neq B'$ **and**
    $PO \neq C'$ **and**
    Per PO B' *B* **and**
    Per PO C' *C* **and**
    Col A B C **and**
    Col PO A C'
  shows *Bet A B C*
⟨*proof*⟩

lemma *per3-preserves-bet2*:
  assumes *Col PO A C* **and**
    $A' \neq C'$ **and**
    Bet A' B' C' **and**
    $PO \neq A'$ **and**
    $PO \neq B'$ **and**
    $PO \neq C'$ **and**
    Per PO A' *A* **and**
    Per PO B' *B* **and**
    Per PO C' *C* **and**
    Col A B C **and**
    Col PO A' C'
  shows *Bet A B C*
⟨*proof*⟩

lemma *symmetry-preserves-per*:
  assumes *Per B P A* **and**

B Midpoint A A′ **and**
B Midpoint P P′
shows *Per B P′ A′*
⟨*proof*⟩

lemma *l13-1-aux*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows
$\exists$ *X Y*. (*R PerpAt X Y A B* $\wedge$ *X Y Perp P Q* $\wedge$ *Coplanar A B C X* $\wedge$ *Coplanar A B C Y*)
⟨*proof*⟩

lemma *l13-1*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows
$\exists$ *X Y*.(*R PerpAt X Y A B* $\wedge$ *X Y Perp P Q*)
⟨*proof*⟩

lemma *per-lt*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
Per A B C
shows *A B Lt A C* $\wedge$ *C B Lt A C*
⟨*proof*⟩

lemma *cong-perp-conga*:
assumes *Cong A B C B* **and**
A C Perp B P
shows *A B P CongA C B P* $\wedge$ *B P TS A C*
⟨*proof*⟩

lemma *perp-per-bet*:
assumes ¬ *Col A B C* **and**

Per A B C **and**
P PerpAt P B A C
shows *Bet A P C*
⟨*proof*⟩

lemma *ts-per-per-ts*:
assumes *A B TS C D* **and**
Per B C A **and**
Per B D A
shows *C D TS A B*
⟨*proof*⟩

lemma *l13-2-1*:
assumes *A B TS C D* **and**
Per B C A **and**
Per B D A **and**
Col C D E **and**
A E Perp C D **and**
C A B CongA D A B
shows *B A C CongA D A E* $\wedge$ *B A D CongA C A E* $\wedge$ *Bet C E D*
⟨*proof*⟩

lemma *triangle-mid-par-lem*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C
shows *A B Par P Q*

⟨*proof*⟩

lemma *triangle-mid-par*:
  assumes ¬ *Col A B C* **and**
    P Midpoint B C **and**
    Q Midpoint A C
  shows *A B ParStrict Q P*
⟨*proof*⟩

lemma *cop4-perp-in2--col*:
  assumes *Coplanar X Y A A′* **and**
    Coplanar X Y A B′ **and**
    Coplanar X Y B A′ **and**
    Coplanar X Y B B′ **and**
    P PerpAt A B X Y **and**
    P PerpAt A′ B′ X Y
  shows *Col A B A′*
⟨*proof*⟩

lemma *l13-2*:
  assumes *A B TS C D* **and**
    Per B C A **and**
    Per B D A **and**
    Col C D E **and**
    A E Perp C D
  shows *B A C CongA D A E* ∧ *B A D CongA C A E* ∧ *Bet C E D*
⟨*proof*⟩

lemma *perp2-refl*:
  assumes *A* ≠ *B*
  shows *P Perp2 A B A B*
⟨*proof*⟩

lemma *perp2-sym*:
  assumes *P Perp2 A B C D*
  shows *P Perp2 C D A B*
⟨*proof*⟩

lemma *perp2-left-comm*:
  assumes *P Perp2 A B C D*
  shows *P Perp2 B A C D*
⟨*proof*⟩

lemma *perp2-right-comm*:
  assumes *P Perp2 A B C D*
  shows *P Perp2 A B D C*
⟨*proof*⟩

lemma *perp2-comm*:
  assumes *P Perp2 A B C D*
  shows *P Perp2 B A D C*
⟨*proof*⟩

lemma *perp2-pseudo-trans*:
  assumes *P Perp2 A B C D* **and**
    P Perp2 C D E F **and**
    ¬ *Col C D P*
  shows *P Perp2 A B E F*
⟨*proof*⟩

lemma *col-cop-perp2--pars-bis*:
  assumes ¬ *Col A B P* **and**
    Col C D P **and**
    Coplanar A B C D **and**
    P Perp2 A B C D
  shows *A B ParStrict C D*

⟨*proof*⟩

lemma *perp2-preserves-bet23*:
assumes *Bet PO A B* **and**
Col PO A′ B′ **and**
¬ *Col PO A A′* **and**
PO Perp2 A A′ B B′
shows *Bet PO A′ B′*
⟨*proof*⟩

lemma *perp2-preserves-bet13*:
assumes *Bet B PO C* **and**
Col PO B′ C′ **and**
¬ *Col PO B B′* **and**
PO Perp2 B C′ C B′
shows *Bet B′ PO C′*
⟨*proof*⟩

lemma *is-image-perp-in*:
assumes $A \neq A'$ **and**
$X \neq Y$ **and**
A A′ Reflect X Y
shows ∃ *P. P PerpAt A A′ X Y*
⟨*proof*⟩

lemma *perp-inter-perp-in-n*:
assumes *A B Perp C D*
shows ∃ *P. Col A B P* ∧ *Col C D P* ∧ *P PerpAt A B C D*
⟨*proof*⟩

lemma *perp2-perp-in*:
assumes *PO Perp2 A B C D* **and**
¬ *Col PO A B* **and**
¬ *Col PO C D*
shows ∃ *P Q. Col A B P* ∧ *Col C D Q* ∧ *Col PO P Q* ∧ *P PerpAt PO P A B* ∧ *Q PerpAt PO Q C D*
⟨*proof*⟩

lemma *l13-8*:
assumes $U \neq PO$ **and**
$V \neq PO$ **and**
Col PO P Q **and**
Col PO U V **and**
Per P U PO **and**
Per Q V PO
shows *PO Out P Q* ⟷ *PO Out U V*
⟨*proof*⟩

lemma *perp-in-rewrite*:
assumes *P PerpAt A B C D*
shows *P PerpAt A P P C* ∨ *P PerpAt A P P D* ∨ *P PerpAt B P P C* ∨ *P PerpAt B P P D*
⟨*proof*⟩

lemma *perp-out-acute*:
assumes *B Out A C′* **and**
A B Perp C C′
shows *Acute A B C*
⟨*proof*⟩

lemma *perp-bet-obtuse*:
assumes $B \neq C'$ **and**
A B Perp C C′ **and**
Bet A B C′
shows *Obtuse A B C*
⟨*proof*⟩

lemma *sac-perm*:
  assumes *Saccheri A B C D*
  shows *Saccheri D C B A*
  ⟨*proof*⟩

lemma *sac-distincts*:
  assumes *Saccheri A B C D*
  shows $A \neq B \wedge B \neq C \wedge C \neq D \wedge A \neq D \wedge A \neq C \wedge B \neq D$
⟨*proof*⟩

lemma *lam-perm*:
  assumes *Lambert A B C D*
  shows *Lambert A D C B*
⟨*proof*⟩

lemma *sac--cong*:
  assumes *Saccheri A B C D*
  shows *Cong A C B D*
⟨*proof*⟩

lemma *sac--conga*:
  assumes *Saccheri A B C D*
  shows *A B C CongA B C D*
⟨*proof*⟩

lemma *lam--pars1234*:
  assumes *Lambert A B C D*
  shows *A B ParStrict C D*
⟨*proof*⟩

lemma *lam--pars1423*:
  assumes *Lambert A B C D*
  shows *A D ParStrict B C*
  ⟨*proof*⟩

lemma *lam--par1234*:
  assumes *Lambert A B C D*
  shows *A B Par C D*
  ⟨*proof*⟩

lemma *lam--par1423*:
  assumes *Lambert A B C D*
  shows *A D Par B C*
  ⟨*proof*⟩

lemma *lam--os*:
  assumes *Lambert A B C D*
  shows *A B OS C D*
  ⟨*proof*⟩

lemma *per2-os--pars*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C
  shows *A B ParStrict C D*
⟨*proof*⟩

lemma *per2-os--ncol123*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C
  shows ¬ *Col A B C*
⟨*proof*⟩

lemma *per2-os--ncol234*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C
  shows ¬ *Col B C D*
⟨*proof*⟩

lemma *sac--pars1234*:
  assumes *Saccheri A B C D*
  shows *A B ParStrict C D*
  ⟨*proof*⟩

lemma *sac--par1234*:
  assumes *Saccheri A B C D*
  shows *A B Par C D*
  ⟨*proof*⟩

lemma *lt-os-per2--lta*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C **and**
    A B Lt C D
  shows *B C D LtA A B C*
⟨*proof*⟩

lemma *lt4321-os-per2--lta*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C **and**
    D C Lt B A
  shows *A B C LtA B C D*
  ⟨*proof*⟩

lemma *lta-os-per2--lt*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C **and**
    B C D LtA A B C
  shows *A B Lt C D*
⟨*proof*⟩

lemma *lta123234-os-per2--lt*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C **and**
    A B C LtA B C D
  shows *D C Lt B A*
  ⟨*proof*⟩

lemma *conga-per2-os--cong*:
  assumes *Per B A D* **and**
    Per A D C **and**
    A D OS B C **and**
    B C D CongA A B C
  shows *Cong A B C D*
⟨*proof*⟩

lemma *mid2-sac--perp-lower*:
  assumes *Saccheri A B C D* **and**
    M Midpoint B C **and**

N Midpoint A D
shows *A D Perp M N*
⟨*proof*⟩

lemma *mid2-sac--perp-upper-R1*:
assumes *Saccheri A B C D* **and**
M Midpoint B C **and**
N Midpoint A D
shows *Cong N B N C*
⟨*proof*⟩

lemma *mid2-sac--perp-upper*:
assumes *Saccheri A B C D* **and**
M Midpoint B C **and**
N Midpoint A D
shows *B C Perp M N*
⟨*proof*⟩

lemma *sac--pars1423*:
assumes *Saccheri A B C D*
shows *A D ParStrict B C*
⟨*proof*⟩

lemma *sac--par1423*:
assumes *Saccheri A B C D*
shows *A D Par B C*
⟨*proof*⟩

lemma *mid2-sac--lam6521*:
assumes *Saccheri A B C D* **and**
M Midpoint B C **and**
N Midpoint A D
shows *Lambert N M B A*
⟨*proof*⟩

lemma *mid2-sac--lam6534*:
assumes *Saccheri A B C D* **and**
M Midpoint B C **and**
N Midpoint A D
shows *Lambert N M C D*
⟨*proof*⟩

lemma *lam6521-mid2--sac*:
assumes *Lambert N M B A* **and**
M Midpoint B C **and**
N Midpoint A D
shows *Saccheri A B C D*
⟨*proof*⟩

lemma *lam6534-mid2--sac*:
assumes *Lambert N M C D* **and**
M Midpoint B C **and**
N Midpoint A D
shows *Saccheri A B C D*
⟨*proof*⟩

lemma *cong-lam--per*:
assumes *Lambert A B C D* **and**
Cong A D B C
shows *Per B C D*
⟨*proof*⟩

lemma *lam-lt--acute*:
assumes *Lambert A B C D* **and**
A D Lt B C
shows *Acute B C D*
⟨*proof*⟩

lemma *lam-lt--obtuse*:
assumes *Lambert A B C D* **and**
B C Lt A D
shows *Obtuse B C D*
⟨*proof*⟩

lemma *ngta*:
shows ¬ *A B C GtA A B C* ⟨*proof*⟩

lemma *lam-per--cong*:
assumes *Lambert A B C D* **and**
Per B C D
shows *Cong A D B C*
⟨*proof*⟩

lemma *acute-lam--lt*:
assumes *Lambert A B C D* **and**
Acute B C D
shows *A D Lt B C*
⟨*proof*⟩

lemma *lam-obtuse--lt*:
assumes *Lambert A B C D* **and**
Obtuse B C D
shows *B C Lt A D*
⟨*proof*⟩

lemma *sac--ex2-mid-mid-lam-conga*:
assumes *Saccheri A B C D*
shows ∃ *M N. M Midpoint B C* ∧ *N Midpoint A D* ∧ *Lambert N M B A* ∧ *A B C CongA M B A*
⟨*proof*⟩

lemma *cong-sac--per-aux1*:
assumes *Saccheri A B C D* **and**
Cong A D B C
shows *Per A B C*
⟨*proof*⟩

lemma *cong-sac--per-aux2*:
assumes *Saccheri A B C D* **and**
Per A B C
shows *Cong A D B C*
⟨*proof*⟩

lemma *cong-sac--per*:
assumes *Saccheri A B C D*
shows *Cong A D B C* ⟷ *Per A B C*
⟨*proof*⟩

lemma *lt-sac--acute-aux1*:
assumes *Saccheri A B C D* **and**
A D Lt B C
shows *Acute A B C*
⟨*proof*⟩

lemma *lt-sac--acute-aux2*:

assumes *Saccheri A B C D* **and**
Acute A B C
shows *A D Lt B C*
⟨*proof*⟩

lemma *lt-sac--acute*:
assumes *Saccheri A B C D*
shows *A D Lt B C* ⟷ *Acute A B C*
⟨*proof*⟩

lemma *lt-sac--obtuse-aux1*:
assumes *Saccheri A B C D* **and**
B C Lt A D
shows *Obtuse A B C*
⟨*proof*⟩

lemma *lt-sac--obtuse-aux2*:
assumes *Saccheri A B C D* **and**
Obtuse A B C
shows *B C Lt A D*
⟨*proof*⟩

lemma *lt-sac--obtuse*:
assumes *Saccheri A B C D*
shows *B C Lt A D* ⟷ *Obtuse A B C*
⟨*proof*⟩

lemma *t22-7--per*:
assumes *Saccheri A B C D* **and**
Bet B P C **and**
Bet A Q D **and**
$A \neq Q$ **and**
$Q \neq D$ **and**
Per P Q A **and**
Cong P Q A B
shows *Per A B C*
⟨*proof*⟩

lemma *t22-7--acute*:
assumes *Saccheri A B C D* **and**
Bet B P C **and**
Bet A Q D **and**
$A \neq Q$ **and**
Per P Q A **and**
P Q Lt A B
shows *Acute A B C*
⟨*proof*⟩

lemma *t22-7--obtuse*:
assumes *Saccheri A B C D* **and**
Bet B P C **and**
Bet A Q D **and**
$A \neq Q$ **and**
Per P Q A **and**
A B Lt P Q
shows *Obtuse A B C*
⟨*proof*⟩

lemma *t22-7--cong*:
assumes *Saccheri A B C D* **and**
Bet B P C **and**
Bet A Q D **and**
$A \neq Q$ **and**
Per P Q A **and**
Per A B C
shows *Cong P Q A B*

⟨*proof*⟩

lemma *t22-7--lt5612*:
  assumes *Saccheri A B C D* **and**
    Bet B P C **and**
    Bet A Q D **and**
    $A \neq Q$ **and**
    $Q \neq D$ **and**
    Per P Q A **and**
    Acute A B C
  shows *P Q Lt A B*
⟨*proof*⟩

lemma *t22-7--lt1256*:
  assumes *Saccheri A B C D* **and**
    Bet B P C **and**
    Bet A Q D **and**
    $A \neq Q$ **and**
    $Q \neq D$ **and**
    Per P Q A **and**
    Obtuse A B C
  shows *A B Lt P Q*
⟨*proof*⟩

lemma *t22-8-aux-1*:
  assumes *Saccheri A B C D* **and**

    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R
  shows $C \neq R$
⟨*proof*⟩

lemma *t22-8-aux-2*:
  assumes*Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**

    Per A S R **and**
    S Out R J **and**
    Cong S J A B
  shows *Saccheri A B J S*
⟨*proof*⟩

lemma *t22-8-aux-3*:
  assumes*Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R **and**
    S Out R J **and**
    Cong S J A B
  shows *Saccheri D C J S*
⟨*proof*⟩

lemma *t22-8-aux-4*:
  assumes*Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R **and**
    S Out R J **and**
    Cong S J A B
  shows *S J OS C B*
⟨*proof*⟩

lemma *t22-8-aux-5*:
  assumes*Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**

    Per A S R **and**
    S Out R J **and**
    Cong S J A B
  shows *Coplanar A B C J*
⟨*proof*⟩

lemma *t22-8-aux*:
  assumes*Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R **and**
    S Out R J **and**
    Cong S J A B
  shows $C \neq R \land$ *Saccheri A B J S* $\land$ *Saccheri D C J S* $\land$ *S J OS C B* $\land$ *Coplanar A B C J*
  ⟨*proof*⟩

lemma *t22-8--per*:
  assumes *Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R **and**
    Cong R S A B
  shows *Per A B C*
⟨*proof*⟩

lemma *t22-8--acute*:
  assumes *Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R **and**
    A B Lt R S
  shows *Acute A B C*
⟨*proof*⟩

lemma *t22-8--obtuse*:
  assumes *Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R **and**
    R S Lt A B
  shows *Obtuse A B C*
⟨*proof*⟩

lemma *t22-8--cong*:
  assumes *Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**
    $D \neq S$ **and**
    Per A S R **and**
    Per A B C
  shows *Cong R S A B*
⟨*proof*⟩

lemma *t22-8--lt1256*:
  assumes *Saccheri A B C D* **and**
    Bet B C R **and**
    Bet A D S **and**

D ≠ *S* **and**
Per A S R **and**
Acute A B C
shows *A B Lt R S*
⟨*proof*⟩

lemma *t22-8--lt5612*:
assumes *Saccheri A B C D* **and**
Bet B C R **and**
Bet A D S **and**
D ≠ *S* **and**
Per A S R **and**
Obtuse A B C
shows *R S Lt A B*
⟨*proof*⟩

lemma *t22-9-aux-1*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Per S R M
shows *Per Q P M*
⟨*proof*⟩

lemma *t22-9-aux-2*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Per Q P M
shows *Per S R M*
⟨*proof*⟩

lemma *t22-9-aux-3*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Acute S R M
shows *Acute Q P M*
⟨*proof*⟩

lemma *t22-9-aux-4*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Acute Q P M
shows
Acute S R M
⟨*proof*⟩

lemma *t22-9-aux*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S
shows (*Per S R M* ⟷ *Per Q P M*) ∧ (*Acute S R M* ⟷ *Acute Q P M*)
⟨*proof*⟩

lemma *t22-9--per-1*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**

Per S R M
shows *Per Q P M*
⟨*proof*⟩

lemma *t22-9--per-2*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Per Q P M
shows *Per S R M*
⟨*proof*⟩

lemma *t22-9--per*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S
shows *Per S R M* ⟷ *Per Q P M*
⟨*proof*⟩

lemma *t22-9--acute-1*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Acute S R M
shows *Acute Q P M*
⟨*proof*⟩

lemma *t22-9--acute-2*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Acute Q P M
shows *Acute S R M*
⟨*proof*⟩

lemma *t22-9--acute*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S
shows *Acute S R M* ⟷ *Acute Q P M*
⟨*proof*⟩

lemma *t22-9--obtuse-1*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Obtuse S R M
shows *Obtuse Q P M*
⟨*proof*⟩

lemma *t22-9--obtuse-2*:
assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S **and**
Obtuse Q P M
shows *Obtuse S R M*
⟨*proof*⟩

lemma *t22-9--obtuse*:

assumes *Lambert N M P Q* **and**
Lambert N M R S **and**
Bet M P R **and**
Bet N Q S
shows *Obtuse S R M* $\longleftrightarrow$ *Obtuse Q P M*
⟨*proof*⟩

lemma *cong2-lam2--cong-conga-1*:
assumes *Lambert N M P Q* **and**
Lambert N' M' P' Q' **and**
Cong N Q N' Q' **and**
Cong P Q P' Q'
shows *Cong N M N' M'*
⟨*proof*⟩

lemma *cong2-lam2--cong-conga-2*:
assumes *Lambert N M P Q* **and**
Lambert N' M' P' Q' **and**
Cong N Q N' Q' **and**
Cong P Q P' Q'
shows *M P Q CongA M' P' Q'*
⟨*proof*⟩

lemma *cong2-lam2--cong-conga*:
assumes *Lambert N M P Q* **and**
Lambert N' M' P' Q' **and**
Cong N Q N' Q' **and**
Cong P Q P' Q'
shows *Cong N M N' M'* $\wedge$ *M P Q CongA M' P' Q'*
⟨*proof*⟩

lemma *cong2-sac2--cong*:
assumes *Saccheri A B C D* **and**
Saccheri A' B' C' D' **and**
Cong A B A' B' **and**
Cong A D A' D'
shows *Cong B C B' C'*
⟨*proof*⟩

lemma *sac--perp1214*:
assumes *Saccheri A B C D*
shows *A B Perp A D*
⟨*proof*⟩

lemma *sac--perp3414*:
assumes *Saccheri A B C D*
shows *C D Perp A D*
⟨*proof*⟩

lemma *cop-sac2--sac*:
assumes *Saccheri A B C D* **and**
Saccheri A B E F **and**
$D \neq F$ **and**
Coplanar A B D F
shows *Saccheri D C E F*
⟨*proof*⟩

lemma *three-hypotheses-aux*:
assumes *Saccheri A B C D* **and**
Saccheri A' B' C' D' **and**
M Midpoint B C **and**

M′ Midpoint B′ C′ **and**
N Midpoint A D **and**
N′ Midpoint A′ D′ **and**
M N Le M′ N′
shows $(Per\ A\ B\ C \longleftrightarrow Per\ A'\ B'\ C') \wedge (Acute\ A\ B\ C \longleftrightarrow Acute\ A'\ B'\ C')$
⟨*proof*⟩

lemma *three-hypotheses-aux-1*:
assumes *Saccheri A B C D* **and**
Saccheri A′ B′ C′ D′ **and**
M Midpoint B C **and**
M′ Midpoint B′ C′ **and**
N Midpoint A D **and**
N′ Midpoint A′ D′ **and**
M N Le M′ N′ **and**
Per A B C
shows *Per A′ B′ C′*
⟨*proof*⟩

lemma *three-hypotheses-aux-2*:
assumes *Saccheri A B C D* **and**
Saccheri A′ B′ C′ D′ **and**
M Midpoint B C **and**
M′ Midpoint B′ C′ **and**
N Midpoint A D **and**
N′ Midpoint A′ D′ **and**
M N Le M′ N′ **and**
Per A′ B′ C′
shows *Per A B C*
⟨*proof*⟩

lemma *three-hypotheses-aux-3*:
assumes *Saccheri A B C D* **and**
Saccheri A′ B′ C′ D′ **and**
M Midpoint B C **and**
M′ Midpoint B′ C′ **and**
N Midpoint A D **and**
N′ Midpoint A′ D′ **and**
M N Le M′ N′ **and**
Acute A B C
shows *Acute A′ B′ C′*
⟨*proof*⟩

lemma *three-hypotheses-aux-4*:
assumes *Saccheri A B C D* **and**
Saccheri A′ B′ C′ D′ **and**
M Midpoint B C **and**
M′ Midpoint B′ C′ **and**
N Midpoint A D **and**
N′ Midpoint A′ D′ **and**
M N Le M′ N′ **and**
Acute A′ B′ C′
shows *Acute A B C*
⟨*proof*⟩

lemma *per-sac--rah*:
assumes *Saccheri A B C D* **and**
Per A B C
shows *HypothesisRightSaccheriQuadrilaterals*
⟨*proof*⟩

lemma *acute-sac--aah*:
assumes *Saccheri A B C D* **and**

Acute A B C
shows *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *obtuse-sac--oah*:
assumes *Saccheri A B C D* **and**
Obtuse A B C
shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *per--ex-saccheri*:
assumes *Per B A D* **and**
$A \neq B$ **and**
$A \neq D$
shows ∃ *C*. *Saccheri A B C D*
⟨*proof*⟩

lemma *ex-saccheri*:
∃ *A B C D*. *Saccheri A B C D*
⟨*proof*⟩

lemma *ex-lambert*:
∃ *A B C D*. *Lambert A B C D*
⟨*proof*⟩

lemma *saccheri-s-three-hypotheses*:
hypothesis-of-acute-saccheri-quadrilaterals ∨
hypothesis-of-right-saccheri-quadrilaterals ∨
hypothesis-of-obtuse-saccheri-quadrilaterals
⟨*proof*⟩

lemma *not-aah*:
assumes *hypothesis-of-right-saccheri-quadrilaterals* ∨
hypothesis-of-obtuse-saccheri-quadrilaterals
shows ¬ *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *not-rah*:
assumes *hypothesis-of-acute-saccheri-quadrilaterals* ∨
hypothesis-of-obtuse-saccheri-quadrilaterals
shows ¬ *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *not-oah*:
assumes *hypothesis-of-acute-saccheri-quadrilaterals* ∨
hypothesis-of-right-saccheri-quadrilaterals
shows ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *lam-per--rah-1*:
assumes *Lambert A B C D* **and**
Per B C D
shows *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *lam-per--rah-2*:
assumes *Lambert A B C D* **and**
hypothesis-of-right-saccheri-quadrilaterals
shows *Per B C D*
⟨*proof*⟩

lemma *lam-per--rah*:
assumes *Lambert A B C D*
shows *Per B C D* ⟷ *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *lam-acute--aah-1*:
  assumes *Lambert A B C D* **and**
    Acute B C D
  shows *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *lam-acute--aah-2*:
  assumes *Lambert A B C D* **and**
    hypothesis-of-acute-saccheri-quadrilaterals
  shows *Acute B C D*
⟨*proof*⟩

lemma *lam-acute--aah*:
  assumes *Lambert A B C D*
  shows *Acute B C D* ⟷ *hypothesis-of-acute-saccheri-quadrilaterals*
  ⟨*proof*⟩

lemma *lam-obtuse--oah-1*:
  assumes *Lambert A B C D* **and**
    Obtuse B C D
  shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *lam-obtuse--oah-2*:
  assumes *Lambert A B C D* **and**
    hypothesis-of-obtuse-saccheri-quadrilaterals
  shows *Obtuse B C D*
⟨*proof*⟩

lemma *lam-obtuse--oah*:
  assumes *Lambert A B C D*
  shows *Obtuse B C D* ⟷ *hypothesis-of-obtuse-saccheri-quadrilaterals*
  ⟨*proof*⟩

lemma *t22-11--per-1*:
  assumes *Saccheri A B C D* **and**
    A B D CongA B D C
  shows *Per A B C*
⟨*proof*⟩

lemma *t22-11--per-2*:
  assumes *Saccheri A B C D* **and**
    Per A B C
  shows *A B D CongA B D C*
⟨*proof*⟩

lemma *t22-11--per*:
  assumes *Saccheri A B C D*
  shows *A B D CongA B D C* ⟷ *Per A B C*
  ⟨*proof*⟩

lemma *t22-11--acute-1*:
  assumes *Saccheri A B C D* **and**
    A B D LtA B D C
  shows *Acute A B C*
⟨*proof*⟩

lemma *t22-11--acute-2*:
  assumes *Saccheri A B C D* **and**
    Acute A B C
  shows *A B D LtA B D C*
⟨*proof*⟩

lemma *t22-11--acute*:
  assumes *Saccheri A B C D*

shows *A B D LtA B D C* ⟷ *Acute A B C*
⟨*proof*⟩

lemma *t22-11--obtuse-1*:
assumes *Saccheri A B C D* **and**
B D C LtA A B D
shows *Obtuse A B C*
⟨*proof*⟩

lemma *t22-11--obtuse-2*:
assumes *Saccheri A B C D* **and**
Obtuse A B C
shows *B D C LtA A B D*
⟨*proof*⟩

lemma *t22-11--obtuse*:
assumes *Saccheri A B C D*
shows *B D C LtA A B D* ⟷ *Obtuse A B C*
⟨*proof*⟩

lemma *t22-12--rah-1*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C **and**
B C A C A B SumA A B C
shows *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-12--rah-2*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C **and**
hypothesis-of-right-saccheri-quadrilaterals
shows *B C A C A B SumA A B C*
⟨*proof*⟩

lemma *t22-12--rah*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C
shows *B C A C A B SumA A B C* ⟷ *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-12--aah-1*:
assumes *Per A B C* **and**
B C A C A B SumA P Q R **and**
Acute P Q R
shows *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-12--aah-2*:
assumes *Per A B C* **and**
B C A C A B SumA P Q R **and**
hypothesis-of-acute-saccheri-quadrilaterals
shows *Acute P Q R*
⟨*proof*⟩

lemma *t22-12--aah*:
assumes *Per A B C* **and**
B C A C A B SumA P Q R
shows *Acute P Q R* ⟷ *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-12--oah-1*:
assumes *Per A B C* **and**
B C A C A B SumA P Q R **and**

Obtuse P Q R
shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-12--oah-2*:
assumes *Per A B C* **and**
B C A C A B SumA P Q R **and**
hypothesis-of-obtuse-saccheri-quadrilaterals
shows *Obtuse P Q R*
⟨*proof*⟩

lemma *t22-12--oah*:
assumes *Per A B C* **and**
B C A C A B SumA P Q R
shows *Obtuse P Q R* ⟷ *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-14-aux*:
assumes ¬ *Col A B C* **and**
Acute A B C **and**
Acute A C B
shows ∃ *A'*. *Bet B A' C* ∧ *Per B A' A* ∧ *Per C A' A* ∧
B A' A CongA C A' A ∧ *A B C CongA A B A'* ∧
B C A CongA A' C A ∧ *A A' TS B C*
⟨*proof*⟩

lemma *t22-14--bet-aux*:
assumes *hypothesis-of-right-saccheri-quadrilaterals* **and**
¬ *Col A B C* **and**
A B C TriSumA P Q R **and**
Acute A B C **and**
Acute A C B
shows *Bet P Q R*
⟨*proof*⟩

lemma *t22-14--bet*:
assumes *hypothesis-of-right-saccheri-quadrilaterals* **and**
A B C TriSumA P Q R
shows *Bet P Q R*
⟨*proof*⟩

lemma *t22-14--sams-nbet-aux*:
assumes *hypothesis-of-acute-saccheri-quadrilaterals* **and**
¬ *Col A B C* **and**
C A B A B C SumA D E F **and**
D E F B C A SumA P Q R **and**
Acute A B C **and**
Acute A C B
shows *SAMS D E F B C A* ∧ ¬ *Bet P Q R*
⟨*proof*⟩

lemma *t22-14--sams-nbet*:
assumes *hypothesis-of-acute-saccheri-quadrilaterals* **and**
¬ *Col A B C* **and**
C A B A B C SumA D E F **and**
D E F B C A SumA P Q R
shows *SAMS D E F B C A* ∧ ¬ *Bet P Q R*
⟨*proof*⟩

lemma *t22-14--nsams-aux*:
assumes *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**

¬ *Col A B C* **and**
C A B A B C SumA D E F **and**
Acute A B C **and**
Acute A C B
shows ¬ *SAMS D E F B C A*
⟨*proof*⟩

lemma *t22-14--nsams*:
assumes *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
¬ *Col A B C* **and**
C A B A B C SumA D E F
shows ¬ *SAMS D E F B C A*
⟨*proof*⟩

lemma *t22-14--rah*:
assumes ¬ *Col A B C* **and**
A B C TriSumA P Q R **and**
Bet P Q R
shows *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-14--aah*:
assumes *C A B A B C SumA D E F* **and**
D E F B C A SumA P Q R **and**
SAMS D E F B C A **and**
¬ *Bet P Q R*
shows *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-14--oah*:
assumes *C A B A B C SumA D E F* **and**
¬ *SAMS D E F B C A*
shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *cong-mid--suma*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C
shows *C A B A B C SumA A C B*
⟨*proof*⟩

lemma *t22-17--rah-1*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C **and**
Per A C B
shows *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-17--rah-2*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C **and**

hypothesis-of-right-saccheri-quadrilaterals
shows *Per A C B*
⟨*proof*⟩

lemma *t22-17--rah*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C
shows *Per A C B* ⟷ *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-17--oah-1*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C **and**
Obtuse A C B
shows *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-17--oah-2*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C **and**
hypothesis-of-obtuse-saccheri-quadrilaterals
shows *Obtuse A C B*
⟨*proof*⟩

lemma *t22-17--oah*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C
shows *Obtuse A C B* ⟷ *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-17--aah-1*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C **and**
Acute A C B
shows *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-17--aah-2*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C **and**
hypothesis-of-acute-saccheri-quadrilaterals
shows *Acute A C B*
⟨*proof*⟩

lemma *t22-17--aah*:
assumes ¬ *Col A B C* **and**
M Midpoint A B **and**
Cong M A M C
shows *Acute A C B* ⟷ *hypothesis-of-acute-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-20*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
A B C B C A SumA D E F
shows *SAMS D E F C A B*
⟨*proof*⟩

lemma *absolute-exterior-angle-theorem*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**

Bet B A B′ **and**
A ≠ B′ **and**
A B C B C A SumA D E F
shows *D E F LeA C A B′*
⟨*proof*⟩

lemma *defect-distincts*:
assumes *Defect A B C D E F*
shows *A ≠ B ∧ B ≠ C ∧ A ≠ C ∧ D ≠ E ∧ E ≠ F*
⟨*proof*⟩

lemma *ex-defect*:
assumes *A ≠ B* **and**
B ≠ C **and**
A ≠ C
shows *∃ D E F. Defect A B C D E F*
⟨*proof*⟩

lemma *conga-defect--defect*:
assumes *Defect A B C D E F* **and**
D E F CongA D′ E′ F′
shows *Defect A B C D′ E′ F′*
⟨*proof*⟩

lemma *defect2--conga*:
assumes *Defect A B C D E F* **and**
Defect A B C D′ E′ F′
shows *D E F CongA D′ E′ F′*
⟨*proof*⟩

lemma *defect-perm-231*:
assumes *Defect A B C D E F*
shows *Defect B C A D E F*
⟨*proof*⟩

lemma *defect-perm-312*:
assumes *Defect A B C D E F*
shows *Defect C A B D E F*
⟨*proof*⟩

lemma *defect-perm-321*:
assumes *Defect A B C D E F*
shows *Defect C B A D E F*
⟨*proof*⟩

lemma *defect-perm-213*:
assumes *Defect A B C D E F*
shows *Defect B A C D E F*
⟨*proof*⟩

lemma *defect-perm-132*:
assumes *Defect A B C D E F*
shows *Defect A C B D E F*
⟨*proof*⟩

lemma *conga3-defect--defect*:
assumes *Defect A B C D E F* **and**
A B C CongA A′ B′ C′ **and**
B C A CongA B′ C′ A′ **and**
C A B CongA C′ A′ B′
shows *Defect A′ B′ C′ D E F*
⟨*proof*⟩

lemma *col-defect--out*:
assumes *Col A B C* **and**
Defect A B C D E F

shows *E Out D F*
⟨*proof*⟩

lemma *rah-defect--out*:
assumes *hypothesis-of-right-saccheri-quadrilaterals* **and**
Defect A B C D E F
shows *E Out D F*
⟨*proof*⟩

lemma *defect-ncol-out--rah*:
assumes ¬ *Col A B C* **and**
Defect A B C D E F **and**
E Out D F
shows *hypothesis-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-16-1*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
Bet A C1 C **and**
Defect A B C1 D E F **and**
Defect B C1 C G H I **and**
D E F G H I SumA K L M
shows *SAMS D E F G H I* ∧ *Defect A B C K L M*
⟨*proof*⟩

lemma *t22-16-1bis*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
Bet A C1 C **and**
Defect A B C1 D E F **and**
Defect B C1 C G H I **and**
Defect A B C K L M
shows *SAMS D E F G H I* ∧ *D E F G H I SumA K L M*
⟨*proof*⟩

lemma *t22-16-2aux*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
Defect A B C D1 D2 D3 **and**
Defect A B D C1 C2 C3 **and**
Defect A D C B1 B2 B3 **and**
Defect C B D A1 A2 A3 **and**
Bet A PO C **and**
Bet B PO D **and**
Col A B C **and**
D1 D2 D3 B1 B2 B3 SumA P Q R
shows *SAMS C1 C2 C3 A1 A2 A3* ∧ *C1 C2 C3 A1 A2 A3 SumA P Q R*
⟨*proof*⟩

lemma *t22-16-2aux1*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
Defect A B C D1 D2 D3 **and**
Defect A B D C1 C2 C3 **and**
Defect A D C B1 B2 B3 **and**
Defect C B D A1 A2 A3 **and**
Bet A PO C **and**
Bet B PO D **and**
Col A B D **and**
D1 D2 D3 B1 B2 B3 SumA P Q R
shows *SAMS C1 C2 C3 A1 A2 A3* ∧ *C1 C2 C3 A1 A2 A3 SumA P Q R*
⟨*proof*⟩

lemma *t22-16-2*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
Defect A B C D1 D2 D3 **and**
Defect A B D C1 C2 C3 **and**
Defect A D C B1 B2 B3 **and**
Defect C B D A1 A2 A3 **and**
Bet A PO C **and**
Bet B PO D **and**
SAMS D1 D2 D3 B1 B2 B3 **and**
D1 D2 D3 B1 B2 B3 SumA P Q R
shows *SAMS C1 C2 C3 A1 A2 A3* ∧ *C1 C2 C3 A1 A2 A3 SumA P Q R*
⟨*proof*⟩

lemma *isosceles-sym* :
assumes *A B C isosceles*
shows *C B A isosceles*
⟨*proof*⟩

lemma *isosceles-conga*:
assumes $A \neq C$ **and**
$A \neq B$ **and**
A B C isosceles
shows *C A B CongA A C B*
⟨*proof*⟩

lemma *conga-isosceles*:
assumes ¬ *Col A B C* **and**
C A B CongA A C B
shows *A B C isosceles*
⟨*proof*⟩

lemma *isosceles-foot--midpoint-conga*:
assumes *A B C isosceles* **and**
Col H A C **and**
H B Perp A C
shows ¬ *Col A B C* ∧ $A \neq H$ ∧ $C \neq H$ ∧ *H Midpoint A C* ∧ *H B A CongA H B C*
⟨*proof*⟩

lemma *equilateral-strict-equilateral*:
assumes *A B C equilateralStrict*
shows *A B C equilateral*
⟨*proof*⟩

lemma *equilateral-cong*:
assumes *A B C equilateral*
shows *Cong A B B C* ∧ *Cong B C C A* ∧ *Cong C A A B*
⟨*proof*⟩

lemma *equilateral-rot*:
assumes *A B C equilateral*
shows *B C A equilateral*
⟨*proof*⟩

lemma *equilateral-swap*:
assumes *A B C equilateral*
shows *B A C equilateral*
⟨*proof*⟩

lemma *equilateral-rot-2*:
assumes *A B C equilateral*
shows *C B A equilateral*
⟨*proof*⟩

lemma *equilateral-swap-2*:

assumes *A B C equilateral*
shows *A C B equilateral*
⟨*proof*⟩

lemma *equilateral-swap-rot*:
assumes *A B C equilateral*
shows *C A B equilateral*
⟨*proof*⟩

lemma *equilateral-isosceles-1*:
assumes *A B C equilateral*
shows *A B C isosceles*
⟨*proof*⟩

lemma *equilateral-isosceles-2*:
assumes *A B C equilateral*
shows *B C A isosceles*
⟨*proof*⟩

lemma *equilateral-isosceles-3*:
assumes *A B C equilateral*
shows *C A B isosceles*
⟨*proof*⟩

lemma *equilateral-strict-neq*:
assumes *A B C equilateralStrict*
shows $A \neq B \wedge B \neq C \wedge A \neq C$
⟨*proof*⟩

lemma *equilateral-strict-swap-1*:
assumes *A B C equilateralStrict*
shows *A C B equilateralStrict*
⟨*proof*⟩

lemma *equilateral-strict-swap-2*:
assumes *A B C equilateralStrict*
shows *B A C equilateralStrict*
⟨*proof*⟩

lemma *equilateral-strict-swap-3*:
assumes *A B C equilateralStrict*
shows *B C A equilateralStrict*
⟨*proof*⟩

lemma *equilateral-strict-swap-4*:
assumes *A B C equilateralStrict*
shows *C A B equilateralStrict*
⟨*proof*⟩

lemma *equilateral-strict-swap-5*:
assumes *A B C equilateralStrict*
shows *C B A equilateralStrict*
⟨*proof*⟩

lemma *equilateral-strict--not-col*:
assumes *A B C equilateralStrict*
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *equilateral-strict-conga-1*:
assumes *A B C equilateralStrict*
shows *C A B CongA A C B*
⟨*proof*⟩

lemma *equilateral-strict-conga-2*:
assumes *A B C equilateralStrict*

shows *B A C CongA A B C*
⟨*proof*⟩

lemma *equilateral-strict-conga-3*:
assumes *A B C equilateralStrict*
shows *C B A CongA B C A*
⟨*proof*⟩

lemma *conga3-equilateral*:
assumes ¬ *Col A B C* **and**
B A C CongA A B C **and**
A B C CongA B C A
shows *A B C equilateral*
⟨*proof*⟩

lemma *lg-exists*:
∃ *l*. (*QCong l* ∧ *l A B*)
⟨*proof*⟩

lemma *lg-cong*:
assumes *QCong l* **and**
l A B **and**
l C D
shows *Cong A B C D*
⟨*proof*⟩

lemma *lg-cong-lg*:
assumes *QCong l* **and**
l A B **and**
Cong A B C D
shows *l C D*
⟨*proof*⟩

lemma *lg-sym*:
assumes *QCong l*
and *l A B*
shows *l B A*
⟨*proof*⟩

lemma *ex-points-lg*:
assumes *QCong l*
shows ∃ *A B*. *l A B*
⟨*proof*⟩

lemma *is-len-cong*:
assumes *TarskiLen A B l* **and**
TarskiLen C D l
shows *Cong A B C D*
⟨*proof*⟩

lemma *is-len-cong-is-len*:
assumes *TarskiLen A B l* **and**
Cong A B C D
shows *TarskiLen C D l*
⟨*proof*⟩

lemma *not-cong-is-len*:
assumes ¬ *Cong A B C D* **and**
TarskiLen A B l
shows ¬ *l C D*
⟨*proof*⟩

lemma *not-cong-is-len1*:
assumes ¬ *Cong A B C D*
and *TarskiLen A B l*
shows ¬ *TarskiLen C D l*

⟨proof⟩

lemma *lg-null-instance*:
assumes *QCongNull l*
shows *l A A*
⟨proof⟩

lemma *lg-null-trivial*:
assumes *QCong l*
and *l A A*
shows *QCongNull l*
⟨proof⟩

lemma *lg-null-dec*:

shows *QCongNull l* ∨ ¬ *QCongNull l*
⟨proof⟩

lemma *ex-point-lg*:
assumes *QCong l*
shows ∃ *B*. *l A B*
⟨proof⟩

lemma *ex-point-lg-out*:
assumes *A* ≠ *P* **and**
QCong l **and**
¬ *QCongNull l*
shows ∃ *B*. (*l A B* ∧ *A Out B P*)
⟨proof⟩

lemma *ex-point-lg-bet*:
assumes *QCong l*
shows ∃ *B*. (*l M B* ∧ *Bet A M B*)
⟨proof⟩

lemma *ex-points-lg-not-col*:
assumes *QCong l*
and ¬ *QCongNull l*
shows ∃ *A B*. (*l A B* ∧ ¬ *Col A B P*)
⟨proof⟩

lemma *ex-eql*:
assumes ∃ *A B*. (*TarskiLen A B l1* ∧ *TarskiLen A B l2*)
shows *l1* = *l2*
⟨proof⟩

lemma *all-eql*:
assumes *TarskiLen A B l1* **and**
TarskiLen A B l2
shows *l1 EqLTarski l2*
⟨proof⟩

lemma *null-len*:
assumes *TarskiLen A A la* **and**
TarskiLen B B lb
shows *la EqLTarski lb*
⟨proof⟩

lemma *eqL-equivalence*:
assumes *QCong la* **and**
QCong lb **and**
QCong lc
shows *la* = *la* ∧ (*la* = *lb* ⟶ *lb* = *la*) ∧ (*la* = *lb* ∧ *lb* = *lc* ⟶ *la* = *lc*)
⟨proof⟩

lemma *ex-lg*:

$\exists$ l. ($QCong$ l $\wedge$ l A B)
⟨proof⟩

lemma *lg-eql-lg*:
assumes *QCong l1* **and**
l1 EqLTarski l2
shows *QCong l2*
⟨proof⟩

lemma *ex-eqL*:
assumes *QCong l1* **and**
QCong l2 **and**
$\exists$ A B. ($l1$ A B $\wedge$ $l2$ A B)
shows *l1 EqLTarski l2*
⟨proof⟩

lemma *ang-exists*:
assumes $A \neq B$ **and**
$C \neq B$
shows $\exists$ a. ($QCongA$ a $\wedge$ a A B C)
⟨proof⟩

lemma *ex-points-eng*:
assumes *QCongA a*
shows $\exists$ A B C. (a A B C)
⟨proof⟩

lemma *ang-conga*:
assumes *QCongA a* **and**
a A B C **and**
a A' B' C'
shows A B C $CongA$ A' B' C'
⟨proof⟩

lemma *is-ang-conga*:
assumes *A B C Ang a* **and**
A' B' C' Ang a
shows A B C $CongA$ A' B' C'
⟨proof⟩

lemma *is-ang-conga-is-ang*:
assumes *A B C Ang a* **and**
A B C $CongA$ A' B' C'
shows A' B' C' Ang a
⟨proof⟩

lemma *not-conga-not-ang*:
assumes *QCongA a* **and**
$\neg$ A B C $CongA$ A' B' C' **and**
a A B C
shows $\neg$ a A' B' C'
⟨proof⟩

lemma *not-conga-is-ang*:
assumes $\neg$ A B C $CongA$ A' B' C' **and**
A B C Ang a
shows $\neg$ a A' B' C'
⟨proof⟩

lemma *not-cong-is-ang1*:
assumes $\neg$ A B C $CongA$ A' B' C' **and**
A B C Ang a
shows $\neg$ A' B' C' Ang a
⟨proof⟩

lemma *ex-eqa*:

assumes $\exists$ *A B C*.(*A B C Ang a1* $\wedge$ *A B C Ang a2*)
shows *a1* = *a2*
⟨*proof*⟩

lemma *all-eqa*:
assumes *A B C Ang a1* **and**
A B C Ang a2
shows *a1* = *a2*
⟨*proof*⟩

lemma *is-ang-distinct*:
assumes *A B C Ang a*
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *null-ang*:
assumes *A B A Ang a1* **and**
C D C Ang a2
shows *a1* = *a2*
⟨*proof*⟩

lemma *flat-ang*:
assumes *Bet A B C* **and**
Bet A' B' C' **and**
A B C Ang a1 **and**
A' B' C' *Ang a2*
shows *a1* = *a2*
⟨*proof*⟩

lemma *ang-distinct*:
assumes *QCongA a* **and**
a A B C
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *ex-ang*:
assumes $B \neq A$ **and**
$B \neq C$
shows $\exists$ *a*. (*QCongA a* $\wedge$ *a A B C*)
⟨*proof*⟩

lemma *anga-exists*:
assumes $A \neq B$ **and**
$C \neq B$ **and**
Acute A B C
shows $\exists$ *a*. (*QCongAAcute a* $\wedge$ *a A B C*)
⟨*proof*⟩

lemma *anga-is-ang*:
assumes *QCongAAcute a*
shows *QCongA a*
⟨*proof*⟩

lemma *ex-points-anga*:
assumes *QCongAAcute a*
shows $\exists$ *A B C*. *a A B C*
⟨*proof*⟩

lemma *anga-conga*:
assumes *QCongAAcute a* **and**
a A B C **and**
a A' B' C'
shows *A B C CongA* A' B' C'
⟨*proof*⟩

lemma *is-anga-to-is-ang*:

assumes *A B C AngAcute a*
shows *A B C Ang a*
⟨*proof*⟩

lemma *is-anga-conga*:
assumes *A B C AngAcute a* **and**
A′ B′ C′ AngAcute a
shows *A B C CongA A′ B′ C′*
⟨*proof*⟩

lemma *is-anga-conga-is-anga*:
assumes *A B C AngAcute a* **and**
A B C CongA A′ B′ C′
shows *A′ B′ C′ AngAcute a*
⟨*proof*⟩

lemma *not-conga-is-anga*:
assumes ¬ *A B C CongA A′ B′ C′* **and**
A B C AngAcute a
shows ¬ *a A′ B′ C′*
⟨*proof*⟩

lemma *not-cong-is-anga1*:
assumes ¬ *A B C CongA A′ B′ C′* **and**
A B C AngAcute a
shows ¬ *A′ B′ C′ AngAcute a*
⟨*proof*⟩

lemma *ex-eqaa*:
assumes ∃ *A B C*. (*A B C AngAcute a1* ∧ *A B C AngAcute a2*)
shows *a1 EqA a2*
⟨*proof*⟩

lemma *all-eqaa*:
assumes *A B C AngAcute a1* **and**
A B C AngAcute a2
shows *a1 EqA a2*
⟨*proof*⟩

lemma *is-anga-distinct*:
assumes *A B C AngAcute a*
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *null-anga*:
assumes *A B A AngAcute a1* **and**
C D C AngAcute a2
shows *a1 EqA a2*
⟨*proof*⟩

lemma *anga-distinct*:
assumes *QCongAAcute a* **and**
a A B C
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *out-is-len-eq*:
assumes *A Out B C* **and**
TarskiLen A B l **and**
TarskiLen A C l
shows $B = C$
⟨*proof*⟩

lemma *out-len-eq*:
assumes *QCong l* **and**
A Out B C **and**

l A B **and**
l A C
shows $B = C$ ⟨*proof*⟩

lemma *ex-anga*:
assumes *Acute A B C*
shows $\exists$ *a*. (*QCongAAcute a* $\wedge$ *a A B C*)
⟨*proof*⟩

lemma *not-null-ang-ang*:
assumes *QCongAnNull a*
shows *QCongA a*
⟨*proof*⟩

lemma *not-null-ang-def-equiv*:
QCongAnNull a $\longleftrightarrow$ (*QCongA a* $\wedge$ ($\exists$ *A B C*. (*a A B C* $\wedge$ $\neg$ *B Out A C*)))
⟨*proof*⟩

lemma *not-flat-ang-def-equiv*:
QCongAnFlat a $\longleftrightarrow$ (*QCongA a* $\wedge$ ($\exists$ *A B C*. (*a A B C* $\wedge$ $\neg$ *Bet A B C*)))
⟨*proof*⟩

lemma *ang-const*:
assumes *QCongA a* **and**
$A \neq B$
shows $\exists$ *C*. *a A B C*
⟨*proof*⟩

lemma *ang-sym*:
assumes *QCongA a* **and**
a A B C
shows *a C B A*
⟨*proof*⟩

lemma *ang-not-null-lg*:
assumes *QCongA a* **and**
QCong l **and**
a A B C **and**
l A B
shows $\neg$ *QCongNull l*
⟨*proof*⟩

lemma *ang-distincts*:
assumes *QCongA a* **and**
a A B C
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *anga-sym*:
assumes *QCongAAcute a* **and**
a A B C
shows *a C B A*
⟨*proof*⟩

lemma *anga-not-null-lg*:
assumes *QCongAAcute a* **and**
QCong l **and**
a A B C **and**
l A B
shows $\neg$ *QCongNull l*
⟨*proof*⟩

lemma *anga-distincts*:
assumes *QCongAAcute a* **and**
a A B C
shows $A \neq B \wedge C \neq B$

⟨*proof*⟩

lemma *ang-const-o*:
assumes ¬ *Col A B P* **and**
QCongA a **and**
QCongAnNull a **and**
QCongAnFlat a
shows ∃ *C*. *a A B C* ∧ *A B OS C P*
⟨*proof*⟩

lemma *anga-const*:
assumes *QCongAAcute a* **and**
$A \neq B$
shows ∃ *C*. *a A B C*
⟨*proof*⟩

lemma *null-anga-null-angaP*:
QCongANullAcute a ⟷ *IsNullAngaP a*
⟨*proof*⟩

lemma *is-null-anga-out*:
assumes
a A B C **and**
QCongANullAcute a
shows *B Out A C*
⟨*proof*⟩

lemma *anga-acute*:
assumes *QCongAAcute a* **and**
a A B C
shows *Acute A B C*
⟨*proof*⟩

lemma *not-null-not-col*:
assumes *QCongAAcute a* **and**
¬ *QCongANullAcute a* **and**
a A B C
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *ang-cong-ang*:
assumes *QCongA a* **and**
a A B C **and**
A B C CongA A' B' C'
shows *a* A' B' C'
⟨*proof*⟩

lemma *is-null-ang-out*:
assumes
a A B C **and**
QCongANull a
shows *B Out A C*
⟨*proof*⟩

lemma *out-null-ang*:
assumes *QCongA a* **and**
a A B C **and**
B Out A C
shows *QCongANull a*
⟨*proof*⟩

lemma *bet-flat-ang*:
assumes *QCongA a* **and**
a A B C **and**
Bet A B C
shows *AngFlat a*

⟨*proof*⟩

lemma *out-null-anga*:
assumes *QCongAAcute a* **and**
a A B C **and**
B Out A C
shows *QCongANullAcute a*
⟨*proof*⟩

lemma *anga-not-flat*:
assumes *QCongAAcute a*
shows *QCongAnFlat a*
⟨*proof*⟩

lemma *anga-const-o*:
assumes ¬ *Col A B P* **and**
¬ *QCongANullAcute a* **and**
QCongAAcute a
shows ∃ *C*. (*a A B C* ∧ *A B OS C P*)
⟨*proof*⟩

lemma *anga-conga-anga*:
assumes *QCongAAcute a* **and**
a A B C **and**
A B C CongA A′ B′ C′
shows *a A′ B′ C′*
⟨*proof*⟩

lemma *anga-out-anga*:
assumes *QCongAAcute a* **and**
a A B C **and**
B Out A A′ **and**
B Out C C′
shows *a A′ B C′*
⟨*proof*⟩

lemma *out-out-anga*:
assumes *QCongAAcute a* **and**
B Out A C **and**
B′ Out A′ C′ **and**
a A B C
shows *a A′ B′ C′*
⟨*proof*⟩

lemma *is-null-all*:
assumes *A* ≠ *B* **and**
QCongANullAcute a
shows *a A B A*
⟨*proof*⟩

lemma *anga-col-out*:
assumes *QCongAAcute a* **and**
a A B C **and**
Col A B C
shows *B Out A C*
⟨*proof*⟩

lemma *ang-not-lg-null*:
assumes *QCong la* **and**
QCong lc **and**
QCongA a **and**
la A B **and**
lc C B **and**
a A B C
shows ¬ *QCongNull la* ∧ ¬ *QCongNull lc*
⟨*proof*⟩

lemma *anga-not-lg-null*:
  assumes
    QCongAAcute a **and**
    la A B **and**
    lc C B **and**
    a A B C
  shows ¬ *QCongNull la* ∧ ¬ *QCongNull lc*
  ⟨*proof*⟩

lemma *anga-col-null*:
  assumes *QCongAAcute a* **and**
    a A B C **and**
    Col A B C
  shows *B Out A C* ∧ *QCongANullAcute a*
  ⟨*proof*⟩

lemma *eqA-preserves-ang*:
  assumes *QCongA a* **and**
    a EqA b
  shows *QCongA b*
⟨*proof*⟩

lemma *eqA-preserves-anga*:
  assumes *QCongAAcute a* **and**
    a EqA b
  shows *QCongAAcute b*
⟨*proof*⟩

lemma *l13-6*:
  assumes *Lcos lc l a* **and**
    Lcos ld l a
  shows *lc EqLTarski ld*
⟨*proof*⟩

lemma *null-lcos-eql*:
  assumes *Lcos lp l a* **and**
    QCongANullAcute a
  shows *l EqLTarski lp*
⟨*proof*⟩

lemma *eql-lcos-null*:
  assumes *Lcos l lp a* **and**
    l EqLTarski lp
  shows *QCongANullAcute a*
⟨*proof*⟩

lemma *lcos-lg-not-null*:
  assumes *Lcos l lp a*
  shows ¬ *QCongNull l* ∧ ¬ *QCongNull lp*
⟨*proof*⟩

lemma *perp-acute-out*:
  assumes *Acute A B C* **and**
    A B Perp C C′ **and**
    Col A B C′
  shows *B Out A C′*
⟨*proof*⟩

lemma *perp-col-out--acute-aux*:
  assumes *A B Perp C C′* **and**
    B Out A C′
  shows *Acute A B C*
  ⟨*proof*⟩

lemma *perp-out--acute*:
assumes $A\ B\ Perp\ C\ C'$ **and**
$Col\ A\ B\ C'$
shows $Acute\ A\ B\ C \longleftrightarrow B\ Out\ A\ C'$
⟨*proof*⟩

lemma *obtuse-not-acute*:
assumes $Obtuse\ A\ B\ C$
shows $\neg\ Acute\ A\ B\ C$
⟨*proof*⟩

lemma *acute-not-obtuse*:
assumes $Acute\ A\ B\ C$
shows $\neg\ Obtuse\ A\ B\ C$
⟨*proof*⟩

lemma *perp-obtuse-bet*:
assumes $A\ B\ Perp\ C\ C'$ **and**
$Col\ A\ B\ C'$ **and**
$Obtuse\ A\ B\ C$
shows $Bet\ A\ B\ C'$
⟨*proof*⟩

lemma *lcos-const0*:
assumes $Lcos\ lp\ l\ a$ **and**
$QCongANullAcute\ a$
shows $\exists\ A\ B\ C.\ l\ A\ B \wedge lp\ B\ C \wedge a\ A\ B\ C$
⟨*proof*⟩

lemma *lcos-const1*:
fixes $P::'p$
assumes $Lcos\ lp\ l\ a$ **and**
$\neg\ QCongANullAcute\ a$
shows $\exists\ A\ B\ C.\ \neg\ Col\ A\ B\ P \wedge A\ B\ OS\ C\ P \wedge l\ A\ B \wedge lp\ B\ C \wedge a\ A\ B\ C$
⟨*proof*⟩

lemma *lcos-const*:
assumes $Lcos\ lp\ l\ a$
shows $\exists\ A\ B\ C.\ lp\ A\ B \wedge l\ B\ C \wedge a\ A\ B\ C$
⟨*proof*⟩

lemma *lcos-lg-distincts*:
assumes $Lcos\ lp\ l\ a$ **and**
$l\ A\ B$ **and**

$a\ A\ B\ C$
shows $A \neq B \wedge C \neq B$
⟨*proof*⟩

lemma *lcos-const-a*:
assumes $Lcos\ lp\ l\ a$
shows $\forall\ B.\ \exists\ A\ C.\ l\ A\ B \wedge lp\ B\ C \wedge a\ A\ B\ C$
⟨*proof*⟩

lemma *lcos-const-ab*:
assumes $Lcos\ lp\ l\ a$ **and**
$l\ A\ B$
shows $\exists\ C.\ lp\ B\ C \wedge a\ A\ B\ C$
⟨*proof*⟩

lemma *lcos-const-cb*:
assumes $Lcos\ lp\ l\ a$ **and**
$lp\ B\ C$
shows $\exists\ A.\ l\ A\ B \wedge a\ A\ B\ C$
⟨*proof*⟩

lemma *lcos-lg-anga*:
assumes *Lcos lp l a*
shows *Lcos lp l a* ∧ *QCong l* ∧ *QCong lp* ∧ *QCongAAcute a*
⟨*proof*⟩

lemma *lcos-eql-lcos*:
assumes *lp1 EqLTarski lp2* **and**
l1 EqLTarski l2 **and**
Lcos lp1 l1 a
shows *Lcos lp2 l2 a*
⟨*proof*⟩

lemma *lcos-not-lg-null*:
assumes *Lcos lp l a*
shows ¬ *QCongNull lp*
⟨*proof*⟩

lemma *lcos-const-o*:
assumes ¬ *Col A B P* **and**
¬ *QCongANullAcute a* **and**
QCong l **and**
QCong lp **and**
QCongAAcute a **and**
l A B **and**
Lcos lp l a
shows ∃ *C*. *A B OS C P* ∧ *a A B C* ∧ *lp B C*
⟨*proof*⟩

lemma *flat-not-acute*:
assumes *Bet A B C*
shows ¬ *Acute A B C*
⟨*proof*⟩

lemma *acute-comp-not-acute*:
assumes *Bet A B C* **and**
Acute A B D
shows ¬ *Acute C B D*
⟨*proof*⟩

lemma *lcos-per*:
assumes *QCongAAcute a* **and**
QCong l **and**
QCong lp **and**
Lcos lp l a **and**
l A C **and**
lp A B **and**
a B A C
shows *Per A B C*
⟨*proof*⟩

lemma *is-null-anga-dec*:

shows *QCongANullAcute a* ∨ ¬ *QCongANullAcute a*
⟨*proof*⟩

lemma *lcos-lg*:
assumes *Lcos lp l a* **and**
A B Perp B C **and**
a B A C **and**
l A C
shows *lp A B*
⟨*proof*⟩

lemma *l13-7*:
assumes *Lcos la l a* **and**

Lcos lb l b **and**
Lcos lab la b **and**
Lcos lba lb a
shows *lab EqLTarski lba*
⟨*proof*⟩

lemma *out-acute*:
assumes *B Out A C*
shows *Acute A B C*
⟨*proof*⟩

lemma *perp-acute*:
assumes *Col A C P* **and**
P PerpAt B P A C
shows *Acute A B P*
⟨*proof*⟩

lemma *null-lcos*:
assumes *QCong l* **and**
¬ *QCongNull l* **and**
QCongANullAcute a
shows *Lcos l l a*
⟨*proof*⟩

lemma *lcos-exists*:
assumes *QCongAAcute a* **and**
QCong l **and**
¬ *QCongNull l*
shows ∃ *lp. Lcos lp l a*
⟨*proof*⟩

lemma *lcos-uniqueness*:
assumes *Lcos l1 l a* **and**
Lcos l2 l a
shows *l1 EqLTarski l2*
⟨*proof*⟩

lemma *lcos-eqa-lcos*:
assumes *Lcos lp l a* **and**
a EqA b
shows *Lcos lp l b*
⟨*proof*⟩

lemma *lcos-eq-refl*:
assumes *QCong la* **and**
¬ *QCongNull la* **and**
QCongAAcute a
shows *EqLcos la a la a*
⟨*proof*⟩

lemma *lcos-eq-sym*:
assumes *EqLcos la a lb b*
shows *EqLcos lb b la a*
⟨*proof*⟩

lemma *lcos-eq-trans*:
assumes *EqLcos la a lb b* **and**
EqLcos lb b lc c
shows *EqLcos la a lc c*
⟨*proof*⟩

lemma *lcos2-comm*:
assumes *Lcos2 lp l a b*
shows *Lcos2 lp l b a*
⟨*proof*⟩

lemma *lcos2-exists*:
assumes *QCong l* **and**
¬ *QCongNull l* **and**
QCongAAcute a **and**
QCongAAcute b
shows ∃ *lp. Lcos2 lp l a b*
⟨*proof*⟩

lemma *lcos2-exists′*:
assumes *QCong l* **and**
¬ *QCongNull l* **and**
QCongAAcute a **and**
QCongAAcute b
shows ∃ *la lab. Lcos la l a* ∧ *Lcos lab la b*
⟨*proof*⟩

lemma *lcos2-eq-refl*:
assumes *QCong l* **and**
¬ *QCongNull l* **and**
QCongAAcute a **and**
QCongAAcute b
shows *EqLcos2 l a b l a b*
⟨*proof*⟩

lemma *lcos2-eq-sym*:
assumes *EqLcos2 l1 a b l2 c d*
shows *EqLcos2 l2 c d l1 a b*
⟨*proof*⟩

lemma *lcos2-uniqueness*:
assumes *Lcos2 l1 l a b* **and**
Lcos2 l2 l a b
shows *l1 EqLTarski l2*
⟨*proof*⟩

lemma *lcos2-eql-lcos2*:
assumes *Lcos2 la lla a b* **and**
lla EqLTarski llb **and**
la EqLTarski lb
shows *Lcos2 lb llb a b*
⟨*proof*⟩

lemma *lcos2-lg-anga*:
assumes *Lcos2 lp l a b*
shows *Lcos2 lp l a b* ∧ *QCong lp* ∧ *QCong l* ∧ *QCongAAcute a* ∧ *QCongAAcute b*
⟨*proof*⟩

lemma *lcos2-eq-trans*:
assumes *EqLcos2 l1 a b l2 c d* **and**
EqLcos2 l2 c d l3 e f
shows *EqLcos2 l1 a b l3 e f*
⟨*proof*⟩

lemma *lcos-eq-lcos2-eq*:
assumes *QCongAAcute c* **and**
EqLcos la a lb b
shows *EqLcos2 la a c lb b c*
⟨*proof*⟩

lemma *lcos2-lg-not-null*:
assumes *Lcos2 lp l a b*
shows ¬ *QCongNull l* ∧ ¬ *QCongNull lp*
⟨*proof*⟩

lemma *lcos3-lcos-1-2-a*:
assumes *Lcos3 lp l a b c*

shows ∃ *la. Lcos la l a* ∧ *Lcos2 lp la b c*
⟨*proof*⟩

lemma *lcos3-lcos-1-2-b*:
assumes ∃ *la. Lcos la l a* ∧ *Lcos2 lp la b c*
shows *Lcos3 lp l a b c*
⟨*proof*⟩

lemma *lcos3-lcos-1-2*:
shows *Lcos3 lp l a b c* ⟷ (∃ *la. Lcos la l a* ∧ *Lcos2 lp la b c*)
⟨*proof*⟩

lemma *lcos3-lcos-2-1-a*:
assumes *Lcos3 lp l a b c*
shows ∃ *lab. Lcos2 lab l a b* ∧ *Lcos lp lab c*
⟨*proof*⟩

lemma *lcos3-lcos-2-1-b*:
assumes ∃ *lab. Lcos2 lab l a b* ∧ *Lcos lp lab c*
shows *Lcos3 lp l a b c*
⟨*proof*⟩

lemma *lcos3-lcos-2-1*:
shows *Lcos3 lp l a b c* ⟷ (∃ *lab. Lcos2 lab l a b* ∧ *Lcos lp lab c*)
⟨*proof*⟩

lemma *lcos3-permut3*:
assumes *Lcos3 lp l a b c*
shows *Lcos3 lp l b a c*
⟨*proof*⟩

lemma *lcos3-permut1*:
assumes *Lcos3 lp l a b c*
shows *Lcos3 lp l a c b*
⟨*proof*⟩

lemma *lcos3-permut2*:
assumes *Lcos3 lp l a b c*
shows *Lcos3 lp l c b a*
⟨*proof*⟩

lemma *lcos3-exists*:
assumes *QCong l* **and**
¬ *QCongNull l* **and**
QCongAAcute a **and**
QCongAAcute b **and**
QCongAAcute c
shows ∃ *lp. Lcos3 lp l a b c*
⟨*proof*⟩

lemma *lcos3-eq-refl*:
assumes *QCong l* **and**
¬ *QCongNull l* **and**
QCongAAcute a **and**
QCongAAcute b **and**
QCongAAcute c
shows *EqLcos3 l a b c l a b c*
⟨*proof*⟩

lemma *lcos3-eq-sym*:
assumes *EqLcos3 l1 a b c l2 d e f*
shows *EqLcos3 l2 d e f l1 a b c*
⟨*proof*⟩

lemma *lcos3-uniqueness*:
assumes *Lcos3 l1 l a b c* **and**

Lcos3 l2 l a b c
shows *l1 EqLTarski l2*
⟨*proof*⟩

lemma *lcos3-eql-lcos3*:
assumes *Lcos3 la lla a b c* **and**
lla EqLTarski llb **and**
la EqLTarski lb
shows *Lcos3 lb llb a b c*
⟨*proof*⟩

lemma *lcos3-lg-anga*:
assumes *Lcos3 lp l a b c*
shows *Lcos3 lp l a b c* ∧ *QCong lp* ∧ *QCong l* ∧ *QCongAAcute a* ∧ *QCongAAcute b* ∧ *QCongAAcute c*
⟨*proof*⟩

lemma *lcos3-lg-not-null*:
assumes *Lcos3 lp l a b c*
shows ¬ *QCongNull l* ∧ ¬ *QCongNull lp*
⟨*proof*⟩

lemma *lcos3-eq-trans*:
assumes *EqLcos3 l1 a b c l2 d e f* **and**
EqLcos3 l2 d e f l3 g h i
shows *EqLcos3 l1 a b c l3 g h i*
⟨*proof*⟩

lemma *lcos-eq-lcos3-eq*:
assumes *QCongAAcute c* **and**
QCongAAcute d **and**
EqLcos la a lb b
shows *EqLcos3 la a c d lb b c d*
⟨*proof*⟩

lemma *lcos2-eq-lcos3-eq*:
assumes *QCongAAcute e* **and**
EqLcos2 la a b lb c d
shows *EqLcos3 la a b e lb c d e*
⟨*proof*⟩

lemma *projp-id*:
assumes $P\ P'$ *Projp A B* **and**
$P\ Q'$ *Projp A B*
shows $P' = Q'$
⟨*proof*⟩

lemma *projp-idem*:
assumes $P\ P'$ *Projp A B*
shows $P'\ P'$ *Projp A B*
⟨*proof*⟩

lemma *col-projp-eq*:
assumes *Col A B P* **and**
$P\ P'$ *Projp A B*
shows $P = P'$
⟨*proof*⟩

lemma *projp-col*:
assumes $P\ P'$ *Projp A B*
shows *Col A B* P'
⟨*proof*⟩

lemma *project-id*:
assumes $P\ P'$ *Proj A B X Y* **and**

Col A B P
shows $P = P'$
⟨*proof*⟩

lemma *project-not-id*:
assumes $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
¬ *Col A B P*
shows $P \neq P'$
⟨*proof*⟩

lemma *project-col*:
assumes *P P Proj A B X Y*
shows *Col A B P*
⟨*proof*⟩

lemma *project-not-col*:
assumes $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$P \neq P'$
shows ¬ *Col A B P*
⟨*proof*⟩

lemma *par-col-project*:
assumes $A \neq B$ **and**
¬ *A B Par X Y* **and**
$P\ P'$ *Par* $X\ Y$ **and**
Col $A\ B\ P'$
shows $P\ P'$ *Proj* $A\ B\ X\ Y$
⟨*proof*⟩

lemma *cong-conga3-cong3*:
assumes ¬ *Col A B C* **and**
Cong $A\ B\ A'\ B'$ **and**
$A\ B\ C$ *CongA3* $A'\ B'\ C'$
shows $A\ B\ C$ *Cong3* $A'\ B'\ C'$
⟨*proof*⟩

lemma *project-par-dir*:
assumes $P \neq P'$ **and**
$P\ P'$ *Proj* $A\ B\ X\ Y$
shows $P\ P'$ *Par* $X\ Y$
⟨*proof*⟩

lemma *project-idem*:
assumes $P\ P'$ *Proj* $A\ B\ X\ Y$
shows $P'\ P'$ *Proj* $A\ B\ X\ Y$
⟨*proof*⟩

lemma *perp-projp*:
assumes P' *PerpAt* $A\ B\ P\ P'$
shows $P\ P'$ *Projp* $A\ B$
⟨*proof*⟩

lemma *proj-distinct*:
assumes $P\ P'$ *Projp* $A\ B$
shows $P' \neq A \vee P' \neq B$
⟨*proof*⟩

lemma *l13-10-aux1*:
assumes *Col PO A B* **and**
Col PO P Q **and**
PO P Perp P A **and**
PO Q Perp Q B **and**
QCong la **and**
QCong lb **and**
QCong lp **and**
QCong lq **and**

la PO A **and**
lb PO B **and**
lp PO P **and**
lq PO Q
shows ∃ *a. QCongAAcute a* ∧ *Lcos lp la a* ∧ *Lcos lq lb a*
⟨*proof*⟩

lemma *l13-10-aux2*:
assumes *Col PO A B* **and**
QCong la **and**
QCong lla **and**
QCong lb **and**
QCong llb **and**
la PO A **and**
lla PO A **and**
lb PO B **and**
llb PO B **and**
A ≠ *PO* **and**
B ≠ *PO*
shows ∃ *a. QCongAAcute a* ∧ *Lcos lla la a* ∧ *Lcos llb lb a*
⟨*proof*⟩

lemma *l13-6-bis*:
assumes *Lcos lp l1 a* **and**
Lcos lp l2 a
shows *l1 EqLTarski l2*
⟨*proof*⟩

lemma *lcos3-lcos2*:
assumes *EqLcos3 l1 a b n l2 c d n*
shows *EqLcos2 l1 a b l2 c d*
⟨*proof*⟩

lemma *lcos2-lcos*:
assumes *EqLcos2 l1 a c l2 b c*
shows *EqLcos l1 a l2 b*
⟨*proof*⟩

lemma *lcos-per-anga*:
assumes *Lcos lp la a* **and**
la PO A **and**
lp PO P **and**
Per A P PO
shows *a A PO P*
⟨*proof*⟩

lemma *lcos-lcos-cop--col*:
assumes *Lcos lp la a* **and**
Lcos lp lb b **and**
la PO A **and**
lb PO B **and**
lp PO P **and**
a A PO P **and**
b B PO P **and**
Coplanar PO A B P
shows *Col A B P*
⟨*proof*⟩

lemma *l13-10-aux3*:
assumes ¬ *Col PO A A′* **and**
B ≠ *PO* **and**
C ≠ *PO* **and**
Col PO A B **and**
Col PO B C **and**
B′ ≠ *PO* **and**
C′ ≠ *PO* **and**

$Col\ PO\ A'\ B'$ **and**
$Col\ PO\ B'\ C'$ **and**
$PO\ Perp2\ B\ C'\ C\ B'$ **and**
$PO\ Perp2\ C\ A'\ A\ C'$ **and**
$Bet\ A\ PO\ B$
shows $Bet\ A'\ PO\ B'$
⟨*proof*⟩

lemma *l13-10-aux4*:
assumes $\neg\ Col\ PO\ A\ A'$ **and**
$B \neq PO$ **and**
$C \neq PO$ **and**
$Col\ PO\ A\ B$ **and**
$Col\ PO\ B\ C$ **and**
$B' \neq PO$ **and**
$C' \neq PO$ **and**
$Col\ PO\ A'\ B'$ **and**
$Col\ PO\ B'\ C'$ **and**
$PO\ Perp2\ B\ C'\ C\ B'$ **and**
$PO\ Perp2\ C\ A'\ A\ C'$ **and**
$Bet\ PO\ A\ B$
shows $PO\ Out\ A'\ B'$
⟨*proof*⟩

lemma *l13-10-aux5*:
assumes $\neg\ Col\ PO\ A\ A'$ **and**
$B \neq PO$ **and**
$C \neq PO$ **and**
$Col\ PO\ A\ B$ **and**
$Col\ PO\ B\ C$ **and**
$B' \neq PO$ **and**
$C' \neq PO$ **and**
$Col\ PO\ A'\ B'$ **and**
$Col\ PO\ B'\ C'$ **and**
$PO\ Perp2\ B\ C'\ C\ B'$ **and**
$PO\ Perp2\ C\ A'\ A\ C'$ **and**
$PO\ Out\ A\ B$
shows $PO\ Out\ A'\ B'$
⟨*proof*⟩

lemma *cop-per2--perp-aux*:
assumes $A \neq B$ **and**
$X \neq Y$ **and**
$B \neq X$ **and**
$Coplanar\ A\ B\ X\ Y$ **and**
$Per\ A\ B\ X$ **and**
$Per\ A\ B\ Y$
shows $A\ B\ Perp\ X\ Y$
⟨*proof*⟩

lemma *cop-per2--perp*:
assumes $A \neq B$ **and**
$X \neq Y$ **and**
$B \neq X \vee B \neq Y$ **and**
$Coplanar\ A\ B\ X\ Y$ **and**
$Per\ A\ B\ X$ **and**
$Per\ A\ B\ Y$
shows $A\ B\ Perp\ X\ Y$
⟨*proof*⟩

lemma *l13-10*:
assumes $\neg\ Col\ PO\ A\ A'$ **and**
$B \neq PO$ **and**
$C \neq PO$ **and**
$Col\ PO\ A\ B$ **and**
$Col\ PO\ B\ C$ **and**

$B' \neq PO$ **and**
$C' \neq PO$ **and**
Col $PO\ A'\ B'$ **and**
Col $PO\ B'\ C'$ **and**
PO *Perp2* $B\ C'\ C\ B'$ **and**
PO *Perp2* $C\ A'\ A\ C'$
shows PO *Perp2* $A\ B'\ B\ A'$
⟨*proof*⟩

lemma *Pj-exists*:
fixes $A\ B\ C$
shows $\exists\ D.\ A\ B$ *Pj* $C\ D$
⟨*proof*⟩

lemma *project-trivial*:
assumes $A \neq B$
and $X \neq Y$
and *Col* $A\ B\ P$
and $\neg\ A\ B$ *Par* $X\ Y$
shows $P\ P$ *Proj* $A\ B\ X\ Y$
⟨*proof*⟩

lemma *pj-col-project*:
assumes $A \neq B$
and $X \neq Y$
and *Col* $P'\ A\ B$
and $\neg\ A\ B$ *Par* $X\ Y$
and $X\ Y$ *Pj* $P\ P'$
shows $P\ P'$ *Proj* $A\ B\ X\ Y$
⟨*proof*⟩

lemma *pj-trivial*:
shows $A\ B$ *Pj* $C\ C$
⟨*proof*⟩

lemma *O-not-positive*:
shows $\neg$ *Ps* $PO\ E\ PO$
⟨*proof*⟩

lemma *col-pos-or-neg*:
assumes $PO \neq E$
and $PO \neq X$
and *Col* $PO\ E\ X$
shows *Ps* $PO\ E\ X \vee$ *Ng* $PO\ E\ X$
⟨*proof*⟩

lemma *length-cong*:
assumes *Length* $PO\ E\ E'\ A\ B\ AB$
shows *Cong* $A\ B\ PO\ AB$
⟨*proof*⟩

lemma *triangular-equality-equiv-a* :
assumes $\forall\ PO\ E\ A.\ PO \neq E \longrightarrow (\forall\ E'\ B\ C\ AB\ BC\ AC.$ *Bet* $A\ B\ C \wedge$ *Length* $PO\ E\ E'\ A\ B\ AB \wedge$
Length $PO\ E\ E'\ B\ C\ BC \wedge$
Length $PO\ E\ E'\ A\ C\ AC$
$\longrightarrow$ *Sum* $PO\ E\ E'\ AB\ BC\ AC)$
shows $\forall\ PO\ E\ E'\ A\ B\ C\ AB\ BC\ AC.\ PO \neq E \wedge$ *Bet* $A\ B\ C \wedge$ *Length* $PO\ E\ E'\ A\ B\ AB \wedge$
Length $PO\ E\ E'\ B\ C\ BC \wedge$ *Length* $PO\ E\ E'\ A\ C\ AC$
$\longrightarrow$ *Sum* $PO\ E\ E'\ AB\ BC\ AC$
⟨*proof*⟩

lemma *triangular-equality-equiv-b* :
assumes $\forall\ PO\ E\ E'\ A\ B\ C\ AB\ BC\ AC.\ PO \neq E \wedge$ *Bet* $A\ B\ C \wedge$ *Length* $PO\ E\ E'\ A\ B\ AB \wedge$

$Length\ PO\ E\ E'\ B\ C\ BC \wedge Length\ PO\ E\ E'\ A\ C\ AC$
$\longrightarrow Sum\ PO\ E\ E'\ AB\ BC\ AC$
shows $\forall\ PO\ E\ A.\ PO \neq E \longrightarrow (\forall\ E'\ B\ C\ AB\ BC\ AC.\ Bet\ A\ B\ C \wedge Length\ PO\ E\ E'\ A\ B\ AB \wedge$
$Length\ PO\ E\ E'\ B\ C\ BC \wedge Length\ PO\ E\ E'\ A\ C\ AC$
$\longrightarrow Sum\ PO\ E\ E'\ AB\ BC\ AC)$
⟨*proof*⟩

lemma *triangular-equality-equiv* :
shows $(\forall\ PO\ E\ A.\ PO \neq E \longrightarrow (\forall\ E'\ B\ C\ AB\ BC\ AC.\ Bet\ A\ B\ C \wedge Length\ PO\ E\ E'\ A\ B\ AB \wedge$
$Length\ PO\ E\ E'\ B\ C\ BC \wedge Length\ PO\ E\ E'\ A\ C\ AC$
$\longrightarrow Sum\ PO\ E\ E'\ AB\ BC\ AC))$
$\longleftrightarrow$
$(\forall\ PO\ E\ E'\ A\ B\ C\ AB\ BC\ AC.\ PO \neq E \wedge Bet\ A\ B\ C \wedge Length\ PO\ E\ E'\ A\ B\ AB \wedge$
$Length\ PO\ E\ E'\ B\ C\ BC \wedge Length\ PO\ E\ E'\ A\ C\ AC$
$\longrightarrow Sum\ PO\ E\ E'\ AB\ BC\ AC)$
⟨*proof*⟩

lemma *sign-dec*:
assumes $Col\ PO\ E\ A$
and $PO \neq E$
shows $A = PO \vee Ps\ PO\ E\ A \vee Ng\ PO\ E\ A$
⟨*proof*⟩

lemma *not-neg-pos*:
assumes $E \neq PO$
and $Col\ PO\ E\ A$
and $\neg\ Ng\ PO\ E\ A$
shows $Ps\ PO\ E\ A \vee A = PO$
⟨*proof*⟩

lemma *Tarski-Pre-Non-Euclidean-aux-pre*:
assumes $\exists\ A\ B\ C\ D\ T.\ \neg\ ((Bet\ A\ D\ T \wedge Bet\ B\ D\ C \wedge A \neq D)$
$\longrightarrow$
$(\exists\ X\ Y.\ Bet\ A\ B\ X \wedge Bet\ A\ C\ Y \wedge Bet\ X\ T\ Y))$
shows $\exists\ A0\ B0\ C0\ D0\ T0.\ (Bet\ A0\ D0\ T0 \wedge Bet\ B0\ D0\ C0 \wedge A0 \neq D0 \wedge$
$(\forall\ X\ Y.\ ((Bet\ A0\ B0\ X \wedge Bet\ A0\ C0\ Y) \longrightarrow \neg\ Bet\ X\ T0\ Y)))$
⟨*proof*⟩

end
end

theory *Tarski-Neutral-Archimedes*

imports
Tarski-Neutral

begin

2 Continuity Axioms

context *Tarski-neutral-dimensionless*

begin

2.1 Definitions

definition *greenberg-s-axiom* ::
bool
(*GreenBergsAxiom*)
where
greenberg-s-axiom $\equiv \forall\ P\ Q\ R\ A\ B\ C.$
$\neg\ Col\ A\ B\ C \wedge Acute\ A\ B\ C \wedge Q \neq R \wedge Per\ P\ Q\ R \longrightarrow (\exists\ S.\ P\ S\ Q\ LtA\ A\ B\ C \wedge Q\ Out\ S\ R)$

definition *aristotle-s-axiom* ::
bool

(*AristotleAxiom*) **where**
aristotle-s-axiom ≡ ∀ *P Q A B C*.
¬ *Col A B C* ∧ *Acute A B C* ⟶
(∃ *X Y*. *B Out A X* ∧ *B Out C Y* ∧ *Per B X Y* ∧ *P Q Lt X Y*)

definition *Axiom1*:: *bool* **where** *Axiom1* ≡ ∀ *A B C D*.
(∃ *I*. *Col I A B* ∧ *Col I C D*) ∨ ¬ (∃ *I*. *Col I A B* ∧ *Col I C D*)

definition *PreGrad* :: *'p* ⇒ *'p* ⇒ *'p* ⇒ *'p* ⇒ *bool* **where**
PreGrad A B C D ≡ (*A* ≠ *B* ∧ *Bet A B C* ∧ *Bet A C D* ∧ *Cong A B C D*)

fun *Sym* :: *'p* ⇒ *'p* ⇒ *'p* ⇒ *'p* **where**
Sym A B C = (*if* (*A* ≠ *B* ∧ *Bet A B C*) *then*
(*SOME x*::*'p*. *PreGrad A B C x*)
else
A)

fun *Gradn* :: [*'p*,*'p*] ⇒ *nat* ⇒*'p***where**
Gradn A B n = (*if* (*A* = *B*) *then*
A
else
(*if* (*n* = *0*) *then*
A
else
(*if* (*n* = *1*) *then*
B
else
(*Sym A B* (*Gradn A B* (*n*−*1*))))))

definition *Grad* :: [*'p*,*'p*,*'p*] ⇒ *bool* **where**
Grad A B C ≡ ∃ *n*. (*n* ≠ *0*) ∧ (*C* = *Gradn A B n*)

inductive *GradI* :: [*'p*,*'p*,*'p*] ⇒ *bool* **for** *A B*
where
gradi-init : *GradI A B B*
| *gradi-stab* : *GradI A B C'* **if**
GradI A B C
and *Bet A C C'*
and *Cong A B C C'*

definition *Reach* :: [*'p*,*'p*,*'p*,*'p*] ⇒ *bool* **where**
Reach A B C D ≡ ∃ *B'*. *Grad A B B'* ∧ *C D Le A B'*

definition *archimedes-axiom* ::
bool
(*ArchimedesAxiom*) **where**
archimedes-axiom ≡ ∀ *A B C D*::*'p*.
A ≠ *B* ⟶ *Reach A B C D*

inductive *GradA* :: [*'p*,*'p*,*'p*,*'p*,*'p*,*'p*] ⇒ *bool* **for** *A B C*
where
grada-init : *GradA A B C D E F* **if**
A B C CongA D E F
| *grada-stab* : *GradA A B C G H I* **if**
GradA A B C D E F
and *SAMS D E F A B C*
and *D E F A B C SumA G H I*

inductive *GradAExp* :: [*'p*,*'p*,*'p*,*'p*,*'p*,*'p*] ⇒ *bool* **for** *A B C*
where
gradaexp-init : *GradAExp A B C D E F* **if**
A B C CongA D E F
| *gradaexp-stab* : *GradAExp A B C G H I* **if**
GradAExp A B C D E F
and *SAMS D E F D E F*
and *D E F D E F SumA G H I*

definition *Grad2* :: ['p,'p,'p,'p,'p,'p] ⇒ *bool* **where**
Grad2 A B C D E F ≡ ∃ *n*. (*n* ≠ *0*) ∧ (*C* = *Gradn A B n*) ∧ (*F* = *Gradn D E n*)

fun *SymR* :: 'p ⇒ 'p ⇒ 'p **where**
SymR A B = (*SOME x*::'p. *B Midpoint A x*)

fun *GradExpn* :: 'p ⇒ 'p ⇒ *nat* ⇒ 'p **where**
```
(GradExpn A B n) = (if (A = B) then
                      A
                        else
                      (if (n = 0) then
                         A
                           else
                         (if (n = 1) then
                            B
                              else
                            (SymR A (GradExpn A B (n−1))))))
```

definition *GradExp* :: 'p ⇒ 'p ⇒ 'p ⇒ *bool* **where**
GradExp A B C ≡ ∃ *n*. (*n* ≠ *0*) ∧ *C* = *GradExpn A B n*

definition *GradExp2* :: ['p,'p,'p,'p,'p,'p] ⇒ *bool* **where**
GradExp2 A B C D E F ≡ ∃ *n*. (*n* ≠ *0*) ∧ (*C* = *GradExpn A B n*) ∧ (*F* = *GradExpn D E n*)

fun *MidR* :: 'p ⇒ 'p ⇒ 'p **where**
MidR A B = (*SOME x*. *x Midpoint A B*)

fun *GradExpInvn* :: 'p ⇒ 'p ⇒ *nat* ⇒ 'p **where**
```
(GradExpInvn A B n) = (if (A = B) then
                         A
                           else
                         (if (n = 0) then
                            B
                              else
                            (if (n = 1) then
                               (MidR A B)
                                 else
                               (MidR A (GradExpInvn A B (n−1))))))
```

definition *GradExpInv* :: 'p ⇒ 'p ⇒ 'p ⇒ *bool* **where**
GradExpInv A B C ≡ ∃ *n*. *B* = *GradExpInvn A C n*

2.2 Propositions

lemma *PreGrad-lem1*:
assumes *A* ≠ *B* **and**
Bet A B C
shows ∃ *x*. *PreGrad A B C x*
⟨*proof*⟩

lemma *PreGrad-uniq*:
assumes *PreGrad A B C x* **and**
PreGrad A B C y
shows *x* = *y*
⟨*proof*⟩

lemma *Diff-Mid--PreGrad*:
assumes *A* ≠ *B* **and**
B Midpoint A C
shows *PreGrad A B B C*
⟨*proof*⟩

lemma *Diff-Mid-Mid-PreGrad*:
assumes *A* ≠ *B* **and**
B Midpoint A C **and**

C Midpoint B D
shows *PreGrad A B C D*
⟨*proof*⟩

lemma *Sym-Diff--Diff*:
assumes *Sym A B C = D* **and**
A ≠ D
shows *A ≠ B*
⟨*proof*⟩

lemma *Sym-Refl*:
Sym A A A = A
⟨*proof*⟩

lemma *Diff-Mid--Sym*:
assumes *A ≠ B* **and**
B Midpoint A C
shows *Sym A B B = C*
⟨*proof*⟩

lemma *Mid-Mid--Sym*:
assumes *A ≠ B* **and**
B Midpoint A C **and**
C Midpoint B D
shows *Sym A B C = D*
⟨*proof*⟩

lemma *Sym-Bet--Bet-Bet*:
assumes *Sym A B C = D* **and**
A ≠ B **and**
Bet A B C
shows *Bet A B D ∧ Bet A C D*
⟨*proof*⟩

lemma *Sym-Bet--Cong*:
assumes *Sym A B C = D* **and**
A ≠ B **and**
Bet A B C
shows *Cong A B C D*
⟨*proof*⟩

lemma *LemSym-aux*:
assumes *A ≠ B* **and**
Bet A B C **and**
Bet A C D **and**
Cong A B C D
shows *Sym A B C = D*
⟨*proof*⟩

lemma *Lem-Gradn-id-n*:
Gradn A A n = A
⟨*proof*⟩

lemma *Lem-Gradn-0*:
Gradn A B 0 = A
⟨*proof*⟩

lemma *Lem-Gradn-1*:
Gradn A B 1 = B
⟨*proof*⟩

lemma *Diff--Gradn-Sym*:
assumes *A ≠ B* **and**
n > 1
shows *Gradn A B n = Sym A B (Gradn A B (n−1))*
⟨*proof*⟩

lemma *Diff--Bet-Gradn-Suc*:
assumes $A \neq B$
shows *Bet A B* (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Diff-Le-Gradn-Suc*:
assumes $A \neq B$
shows *A B Le A* (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Diff--Bet-Gradn*:
assumes $A \neq B$ **and**
$n \neq 0$
shows *Bet A B* (*Gradn A B n*)
⟨*proof*⟩

lemma *Diff-Le-Gradn-n*:
assumes $A \neq B$ **and**
$n \neq 0$
shows *A B Le A* (*Gradn A B n*)
⟨*proof*⟩

lemma *Diff-Bet-Gradn-Suc-Gradn-Suc2*:
assumes $A \neq B$
shows *Bet A* (*Gradn A B* (*Suc n*)) (*Gradn A B* (*Suc* (*Suc n*)))
⟨*proof*⟩

lemma *Diff--Bet-Gradn-Gradn-SucA*:
assumes $A \neq B$
shows *A* (*Gradn A B* (*Suc n*)) *Le A* (*Gradn A B* (*Suc* (*Suc n*)))
⟨*proof*⟩

lemma *Diff--Bet-Gradn-Gradn-Suc*:
assumes $A \neq B$
shows *Bet A* (*Gradn A B n*) (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Bet-Gradn-Gradn-Suc*:
shows *Bet A* (*Gradn A B n*) (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Gradn-Le-Gradn-Suc*:
shows *A* (*Gradn A B n*) *Le A* (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Bet-Gradn-Suc-Gradn-Suc2*:
shows *Bet B* (*Gradn A B* (*Suc n*)) (*Gradn A B* (*Suc*(*Suc n*)))
⟨*proof*⟩

lemma *Gradn-Suc-Le-Gradn-Suc2*:
shows *B* (*Gradn A B* (*Suc n*)) *Le B* (*Gradn A B* (*Suc*(*Suc n*)))
⟨*proof*⟩

lemma *Diff-Le--Bet-Gradn-Plus*:
assumes $A \neq B$ **and**
$n \leq m$
shows *Bet A* (*Gradn A B n*) (*Gradn A B* ($k + n$))
⟨*proof*⟩

lemma *Diff-Le-Gradn-Plus*:
assumes $A \neq B$ **and**
$n \leq m$
shows *A* (*Gradn A B n*) *Le A* (*Gradn A B* ($k + n$))
⟨*proof*⟩

lemma *Diff-Le-Bet--Gradn-Gradn*:
assumes $A \neq B$ **and**
$n \leq m$
shows *Bet A* (*Gradn A B n*) (*Gradn A B m*)
⟨*proof*⟩

lemma *Diff-Le-Gradn*:
assumes $A \neq B$ **and**
$n \leq m$
shows *A* (*Gradn A B n*) *Le A* (*Gradn A B m*)
⟨*proof*⟩

lemma *Diff--Cong-Gradn-Suc-Gradn-Suc2*:
assumes $A \neq B$
shows *Cong A B* (*Gradn A B* (*Suc n*)) (*Gradn A B* (*Suc* (*Suc n*)))
⟨*proof*⟩

lemma *Cong-Gradn-Suc-Gradn-Suc2*:
shows *Cong A B* (*Gradn A B* (*Suc n*)) (*Gradn A B* (*Suc* (*Suc n*)))
⟨*proof*⟩

lemma *Cong-Gradn-Gradn-Suc*:
shows *Cong a b* (*Gradn a b n*) (*Gradn a b* (*Suc n*))
⟨*proof*⟩

lemma *Diff-Bet-Bet-Cong-Gradn-Suc*:
assumes $A \neq B$ **and**
Bet A B C **and**
Bet A (*Gradn A B n*) *C* **and**
Cong A B (*Gradn A B n*) *C*
shows $C =$ (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *grad-rec-0-1*:
shows *Cong a b* (*Gradn a b 0*) (*Gradn a b 1*)
⟨*proof*⟩

lemma *grad-rec-1-2*:
shows *Cong a b* (*Gradn a b 1*) (*Gradn a b 2*)
⟨*proof*⟩

lemma *grad-rec-2-3*:
shows *Cong a b* (*Gradn a b 2*) (*Gradn a b 3*)
⟨*proof*⟩

lemma *grad-rec-a-a*:
shows (*Gradn a a n*) $=$ *a*
⟨*proof*⟩

lemma *Gradn-uniq-aux-1*:
assumes $A \neq B$
shows *Gradn A B n* $\neq$ *Gradn A B* (*Suc n*)
⟨*proof*⟩

lemma *Gradn-uniq-aux-1-aa*:
assumes $A \neq B$
shows *Gradn A B* ($k + n$) $\neq$ *Gradn A B* ($k +$ (*Suc n*))
⟨*proof*⟩

lemma *Gradn-uniq-aux-1-bb*:
assumes $A \neq B$
shows *Gradn A B* ($k + n$) $\neq$ *Gradn A B* ($k +$ (*Suc* (*Suc n*)))
⟨*proof*⟩

lemma *Gradn-aux-1-0*:
assumes $A \neq B$

shows *Gradn A B* (*Suc n*) ≠ *A*
⟨*proof*⟩

lemma *Gradn-aux-1-1*:
assumes *A* ≠ *B* **and**
n ≠ *0*
shows *Gradn A B* (*Suc n*) ≠ *B*
⟨*proof*⟩

lemma *Gradn-aux-1-1-bis*:
assumes *A* ≠ *B* **and**
n ≠ *1*
shows *Gradn A B n* ≠ *B*
⟨*proof*⟩

lemma *Gradn-aux-1-2*:
assumes *A* ≠ *B* **and**
Gradn A B n = *A*
shows *n* = *0*
⟨*proof*⟩

lemma *Gradn-aux-1-3*:
assumes *A* ≠ *B* **and**
Gradn A B n = *B*
shows *n* = *1*
⟨*proof*⟩

lemma *Gradn-uniq-aux-2-a*:
assumes *A* ≠ *B* **and**
n ≠ *0*
shows *Gradn A B 0* ≠ *Gradn A B n*
⟨*proof*⟩

lemma *Gradn-uniq-aux-2*:
assumes *A* ≠ *B* **and**
n < *m*
shows *Gradn A B n* ≠ *Gradn A B m*
⟨*proof*⟩

lemma *Gradn-uniq*:
assumes *A* ≠ *B* **and**
Gradn A B n = *Gradn A B m*
shows *n* = *m*
⟨*proof*⟩

lemma *Gradn-le-suc-1*:
shows *A* (*Gradn A B n*) *Le A* (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Gradn-le-1*:
assumes *m* ≤ *n*
shows *A* (*Gradn A B m*) *Le A* (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Gradn-le-suc-2*:
shows *B* (*Gradn A B* (*Suc n*)) *Le B* (*Gradn A B* (*Suc*(*Suc n*)))
⟨*proof*⟩

lemma *grad-equiv-coq-1*:
shows *Grad A B B*
⟨*proof*⟩

lemma *grad-aab--ab*:
assumes *Grad A A B*
shows *A* = *B*
⟨*proof*⟩

lemma *grad-stab*:
assumes *Grad A B C* **and**
Bet A C C' **and**
Cong A B C C'
shows *Grad A B C'*
⟨*proof*⟩

lemma *grad--bet*:
assumes *Grad A B C*
shows *Bet A B C*
⟨*proof*⟩

lemma *grad--col*:
assumes *Grad A B C*
shows *Col A B C*
⟨*proof*⟩

lemma *grad-neq--neq13*:
assumes *Grad A B C* **and**
$A \neq B$
shows $A \neq C$
⟨*proof*⟩

lemma *grad-neq--neq12*:
assumes *Grad A B C* **and**
$A \neq C$
shows $A \neq B$
⟨*proof*⟩

lemma *grad112--eq*:
assumes *Grad A A B*
shows $A = B$
⟨*proof*⟩

lemma *grad121--eq*:
assumes *Grad A B A*
shows $A = B$
⟨*proof*⟩

lemma *grad--le*:
assumes *Grad A B C*
shows *A B Le A C*
⟨*proof*⟩

lemma *grad2-init*:
shows *Grad2 A B B C D D*
⟨*proof*⟩

lemma *Grad2-stab*:
assumes *Grad2 A B C D E F* **and**
Bet A C C' **and**
Cong A B C C' **and**
Bet D F F' **and**
Cong D E F F'
shows *Grad2 A B C' D E F'*
⟨*proof*⟩

lemma *bet-cong2-grad--grad2-aux-1*:
assumes $C = (Gradn\ A\ B\ 0)$ **and**
Bet D E F **and**
Cong A B D E **and**
Cong B C E F
shows $F = Gradn\ D\ E\ 2$
⟨*proof*⟩

lemma *bet-cong2-grad--grad2-aux-2*:
assumes *Bet D E F* **and**
Cong A B D E **and**
Cong B (Gradn A B (Suc n)) E F
shows *F = Gradn D E (Suc n)*
⟨*proof*⟩

lemma *bet-cong2-grad--grad2-aux*:
assumes $n \neq 0$ **and**
C = (Gradn A B n) **and**
Bet D E F **and**
Cong A B D E **and**
Cong B C E F
shows *F = Gradn D E n*
⟨*proof*⟩

lemma *bet-cong2-grad--grad2*:
assumes *Grad A B C* **and**
Bet D E F **and**
Cong A B D E **and**
Cong B C E F
shows *Grad2 A B C D E F*
⟨*proof*⟩

lemma *grad2--grad123*:
assumes *Grad2 A B C D E F*
shows *Grad A B C*
⟨*proof*⟩

lemma *grad2--grad456*:
assumes *Grad2 A B C D E F*
shows *Grad D E F*
⟨*proof*⟩

lemma *grad-sum-aux-R1*:
assumes
A C Le A D **and**
Cong A D A E **and**
Cong A C A E' **and**
A Out E E'
shows *Bet A E' E*
⟨*proof*⟩

lemma *grad-sum-aux-0*:
assumes $A \neq B$ **and**
D = Gradn A B (Suc(Suc n)) **and**
Cong A D A E **and**
C = Gradn A B (Suc n) **and**
Cong A C A E' **and**
A Out E E'
shows *Bet A E' E*
⟨*proof*⟩

lemma *grad-sum-aux-1*:
assumes $A \neq B$ **and**
D = Gradn A B (Suc(Suc n)) **and**
Bet A C E **and**
Cong A D C E **and**
F = Gradn A B (Suc n) **and**
Bet A C E' **and**
Cong A F C E' **and**
$C \neq A$
shows *Bet A E' E*
⟨*proof*⟩

lemma *grad-sum-aux-1A*:
  assumes $A \neq B$ **and**
    $C = Gradn\ A\ B\ (Suc\ (Suc\ 0))$
  shows *B Midpoint A C*
⟨*proof*⟩

lemma *grad-sum-aux-2*:
  assumes $A \neq B$ **and**
    $D = Gradn\ A\ B\ (Suc(Suc\ n))$ **and**
    Bet A C E **and**
    Cong A D C E **and**
    $F = Gradn\ A\ B\ (Suc\ n)$ **and**
    Bet A C E' **and**
    Cong A F C E' **and**
    $C \neq A$
  shows *Cong A B E' E*
⟨*proof*⟩

lemma *grad-sum-aux*:
  assumes $A \neq B$ **and**
    $C = Gradn\ A\ B\ (Suc\ n)$ **and**
    $D = Gradn\ A\ B\ (Suc\ m)$ **and**
    Bet A C E **and**
    Cong A D C E
  shows $E = Gradn\ A\ B\ ((Suc\ n) + (Suc\ m))$
⟨*proof*⟩

lemma *grad-sum*:
  assumes *Grad A B C* **and**
    Grad A B D **and**
    Bet A C E **and**
    Cong A D C E
  shows *Grad A B E*
⟨*proof*⟩

lemma *SymR-ex*:
  assumes *B Midpoint A C*
  shows $C = SymR\ A\ B$
⟨*proof*⟩

lemma *SymR--midp*:
  assumes $C = SymR\ A\ B$
  shows *B Midpoint A C*
  ⟨*proof*⟩

lemma *SymR-uniq*:
  assumes $C = SymR\ A\ B$ **and**
    $D = SymR\ A\ B$
  shows $C = D$
  ⟨*proof*⟩

lemma *GradExpn-1*:
  shows $GradExpn\ A\ A\ n = A$
  ⟨*proof*⟩

lemma *GradExpn-2*:
  shows $Bet\ A\ B\ (GradExpn\ A\ B\ (Suc\ n))$
⟨*proof*⟩

lemma *GradExpn-3*:
  shows $Bet\ A\ (GradExpn\ A\ B\ (Suc\ n))\ (GradExpn\ A\ B\ (Suc(Suc\ n)))$
⟨*proof*⟩

lemma *GradExpn-4*:
  shows $Cong\ A\ (GradExpn\ A\ B\ (Suc\ n))\ (GradExpn\ A\ B\ (Suc\ n))\ (GradExpn\ A\ B\ (Suc(Suc\ n)))$
⟨*proof*⟩

lemma *gradexp-init*:
shows *GradExp A B B*
⟨*proof*⟩

lemma *gradexp-stab-aux-0*:
assumes $C = GradExpn\ A\ B\ 0$ **and**
Bet A C C' **and**
Cong A C C C'
shows $C' = GradExpn\ A\ B\ 0$
⟨*proof*⟩

lemma *gradexp-stab-aux-n*:
assumes $C = GradExpn\ A\ B\ (Suc\ n)$ **and**
Bet A C C' **and**
Cong A C C C'
shows $C' = GradExpn\ A\ B\ (Suc(Suc\ n))$
⟨*proof*⟩

lemma *gradexp-stab*:
assumes *GradExp A B C* **and**
Bet A C C' **and**
Cong A C C C'
shows *GradExp A B C'*
⟨*proof*⟩

lemma *gradexp--grad-aux-1*:
shows $\forall\ C.\ (C = (GradExpn\ A\ A\ (Suc\ n)) \longrightarrow Grad\ A\ A\ C)$
⟨*proof*⟩

lemma *gradexp--grad-aux*:
assumes $A \neq B$
shows $\forall\ C.\ (C = (GradExpn\ A\ B\ (Suc\ n)) \longrightarrow Grad\ A\ B\ C)$
⟨*proof*⟩

lemma *gradexp--grad*:
assumes *GradExp A B C*
shows *Grad A B C*
⟨*proof*⟩

lemma *gradexp-le--reach*:
assumes *GradExp A B B'* **and**
C D Le A B'
shows *Reach A B C D*
⟨*proof*⟩

lemma *grad--ex-gradexp-le-aux-0*:
assumes $A = B$
shows $\exists\ D.\ GradExp\ A\ B\ D \wedge A\ (Gradn\ A\ B\ n)\ Le\ A\ D$
⟨*proof*⟩

lemma *grad--ex-gradexp-le-aux-1*:
assumes $A \neq B$
shows $\exists\ D.\ GradExp\ A\ B\ D \wedge A\ (Gradn\ A\ B\ n)\ Le\ A\ D$
⟨*proof*⟩

lemma *grad--ex-gradexp-le-aux*:
shows $\exists\ D.\ GradExp\ A\ B\ D \wedge A\ (Gradn\ A\ B\ n)\ Le\ A\ D$
⟨*proof*⟩

lemma *grad--ex-gradexp-le*:
assumes *Grad A B C*
shows $\exists\ D.\ GradExp\ A\ B\ D \wedge A\ C\ Le\ A\ D$
⟨*proof*⟩

lemma *reach--ex-gradexp-le*:

assumes *Reach A B C D*
shows $\exists$ *B'*. *GradExp A B B'* $\wedge$ *C D Le A B'*
⟨*proof*⟩

lemma *gradexp2-init*:
shows *GradExp2 A B B C D D*
⟨*proof*⟩

lemma *GradExp2-stab*:
assumes *GradExp2 A B C D E F* **and**
Bet A C C' **and**
Cong A C C C' **and**
Bet D F F' **and**
Cong D F F F'
shows *GradExp2 A B C' D E F'*
⟨*proof*⟩

lemma *gradexp2--gradexp123*:
assumes *GradExp2 A B C D E F*
shows *GradExp A B C*
⟨*proof*⟩

lemma *gradexp2--gradexp456*:
assumes *GradExp2 A B C D E F*
shows *GradExp D E F*
⟨*proof*⟩

lemma *MidR-uniq*:
assumes *C* = *MidR A B* **and**
D = *MidR A B*
shows *C* = *D*
⟨*proof*⟩

lemma *MidR-ex-0*:
shows (*MidR A B*) *Midpoint A B*
⟨*proof*⟩

lemma *MidR-ex-1*:
assumes *C* = (*MidR A B*)
shows *C Midpoint A B*
⟨*proof*⟩

lemma *MidR-ex-aux*:
assumes *C Midpoint A B*
shows *C* = (*SOME x. x Midpoint A B*)
⟨*proof*⟩

lemma *MidR-ex*:
assumes *C Midpoint A B*
shows *C* = (*MidR A B*)
⟨*proof*⟩

lemma *gradexpinv-init-aux*:
shows *B* = *GradExpInvn A B 0*
⟨*proof*⟩

lemma *gradexpinv-init*:
shows *GradExpInv A B B*
⟨*proof*⟩

lemma *gradexpinv-stab-aux-0*:
assumes *B* = *GradExpInvn A C 0* **and**
Bet A B' B **and**
Cong A B' B' B
shows *B'* = *GradExpInvn A C* (*Suc 0*)
⟨*proof*⟩

lemma *gradexpinv-stab-aux-n*:
assumes *Bet A B′ B* **and**
Cong A B′ B′ B **and**
B = GradExpInvn A C (Suc n)
shows *B′ = GradExpInvn A C (Suc (Suc n))*
⟨*proof*⟩

lemma *gradexpinv-stab*:
assumes *Bet A B′ B* **and**
Cong A B′ B′ B **and**
GradExpInv A B C
shows *GradExpInv A B′ C*
⟨*proof*⟩

lemma *gradexp--gradexpinv-aux-1-0*:
assumes *C = GradExpn A B (Suc 0)*
shows *B = GradExpInvn A C 0*
⟨*proof*⟩

lemma *SymR-MidR*:
assumes *A = SymR B C*
shows *C = MidR A B*
⟨*proof*⟩

lemma *MidR-comm*:
assumes *C = MidR A B*
shows *C = MidR B A*
⟨*proof*⟩

lemma *MidR-SymR*:
assumes *C = MidR A B*
shows *A = SymR B C*
⟨*proof*⟩

lemma *MidR-AA*:
shows *A = MidR A A*
⟨*proof*⟩

lemma *MidR-AB*:
assumes *A = MidR A B*
shows *A = B*
⟨*proof*⟩

lemma *gradexp--gradexpinv-aux-1-n-aa*:
shows *GradExpInvn A′ (MidR A′ C′) n = GradExpInvn A′ C′ (Suc n)*
⟨*proof*⟩

lemma *gradexp--gradexpinv-aux-1-n-a*:
assumes ∀ *A B C*. (*C = GradExpn A B (Suc n)*) ⟶ *B = GradExpInvn A C n*
shows ∀ *A′ B′ C′*. (*C′ = GradExpn A′ B′ (Suc(Suc n))*) ⟶ *B′ = GradExpInvn A′ C′ (Suc n)*
⟨*proof*⟩

lemma *gradexp--gradexpinv-aux-1-b*:
shows ∀ *A B C*. *C = GradExpn A B (Suc n)* ⟶ *B = GradExpInvn A C n*
⟨*proof*⟩

lemma *gradexp--gradexpinv-aux-1*:
assumes *C = GradExpn A B (Suc n)*
shows *B = GradExpInvn A C n*
⟨*proof*⟩

lemma *gradexp--gradexpinv-aux*:
assumes *GradExp A B C*
shows *GradExpInv A B C*
⟨*proof*⟩

lemma *gradexpinv--gradexp-aux-1-a-0*:
assumes $B' = MidR\ A'\ (GradExpInvn\ A'\ C'\ (Suc\ 0))$
shows $C' = SymR\ A'\ (GradExpn\ A'\ B'\ (Suc(Suc\ 0)))$
⟨*proof*⟩

lemma *sym-mid*:
shows $SymR\ A\ (MidR\ A\ B) = B$
⟨*proof*⟩

lemma *gradexpn-suc-suc*:
shows $GradExpn\ A\ B\ (Suc\ n) = GradExpn\ A\ (MidR\ A\ B)\ (Suc(Suc\ n))$
⟨*proof*⟩

lemma *gradexpinv--gradexp-aux-1-a-n*:
assumes $\forall\ A\ B\ C.\ (B = MidR\ A\ (GradExpInvn\ A\ C\ (Suc\ n))) \longrightarrow$
$C = SymR\ A\ (GradExpn\ A\ B\ (Suc(Suc\ n)))$
shows $B' = MidR\ A'\ (GradExpInvn\ A'\ C'\ (Suc(Suc\ n))) \longrightarrow$
$C' = SymR\ A'\ (GradExpn\ A'\ B'\ (Suc(Suc(Suc\ n))))$
⟨*proof*⟩

lemma *gradexpinv--gradexp-aux-1-a*:
shows $\forall\ A\ B\ C.\ B = MidR\ A\ (GradExpInvn\ A\ C\ (Suc\ n)) \longrightarrow$
$C = SymR\ A\ (GradExpn\ A\ B\ (Suc(Suc\ n)))$
⟨*proof*⟩

lemma *gradexpinv--gradexp-aux-1-n*:
assumes $B = GradExpInvn\ A\ C\ n \longrightarrow C = GradExpn\ A\ B\ (Suc\ n)$
shows $B' = GradExpInvn\ A'\ C'\ (Suc\ n) \longrightarrow C' = GradExpn\ A'\ B'\ (Suc(Suc\ n))$
⟨*proof*⟩

lemma *gradexpinv--gradexp-aux-1*:
shows $B = GradExpInvn\ A\ C\ n \longrightarrow C = GradExpn\ A\ B\ (Suc\ n)$
⟨*proof*⟩

lemma *gradexpinv--gradexp-aux*:
assumes $GradExpInv\ A\ B\ C$
shows $GradExp\ A\ B\ C$
⟨*proof*⟩

lemma *gradexp--gradexpinv*:
shows $GradExp\ A\ B\ C \longleftrightarrow GradExpInv\ A\ B\ C$
⟨*proof*⟩

lemma *reach--ex-gradexp-lt-aux*:
shows $\forall\ A\ B\ C\ P\ Q\ R.\ ((A \neq B \wedge A\ B\ Le\ P\ R \wedge R = GradExpn\ P\ Q\ (Suc\ n)) \longrightarrow$
$(\exists\ C.\ GradExp\ A\ C\ B \wedge A\ C\ Lt\ P\ Q))$
⟨*proof*⟩

lemma *reach--grad-min-1*:
assumes $A \neq B$ **and**
$Bet\ A\ B\ C$ **and**
$A\ C\ Le\ A\ (Gradn\ A\ B\ (Suc\ 0))$
shows $\exists\ D\ E.\ (Bet\ A\ D\ C \wedge Grad\ A\ B\ D \wedge E \neq C \wedge Bet\ A\ C\ E \wedge Bet\ A\ D\ E \wedge Cong\ A\ B\ D\ E)$
⟨*proof*⟩

lemma *reach--grad-min-n*:
assumes $A \neq B$ **and**
$Bet\ A\ B\ C$ **and**
$A\ C\ Le\ A\ (Gradn\ A\ B\ (Suc\ n)) \longrightarrow$
$(\exists\ D\ E.\ (Bet\ A\ D\ C \wedge Grad\ A\ B\ D \wedge E \neq C \wedge Bet\ A\ C\ E \wedge Bet\ A\ D\ E \wedge Cong\ A\ B\ D\ E))$
shows $A\ C\ Le\ A\ (Gradn\ A\ B\ (Suc(Suc\ n))) \longrightarrow$
$(\exists\ D\ E.\ (Bet\ A\ D\ C \wedge Grad\ A\ B\ D \wedge E \neq C \wedge Bet\ A\ C\ E \wedge Bet\ A\ D\ E \wedge Cong\ A\ B\ D\ E))$
⟨*proof*⟩

lemma *reach--grad-min-aux*:
assumes $A \neq B$ **and**
Bet A B C
shows (*Grad A B* (*Gradn A B* (*Suc n*)) ∧ *A C Le A* (*Gradn A B* (*Suc n*)))
⟶ (∃ *D E*. (*Bet A D C* ∧ *Grad A B D* ∧ $E \neq C$ ∧ *Bet A C E* ∧ *Bet A D E* ∧ *Cong A B D E*))
⟨*proof*⟩

lemma *reach--grad-min*:
assumes $A \neq B$ **and**
Bet A B C **and**
Reach A B A C
shows ∃ *D E*. (*Bet A D C* ∧ *Grad A B D* ∧ $E \neq C$ ∧ *Bet A C E* ∧ *Bet A D E* ∧ *Cong A B D E*)
⟨*proof*⟩

lemma *reach--ex-gradexp-lt*:
assumes $A \neq B$ **and**
Reach P Q A B
shows ∃ *C*. *GradExp A C B* ∧ *A C Lt P Q*
⟨*proof*⟩

lemma *t22-18-aux-0*:
assumes *Bet A0 D1 A1* **and**
Cong E0 E1 A1 D1 **and**
D = *Gradn A0 D1* (*Suc 0*)
shows ∃ *A E*. (*Grad2 A0 A1 A E0 E1 E* ∧ *Cong E0 E A D* ∧ *Bet A0 D A*)
⟨*proof*⟩

lemma *t22-18-aux-n*:
assumes ∀ *A0 D1 A1 E0 E1 D*.
(*Bet A0 D1 A1* ∧ *Cong E0 E1 A1 D1* ∧ *D* = *Gradn A0 D1* (*Suc n*)) ⟶
(∃ *A E*. (*Grad2 A0 A1 A E0 E1 E* ∧ *Cong E0 E A D* ∧ *Bet A0 D A*))
shows ∀ *A0 D1 A1 E0 E1 D*.
(*Bet A0 D1 A1* ∧ *Cong E0 E1 A1 D1* ∧ *D* = *Gradn A0 D1* (*Suc*(*Suc n*))) ⟶
(∃ *A E*. (*Grad2 A0 A1 A E0 E1 E* ∧ *Cong E0 E A D* ∧ *Bet A0 D A*))
⟨*proof*⟩

lemma *t22-18-aux0*:
shows ∀ *A0 D1 A1 E0 E1 D*.
(*Bet A0 D1 A1* ∧ *Cong E0 E1 A1 D1* ∧ *D* = *Gradn A0 D1* (*Suc n*)) ⟶
(∃ *A E*. (*Grad2 A0 A1 A E0 E1 E* ∧ *Cong E0 E A D* ∧ *Bet A0 D A*))
⟨*proof*⟩

lemma *t22-18-aux1*:
assumes *Bet A0 D1 A1* **and**
Cong E0 E1 A1 D1 **and**
Grad A0 D1 D
shows ∃ *A E*. (*Grad2 A0 A1 A E0 E1 E* ∧ *Cong E0 E A D* ∧ *Bet A0 D A*)
⟨*proof*⟩

lemma *t22-18-aux2-0*:
assumes *Saccheri A0 B0 B1 A1* **and**
A = *Gradn A0 A1* (*Suc 0*) **and**
E = *Gradn B0 B1* (*Suc 0*) **and**
Saccheri A0 B0 B A
shows *B0 B Le B0 E*
⟨*proof*⟩

lemma *t22-18-aux2-Sucn*:
assumes ∀ *A A0 A1 B B0 B1 E*.
Saccheri A0 B0 B1 A1 ∧ *A* = *Gradn A0 A1* (*Suc n*) ∧
E = *Gradn B0 B1* (*Suc n*) ∧ *Saccheri A0 B0 B A* ⟶ *B0 B Le B0 E*
shows ∀ *A A0 A1 B B0 B1 E*.

Saccheri A0 B0 B1 A1 ∧ *A* = *Gradn A0 A1* (*Suc*(*Suc n*)) ∧
E = *Gradn B0 B1* (*Suc*(*Suc n*)) ∧ *Saccheri A0 B0 B A* ⟶ *B0 B Le B0 E*
⟨*proof*⟩

lemma *t22-18-aux2-n*:
shows ∀ *A A0 A1 B B0 B1 E*.
Saccheri A0 B0 B1 A1 ∧ *A* = *Gradn A0 A1* (*Suc n*) ∧
E = *Gradn B0 B1* (*Suc n*) ∧ *Saccheri A0 B0 B A*
⟶ *B0 B Le B0 E*
⟨*proof*⟩

lemma *t22-18-aux2*:
assumes *Saccheri A0 B0 B1 A1* **and**
Grad2 A0 A1 A B0 B1 E **and**
Saccheri A0 B0 B A
shows *B0 B Le B0 E*
⟨*proof*⟩

lemma *t22-18*:
assumes *archimedes-axiom*
shows ∀ *A0 B0 B1 A1*. *Saccheri A0 B0 B1 A1* ⟶ ¬ (*B0 B1 Lt A1 A0*)
⟨*proof*⟩

lemma *t22-19*:
assumes *archimedes-axiom*
shows ∀ *A B C D*. *Saccheri A B C D* ⟶ ¬ *Obtuse A B C*
⟨*proof*⟩

lemma *archi--obtuse-case-elimination*:
assumes *archimedes-axiom*
shows ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *t22-23-aux*:
assumes ¬ *Col A M N* **and**
Per B C A **and**
A ≠ *C* **and**
M Midpoint A B **and**
Per M N A **and**
Col A C N **and**
M Midpoint N L
shows *Bet A N C* ∧ *Lambert N L B C* ∧ *Cong B L A N*
⟨*proof*⟩

lemma *t22-23*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
shows ∀ *A B C M N L*.
¬ *Col A M N* ∧ *Per B C A* ∧ *A* ≠ *C* ∧ *M Midpoint A B* ∧
Per M N A ∧ *Col A C N* ∧ *M Midpoint N L* ⟶
(*Bet A N C* ∧ *N C Le A N* ∧ *L N Le B C*)
⟨*proof*⟩

lemma *t22-24-aux-0-a*:
assumes
¬ *Col A B0 C0* **and**
A C0 Perp B0 C0 **and**
B = *GradExpn A B0* (*Suc 0*) **and**
E = *GradExpn B0 C0* (*Suc 0*) **and**
A C0 Perp B C **and**
Col A C0 C
shows *B0 E Le B C*
⟨*proof*⟩

lemma *t22-24-aux-0*:

shows ∀ *A B0 C0 B C E*.
¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
(*B* = *GradExpn A B0* (*Suc 0*)) ∧ (*E* = *GradExpn B0 C0* (*Suc 0*)) ∧
A C0 Perp B C ∧ *Col A C0 C* ⟶
B0 E Le B C
⟨*proof*⟩

lemma *t22-24-aux-suc*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals* **and**
∀ *A B0 C0 B C E*.
¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
(*B* = *GradExpn A B0* (*Suc n*)) ∧ (*E* = *GradExpn B0 C0* (*Suc n*)) ∧
A C0 Perp B C ∧ *Col A C0 C* ⟶
B0 E Le B C
shows
∀ *A B0 C0 B C E*.
¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
(*B* = *GradExpn A B0* (*Suc*(*Suc n*))) ∧ (*E* = *GradExpn B0 C0* (*Suc*(*Suc n*))) ∧
A C0 Perp B C ∧ *Col A C0 C* ⟶
B0 E Le B C
⟨*proof*⟩

lemma *t22-24-aux-n*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
shows ∀ *A B0 C0 B C E*.
(¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
(*B* = *GradExpn A B0* (*Suc n*)) ∧ (*E* = *GradExpn B0 C0* (*Suc n*)) ∧
A C0 Perp B C ∧ *Col A C0 C*) ⟶
B0 E Le B C
⟨*proof*⟩

lemma *t22-24-aux*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
shows ∀ *A B0 B00 C0 B C E*.
¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧ *B0* = *B00* ∧
GradExp2 A B0 B B00 C0 E ∧ *A C0 Perp B C* ∧ *Col A C0 C* ⟶
B0 E Le B C
⟨*proof*⟩

lemma *t22-24-aux1-0*:
shows∀ *A B0 C0 E*.
(¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
E = *GradExpn B0 C0* (*Suc 0*)) ⟶
(∃ *B C*. *GradExp2 A B0 B B0 C0 E* ∧ *A C0 Perp B C* ∧ *Col A C0 C*)
⟨*proof*⟩

lemma *t22-24-aux1-suc*:
assumes ∀ *A B0 C0 E*.
(¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
E = *GradExpn B0 C0* (*Suc*(*n*))) ⟶
(∃ *B C*. *GradExp2 A B0 B B0 C0 E* ∧ *A C0 Perp B C* ∧ *Col A C0 C*)
shows ∀ *A B0 C0 E*.
(¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
E = *GradExpn B0 C0* (*Suc*(*Suc*(*n*)))) ⟶
(∃ *B C*. *GradExp2 A B0 B B0 C0 E* ∧ *A C0 Perp B C* ∧ *Col A C0 C*)
⟨*proof*⟩

lemma *t22-24-aux1-n*:
shows ∀ *A B0 C0 E*.
(¬ *Col A B0 C0* ∧ *A C0 Perp B0 C0* ∧
E = *GradExpn B0 C0* (*Suc*(*n*))) ⟶
(∃ *B C*. *GradExp2 A B0 B B0 C0 E* ∧ *A C0 Perp B C* ∧ *Col A C0 C*)
⟨*proof*⟩

lemma *t22-24-aux1*:
  assumes ¬ *Col A B0 C0* **and**
    A C0 Perp B0 C0 **and**
    GradExp B0 C0 E
  shows ∃ *B C*. *GradExp2 A B0 B B0 C0 E* ∧ *A C0 Perp B C* ∧ *Col A C0 C*
⟨*proof*⟩

lemma *t22-24*:
  assumes *archimedes-axiom*
  shows *aristotle-s-axiom*
⟨*proof*⟩

2.3 Equivalence Grad / GradI (inductive)

lemma *Grad1--GradI*:
  shows *GradI A B* (*Gradn A B 1*)
⟨*proof*⟩

lemma *Gradn--GradI*:
  shows *GradI A B* (*Gradn A B* (*Suc n*))
⟨*proof*⟩

lemma *Grad--GradI*:
  assumes *Grad A B C*
  shows *GradI A B C*
⟨*proof*⟩

lemma *GradIAAB--not*:
  assumes *GradI A A B*
  shows *A* = *B*
⟨*proof*⟩

lemma *GradI--Grad*:
  assumes *GradI A B C*
  shows *Grad A B C*
⟨*proof*⟩

theorem *Grad-GradI*:
  shows *Grad A B C* ⟷ *GradI A B C*
  ⟨*proof*⟩

2.4 GradA

lemma *grada-distincts*:
  assumes *GradA A B C D E F*
  shows *A* ≠ *B* ∧ *C* ≠ *B* ∧ *D* ≠ *E* ∧ *F* ≠ *E*
⟨*proof*⟩

lemma *grada-ABC*:
  assumes *A* ≠ *B* **and**
    B ≠ *C*
  shows *GradA A B C A B C*
⟨*proof*⟩

lemma *gradaexp-distincts*:
  assumes *GradAExp A B C D E F*
  shows *A* ≠ *B* ∧ *C* ≠ *B* ∧ *D* ≠ *E* ∧ *F* ≠ *E*
⟨*proof*⟩

lemma *gradaexp-ABC*:
  assumes *A* ≠ *B* **and**
    B ≠ *C*
  shows *GradAExp A B C A B C*
⟨*proof*⟩

lemma *conga2-grada--grada-aux1*:
  assumes *GradA A B C D E F* **and**
    A B C CongA A′ B′ C′
  shows *GradA A′ B′ C′ D E F*
⟨*proof*⟩

lemma *conga2-grada--grada-aux2*:
  assumes *GradA A B C D E F* **and**
    D E F CongA D′ E′ F′
  shows *GradA A B C D′ E′ F′*
⟨*proof*⟩

lemma *conga2-grada--grada*:
  assumes *GradA A B C D E F* **and**
    A B C CongA A′ B′ C′ **and**
    D E F CongA D′ E′ F′
  shows *GradA A′ B′ C′ D′ E′ F′*
  ⟨*proof*⟩

lemma *grada--lea*:
  assumes *GradA A B C D E F*
  shows *A B C LeA D E F*
⟨*proof*⟩

lemma *grada-out--out*:
  assumes *E Out D F* **and**
    GradA A B C D E F
  shows *B Out A C*
⟨*proof*⟩

lemma *grada2-sams-suma--grada-aux*:
  shows ∀ *A B C D E F G H I K L M*.
        GradA A B C D E F ∧ *GradA A B C G H I* ∧
        SAMS D E F G H I ∧ *D E F G H I SumA K L M* ⟶ *GradA A B C K L M*
⟨*proof*⟩

lemma *grada2-sams-suma--grada*:
  assumes *GradA A B C D E F* **and**
    GradA A B C G H I **and**
    SAMS D E F G H I **and**
    D E F G H I SumA K L M
  shows *GradA A B C K L M*
  ⟨*proof*⟩

lemma *gradaexp--grada*:
  assumes *GradAExp A B C D E F*
  shows *GradA A B C D E F*
⟨*proof*⟩

lemma *acute-archi-aux*:
  assumes *Per PO A B* **and**
    PO ≠ *A* **and**
    B ≠ *A* **and**
    C ≠ *D* **and**
    D ≠ *E* **and**
    Bet A C D **and**
    Bet C D E **and**
    Bet D E B **and**
    C PO D CongA D PO E
  shows *C D Lt D E*
⟨*proof*⟩

lemma *acute-archi-aux1*:
  assumes *Per PO A0 B* **and**
    B ≠ *A0* **and**
    Bet A0 A1 B **and**

GradA A0 PO A1 P Q R **and**
A0 ≠ *A1*
shows *A0 PO B LeA P Q R* ∨ (∃ *A*. *Bet A0 A1 A* ∧ *Bet A0 A B* ∧ *P Q R CongA A0 PO A*)
⟨*proof*⟩

lemma *acute-archi-aux2-1-a*:
assumes *Per PO A0 B* **and**
PO ≠ *A0* **and**
B ≠ *A0* **and**
Bet A0 A1 B **and**
A0 ≠ *A1* **and** ¬ *Col PO A0 B* **and** ¬ *Col A0 PO A1* **and** *PO* ≠ *A1* **and** *PO* ≠ *B*
shows ∃ *P Q R*. (*GradA A0 PO A1 P Q R* ∧ (*A0 PO B LeA P Q R* ∨
(∃ *A'*. *Bet A0 A1 A'* ∧ *Bet A0 A' B* ∧ *P Q R CongA A0 PO A'* ∧ *A0 A1 Le A0 A'* ∧
(∃ *A*. *Bet A0 A A'* ∧ *A0 PO A' CongA A0 PO A1* ∧ *A0 A1 Le A A'*))))
⟨*proof*⟩

lemma *acute-archi-aux2-1*:
assumes *Per PO A B* **and**
PO ≠ *A* **and**
B ≠ *A* **and**
Bet A B0 B **and**
A ≠ *B0* **and** ¬ *Col PO A B* **and** ¬ *Col A PO B0* **and** *PO* ≠ *B0* **and** *PO* ≠ *B*
shows ∃ *P Q R*. (*GradA A PO B0 P Q R* ∧ (*A PO B LeA P Q R* ∨
(∃ *A'*. *Bet A B0 A'* ∧ *Bet A A' B* ∧ *P Q R CongA A PO A'* ∧ *A B0 Le A A'* ∧
(∃ *A0*. *Bet A A0 A'* ∧ *A0 PO A' CongA A PO B0* ∧ *A B0 Le A0 A'*))))
⟨*proof*⟩

lemma *acute-archi-aux2-2*:
assumes *Per PO A0 B* **and**
PO ≠ *A0* **and**
B ≠ *A0* **and**
Bet A0 A1 B **and**
A0 ≠ *A1* **and**
Grad A0 A1 C **and**
Bet A0 C C' **and**
Cong A0 A1 C C' **and**
¬ *Col PO A0 B* **and**
¬ *Col A0 PO A1* **and**
PO ≠ *A1* **and**
PO ≠ *B* **and**
Per PO A0 B ∧ *PO* ≠ *A0* ∧
B ≠ *A0* ∧ *Bet A0 A1 B* ∧
A0 ≠ *A1* ∧
¬ *Col PO A0 B* ∧
¬ *Col A0 PO A1* ∧
PO ≠ *A1* ⟶ (∃ *P Q R*.
(*GradA A0 PO A1 P Q R* ∧
(*A0 PO B LeA P Q R* ∨
(∃ *A'*.
Bet A0 A1 A' ∧
Bet A0 A' B ∧
P Q R CongA A0 PO A' ∧
A0 C Le A0 A' ∧ (∃ *A*. (*Bet A0 A A'* ∧ *A PO A' CongA A0 PO A1* ∧ *A0 A1 Le A A'*))))))
shows ∃ *P Q R*. (*GradA A0 PO A1 P Q R* ∧
(*A0 PO B LeA P Q R* ∨
(∃ *A'*. *Bet A0 A1 A'* ∧ *Bet A0 A' B* ∧ *P Q R CongA A0 PO A'* ∧ *A0 C' Le A0 A'* ∧
(∃ *A*. *Bet A0 A A'* ∧ *A PO A' CongA A0 PO A1* ∧ *A0 A1 Le A A'*))))
⟨*proof*⟩

lemma *acute-archi-aux2*:
assumes *Per PO A0 B* **and**
PO ≠ *A0* **and**
B ≠ *A0* **and**
Bet A0 A1 B **and**
A0 ≠ *A1* **and**

Grad A0 A1 C
shows ∃ *P Q R*. (*GradA A0 PO A1 P Q R* ∧ (*A0 PO B LeA P Q R* ∨
(∃ *A'*. *Bet A0 A1 A'* ∧ *Bet A0 A' B* ∧ *P Q R CongA A0 PO A'* ∧ *A0 C Le A0 A'* ∧
(∃ *A*. *Bet A0 A A'* ∧ *A PO A' CongA A0 PO A1* ∧ *A0 A1 Le A A'*))))
⟨*proof*⟩

lemma *archi-in-acute-angles*:
assumes *archimedes-axiom*
shows ∀ *A B C D E F*. ¬ *Col A B C* ∧ *Acute D E F*
⟶ (∃ *P Q R*. *GradA A B C P Q R* ∧ *D E F LeA P Q R*)
⟨*proof*⟩

lemma *angles-archi-aux*:
assumes *GradA A B C D E F* **and**
GradA A B C G H I **and**
¬ *SAMS D E F G H I*
shows ∃ *P Q R*. *GradA A B C P Q R* ∧ ¬ *SAMS P Q R A B C*
⟨*proof*⟩

lemma *angles-archi-aux1*:
assumes *archimedes-axiom*
shows ∀ *A B C D E F*.
¬ *Col A B C* ∧ ¬ *Bet D E F* ⟶
(∃ *P Q R*. *GradA A B C P Q R* ∧ (*D E F LeA P Q R* ∨ ¬ *SAMS P Q R A B C*))
⟨*proof*⟩

lemma *archi-in-angles*:
assumes *archimedes-axiom*
shows ∀ *A B C*. ∀ *D* ::'*p*. ∀ *E* ::'*p*. ∀ *F* ::'*p*. (¬ *Col A B C* ∧ *D* ≠ *E* ∧ *F* ≠ *E*) ⟶
(∃ *P Q R*. *GradA A B C P Q R* ∧ (*D E F LeA P Q R* ∨ ¬ *SAMS P Q R A B C*))
⟨*proof*⟩

lemma *archi--grada-destruction*:
assumes *archimedes-axiom*
shows ∀ *A B C*. ¬ *Col A B C* ⟶
(∃ *P Q R*. *GradA A B C P Q R* ∧ ¬ *SAMS P Q R A B C*)
⟨*proof*⟩

lemma *gradaexp-destruction-aux*:
assumes *GradA A B C P Q R*
shows ∃ *S T U*. *GradAExp A B C S T U* ∧ (*Obtuse S T U* ∨ *P Q R LeA S T U*)
⟨*proof*⟩

lemma *archi--gradaexp-destruction*:
assumes *archimedes-axiom*
shows ∀ *A B C*. ¬ *Col A B C* ⟶ (∃ *P Q R*. *GradAExp A B C P Q R* ∧ *Obtuse P Q R*)
⟨*proof*⟩

2.5 Equivalence Greenberg - Aristotle

lemma *aristotle--greenberg*:
assumes *aristotle-s-axiom*
shows *greenberg-s-axiom*
⟨*proof*⟩

lemma *greenberg--aristotle*:
assumes *greenberg-s-axiom*
shows *aristotle-s-axiom*
⟨*proof*⟩

theorem *equiv-aristotle---greenberg*:

shows *aristotle-s-axiom* ⟷ *greenberg-s-axiom*
⟨*proof*⟩

end
end

theory *Tarski-Postulate-Parallels*

imports
Tarski-Neutral-Archimedes

begin

context *Tarski-neutral-dimensionless*

begin

3 Parallel's Postulate

3.1 Definitions

definition *tarski-s-parallel-postulate* ::
bool
(*TarskiSParallelPostulate*)
where
tarski-s-parallel-postulate ≡

∀ *A B C D T*.
Bet A D T ∧ *Bet B D C* ∧ *A* ≠ *D*
⟶
(∃ *X Y*. *Bet A B X* ∧ *Bet A C Y* ∧ *Bet X T Y*)

definition *euclid-5* ::
bool (*Euclid5*)
where
euclid-5 ≡

∀ *P Q R S T U*.
BetS P T Q ∧ *BetS R T S* ∧ *BetS Q U R* ∧ ¬ *Col P Q S* ∧ *Cong P T Q T* ∧ *Cong R T S T*
⟶
(∃ *I*. *BetS S Q I* ∧ *BetS P U I*)

definition *euclid-s-parallel-postulate* ::
bool (*EuclidSParallelPostulate*)
where
euclid-s-parallel-postulate ≡

∀ *A B C D P Q R*.
B C OS A D ∧ *SAMS A B C B C D* ∧ *A B C B C D SumA P Q R* ∧ ¬ *Bet P Q R*
⟶
(∃ *Y*. *B Out A Y* ∧ *C Out D Y*)

definition *playfair-s-postulate* ::
bool
(*PlayfairSPostulate*)
where
playfair-s-postulate ≡

∀ *A1 A2 B1 B2 C1 C2 P*.
A1 A2 Par B1 B2 ∧ *Col P B1 B2* ∧ *A1 A2 Par C1 C2* ∧ *Col P C1 C2*
⟶
Col C1 B1 B2 ∧ *Col C2 B1 B2*

definition *decidability-of-intersection* ::
bool

(*DecidabilityIntersection*)
where
decidability-of-intersection ≡

$\forall\ A\ B\ C\ D.$

$(\exists\ I.\ Col\ I\ A\ B \wedge Col\ I\ C\ D) \vee \neg\ (\exists\ I.\ Col\ I\ A\ B \wedge Col\ I\ C\ D)$

definition *alternate-interior-angles-postulate* ::
bool
(*AlternateInteriorAnglesPostulate*)
where
alternate-interior-angles-postulate ≡

$\forall\ A\ B\ C\ D.$
$A\ C\ TS\ B\ D \wedge A\ B\ Par\ C\ D$
$\longrightarrow$
$B\ A\ C\ CongA\ D\ C\ A$

definition *consecutive-interior-angles-postulate* ::
bool
(*ConsecutiveInteriorAnglesPostulate*)
where
consecutive-interior-angles-postulate ≡

$\forall\ A\ B\ C\ D.$
$B\ C\ OS\ A\ D \wedge A\ B\ Par\ C\ D$
$\longrightarrow$
$A\ B\ C\ SuppA\ B\ C\ D$

definition *alternative-playfair-s-postulate* ::
bool
(*AlternativePlayfairSPostulate*)
where
alternative-playfair-s-postulate ≡

$\forall\ A1\ A2\ B1\ B2\ C1\ C2\ P.$
$P\ Perp2\ A1\ A2\ B1\ B2 \wedge \neg\ Col\ A1\ A2\ P \wedge Col\ P\ B1\ B2 \wedge$
$Coplanar\ A1\ A2\ B1\ B2 \wedge A1\ A2\ Par\ C1\ C2 \wedge Col\ P\ C1\ C2$
$\longrightarrow$
$Col\ C1\ B1\ B2 \wedge Col\ C2\ B1\ B2$

definition *proclus-postulate* ::
bool
(*ProclusPostulate*)
where
proclus-postulate ≡

$\forall\ A\ B\ C\ D\ P\ Q.$
$A\ B\ Par\ C\ D \wedge Col\ A\ B\ P \wedge \neg\ Col\ A\ B\ Q \wedge Coplanar\ C\ D\ P\ Q$
$\longrightarrow$
$(\exists\ Y.\ Col\ P\ Q\ Y \wedge Col\ C\ D\ Y)$

definition *triangle-postulate* ::
bool
(*TrianglePostulate*)
where
triangle-postulate ≡

$\forall\ A\ B\ C\ D\ E\ F.$
$A\ B\ C\ TriSumA\ D\ E\ F$
$\longrightarrow$
$Bet\ D\ E\ F$

definition *bachmann-s-lotschnittaxiom* ::
bool

(*BachmannsLotschnittaxiom*)
where
bachmann-s-lotschnittaxiom ≡

$\forall\ P\ Q\ R\ P1\ R1.$
$(P \neq Q \wedge Q \neq R \wedge Per\ P\ Q\ R \wedge Per\ Q\ P\ P1 \wedge Per\ Q\ R\ R1 \wedge$
$Coplanar\ P\ Q\ R\ P1 \wedge Coplanar\ P\ Q\ R\ R1$
$\longrightarrow$
$(\exists\ S.\ Col\ P\ P1\ S \wedge Col\ R\ R1\ S))$

definition *legendre-s-parallel-postulate* ::
bool
(*LegendresParallelPostulate*)
where
legendre-s-parallel-postulate ≡

$\exists\ A\ B\ C.$

$\neg\ Col\ A\ B\ C \wedge$
$Acute\ A\ B\ C \wedge$
$(\forall\ T.\ T\ InAngle\ A\ B\ C \longrightarrow (\exists\ X\ Y.\ B\ Out\ A\ X \wedge B\ Out\ C\ Y \wedge Bet\ X\ T\ Y))$

definition *weak-inverse-projection-postulate* ::
bool
(*WeakInverseProjectionPostulate*)
where
weak-inverse-projection-postulate ≡

$\forall\ A\ B\ C\ D\ E\ F\ P\ Q.$
$((Acute\ A\ B\ C \wedge Per\ D\ E\ F \wedge A\ B\ C\ A\ B\ C\ SumA\ D\ E\ F \wedge$
$B\ Out\ A\ P \wedge P \neq Q \wedge Per\ B\ P\ Q \wedge Coplanar\ A\ B\ C\ Q)$
$\longrightarrow$
$(\exists\ Y.\ B\ Out\ C\ Y \wedge Col\ P\ Q\ Y))$

definition *weak-triangle-circumscription-principle* ::
bool
(*WeakTriangleCircumscriptionPrinciple*)
where
weak-triangle-circumscription-principle ≡

$\forall\ A\ B\ C\ A1\ A2\ B1\ B2.$
$(\neg\ Col\ A\ B\ C \wedge Per\ A\ C\ B \wedge A1\ A2\ PerpBisect\ B\ C \wedge$
$B1\ B2\ PerpBisect\ A\ C \wedge Coplanar\ A\ B\ C\ A1 \wedge Coplanar\ A\ B\ C\ A2 \wedge$
$Coplanar\ A\ B\ C\ B1 \wedge Coplanar\ A\ B\ C\ B2$
$\longrightarrow$
$(\exists\ I.\ Col\ A1\ A2\ I \wedge Col\ B1\ B2\ I))$

definition *weak-tarski-s-parallel-postulate* ::
bool
(*WeakTarskiParallelPostulate*)
where
weak-tarski-s-parallel-postulate ≡

$\forall\ A\ B\ C\ T.$
$(Per\ A\ B\ C \wedge T\ InAngle\ A\ B\ C$
$\longrightarrow$
$(\exists\ X\ Y.\ B\ Out\ A\ X \wedge B\ Out\ C\ Y \wedge Bet\ X\ T\ Y))$

definition *existential-playfair-s-postulate* ::
bool
(*ExistentialPlayfairPostulate*)
where
existential-playfair-s-postulate ≡

$\exists\ A1\ A2\ P.$

¬ *Col A1 A2 P* ∧

(∀ *B1 B2 C1 C2*.
A1 A2 Par B1 B2 ∧ *Col P B1 B2* ∧ *A1 A2 Par C1 C2* ∧ *Col P C1 C2*
⟶
(*Col C1 B1 B2* ∧ *Col C2 B1 B2*))

definition *postulate-of-right-saccheri-quadrilaterals* ::
bool
(*PostulateRightSaccheriQuadrilaterals*)
where
postulate-of-right-saccheri-quadrilaterals ≡

∀ *A B C D*.
Saccheri A B C D
⟶
Per A B C

definition *postulate-of-existence-of-a-right-saccheri-quadrilateral* ::
bool
(*PostulateExistenceRightSaccheriQuadrilateral*)
where
postulate-of-existence-of-a-right-saccheri-quadrilateral ≡

∃ *A B C D*.

Saccheri A B C D ∧ *Per A B C*

definition *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights* ::
bool
(*PostulateExistenceTriangleAnglesSumTwoRights*)
where
postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights ≡

∃ *A B C D E F*.

¬ *Col A B C* ∧ *A B C TriSumA D E F* ∧ *Bet D E F*

definition *inverse-projection-postulate* ::
bool
(*InverseProjectionPostulate*)
where
inverse-projection-postulate ≡

∀ *A B C P Q*.
Acute A B C ∧ *B Out A P* ∧ *P* ≠ *Q* ∧ *Per B P Q* ∧ *Coplanar A B C Q*
⟶
(∃ *Y*. *B Out C Y* ∧ *Col P Q Y*)

definition *alternative-proclus-postulate* ::
bool
(*AlternativeProclusPostulate*)
where
alternative-proclus-postulate ≡

∀ *A B C D P Q*.
P Perp2 A B C D ∧ ¬ *Col C D P* ∧ *Coplanar A B C D* ∧ *Col A B P* ∧
¬ *Col A B Q* ∧ *Coplanar C D P Q*
⟶
(∃ *Y*. (*Col P Q Y* ∧ *Col C D Y*))

definition *strong-parallel-postulate* ::
bool
(*StrongParallelPostulate*)
where
strong-parallel-postulate ≡

$\forall\ P\ Q\ R\ S\ T\ U.$
$BetS\ P\ T\ Q \wedge BetS\ R\ T\ S \wedge \neg\ Col\ P\ R\ U \wedge Coplanar\ P\ Q\ R\ U \wedge$
$Cong\ P\ T\ Q\ T \wedge Cong\ R\ T\ S\ T$
$\longrightarrow$
$(\exists\ I.\ (Col\ S\ Q\ I \wedge Col\ P\ U\ I))$

definition *triangle-circumscription-principle* ::
bool
(*TriangleCircumscriptionPrinciple*)
where
triangle-circumscription-principle ≡

$\forall\ A\ B\ C.$
$\neg\ Col\ A\ B\ C$
$\longrightarrow$
$(\exists\ D.\ Cong\ A\ D\ B\ D \wedge Cong\ A\ D\ C\ D \wedge Coplanar\ A\ B\ C\ D)$

definition *thales-converse-postulate*::
bool
(*ThalesConversePostulate*) **where**
thales-converse-postulate ≡

$\forall\ A\ B\ C\ M.$
$M\ Midpoint\ A\ B \wedge Per\ A\ C\ B$
$\longrightarrow$
$Cong\ M\ A\ M\ C$

definition *existential-thales-postulate* ::
bool
(*ExistentialThalesPostulate*) **where**
existential-thales-postulate ≡

$\exists\ A\ B\ C\ M.$

$\neg\ Col\ A\ B\ C \wedge M\ Midpoint\ A\ B \wedge Cong\ M\ A\ M\ C \wedge Per\ A\ C\ B$

definition *thales-postulate*::
bool
(*ThalesPostulate*) **where**
thales-postulate ≡

$\forall\ A\ B\ C\ M.$
$M\ Midpoint\ A\ B \wedge Cong\ M\ A\ M\ C$
$\longrightarrow$
$Per\ A\ C\ B$

definition *posidonius-postulate* ::
bool
(*PosidoniusPostulate*) **where**
posidonius-postulate ≡

$\exists\ A1\ A2\ B1\ B2.$

$\neg\ Col\ A1\ A2\ B1 \wedge B1 \neq B2 \wedge Coplanar\ A1\ A2\ B1\ B2 \wedge$
$(\forall\ A3\ A4\ B3\ B4.$
$Col\ A1\ A2\ A3 \wedge Col\ B1\ B2\ B3 \wedge A1\ A2\ Perp\ A3\ B3 \wedge$
$Col\ A1\ A2\ A4 \wedge Col\ B1\ B2\ B4 \wedge A1\ A2\ Perp\ A4\ B4$
$\longrightarrow$
$Cong\ A3\ B3\ A4\ B4)$

definition *postulate-of-right-lambert-quadrilaterals* ::
bool
(*PostulateOfRightLambertQuadrilaterals*) **where**
postulate-of-right-lambert-quadrilaterals ≡

∀ *A B C D*.
Lambert A B C D
⟶
Per B C D

definition *postulate-of-existence-of-a-right-lambert-quadrilateral* ::
bool
(*PostulateExistenceRightLambertQuadrilateral*) **where**
postulate-of-existence-of-a-right-lambert-quadrilateral ≡

∃ *A B C D*.

Lambert A B C D ∧ *Per B C D*

definition *postulate-of-existence-of-similar-triangles* ::
bool
(*PostulateOfExistenceOfSimilarTriangles*) **where**
postulate-of-existence-of-similar-triangles ≡

∃ *A B C D E F*.

¬ *Col A B C* ∧ ¬ *Cong A B D E* ∧ *A B C CongA D E F* ∧
B C A CongA E F D ∧ *C A B CongA F D E*

definition *midpoint-converse-postulate* ::
bool
(*MidpointConversePostulate*) **where**
midpoint-converse-postulate ≡

∀ *A B C P Q*.
¬ *Col A B C* ∧ *P Midpoint B C* ∧ *A B Par Q P* ∧ *Col A C Q*
⟶
Q Midpoint A C

definition *postulate-of-transitivity-of-parallelism*::
bool
(*PostulateOfTransitivityOfParallelism*) **where**
postulate-of-transitivity-of-parallelism ≡

∀ *A1 A2 B1 B2 C1 C2*.
A1 A2 Par B1 B2 ∧ *B1 B2 Par C1 C2*
⟶
A1 A2 Par C1 C2

definition *perpendicular-transversal-postulate* ::
bool
(*PerpendicularTransversalPostulate*) **where**
perpendicular-transversal-postulate ≡

∀ *A B C D P Q*.
A B Par C D ∧ *A B Perp P Q* ∧ *Coplanar C D P Q*
⟶
C D Perp P Q

definition *postulate-of-parallelism-of-perpendicular-transversals* ::
bool
(*PostulateOfParallelismOfPerpendicularTransversals*) **where**
postulate-of-parallelism-of-perpendicular-transversals ≡

∀ *A1 A2 B1 B2 C1 C2 D1 D2*.
A1 A2 Par B1 B2 ∧ *A1 A2 Perp C1 C2* ∧ *B1 B2 Perp D1 D2* ∧
Coplanar A1 A2 C1 D1 ∧ *Coplanar A1 A2 C1 D2* ∧
Coplanar A1 A2 C2 D1 ∧ *Coplanar A1 A2 C2 D2*
⟶
C1 C2 Par D1 D2

definition *universal-posidonius-postulate* ::
bool
(*UniversalPosidoniusPostulate*) **where**
universal-posidonius-postulate ≡

∀ *A1 A2 A3 A4 B1 B2 B3 B4*.
A1 A2 Par B1 B2 ∧ *Col A1 A2 A3* ∧ *Col B1 B2 B3* ∧ *A1 A2 Perp A3 B3* ∧
Col A1 A2 A4 ∧ *Col B1 B2 B4* ∧ *A1 A2 Perp A4 B4*
⟶
Cong A3 B3 A4 B4

definition *alternative-strong-parallel-postulate* ::
bool
(*AlternativeStrongParallelPostulate*) **where**
alternative-strong-parallel-postulate ≡

∀ *A B C D P Q R*.
B C OS A D ∧ *A B C B C D SumA P Q R* ∧ ¬ *Bet P Q R*
⟶
(∃ *Y*. *Col B A Y* ∧ *Col C D Y*)

definition *Postulate01* :: *bool* **where**
Postulate01 ≡ *tarski-s-parallel-postulate*

definition *Postulate02* :: *bool* **where**
Postulate02 ≡ *playfair-s-postulate*

definition *Postulate03* :: *bool* **where**
Postulate03 ≡ *triangle-postulate*

definition *Postulate04* :: *bool* **where**
Postulate04 ≡ *bachmann-s-lotschnittaxiom*

definition *Postulate05* :: *bool* **where**
Postulate05 ≡ *postulate-of-transitivity-of-parallelism*

definition *Postulate06* :: *bool* **where**
Postulate06 ≡ *midpoint-converse-postulate*

definition *Postulate07* :: *bool* **where**
Postulate07 ≡ *alternate-interior-angles-postulate*

definition *Postulate08* :: *bool* **where**
Postulate08 ≡ *consecutive-interior-angles-postulate*

definition *Postulate09* :: *bool* **where**
Postulate09 ≡ *perpendicular-transversal-postulate*

definition *Postulate10* :: *bool* **where**
Postulate10 ≡ *postulate-of-parallelism-of-perpendicular-transversals*

definition *Postulate11* :: *bool* **where**
Postulate11 ≡ *universal-posidonius-postulate*

definition *Postulate12* :: *bool* **where**
Postulate12 ≡ *alternative-playfair-s-postulate*

definition *Postulate13* :: *bool* **where**
Postulate13 ≡ *proclus-postulate*

definition *Postulate14* :: *bool* **where**
Postulate14 ≡ *alternative-proclus-postulate*

definition *Postulate15* :: *bool* **where**
Postulate15 ≡ *triangle-circumscription-principle*

definition *Postulate16* :: *bool* **where**
Postulate16 ≡ *inverse-projection-postulate*

definition *Postulate17* :: *bool* **where**
Postulate17 ≡ *euclid-5*

definition *Postulate18* :: *bool* **where**
Postulate18 ≡ *strong-parallel-postulate*

definition *Postulate19* :: *bool* **where**
Postulate19 ≡ *alternative-strong-parallel-postulate*

definition *Postulate20* :: *bool* **where**
Postulate20 ≡ *euclid-s-parallel-postulate*

definition *Postulate21* :: *bool* **where**
Postulate21 ≡ *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*

definition *Postulate22* :: *bool* **where**
Postulate22 ≡ *posidonius-postulate*

definition *Postulate23* :: *bool* **where**
Postulate23 ≡ *postulate-of-existence-of-similar-triangles*

definition *Postulate24* :: *bool* **where**
Postulate24 ≡ *thales-postulate*

definition *Postulate25* :: *bool* **where**
Postulate25 ≡ *thales-converse-postulate*

definition *Postulate26* :: *bool* **where**
Postulate26 ≡ *existential-thales-postulate*

definition *Postulate27* :: *bool* **where**
Postulate27 ≡ *postulate-of-right-saccheri-quadrilaterals*

definition *Postulate28* :: *bool* **where**

Postulate28 ≡ *postulate-of-existence-of-a-right-saccheri-quadrilateral*

definition *Postulate29* :: *bool* **where**
Postulate29 ≡ *postulate-of-right-lambert-quadrilaterals*

definition *Postulate30* :: *bool* **where**
Postulate30 ≡ *postulate-of-existence-of-a-right-lambert-quadrilateral*

definition *Postulate31* :: *bool* **where**
Postulate31 ≡ *weak-inverse-projection-postulate*

definition *Postulate32* :: *bool* **where**
Postulate32 ≡ *weak-tarski-s-parallel-postulate*

definition *Postulate33* :: *bool* **where**
Postulate33 ≡ *weak-triangle-circumscription-principle*

definition *Postulate34* :: *bool* **where**
Postulate34 ≡ *legendre-s-parallel-postulate*

definition *Postulate35* :: *bool* **where**
Postulate35 ≡ *existential-playfair-s-postulate*

3.2 Propositions

lemma *euclid-5--original-euclid*:
assumes *Euclid5*
shows *EuclidSParallelPostulate*
⟨*proof*⟩

lemma *tarski-s-euclid-implies-euclid-5*:
assumes *TarskiSParallelPostulate*
shows *Euclid5*
⟨*proof*⟩

lemma *tarski-s-implies-euclid-s-parallel-postulate*:
assumes *TarskiSParallelPostulate*
shows *EuclidSParallelPostulate*
⟨*proof*⟩

theorem *tarski-s-euclid-implies-playfair-s-postulate*:
assumes *TarskiSParallelPostulate*
shows *PlayfairSPostulate*
⟨*proof*⟩

lemma *tarski-s-euclid-remove-degenerated-cases*:
assumes $\forall\ A\ B\ C\ D\ T.\ A \neq B \wedge A \neq C \wedge A \neq D \wedge$
$A \neq T \wedge B \neq C \wedge B \neq D \wedge B \neq T \wedge C \neq D \wedge$
$C \neq T \wedge D \neq T \wedge \neg\ Col\ A\ B\ C \wedge Bet\ A\ D\ T \wedge$
$Bet\ B\ D\ C \wedge \neg\ Col\ B\ C\ T \longrightarrow$
$(\exists\ x\ y.\ (Bet\ A\ B\ x \wedge Bet\ A\ C\ y \wedge Bet\ x\ T\ y))$
shows $\forall\ A\ B\ C\ D\ T.\ Bet\ A\ D\ T \wedge Bet\ B\ D\ C \wedge A \neq D \longrightarrow$
$(\exists\ x\ y.\ (Bet\ A\ B\ x \wedge Bet\ A\ C\ y \wedge Bet\ x\ T\ y))$
⟨*proof*⟩

lemma *alternate-interior--consecutive-interior*:
assumes *AlternateInteriorAnglesPostulate*
shows *ConsecutiveInteriorAnglesPostulate*
⟨*proof*⟩

lemma *alternate-interior--playfair-aux-1*:
assumes *A1 A2 Par C1 C2* **and**
Col P C1 C2 **and**
Col P P1 P2 **and**
P1 P2 Perp A1 A2 **and**
Col Q A1 A2
shows *Coplanar P Q A1 C1*
⟨*proof*⟩

lemma *alternate-interior--playfair-aux-2*:
assumes *A1 A2 Par C1 C2* **and**
Col P C1 C2 **and**
Col Q A1 A2 **and**
Col B1 B2 B3 **and**
¬ *Col A1 A2 P* **and**
Col P B1 B2 **and**
Coplanar A1 A2 B1 B2 **and**
A1 A2 ParStrict B1 B2
shows *Coplanar P Q C1 B3*
⟨*proof*⟩

lemma *alternate-interior--playfair-aux*:
assumes *alternate-interior-angles-postulate*
shows ∀ *A1 A2 B1 B2 C1 C2 P*.
(*P Perp2 A1 A2 B1 B2* ∧ ¬ *Col A1 A2 P* ∧ *Col P B1 B2* ∧
Coplanar A1 A2 B1 B2 ∧
A1 A2 Par C1 C2 ∧ *Col P C1 C2*) ⟶
Col C1 B1 B2
⟨*proof*⟩

lemma *alternate-interior--playfair-bis*:
assumes *alternate-interior-angles-postulate*
shows *alternative-playfair-s-postulate*
⟨*proof*⟩

lemma *alternate-interior--proclus-aux*:
assumes *greenberg-s-axiom* **and**
alternate-interior-angles-postulate
shows ∀ *A C D P Q*. (*P A ParStrict C D* ∧ *C D Perp P C* ∧
P A OS C Q ∧ *P C OS Q A* ∧ *P C OS Q D*)
⟶
(∃ *Y*. *Col P Q Y* ∧ *Col C D Y*)
⟨*proof*⟩

lemma *alternate-interior--proclus*:
assumes *greenberg-s-axiom* **and**
alternate-interior-angles-postulate
shows *proclus-postulate*
⟨*proof*⟩

lemma *alternate-interior--triangle*:
assumes *alternate-interior-angles-postulate*
shows *triangle-postulate*
⟨*proof*⟩

lemma *bachmann-s-lotschnittaxiom-aux-R1*:
assumes *bachmann-s-lotschnittaxiom*
shows ∀ *A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD*.

IAB ≠ *IAC* ∧ *IAB* ≠ *IBD* ∧
A1 A2 Perp B1 B2 ∧
A1 A2 Perp C1 C2 ∧
B1 B2 Perp D1 D2 ∧
Col A1 A2 IAB ∧
Col B1 B2 IAB ∧ *Col A1 A2 IAC* ∧
Col C1 C2 IAC ∧ *Col B1 B2 IBD* ∧ *Col D1 D2 IBD* ∧
Coplanar IAB IAC IBD C1 ∧
Coplanar IAB IAC IBD C2 ∧
Coplanar IAB IAC IBD D1 ∧ *Coplanar IAB IAC IBD D2* ⟶
(∃ *I*. *Col C1 C2 I* ∧ *Col D1 D2 I*)
⟨*proof*⟩

lemma *bachmann-s-lotschnittaxiom-aux-R2*:
assumes ∀ *A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD*.
IAB ≠ *IAC* ∧ *IAB* ≠ *IBD* ∧
A1 A2 Perp B1 B2 ∧
A1 A2 Perp C1 C2 ∧
B1 B2 Perp D1 D2 ∧
Col A1 A2 IAB ∧
Col B1 B2 IAB ∧ *Col A1 A2 IAC* ∧
Col C1 C2 IAC ∧ *Col B1 B2 IBD* ∧ *Col D1 D2 IBD* ∧
Coplanar IAB IAC IBD C1 ∧
Coplanar IAB IAC IBD C2 ∧
Coplanar IAB IAC IBD D1 ∧ *Coplanar IAB IAC IBD D2* ⟶
(∃ *I*. *Col C1 C2 I* ∧ *Col D1 D2 I*)
shows *bachmann-s-lotschnittaxiom*
⟨*proof*⟩

lemma *bachmann-s-lotschnittaxiom-aux*:
shows *bachmann-s-lotschnittaxiom* ⟷
(∀ *A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD*.
IAB ≠ *IAC* ∧ *IAB* ≠ *IBD* ∧
A1 A2 Perp B1 B2 ∧
A1 A2 Perp C1 C2 ∧
B1 B2 Perp D1 D2 ∧
Col A1 A2 IAB ∧
Col B1 B2 IAB ∧ *Col A1 A2 IAC* ∧
Col C1 C2 IAC ∧ *Col B1 B2 IBD* ∧ *Col D1 D2 IBD* ∧
Coplanar IAB IAC IBD C1 ∧
Coplanar IAB IAC IBD C2 ∧
Coplanar IAB IAC IBD D1 ∧ *Coplanar IAB IAC IBD D2* ⟶
(∃ *I*. *Col C1 C2 I* ∧ *Col D1 D2 I*))
⟨*proof*⟩

lemma *bachmann-s-lotschnittaxiom--legendre-s-parallel-postulate*:
assumes *bachmann-s-lotschnittaxiom*
shows *legendre-s-parallel-postulate*
⟨*proof*⟩

lemma *bachmann-s-lotschnittaxiom--weak-inverse-projection-postulate*:
assumes *bachmann-s-lotschnittaxiom*
shows *weak-inverse-projection-postulate*
⟨*proof*⟩

lemma *bachmann-s-lotschnittaxiom--weak-triangle-circumscription-principle*:
assumes *bachmann-s-lotschnittaxiom*
shows *weak-triangle-circumscription-principle*
⟨*proof*⟩

lemma *consecutive-interior--alternate-interior*:
assumes *consecutive-interior-angles-postulate*
shows *alternate-interior-angles-postulate*
⟨*proof*⟩

lemma *existential-playfair--rah-1*:
postulate-of-right-saccheri-quadrilaterals $\longleftrightarrow$
hypothesis-of-right-saccheri-quadrilaterals
⟨*proof*⟩

lemma *existential-playfair--rah*:
assumes *existential-playfair-s-postulate*
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *existential-saccheri--rah*:
assumes *postulate-of-existence-of-a-right-saccheri-quadrilateral*
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *existential-triangle--rah*:
assumes *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *inverse-projection-postulate--proclus-bis*:
assumes *inverse-projection-postulate*
shows *alternative-proclus-postulate*
⟨*proof*⟩

lemma *strong-parallel-postulate-implies-inter-dec*:

shows *decidability-of-intersection*
⟨*proof*⟩

lemma *impossible-case-1*:
assumes *Bet A B x* **and**
Bet C y A **and**
B C ParStrict x y
shows *False*
⟨*proof*⟩

lemma *impossible-case-2*:
assumes $A \neq D$ **and**
$B \neq D$ **and**

$\neg$ *Col A B C* **and**
Col A B x **and**
Bet A D T **and**

Bet B D C **and**
Bet y A C **and**
Bet x T y
shows *False*
⟨*proof*⟩

lemma *impossible-case-3*:
assumes
Bet A D T **and**
$\neg$ *Col B C T* **and**
Bet B D C **and**
Bet B x A **and**
Bet x T y **and**
B C ParStrict x y
shows *False*
⟨*proof*⟩

lemma *impossible-case-4-1*:
assumes $A \neq D$ **and**

$C \neq D$ **and**
¬ *Col A B C* **and**
Col A C y **and**
Bet A D T **and**

Bet B D C **and**
A Out B x **and**
Bet T y x
shows *False*
⟨*proof*⟩

lemma *impossible-case-4-2*:
assumes ¬ *Col A B C* **and**
Col A C y **and**
Bet A D T **and**
¬ *Col B C T* **and**
Bet B D C **and**
Bet B A x **and**
Bet T y x **and**
B C ParStrict x y
shows *False*
⟨*proof*⟩

lemma *impossible-case-4*:
assumes $A \neq D$ **and**

$C \neq D$ **and**
$D \neq T$ **and**
¬ *Col A B C* **and**
Col A C y **and**
Bet A D T **and**
¬ *Col B C T* **and**
Bet B D C **and**
Col A B x **and**
Bet T y x **and**
B C ParStrict x y
shows *False*
⟨*proof*⟩

lemma *impossible-two-sides-not-col*:
assumes $A \neq D$ **and**

$C \neq D$ **and**

¬ *Col A B C* **and**
Bet A D T **and**

Bet B D C **and**
Bet B Y T
shows ¬ *Col A C Y*
⟨*proof*⟩

lemma *triangle-circumscription-implies-tarski-s-euclid-aux1*:
assumes *triangle-circumscription-principle* **and**
$B \neq D$ **and**
$C \neq D$ **and**
$D \neq T$ **and**
$T \neq X$ **and**
¬ *Col A B C* **and**
Col A B M1 **and**
Bet A D T **and**
¬ *Col B C T* **and**
Bet B D C **and**
Col T Y Z **and**
Bet Y T X **and**

Bet Y M1 Z1 **and**
Cong Y T T X **and**
Cong Y M1 M1 Z1 **and**
B C Perp T Z **and**
A B Perp Y Z1
shows $\exists x.$ *Col A B x* $\wedge$ *B C ParStrict x T* $\wedge$ *Cong X x Y x*
⟨*proof*⟩

lemma *triangle-circumscription-implies-tarski-s-euclid-aux*:
assumes *triangle-circumscription-principle* **and**
$B \neq D$ **and**
$C \neq D$ **and**
$D \neq T$ **and**
$T \neq X$ **and**
$\neg$ *Col A B C* **and**
Col A B M1 **and**
Col A C M2 **and**
Bet A D T **and**
$\neg$ *Col B C T* **and**
Bet B D C **and**
Col T Y Z **and**
Bet Y T X **and**
Bet Y M1 Z1 **and**
Bet Y M2 Z2 **and**
Cong Y T T X **and**
Cong Y M1 M1 Z1 **and**
Cong Y M2 M2 Z2 **and**
B C Perp T Z **and**
A B Perp Y Z1 **and**
A C Perp Y Z2
shows $\exists$ *x y. Bet A B x* $\wedge$ *Bet A C y* $\wedge$ *Bet x T y*
⟨*proof*⟩

lemma *triangle-circumscription-implies-tarski-s-euclid*:
assumes *triangle-circumscription-principle*
shows *tarski-s-parallel-postulate*
⟨*proof*⟩

lemma *thales-converse-postulate--thales-existence*:
assumes *thales-converse-postulate*
shows *existential-thales-postulate*
⟨*proof*⟩

lemma *thales-converse-postulate--weak-triangle-circumscription-principle*:
assumes *thales-converse-postulate*
shows *weak-triangle-circumscription-principle*
⟨*proof*⟩

lemma *thales-existence--rah*:
assumes *existential-thales-postulate*
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *thales-postulate--thales-converse-postulate*:
assumes *thales-postulate*
shows *thales-converse-postulate*

⟨*proof*⟩

lemma *triangle--existential-triangle*:
assumes *triangle-postulate*
shows *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*
⟨*proof*⟩

lemma *legendre-aux-tr*:
assumes *greenberg-s-axiom* **and**
triangle-postulate
shows (∀ *A B C P Q*. ¬(
Q A Perp P Q ∧ *P B Perp P Q* ∧ *Q A ParStrict P B* ∧
Q A ParStrict P C ∧ *P B OS Q C* ∧ *P Q OS C A* ∧ *P Q OS C B*))
⟨*proof*⟩

lemma *legendre-aux1-tr*:
assumes *greenberg-s-axiom* **and**
triangle-postulate
shows ∀ *A1 A2 B1 B2 C1 C2 P*.
(*P Perp2 A1 A2 B1 B2* ∧ ¬ *Col A1 A2 P* ∧
Col P B1 B2 ∧ *Coplanar A1 A2 B1 B2* ∧
A1 A2 Par C1 C2 ∧ *Col P C1 C2* ∧
¬ *B1 B2 TS A1 C1*)
⟶
Col C1 B1 B2
⟨*proof*⟩

lemma *legendre-aux2-tr*:
assumes *greenberg-s-axiom* **and**
triangle-postulate
shows ∀ *A1 A2 B1 B2 C1 C2 P*.
(*P Perp2 A1 A2 B1 B2* ∧ ¬ *Col A1 A2 P* ∧
Col P B1 B2 ∧ *Coplanar A1 A2 B1 B2* ∧
A1 A2 Par C1 C2 ∧ *Col P C1 C2*)
⟶
Col C1 B1 B2
⟨*proof*⟩

lemma *triangle--playfair-bis*:
assumes *greenberg-s-axiom* **and**
triangle-postulate
shows *alternative-playfair-s-postulate*
⟨*proof*⟩

lemma *rah--existential-saccheri*:
assumes *postulate-of-right-saccheri-quadrilaterals*
shows *postulate-of-existence-of-a-right-saccheri-quadrilateral*
⟨*proof*⟩

lemma *rah--posidonius-aux*:
assumes *postulate-of-right-saccheri-quadrilaterals*
shows ∀ *A1 A2 A3 B1 B2 B3*.
(*Per A2 A1 B1* ∧ *Per A1 A2 B2* ∧
Cong A1 B1 A2 B2 ∧ *A1 A2 OS B1 B2* ∧
Col A1 A2 A3 ∧ *Col B1 B2 B3* ∧
A1 A2 Perp A3 B3)
⟶
Cong A3 B3 A1 B1
⟨*proof*⟩

lemma *rah--posidonius*:
assumes *postulate-of-right-saccheri-quadrilaterals*
shows *posidonius-postulate*
⟨*proof*⟩

lemma *rah--rectangle-principle*:
assumes *postulate-of-right-saccheri-quadrilaterals*
shows *postulate-of-right-lambert-quadrilaterals*
⟨*proof*⟩

lemma *rah--similar*:
assumes *postulate-of-right-saccheri-quadrilaterals*
shows *postulate-of-existence-of-similar-triangles*
⟨*proof*⟩

lemma *rah--thales-postulate*:
assumes *postulate-of-right-saccheri-quadrilaterals*
shows *thales-postulate*
⟨*proof*⟩

lemma *rah--triangle*:
assumes *postulate-of-right-saccheri-quadrilaterals*
shows *triangle-postulate*
⟨*proof*⟩

lemma *rectangle-principle--rectangle-existence*:
assumes *postulate-of-right-lambert-quadrilaterals*
shows *postulate-of-existence-of-a-right-lambert-quadrilateral*
⟨*proof*⟩

lemma *similar--rah-aux*:
assumes ¬ *Col A B C* **and**
A B C CongA D E F **and**
B C A CongA E F D **and**
C A B CongA F D E **and**
B C A LeA A B C **and**
D E Lt A B
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *similar--rah*:
assumes *postulate-of-existence-of-similar-triangles*
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *impossible-case-5*:
shows ∀ *P Q R S U I*.
¬ (*BetS Q U R* ∧ ¬ *Col P Q S* ∧
¬ *Col P R U* ∧ *P R Par Q S* ∧
Bet S Q I ∧ *Bet U I P*)
⟨*proof*⟩

lemma *impossible-case-6*:
shows ∀ *P Q R S U I*.
¬ (*BetS Q U R* ∧ ¬ *Col P Q S* ∧

P S Par Q R ∧ *Bet S Q I* ∧ *Bet I P U*)
⟨*proof*⟩

lemma *impossible-case-7*:
shows ∀ *P Q R S U I*.
¬ (*BetS Q U R* ∧ ¬ *Col P Q S* ∧
¬ *Col P R U* ∧ *P R Par Q S* ∧
P S Par Q R ∧ *Col P U I* ∧ *Bet Q I S*)
⟨*proof*⟩

lemma *impossible-case-8*:
shows ∀ *P Q R S U I*.
¬ (*BetS Q U R* ∧ ¬ *Col P Q S* ∧
P R Par Q S ∧ *P S Par Q R* ∧
Col P U I ∧ *Bet I S Q*)
⟨*proof*⟩

lemma *strong-parallel-postulate-implies-tarski-s-euclid-aux*:
assumes *strong-parallel-postulate* **and**
$A \neq B$ **and**
$A \neq D$ **and**
$A \neq T$ **and**
$B \neq D$ **and**
$B \neq T$ **and**
$D \neq T$ **and**
¬ *Col A B T* **and**
Bet A D T
shows ∃ B' B'' *MB X*.
(*Bet A B X* ∧ *T X ParStrict B D* ∧
BetS B MB T ∧ *BetS* B' *MB* B'' ∧
Cong B MB T MB ∧ *Cong* B' *MB* B'' *MB* ∧
Col B B' *D* ∧ *Bet* B'' *T X* ∧
$B \neq B'$ ∧ $B'' \neq T$)
⟨*proof*⟩

lemma *strong-parallel-postulate-implies-tarski-s-euclid* :
assumes *strong-parallel-postulate*
shows *tarski-s-parallel-postulate*
⟨*proof*⟩

lemma *midpoint-converse-postulate-implies-playfair*:
assumes *midpoint-converse-postulate*
shows *playfair-s-postulate*
⟨*proof*⟩

lemma *playfair-implies-par-trans*:
assumes *playfair-s-postulate*
shows *postulate-of-transitivity-of-parallelism*
⟨*proof*⟩

lemma *par-trans-implies-playfair*:
assumes *postulate-of-transitivity-of-parallelism*
shows *playfair-s-postulate*
⟨*proof*⟩

lemma *par-perp-perp-implies-par-perp-2-par*:
assumes *perpendicular-transversal-postulate*
shows *postulate-of-parallelism-of-perpendicular-transversals*
⟨*proof*⟩

lemma *par-perp-2-par-implies-par-perp-perp*:
assumes *postulate-of-parallelism-of-perpendicular-transversals*
shows *perpendicular-transversal-postulate*

⟨*proof*⟩

lemma *par-perp-perp-implies-playfair*:
assumes *perpendicular-transversal-postulate*
shows *playfair-s-postulate*
⟨*proof*⟩

lemma *playfair--universal-posidonius-postulate*:
assumes *playfair-s-postulate*
shows *universal-posidonius-postulate*
⟨*proof*⟩

lemma *weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom-aux*:
assumes *weak-inverse-projection-postulate*
shows ∀ *A1 A2 B1 B2 C1 C2 Q P R M*.
(*A1 A2 Perp B1 B2* ∧ *A1 A2 Perp C1 C2* ∧
Col A1 A2 Q ∧ *Col B1 B2 Q* ∧
Col A1 A2 P ∧ *Col C1 C2 P* ∧
Col B1 B2 R ∧ *Coplanar Q P R C1* ∧
Coplanar Q P R C2 ∧ ¬ *Col Q P R* ∧
M InAngle P Q R ∧ *M Q P CongA M Q R*)
⟶
(*B1 B2 ParStrict C1 C2* ∧
(∃ *S*. *Q Out M S* ∧ *Col C1 C2 S*))
⟨*proof*⟩

lemma *weak-inverse-projection-postulate--bachmann-s-lotschnittaxiom*:
assumes *weak-inverse-projection-postulate*
shows *bachmann-s-lotschnittaxiom*
⟨*proof*⟩

lemma *weak-triangle-circumscription-principle--bachmann-s-lotschnittaxiom*:
assumes *weak-triangle-circumscription-principle*
shows *bachmann-s-lotschnittaxiom*
⟨*proof*⟩

lemma *universal-posidonius-postulate--perpendicular-transversal-postulate-aux-lem*:
fixes *A1 A2 B1 B2 C1 C2 D1 D2 IAB IAC IBD*
assumes
IAB ≠ *IAC* **and**
IAB ≠ *IBD* **and**
A1 A2 Perp B1 B2 **and**
A1 A2 Perp C1 C2 **and**
B1 B2 Perp D1 D2 **and**
Col A1 A2 IAB **and**
Col B1 B2 IAB **and**
Col A1 A2 IAC **and**
Col C1 C2 IAC **and**
Col B1 B2 IBD **and**
Col D1 D2 IBD **and**
Coplanar IAB IAC IBD C1 **and**
Coplanar IAB IAC IBD C2 **and**
Coplanar IAB IAC IBD D1 **and**
Coplanar IAB IAC IBD D2 **and**
postulate-of-right-saccheri-quadrilaterals
shows ∃ *I*. *Col C1 C2 I* ∧ *Col D1 D2 I*
⟨*proof*⟩

lemma *universal-posidonius-postulate--perpendicular-transversal-postulate-aux*:
assumes *universal-posidonius-postulate*
shows ∀ *E F G H R P*.
E G Perp R P ∧ *Coplanar F H P R* ∧ *Col E G R* ∧ *Saccheri E F H G* ⟶
F H Perp P R
⟨*proof*⟩

lemma *universal-posidonius-postulate--perpendicular-transversal-postulate*:
assumes *universal-posidonius-postulate*
shows *perpendicular-transversal-postulate*
⟨*proof*⟩

lemma *playfair--alternate-interior*:
assumes *playfair-s-postulate*
shows *alternate-interior-angles-postulate*
⟨*proof*⟩

lemma *playfair-bis--playfair*:
assumes *alternative-playfair-s-postulate*
shows *playfair-s-postulate*
⟨*proof*⟩

lemma *playfair-s-postulate-implies-midpoint-converse-postulate*:
assumes *playfair-s-postulate*
shows *midpoint-converse-postulate*
⟨*proof*⟩

lemma *inter-dec-plus-par-perp-perp-imply-triangle-circumscription*:
assumes
perpendicular-transversal-postulate
shows *triangle-circumscription-principle*
⟨*proof*⟩

lemma *original-euclid--original-spp*:
assumes *euclid-s-parallel-postulate*
shows *alternative-strong-parallel-postulate*
⟨*proof*⟩

lemma *original-spp--inverse-projection-postulate*:
assumes *alternative-strong-parallel-postulate*
shows *inverse-projection-postulate*
⟨*proof*⟩

lemma *proclus-bis--proclus*:
assumes *alternative-proclus-postulate*
shows *proclus-postulate*
⟨*proof*⟩

lemma *proclus-s-postulate-implies-strong-parallel-postulate*:
assumes *proclus-postulate*
shows *strong-parallel-postulate*
⟨*proof*⟩

lemma *rectangle-existence--rah*:
assumes *postulate-of-existence-of-a-right-lambert-quadrilateral*
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *posidonius-postulate--rah*:
assumes *posidonius-postulate*
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *playfair--existential-playfair*:
assumes *playfair-s-postulate*
shows *existential-playfair-s-postulate*
⟨*proof*⟩

3.3 Equivalences

lemma *proclus--aristotle*:
  assumes *proclus-postulate*
  shows *aristotle-s-axiom*
⟨*proof*⟩

lemma *aristotle--obtuse-case-elimination*:
  assumes *aristotle-s-axiom*
  shows ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *aristotle--acute-or-right*:
  assumes *aristotle-s-axiom*
  shows *hypothesis-of-acute-saccheri-quadrilaterals*
    ∨
  hypothesis-of-right-saccheri-quadrilaterals
⟨*proof*⟩

lemma *Axiom1ProofIsabelleHOL*:
  shows *Axiom1*
  ⟨*proof*⟩

theorem *equivalent-postulates-without-decidability-of-intersection-of-lines*:
  shows (*Postulate07* ⟷ *Postulate12*) ∧
    (*Postulate12* ⟷ *Postulate08*) ∧
    (*Postulate08* ⟷ *Postulate06*) ∧
    (*Postulate06* ⟷ *Postulate09*) ∧
    (*Postulate09* ⟷ *Postulate02*) ∧
    (*Postulate02* ⟷ *Postulate11*) ∧
    (*Postulate11* ⟷ *Postulate10*) ∧
    (*Postulate10* ⟷ *Postulate05*)
⟨*proof*⟩

lemma *Cycle-1*:
  shows (*Postulate01* ⟶ *Postulate02*) ∧
    (*Postulate09* ⟶ *Postulate15*) ∧
    (*Postulate15* ⟶ *Postulate01*)
⟨*proof*⟩

lemma *Cycle-2*:
  shows (*Postulate01* ⟷ *Postulate13*) ∧
    (*Postulate13* ⟷ *Postulate14*) ∧
    (*Postulate14* ⟷ *Postulate16*) ∧
    (*Postulate16* ⟷ *Postulate17*) ∧
    (*Postulate17* ⟷ *Postulate18*) ∧
    (*Postulate18* ⟷ *Postulate19*) ∧
    (*Postulate19* ⟷ *Postulate20*) ∧
    (*Postulate20* ⟷ *Postulate01*)
⟨*proof*⟩

lemma *Cycle-3*:
  shows (*Postulate03* ⟷ *Postulate21*) ∧
    (*Postulate21* ⟷ *Postulate22*) ∧
    (*Postulate22* ⟷ *Postulate23*) ∧
    (*Postulate23* ⟷ *Postulate24*) ∧
    (*Postulate24* ⟷ *Postulate25*) ∧
    (*Postulate25* ⟷ *Postulate26*) ∧
    (*Postulate26* ⟷ *Postulate27*) ∧
    (*Postulate27* ⟷ *Postulate28*) ∧
    (*Postulate28* ⟷ *Postulate29*) ∧
    (*Postulate29* ⟷ *Postulate30*) ∧
    (*Postulate30* ⟷ *Postulate03*)
⟨*proof*⟩

lemma *InterCycle1*:
assumes *greenberg-s-axiom* **and** *Postulate03*
shows *Postulate12*
⟨*proof*⟩

lemma *InterCycle2*:
assumes *Postulate07*
shows *Postulate03*
⟨*proof*⟩

lemma *InterCycle3*:
assumes *Postulate01*
shows *Postulate02*
⟨*proof*⟩

lemma *InterCycle4*:
assumes *Postulate09*
shows *Postulate15*
⟨*proof*⟩

lemma *InterAx1-R1*:
assumes *Postulate13*
shows *aristotle-s-axiom*
⟨*proof*⟩

lemma *InterAx1*:
assumes *Postulate01*
shows *aristotle-s-axiom*
⟨*proof*⟩

lemma *InterAx3*:
assumes *greenberg-s-axiom* **and** *Postulate02*
shows *Postulate01*
⟨*proof*⟩

lemma *InterAx4*:
assumes *Postulate01*
shows *Axiom1*
⟨*proof*⟩

lemma *InterAx5*:
assumes *Postulate02*
shows *Postulate01*
⟨*proof*⟩

lemma *PPR-Theorem-4-bis*:
assumes *Postulate01*
shows *greenberg-s-axiom*
⟨*proof*⟩

lemma *PPR-Proposition-6*:
assumes *archimedes-axiom*
shows *aristotle-s-axiom*
⟨*proof*⟩

lemma *InterCycle1bis*:
assumes *Postulate01*
shows *Postulate03* ⟶ *Postulate12*
⟨*proof*⟩

lemma *weak-inverse-projection-postulate--weak-tarski-s-parallel-postulate*:
assumes *weak-inverse-projection-postulate*
shows *weak-tarski-s-parallel-postulate*
⟨*proof*⟩

lemma *weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate-aux* :
assumes *weak-tarski-s-parallel-postulate*
shows ∀ *A B C P T*.
Per A B C ∧ *T InAngle A B C* ∧
P ≠ *T* ∧ *P B A CongA P B C* ∧ *Per B P T* ∧ *Coplanar A B C P*
⟶
((∃ *X*. (*B Out A X* ∧ *Col T P X*)) ∨ (∃ *Y*. (*B Out C Y* ∧ *Col T P Y*)))
⟨*proof*⟩

lemma *weak-tarski-s-parallel-postulate--weak-inverse-projection-postulate*:
assumes *weak-tarski-s-parallel-postulate*
shows *weak-inverse-projection-postulate*
⟨*proof*⟩

lemma *P31-P32*:
shows *Postulate31* ⟷ *Postulate32*
⟨*proof*⟩

lemma *P31-P04*:
shows *Postulate31* ⟷ *Postulate04*
⟨*proof*⟩

lemma *P04-P33*:
shows *Postulate04* ⟷ *Postulate33*
⟨*proof*⟩

lemma *equivalent-postulates-without-any-continuity-bis*:
shows *Postulate04* ⟷ *Postulate33* ∧
Postulate31 ⟷ *Postulate04* ∧
Postulate31 ⟷ *Postulate32*
⟨*proof*⟩

lemma *P4-P34*:
assumes *Postulate04*
shows *Postulate34*
⟨*proof*⟩

lemma *P01--P35*:
assumes *Postulate01*
shows *Postulate35*
⟨*proof*⟩

lemma *P35--P01*:
assumes *greenberg-s-axiom* **and**
Postulate01
shows *Postulate35*
⟨*proof*⟩

theorem *stronger-legendre-s-first-theorem*:
assumes *aristotle-s-axiom*
shows ∀ *A B C D E F*. *C A B A B C SumA D E F* ⟶ *SAMS D E F B C A*
⟨*proof*⟩

theorem *legendre-s-first-theorem*:
assumes *archimedes-axiom*
shows ∀ *A B C D E F*. *C A B A B C SumA D E F* ⟶ *SAMS D E F B C A*
⟨*proof*⟩

theorem *legendre-s-second-theorem*:
assumes *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*
shows *triangle-postulate*
⟨*proof*⟩

lemma *legendre-s-third-theorem-aux*:

assumes *aristotle-s-axiom* **and**
triangle-postulate
shows *euclid-s-parallel-postulate*
⟨*proof*⟩

theorem *legendre-s-third-theorem*:
assumes *archimedes-axiom* **and**
triangle-postulate
shows *euclid-s-parallel-postulate*
⟨*proof*⟩

lemma *legendre-aux*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
shows ∀ *A B C D B1 C1 P Q R S T U V W X*.
¬ *Col A B C* ∧ *A C B CongA C B D* ∧
Cong A C B D ∧ *B C TS A D* ∧ *A Out B B1* ∧ *A Out C C1* ∧ *Bet B1 D C1* ∧
Defect A B C P Q R ∧ *Defect A B1 C1 S T U* ∧ *P Q R P Q R SumA V W X* ⟶
(*SAMS P Q R P Q R* ∧ *V W X LeA S T U*)
⟨*proof*⟩

lemma *legendre-aux1*:
shows ∀ *A B C B′ C′*.
¬ *Col A B C* ∧ *A Out B B′* ∧ *A Out C C′* ⟶
(∃ *D′*. *D′ InAngle B A C* ∧ *A C′ B′ CongA C′ B′ D′* ∧
Cong A C′ B′ D′ ∧ *B′ C′ TS A D′*)
⟨*proof*⟩

lemma *legendre-aux2*:
assumes ¬ *hypothesis-of-obtuse-saccheri-quadrilaterals*
shows ∀ *A B C*.
¬ *Col A B C* ∧ *Acute B A C* ∧
(∀ *T*. *T InAngle B A C* ⟶ (∃ *X Y*. *A Out B X* ∧ *A Out C Y* ∧ *Bet X T Y*))
⟶
(∀ *P Q R S T U*. *Defect A B C P Q R* ∧ *GradAExp P Q R S T U* ⟶
(∃ *B′ C′ P′ Q′ R′*. (*A Out B B′* ∧ *A Out C C′* ∧
Defect A B′ C′ P′ Q′ R′ ∧ *S T U LeA P′ Q′ R′*)))
⟨*proof*⟩

lemma *legendre-s-fourth-theorem-aux*:
assumes *archimedes-axiom* **and**
legendre-s-parallel-postulate
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

theorem *legendre-s-fourth-theorem*:
assumes *archimedes-axiom* **and**
legendre-s-parallel-postulate
shows *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*
⟨*proof*⟩

lemma *P34-P21*:
assumes *archimedes-axiom* **and**
Postulate34
shows *Postulate21*
⟨*proof*⟩

lemma *P34-P27*:
assumes *archimedes-axiom* **and**
Postulate34
shows *Postulate27*
⟨*proof*⟩

lemma *P25-33*:
assumes *Postulate25*
shows *Postulate33*
⟨*proof*⟩

lemma *P23-33*:
  assumes *Postulate23*
  shows *Postulate33*
⟨*proof*⟩

lemma *P01-35*:
  assumes *Postulate01*
  shows *Postulate35*
  ⟨*proof*⟩

lemma *P35-27*:
  assumes *Postulate35*
  shows *Postulate27*
  ⟨*proof*⟩

lemma *Thm10-1*:
  assumes *archimedes-axiom*
  shows (*Postulate01* ⟷ *Postulate02*) ∧
    (*Postulate01* ⟷ *Postulate03*) ∧
    (*Postulate01* ⟷ *Postulate04*) ∧
    (*Postulate01* ⟷ *Postulate05*) ∧
    (*Postulate01* ⟷ *Postulate06*) ∧
    (*Postulate01* ⟷ *Postulate07*) ∧
    (*Postulate01* ⟷ *Postulate08*) ∧
    (*Postulate01* ⟷ *Postulate09*) ∧
    (*Postulate01* ⟷ *Postulate10*)
  ⟨*proof*⟩

lemma *Thm10-2*:
  assumes *archimedes-axiom*
  shows (*Postulate01* ⟷ *Postulate11*) ∧
    (*Postulate01* ⟷ *Postulate12*) ∧
    (*Postulate01* ⟷ *Postulate13*) ∧
    (*Postulate01* ⟷ *Postulate14*) ∧
    (*Postulate01* ⟷ *Postulate15*) ∧
    (*Postulate01* ⟷ *Postulate16*) ∧
    (*Postulate01* ⟷ *Postulate17*) ∧
    (*Postulate01* ⟷ *Postulate18*) ∧
    (*Postulate01* ⟷ *Postulate19*) ∧
    (*Postulate01* ⟷ *Postulate20*)
  ⟨*proof*⟩

lemma *Thm10-3*:
  assumes *archimedes-axiom*
  shows (*Postulate01* ⟷ *Postulate21*) ∧
    (*Postulate01* ⟷ *Postulate22*) ∧
    (*Postulate01* ⟷ *Postulate23*) ∧
    (*Postulate01* ⟷ *Postulate24*) ∧
    (*Postulate01* ⟷ *Postulate25*) ∧
    (*Postulate01* ⟷ *Postulate26*) ∧
    (*Postulate01* ⟷ *Postulate27*) ∧
    (*Postulate01* ⟷ *Postulate28*) ∧
    (*Postulate01* ⟷ *Postulate29*) ∧
    (*Postulate01* ⟷ *Postulate30*)
  ⟨*proof*⟩

lemma *Thm10-4*:
  assumes *archimedes-axiom*
  shows (*Postulate01* ⟷ *Postulate31*) ∧
    (*Postulate01* ⟷ *Postulate32*) ∧
    (*Postulate01* ⟷ *Postulate33*) ∧
    (*Postulate01* ⟷ *Postulate34*) ∧
    (*Postulate01* ⟷ *Postulate35*)
  ⟨*proof*⟩

theorem *Thm10*:
assumes *archimedes-axiom*
shows (*Postulate01* ⟷ *Postulate02*) ∧
(*Postulate01* ⟷ *Postulate03*) ∧
(*Postulate01* ⟷ *Postulate04*) ∧
(*Postulate01* ⟷ *Postulate05*) ∧
(*Postulate01* ⟷ *Postulate06*) ∧
(*Postulate01* ⟷ *Postulate07*) ∧
(*Postulate01* ⟷ *Postulate08*) ∧
(*Postulate01* ⟷ *Postulate09*) ∧
(*Postulate01* ⟷ *Postulate10*) ∧
(*Postulate01* ⟷ *Postulate11*) ∧
(*Postulate01* ⟷ *Postulate12*) ∧
(*Postulate01* ⟷ *Postulate13*) ∧
(*Postulate01* ⟷ *Postulate14*) ∧
(*Postulate01* ⟷ *Postulate15*) ∧
(*Postulate01* ⟷ *Postulate16*) ∧
(*Postulate01* ⟷ *Postulate17*) ∧
(*Postulate01* ⟷ *Postulate18*) ∧
(*Postulate01* ⟷ *Postulate19*) ∧
(*Postulate01* ⟷ *Postulate20*) ∧
(*Postulate01* ⟷ *Postulate21*) ∧
(*Postulate01* ⟷ *Postulate22*) ∧
(*Postulate01* ⟷ *Postulate23*) ∧
(*Postulate01* ⟷ *Postulate24*) ∧
(*Postulate01* ⟷ *Postulate25*) ∧
(*Postulate01* ⟷ *Postulate26*) ∧
(*Postulate01* ⟷ *Postulate27*) ∧
(*Postulate01* ⟷ *Postulate28*) ∧
(*Postulate01* ⟷ *Postulate29*) ∧
(*Postulate01* ⟷ *Postulate30*) ∧
(*Postulate01* ⟷ *Postulate31*) ∧
(*Postulate01* ⟷ *Postulate32*) ∧
(*Postulate01* ⟷ *Postulate33*) ∧
(*Postulate01* ⟷ *Postulate34*) ∧
(*Postulate01* ⟷ *Postulate35*)
⟨*proof*⟩

theorem *equivalent-postulates-assuming-greenberg-s-axiom*:
assumes *greenberg-s-axiom*
shows (*tarski-s-parallel-postulate* ⟷ *alternate-interior-angles-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *alternative-playfair-s-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *alternative-playfair-s-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *alternative-proclus-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *alternative-strong-parallel-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *consecutive-interior-angles-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *euclid-5*) ∧
(*tarski-s-parallel-postulate* ⟷ *euclid-s-parallel-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *existential-playfair-s-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *existential-thales-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *inverse-projection-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *midpoint-converse-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *perpendicular-transversal-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *postulate-of-transitivity-of-parallelism*) ∧
(*tarski-s-parallel-postulate* ⟷ *playfair-s-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *posidonius-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *universal-posidonius-postulate*) ∧
(*tarski-s-parallel-postulate*
⟷ *postulate-of-existence-of-a-right-lambert-quadrilateral*) ∧
(*tarski-s-parallel-postulate*
⟷ *postulate-of-existence-of-a-right-saccheri-quadrilateral*) ∧
(*tarski-s-parallel-postulate*
⟷ *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*) ∧
(*tarski-s-parallel-postulate* ⟷ *postulate-of-existence-of-similar-triangles*) ∧
(*tarski-s-parallel-postulate*

⟷ *postulate-of-parallelism-of-perpendicular-transversals*) ∧
(*tarski-s-parallel-postulate* ⟷ *postulate-of-right-lambert-quadrilaterals*) ∧
(*tarski-s-parallel-postulate* ⟷ *postulate-of-right-saccheri-quadrilaterals*) ∧
(*tarski-s-parallel-postulate* ⟷ *postulate-of-transitivity-of-parallelism*) ∧
(*tarski-s-parallel-postulate* ⟷ *proclus-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *strong-parallel-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *tarski-s-parallel-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *thales-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *thales-converse-postulate*) ∧
(*tarski-s-parallel-postulate* ⟷ *triangle-circumscription-principle*) ∧
(*tarski-s-parallel-postulate* ⟷ *triangle-postulate*)
⟨*proof*⟩

theorem *equivalent-postulates-assuming-archimesdes-axiom*:
assumes *archimedes-axiom*
shows *alternate-interior-angles-postulate* ⟷ *alternative-playfair-s-postulate* ∧
alternative-playfair-s-postulate ⟷ *alternative-proclus-postulate* ∧
alternative-proclus-postulate ⟷ *alternative-strong-parallel-postulate* ∧
alternative-strong-parallel-postulate ⟷ *bachmann-s-lotschnittaxiom* ∧
bachmann-s-lotschnittaxiom ⟷ *consecutive-interior-angles-postulate* ∧
consecutive-interior-angles-postulate ⟷ *euclid-5* ∧
euclid-5 ⟷ *euclid-s-parallel-postulate* ∧
euclid-s-parallel-postulate ⟷ *existential-playfair-s-postulate* ∧
existential-playfair-s-postulate ⟷ *existential-thales-postulate* ∧
existential-thales-postulate ⟷ *inverse-projection-postulate* ∧
inverse-projection-postulate ⟷ *legendre-s-parallel-postulate* ∧
legendre-s-parallel-postulate ⟷ *midpoint-converse-postulate* ∧
midpoint-converse-postulate ⟷ *perpendicular-transversal-postulate* ∧
perpendicular-transversal-postulate ⟷ *postulate-of-transitivity-of-parallelism* ∧
postulate-of-transitivity-of-parallelism ⟷ *playfair-s-postulate* ∧
playfair-s-postulate ⟷ *posidonius-postulate* ∧
posidonius-postulate ⟷ *universal-posidonius-postulate* ∧
universal-posidonius-postulate ⟷
postulate-of-existence-of-a-right-lambert-quadrilateral ∧
postulate-of-existence-of-a-right-lambert-quadrilateral ⟷
postulate-of-existence-of-a-right-saccheri-quadrilateral ∧
postulate-of-existence-of-a-right-saccheri-quadrilateral ⟷
postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights ∧
postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights ⟷
postulate-of-existence-of-similar-triangles ∧
postulate-of-existence-of-similar-triangles ⟷
postulate-of-parallelism-of-perpendicular-transversals ∧
postulate-of-parallelism-of-perpendicular-transversals ⟷
postulate-of-right-lambert-quadrilaterals ∧
postulate-of-right-lambert-quadrilaterals ⟷ *postulate-of-right-saccheri-quadrilaterals* ∧
postulate-of-right-saccheri-quadrilaterals ⟷ *postulate-of-transitivity-of-parallelism* ∧
postulate-of-transitivity-of-parallelism ⟷ *proclus-postulate* ∧
proclus-postulate ⟷ *strong-parallel-postulate* ∧
strong-parallel-postulate ⟷ *tarski-s-parallel-postulate* ∧
tarski-s-parallel-postulate ⟷ *thales-postulate* ∧
thales-postulate ⟷ *thales-converse-postulate* ∧
thales-converse-postulate ⟷ *triangle-circumscription-principle* ∧
triangle-circumscription-principle ⟷ *triangle-postulate* ∧
triangle-postulate ⟷ *weak-inverse-projection-postulate* ∧
weak-inverse-projection-postulate ⟷ *weak-tarski-s-parallel-postulate* ∧
weak-tarski-s-parallel-postulate ⟷ *weak-triangle-circumscription-principle*
⟨*proof*⟩

4 Szmielew: hyperbolic plane postulate

4.1 Definition

definition *hyperbolic-plane-postulate* ::
bool
(*HyperbolicPlanePostulate*) **where**
hyperbolic-plane-postulate ≡

$\forall$ *A1 A2 P*.
$\neg$ *Col A1 A2 P*
$\longrightarrow$
($\exists$ *B1 B2 C1 C2*.
A1 A2 Par B1 B2 $\wedge$ *Col P B1 B2* $\wedge$ *A1 A2 Par C1 C2* $\wedge$ *Col P C1 C2* $\wedge$ $\neg$ *Col C1 B1 B2*)

4.2 Propositions

lemma *hpp--nP35*:
assumes *greenberg-s-axiom*
shows *hyperbolic-plane-postulate* $\longleftrightarrow$ $\neg$ *Postulate35*
⟨*proof*⟩

lemma *aah--hpp*:
assumes *hypothesis-of-acute-saccheri-quadrilaterals*
shows *hyperbolic-plane-postulate*
⟨*proof*⟩

theorem *szmielew-s-theorem*:
assumes *aristotle-s-axiom*
shows $\forall$ *P*:: *bool*.
(*playfair-s-postulate* $\longrightarrow$ *P*) $\wedge$ (*hyperbolic-plane-postulate* $\longrightarrow$ $\neg$ *P*)
$\longrightarrow$
(*P* $\longleftrightarrow$ *playfair-s-postulate*)
⟨*proof*⟩

end
end

theory *Tarski-Neutral-Continuity*

imports
Tarski-Neutral

begin

context *Tarski-neutral-dimensionless*

begin

5 Continuity

5.1 Definitions

definition *OnCircle* ::
['p,'p,'p] $\Rightarrow$ *bool*
(- *OnCircle* - - [*99*,*99*,*99*] *50*)
where
P OnCircle A B $\equiv$ *Cong A P A B*

definition *InCircle* ::
['p,'p,'p] $\Rightarrow$ *bool*
(- *InCircle* - - [*99*,*99*,*99*] *50*)
where
P InCircle A B $\equiv$ *A P Le A B*

definition *OutCircle* ::
['p,'p,'p] $\Rightarrow$ *bool*
(- *OutCircle* - - [*99*,*99*,*99*] *50*)
where
P OutCircle A B $\equiv$ *A B Le A P*

definition *InCircleS* ::
$['p,'p,'p] \Rightarrow bool$
(- *InCircleS* - - [*99*,*99*,*99*] *50*)
where
P InCircleS A B $\equiv$ *A P Lt A B*

definition *OutCircleS* ::
$['p,'p,'p] \Rightarrow bool$
(- *OutCircleS* - - [*99*,*99*,*99*] *50*)
where
P OutCircleS A B $\equiv$ *A B Lt A P*

definition *Diam* ::
$['p,'p,'p,'p] \Rightarrow bool$
(*Diam* - - - - [*99*,*99*,*99*,*99*] *50*)
where
Diam A B PO P $\equiv$ *Bet A PO B* $\wedge$ *A OnCircle PO P* $\wedge$ *B OnCircle PO P*

definition *EqC* ::
$['p,'p,'p,'p] \Rightarrow bool$
(*EqC* - - - - [*99*,*99*,*99*,*99*] *50*)
where
EqC A B C D $\equiv$
$\forall$ *X*. (*X OnCircle A B* $\longleftrightarrow$ *X OnCircle C D*)

definition *InterCCAt* ::
$['p,'p,'p,'p,'p,'p] \Rightarrow bool$
(*InterCCAt* - - - - - - [*99*,*99*,*99*,*99*,*99*,*99*] *50*)
where
InterCCAt A B C D P Q $\equiv$
$\neg$ *EqC A B C D* $\wedge$
P $\neq$ *Q* $\wedge$ *P OnCircle C D* $\wedge$ *Q OnCircle C D* $\wedge$ *P OnCircle A B* $\wedge$ *Q OnCircle A B*

definition *InterCC* ::
$['p,'p,'p,'p] \Rightarrow bool$
(*InterCC* - - - - [*99*,*99*,*99*,*99*] *50*)
where
InterCC A B C D $\equiv$ $\exists$ *P Q*. *InterCCAt A B C D P Q*

definition *TangentCC* ::
$['p,'p,'p,'p] \Rightarrow bool$
(*TangentCC* - - - - [*99*,*99*,*99*,*99*] *50*)
where
TangentCC A B C D $\equiv$
$\exists$ *X*. (*X OnCircle A B* $\wedge$ *X OnCircle C D* $\wedge$
($\forall$ *Y*. (*Y OnCircle A B* $\wedge$ *Y OnCircle C D*) $\longrightarrow$ *X* = *Y*))

definition *Tangent* ::
$['p,'p,'p,'p] \Rightarrow bool$
(*Tangent* - - - - [*99*,*99*,*99*,*99*] *50*)
where

Tangent A B PO P ≡
∃ *X*. (*Col A B X* ∧ *X OnCircle PO P* ∧
(∀ *Y*. (*Col A B Y* ∧ *Y OnCircle PO P*) ⟶ *X* = *Y*))

definition *TangentAt* ::
[′*p*,′*p*,′*p*,′*p*,′*p*] ⇒ *bool*
(*TangentAt* - - - - - [*99*,*99*,*99*,*99*,*99*] *50*)
where
TangentAt A B PO P T ≡ *Tangent A B PO P* ∧ *Col A B T* ∧ *T OnCircle PO P*

definition *Concyclic* ::
[′*p*,′*p*,′*p*,′*p*] ⇒ *bool*
(*Concyclic* - - - - [*99*,*99*,*99*,*99*] *50*)
where
Concyclic A B C D ≡ *Coplanar A B C D* ∧
(∃ *PO P*. *A OnCircle PO P* ∧ *B OnCircle PO P* ∧
C OnCircle PO P ∧ *D OnCircle PO P*)

definition *ConcyclicGen* ::
[′*p*,′*p*,′*p*,′*p*] ⇒ *bool*
(*ConcyclicGen* - - - - [*99*,*99*,*99*,*99*] *50*)
where
ConcyclicGen A B C D ≡
Concyclic A B C D ∨
(*Col A B C* ∧ *Col A B D* ∧ *Col A C D* ∧ *Col B C D*)

definition *Concyclic2* ::
[′*p*,′*p*,′*p*,′*p*] ⇒ *bool*
(*Concyclic2* - - - - [*99*,*99*,*99*,*99*] *50*)
where
Concyclic2 A B C D ≡ *Coplanar A B C D* ∧
(∃ *P*. *Cong P A P B* ∧ *Cong P A P C* ∧ *Cong P A P D*)

definition *segment-circle* :: *bool*
(*SegmentCircle 50*)
where
segment-circle ≡ ∀ *A B P Q*. (*P InCircle A B* ∧ *Q OutCircle A B*) ⟶
(∃ *Z*. *Bet P Z Q* ∧ *Z OnCircle A B*)

definition *one-point-line-circle* :: *bool*
(*OnePointLineCircle 50*)
where
one-point-line-circle ≡ ∀ *A B U V P*.
Col U V P ∧ *U* ≠ *V* ∧ *Bet A P B* ⟶ (∃ *Z*. *Col U V Z* ∧ *Z OnCircle A B*)

definition *two-points-line-circle* :: *bool*
(*TwoPointLineCircle 50*)
where
TwoPointLineCircle ≡ ∀ *A B U V P*.
(*Col U V P* ∧ *U* ≠ *V* ∧ *Bet A P B*) ⟶ (∃ *Z1 Z2*. *Col U V Z1* ∧ *Z1 OnCircle A B* ∧
Col U V Z2 ∧ *Z2 OnCircle A B* ∧
Bet Z1 P Z2 ∧ (*P* ≠ *B* ⟶ *Z1* ≠ *Z2*))

definition *circle-circle* :: *bool*
(*CircleCircle 50*)
where
circle-circle $\equiv \forall\ A\ B\ C\ D\ P\ Q.$
$(P\ OnCircle\ C\ D\ \wedge\ Q\ OnCircle\ C\ D \wedge P\ InCircle\ A\ B \wedge Q\ OutCircle\ A\ B)$
$\longrightarrow (\exists\ Z.\ Z\ OnCircle\ A\ B \wedge Z\ OnCircle\ C\ D)$

definition *circle-circle-bis* :: *bool*
(*CircleCircleBis 50*)
where
circle-circle-bis $\equiv \forall\ A\ B\ C\ D\ P\ Q.$
$(P\ OnCircle\ C\ D \wedge P\ InCircle\ A\ B \wedge Q\ OnCircle\ A\ B \wedge Q\ InCircle\ C\ D)$
$\longrightarrow (\exists\ Z.\ Z\ OnCircle\ A\ B \wedge Z\ OnCircle\ C\ D)$

definition *circle-circle-axiom* :: *bool*
(*CircleCircleAxiom 50*)
where
circle-circle-axiom $\equiv \forall\ A\ B\ C\ D\ B'\ D'.$
$(Cong\ A\ B'\ A\ B \wedge Cong\ C\ D'\ C\ D \wedge Bet\ A\ D'\ B \wedge Bet\ C\ B'\ D)$
$\longrightarrow (\exists\ Z.\ Cong\ A\ Z\ A\ B \wedge Cong\ C\ Z\ C\ D)$

definition *circle-circle-two* :: *bool*
(*CircleCircleTwo 50*)
where
circle-circle-two $\equiv \forall\ A\ B\ C\ D\ P\ Q.$
$(P\ OnCircle\ C\ D \wedge Q\ OnCircle\ C\ D \wedge P\ InCircle\ A\ B \wedge Q\ OutCircle\ A\ B)$
$\longrightarrow (\exists\ Z1\ Z2.\ Z1\ OnCircle\ A\ B \wedge Z1\ OnCircle\ C\ D \wedge$
$Z2\ OnCircle\ A\ B \wedge Z2\ OnCircle\ \ C\ D \wedge$
$((P\ InCircleS\ A\ B \wedge Q\ OutCircleS\ A\ B) \longrightarrow Z1 \neq Z2))$

definition *euclid-s-prop-1-22* :: *bool*
(*EuclidsProp122 50*)
where
euclid-s-prop-1-22 $\equiv \forall\ A\ B\ C\ D\ E\ F\ A'\ B'\ C'\ D'\ E'\ F'.$
$(A\ B\ C\ D\ SumS\ E'\ F' \wedge A\ B\ E\ F\ SumS\ C'\ D' \wedge C\ D\ E\ F\ SumS\ A'\ B' \wedge$
$E\ F\ Le\ E'\ F' \wedge C\ D\ Le\ C'\ D' \wedge A\ B\ Le\ A'\ B') \longrightarrow$
$(\exists\ P\ Q\ R.\ Cong\ P\ Q\ A\ B \wedge Cong\ P\ R\ C\ D \wedge Cong\ Q\ R\ E\ F)$

definition *Nested* ::
$[(nat \Rightarrow 'p \Rightarrow bool),(nat \Rightarrow 'p \Rightarrow bool)] \Rightarrow bool$
(*Nested - - [99,99] 50*)
where
Nested A B $\equiv$
$(\forall\ n.\ (\exists\ An\ Bn.\ A\ n\ An \wedge B\ n\ Bn)) \wedge$
$(\forall\ n\ An\ Am\ Bm\ Bn.$
$(A\ n\ An \wedge A\ (Suc\ n)\ Am \wedge B\ (Suc\ n)\ Bm \wedge B\ n\ Bn)$
$\longrightarrow$
$(Bet\ An\ Am\ Bm \wedge Bet\ Am\ Bm\ Bn \wedge Am \neq Bm))$

definition *CantorsAxiom* :: *bool*
(*CantorsAxiom 50*)
where

CantorsAxiom ≡ ∀ *A B*. *Nested A B* ⟶
(∃ *X*. ∀ *n An Bn*. (*A n An* ∧ *B n Bn*) ⟶ *Bet An X Bn*)

definition *DedekindsAxiom* :: *bool*
(*DedekindsAxiom 50*)
where
DedekindsAxiom ≡ ∀ *Alpha Beta*.
(∃ *A*. ∀ *X Y*. (*Alpha X* ∧ *Beta Y*) ⟶ *Bet A X Y*) ⟶
(∃ *B*. ∀ *X Y*. (*Alpha X* ∧ *Beta Y*) ⟶ *Bet X B Y*)

definition *DedekindVariant* :: *bool*
(*DedekindVariant 50*)
where
DedekindVariant ≡ ∀ *Alpha Beta* :: ′*p* ⇒ *bool*. ∀ *A C*.
(*Alpha A* ∧ *Beta C* ∧
(∀ *P*. *A Out P C* ⟶ (*Alpha P* ∨ *Beta P*)) ∧
(∀ *X Y*. (*Alpha X* ∧ *Beta Y*) ⟶ (*Bet A X Y* ∧ *X* ≠ *Y*)))
⟶
(∃ *B*. ∀ *X Y*. (*Alpha X* ∧ *Beta Y*) ⟶ *Bet X B Y*)

definition *Alpha′Fun* :: ′*p* ⇒ (′*p* ⇒ *bool*) ⇒ (′*p* ⇒ *bool*)
where
Alpha′Fun A Alpha ≡ λ *X′*::′*p*. *X′* = *A* ∨ (∃ *X*::′*p*. *Alpha X* ∧ *Bet A X′ X*)

definition *Beta′Fun* :: ′*p* ⇒ ′*p* ⇒ (′*p* ⇒ *bool*) ⇒ (′*p* ⇒ *bool*)
where
Beta′Fun A C Alpha ≡ λ *Y′*::′*p*. *A Out Y′ C* ∧ ¬ (∃ *X*::′*p*. *Alpha X* ∧ *Bet A Y′ X*)

definition *Eq* ::
[′*p*,′*p*] ⇒ *bool*
(- *Eq* - [*99*,*99*] *50*)
where
A Eq B ≡ *A* = *B*

inductive *FOF* :: *bool* ⇒ *bool*
where
eq-fof : *FOF* (*A Eq B*) **for** *A B* :: ′*p*
| *bet-fof* : *FOF* (*Bet A B C*) **for** *A B C* :: ′*p*
| *cong-fof* : *FOF* (*Cong A B C D*) **for** *A B C D* :: ′*p*
| *not-fof* : *FOF* (¬ *P*) **if** *FOF P* **for** *P* :: *bool*
| *and-fof* : *FOF* (*P* ∧ *Q*) **if** *FOF P* **and** *FOF Q* **for** *P Q* :: *bool*
| *or-fof* : *FOF* (*P* ∨ *Q*) **if** *FOF P* **and** *FOF Q* **for** *P Q* :: *bool*
| *implies-fof*: *FOF* (*P* ⟶ *Q*) **if** *FOF P* **and** *FOF Q* **for** *P Q* :: *bool*
| *forall-fof* : *FOF* (∀ *A*::′*p*. *P A*) **if** ∀ *A*. *FOF* (*P A*) **for** *P* :: ′*p* ⇒ *bool*

| *exists-fof* : *FOF* (∃ *A*::′*p*. *P A*) **if** *FOF* (*P A*) **for** *P* :: ′*p* ⇒ *bool*

definition *FirstOrderDedekind* :: *bool*
(*FirstOrderDedekind 50*)
where
FirstOrderDedekind ≡
∀ *Alpha Beta* :: ′*p*⇒*bool*.
(∀ *X* ::′*p*. *FOF* (*Alpha X*)) ∧
(∀ *Y* ::′*p*. *FOF* (*Beta Y*)) ∧
(∃ *A*. ∀ *X Y*. (*Alpha X* ∧ *Beta Y*) ⟶ *Bet A X Y*)
⟶
(∃ *B*. ∀ *X Y*. (*Alpha X* ∧ *Beta Y*) ⟶ *Bet X B Y*)

definition *AlphaFun*:: ′*p* ⇒ ′*p* ⇒ ′*p* ⇒ ′*p* ⇒
′*p* ⇒ ′*p* ⇒ (′*p* ⇒ *bool*)
where
AlphaFun A B C D P Q ≡
λ *X*. *Bet P X Q* ∧ (∃ *X0*. *X0 OnCircle C D* ∧ *C Out X X0* ∧ *X0 InCircle A B*)

definition *BetaFun*:: ′*p* ⇒ ′*p* ⇒ ′*p* ⇒ ′*p* ⇒
′*p* ⇒′*p* ⇒ (′*p* ⇒ *bool*)

where
BetaFun A B C D P Q ≡
λ *Y*. *Bet P Y Q* ∧ (∃ *Y0*. *Y0 OnCircle C D* ∧ *C Out Y Y0* ∧ *Y0 OutCircle A B*)

definition *DedekindLemFun*:: ′*p* ⇒ ′*p* ⇒ ′*p* ⇒ ′*p* ⇒
′*p* ⇒′*p* ⇒′*p*⇒′*p*⇒′*p*⇒*bool*
where
DedekindLemFun ≡
λ *A B C D P Q*. λ *R*. λ *X Y*. (*Bet P X Q* ∧
(∃ *X0*. *X0 OnCircle C D* ∧ *C Out X X0* ∧ *X0 InCircle A B*) ∧
Bet P Y Q ∧
(∃ *Y0*. *Y0 OnCircle C D* ∧ *C Out Y Y0* ∧ *Y0 OutCircle A B*))
⟶
Bet X R Y

5.2 Propositions

lemma *inc112*:
shows *A InCircle A B*
⟨*proof*⟩

lemma *onc212*:
shows *B OnCircle A B*
⟨*proof*⟩

lemma *onc-sym*:
assumes *P OnCircle A B*
shows *B OnCircle A P*
⟨*proof*⟩

lemma *ninc--outcs*:
assumes ¬ (*X InCircle PO P*)
shows *X OutCircleS PO P*
⟨*proof*⟩

lemma *inc--outc-1*:
assumes *P InCircle A B*
shows *B OutCircle A P*
⟨*proof*⟩

lemma *inc--outc-2*:
assumes *B OutCircle A P*
shows *P InCircle A B*
⟨*proof*⟩

lemma *inc--outc*:
shows *P InCircle A B* ⟷ *B OutCircle A P*
⟨*proof*⟩

lemma *incs--outcs-1*:
assumes *P InCircleS A B*
shows *B OutCircleS A P*
⟨*proof*⟩

lemma *incs--outcs-2*:
assumes *B OutCircleS A P*
shows *P InCircleS A B*
⟨*proof*⟩

lemma *incs--outcs*:
shows *P InCircleS A B* ⟷ *B OutCircleS A P*
⟨*proof*⟩

lemma *onc--inc*:
assumes *P OnCircle A B*
shows *P InCircle A B*

⟨*proof*⟩

lemma *onc--outc*:
assumes *P OnCircle A B*
shows *P OutCircle A B*
⟨*proof*⟩

lemma *inc-outc--onc*:
assumes *P InCircle A B* **and**
P OutCircle A B
shows *P OnCircle A B*
⟨*proof*⟩

lemma *incs--inc*:
assumes *P InCircleS A B*
shows *P InCircle A B*
⟨*proof*⟩

lemma *outcs--outc*:
assumes *P OutCircleS A B*
shows *P OutCircle A B*
⟨*proof*⟩

lemma *incs--noutc-1*:
assumes *P InCircleS A B*
shows ¬ *P OutCircle A B*
⟨*proof*⟩

lemma *incs--noutc-2*:
assumes ¬ *P OutCircle A B*
shows *P InCircleS A B*
⟨*proof*⟩

lemma *incs--noutc*:
shows *P InCircleS A B* ⟷ ¬ *P OutCircle A B*
⟨*proof*⟩

lemma *outcs--ninc-1*:
assumes *P OutCircleS A B*
shows ¬ *P InCircle A B*
⟨*proof*⟩

lemma *outcs--ninc-2*:
assumes ¬ *P InCircle A B*
shows *P OutCircleS A B*
⟨*proof*⟩

lemma *outcs--ninc*:
shows *P OutCircleS A B* ⟷ ¬ *P InCircle A B*
⟨*proof*⟩

lemma *inc--noutcs-1*:
assumes *P InCircle A B*
shows ¬ *P OutCircleS A B*
⟨*proof*⟩

lemma *inc--noutcs-2*:
assumes ¬ *P OutCircleS A B*
shows *P InCircle A B*
⟨*proof*⟩

lemma *inc--noutcs*:
P InCircle A B ⟷ ¬ *P OutCircleS A B*
⟨*proof*⟩

lemma *outc--nincs-1*:

assumes *P OutCircle A B*
shows ¬ *P InCircleS A B*
⟨*proof*⟩

lemma *outc--nincs-2*:
assumes ¬ *P InCircleS A B*
shows *P OutCircle A B*
⟨*proof*⟩

lemma *outc--nincs*:
shows *P OutCircle A B* ⟷ ¬ *P InCircleS A B*
⟨*proof*⟩

lemma *inc-eq*:
assumes *P InCircle A A*
shows *A* = *P*
⟨*proof*⟩

lemma *outc-eq*:
assumes *A OutCircle A B*
shows *A* = *B*
⟨*proof*⟩

lemma *onc2--cong*:
assumes *A OnCircle PO P* **and**
B OnCircle PO P
shows *Cong PO A PO B*
⟨*proof*⟩

lemma *bet-inc2--incs*:
assumes *X* ≠ *U* **and**
X ≠ *V* **and**
Bet U X V **and**
U InCircle PO P **and**
V InCircle PO P
shows *X InCircleS PO P*
⟨*proof*⟩

lemma *bet-incs2--incs*:
assumes *Bet U X V* **and**
U InCircleS PO P **and**
V InCircleS PO P
shows *X InCircleS PO P*
⟨*proof*⟩

lemma *bet-inc2--inc*:
assumes *U InCircle A B* **and**
V InCircle A B **and**
Bet U P V
shows *P InCircle A B*
⟨*proof*⟩

lemma *col-inc-onc2--bet*:
assumes *U* ≠ *V* **and**
U OnCircle A B **and**
V OnCircle A B **and**
Col U V P **and**
P InCircle A B
shows *Bet U P V*
⟨*proof*⟩

lemma *onc2-out--outcs*:
  assumes $U \neq V$ **and**
    U OnCircle A B **and**
    V OnCircle A B **and**
    P Out U V
  shows *P OutCircleS A B*
  ⟨*proof*⟩

lemma *col-inc2-outcs--out*:
  assumes *U InCircle A B* **and**
    V InCircle A B **and**
    Col U V P **and**
    P OutCircleS A B
  shows *P Out U V*
  ⟨*proof*⟩

lemma *col-onc2--mid*:
  assumes $U \neq V$ **and**
    U OnCircle A B **and**
    V OnCircle A B **and**
    Col U V A
  shows *A Midpoint U V*
  ⟨*proof*⟩

lemma *chord-completion*:
  assumes *U OnCircle A B* **and**
    P InCircle A B
  shows ∃ *V*. *V OnCircle A B* ∧ *Bet U P V*
⟨*proof*⟩

lemma *outcs-exists*:
  shows ∃ *Q*. *Q OutCircleS PO P*
  ⟨*proof*⟩

lemma *outcs-exists1*:
  assumes $X \neq PO$
  shows ∃ *Q*. *PO Out X Q* ∧ *Q OutCircleS PO P*
⟨*proof*⟩

lemma *incs-exists*:
  assumes $PO \neq P$
  shows ∃ *Q*. *Q InCircleS PO P*
  ⟨*proof*⟩

lemma *incs-exists1*:
  assumes $X \neq PO$ **and**
    $P \neq PO$
  shows ∃ *Q*. *PO Out X Q* ∧ *Q InCircleS PO P*
⟨*proof*⟩

lemma *onc-exists*:
assumes $X \neq PO$ **and**
$PO \neq P$
shows $\exists\ Q.\ Q\ OnCircle\ PO\ P \wedge PO\ Out\ X\ Q$
⟨*proof*⟩

lemma *diam-points*:
shows $\exists\ Q1\ Q2.\ Bet\ Q1\ PO\ Q2 \wedge Col\ Q1\ Q2\ X \wedge Q1\ OnCircle\ PO\ P \wedge Q2\ OnCircle\ PO\ P$
⟨*proof*⟩

lemma *symmetric-oncircle*:
assumes $PO\ Midpoint\ X\ Y$ **and**
$X\ OnCircle\ PO\ P$
shows $Y\ OnCircle\ PO\ P$
⟨*proof*⟩

lemma *mid-onc2--per*:
assumes $U\ OnCircle\ PO\ P$ **and**
$V\ OnCircle\ PO\ P$ **and**
$X\ Midpoint\ U\ V$
shows $Per\ PO\ X\ U$
⟨*proof*⟩

lemma *mid-onc2--perp*:
assumes $PO \neq X$ **and**
$A \neq B$ **and**
$A\ OnCircle\ PO\ P$ **and**
$B\ OnCircle\ PO\ P$ **and**
$X\ Midpoint\ A\ B$
shows $PO\ X\ Perp\ A\ B$
⟨*proof*⟩

lemma *col-onc2-perp--mid*:
assumes $PO \neq X$ **and**
$A \neq B$ **and**
$Col\ A\ B\ X$ **and**
$A\ OnCircle\ PO\ P$ **and**
$B\ OnCircle\ PO\ P$ **and**
$PO\ X\ Perp\ A\ B$
shows $X\ Midpoint\ A\ B$
⟨*proof*⟩

lemma *circle-circle-os*:
assumes $I\ OnCircle\ A\ B$ **and**
$I\ OnCircle\ C\ D$ **and**
$\neg\ Col\ A\ C\ I$ **and**
$\neg\ Col\ A\ C\ P$
shows $\exists\ Z.\ Z\ OnCircle\ A\ B \wedge Z\ OnCircle\ C\ D \wedge A\ C\ OS\ P\ Z$
⟨*proof*⟩

lemma *circle-circle-cop*:
assumes I *OnCircle* A B **and**
I *OnCircle* C D
shows $\exists$ Z. Z *OnCircle* A B $\wedge$ Z *OnCircle* C D $\wedge$ *Coplanar* A C P Z
⟨*proof*⟩

lemma *line-circle-two-points*:
assumes $U \neq V$ **and**
Col U V W **and**
U *OnCircle* PO P **and**
V *OnCircle* PO P **and**
W *OnCircle* PO P
shows $W = U \vee W = V$
⟨*proof*⟩

lemma *onc2-mid--incs*:
assumes $U \neq V$ **and**
U *OnCircle* PO P **and**
V *OnCircle* PO P **and**
M *Midpoint* U V
shows M *InCircleS* PO P
⟨*proof*⟩

lemma *circle-cases*:
shows X *OnCircle* PO P $\vee$ X *InCircleS* PO P $\vee$ X *OutCircleS* PO P
⟨*proof*⟩

lemma *inc--radius*:
assumes X *InCircle* PO P
shows $\exists$ Y. Y *OnCircle* PO P $\wedge$ *Bet* PO X Y
⟨*proof*⟩

lemma *inc-onc2-out--eq*:
assumes A *InCircle* PO P **and**
B *OnCircle* PO P **and**
C *OnCircle* PO P **and**
A *Out* B C
shows $B = C$
⟨*proof*⟩

lemma *onc-not-center*:
assumes $PO \neq P$ **and**
A *OnCircle* PO P
shows $A \neq PO$
⟨*proof*⟩

lemma *onc2-per--mid*:
assumes $U \neq V$ **and**
$M \neq U$ **and**
U *OnCircle* PO P **and**
V *OnCircle* PO P **and**
Col M U V **and**
Per PO M U
shows M *Midpoint* U V
⟨*proof*⟩

lemma *cong-chord-cong-center*:
assumes *A OnCircle PO P* **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
D OnCircle PO P **and**
M Midpoint A B **and**
N Midpoint C D **and**
Cong A B C D
shows *Cong PO N PO M*
⟨*proof*⟩

lemma *cong-chord-cong-center1*:
assumes $A \neq B$ **and**
$C \neq D$ **and**
$M \neq A$ **and**
$N \neq C$
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
D OnCircle PO P **and**
Col M A B **and**
Col N C D **and**
Per PO M A **and**
Per PO N C **and**
Cong A B C D
shows *Cong PO N PO M*
⟨*proof*⟩

lemma *onc-sym--onc*:
assumes *Bet PO A B* **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
X OnCircle PO P **and**
X Y ReflectL A B
shows *Y OnCircle PO P*
⟨*proof*⟩

lemma *mid-onc--diam*:
assumes *A OnCircle PO P* **and**
PO Midpoint A B
shows *Diam A B PO P*
⟨*proof*⟩

lemma *chord-le-diam*:
assumes *Diam A B PO P* **and**
U OnCircle PO P **and**
V OnCircle PO P
shows *U V Le A B*
⟨*proof*⟩

lemma *chord-lt-diam*:
assumes ¬ *Col PO U V* **and**
Diam A B PO P **and**
U OnCircle PO P **and**
V OnCircle PO P
shows *U V Lt A B*
⟨*proof*⟩

lemma *inc2-le-diam*:
assumes *Diam A B PO P* **and**

U InCircle PO P **and**
V InCircle PO P
shows *U V Le A B*
⟨*proof*⟩

lemma *onc-col-diam--eq*:
assumes *Diam A B PO P* **and**
X OnCircle PO P **and**
Col A B X
shows $X = A \lor X = B$
⟨*proof*⟩

lemma *bet-onc-le-a*:
assumes *Diam A B PO P* **and**
Bet B PO T **and**
X OnCircle PO P
shows *T A Le T X*
⟨*proof*⟩

lemma *bet-onc-lt-a* :
assumes *Diam A B PO P* **and**
$PO \neq P$ **and**
$PO \neq T$ **and**
$X \neq A$ **and**
Bet B PO T **and**
X OnCircle PO P
shows *T A Lt T X*
⟨*proof*⟩

lemma *bet-onc-le-b*:
assumes *Diam A B PO P* **and**
Bet A PO T **and**
X OnCircle PO P
shows *T X Le T A*
⟨*proof*⟩

lemma *bet-onc-lt-b*:
assumes *Diam A B PO P* **and**
$T \neq PO$ **and**
$X \neq A$ **and**
Bet A PO T **and**
X OnCircle PO P
shows *T X Lt T A*
⟨*proof*⟩

lemma *incs2-lt-diam*:
assumes *Diam A B PO P* **and**
U InCircleS PO P **and**
V InCircleS PO P
shows *U V Lt A B*
⟨*proof*⟩

lemma *incs-onc-diam--lt*:
assumes *Diam A B PO P* **and**
U InCircleS PO P **and**
V OnCircle PO P
shows *U V Lt A B*
⟨*proof*⟩

lemma *diam-cong-incs--outcs*:
assumes *Diam A B PO P* **and**
Cong A B U V **and**
U InCircleS PO P
shows *V OutCircleS PO P*
⟨*proof*⟩

lemma *diam-uniqueness*:
assumes *Diam A B PO P* **and**
Cong A X A B **and**
X OnCircle PO P
shows $X = B$
⟨*proof*⟩

lemma *onc3--ncol*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P
shows ¬ *Col A B C*
⟨*proof*⟩

lemma *diam-exists*:
shows ∃ *A B*. *Diam A B PO P* ∧ *Col A B T*
⟨*proof*⟩

lemma *chord-intersection*:
assumes *A OnCircle PO P* **and**
B OnCircle PO P **and**
X OnCircle PO P **and**
Y OnCircle PO P **and**
A B TS X Y
shows *X Y TS A B*
⟨*proof*⟩

lemma *ray-cut-chord*:
assumes *Diam A B PO P* **and**
X OnCircle PO P **and**
Y OnCircle PO P **and**
A B TS X Y **and**
X Y OS PO A
shows *X Y TS PO B*
⟨*proof*⟩

lemma *center-col--diam*:
assumes $A \neq B$ **and**
Col PO A B **and**
A OnCircle PO P **and**
B OnCircle PO P
shows *Diam A B PO P*
⟨*proof*⟩

lemma *diam--midpoint*:
assumes *Diam A B PO P*
shows *PO Midpoint A B*
⟨*proof*⟩

lemma *diam-sym*:
assumes *Diam A B PO P*
shows *Diam B A PO P*
⟨*proof*⟩

lemma *diam-end-uniqueness*:
assumes *Diam A B PO P* **and**
Diam A C PO P
shows $B = C$
⟨*proof*⟩

lemma *center-onc2-mid--ncol*:
assumes
¬ *Col PO A B* **and**

M Midpoint A B
shows ¬ *Col A PO M*
⟨*proof*⟩

lemma *bet-chord--diam-or-ncol*:
assumes $A \neq B$ **and**
$T \neq A$ **and**
$T \neq B$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
Bet A T B
shows *Diam A B PO P* ∨ ¬ *Col PO A T* ∧ ¬ *Col PO B T*
⟨*proof*⟩

lemma *mid-chord--diam-or-ncol*:
assumes $A \neq B$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
T Midpoint A B
shows *Diam A B PO P* ∨ ¬*Col PO A T* ∧ ¬*Col PO B T*
⟨*proof*⟩

lemma *cop-mid-onc2-perp--col*:
assumes $A \neq B$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
X Midpoint A B **and**
X Y Perp A B **and**
Coplanar PO A B Y
shows *Col X Y PO*
⟨*proof*⟩

lemma *cong2-cop2-onc3--eq*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
Coplanar A B C PO **and**
Cong X A X B **and**
Cong X A X C **and**
Coplanar A B C X
shows $X = PO$
⟨*proof*⟩

lemma *tree-points-onc-cop*:
assumes $PO \neq P$
shows ∃ *A B C*. $A \neq B \wedge A \neq C \wedge B \neq C \wedge$
A OnCircle PO P ∧ *B OnCircle PO P* ∧ *C OnCircle PO P* ∧
Coplanar A B C PO
⟨*proof*⟩

lemma *tree-points-onc-cop2*:
assumes $PO \neq P$
shows ∃ *A B C*. $A \neq B \wedge A \neq C \wedge B \neq C \wedge$
A OnCircle PO P ∧ *B OnCircle PO P* ∧ *C OnCircle PO P* ∧
Coplanar A B C PO ∧ *Coplanar A B C Q*
⟨*proof*⟩

lemma *tree-points-onc*:
assumes $PO \neq P$
shows ∃ *A B C*.
$A \neq B \wedge A \neq C \wedge B \neq C$ ∧ *A OnCircle PO P* ∧ *B OnCircle PO P* ∧ *C OnCircle PO P*

⟨proof⟩

lemma *bet-cop-onc2--ex-onc-os-out*:
  assumes $A \neq I$ **and**
    $B \neq I$ **and**
    $\neg$ *Col A B C* **and**
    $\neg$ *Col A B PO* **and**
    A OnCircle PO P **and**
    B OnCircle PO P **and**
    Bet A I B **and**
    Coplanar A B C PO
  shows $\exists$ *C1*. *C1 Out PO I* $\wedge$ *C1 OnCircle PO P* $\wedge$ *A B OS C C1*
⟨proof⟩

lemma *eqc-chara-1*:
  assumes *EqC A B C D*
  shows $A = C \wedge$ *Cong A B C D*
⟨proof⟩

lemma *eqc-chara-2*:
  assumes $A = C$ **and**
    Cong A B C D
  shows *EqC A B C D*
  ⟨proof⟩

lemma *eqc-chara*:
  shows *EqC A B C D* $\longleftrightarrow$ ($A = C \wedge$ *Cong A B C D*)
  ⟨proof⟩

lemma *neqc-chara-1*:
  assumes $A \neq B$ **and**
    $\neg$ *EqC A B C D*
  shows $\exists$ *X*. *X OnCircle A B* $\wedge$ $\neg$ *X OnCircle C D*
⟨proof⟩

lemma *neqc-chara-2*:
  assumes
    X OnCircle A B **and**
    $\neg$ *X OnCircle C D*
  shows $\neg$ *EqC A B C D*
  ⟨proof⟩

lemma *neqc-chara*:
  assumes $A \neq B$
  shows $\neg$ *EqC A B C D* $\longleftrightarrow$ ($\exists$ *X*. *X OnCircle A B* $\wedge$ $\neg$ *X OnCircle C D*)
  ⟨proof⟩

lemma *eqc-refl*:
  shows *EqC A B A B*
  ⟨proof⟩

lemma *eqc-sym*:
  assumes *EqC A B C D*
  shows *EqC C D A B*
  ⟨proof⟩

lemma *eqc-trans*:
  assumes *EqC A B C D* **and**
    EqC C D E F
  shows *EqC A B E F*

⟨*proof*⟩

lemma *cop2-onc6--eqc*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
Coplanar A B C PO **and**
A OnCircle O′ P′ **and**
B OnCircle O′ P′ **and**
C OnCircle O′ P′ **and**
Coplanar A B C O′
shows *EqC PO P O′ P′*
⟨*proof*⟩

lemma *concyclic-aux*:
assumes *Concyclic A B C D*
shows ∃ *PO P. A OnCircle PO P* ∧ *B OnCircle PO P* ∧ *C OnCircle PO P* ∧
D OnCircle PO P ∧ *Coplanar A B C PO*
⟨*proof*⟩

lemma *concyclic-perm-1*:
assumes *Concyclic A B C D*
shows *Concyclic B C D A*
⟨*proof*⟩

lemma *concyclic-gen-perm-1*:
assumes *ConcyclicGen A B C D*
shows *ConcyclicGen B C D A*
⟨*proof*⟩

lemma *concyclic-perm-2*:
assumes *Concyclic A B C D*
shows *Concyclic B A C D*
⟨*proof*⟩

lemma *concyclic-gen-perm-2*:
assumes *ConcyclicGen A B C D*
shows *ConcyclicGen B A C D*
⟨*proof*⟩

lemma *concyclic-trans-1*:
assumes ¬ *Col P Q R* **and**
Concyclic P Q R A **and**
Concyclic P Q R B
shows *Concyclic Q R A B*
⟨*proof*⟩

lemma *concyclic-gen-trans-1*:
assumes ¬ *Col P Q R* **and**
ConcyclicGen P Q R A **and**
ConcyclicGen P Q R B
shows *ConcyclicGen Q R A B*
⟨*proof*⟩

lemma *concyclic-pseudo-trans*:
assumes ¬ *Col P Q R* **and**
Concyclic P Q R A **and**
Concyclic P Q R B **and**
Concyclic P Q R C **and**

Concyclic P Q R D
shows *Concyclic A B C D*
⟨*proof*⟩

lemma *concyclic-gen-pseudo-trans*:
assumes ¬ *Col P Q R* **and**
ConcyclicGen P Q R A
ConcyclicGen P Q R B
ConcyclicGen P Q R C
ConcyclicGen P Q R D
shows *ConcyclicGen A B C D*
⟨*proof*⟩

lemma *Concyclic--Concyclic2*:
assumes *Concyclic A B C D*
shows *Concyclic2 A B C D*
⟨*proof*⟩

lemma *Concyclic2--Concyclic*:
assumes *Concyclic2 A B C D*
shows *Concyclic A B C D*
⟨*proof*⟩

lemma *TangentCC-Col*:
assumes *TangentCC A B C D* **and**
X OnCircle A B **and**
X OnCircle C D
shows *Col X A C*
⟨*proof*⟩

lemma *tangent-neq*:
assumes $PO \neq P$ **and**
Tangent A B PO P
shows $A \neq B$
⟨*proof*⟩

lemma *diam-not-tangent*:
assumes $P \neq PO$ **and**
Col PO A B
shows ¬ *Tangent A B PO P*
⟨*proof*⟩

lemma *tangent-out*:
assumes $X \neq T$ **and**
Col A B X **and**
TangentAt A B PO P T
shows *X OutCircleS PO P*
⟨*proof*⟩

lemma *tangentat-perp*:
assumes $PO \neq P$ **and**
TangentAt A B PO P T
shows *A B Perp PO T*
⟨*proof*⟩

lemma *tangency-chara-R1*:
assumes $P \neq PO$ **and**
$\exists\ X.\ (X\ OnCircle\ PO\ P \wedge X\ PerpAt\ A\ B\ PO\ X)$
shows *Tangent A B PO P*
⟨*proof*⟩

lemma *tangency-chara-R2*:
assumes $P \neq PO$ **and**
Tangent A B PO P
shows $\exists\ X.\ (X\ OnCircle\ PO\ P \wedge X\ PerpAt\ A\ B\ PO\ X)$
⟨*proof*⟩

lemma *tangency-chara*:
assumes $P \neq PO$
shows $(\exists\ X.\ (X\ OnCircle\ PO\ P \wedge X\ PerpAt\ A\ B\ PO\ X)) \longleftrightarrow Tangent\ A\ B\ PO\ P$
⟨*proof*⟩

lemma *tangency-chara2-R1*:
assumes *Q OnCircle PO P* **and**
Col Q A B **and**
$(\forall\ X.\ Col\ A\ B\ X \longrightarrow X = Q \vee X\ OutCircleS\ PO\ P)$
shows *Tangent A B PO P*
⟨*proof*⟩

lemma *tangency-chara2-R2*:
assumes *Q OnCircle PO P* **and**
Col Q A B **and**
Tangent A B PO P
shows $\forall\ X.\ Col\ A\ B\ X \longrightarrow X = Q \vee X\ OutCircleS\ PO\ P$
⟨*proof*⟩

lemma *tangency-chara2*:
assumes *Q OnCircle PO P* **and**
Col Q A B
shows $(\forall\ X.\ Col\ A\ B\ X \longrightarrow X = Q \vee X\ OutCircleS\ PO\ P) \longleftrightarrow Tangent\ A\ B\ PO\ P$
⟨*proof*⟩

lemma *tangency-chara3-R1*:
assumes $A \neq B$ **and**
Q OnCircle PO P **and**
Col Q A B **and**
$\forall\ X.\ (Col\ A\ B\ X \longrightarrow X\ OutCircle\ PO\ P)$
shows *Tangent A B PO P*
⟨*proof*⟩

lemma *tangency-chara3-R2*:
assumes *Q OnCircle PO P* **and**
Col Q A B **and**
Tangent A B PO P
shows $\forall\ X.\ (Col\ A\ B\ X \longrightarrow X\ OutCircle\ PO\ P)$
⟨*proof*⟩

lemma *tangency-chara3*:
assumes $A \neq B$ **and**
Q OnCircle PO P **and**
Col Q A B
shows $(\forall\ X.\ (Col\ A\ B\ X \longrightarrow X\ OutCircle\ PO\ P)) \longleftrightarrow Tangent\ A\ B\ PO\ P$
⟨*proof*⟩

lemma *intercc--neq*:
assumes *InterCC A B C D*
shows $A \neq C$
⟨*proof*⟩

lemma *tangentcc--neq*:
assumes $A \neq B$ **and**
TangentCC A B C D
shows $A \neq C$
⟨*proof*⟩

lemma *interccat--neq*:
assumes *InterCCAt A B C D P Q*
shows $A \neq C$
⟨*proof*⟩

lemma *interccat--ncol*:
assumes *InterCCAt A B C D P Q*
shows ¬ *Col A C P*
⟨*proof*⟩

lemma *cop-onc2--oreq*:
assumes *InterCCAt A B C D P Q* **and**
Coplanar A C P Q **and**
Z OnCircle A B **and**
Z OnCircle C D **and**
Coplanar A C P Z
shows $Z = P \vee Z = Q$
⟨*proof*⟩

lemma *tangent-construction*:
assumes *segment-circle* **and**
X OutCircle PO P
shows ∃ *Y*. *Tangent X Y PO P*
⟨*proof*⟩

lemma *dedekind-equiv-R1*:
assumes *DedekindsAxiom*
shows *DedekindVariant*
⟨*proof*⟩

lemma *dedekind-equiv-R2*:
assumes *DedekindVariant*
shows *DedekindsAxiom*
⟨*proof*⟩

lemma *dedekind-equiv*:
shows *DedekindsAxiom* ⟷ *DedekindVariant*
⟨*proof*⟩

lemma *dedekind--fod*:
assumes *DedekindsAxiom*
shows *FirstOrderDedekind*
⟨*proof*⟩

lemma *circle-circle-aux*:
assumes ∀ *A B C D P Q*. (*P OnCircle C D* ∧ *Q OnCircle C D* ∧
P InCircleS A B ∧ *Q OutCircleS A B* ∧
(*A C OS P Q* ∨ (*Col P A C* ∧ ¬ *Col Q A C*) ∨ (¬ *Col P A C* ∧ *Col Q A C*)) ⟶
(∃ *Z*. *Z OnCircle A B* ∧ *Z OnCircle C D*))
shows *circle-circle*
⟨*proof*⟩

lemma *fod--circle-circle*:
assumes *FirstOrderDedekind*
shows *CircleCircle*
⟨*proof*⟩

lemma *nested--diff0*:
assumes *Nested A B*
shows $\forall$ *n An Bn. A n An* $\wedge$ *B n Bn* $\longrightarrow$ *An* $\neq$ *Bn*
⟨*proof*⟩

lemma *nested-sym*:
assumes *Nested A B*
shows *Nested B A*
⟨*proof*⟩

lemma *nested--ex-left*:
assumes *Nested A B*
shows $\exists$ *An. A n An*
⟨*proof*⟩

lemma *nested--ex-right*:
assumes *Nested A B*
shows $\exists$ *Bn. B n Bn*
⟨*proof*⟩

lemma *nested-aux1-1*:
fixes *n*::*nat*
assumes *Nested A B*
shows $\forall$ *Am Bm. A n An* $\wedge$ *A* (*Suc n*) *Am* $\wedge$ *B* (*Suc n*) *Bm* $\longrightarrow$ *Bet An Am Bm*
⟨*proof*⟩

lemma *nested-aux1-2*:
fixes *n*::*nat*
assumes *Nested A B*
shows ($n < m$ $\wedge$ (*Suc n*) $\leq$ *m*) $\longrightarrow$ (($\forall$ *Am Bm. A n An* $\wedge$ *A m Am* $\wedge$ *B m Bm* $\longrightarrow$ *Bet An Am Bm*))
⟨*proof*⟩

lemma *nested-aux1*:
fixes *n*::*nat*
assumes *Nested A B* **and**
$n < m$
shows $\forall$ *Am Bm. A n An* $\wedge$ *A m Am* $\wedge$ *B m Bm* $\longrightarrow$ *Bet An Am Bm*
⟨*proof*⟩

lemma *nested-aux2*:
assumes *Nested A B* **and**
$n < m$ **and**
A n An **and**
A m Am **and**
B n Bn
shows *Bet An Am Bn*
⟨*proof*⟩

lemma *nested--bet*:
assumes *Nested A B* **and**
$n < n1$ **and**
A n An **and**
A n1 An1 **and**
B m Bm
shows *Bet An An1 Bm*
⟨*proof*⟩

lemma *nested--diff*:
assumes *Nested A B* **and**
A n An **and**
B m Bm

shows *An* ≠ *Bm*
⟨*proof*⟩

lemma *dedekind--cantor*:
assumes *DedekindsAxiom*
shows *CantorsAxiom*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-Continuous*

imports
Tarski-Neutral-Continuity

begin

6 Tarski Neutral Continuous

6.1 Tarski's axiom system for Neutral Continuous

locale *Tarski-Neutral-Continuous* = *Tarski-neutral-dimensionless* +
assumes *continuity* : $\forall$ *Alpha Beta*.
$(\exists\ A.\ \forall\ X\ Y.\ (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ A\ X\ Y) \longrightarrow$
$(\exists\ B.\ \forall\ X\ Y.\ (Alpha\ X \wedge Beta\ Y) \longrightarrow Bet\ X\ B\ Y)$

context *Tarski-Neutral-Continuous*

begin

6.2 Definitions

6.3 Propositions

theorem *continuity-DedekindAxiom* :
shows *DedekindsAxiom*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-Archimedes-Continuity*

imports
Tarski-Neutral-Archimedes
Tarski-Neutral-Continuity
begin

context *Tarski-neutral-dimensionless*

begin

7 Archimedes continuity

7.1 Definitions

definition *AlphaTmp* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
where
$AlphaTmp\ A\ B \equiv \lambda\ X.\ X = A \vee (A\ Out\ B\ X \wedge Reach\ A\ B\ A\ X)$

definition *BetaTmp* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
where

BetaTmp A B ≡ λ *X*. *A Out B X* ∧ ¬ *Reach A B A X*

definition *NestedBis* :: (*nat* ⇒ ′*p* ⇒ *bool*) ⇒ (*nat* ⇒ ′*p* ⇒ *bool*)=>*bool*
where
NestedBis A B ≡
(∀ *n*. ∃ *An Bn*. *A n An* ∧ *B n Bn*) ∧
(∀ *n An An*′. *A n An* ∧ *A n An*′ ⟶ *An* = *An*′) ∧
(∀ *n Bn Bn*′. *B n Bn* ∧ *B n Bn*′ ⟶ *Bn* = *Bn*′) ∧
(∀ *n An Am Bm Bn*. *A n An* ∧ *A* (*Suc n*) *Am* ∧ *B* (*Suc n*) *Bm* ∧ *B n Bn* ⟶
Bet An Am Bm ∧ *Bet Am Bm Bn* ∧ *Am* ≠ *Bm*)

definition *CantorVariant* :: *bool*
where
CantorVariant ≡ ∀ *A B*. *NestedBis A B* ⟶
(∃ *X*. ∀ *n An Bn*. *A n An* ∧ *B n Bn* ⟶ *Bet An X Bn*)

definition *inj*:: (′*p* ⇒ ′*p*) ⇒ *bool*
where
inj f ≡ ∀ *A B*::′*p*. *f A* = *f B* ⟶ *A* = *B*

definition *pres-bet*:: (′*p*⇒′*p*) ⇒*bool*
where
pres-bet f ≡ ∀ *A B C*. *Bet A B C* ⟶ *Bet* (*f A*) (*f B*) (*f C*)

definition *pres-cong* :: (′*p*⇒′*p*) ⇒*bool*
where
pres-cong f ≡ ∀ *A B C D*. *Cong A B C D* ⟶ *Cong* (*f A*) (*f B*) (*f C*) (*f D*)

definition *extension*:: (′*p*⇒′*p*) ⇒ *bool*
where
extension f ≡ *inj f* ∧ *pres-bet f* ∧ *pres-cong f*

definition *inj-line* :: (′*p*⇒′*p*)⇒′*p*⇒′*p*⇒*bool*
where
inj-line f P Q ≡ ∀ *A B*. *Col P Q A* ∧ *Col P Q B* ∧ *f A* = *f B* ⟶ *A* = *B*

definition *pres-bet-line* :: (′*p*⇒′*p*)⇒′*p*⇒′*p*⇒*bool*
where
pres-bet-line f P Q ≡
∀ *A B C*. *Col P Q A* ∧ *Col P Q B* ∧ *Col P Q C* ∧ *Bet A B C*
⟶
Bet (*f A*)(*f B*)(*f C*)

definition *pres-cong-line* :: (′*p*⇒′*p*)⇒′*p*⇒′*p*⇒*bool*
where
pres-cong-line f P Q ≡
∀ *A B C D*. *Col P Q A* ∧ *Col P Q B* ∧ *Col P Q C* ∧ *Col P Q D* ∧ *Cong A B C D*
⟶
Cong (*f A*) (*f B*) (*f C*) (*f D*)

definition *line-extension* :: (′*p*⇒′*p*)⇒′*p*⇒′*p*⇒*bool*
where
line-extension f P Q ≡ *P* ≠ *Q* ∧ *inj-line f P Q* ∧ *pres-bet-line f P Q* ∧ *pres-cong-line f P Q*

definition *line-completeness* :: *bool*
where
line-completeness ≡ *archimedes-axiom* ∧
(∀ *f*. ∀ *P Q*. *line-extension f P Q*
⟶
(∀ *A*. *Col* (*f P*) (*f Q*) *A* ⟶ (∃ *B*. *Col P Q B* ∧ *f B* = *A*)))

7.2 Propositions

lemma *archimedes-aux* :

assumes ∀ *A B C*. *A Out B C* ⟶ *Reach A B A C*
shows *ArchimedesAxiom*
⟨*proof*⟩

lemma *dedekind-variant--archimedes*:
assumes *DedekindVariant*
shows *archimedes-axiom*
⟨*proof*⟩

lemma *dedekind--archimedes*:
assumes *DedekindsAxiom*
shows *archimedes-axiom*
⟨*proof*⟩

lemma *nested-bis--nested*:
assumes *NestedBis A B*
shows *Nested A B*
⟨*proof*⟩

lemma *cantor--cantor-variant-1*:
assumes *CantorsAxiom*
shows *CantorVariant*
⟨*proof*⟩

lemma *cantor--cantor-variant-2*:
assumes *CantorVariant*
shows *CantorsAxiom*
⟨*proof*⟩

lemma *cantor--cantor-variant*:
shows *CantorsAxiom* ⟷ *CantorVariant*
⟨*proof*⟩

lemma *inj-line-symmetry*:
assumes *inj-line f P Q*
shows *inj-line f Q P*
⟨*proof*⟩

lemma *pres-bet-line-symmetry*:
assumes *pres-bet-line f P Q*
shows *pres-bet-line f Q P*
⟨*proof*⟩

lemma *pres-cong-line-symmetry*:
assumes *pres-cong-line f P Q*
shows *pres-cong-line f Q P*
⟨*proof*⟩

lemma *line-extension-symmetry*:
assumes *line-extension f P Q*
shows *line-extension f Q P*
⟨*proof*⟩

lemma *inj-line-stability*:
assumes *Col P Q R* **and**
$P \neq R$ **and**
inj-line f P Q
shows *inj-line f P R*
⟨*proof*⟩

lemma *pres-bet-line-stability*:
assumes *Col P Q R* **and**
$P \neq R$ **and**
pres-bet-line f P Q
shows *pres-bet-line f P R*
⟨*proof*⟩

lemma *pres-cong-line-stability*:
assumes *Col P Q R* **and**
$P \neq R$ **and**
pres-cong-line f P Q
shows *pres-cong-line f P R*
⟨*proof*⟩

lemma *line-extension-stability*:
assumes *Col P Q R* **and**
$P \neq R$ **and**
line-extension f P Q
shows *line-extension f P R*
⟨*proof*⟩

lemma *line-extension-reverse-bet*:
assumes *line-extension f P Q* **and**
Col P Q A **and**
Col P Q B **and**
Col P Q C **and**
Bet (*f A*) (*f B*) (*f C*)
shows *Bet A B C*
⟨*proof*⟩

lemma *pres-bet-line--col*:
assumes $P \neq Q$ **and**
pres-bet-line f P Q **and**
Col P Q A **and**
Col P Q B **and**
Col P Q C
shows *Col* (*f A*) (*f B*) (*f C*)
⟨*proof*⟩

lemma *col2-diff-inj-line--diff*:
assumes *inj-line f P Q* **and**
Col P Q A **and**
Col P Q B **and**
$A \neq B$
shows $f\ A \neq f\ B$
⟨*proof*⟩

lemma *extension--line-extension*:
assumes $P \neq Q$ **and**
extension f
shows *line-extension f P Q*
⟨*proof*⟩

lemma *extension-reverse-bet*:
assumes *extension f* **and**
Bet (*f A*) (*f B*) (*f C*)
shows *Bet A B C*
⟨*proof*⟩

lemma *extension-reverse-col*:
assumes *extension f* **and**
Col (*f A*) (*f B*) (*f C*)
shows *Col A B C*
⟨*proof*⟩

lemma *line-completeness-aux*:
assumes *line-completeness* **and**
archimedes-axiom **and**
extension f **and**
¬ *Col P Q R* **and**

Coplanar (*f P*) (*f Q*) (*f R*) *A*
shows ∃ *B*. *Coplanar P Q R B* ∧ *f B* = *A*
⟨*proof*⟩

lemma *segment-circle--one-point-line-circle-R1*:
assumes *SegmentCircle*
shows *OnePointLineCircle*
⟨*proof*⟩

lemma *segment-circle--one-point-line-circle-R2*:
assumes *OnePointLineCircle*
shows *SegmentCircle*
⟨*proof*⟩

lemma *segment-circle--one-point-line-circle*:
shows *SegmentCircle* ⟷ *OnePointLineCircle*
⟨*proof*⟩

lemma *one-point-line-circle--two-points-line-circle-R1* :
assumes *one-point-line-circle*
shows *two-points-line-circle*
⟨*proof*⟩

lemma *one-point-line-circle--two-points-line-circle-R2* :
assumes *two-points-line-circle*
shows *one-point-line-circle*
⟨*proof*⟩

lemma *one-point-line-circle--two-points-line-circle* :
shows *one-point-line-circle* ⟷ *two-points-line-circle*
⟨*proof*⟩

lemma *circle-circle-bis--circle-circle-axiom-R1*:
assumes *circle-circle-bis*
shows *circle-circle-axiom*
⟨*proof*⟩

lemma *circle-circle-bis--circle-circle-axiom-R2*:
assumes *circle-circle-axiom*
shows *circle-circle-bis*
⟨*proof*⟩

lemma *circle-circle-bis--circle-circle-axiom*:
shows *circle-circle-bis* ⟷ *circle-circle-axiom*
⟨*proof*⟩

lemma *circle-circle--circle-circle-bis*:
assumes *circle-circle*
shows *circle-circle-bis*
⟨*proof*⟩

lemma *circle-circle-bis--one-point-line-circle-aux*:
assumes *circle-circle-bis* **and**
Col U V P **and** *U* ≠ *V* **and** *Bet A P B* **and** ¬ *Per A U V*
shows ∃ *Z*. *Col U V Z* ∧ *Z OnCircle A B*
⟨*proof*⟩

lemma *circle-circle-bis--one-point-line-circle*:
assumes *circle-circle-bis*
shows *one-point-line-circle*
⟨*proof*⟩

lemma *circle-circle--circle-circle-two-R1* :
  assumes *circle-circle*
  shows *circle-circle-two*
⟨*proof*⟩

lemma *circle-circle--circle-circle-two-R2* :
  assumes *circle-circle-two*
  shows *circle-circle*
⟨*proof*⟩

lemma *circle-circle--circle-circle-two* :
  shows *circle-circle* ⟷ *circle-circle-two*
  ⟨*proof*⟩

lemma *euclid-22-aux*:
  assumes *A B C D SumS E′ F′* **and**
    C D E F SumS A′ B′ **and**
    E F Le E′ F′ **and**
    A B Le A′ B′ **and**
    A Out B C1 **and**
    Cong A C1 C D **and**
    Bet B A C2 **and**
    Cong A C2 C D **and**
    B Out A E1 **and**
    Cong B E1 E F
  shows *Bet C1 E1 C2*
⟨*proof*⟩

lemma *circle-circle-bis--euclid-22*:
  assumes *circle-circle-bis*
  shows *euclid-s-prop-1-22*
⟨*proof*⟩

lemma *triangle-inequality1*:
  assumes *A B B C SumS D E*
  shows *A C Le D E*
⟨*proof*⟩

lemma *euclid-22--circle-circle*:
  assumes *euclid-s-prop-1-22*
  shows *circle-circle*
⟨*proof*⟩

theorem *equivalent-variants-of-circle-circle*:
  shows (*circle-circle* ⟷ *circle-circle-two*) ∧
    (*circle-circle-two* ⟷ *circle-circle-bis*) ∧
    (*circle-circle-bis* ⟷ *circle-circle-axiom*)∧
    (*circle-circle-axiom* ⟷ *euclid-s-prop-1-22*)
⟨*proof*⟩

theorem *equivalent-variants-of-line-circle*:
  shows (*segment-circle* ⟷ *one-point-line-circle*) ∧
        (*one-point-line-circle* ⟷ *two-points-line-circle*)
⟨*proof*⟩

end
end

theory *Hilbert-Neutral*

imports
  Tarski-Neutral

begin

8 Hilbert - Geometry - neutral dimension less

8.1 Axioms

locale *Hilbert-neutral-dimensionless-pre* =
  fixes
    $IncidL :: 'p \Rightarrow 'b \Rightarrow bool$ **and**
    $IncidP :: 'p \Rightarrow 'c \Rightarrow bool$ **and**
    $EqL ::'b \Rightarrow 'b \Rightarrow bool$ **and**
    $EqP ::'c \Rightarrow 'c \Rightarrow bool$ **and**
    $IsL ::'b \Rightarrow bool$ **and**
    $IsP ::'c \Rightarrow bool$ **and**
    $BetH ::'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$ **and**
    $CongH::'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$ **and**
    $CongaH::'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$

context *Hilbert-neutral-dimensionless-pre*

begin

8.2 Definitions

definition $ColH :: 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
  where
    $ColH\ A\ B\ C \equiv (\exists\ l.\ IsL\ l \wedge IncidL\ A\ l \wedge IncidL\ B\ l \wedge IncidL\ C\ l)$

definition $IncidLP :: 'b\Rightarrow'c\Rightarrow bool$ **where**
  $IncidLP\ l\ p \equiv IsL\ l \wedge IsP\ p \wedge (\forall\ A.\ IncidL\ A\ l \longrightarrow IncidP\ A\ p)$

definition $cut :: 'b\Rightarrow'p\Rightarrow'p\Rightarrow bool$ **where**
  $cut\ l\ A\ B \equiv IsL\ l \wedge \neg\ IncidL\ A\ l \wedge \neg\ IncidL\ B\ l \wedge (\exists\ I.\ IncidL\ I\ l \wedge BetH\ A\ I\ B)$

definition $outH :: 'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$ **where**
  $outH\ P\ A\ B \equiv BetH\ P\ A\ B \vee BetH\ P\ B\ A \vee (P \neq A \wedge A = B)$

definition *disjoint* ::
  $'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$ **where**
  $disjoint\ A\ B\ C\ D \equiv \neg\ (\exists\ P.\ BetH\ A\ P\ B \wedge BetH\ C\ P\ D)$

definition $same\text{-}side :: 'p\Rightarrow'p\Rightarrow'b\Rightarrow bool$ **where**
  $same\text{-}side\ A\ B\ l \equiv IsL\ l \wedge (\exists\ P.\ cut\ l\ A\ P \wedge cut\ l\ B\ P)$

definition $same\text{-}side' ::$
  $'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$ **where**
  $same\text{-}side'\ A\ B\ X\ Y \equiv$
    $X \neq Y \wedge$
    $(\forall\ l::'b.\ (IsL\ l \wedge IncidL\ X\ l \wedge IncidL\ Y\ l) \longrightarrow same\text{-}side\ A\ B\ l)$

definition $Para :: 'b \Rightarrow 'b \Rightarrow bool$ **where**
  $Para\ l\ m \equiv IsL\ l \wedge IsL\ m \wedge$
      $(\neg(\exists\ X.\ IncidL\ X\ l \wedge IncidL\ X\ m)) \wedge (\exists\ p.\ IncidLP\ l\ p \wedge IncidLP\ m\ p)$

definition $Bet :: 'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$
  where
    $Bet\ A\ B\ C \equiv BetH\ A\ B\ C \vee A = B \vee B = C$

definition $Cong :: 'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$
  where
    $Cong\ A\ B\ C\ D \equiv (CongH\ A\ B\ C\ D \wedge A \neq B \wedge C \neq D) \vee (A = B \wedge C = D)$

definition $ParaP :: 'p\Rightarrow'p\Rightarrow'p\Rightarrow'p\Rightarrow bool$
  where
    $ParaP\ A\ B\ C\ D \equiv \forall\ l\ m.$
      $IsL\ l \wedge IsL\ m \wedge IncidL\ A\ l \wedge IncidL\ B\ l \wedge IncidL\ C\ m \wedge IncidL\ D\ m$
      $\longrightarrow$
      $Para\ l\ m$

definition *is-line* :: $'p \Rightarrow 'p \Rightarrow 'b \Rightarrow bool$
where
is-line A B l $\equiv$ (*IsL l* $\wedge$ $A \neq B$ $\wedge$ *IncidL A l* $\wedge$ *IncidL B l*)

definition *cut'* ::
$'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
where
cut' A B X Y $\equiv$ $X \neq Y$ $\wedge$ ($\forall$ *l*. (*IsL l* $\wedge$ *IncidL X l* $\wedge$ *IncidL Y l*) $\longrightarrow$ *cut l A B*)

definition *Midpoint* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
(- *Midpoint* - - [*99*,*99*,*99*] *50*)
where *M Midpoint A B* $\equiv$ *Bet A M B* $\wedge$ *Cong A M M B*

end

locale *Hilbert-neutral-dimensionless* = *Hilbert-neutral-dimensionless-pre IncidL IncidP EqL EqP IsL IsP BetH CongH CongaH*
for
IncidL :: $'p \Rightarrow 'b \Rightarrow bool$ **and**
IncidP :: $'p \Rightarrow 'c \Rightarrow bool$ **and**
EqL :: $'b \Rightarrow 'b \Rightarrow bool$ **and**
EqP :: $'c \Rightarrow 'c \Rightarrow bool$ **and**
IsL :: $'b \Rightarrow bool$ **and**
IsP :: $'c \Rightarrow bool$ **and**
BetH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongaH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ +
fixes *PP PQ PR* :: $'p$
assumes
EqL-refl: *IsL l* $\longrightarrow$ *EqL l l* **and**
EqL-sym: *IsL l1* $\wedge$ *IsL l2* $\wedge$ *EqL l1 l2* $\longrightarrow$ *EqL l2 l1* **and**
EqL-trans: (*EqL l1 l2* $\wedge$ *EqL l2 l3*) $\longrightarrow$ *EqL l1 l3* **and**
EqP-refl: *IsP p* $\longrightarrow$ *EqP p p* **and**
EqP-sym: *EqP p1 p2* $\longrightarrow$ *EqP p2 p1* **and**
EqP-trans: (*EqP p1 p2* $\wedge$ *EqP p2 p3*) $\longrightarrow$ *EqP p1 p3* **and**
IncidL-morphism: (*IsL l* $\wedge$ *IsL m* $\wedge$ *IncidL P l* $\wedge$ *EqL l m*) $\longrightarrow$ *IncidL P m* **and**
IncidP-morphism: (*IsP p* $\wedge$ *IsP q* $\wedge$ *IncidP M p* $\wedge$ *EqP p q*) $\longrightarrow$ *IncidP M q* **and**
Is-line:*IncidL P l* $\longrightarrow$ *IsL l* **and**
Is-plane:*IncidP P p* $\longrightarrow$ *IsP p*
assumes

line-existence: $A \neq B$ $\longrightarrow$ ($\exists$ *l*. *IsL l* $\wedge$ (*IncidL A l* $\wedge$ *IncidL B l*)) **and**
line-uniqueness: $A \neq B$ $\wedge$ *IsL l* $\wedge$ *IsL m* $\wedge$
IncidL A l $\wedge$ *IncidL B l* $\wedge$ *IncidL A m* $\wedge$ *IncidL B m* $\longrightarrow$
EqL l m **and**
two-points-on-line: $\forall$ *l*. *IsL l* $\longrightarrow$ ($\exists$ *A B*. *IncidL A l* $\wedge$ *IncidL B l* $\wedge$ $A \neq B$)
assumes
lower-dim-2: $PP \neq PQ$ $\wedge$ $PQ \neq PR$ $\wedge$ $PP \neq PR$ $\wedge$ $\neg$ *ColH PP PQ PR* **and**

plan-existence: $\forall$ *A B C*. (($\neg$ *ColH A B C*) $\longrightarrow$
($\exists$ *p*. *IsP p* $\wedge$ *IncidP A p* $\wedge$ *IncidP B p* $\wedge$ *IncidP C p*)) **and**
one-point-on-plane: $\forall$ *p*. $\exists$ *A*. *IsP p* $\longrightarrow$ *IncidP A p* **and**
plane-uniqueness: $\neg$ *ColH A B C* $\wedge$ *IsP p* $\wedge$ *IsP q* $\wedge$
IncidP A p $\wedge$ *IncidP B p* $\wedge$ *IncidP C p* $\wedge$ *IncidP A q* $\wedge$ *IncidP B q* $\wedge$ *IncidP C q* $\longrightarrow$
EqP p q **and**
line-on-plane: $\forall$ *A B l p*. $A \neq B$ $\wedge$ *IsL l* $\wedge$ *IsP p* $\wedge$
IncidL A l $\wedge$ *IncidL B l* $\wedge$ *IncidP A p* $\wedge$ *IncidP B p* $\longrightarrow$ *IncidLP l p*
assumes
between-diff: *BetH A B C* $\longrightarrow$ $A \neq C$ **and**
between-col: *BetH A B C* $\longrightarrow$ *ColH A B C* **and**
between-comm: *BetH A B C* $\longrightarrow$ *BetH C B A* **and**
between-out: $A \neq B$ $\longrightarrow$ ($\exists$ *C*. *BetH A B C*) **and**
between-only-one: *BetH A B C* $\longrightarrow$ $\neg$ *BetH B C A* **and**
pasch: $\neg$ *ColH A B C* $\wedge$ *IsL l* $\wedge$ *IsP p* $\wedge$
IncidP A p $\wedge$ *IncidP B p* $\wedge$ *IncidP C p* $\wedge$ *IncidLP l p* $\wedge$ $\neg$ *IncidL C l* $\wedge$
(*cut l A B*)

$\longrightarrow$
  $(cut\ l\ A\ C) \vee (cut\ l\ B\ C)$
assumes
  cong-permr: $CongH\ A\ B\ C\ D \longrightarrow CongH\ A\ B\ D\ C$ **and**
  cong-existence: $\bigwedge A\ B\ A'\ P{::}'p.\ A \neq B \wedge A' \neq P \wedge IsL\ l \wedge$
  $IncidL\ A'\ l \wedge IncidL\ P\ l \longrightarrow$
  $(\exists\ B'.\ (IncidL\ B'\ l \wedge outH\ A'\ P\ B' \wedge CongH\ A'\ B'\ A\ B))$ **and**
  cong-pseudo-transitivity:
  $CongH\ A\ B\ C\ D \wedge CongH\ A\ B\ E\ F \longrightarrow CongH\ C\ D\ E\ F$ **and**
  addition :
  $ColH\ A\ B\ C \wedge ColH\ A'\ B'\ C' \wedge$
  $disjoint\ A\ B\ B\ C \wedge disjoint\ A'\ B'\ B'\ C' \wedge$
  $CongH\ A\ B\ A'\ B' \wedge CongH\ B\ C\ B'\ C' \longrightarrow$
  $CongH\ A\ C\ A'\ C'$ **and**
  conga-refl: $\neg\ ColH\ A\ B\ C \longrightarrow CongaH\ A\ B\ C\ A\ B\ C$ **and**
  conga-comm : $\neg\ ColH\ A\ B\ C \longrightarrow CongaH\ A\ B\ C\ C\ B\ A$ **and**
  conga-permlr: $CongaH\ A\ B\ C\ D\ E\ F \longrightarrow CongaH\ C\ B\ A\ F\ E\ D$ **and**
  conga-out-conga: $(CongaH\ A\ B\ C\ D\ E\ F \wedge$
  $outH\ B\ A\ A' \wedge outH\ B\ C\ C' \wedge outH\ E\ D\ D' \wedge outH\ E\ F\ F') \longrightarrow$
  $CongaH\ A'\ B\ C'\ D'\ E\ F'$ **and**
  cong-4-existence:
  $\neg\ ColH\ P\ PO\ X \wedge \neg\ ColH\ A\ B\ C \longrightarrow$
    $(\exists\ Y.\ (CongaH\ A\ B\ C\ X\ PO\ Y \wedge same\text{-}side'\ P\ Y\ PO\ X))$ **and**
  cong-4-uniqueness:
  $\neg\ ColH\ P\ PO\ X \wedge \neg\ ColH\ A\ B\ C \wedge$
  $CongaH\ A\ B\ C\ X\ PO\ Y \wedge CongaH\ A\ B\ C\ X\ PO\ Y' \wedge$
  $same\text{-}side'\ P\ Y\ PO\ X \wedge same\text{-}side'\ P\ Y'\ PO\ X \longrightarrow$
  $outH\ PO\ Y\ Y'$ **and**
  cong-5 : $\neg\ ColH\ A\ B\ C \wedge \neg\ ColH\ A'\ B'\ C' \wedge$
  $CongH\ A\ B\ A'\ B' \wedge CongH\ A\ C\ A'\ C' \wedge$
  $CongaH\ B\ A\ C\ B'\ A'\ C' \longrightarrow$
  $CongaH\ A\ B\ C\ A'\ B'\ C'$

context *Hilbert-neutral-dimensionless*

begin

8.3 Propositions

lemma *betH-distincts*:
  assumes $BetH\ A\ B\ C$
  shows $A \neq B \wedge B \neq C \wedge A \neq C$
  ⟨*proof*⟩

lemma *congH-perm*:
  assumes $A \neq B$
  shows $CongH\ A\ B\ B\ A$
⟨*proof*⟩

lemma *congH-refl*:
  assumes $A \neq B$
  shows $CongH\ A\ B\ A\ B$
  ⟨*proof*⟩

lemma *congH-sym*:
  assumes $A \neq B$ **and**
  $CongH\ A\ B\ C\ D$
  shows $CongH\ C\ D\ A\ B$
  ⟨*proof*⟩

lemma *colH-permut-231*:
  assumes $ColH\ A\ B\ C$
  shows $ColH\ B\ C\ A$
  ⟨*proof*⟩

lemma *colH-permut-312*:
  assumes *ColH A B C*
  shows *ColH C A B*
  ⟨*proof*⟩

lemma *colH-permut-213*:
  assumes *ColH A B C*
  shows *ColH B A C*
  ⟨*proof*⟩

lemma *colH-permut-132*:
  assumes *ColH A B C*
  shows *ColH A C B*
  ⟨*proof*⟩

lemma *colH-permut-321*:
  assumes *ColH A B C*
  shows *ColH C B A*
  ⟨*proof*⟩

lemma *other-point-exists*:
  fixes $A::'p$
  shows $\exists\ B.\ A \neq B$
  ⟨*proof*⟩

lemma *colH-trivial111*:
  shows *ColH A A A*
  ⟨*proof*⟩

lemma *colH-trivial112*:
  shows *ColH A A B*
  ⟨*proof*⟩

lemma *colH-trivial122*:
  shows *ColH A B B*
  ⟨*proof*⟩

lemma *colH-trivial121*:
  shows *ColH A B A*
  ⟨*proof*⟩

lemma *colH-dec*:
  shows *ColH A B C* ∨ ¬ *ColH A B C*
  ⟨*proof*⟩

lemma *colH-trans*:
  assumes $X \neq Y$ **and**
    ColH X Y A **and**
    ColH X Y B **and**
    ColH X Y C
  shows *ColH A B C*
⟨*proof*⟩

lemma *bet-colH*:
  assumes *Bet A B C*
  shows *ColH A B C*
  ⟨*proof*⟩

lemma *ncolH-exists*:
  assumes $A \neq B$
  shows ∃ *C*. ¬ *ColH A B C*
  ⟨*proof*⟩

lemma *ncolH-distincts*:
  assumes ¬ *ColH A B C*

shows $A \neq B \wedge B \neq C \wedge A \neq C$
⟨*proof*⟩

lemma *betH-expand*:
assumes *BetH A B C*
shows $BetH\ A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C \wedge ColH\ A\ B\ C$
⟨*proof*⟩

lemma *inter-uniquenessH*:
assumes $A' \neq B'$ **and**
$\neg$ *ColH A B A'* **and**
ColH A B X **and**
ColH A B Y **and**
ColH A' B' X **and**
ColH A' B' Y
shows $X = Y$
⟨*proof*⟩

lemma *inter-incid-uniquenessH*:
assumes $\neg$ *IncidL P l* **and**
IncidL P m **and**
IncidL X l **and**
IncidL Y l **and**
IncidL X m **and**
IncidL Y m
shows $X = Y$
⟨*proof*⟩

lemma *between-only-one'*:
assumes *BetH A B C*
shows $\neg$ *BetH B A C*
⟨*proof*⟩

lemma *betH-colH*:
assumes *BetH A B C*
shows $ColH\ A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C$
⟨*proof*⟩

lemma *cut-comm*:
assumes *cut l A B*
shows *cut l B A*
⟨*proof*⟩

lemma *line-on-plane'*:
assumes $A \neq B$ **and**
IncidP A p **and**
IncidP B p **and**
ColH A B C
shows *IncidP C p*
⟨*proof*⟩

lemma *inner-pasch-aux*:
assumes $\neg$ *ColH B C P* **and**
Bet A P C **and**
Bet B Q C
shows $\exists\ X.\ Bet\ P\ X\ B \wedge Bet\ Q\ X\ A$
⟨*proof*⟩

lemma *betH-trans1*:
assumes *BetH A B C* **and**
BetH B C D
shows *BetH A C D*
⟨*proof*⟩

lemma *betH-trans2*:
assumes *BetH A B C* **and**

BetH B C D
shows *BetH A B D*
⟨*proof*⟩

lemma *betH-trans*:
assumes *BetH A B C* **and**
BetH B C D
shows *BetH A B D* ∧ *BetH A C D*
⟨*proof*⟩

lemma *not-cut3*:
assumes *IncidP A p* **and**
IncidP B p **and**
IncidP C p **and**
IncidLP l p **and**
¬ *IncidL A l* **and**
¬ *ColH A B C* **and**
¬ *cut l A B* **and**
¬ *cut l A C*
shows ¬ *cut l B C*
⟨*proof*⟩

lemma *betH-trans0*:
assumes *BetH A B C* **and**
BetH A C D
shows *BetH B C D* ∧ *BetH A B D*
⟨*proof*⟩

lemma *betH-outH2--betH*:
assumes *BetH A B C* **and**
outH B C C' **and**
outH B A A'
shows *BetH A' B C'*
⟨*proof*⟩

lemma *cong-existence'*:
fixes *A B*::'*p*
assumes $A \neq B$ **and**
IncidL M l
shows ∃ *A' B'*. *IncidL A' l* ∧ *IncidL B' l* ∧ *BetH A' M B'* ∧ *CongH M A' A B* ∧ *CongH M B' A B*
⟨*proof*⟩

lemma *betH-to-bet*:
assumes *BetH A B C*
shows *Bet A B C* ∧ $A \neq B$ ∧ $B \neq C$ ∧ $A \neq C$
⟨*proof*⟩

lemma *betH-line*:
assumes *BetH A B C*
shows ∃ *l*. *IncidL A l* ∧ *IncidL B l* ∧ *IncidL C l*
⟨*proof*⟩

lemma *bet-identity*:
assumes *Bet A B A*
shows $A = B$
⟨*proof*⟩

lemma *morph*:
assumes *IsL l* **and**
IsL m **and**
EqL l m
shows ∀ *A*. *IncidL A l* ⟷ *IncidL A m*

⟨*proof*⟩

lemma *point3-online-exists*:
assumes *IncidL A l* **and**
IncidL B l
shows $\exists$ *C*. *IncidL C l* $\wedge$ *C* $\neq$ *A* $\wedge$ *C* $\neq$ *B*
⟨*proof*⟩

lemma *not-betH121*:
shows $\neg$ *BetH A B A*
⟨*proof*⟩

lemma *cong-identity*:
assumes *Cong A B C C*
shows *A* = *B*
⟨*proof*⟩

lemma *cong-inner-transitivity*:
assumes *Cong A B C D* **and**
Cong A B E F
shows *Cong C D E F*
⟨*proof*⟩

lemma *other-point-on-line*:
assumes *IncidL A l*
shows $\exists$ *B*. *A* $\neq$ *B* $\wedge$ *IncidL B l*
⟨*proof*⟩

lemma *bet-disjoint*:
assumes *BetH A B C*
shows *disjoint A B B C*
⟨*proof*⟩

lemma *addition-betH*:
assumes *BetH A B C* **and**
BetH A' B' C' **and**
CongH A B A' B' **and**
CongH B C B' C'
shows *CongH A C A' C'*
⟨*proof*⟩

lemma *outH-trivial*:
assumes *A* $\neq$ *B*
shows *outH A B B*
⟨*proof*⟩

lemma *same-side-refl*:
assumes *IsL l* **and**
$\neg$ *IncidL A l*
shows *same-side A A l*
⟨*proof*⟩

lemma *same-side-prime-refl*:
assumes $\neg$ *ColH A B C*
shows *same-side' C C A B*
⟨*proof*⟩

lemma *outH-expand*:
assumes *outH A B C*
shows *outH A B C* $\wedge$ *ColH A B C* $\wedge$ *A* $\neq$ *C* $\wedge$ *A* $\neq$ *B*
⟨*proof*⟩

lemma *construction-uniqueness*:
  assumes *BetH A B D* **and**
    BetH A B E **and**
    CongH B D B E
  shows $D = E$
⟨*proof*⟩

lemma *out-distinct*:
  assumes *outH A B C*
  shows $A \neq B \wedge A \neq C$
  ⟨*proof*⟩

lemma *out-same-side*:
  assumes *IncidL A l* **and**
    ¬ *IncidL B l* **and**
    outH A B C
  shows *same-side B C l*
⟨*proof*⟩

lemma *between-one*:
  assumes $A \neq B$ **and**
    $A \neq C$ **and**
    $B \neq C$ **and**
    ColH A B C
  shows *BetH A B C* ∨ *BetH B C A* ∨ *BetH B A C*
⟨*proof*⟩

lemma *betH-dec*:

  shows *BetH A B C* ∨ ¬ *BetH A B C*
  ⟨*proof*⟩

lemma *cut2-not-cut*:
  assumes ¬ *ColH A B C* **and**
    cut l A B **and**
    cut l A C
  shows ¬ *cut l B C*
⟨*proof*⟩

lemma *strong-pasch*:
  assumes ¬ *ColH A B C* **and**
    IncidP A p **and**
    IncidP B p **and**
    IncidP C p **and**
    IncidLP l p **and**
    ¬ *IncidL C l* **and**
    cut l A B
  shows (*cut l A C* ∧ ¬ *cut l B C*) ∨ (*cut l B C* ∧ ¬ *cut l A C*)
  ⟨*proof*⟩

lemma *out2-out*:
  assumes $C \neq D$ **and**
    BetH A B C **and**
    BetH A B D
  shows *BetH B C D* ∨ *BetH B D C*
⟨*proof*⟩

lemma *out2-out1*:
  assumes $C \neq D$ **and**
    BetH A B C **and**
    BetH A B D
  shows *BetH A C D* ∨ *BetH A D C*
  ⟨*proof*⟩

lemma *betH2-out*:
  assumes $B \neq C$ **and**

BetH A B D **and**
BetH A C D
shows *BetH A B C* ∨ *BetH A C B*
⟨*proof*⟩

lemma *segment-construction*:
shows ∃ *E*. *Bet A B E* ∧ *Cong B E C D*
⟨*proof*⟩

lemma *lower-dim-e*:
shows ∃ *A B C*. ¬ (*Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*)
⟨*proof*⟩

lemma *outH-dec*:
shows *outH A B C* ∨ ¬ *outH A B C*
⟨*proof*⟩

lemma *out-construction*:
assumes *X* ≠ *Y* **and**
A ≠ *B*
shows ∃ *C*. *CongH A C X Y* ∧ *outH A B C*
⟨*proof*⟩

lemma *segment-constructionH*:
assumes *A* ≠ *B* **and**
C ≠ *D*
shows ∃ *E*. *BetH A B E* ∧ *CongH B E C D*
⟨*proof*⟩

lemma *EqL-dec*:
shows *EqL l m* ∨ ¬ *EqL l m*
⟨*proof*⟩

lemma *cut-exists*:
assumes *IsL l* **and**
¬ *IncidL A l*
shows ∃ *B*. *cut l A B*
⟨*proof*⟩

lemma *outH-col*:
assumes *outH A B C*
shows *ColH A B C*
⟨*proof*⟩

lemma *cut-distinct*:
assumes *cut l A B*
shows *A* ≠ *B*
⟨*proof*⟩

lemma *same-side-not-cut*:
assumes *same-side A B l*
shows ¬ *cut l A B*
⟨*proof*⟩

lemma *IncidLP-morphism*:
assumes
IsL m **and**
IsP q **and**
IncidLP l p **and**
EqL l m **and**
EqP p q
shows *IncidLP m q*
⟨*proof*⟩

lemma *same-side--plane*:
assumes *same-side A B l*

shows ∃ *p*. *IncidP A p* ∧ *IncidP B p* ∧ *IncidLP l p*
⟨*proof*⟩

lemma *same-side-prime--plane*:
assumes *same-side′ A B C D*
shows ∃ *p*. *IncidP A p* ∧ *IncidP B p* ∧ *IncidP C p* ∧ *IncidP D p*
⟨*proof*⟩

lemma *cut-same-side-cut*:
assumes *cut l P X* **and**
same-side X Y l
shows *cut l P Y*
⟨*proof*⟩

lemma *isosceles-congaH*:
assumes ¬ *ColH A B C* **and**
CongH A B A C
shows *CongaH A B C A C B*
⟨*proof*⟩

lemma *cong-distincts*:
assumes *A* ≠ *B* **and**
Cong A B C D
shows *C* ≠ *D*
⟨*proof*⟩

lemma *cong-sym*:
assumes *Cong A B C D*
shows *Cong C D A B*
⟨*proof*⟩

lemma *cong-trans*:
assumes *Cong A B C D* **and**
Cong C D E F
shows *Cong A B E F*
⟨*proof*⟩

lemma *betH-not-congH*:
assumes *BetH A B C*
shows ¬ *CongH A B A C*
⟨*proof*⟩

lemma *congH-permlr*:
assumes
C ≠ *D* **and**
CongH A B C D
shows *CongH B A D C*
⟨*proof*⟩

lemma *congH-perms*:
assumes *A* ≠ *B* **and**
C ≠ *D* **and**
CongH A B C D
shows *CongH B A C D* ∧ *CongH A B D C* ∧ *CongH C D A B* ∧
CongH D C A B ∧ *CongH C D B A* ∧ *CongH D C B A* ∧ *CongH B A D C*
⟨*proof*⟩

lemma *congH-perml*:
assumes
C ≠ *D* **and**
CongH A B C D
shows *CongH B A C D*
⟨*proof*⟩

lemma *bet-cong3-bet*:
assumes $A' \neq B'$ **and**
$B' \neq C'$ **and**
$A' \neq C'$ **and**
BetH $A\ B\ C$ **and**
ColH $A'\ B'\ C'$ **and**
CongH $A\ B\ A'\ B'$ **and**
CongH $B\ C\ B'\ C'$ **and**
CongH $A\ C\ A'\ C'$
shows *BetH* $A'\ B'\ C'$
⟨*proof*⟩

lemma *betH-congH3-outH-betH*:
assumes *BetH* $A\ B\ C$ **and**
outH $A'\ B'\ C'$ **and**
CongH $A\ C\ A'\ C'$ **and**
CongH $A\ B\ A'\ B'$
shows *BetH* $A'\ B'\ C'$
⟨*proof*⟩

lemma *outH-sym*:
assumes $A \neq C$ **and**
outH $A\ B\ C$
shows *outH* $A\ C\ B$
⟨*proof*⟩

lemma *soustraction-betH*:
assumes *BetH* $A\ B\ C$ **and**
BetH $A'\ B'\ C'$ **and**
CongH $A\ B\ A'\ B'$ **and**
CongH $A\ C\ A'\ C'$
shows *CongH* $B\ C\ B'\ C'$
⟨*proof*⟩

lemma *ncolH-expand*:
assumes ¬ *ColH* $A\ B\ C$
shows ¬ *ColH* $A\ B\ C \wedge A \neq B \wedge B \neq C \wedge A \neq C$
⟨*proof*⟩

lemma *betH-outH--outH*:
assumes *BetH* $A\ B\ C$ **and**
outH $B\ C\ D$
shows *outH* $A\ C\ D$
⟨*proof*⟩

lemma *th12*:
assumes ¬ *ColH* $A\ B\ C$ **and**
¬ *ColH* $A'\ B'\ C'$ **and**
CongH $A\ B\ A'\ B'$ **and**
CongH $A\ C\ A'\ C'$ **and**
CongaH $B\ A\ C\ B'\ A'\ C'$
shows *CongaH* $A\ B\ C\ A'\ B'\ C'$ ∧ *CongaH* $A\ C\ B\ A'\ C'\ B'$ ∧ *CongH* $B\ C\ B'\ C'$
⟨*proof*⟩

lemma *th14*:
assumes ¬ *ColH* $A\ B\ C$ **and**
¬ *ColH* $A'\ B'\ C'$ **and**
CongaH $A\ B\ C\ A'\ B'\ C'$ **and**
BetH $A\ B\ D$ **and**
BetH $A'\ B'\ D'$
shows *CongaH* $C\ B\ D\ C'\ B'\ D'$
⟨*proof*⟩

lemma *congH-colH-betH*:

assumes $A \neq B$ **and**
$A \neq I$ **and**
$B \neq I$ **and**
CongH I A I B **and**
ColH I A B
shows *BetH A I B*
⟨*proof*⟩

lemma *plane-separation*:
assumes ¬ *ColH A X Y* **and**
¬ *ColH B X Y* **and**
IncidP A p **and**
IncidP B p **and**
IncidP X p **and**
IncidP Y p
shows *cut′ A B X Y* ∨ *same-side′ A B X Y*
⟨*proof*⟩

lemma *same-side-comm*:
assumes *same-side A B l*
shows *same-side B A l*
⟨*proof*⟩

lemma *same-side-not-incid*:
assumes *same-side A B l*
shows ¬ *IncidL A l* ∧ ¬ *IncidL B l*
⟨*proof*⟩

lemma *out-same-side′*:
assumes $X \neq Y$ **and**
IncidL X l **and**
IncidL Y l **and**
IncidL A l **and**
¬ *IncidL B l* **and**
outH A B C
shows *same-side′ B C X Y*
⟨*proof*⟩

lemma *same-side-trans*:
assumes *same-side A B l* **and**
same-side B C l
shows *same-side A C l*
⟨*proof*⟩

lemma *colH-IncidL--IncidL*:
assumes $A \neq B$ **and**
IncidL A l **and**
IncidL B l **and**
ColH A B C
shows *IncidL C l*
⟨*proof*⟩

lemma *IncidL-not-IncidL--not-colH*:
assumes $A \neq B$ **and**
IncidL A l **and**
IncidL B l **and**
¬ *IncidL C l*
shows ¬ *ColH A B C*
⟨*proof*⟩

lemma *same-side-prime-not-colH*:
assumes *same-side′ A B C D*
shows ¬ *ColH A C D* ∧ ¬ *ColH B C D*
⟨*proof*⟩

lemma *OS2--TS*:

assumes *same-side'* Y Z PO X **and**
same-side' X Y PO Z
shows *cut'* X Z PO Y
⟨*proof*⟩

lemma *th15-aux-1*:
assumes ¬ *ColH* H PO L **and**
¬ *ColH* H' O' L' **and**
¬ *ColH* K PO L **and**
¬ *ColH* K' O' L' **and**
¬ *ColH* H PO K **and**
¬ *ColH* H' O' K' **and**
same-side' H K PO L **and**
same-side' H' K' O' L' **and**
cut' K L PO H **and**
CongaH H PO L H' O' L' **and**
CongaH K PO L K' O' L'
shows *CongaH* H PO K H' O' K'
⟨*proof*⟩

lemma *th15-aux*:
assumes ¬ *ColH* H PO L **and**
¬ *ColH* H' O' L' **and**
¬ *ColH* K PO L **and**
¬ *ColH* K' O' L' **and**
¬ *ColH* H PO K **and**
¬ *ColH* H' O' K' **and**
same-side' H K PO L **and**
same-side' H' K' O' L' **and**
CongaH H PO L H' O' L' **and**
CongaH K PO L K' O' L'
shows *CongaH* H PO K H' O' K'
⟨*proof*⟩

lemma *th15*:
assumes ¬ *ColH* H PO L **and**
¬ *ColH* H' O' L' **and**
¬ *ColH* K PO L **and**
¬ *ColH* K' O' L' **and**
¬ *ColH* H PO K **and**
¬ *ColH* H' O' K' **and**
(*cut'* H K PO L ∧ *cut'* H' K' O' L') ∨ (*same-side'* H K PO L ∧ *same-side'* H' K' O' L') **and**
CongaH H PO L H' O' L' **and**
CongaH K PO L K' O' L'
shows *CongaH* H PO K H' O' K'
⟨*proof*⟩

lemma *th17*:
assumes ¬ *ColH* X Y *Z1* **and**
¬ *ColH* X Y *Z2* **and**
ColH X I Y **and**
BetH *Z1* I *Z2* **and**
CongH X *Z1* X *Z2* **and**
CongH Y *Z1* Y *Z2*
shows *CongaH* X Y *Z1* X Y *Z2*
⟨*proof*⟩

lemma *congaH-existence-congH*:
assumes $U \neq V$ **and**
¬ *ColH* P PO X **and**
¬ *ColH* A B C
shows ∃ Y. (*CongaH* A B C X PO Y ∧ *same-side'* P Y PO X ∧ *CongH* PO Y U V)
⟨*proof*⟩

lemma *th18-aux*:
assumes ¬ *ColH* A B C **and**

¬ *ColH* A' B' C' **and**
CongH A B A' B' **and**
CongH A C A' C' **and**
CongH B C B' C'
shows *CongaH* B A C B' A' C'
⟨*proof*⟩

lemma *th19*:
assumes ¬ *ColH PO A B* **and**
¬ *ColH O1 A1 B1* **and**
¬ *ColH O2 A2 B2* **and**
CongaH A PO B A1 O1 B1 **and**
CongaH A PO B A2 O2 B2
shows *CongaH A1 O1 B1 A2 O2 B2*
⟨*proof*⟩

lemma *congaH-sym*:
assumes ¬ *ColH A B C* **and**
¬ *ColH D E F* **and**
CongaH A B C D E F
shows *CongaH D E F A B C*
⟨*proof*⟩

lemma *congaH-commr*:
assumes ¬ *ColH A B C* **and**
¬ *ColH D E F* **and**
CongaH A B C D E F
shows *CongaH A B C F E D*
⟨*proof*⟩

lemma *cong-preserves-col*:
assumes *BetH A B C* **and**
CongH A B A' B' **and**
CongH B C B' C' **and**
CongH A C A' C'
shows *ColH* A' B' C'
⟨*proof*⟩

lemma *cong-preserves-col-stronger*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
ColH A B C **and**
CongH A B A' B' **and**
CongH B C B' C' **and**
CongH A C A' C'
shows *ColH* A' B' C'
⟨*proof*⟩

lemma *betH-congH2--False*:
assumes *BetH A B C* **and**
BetH A' C' B' **and**
CongH A B A' B' **and**
CongH A C A' C'
shows *False*
⟨*proof*⟩

lemma *cong-preserves-bet*:
assumes $A' \neq B'$ **and**
$B' \neq C'$ **and**
$A'{\neq}C'$ **and**
BetH A B C **and**
CongH A B A' B' **and**
CongH B C B' C' **and**
CongH A C A' C'

shows *BetH* A' B' C'
⟨*proof*⟩

lemma *axiom-five-segmentsH*:
assumes $A \neq D$ **and**
$A' \neq D'$ **and**
$B \neq D$ **and**
$B' \neq D'$ **and**
$C \neq D$ **and**
$C' \neq D'$ **and**
CongH A B A' B' **and**
CongH B C B' C' **and**
CongH A D A' D' **and**
CongH B D B' D' **and**
BetH A B C **and**
BetH A' B' C' **and**
$A' \neq D'$
shows *CongH* C D C' D'
⟨*proof*⟩

lemma *five-segment*:
assumes *Cong* A B A' B' **and**
Cong B C B' C' **and**
Cong A D A' D' **and**
Cong B D B' D' **and**
Bet A B C **and**
Bet A' B' C' **and**
$A \neq B$
shows *Cong* C D C' D'
⟨*proof*⟩

lemma *bet-comm*:
assumes *Bet* A B C
shows *Bet* C B A
⟨*proof*⟩

lemma *bet-trans*:
assumes *Bet* A B D **and**
Bet B C D
shows *Bet* A B C
⟨*proof*⟩

lemma *cong-transitivity*:
assumes *Cong* A B E F **and**
Cong C D E F
shows *Cong* A B C D
⟨*proof*⟩

lemma *cong-permT*:
shows *Cong* A B B A
⟨*proof*⟩

lemma *pasch-general-case*:
assumes *Bet* A P C **and**
Bet B Q C **and**
$A \neq P$ **and**
$P \neq C$ **and**
$B \neq Q$ **and**
$Q \neq C$ **and**
$\neg$ (*Bet* A B C $\vee$ *Bet* B C A $\vee$ *Bet* C A B)
shows $\exists$ x. *Bet* P x B $\wedge$ *Bet* Q x A
⟨*proof*⟩

lemma *lower-dim-l*:
shows $\neg$ (*Bet* *PP* *PQ* *PR* $\vee$ *Bet* *PQ* *PR* *PP* $\vee$ *Bet* *PR* *PP* *PQ*)
⟨*proof*⟩

lemma *ColH-bets*:
  assumes *ColH A B C*
  shows *Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*
  ⟨*proof*⟩

end
end

theory *Tarski-Neutral-Hilbert*

imports
  Tarski-Neutral
  Hilbert-Neutral

begin

9 Tarski neutral dimensionless - Hilbert

context *Tarski-neutral-dimensionless*

begin

9.1 Definition

definition *isLine*:: *'p×'p⇒bool* **where**
  isLine l ≡ (*fst l* ≠ *snd l*)

definition *Line* :: *'p ⇒'p ⇒'p×'p* **where**
  Line A B = (*if A* ≠ *B then* (*Pair A B*) *else undefined*)

definition *IncidentL* ::*'p⇒'p×'p⇒bool* **where**
  IncidentL A l ≡ *isLine l* ∧ *Col A* (*fst l*) (*snd l*)

definition *EqTL*:: *'p×'p⇒'p×'p⇒bool*
  (- =*l*= - [*99*,*99*] *50*) **where**
  l1 =*l*= *l2* ≡ *isLine l1* ∧ *isLine l2* ∧ (∀ *X*. (*IncidentL X l1* ⟷ *IncidentL X l2*))

definition *Col-H*:: *'p⇒'p⇒'p⇒bool* **where**
  Col-H A B C ≡ ∃ *l*. (*isLine l* ∧ *IncidentL A l* ∧ *IncidentL B l* ∧ *IncidentL C l*)

definition *isPlane* :: *'p×'p×'p⇒bool* **where**
  isPlane pl ≡ (∀ *P Q R*::*'p*. *pl* = (*P*,*Q*,*R*) ⟶ ¬ *Col P Q R*)

definition *Plane* :: *'p⇒'p⇒'p⇒'p×'p×'p* **where**
  Plane A B C = (*A*,*B*,*C*)

definition *IncidentP* ::*'p⇒'p×'p×'p⇒bool*
  where
    IncidentP A pl ≡ (*isPlane pl*) ∧ (∃ *P Q R*. *pl* = *Plane P Q R* ∧ *Coplanar A P Q R*)

definition *EqTP*:: *'p×'p×'p⇒'p×'p×'p⇒bool* (- =*p*= - [*99*,*99*] *50*) **where**
  p1 =*p*= *p2* ≡ *isPlane p1* ∧ *isPlane p2* ∧ (∀ *X*. (*IncidentP X p1* ⟷ *IncidentP X p2*))

definition *IncidentLP* ::*'p×'p⇒'p×'p×'p⇒bool* **where**
  IncidentLP l p ≡ *isLine l* ∧ *isPlane p* ∧ (∀ *A*. *IncidentL A l* ⟶ *IncidentP A p*)

definition *Between-H* :: *'p⇒'p⇒'p⇒bool* **where**
  Between-H A B C ≡ *Bet A B C* ∧ *A* ≠ *B* ∧ *B* ≠ *C* ∧ *A* ≠ *C*

definition *cut-H* :: *'p×'p⇒'p⇒'p⇒bool* **where**
  cut-H l A B ≡ *isLine l* ∧ ¬ *IncidentL A l* ∧ ¬ *IncidentL B l* ∧
                (∃ *I*. *IncidentL I l* ∧ *Between-H A I B*)

definition *outH* :: *'p⇒'p⇒'p⇒bool* **where**

outH P A B ≡ *Between-H P A B* ∨ *Between-H P B A* ∨ (*P* ≠ *A* ∧ *A* = *B*)

definition *same-side-scott* :: *′p⇒′p⇒′p⇒bool*
where *same-side-scott E A B* ≡ *E* ≠ *A* ∧ *E* ≠ *B* ∧ *Col-H E A B* ∧ ¬ *Between-H A E B*

definition *disjoint-H* :: *′p⇒′p⇒′p⇒′p⇒bool*
where *disjoint-H A B C D* ≡ ¬ (∃ *P*. *Between-H A P B* ∧ *Between-H C P D*)

definition *same-side-H* :: *′p⇒′p⇒′p×′p⇒bool*
where *same-side-H A B l* ≡ *isLine l* ∧ (∃ *P*. *cut-H l A P* ∧ *cut-H l B P*)

definition *same-side′-H* :: *′p⇒′p⇒′p⇒′p⇒bool*
where
same-side′-H A B X Y ≡ *X* ≠ *Y* ∧ (∀ *l*. *isLine l* ∧ (*IncidentL X l* ∧ *IncidentL Y l*)
⟶ *same-side-H A B l*)

definition *CongA-H* :: *′p⇒′p⇒′p⇒′p⇒′p⇒′p⇒bool* **where**
CongA-H A B C D E F ≡ *A B C CongA D E F*

definition *Para-H* :: *′p×′p* ⇒ *′p×′p* ⇒*bool* **where**
Para-H l m ≡ *isLine l* ∧ *isLine m* ∧ (¬(∃ *X*. *IncidentL X l* ∧ *IncidentL X m*)) ∧
(∃ *p*. *IncidentLP l p* ∧ *IncidentLP m p*)

9.2 Propositions

lemma *EqL--diff-left*:
assumes *l1* =*l*= *l1*
shows *fst l1* ≠ *snd l1*
⟨*proof*⟩

lemma *EqL--diff-right*:
assumes *l1* =*l*= *l2*
shows (*fst l2*) ≠ (*snd l2*)
⟨*proof*⟩

lemma *axiom-line-existence*:
assumes *A* ≠ *B*
shows ∃ *l*. *isLine l* ∧ *IncidentL A l* ∧ *IncidentL B l*
⟨*proof*⟩

lemma *incident-eq*:
assumes *A* ≠ *B* **and**
IncidentL A l **and**
IncidentL B l
shows *Line A B* =*l*= *l*
⟨*proof*⟩

lemma *eq-transitivity*:
assumes *l* =*l*= *m* **and**
m =*l*= *n*
shows *l* =*l*= *n*
⟨*proof*⟩

lemma *eq-reflexivity*:
assumes *isLine l*
shows *l* =*l*= *l*
⟨*proof*⟩

lemma *eq-symmetry*:
assumes *l* =*l*= *m*
shows *m* =*l*= *l*
⟨*proof*⟩

lemma *eq-incident*:

assumes *l =l= m*
shows *IncidentL A l* ⟷ *IncidentL A m*
⟨*proof*⟩

lemma *axiom-Incidl-morphism*:
assumes *IncidentL P l* **and**
l =l= m
shows *IncidentL P m*
⟨*proof*⟩

lemma *axiom-line-uniqueness*:
assumes $A \neq B$ **and**
IncidentL A l **and** *IncidentL B l* **and**
IncidentL A m **and** *IncidentL B m*
shows *l =l= m*
⟨*proof*⟩

lemma *axiom-two-points-on-line*:
assumes *isLine l*
shows ∃ *A B. IncidentL B l* ∧ *IncidentL A l* ∧ $A \neq B$
⟨*proof*⟩

lemma *cols-coincide-1*:
assumes *Col-H A B C*
shows *Col A B C*
⟨*proof*⟩

lemma *cols-coincide-2*:
assumes *Col A B C*
shows *Col-H A B C*
⟨*proof*⟩

lemma *cols-coincide*:
shows *Col A B C* ⟷ *Col-H A B C*
⟨*proof*⟩

lemma *ncols-coincide*:
shows ¬ *Col A B C* ⟷ ¬ *Col-H A B C*
⟨*proof*⟩

lemma *lower-dim'*:
shows ∃ *PA PB PC*. $PA \neq PB$ ∧ $PB \neq PC$ ∧
$PA \neq PC$ ∧ ¬ *Col-H PA PB PC*
⟨*proof*⟩

lemma *axiom-plane-existence*:
assumes ¬ *Col-H A B C*
shows ∃ *p. IncidentP A p* ∧ *IncidentP B p* ∧ *IncidentP C p*
⟨*proof*⟩

lemma *incidentp-eqp*:
assumes ¬ *Col-H A B C* **and**
IncidentP A p **and**
IncidentP B p **and**
IncidentP C p
shows (*Plane A B C*) *=p= p*
⟨*proof*⟩

lemma *eqp-transitivity*:

assumes *p* =*p*= *q* **and**
q =*p*= *r*
shows *p* =*p*= *r*
⟨*proof*⟩

lemma *eqp-reflexivity*:
assumes *isPlane p*
shows *p* =*p*= *p*
⟨*proof*⟩

lemma *eqp-symmetry*:
assumes *p* =*p*= *q*
shows *q* =*p*= *p*
⟨*proof*⟩

lemma *eqp-incidentp*:
assumes *p* =*p*= *q*
shows *IncidentP A p* ⟷ *IncidentP A q*
⟨*proof*⟩

lemma *axiom-Incidp-morphism* :
assumes *IncidentP M p* **and**
EqTP p q
shows *IncidentP M q*
⟨*proof*⟩

lemma *axiom-plane-uniqueness*:
assumes ¬ *Col-H A B C* **and**
IncidentP A p **and**
IncidentP B p **and**
IncidentP C p **and**
IncidentP A q **and**
IncidentP B q **and**
IncidentP C q
shows *p* =*p*= *q*
⟨*proof*⟩

lemma *axiom-one-point-on-plane*:
assumes *isPlane p*
shows ∃ *A*. *IncidentP A p*
⟨*proof*⟩

lemma *axiom-line-on-plane*:
assumes *A* ≠ *B* **and**
IncidentL A l **and**
IncidentL B l **and**
IncidentP A p **and**
IncidentP B p
shows *IncidentLP l p*
⟨*proof*⟩

lemma *axiom-between-col*:
assumes *Between-H A B C*
shows *Col-H A B C*
⟨*proof*⟩

lemma *axiom-between-diff*:
  assumes *Between-H A B C*
  shows $A \neq C$
  ⟨*proof*⟩

lemma *axiom-between-comm*:
  assumes *Between-H A B C*
  shows *Between-H C B A*
  ⟨*proof*⟩

lemma *axiom-between-out*:
  assumes $A \neq B$
  shows ∃ *C*. *Between-H A B C*
  ⟨*proof*⟩

lemma *axiom-between-only-one*:
  assumes *Between-H A B C*
  shows ¬ *Between-H B C A*
  ⟨*proof*⟩

lemma *between-one*:
  assumes $A \neq B$ **and**
    $A \neq C$ **and**
    $B \neq C$ **and**
    Col A B C
  shows *Between-H A B C* ∨ *Between-H B C A* ∨ *Between-H B A C*
  ⟨*proof*⟩

lemma *axiom-between-one*:
  assumes $A \neq B$ **and**
    $A \neq C$ **and**
    $B \neq C$ **and**
    Col-H A B C
  shows *Between-H A B C* ∨ *Between-H B C A* ∨ *Between-H B A C*
  ⟨*proof*⟩

lemma *cut-two-sides*:
  shows *cut-H l A B* ⟷ (*fst l*) (*snd l*) *TS A B*
⟨*proof*⟩

lemma *cop-plane-aux*:
  assumes *Coplanar A B C D* **and**
    $A \neq B$
  shows ∃ *p*. *IncidentP A p* ∧ *IncidentP B p* ∧ *IncidentP C p* ∧ *IncidentP D p*
⟨*proof*⟩

lemma *cop-plane*:
  assumes *Coplanar A B C D*
  shows ∃ *p*. *IncidentP A p* ∧ *IncidentP B p* ∧ *IncidentP C p* ∧ *IncidentP D p*
  ⟨*proof*⟩

lemma *plane-cop*:
  assumes *IncidentP A p* **and**
    IncidentP B p **and**
    IncidentP C p **and**
    IncidentP D p

shows *Coplanar A B C D*
⟨*proof*⟩

lemma *axiom-pasch*:
assumes ¬ *Col-H A B C* **and**
IncidentP A p **and**
IncidentP B p **and**
IncidentP C p **and**
IncidentLP l p **and**
¬ *IncidentL C l* **and**
cut-H l A B
shows *cut-H l A C* ∨ *cut-H l B C*
⟨*proof*⟩

lemma *Incid-line*:
assumes $A \neq B$ **and**
IncidentL A l **and**
IncidentL B l **and**
Col P A B
shows *IncidentL P l*
⟨*proof*⟩

lemma *out-outH*:
assumes *P Out A B*
shows *outH P A B*
⟨*proof*⟩

lemma *axiom-hcong-1-existence*:
assumes $A \neq B$ **and**
$A' \neq P$ **and**
IncidentL A' *l* **and**
IncidentL P l
shows ∃ B'. *IncidentL* B' *l* ∧ *outH* A' *P* B' ∧ *Cong* A' B' *A B*
⟨*proof*⟩

lemma *axiom-hcong-1-uniqueness*:
assumes
IncidentL M l **and**
IncidentL A' *l* **and**

IncidentL A'' *l* **and**

Between-H A' *M* B' **and**
Cong M A' *A B* **and**
Cong M B' *A B* **and**
Between-H A'' *M* B'' **and**
Cong M A'' *A B* **and**
Cong M B'' *A B*
shows $(A' = A'' \wedge B' = B'') \vee (A' = B'' \wedge B' = A'')$
⟨*proof*⟩

lemma *axiom-hcong-scott*:
assumes $A \neq C$ **and**
$P \neq Q$
shows ∃ *B*. *same-side-scott A B C* ∧ *Cong P Q A B*
⟨*proof*⟩

lemma *col-disjoint-bet*:
  assumes *Col-H A B C* **and**
    disjoint-H A B B C
  shows *Bet A B C*
⟨*proof*⟩

lemma *axiom-hcong-3*:
  assumes *Col-H A B C* **and**
    Col-H A′ B′ C′ **and**
    disjoint-H A B B C **and**
    disjoint-H A′ B′ B′ C′ **and**
    Cong A B A′ B′ **and**
    Cong B C B′ C′
  shows *Cong A C A′ C′*
  ⟨*proof*⟩

lemma *exists-not-incident*:
  assumes $A \neq B$
  shows $\exists\ C.\ \neg$ *IncidentL C* (*Line A B*)
  ⟨*proof*⟩

lemma *same-side-one-side*:
  assumes *same-side-H A B l*
  shows (*fst l*) (*snd l*) *OS A B*
  ⟨*proof*⟩

lemma *one-side-same-side*:
  assumes (*fst l*)(*snd l*) *OS A B*
  shows *same-side-H A B l*
⟨*proof*⟩

lemma *OS-distinct*:
  assumes *P Q OS A B*
  shows $P \neq Q$
  ⟨*proof*⟩

lemma *OS-same-side′*:
  assumes *P Q OS A B*
  shows *same-side′-H A B P Q*
⟨*proof*⟩

lemma *same-side-OS*:
  assumes *same-side′-H P Q A B*
  shows *A B OS P Q*
⟨*proof*⟩

lemma *outH-out*:
  assumes *outH P A B*
  shows *P Out A B*
  ⟨*proof*⟩

lemma *incident-col*:
  assumes *IncidentL M l*
  shows *Col M* (*fst l*)(*snd l*)
  ⟨*proof*⟩

lemma *col-incident*:
  assumes (*fst l*) $\neq$ (*snd l*) **and**

Col M (fst l)(snd l)
shows *IncidentL M l*
⟨*proof*⟩

lemma *Bet-Between-H*:
assumes *Bet A B C* **and**
$A \neq B$ **and**
$B \neq C$
shows *Between-H A B C*
⟨*proof*⟩

lemma *axiom-cong-5'*:
assumes
¬ *Col-H A' B' C'* **and**
Cong A B A' B' **and**
Cong A C A' C' **and**
B A C CongA B' A' C'
shows *A B C CongA A' B' C'*
⟨*proof*⟩

lemma *axiom-cong-5'-bis*:
assumes ¬ *Col-H A B C* **and**

Cong A B A' B' **and**
Cong A C A' C' **and**
B A C CongA B' A' C'
shows *A B C CongA A' B' C'*
⟨*proof*⟩

lemma *axiom-hcong-4-existence*:
assumes ¬ *Col-H P PO X* **and**
¬ *Col-H A B C*
shows ∃ *Y*. (*A B C CongA X PO Y* ∧ *same-side'-H P Y PO X*)
⟨*proof*⟩

lemma *same-side-trans*:
assumes *same-side-H A B l* **and**
same-side-H B C l
shows *same-side-H A C l*
⟨*proof*⟩

lemma *same-side-sym*:
assumes *same-side-H A B l*
shows *same-side-H B A l*
⟨*proof*⟩

lemma *axiom-hcong-4-uniqueness*:
assumes ¬ *Col-H P PO X* **and**
¬ *Col-H A B C* **and**
A B C CongA X PO Y **and**
A B C CongA X PO Y' **and**
same-side'-H P Y PO X **and**
same-side'-H P Y' PO X
shows *outH PO Y Y'*
⟨*proof*⟩

lemma *axiom-conga-comm*:
assumes ¬ *Col-H A B C*
shows *A B C CongA C B A*
⟨*proof*⟩

lemma *axiom-congaH-outH-congaH*:
assumes *A B C CongA D E F* **and**
Between-H B A A' ∨ *Between-H B A' A* ∨ $B \neq A \wedge A = A'$ **and**
Between-H B C C' ∨ *Between-H B C' C* ∨ $B \neq C \wedge C = C'$ **and**
Between-H E D D' ∨ *Between-H E D' D* ∨ $E \neq D \wedge D = D'$ **and**

$Between\text{-}H\ E\ F\ F' \vee Between\text{-}H\ E\ F'\ F \vee E \neq F \wedge F = F'$
shows $A'\ B\ C'\ CongA\ D'\ E\ F'$
⟨*proof*⟩

lemma *axiom-conga-permlr*:
assumes $A\ B\ C\ CongA\ D\ E\ F$
shows $C\ B\ A\ CongA\ F\ E\ D$
⟨*proof*⟩

lemma *axiom-conga-refl*:
assumes $\neg\ Col\text{-}H\ A\ B\ C$
shows $A\ B\ C\ CongA\ A\ B\ C$
⟨*proof*⟩

end
end

theory *Tarski-Neutral-2D*

imports
Tarski-Neutral

begin

10 Tarski's axiom system for neutral geometry: 2D

10.1 Definitions

locale *Tarski-neutral-2D* = *Tarski-neutral-dimensionless* +
assumes *upper-dim*: $\forall\ a\ b\ c\ p\ q.$
$p \neq q \wedge$
$Cong\ a\ p\ a\ q \wedge$
$Cong\ b\ p\ b\ q \wedge$
$Cong\ c\ p\ c\ q$
$\longrightarrow$
$(Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b)$

10.2 Propositions

context *Tarski-neutral-2D*

begin

lemma *all-coplanar*:
$Coplanar\ A\ B\ C\ D$
⟨*proof*⟩

lemma *per2--col*:
assumes $Per\ A\ X\ C$ **and**
$X \neq C$ **and**
$Per\ B\ X\ C$
shows $Col\ A\ B\ X$
⟨*proof*⟩

lemma *perp2--col*:
assumes $X\ Y\ Perp\ A\ B$ **and**
$X\ Z\ Perp\ A\ B$
shows $Col\ X\ Y\ Z$
⟨*proof*⟩

lemma *l12-9-2D*:
assumes $A1\ A2\ Perp\ C1\ C2$ **and**
$B1\ B2\ Perp\ C1\ C2$
shows $A1\ A2\ Par\ B1\ B2$
⟨*proof*⟩

lemma *perp-in2--col*:
assumes *P PerpAt A B X Y* **and**
P PerpAt A′ B′ X Y
shows *Col A B A′*
⟨*proof*⟩

lemma *perp2-trans*:
assumes *P Perp2 A B C D* **and**
P Perp2 C D E F
shows *P Perp2 A B E F*
⟨*proof*⟩

lemma *perp2-par*:
assumes *PO Perp2 A B C D*
shows *A B Par C D*
⟨*proof*⟩

lemma *not-par-strict-inter-exists*:
assumes ¬ *A1 B1 ParStrict A2 B2*
shows ∃ *X*. *Col X A1 B1* ∧ *Col X A2 B2*
⟨*proof*⟩

lemma *not-par-inter-exists*:
assumes ¬ *A1 B1 Par A2 B2*
shows ∃ *X*. *Col X A1 B1* ∧ *Col X A2 B2*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-Continuity-2D*

imports
Tarski-Neutral-2D
Tarski-Neutral-Continuity
begin

context *Tarski-neutral-2D*

begin

11 Tarski Neutral Continuity 2D

11.1 Definitions

11.2 Propositions

lemma *mid-onc2-perp--col*:
assumes $A \neq B$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
X Midpoint A B **and**
X Y Perp A B
shows *Col X Y PO*
⟨*proof*⟩

lemma *mid2-onc4--eq*:
assumes $B \neq C$ **and**
$A \neq B$ **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**

D OnCircle PO P **and**
X Midpoint A C **and**
X Midpoint B D
shows *X = PO*
⟨*proof*⟩

lemma *cong2-onc3--eq*:
assumes *A ≠ B* **and**
A ≠ C **and**
B ≠ C **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
Cong X A X B **and**
Cong X A X C
shows *X = PO*
⟨*proof*⟩

lemma *onc2-mid-cong-col*:
assumes *U ≠ V* **and**
U OnCircle PO P **and**
V OnCircle PO P **and**
M Midpoint U V **and**
Cong U X V X
shows *Col PO X M*
⟨*proof*⟩

lemma *cong-onc3-cases*:
assumes *Cong A X A Y* **and**
A OnCircle PO P **and**
X OnCircle PO P **and**
Y OnCircle PO P
shows *X = Y ∨ X Y ReflectL PO A*
⟨*proof*⟩

lemma *bet-cong-onc3-cases*:
assumes *T ≠ PO* **and**
Bet A PO T **and**
Cong T X T Y **and**
A OnCircle PO P **and**
X OnCircle PO P **and**
Y OnCircle PO P
shows *X = Y ∨ X Y ReflectL PO A*
⟨*proof*⟩

lemma *prop-7-8*:
assumes *Diam A B PO P* **and**
Bet A PO T **and**
X OnCircle PO P **and**
Y OnCircle PO P **and**
A PO X LeA A PO Y
shows *T Y Le T X*
⟨*proof*⟩

lemma *Prop-7-8-uniqueness*:
assumes *T ≠ PO* **and**
X ≠ Y **and**

Cong T X T Y **and**
Cong T X T Z **and**

X OnCircle PO P **and**
Y OnCircle PO P **and**
Z OnCircle PO P

shows $Z = X \lor Z = Y$
⟨*proof*⟩

lemma *chords-midpoints-col-par*:
assumes
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
D OnCircle PO P **and**
M Midpoint A B **and**
N Midpoint C D **and**
Col PO M N **and**
$\neg$ *Col PO A B* **and**
$\neg$ *Col PO C D*
shows *A B Par C D*
⟨*proof*⟩

lemma *onc3-mid2--ncol*:
assumes
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
A' *Midpoint A C* **and**
B' *Midpoint B C* **and**
$\neg$ *Col A B C*
shows $\neg$ *Col PO* A' B' $\lor A' = PO \lor B' = PO$
⟨*proof*⟩

lemma *onc4-cong2--eq*:
assumes $A \neq B$ **and**
$C \neq D$ **and**
$\neg$ *A B Par C D* **and**
A OnCircle PO P **and**
B OnCircle PO P **and**
C OnCircle PO P **and**
D OnCircle PO P **and**
Cong A X B X **and**
Cong C X D X
shows $PO = X$
⟨*proof*⟩

lemma *onc2--oreq*:
assumes *InterCCAt A B C D P Q* **and**
Z OnCircle A B **and**
Z OnCircle C D
shows $Z = P \lor Z = Q$
⟨*proof*⟩

end
end

theory *Tarski-Neutral-3D*

imports
Tarski-Neutral

begin

12 Tarski's axiom system for neutral geometry: 3D

12.1 Definitions

locale *Tarski-neutral-3D* = *Tarski-neutral-dimensionless* +

fixes *TS1* **and** *TS2* **and** *TS3* **and** *TS4*
assumes *lower-dim-3*: $\neg$ ($\exists$ *X*.
((*Bet TS1 TS2 X* $\vee$ *Bet TS2 X TS1* $\vee$ *Bet X TS1 TS2*) $\wedge$
(*Bet TS3 TS4 X* $\vee$ *Bet TS4 X TS3* $\vee$ *Bet X TS3 TS4*) $\vee$
(*Bet TS1 TS3 X* $\vee$ *Bet TS3 X TS1* $\vee$ *Bet X TS1 TS3*) $\wedge$
(*Bet TS2 TS4 X* $\vee$ *Bet TS4 X TS2* $\vee$ *Bet X TS2 TS4*) $\vee$
(*Bet TS1 TS4 X* $\vee$ *Bet TS4 X TS1* $\vee$ *Bet X TS1 TS4*) $\wedge$
(*Bet TS2 TS3 X* $\vee$ *Bet TS3 X TS2* $\vee$ *Bet X TS2 TS3*)))
assumes *upper-dim-3*: $\forall$ *A B C P Q R*.
$P \neq Q \wedge Q \neq R \wedge P \neq R \wedge$
Cong A P A Q $\wedge$ *Cong B P B Q* $\wedge$ *Cong C P C Q* $\wedge$
Cong A P A R $\wedge$ *Cong B P B R* $\wedge$ *Cong C P C R* $\longrightarrow$
(*Bet A B C* $\vee$ *Bet B C A* $\vee$ *Bet C A B*)

context *Tarski-neutral-3D*

begin

12.2 Propositions

lemma *not-coplanar-S1-S2-S3-S4*:
shows $\neg$ *Coplanar TS1 TS2 TS3 TS4*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-3D-Hilbert*

imports
Tarski-Neutral-Hilbert
Tarski-Neutral-3D

begin

13 Tarski neutral dimensionless - Hilbert

context *Tarski-neutral-3D*

begin

13.1 Definition

13.2 Propositions

lemma *lower-dim-3′*:
shows $\neg$ ($\exists$ *p*. *isPlane p* $\wedge$ *IncidentP TS1 p* $\wedge$ *IncidentP TS2 p* $\wedge$
IncidentP TS3 p $\wedge$ *IncidentP TS4 p*)
⟨*proof*⟩

end
end

theory *Tarski-Euclidean*

imports
Tarski-Neutral
Tarski-Postulate-Parallels

begin

14 Tarski Euclidean

14.1 Tarski's axiom system for Euclidean

locale *Tarski-Euclidean* = *Tarski-neutral-dimensionless* +
assumes *tarski-s-parallel-postulate*:
$\forall\ A\ B\ C\ D\ T.\ (Bet\ A\ D\ T \wedge Bet\ B\ D\ C \wedge A \neq D)$
$\longrightarrow$
$(\exists\ X\ Y.\ Bet\ A\ B\ X \wedge Bet\ A\ C\ Y \wedge Bet\ X\ T\ Y)$

context *Tarski-Euclidean*

begin

14.2 Definitions

14.3 Propositions

theorem *tarski-s-parallel-postulate-thm*:
shows *tarski-s-parallel-postulate*
⟨*proof*⟩

lemma *Post02*:
shows *Postulate02*
⟨*proof*⟩

theorem *playfair-s-postulate-thm*:
shows *playfair-s-postulate*
⟨*proof*⟩

lemma *Post03*:
shows *Postulate03*
⟨*proof*⟩

theorem *triangle-postulate-thm*:
shows *triangle-postulate*
⟨*proof*⟩

lemma *Post04*:
shows *Postulate04*
⟨*proof*⟩

theorem *bachmann-s-lotschnittaxiom-thm*:
shows *bachmann-s-lotschnittaxiom*
⟨*proof*⟩

lemma *Post05*:
shows *Postulate05*
⟨*proof*⟩

theorem *postulate-of-transitivity-of-parallelism-thm*:
shows *postulate-of-transitivity-of-parallelism*
⟨*proof*⟩

lemma *Post06*:
shows *Postulate06*
⟨*proof*⟩

theorem *midpoint-converse-postulate-thm*:
shows *midpoint-converse-postulate*
⟨*proof*⟩

lemma *Post07*:
shows *Postulate07*
⟨*proof*⟩

theorem *alternate-interior-angles-postulate-thm*:

shows *alternate-interior-angles-postulate*
⟨*proof*⟩

lemma *Post08*:
shows *Postulate08*
⟨*proof*⟩

theorem *consecutive-interior-angles-postulate-thm*:
shows *consecutive-interior-angles-postulate*
⟨*proof*⟩

lemma *Post09*:
shows *Postulate09*
⟨*proof*⟩

theorem *perpendicular-transversal-postulate-thm*:
shows *perpendicular-transversal-postulate*
⟨*proof*⟩

lemma *Post10*:
shows *Postulate10*
⟨*proof*⟩

theorem *postulate-of-parallelism-of-perpendicular-transversals-thm*:
shows *postulate-of-parallelism-of-perpendicular-transversals*
⟨*proof*⟩

lemma *Post11*:
shows *Postulate11*
⟨*proof*⟩

theorem *universal-posidonius-postulate-thm*:
shows *universal-posidonius-postulate*
⟨*proof*⟩

lemma *Post12*:
shows *Postulate12*
⟨*proof*⟩

theorem *alternative-playfair-s-postulate-thm*:
shows *alternative-playfair-s-postulate*
⟨*proof*⟩

lemma *Post13*:
shows *Postulate13*
⟨*proof*⟩

theorem *proclus-postulate-thm*:
shows *proclus-postulate*
⟨*proof*⟩

lemma *Post14*:
shows *Postulate14*
⟨*proof*⟩

theorem *alternative-proclus-postulate-thm*:
shows *alternative-proclus-postulate*
⟨*proof*⟩

lemma *Post15*:
shows *Postulate15*
⟨*proof*⟩

theorem *triangle-circumscription-principle-thm*:
shows *triangle-circumscription-principle*
⟨*proof*⟩

lemma *Post16*:
shows *Postulate16*
⟨*proof*⟩

theorem *inverse-projection-postulate-thm*:
shows *inverse-projection-postulate*
⟨*proof*⟩

lemma *Post17*:
shows *Postulate17*
⟨*proof*⟩

theorem *euclid-5-thm*:
shows *euclid-5*
⟨*proof*⟩

lemma *Post18*:
shows *Postulate18*
⟨*proof*⟩

theorem *strong-parallel-postulate-thm*:
shows *strong-parallel-postulate*
⟨*proof*⟩

lemma *Post19*:
shows *Postulate19*
⟨*proof*⟩

theorem *alternative-strong-parallel-postulate-thm*:
shows *alternative-strong-parallel-postulate*
⟨*proof*⟩

lemma *Post20*:
shows *Postulate20*
⟨*proof*⟩

theorem *euclid-s-parallel-postulate-thm*:
shows *euclid-s-parallel-postulate*
⟨*proof*⟩

lemma *Post21*:
shows *Postulate21*
⟨*proof*⟩

theorem *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights-thm*:
shows *postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights*
⟨*proof*⟩

lemma *Post22*:
shows *Postulate22*
⟨*proof*⟩

theorem *posidonius-postulate-thm*:
shows *posidonius-postulate*
⟨*proof*⟩

lemma *Post23*:
assumes *Postulate01*
shows *Postulate23*
⟨*proof*⟩

theorem *postulate-of-existence-of-similar-triangles-thm*:
shows *postulate-of-existence-of-similar-triangles*
⟨*proof*⟩

lemma *Post24*:
shows *Postulate24*
⟨*proof*⟩

theorem *thales-postulate-thm*:
shows *thales-postulate*
⟨*proof*⟩

lemma *Post25*:
shows *Postulate25*
⟨*proof*⟩

theorem *thales-converse-postulate-thm*:
shows *thales-converse-postulate*
⟨*proof*⟩

lemma *Post26*:
shows *Postulate26*
⟨*proof*⟩

theorem *existential-thales-postulate-thm*:
shows *existential-thales-postulate*
⟨*proof*⟩

lemma *Post27*:
shows *Postulate27*
⟨*proof*⟩

theorem *postulate-of-right-saccheri-quadrilaterals-thm*:
shows *postulate-of-right-saccheri-quadrilaterals*
⟨*proof*⟩

lemma *Post28*:
shows *Postulate28*
⟨*proof*⟩

theorem *postulate-of-existence-of-a-right-saccheri-quadrilateral-thm*:
shows *postulate-of-existence-of-a-right-saccheri-quadrilateral*
⟨*proof*⟩

lemma *Post29*:
shows *Postulate29*
⟨*proof*⟩

theorem *postulate-of-right-lambert-quadrilaterals-thm*:
shows *postulate-of-right-lambert-quadrilaterals*
⟨*proof*⟩

lemma *Post30*:
shows *Postulate30*
⟨*proof*⟩

theorem *postulate-of-existence-of-a-right-lambert-quadrilateral-thm*:
shows *postulate-of-existence-of-a-right-lambert-quadrilateral*
⟨*proof*⟩

lemma *Post31*:
shows *Postulate31*
⟨*proof*⟩

theorem *weak-inverse-projection-postulate-thm*:
shows *weak-inverse-projection-postulate*
⟨*proof*⟩

lemma *Post32*:
shows *Postulate32*

⟨*proof*⟩

theorem *weak-tarski-s-parallel-postulate-thm*:
shows *weak-tarski-s-parallel-postulate*
⟨*proof*⟩

lemma *Post33*:
shows *Postulate33*
⟨*proof*⟩

theorem *weak-triangle-circumscription-principle-thm*:
shows *weak-triangle-circumscription-principle*
⟨*proof*⟩

lemma *Post34*:
shows *Postulate34*
⟨*proof*⟩

theorem *legendre-s-parallel-postulate-thm*:
shows *legendre-s-parallel-postulate*
⟨*proof*⟩

theorem *TarskiPP*:
shows *tarski-s-parallel-postulate*
⟨*proof*⟩

lemma *parallel-uniqueness*:
assumes *A1 A2 Par B1 B2* **and**
Col P B1 B2 **and**
A1 A2 Par C1 C2 **and**
Col P C1 C2
shows *Col C1 B1 B2* ∧ *Col C2 B1 B2*
⟨*proof*⟩

lemma *par-trans*:
assumes *A1 A2 Par B1 B2* **and**
B1 B2 Par C1 C2
shows *A1 A2 Par C1 C2*
⟨*proof*⟩

lemma *l12-16*:
assumes *A1 A2 Par B1 B2* **and**
Coplanar B1 B2 C1 C2 **and**
X Inter A1 A2 C1 C2
shows ∃ *Y*. *Y Inter B1 B2 C1 C2*
⟨*proof*⟩

lemma *par-dec*:
shows *A B Par C D* ∨ ¬ *A B Par C D*
⟨*proof*⟩

lemma *par-not-par*:
assumes *A B Par C D* **and**
¬ *A B Par P Q*
shows ¬ *C D Par P Q*
⟨*proof*⟩

lemma *cop-par--inter*:
assumes *A B Par C D* **and**
¬ *A B Par P Q* **and**
Coplanar C D P Q
shows ∃ *Y*. *Col P Q Y* ∧ *Col C D Y*
⟨*proof*⟩

lemma *l12-19*:
assumes ¬ *Col A B C* **and**

$A\ B\ Par\ C\ D$ **and**
$B\ C\ Par\ D\ A$
shows $Cong\ A\ B\ C\ D \wedge Cong\ B\ C\ D\ A \wedge B\ D\ TS\ A\ C \wedge A\ C\ TS\ B\ D$
⟨*proof*⟩

lemma *l12-20-bis*:
assumes $A\ B\ Par\ C\ D$ **and**
$Cong\ A\ B\ C\ D$ **and**
$B\ D\ TS\ A\ C$
shows $B\ C\ Par\ D\ A \wedge Cong\ B\ C\ D\ A \wedge A\ C\ TS\ B\ D$
⟨*proof*⟩

lemma *l12-20*:
assumes $A\ B\ Par\ C\ D$ **and**
$Cong\ A\ B\ C\ D$ **and**
$A\ C\ TS\ B\ D$
shows $B\ C\ Par\ D\ A \wedge Cong\ B\ C\ D\ A \wedge A\ C\ TS\ B\ D$
⟨*proof*⟩

lemma *l12-21-a*:
assumes $A\ C\ TS\ B\ D$ **and**
$A\ B\ Par\ C\ D$
shows $B\ A\ C\ CongA\ D\ C\ A$
⟨*proof*⟩

lemma *l12-21*:
assumes $A\ C\ TS\ B\ D$
shows $B\ A\ C\ CongA\ D\ C\ A \longleftrightarrow A\ B\ Par\ C\ D$
⟨*proof*⟩

lemma *l12-22-a*:
assumes $P\ Out\ A\ C$ **and**
$P\ A\ OS\ B\ D$ **and**
$A\ B\ Par\ C\ D$
shows $B\ A\ P\ CongA\ D\ C\ P$
⟨*proof*⟩

lemma *l12-22*:
assumes $P\ Out\ A\ C$ **and**
$P\ A\ OS\ B\ D$
shows $B\ A\ P\ CongA\ D\ C\ P \longleftrightarrow A\ B\ Par\ C\ D$
⟨*proof*⟩

lemma *l12-23*:
assumes $\neg\ Col\ A\ B\ C$
shows $\exists\ B'\ C'.\ A\ C\ TS\ B\ B' \wedge A\ B\ TS\ C\ C' \wedge Bet\ B'\ A\ C' \wedge$
$A\ B\ C\ CongA\ B\ A\ C' \wedge A\ C\ B\ CongA\ C\ A\ B'$
⟨*proof*⟩

lemma *cop2-npar--inter*:
assumes $Coplanar\ A\ B\ X\ Y$ **and**
$Coplanar\ A'\ B'\ X\ Y$ **and**
$\neg\ A\ B\ Par\ A'\ B'$
shows $\exists\ P.\ Col\ P\ X\ Y \wedge (Col\ P\ A\ B \vee Col\ P\ A'\ B')$
⟨*proof*⟩

lemma *not-par-one-not-par*:
assumes $\neg\ A\ B\ Par\ A'\ B'$
shows $\neg\ A\ B\ Par\ X\ Y \vee \neg\ A'\ B'\ Par\ X\ Y$
⟨*proof*⟩

lemma *col-par-par-col*:
assumes $Col\ A\ B\ C$ **and**
$A\ B\ Par\ A'\ B'$ **and**
$B\ C\ Par\ B'\ C'$
shows $Col\ A'\ B'\ C'$

⟨*proof*⟩

lemma *cop-par-perp--perp*:
assumes *A B Par C D* **and**
A B Perp P Q **and**
Coplanar C D P Q
shows *C D Perp P Q*
⟨*proof*⟩

lemma *cop4-par-perp2--par*:
assumes *A B Par C D* **and**
A B Perp E F **and**
C D Perp G H **and**
Coplanar A B E G **and**
Coplanar A B E H **and**
Coplanar A B F G **and**
Coplanar A B F H
shows *E F Par G H*
⟨*proof*⟩

lemma *par-cong-mid-ts*:
assumes $A\ B\ ParStrict\ A'\ B'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$A\ A'\ TS\ B\ B'$
shows $\exists\ M.\ \ M\ Midpoint\ A\ A' \wedge M\ Midpoint\ B\ B'$
⟨*proof*⟩

lemma *par-cong-mid-os*:
assumes $A\ B\ ParStrict\ A'\ B'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$A\ A'\ OS\ B\ B'$
shows $\exists\ M.\ M\ Midpoint\ A\ B' \wedge M\ Midpoint\ B\ A'$
⟨*proof*⟩

lemma *par-strict-cong-mid*:
assumes $A\ B\ ParStrict\ A'\ B'$ **and**
$Cong\ A\ B\ A'\ B'$
shows $\exists\ M.\ (M\ Midpoint\ A\ A' \wedge M\ Midpoint\ B\ B') \vee (M\ Midpoint\ A\ B' \wedge M\ Midpoint\ B\ A')$
⟨*proof*⟩

lemma *par-strict-cong-mid1*:
assumes $A\ B\ ParStrict\ A'\ B'$ **and**
$Cong\ A\ B\ A'\ B'$
shows $(A\ A'\ TS\ B\ B' \wedge (\exists\ M.\ M\ Midpoint\ A\ A' \wedge M\ Midpoint\ B\ B')) \vee$
$(A\ A'\ OS\ B\ B' \wedge (\exists\ M.\ M\ Midpoint\ A\ B' \wedge M\ Midpoint\ B\ A'))$
⟨*proof*⟩

lemma *par-cong-mid*:
assumes $A\ B\ Par\ A'\ B'$ **and**
$Cong\ A\ B\ A'\ B'$
shows $\exists\ M.\ (M\ Midpoint\ A\ A' \wedge M\ Midpoint\ B\ B') \vee$
$(M\ Midpoint\ A\ B' \wedge M\ Midpoint\ B\ A')$
⟨*proof*⟩

lemma *ts-cong-par-cong-par*:
assumes $A\ A'\ TS\ B\ B'$ **and**
$Cong\ A\ B\ A'\ B'$ **and**
$A\ B\ Par\ A'\ B'$
shows $Cong\ A\ B'\ A'\ B \wedge A\ B'\ Par\ A'\ B$
⟨*proof*⟩

lemma *plgs-cong*:
assumes *ParallelogramStrict A B C D*
shows *Cong A B C D* ∧ *Cong A D B C*
⟨*proof*⟩

lemma *plg-cong*:
assumes *Parallelogram A B C D*
shows *Cong A B C D* ∧ *Cong A D B C*
⟨*proof*⟩

lemma *rmb-cong*:
assumes *Rhombus A B C D*
shows *Cong A B B C* ∧ *Cong A B C D* ∧ *Cong A B D A*
⟨*proof*⟩

lemma *rmb-per*:
assumes *M Midpoint A C* **and**
Rhombus A B C D
shows *Per A M D*
⟨*proof*⟩

lemma *per-rmb*:
assumes *Plg A B C D* **and**
M Midpoint A C **and**
Per A M B
shows *Rhombus A B C D*
⟨*proof*⟩

lemma *perp-rmb*:
assumes *Plg A B C D* **and**
A C Perp B D
shows *Rhombus A B C D*
⟨*proof*⟩

lemma *plg-conga1*:
assumes $A \neq B$ **and**
$A \neq C$ **and**
Plg A B C D
shows *B A C CongA D C A*
⟨*proof*⟩

lemma *os-cong-par-cong-par*:
assumes *A A′ OS B B′* **and**
Cong A B A′ B′ **and**
A B Par A′ B′
shows *Cong A A′ B B′* ∧ *A A′ Par B B′*
⟨*proof*⟩

lemma *plgs-permut*:
assumes *ParallelogramStrict A B C D*
shows *ParallelogramStrict B C D A*
⟨*proof*⟩

lemma *plg-permut*:
assumes *Parallelogram A B C D*
shows *Parallelogram B C D A*
⟨*proof*⟩

lemma *plgs-mid*:
assumes *ParallelogramStrict A B C D*
shows ∃ *M*. *M Midpoint A C* ∧ *M Midpoint B D*
⟨*proof*⟩

lemma *plg-mid*:
assumes *Parallelogram A B C D*
shows ∃ *M*. *M Midpoint A C* ∧ *M Midpoint B D*
⟨*proof*⟩

lemma *plg-mid-2*:
assumes *Parallelogram A B C D* **and**
I Midpoint A C

shows *I Midpoint B D*
⟨*proof*⟩

lemma *plgs-not-comm-1*:
assumes *ParallelogramStrict A B C D*
shows ¬ *ParallelogramStrict A B D C*
⟨*proof*⟩

lemma *plgs-not-comm-2*:
assumes *ParallelogramStrict A B C D*
shows ¬ *ParallelogramStrict B A C D*
⟨*proof*⟩

lemma *plgs-not-comm*:
assumes *ParallelogramStrict A B C D*
shows ¬ *ParallelogramStrict A B D C* ∧ ¬ *ParallelogramStrict B A C D*
⟨*proof*⟩

lemma *plg-not-comm-R1*:
assumes $A \neq B$ **and**
Parallelogram A B C D
shows ¬ *Parallelogram A B D C*
⟨*proof*⟩

lemma *plg-not-comm-R2*:
assumes $A \neq B$ **and**
Parallelogram A B C D
shows ¬ *Parallelogram B A C D*
⟨*proof*⟩

lemma *plg-not-comm*:
assumes $A \neq B$ **and**
Parallelogram A B C D
shows ¬ *Parallelogram A B D C* ∧ ¬ *Parallelogram B A C D*
⟨*proof*⟩

lemma *parallelogram-to-plg*:
assumes *Parallelogram A B C D*
shows *Plg A B C D*
⟨*proof*⟩

lemma *parallelogram-equiv-plg*:
shows *Parallelogram A B C D* ⟷ *Plg A B C D*
⟨*proof*⟩

lemma *plg-conga*:
assumes $A \neq B$ **and**

$B \neq C$ **and**
Parallelogram A B C D
shows *A B C CongA C D A* ∧ *B C D CongA D A B*
⟨*proof*⟩

lemma *half-plgs-R1*:
assumes *ParallelogramStrict A B C D* **and**
P Midpoint A B **and**
Q Midpoint C D **and**
M Midpoint A C
shows *P Q Par A D* ∧ *Cong A D P Q*
⟨*proof*⟩

lemma *half-plgs-R2*:
assumes *ParallelogramStrict A B C D* **and**
P Midpoint A B **and**
Q Midpoint C D **and**
M Midpoint A C

shows *M Midpoint P Q*
⟨*proof*⟩

lemma *half-plgs*:
assumes *ParallelogramStrict A B C D* **and**
P Midpoint A B **and**
Q Midpoint C D **and**
M Midpoint A C
shows *P Q Par A D* ∧ *M Midpoint P Q* ∧ *Cong A D P Q*
⟨*proof*⟩

lemma *plgs-two-sides*:
assumes *ParallelogramStrict A B C D*
shows *A C TS B D* ∧ *B D TS A C*
⟨*proof*⟩

lemma *plgs-par-strict*:
assumes *ParallelogramStrict A B C D*
shows *A B ParStrict C D* ∧ *A D ParStrict B C*
⟨*proof*⟩

lemma *plgs-half-plgs-aux*:
assumes *ParallelogramStrict A B C D* **and**
P Midpoint A B **and**
Q Midpoint C D
shows *ParallelogramStrict A P Q D*
⟨*proof*⟩

lemma *plgs-comm2*:
assumes *ParallelogramStrict A B C D*
shows *ParallelogramStrict B A D C*
⟨*proof*⟩

lemma *plgf-comm2*:
assumes *ParallelogramFlat A B C D*
shows *ParallelogramFlat B A D C*
⟨*proof*⟩

lemma *plg-comm2*:
assumes *Parallelogram A B C D*
shows *Parallelogram B A D C*
⟨*proof*⟩

lemma *par-preserves-conga-os*:
assumes *A B Par C D* **and**
Bet A D P **and**
$D \neq P$ **and**
A D OS B C
shows *B A P CongA C D P*
⟨*proof*⟩

lemma *cong3-par2-par*:
assumes $A \neq C$ **and**
B A C Cong3 B' A' C' **and**
B A Par B' A' **and**
B C Par B' C'
shows *A C Par* A' C' ∨ ¬ *B* B' *ParStrict A* A' ∨ ¬ *B* B' *ParStrict C* C'
⟨*proof*⟩

lemma *square-perp-rectangle*:
assumes *Rectangle A B C D* **and**
A C Perp B D
shows *Square A B C D*
⟨*proof*⟩

lemma *plgs-half-plgs*:

assumes *ParallelogramStrict A B C D* **and**
P Midpoint A B **and**
Q Midpoint C D
shows *ParallelogramStrict A P Q D* ∧ *ParallelogramStrict B P Q C*
⟨*proof*⟩

lemma *parallel-2-plg*:
assumes ¬ *Col A B C* **and**
A B Par C D **and**
A D Par B C
shows *ParallelogramStrict A B C D*
⟨*proof*⟩

lemma *par-2-plg*:
assumes ¬ *Col A B C* **and**
A B Par C D **and**
A D Par B C
shows *Parallelogram A B C D*
⟨*proof*⟩

lemma *plg-cong-1*:
assumes *Parallelogram A B C D*
shows *Cong A B C D*
⟨*proof*⟩

lemma *plg-cong-2*:
assumes *Parallelogram A B C D*
shows *Cong A D B C*
⟨*proof*⟩

lemma *plgs-cong-1*:
assumes *ParallelogramStrict A B C D*
shows *Cong A B C D*
⟨*proof*⟩

lemma *plgs-cong-2*:
assumes *ParallelogramStrict A B C D*
shows *Cong A D B C*
⟨*proof*⟩

lemma *Plg-perm*:
assumes *Parallelogram A B C D*
shows *Parallelogram A B C D* ∧ *Parallelogram B C D A* ∧
Parallelogram C D A B ∧*Parallelogram D A B C* ∧
Parallelogram A D C B ∧ *Parallelogram D C B A* ∧
Parallelogram C B A D ∧ *Parallelogram B A D C*
⟨*proof*⟩

lemma *plg-not-comm-1*:
assumes $A \neq B$ **and**
Parallelogram A B C D
shows ¬ *Parallelogram A B D C*
⟨*proof*⟩

lemma *plg-not-comm-2*:
assumes $A \neq B$ **and**
Parallelogram A B C D
shows ¬ *Parallelogram B A C D*
⟨*proof*⟩

lemma *parallelogram-strict-midpoint*:
assumes *ParallelogramStrict A B C D* **and**
Col I A C **and**
Col I B D
shows *I Midpoint A C* ∧ *I Midpoint B D*

⟨*proof*⟩

lemma *rmb-perp*:
assumes $A \neq C$ **and**
$B \neq D$ **and**
Rhombus A B C D
shows *A C Perp B D*
⟨*proof*⟩

lemma *rect-permut*:
assumes *Rectangle A B C D*
shows *Rectangle B C D A*
⟨*proof*⟩

lemma *rect-comm2*:
assumes *Rectangle A B C D*
shows *Rectangle B A D C*
⟨*proof*⟩

lemma *rect-per1*:
assumes *Rectangle A B C D*
shows *Per B A D*
⟨*proof*⟩

lemma *rect-per2*:
assumes *Rectangle A B C D*
shows *Per A B C*
⟨*proof*⟩

lemma *rect-per3*:
assumes *Rectangle A B C D*
shows *Per B C D*
⟨*proof*⟩

lemma *rect-per4*:
assumes *Rectangle A B C D*
shows *Per A D C*
⟨*proof*⟩

lemma *plg-per-rect1*:
assumes *Plg A B C D* **and**
Per D A B
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *plg-per-rect2*:
assumes *Plg A B C D* **and**
Per C B A
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *plg-per-rect3*:
assumes *Plg A B C D* **and**
Per A D C
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *plg-per-rect4*:
assumes *Plg A B C D* **and**
Per B C D
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *plg-per-rect*:
assumes *Plg A B C D* **and**
Per D A B ∨ *Per C B A* ∨ *Per A D C* ∨ *Per B C D*

shows *Rectangle A B C D*
⟨*proof*⟩

lemma *rect-per*:
assumes *Rectangle A B C D*
shows *Per B A D* ∧ *Per A B C* ∧ *Per B C D* ∧ *Per A D C*
⟨*proof*⟩

lemma *plgf-rect-id*:
assumes *ParallelogramFlat A B C D* **and**
Rectangle A B C D
shows (*A* = *D* ∧ *B* = *C*) ∨ (*A* = *B* ∧ *D* = *C*)
⟨*proof*⟩

lemma *cop-perp3--perp*:
assumes *Coplanar A B C D* **and**
A B Perp B C **and**
B C Perp C D **and**
C D Perp D A
shows *D A Perp A B*
⟨*proof*⟩

lemma *cop-perp3--rect*:
assumes *Coplanar A B C D* **and**
A B Perp B C **and**
B C Perp C D **and**
C D Perp D A
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *conga-to-par-os*:
assumes *Bet A D P* **and**
A D OS B C **and**
B A P CongA C D P
shows *A B Par C D*
⟨*proof*⟩

lemma *plg-par*:
assumes *A* ≠ *B* **and**
B ≠ *C* **and**
Parallelogram A B C D
shows *A B Par C D* ∧ *A D Par B C*
⟨*proof*⟩

lemma *plg-par-1*:
assumes *A* ≠ *B* **and**
B ≠ *C* **and**
Parallelogram A B C D
shows *A B Par C D*
⟨*proof*⟩

lemma *plg-par-2*:
assumes *A* ≠ *B* **and**
B ≠ *C* **and**
Parallelogram A B C D
shows *A D Par B C*
⟨*proof*⟩

lemma *plgs-pars-1*:
assumes *ParallelogramStrict A B C D*
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *plgs-pars-2*:
assumes *ParallelogramStrict A B C D*
shows *A D ParStrict B C*

⟨*proof*⟩

lemma *par-cong-cong*:
assumes *A B Par C D* **and**
Cong A B C D
shows *Cong A C B D* ∨ *Cong A D B C*
⟨*proof*⟩

lemma *col-cong-cong*:
assumes *Col A B C* **and**
Col A B D **and**
Cong A B C D
shows *Cong A C B D* ∨ *Cong A D B C*
⟨*proof*⟩

lemma *par-cong-cop*:
assumes *A B Par C D*
shows *Coplanar A B C D*
⟨*proof*⟩

lemma *par-cong-plg* :
assumes *A B Par C D* **and**
Cong A B C D
shows *Plg A B C D* ∨ *Plg A B D C*
⟨*proof*⟩

lemma *par-cong-plg-2* :
assumes *A B Par C D* **and**
Cong A B C D
shows *Parallelogram A B C D* ∨ *Parallelogram A B D C*
⟨*proof*⟩

lemma *par-cong3-rect*:
assumes $A \neq C \vee B \neq D$ **and**
A B Par C D **and**
Cong A B C D **and**
Cong A D B C **and**
Cong A C B D
shows *Rectangle A B C D* ∨ *Rectangle A B D C*
⟨*proof*⟩

lemma *pars-par-pars*:
assumes *A B ParStrict C D* **and**
A D Par B C
shows *A D ParStrict B C*
⟨*proof*⟩

lemma *pars-par-plg*:
assumes *A B ParStrict C D* **and**
A D Par B C
shows *Plg A B C D*
⟨*proof*⟩

lemma *not-par-pars-not-cong*:
assumes *P Out A B* **and**
P Out A' B' **and**
A A' ParStrict B B'
shows ¬ *Cong A A' B B'*
⟨*proof*⟩

lemma *plg-uniqueness*:
assumes *Parallelogram A B C D* **and**
Parallelogram A B C D'
shows $D = D'$
⟨*proof*⟩

lemma *plgs-trans-trivial*:
assumes *ParallelogramStrict A B C D* **and**
ParallelogramStrict C D A B'
shows *Parallelogram A B B' A*
⟨*proof*⟩

lemma *par-strict-trans*:
assumes *A B ParStrict C D* **and**
C D ParStrict E F
shows *A B Par E F*
⟨*proof*⟩

lemma *plgs-pseudo-trans*:
assumes *ParallelogramStrict A B C D* **and**
ParallelogramStrict C D E F
shows *Parallelogram A B F E*
⟨*proof*⟩

lemma *plgf-plgs-trans*:
assumes $A \neq B$ **and**
ParallelogramFlat A B C D **and**
ParallelogramStrict C D E F
shows *ParallelogramStrict A B F E*
⟨*proof*⟩

lemma *plgf-plgf-plgf*:
assumes $A \neq B$ **and**
ParallelogramFlat A B C D **and**
ParallelogramFlat C D E F
shows *ParallelogramFlat A B F E*
⟨*proof*⟩

lemma *plg-pseudo-trans*:
assumes *Parallelogram A B C D* **and**
Parallelogram C D E F
shows *Parallelogram A B F E* $\vee$ $(A = B \wedge C = D \wedge E = F \wedge A = E)$
⟨*proof*⟩

lemma *Square-Rhombus*:
assumes *Square A B C D*
shows *Rhombus A B C D*
⟨*proof*⟩

lemma *plgs-in-angle*:
assumes *ParallelogramStrict A B C D*
shows *D InAngle A B C*
⟨*proof*⟩

lemma *par-par-cong-cong-parallelogram*:
assumes $B \neq D$ **and**
Cong A B C D **and**
Cong B C D A **and**
B C Par A D **and**
A B Par C D
shows *Parallelogram A B C D*
⟨*proof*⟩

lemma *degenerated-rect-eq*:
assumes *Rectangle A B B C*
shows $A = C$
⟨*proof*⟩

lemma *rect-2-rect*:
assumes $A \neq B$ **and**
Rectangle A B C1 D1 **and**

Rectangle A B C2 D2
shows *Rectangle C1 D1 D2 C2*
⟨*proof*⟩

lemma *ncol123-plg--plgs*:
assumes ¬ *Col A B C* **and**
Parallelogram A B C D
shows *ParallelogramStrict A B C D*
⟨*proof*⟩

lemma *ncol124-plg--plgs*:
assumes ¬ *Col A B D* **and**
Parallelogram A B C D
shows *ParallelogramStrict A B C D*
⟨*proof*⟩

lemma *ncol134-plg--plgs*:
assumes ¬ *Col A C D* **and**
Parallelogram A B C D
shows *ParallelogramStrict A B C D*
⟨*proof*⟩

lemma *ncol234-plg--plgs*:
assumes ¬ *Col B C D* **and**
Parallelogram A B C D
shows *ParallelogramStrict A B C D*
⟨*proof*⟩

lemma *ncol123-plg--pars1234*:
assumes ¬ *Col A B C* **and**
Parallelogram A B C D
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *ncol124-plg--pars1234*:
assumes ¬ *Col A B D* **and**
Parallelogram A B C D
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *ncol134-plg--pars1234*:
assumes ¬ *Col A C D* **and**
Parallelogram A B C D
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *ncol234-plg--pars1234*:
assumes ¬ *Col B C D* **and**
Parallelogram A B C D
shows *A B ParStrict C D*
⟨*proof*⟩

lemma *ncol123-plg--pars1423*:
assumes ¬ *Col A B C* **and**
Parallelogram A B C D
shows *A D ParStrict B C*
⟨*proof*⟩

lemma *ncol124-plg--pars1423*:
assumes ¬ *Col A B D* **and**
Parallelogram A B C D
shows *A D ParStrict B C*
⟨*proof*⟩

lemma *ncol134-plg--pars1423*:
assumes ¬ *Col A C D* **and**

Parallelogram A B C D
shows *A D ParStrict B C*
⟨*proof*⟩

lemma *ncol234-plg--pars1423*:
assumes ¬ *Col B C D* **and**
Parallelogram A B C D
shows *A D ParStrict B C*
⟨*proof*⟩

lemma *sac-plg*:
assumes *Saccheri A B C D*
shows *Parallelogram A B C D*
⟨*proof*⟩

lemma *sac-rectangle*:
assumes *Saccheri A B C D*
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *exists-square*:
assumes $A \neq B$
shows ∃ *C D*. *Square A B C D*
⟨*proof*⟩

lemma *euclidT*:
assumes *Bet A D T* **and**
Bet B D C **and**
$A \neq D$
shows ∃ *X Y*. *Bet A B X* ∧ *Bet A C Y* ∧ *Bet X T Y*
⟨*proof*⟩

lemma *eqv-refl*:
shows *A B EqV A B*
⟨*proof*⟩

lemma *eqv-sym*:
assumes *A B EqV C D*
shows *C D EqV A B*
⟨*proof*⟩

lemma *eqv-trans*:
assumes *A B EqV C D* **and**
C D EqV E F
shows *A B EqV E F*
⟨*proof*⟩

lemma *eqv-comm*:
assumes *A B EqV C D*
shows *B A EqV D C*
⟨*proof*⟩

lemma *vector-construction*:
shows ∃ *D*. *A B EqV C D*
⟨*proof*⟩

lemma *vector-construction-uniqueness*:
assumes *A B EqV C D* **and**
A B EqV C D'
shows $D = D'$
⟨*proof*⟩

lemma *null-vector*:
  assumes $A\ A\ EqV\ B\ C$
  shows $B = C$
  ⟨*proof*⟩

lemma *vector-uniqueness*:
  assumes $A\ B\ EqV\ A\ C$
  shows $B = C$
  ⟨*proof*⟩

lemma *eqv-trivial*:
  shows $A\ A\ EqV\ B\ B$
  ⟨*proof*⟩

lemma *eqv-permut*:
  assumes $A\ B\ EqV\ C\ D$
  shows $A\ C\ EqV\ B\ D$
  ⟨*proof*⟩

lemma *eqv-par*:
  assumes $A \neq B$ **and**
    $A\ B\ EqV\ C\ D$
  shows $A\ B\ Par\ C\ D$
  ⟨*proof*⟩

lemma *eqv-opp-null*:
  assumes $A\ B\ EqV\ B\ A$
  shows $A = B$
  ⟨*proof*⟩

lemma *eqv-sum*:
  assumes $A\ B\ EqV\ A'\ B'$ **and**
    $B\ C\ EqV\ B'\ C'$
  shows $A\ C\ EqV\ A'\ C'$
  ⟨*proof*⟩

lemma *null-sum*:
  shows $A\ B\ B\ A\ SumV\ C\ C$
  ⟨*proof*⟩

lemma *chasles*:
  shows $A\ B\ B\ C\ SumV\ A\ C$
  ⟨*proof*⟩

lemma *eqv-mid* :
  assumes $A\ B\ EqV\ B\ C$
  shows $B\ Midpoint\ A\ C$
  ⟨*proof*⟩

lemma *mid-eqv*:
  assumes $A\ Midpoint\ B\ C$
  shows $B\ A\ EqV\ A\ C$
  ⟨*proof*⟩

lemma *sum-sym*:
  assumes $A\ B\ C\ D\ SumV\ E\ F$
  shows $C\ D\ A\ B\ SumV\ E\ F$
⟨*proof*⟩

lemma *opposite-sum*:
  assumes $A\ B\ C\ D\ SumV\ E\ F$
  shows $B\ A\ D\ C\ SumV\ F\ E$
⟨*proof*⟩

lemma *null-sum-eq*:
  assumes $A\ B\ B\ C\ SumV\ D\ D$

shows $A = C$
⟨*proof*⟩

lemma *is-to-ise*:
assumes $A\ B\ C\ D\ SumV\ E\ F$
shows $A\ B\ C\ D\ SumVExists\ E\ F$
⟨*proof*⟩

lemma *ise-to-is*:
assumes $A\ B\ C\ D\ SumVExists\ E\ F$
shows $A\ B\ C\ D\ SumV\ E\ F$
⟨*proof*⟩

lemma *sum-exists*:
shows $\exists\ E\ F.\ A\ B\ C\ D\ SumV\ E\ F$
⟨*proof*⟩

lemma *sum-uniqueness-pappus*:

assumes $A\ B\ C\ D\ SumV\ E\ F$ **and**
$A\ B\ C\ D\ SumV\ E'\ F'$
shows $E\ F\ EqV\ E'\ F'$
⟨*proof*⟩

lemma *same-dir-refl*:
shows $A\ B\ SameDir\ A\ B$
⟨*proof*⟩

lemma *same-dir-ts*:
assumes $A\ B\ SameDir\ C\ D$
shows $\exists\ M.\ Bet\ A\ M\ D \wedge Bet\ B\ M\ C$
⟨*proof*⟩

lemma *one-side-col-out*:
assumes $Col\ A\ X\ Y$ **and**
$A\ B\ OS\ X\ Y$
shows $A\ Out\ X\ Y$
⟨*proof*⟩

lemma *par-ts-same-dir*:
assumes $A\ B\ ParStrict\ C\ D$ **and**
$\exists\ M.\ Bet\ A\ M\ D \wedge Bet\ B\ M\ C$
shows $A\ B\ SameDir\ C\ D$
⟨*proof*⟩

lemma *same-dir-out*:
assumes $A\ B\ SameDir\ A\ C$
shows $A\ Out\ B\ C \vee (A = B \wedge A = C)$
⟨*proof*⟩

lemma *same-dir-out1*:
assumes $A\ B\ SameDir\ B\ C$
shows $A\ Out\ B\ C \vee (A = B \wedge A = C)$
⟨*proof*⟩

lemma *same-dir-null*:
assumes $A\ A\ SameDir\ B\ C$
shows $B = C$
⟨*proof*⟩

lemma *plgs-out-plgs*:
assumes $ParallelogramStrict\ A\ B\ C\ D$ **and**
$A\ Out\ B\ B'$ **and**
$D\ Out\ C\ C'$ **and**
$Cong\ A\ B'\ D\ C'$
shows $ParallelogramStrict\ A\ B'\ C'\ D$

⟨*proof*⟩

lemma *plgs-plgs-bet*:
assumes *ParallelogramStrict A B C D* **and**
Bet A B B′ **and**
ParallelogramStrict A B′ C′ D
shows *Bet D C C′*
⟨*proof*⟩

lemma *plgf-plgf-bet*:
assumes *ParallelogramFlat A B C D* **and**
Bet A B B′ **and**
ParallelogramFlat A B′ C′ D
shows *Bet D C C′*
⟨*proof*⟩

lemma *plg-plg-bet*:
assumes *Parallelogram A B C D* **and**
Bet A B B′ **and**
Parallelogram A B′ C′ D
shows *Bet D C C′*
⟨*proof*⟩

lemma *plgf-out-plgf*:
assumes *ParallelogramFlat A B C D* **and**
A Out B B′ **and**
D Out C C′ **and**
Cong A B′ D C′
shows *ParallelogramFlat A B′ C′ D*
⟨*proof*⟩

lemma *plg-out-plg*:
assumes *Parallelogram A B C D* **and**
A Out B B′ **and**
D Out C C′ **and**
Cong A B′ D C′
shows *Parallelogram A B′ C′ D*
⟨*proof*⟩

lemma *same-dir-sym*:
assumes *A B SameDir C D*
shows *C D SameDir A B*
⟨*proof*⟩

lemma *same-dir-trans*:
assumes *A B SameDir C D* **and**
C D SameDir E F
shows *A B SameDir E F*
⟨*proof*⟩

lemma *same-dir-comm*:
assumes *A B SameDir C D*
shows *B A SameDir D C*
⟨*proof*⟩

lemma *bet-same-dir1*:
assumes $A \neq B$ **and**

Bet A B C
shows *A B SameDir A C*
⟨*proof*⟩

lemma *bet-same-dir2*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Bet A B C

shows *A B SameDir B C*
⟨*proof*⟩

lemma *plg-opp-dir*:
assumes *Parallelogram A B C D*
shows *A B SameDir D C*
⟨*proof*⟩

lemma *same-dir-dec*:
shows *A B SameDir C D* ∨ ¬ *A B SameDir C D*
⟨*proof*⟩

lemma *same-or-opp-dir*:
assumes *A B Par C D*
shows *A B SameDir C D* ∨ *A B OppDir C D*
⟨*proof*⟩

lemma *same-dir-id*:
assumes *A B SameDir B A*
shows $A = B$
⟨*proof*⟩

lemma *opp-dir-id*:
assumes *A B OppDir A B*
shows $A = B$
⟨*proof*⟩

lemma *same-dir-to-null*:
assumes *A B SameDir C D* **and**
A B SameDir D C
shows $A = B \wedge C = D$
⟨*proof*⟩

lemma *opp-dir-to-null*:
assumes *A B OppDir C D* **and**
A B OppDir D C
shows $A = B \wedge C = D$
⟨*proof*⟩

lemma *same-not-opp-dir*:
assumes $A \neq B$ **and**
A B SameDir C D
shows ¬ *A B OppDir C D*
⟨*proof*⟩

lemma *opp-not-same-dir*:
assumes $A \neq B$ **and**
A B OppDir C D
shows ¬ *A B SameDir C D*
⟨*proof*⟩

lemma *vector-same-dir-cong*:
assumes $A \neq B$ **and**
$C \neq D$
shows ∃ *X Y*. *A B SameDir X Y* ∧ *Cong X Y C D*
⟨*proof*⟩

lemma *project-par*:
assumes *P P′ Proj A B X Y* **and**
Q Q′ Proj A B X Y **and**
P Q Par X Y
shows $P' = Q'$
⟨*proof*⟩

lemma *ker-col*:

assumes $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$Q\ P'$ *Proj* $A\ B\ X\ Y$
shows *Col* $P\ Q\ P'$
⟨*proof*⟩

lemma *ker-par*:
assumes $P \neq Q$ **and**
$P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$Q\ P'$ *Proj* $A\ B\ X\ Y$
shows $P\ Q$ *Par* $X\ Y$
⟨*proof*⟩

lemma *project-uniqueness*:
assumes $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$P\ Q'$ *Proj* $A\ B\ X\ Y$
shows $P' = Q'$
⟨*proof*⟩

lemma *project-col-eq*:
assumes $P \neq P'$ **and**
Col $P\ Q\ P'$ **and**
$P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$Q\ Q'$ *Proj* $A\ B\ X\ Y$
shows $P' = Q'$
⟨*proof*⟩

lemma *project-preserves-bet*:
assumes *Bet* $P\ Q\ R$ **and**
$P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$Q\ Q'$ *Proj* $A\ B\ X\ Y$ **and**
$R\ R'$ *Proj* $A\ B\ X\ Y$
shows *Bet* $P'\ Q'\ R'$
⟨*proof*⟩

lemma *triangle-par*:
assumes $\neg$ *Col* $A\ B\ C$ **and**
$A\ B$ *Par* $A'\ B'$ **and**
$B\ C$ *Par* $B'\ C'$ **and**
$A\ C$ *Par* $A'\ C'$
shows $A\ B\ C$ *CongA* $A'\ B'\ C'$
⟨*proof*⟩

lemma *par3-conga3* :
assumes $\neg$ *Col* $A\ B\ C$ **and**
$A\ B$ *Par* $A'\ B'$ **and**
$B\ C$ *Par* $B'\ C'$ **and**
$A\ C$ *Par* $A'\ C'$
shows $A\ B\ C$ *CongA3* $A'\ B'\ C'$
⟨*proof*⟩

lemma *project-par-eqv*:
assumes $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$Q\ Q'$ *Proj* $A\ B\ X\ Y$ **and**
$P\ Q$ *Par* $A\ B$
shows $P\ Q$ *EqV* $P'\ Q'$
⟨*proof*⟩

lemma *eqv-project-eq-eq*:
assumes $P\ Q$ *EqV* $R\ S$ **and**
$P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$Q\ Q'$ *Proj* $A\ B\ X\ Y$ **and**
$R\ P'$ *Proj* $A\ B\ X\ Y$ **and**
$S\ S'$ *Proj* $A\ B\ X\ Y$
shows $Q' = S'$
⟨*proof*⟩

lemma *eqv-eq-project*:
assumes $P\ Q\ EqV\ R\ S$ **and**
$P\ P'\ Proj\ A\ B\ X\ Y$ **and**
$Q\ Q'\ Proj\ A\ B\ X\ Y$ **and**
$R\ P'\ Proj\ A\ B\ X\ Y$
shows $S\ Q'\ Proj\ A\ B\ X\ Y$
⟨*proof*⟩

lemma *eqv-cong*:
assumes $A\ B\ EqV\ C\ D$
shows $Cong\ A\ B\ C\ D$
⟨*proof*⟩

lemma *project-preserves-eqv*:
assumes $P\ Q\ EqV\ R\ S$ **and**
$P\ P'\ Proj\ A\ B\ X\ Y$ **and**
$Q\ Q'\ Proj\ A\ B\ X\ Y$ **and**
$R\ R'\ Proj\ A\ B\ X\ Y$ **and**
$S\ S'\ Proj\ A\ B\ X\ Y$
shows $P'\ Q'\ EqV\ R'\ S'$
⟨*proof*⟩

lemma *cop-par--perp2*:
assumes $Coplanar\ A\ B\ C\ P$ **and**
$A\ B\ Par\ C\ D$
shows $P\ Perp2\ A\ B\ C\ D$
⟨*proof*⟩

lemma *l13-11*:
assumes $\neg\ Col\ PO\ A\ A'$ **and**
$B \neq PO$ **and**
$C \neq PO$ **and**
$Col\ PO\ A\ B$ **and**
$Col\ PO\ B\ C$ **and**
$B' \neq PO$ **and**
$C' \neq PO$ **and**
$Col\ PO\ A'\ B'$ **and**
$Col\ PO\ B'\ C'$ **and**
$B\ C'\ Par\ C\ B'$ **and**
$C\ A'\ Par\ A\ C'$
shows $A\ B'\ Par\ B\ A'$
⟨*proof*⟩

lemma *l13-14*:
assumes $PO\ A\ ParStrict\ O'\ A'$ **and**
$Col\ PO\ A\ B$ **and**
$Col\ PO\ B\ C$ **and**
$Col\ PO\ A\ C$ **and**
$Col\ O'\ A'\ B'$ **and**
$Col\ O'\ B'\ C'$ **and**
$Col\ O'\ A'\ C'$ **and**
$A\ C'\ Par\ A'\ C$ **and**
$B\ C'\ Par\ B'\ C$
shows $A\ B'\ Par\ A'\ B$
⟨*proof*⟩

lemma *l13-15-1*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$\neg\ PO\ B\ Par\ A\ C$ **and**
$Coplanar\ PO\ B\ A\ C$ **and**
$A\ B\ ParStrict\ A'\ B'$ **and**
$A\ C\ ParStrict\ A'\ C'$**and**
$Col\ PO\ A\ A'$ **and**
$Col\ PO\ B\ B'$ **and**
$Col\ PO\ C\ C'$
shows $B\ C\ Par\ B'\ C'$

⟨*proof*⟩

lemma *l13-15-2-aux*:
  assumes ¬ *Col A B C* **and**
    ¬ *PO A Par B C* **and**
    PO B Par A C **and**
    A B ParStrict A′ B′ **and**
    A C ParStrict A′ C′ **and**
    Col PO A A′ **and**
    Col PO B B′ **and**
    Col PO C C′
  shows *B C Par B′ C′*
⟨*proof*⟩

lemma *l13-15-2*:
  assumes ¬ *Col A B C* **and**
    PO B Par A C **and**
    A B ParStrict A′ B′ **and**
    A C ParStrict A′ C′ **and**
    Col PO A A′ **and**
    Col PO B B′ **and**
    Col PO C C′
  shows *B C Par B′ C′*
⟨*proof*⟩

lemma *l13-15*:
  assumes ¬ *Col A B C* **and**
    Coplanar PO B A C **and**
    A B ParStrict A′ B′ **and**
    A C ParStrict A′ C′ **and**
    Col PO A A′ **and**
    Col PO B B′ **and**
    Col PO C C′
  shows *B C Par B′ C′*
  ⟨*proof*⟩

lemma *l13-15-par*:
  assumes ¬ *Col A B C* **and**
    A B ParStrict A′ B′ **and**
    A C ParStrict A′ C′ **and**
    A A′ Par B B′ **and**
    A A′ Par C C′
  shows *B C Par B′ C′*
⟨*proof*⟩

lemma *l13-18-2*:
  assumes ¬ *Col A B C* **and**
    A B ParStrict A′ B′ **and**
    A C ParStrict A′ C′ **and**
    B C ParStrict B′ C′ **and**
    Col PO A A′ **and**
    Col PO B B′
  shows *Col PO C C′*
⟨*proof*⟩

lemma *l13-18-3-R1*:
  assumes ¬ *Col A B C* **and**
    A B ParStrict A′ B′ **and**
    A C ParStrict A′ C′ **and**
    B C ParStrict B′ C′ **and**
    A A′ Par B B′
  shows *C C′ Par A A′*
⟨*proof*⟩

lemma *l13-18-3-R2*:
  assumes ¬ *Col A B C* **and**

$A\ B\ ParStrict\ A'\ B'$ **and**
$A\ C\ ParStrict\ A'\ C'$ **and**
$B\ C\ ParStrict\ B'\ C'$ **and**
$A\ A'\ Par\ B\ B'$
shows $C\ C'\ Par\ B\ B'$
⟨*proof*⟩

lemma *l13-18-3*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$A\ B\ ParStrict\ A'\ B'$ **and**
$A\ C\ ParStrict\ A'\ C'$ **and**
$B\ C\ ParStrict\ B'\ C'$ **and**
$A\ A'\ Par\ B\ B'$
shows $C\ C'\ Par\ A\ A' \wedge C\ C'\ Par\ B\ B'$
⟨*proof*⟩

lemma *l13-18*:
assumes $\neg\ Col\ A\ B\ C$ **and**
$A\ B\ ParStrict\ A'\ B'$ **and**
$A\ C\ ParStrict\ A'\ C'$
shows $(B\ C\ ParStrict\ B'\ C' \wedge Col\ PO\ A\ A' \wedge Col\ PO\ B\ B' \longrightarrow Col\ PO\ C\ C')$
$\wedge ((B\ C\ ParStrict\ B'\ C' \wedge A\ A'\ Par\ B\ B') \longrightarrow (C\ C'\ Par\ A\ A' \wedge C\ C'\ Par\ B\ B'))$
$\wedge (A\ A'\ Par\ B\ B' \wedge A\ A'\ Par\ C\ C' \longrightarrow B\ C\ Par\ B'\ C')$
⟨*proof*⟩

lemma *l13-19-aux*:
assumes $\neg\ Col\ PO\ A\ B$ **and**
$A \neq A'$ **and**
$A \neq C$ **and**
$PO \neq A$ **and**
$PO \neq A'$ **and**
$PO \neq C$ **and**
$PO \neq C'$ **and**
$PO \neq B$ **and**
$PO \neq B'$ **and**
$PO \neq D$ **and**
$PO \neq D'$ **and**
$Col\ PO\ A\ C$ **and**
$Col\ PO\ A\ A'$ **and**
$Col\ PO\ A\ C'$ **and**
$Col\ PO\ B\ D$ **and**
$Col\ PO\ B\ B'$ **and**
$Col\ PO\ B\ D'$ **and**
$\neg\ A\ B\ Par\ C\ D$ **and**
$A\ B\ Par\ A'\ B'$ **and**
$A\ D\ Par\ A'\ D'$ **and**
$B\ C\ Par\ B'\ C'$
shows $C\ D\ Par\ C'\ D'$
⟨*proof*⟩

lemma *l13-19*:
assumes $\neg\ Col\ PO\ A\ B$ **and**
$PO \neq A$ **and**
$PO \neq A'$ **and**
$PO \neq C$ **and**
$PO \neq C'$ **and**
$PO \neq B$ **and**
$PO \neq B'$ **and**
$PO \neq D$ **and**
$PO \neq D'$ **and**
$Col\ PO\ A\ C$ **and**
$Col\ PO\ A\ A'$ **and**
$Col\ PO\ A\ C'$ **and**
$Col\ PO\ B\ D$ **and**
$Col\ PO\ B\ B'$ **and**
$Col\ PO\ B\ D'$ **and**

$A\ B\ Par\ A'\ B'$ **and**
$A\ D\ Par\ A'\ D'$ **and**
$B\ C\ Par\ B'\ C'$
shows $C\ D\ Par\ C'\ D'$
⟨*proof*⟩

lemma *l13-19-par-aux*:
assumes $X \neq A$ **and**

$Y \neq B$ **and**

$Col\ X\ A\ C$ **and**
$Col\ X\ A\ A'$ **and**
$Col\ X\ A\ C'$ **and**
$Col\ Y\ B\ D$ **and**
$Col\ Y\ B\ B'$ **and**
$Col\ Y\ B\ D'$ **and**
$A \neq C$ **and**
$B \neq D$ **and**
$A \neq A'$ **and**
$X\ A\ ParStrict\ Y\ B$ **and**
$\neg\ A\ B\ Par\ C\ D$ **and**
$A\ B\ Par\ A'\ B'$ **and**
$A\ D\ Par\ A'\ D'$ **and**
$B\ C\ Par\ B'\ C'$
shows $C\ D\ Par\ C'\ D'$
⟨*proof*⟩

lemma *l13-19-par*:
assumes $X \neq A$ **and**
$X \neq A'$ **and**

$Y \neq B$ **and**
$Y \neq B'$ **and**

$Col\ X\ A\ C$ **and**
$Col\ X\ A\ A'$ **and**
$Col\ X\ A\ C'$ **and**
$Col\ Y\ B\ D$ **and**
$Col\ Y\ B\ B'$ **and**
$Col\ Y\ B\ D'$ **and**
$X\ A\ ParStrict\ Y\ B$ **and**
$A\ B\ Par\ A'\ B'$ **and**
$A\ D\ Par\ A'\ D'$ **and**
$B\ C\ Par\ B'\ C'$
shows $C\ D\ Par\ C'\ D'$
⟨*proof*⟩

lemma *sum-to-sump*:
assumes $Sum\ PO\ E\ E'\ A\ B\ C$
shows $Sump\ PO\ E\ E'\ A\ B\ C$
⟨*proof*⟩

lemma *sump-to-sum*:
assumes $Sump\ PO\ E\ E'\ A\ B\ C$
shows $Sum\ PO\ E\ E'\ A\ B\ C$
⟨*proof*⟩

lemma *project-col-project*:
assumes $A \neq C$
and $Col\ A\ B\ C$
and $P\ P'\ Proj\ A\ B\ X\ Y$
shows $P\ P'\ Proj\ A\ C\ X\ Y$
⟨*proof*⟩

lemma *pj-uniqueness*:

assumes ¬ *Col PO E E'*
and *Col PO E A*
and *Col PO E' A'*
and *Col PO E' A''*
and *E E' Pj A A'*
and *E E' Pj A A''*
shows *A' = A''*
⟨*proof*⟩

lemma *pj-right-comm*:
assumes *A B Pj C D*
shows *A B Pj D C*
⟨*proof*⟩

lemma *pj-left-comm*:
assumes *A B Pj C D*
shows *B A Pj C D*
⟨*proof*⟩

lemma *pj-comm*:
assumes *A B Pj C D*
shows *B A Pj D C*
⟨*proof*⟩

lemma *grid-not-par-1*:
assumes *grid-ok*: ¬ *Col PO E E'*
shows ¬ *PO E Par E E'*
⟨*proof*⟩

lemma *grid-not-par-2*:
assumes *grid-ok*: ¬ *Col PO E E'*
shows ¬ *PO E Par PO E'*
⟨*proof*⟩

lemma *grid-not-par-3*:
assumes *grid-ok*: ¬ *Col PO E E'*
shows ¬ *PO E' Par E E'*
⟨*proof*⟩

lemma *grid-not-par-4*:
assumes *grid-ok*: ¬ *Col PO E E'*
shows *PO* ≠ *E*
⟨*proof*⟩

lemma *grid-not-par-5*:
assumes *grid-ok*: ¬ *Col PO E E'*
shows *PO* ≠ *E'*
⟨*proof*⟩

lemma *grid-not-par-6*:
assumes *grid-ok*: ¬ *Col PO E E'*
shows *E* ≠ *E'*
⟨*proof*⟩

lemma *grid-not-par*:
assumes *grid-ok*: ¬ *Col PO E E'*
shows ¬ *PO E Par E E'* ∧ ¬ *PO E Par PO E'* ∧
¬ *PO E' Par E E'* ∧ *PO* ≠ *E* ∧ *PO* ≠ *E'* ∧ *E* ≠ *E'*
⟨*proof*⟩

lemma *proj-id*:
assumes *grid-ok*: ¬ *Col PO E E'*
and *A A' Proj PO E' E E'*
and *Col PO E A*
and *Col PO E A'*
shows *A = PO*

⟨proof⟩

lemma *sum-uniqueness*:
  assumes *grid-ok*: ¬ *Col PO E E′*
    and *Sum PO E E′ A B C1*
    and *Sum PO E E′ A B C2*
  shows *C1* = *C2*
⟨proof⟩

lemma *opp0*:
  assumes *grid-ok*: ¬ *Col PO E E′*
  shows *Opp PO E E′ PO PO*
  ⟨proof⟩

lemma *sum-O-O*:
  assumes *grid-ok*: ¬ *Col PO E E′*
  shows *Sum PO E E′ PO PO PO*
  ⟨proof⟩

lemma *sum-par-strict-a*:
  assumes *grid-ok*: ¬ *Col PO E E′*
    and *Ar2 PO E E′ A B C*
    and *A* ≠ *PO*
    and *E E′ Pj A A′*

  shows *A′* ≠ *PO*
  ⟨proof⟩

lemma *sum-par-strict-b*:
  assumes *grid-ok*: ¬ *Col PO E E′*
    and *Ar2 PO E E′ A B C*
    and *A* ≠ *PO*
    and *E E′ Pj A A′*
    and *Col PO E′ A′*
    and *PO E Pj A′ C′*
    and *PO E′ Pj B C′*
    and *E′ E Pj C′ C*
  shows (*PO E ParStrict A′ C′* ∨ *B* = *PO*)
⟨proof⟩

lemma *sum-par-strict*:
  assumes *grid-ok*: ¬ *Col PO E E′*
    and *Ar2 PO E E′ A B C*
    and *A* ≠ *PO*
    and *E E′ Pj A A′*
    and *Col PO E′ A′*
    and *PO E Pj A′ C′*
    and *PO E′ Pj B C′*
    and *E′ E Pj C′ C*
  shows *A′* ≠ *PO* ∧ (*PO E ParStrict A′ C′* ∨ *B* = *PO*)
  ⟨proof⟩

end
end

theory *Tarski-Euclidean-2D*

imports
  Tarski-Neutral
  Tarski-Postulate-Parallels
  Tarski-Euclidean
  Tarski-Neutral-2D

begin

15 Tarski Euclidean 2D

15.1 Tarski's axiom system for Euclidean 2D

locale *Tarski-Euclidean-2D* = *Tarski-Euclidean* +
assumes *upper-dim*: $\forall\ a\ b\ c\ p\ q.$
$p \neq q \wedge$
$Cong\ a\ p\ a\ q \wedge$
$Cong\ b\ p\ b\ q \wedge$
$Cong\ c\ p\ c\ q$
$\longrightarrow$
$(Bet\ a\ b\ c \vee Bet\ b\ c\ a \vee Bet\ c\ a\ b)$

sublocale *Tarski-Euclidean-2D* ⊆ *Tarski-neutral-2D*
⟨*proof*⟩

context *Tarski-Euclidean-2D*

begin

15.2 Definitions

15.3 Propositions

lemma *l12-16-2D*:
assumes *A1 A2 Par B1 B2* **and**
X Inter A1 A2 C1 C2
shows $\exists\ Y.\ Y$ *Inter B1 B2 C1 C2*
⟨*proof*⟩

lemma *par-inter*:
assumes *A B Par C D* **and**
$\neg$ *A B Par P Q*
shows $\exists\ Y.\ Col\ P\ Q\ Y \wedge Col\ C\ D\ Y$
⟨*proof*⟩

lemma *not-par-inter*:
assumes $\neg\ A\ B\ Par\ A'\ B'$
shows $\exists\ P.\ Col\ P\ X\ Y \wedge (Col\ P\ A\ B \vee Col\ P\ A'\ B')$
⟨*proof*⟩

lemma *par-perp--perp*:
assumes *A B Par C D* **and**
A B Perp P Q
shows *C D Perp P Q*
⟨*proof*⟩

lemma *par-perp2--par*:
assumes *A B Par C D* **and**
A B Perp E F **and**
C D Perp G H
shows *E F Par G H*
⟨*proof*⟩

lemma *perp3--perp*:
assumes *A B Perp B C* **and**
B C Perp C D **and**
C D Perp D A
shows *D A Perp A B*
⟨*proof*⟩

lemma *perp3--rect*:
assumes *A B Perp B C* **and**
B C Perp C D **and**
C D Perp D A
shows *Rectangle A B C D*
⟨*proof*⟩

lemma *projp-is-project*:
  assumes $P\ P'$ *Projp* $A\ B$
  shows $\exists\ X\ Y.\ P\ P'$ *Proj* $A\ B\ X\ Y$
⟨*proof*⟩

lemma *projp-is-project-perp*:
  assumes $P\ P'$ *Projp* $A\ B$
  shows $\exists\ X\ Y.\ P\ P'$ *Proj* $A\ B\ X\ Y \wedge A\ B$ *Perp* $X\ Y$
⟨*proof*⟩

lemma *projp-to-project*:
  assumes $A\ B$ *Perp* $X\ Y$ **and**
    $P\ P'$ *Projp* $A\ B$
  shows $P\ P'$ *Proj* $A\ B\ X\ Y$
⟨*proof*⟩

lemma *project-to-projp*:
  assumes $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
    $A\ B$ *Perp* $X\ Y$
  shows $P\ P'$ *Projp* $A\ B$
⟨*proof*⟩

lemma *projp-project-to-perp*:
  assumes $P \neq P'$ **and**
    $P\ P'$ *Projp* $A\ B$ **and**
    $P\ P'$ *Proj* $A\ B\ X\ Y$
  shows $A\ B$ *Perp* $X\ Y$
  ⟨*proof*⟩

lemma *project-par-project*:
  assumes $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
    $X\ Y$ *Par* $X'\ Y'$
  shows $P\ P'$ *Proj* $A\ B\ X'\ Y'$
  ⟨*proof*⟩

lemma *project-project-par* :
  assumes $P \neq P'$ **and**
    $P\ P'$ *Proj* $A\ B\ X\ Y$ **and**
    $P\ P'$ *Proj* $A\ B\ X'\ Y'$
  shows $X\ Y$ *Par* $X'\ Y'$
⟨*proof*⟩

lemma *projp-preserves-bet*:
  assumes *Bet* $A\ B\ C$ **and**
    $A\ A'$ *Projp* $X\ Y$ **and**
    $B\ B'$ *Projp* $X\ Y$ **and**
    $C\ C'$ *Projp* $X\ Y$
  shows *Bet* $A'\ B'\ C'$
⟨*proof*⟩

lemma *projp-preserves-eqv*:
  assumes $A\ B$ *EqV* $C\ D$ **and**
    $A\ A'$ *Projp* $X\ Y$ **and**
    $B\ B'$ *Projp* $X\ Y$ **and**
    $C\ C'$ *Projp* $X\ Y$ **and**
    $D\ D'$ *Projp* $X\ Y$
  shows $A'\ B'$ *EqV* $C'\ D'$
⟨*proof*⟩

lemma *projp2-col*:
  assumes $P\ A$ *Projp* $B\ C$ **and**
    $Q\ A$ *Projp* $B\ C$
  shows *Col* $A\ P\ Q$
⟨*proof*⟩

lemma *projp-projp-perp*:
assumes $P1 \neq P2$ **and**
P1 P Projp Q1 Q2 **and**
P2 P Projp Q1 Q2
shows *P1 P2 Perp Q1 Q2*
⟨*proof*⟩

lemma *perp-projp2-eq*:
assumes *A A′ Projp C D* **and**
B B′ Projp C D **and**
A B Perp C D
shows $A' = B'$
⟨*proof*⟩

lemma *col-par-projp2-eq*:
assumes *Col L11 L12 P* **and**
L11 L12 Par L21 L22 **and**
P P′ Projp L21 L22 **and**
P′ P′′ Projp L11 L12
shows $P = P''$
⟨*proof*⟩

lemma *col-2-par-projp2-cong*:
assumes *Col L11 L12 A′* **and**
Col L11 L12 B′ **and**
L11 L12 Par L21 L22 **and**
A′ A Projp L21 L22 **and**
B′ B Projp L21 L22
shows *Cong A B A′ B′*
⟨*proof*⟩

lemma *project-existence*:
assumes $X \neq Y$
and $A \neq B$
and ¬ *X Y Par A B*
shows ∃ *P′*. *P P′ Proj A B X Y*
⟨*proof*⟩

lemma *sum-exists*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Col PO E A*
and *Col PO E B*
shows ∃ *C*. *Sum PO E E′ A B C*
⟨*proof*⟩

lemma *opp-exists*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Col PO E A*
shows ∃ *MA*. *Opp PO E E′ A MA*
⟨*proof*⟩

lemma *sum-A-O*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Col PO E A*
shows *Sum PO E E′ A PO A*
⟨*proof*⟩

lemma *sum-O-B*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Col PO E B*
shows *Sum PO E E′ PO B B*
⟨*proof*⟩

lemma *opp0-uniqueness*:
assumes *grid-ok*: ¬ *Col PO E E′*

and *Opp PO E E′ PO M*
shows *M = PO*
⟨*proof*⟩

lemma *proj-pars*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *A ≠ PO*
and *Col PO E A*
and *PO E Par A′ C′*
and *A A′ Proj PO E′ E E′*
shows *PO E ParStrict A′ C′*
⟨*proof*⟩

lemma *sum-O-B-eq*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ PO B C*
shows *B = C*
⟨*proof*⟩

lemma *sum-A-O-eq*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A PO C*
shows *A = C*
⟨*proof*⟩

lemma *sum-A-B-A*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A B A*
shows *B = PO*
⟨*proof*⟩

lemma *sum-A-B-B*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A B B*
shows *A = PO*
⟨*proof*⟩

lemma *sum-uniquenessB*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A X C*
and *Sum PO E E′ A Y C*
shows *X = Y*
⟨*proof*⟩

lemma *sum-uniquenessA*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ X B C*
and *Sum PO E E′ Y B C*
shows *X = Y*
⟨*proof*⟩

lemma *sum-B-null*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A B A*
shows *B = PO*
⟨*proof*⟩

lemma *sum-A-null*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A B B*
shows *A = PO*
⟨*proof*⟩

lemma *sum-plg*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A B C*

and $(A \neq PO\ ) \vee (\ B \neq PO)$
shows $\exists\ A'\ C'.\ Plg\ PO\ B\ C'\ A' \wedge Plg\ C'\ A'\ A\ C$
⟨*proof*⟩

lemma *sum-cong*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Sum\ PO\ E\ E'\ A\ B\ C$
and $(A \neq PO \vee B \neq PO)$
shows $ParallelogramFlat\ PO\ A\ C\ B$
⟨*proof*⟩

lemma *sum-cong2*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Sum\ PO\ E\ E'\ A\ B\ C$
and $(A \neq PO \vee B \neq PO)$
shows $Cong\ PO\ A\ B\ C \wedge Cong\ PO\ B\ A\ C$
⟨*proof*⟩

lemma *sum-comm*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Sum\ PO\ E\ E'\ A\ B\ C$
shows $Sum\ PO\ E\ E'\ B\ A\ C$
⟨*proof*⟩

lemma *cong-sum*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $PO \neq C \vee B \neq A$
and $Ar2\ PO\ E\ E'\ A\ B\ C$
and $Cong\ PO\ A\ B\ C$
and $Cong\ PO\ B\ A\ C$
shows $Sum\ PO\ E\ E'\ A\ B\ C$
⟨*proof*⟩

lemma *sum-iff-cong-a*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Ar2\ PO\ E\ E'\ A\ B\ C$
and $PO \neq C \vee B \neq A$
and $Cong\ PO\ A\ B\ C$
and $Cong\ PO\ B\ A\ C$
shows $Sum\ PO\ E\ E'\ A\ B\ C$
⟨*proof*⟩

lemma *sum-iff-cong-b*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $PO \neq C \vee B \neq A$
and $Sum\ PO\ E\ E'\ A\ B\ C$
shows $Cong\ PO\ A\ B\ C \wedge Cong\ PO\ B\ A\ C$
⟨*proof*⟩

lemma *sum-iff-cong*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Ar2\ PO\ E\ E'\ A\ B\ C$
and $PO \neq C \vee B \neq A$
shows $(Cong\ PO\ A\ B\ C \wedge Cong\ PO\ B\ A\ C) \longleftrightarrow Sum\ PO\ E\ E'\ A\ B\ C$
⟨*proof*⟩

lemma *opp-comm*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Opp\ PO\ E\ E'\ X\ Y$
shows $Opp\ PO\ E\ E'\ Y\ X$
⟨*proof*⟩

lemma *opp-uniqueness*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Opp\ PO\ E\ E'\ A\ MA1$
and $Opp\ PO\ E\ E'\ A\ MA2$

shows *MA1* = *MA2*
⟨*proof*⟩

lemma *proj-preserves-sum*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Sum PO E E′ A B C*
and *Ar1 PO E′ A′ B′ C′*
and *E E′ Pj A A′*
and *E E′ Pj B B′*
and *E E′ Pj C C′*
shows *Sum PO E′ E A′ B′ C′*
⟨*proof*⟩

lemma *sum-assoc-1*:
assumes *Sum PO E E′ A B AB*
and *Sum PO E E′ B C BC*
and *Sum PO E E′ A BC ABC*
shows *Sum PO E E′ AB C ABC*
⟨*proof*⟩

lemma *sum-assoc-2*:
assumes *Sum PO E E′ A B AB*
and *Sum PO E E′ B C BC*
and *Sum PO E E′ AB C ABC*
shows *Sum PO E E′ A BC ABC*
⟨*proof*⟩

lemma *sum-assoc*:
assumes *Sum PO E E′ A B AB*
and *Sum PO E E′ B C BC*
shows *Sum PO E E′ A BC ABC* ≡ *Sum PO E E′ AB C ABC*
⟨*proof*⟩

lemma *sum-y-axis-change*:
assumes *Sum PO E E′ A B C*
and ¬ *Col PO E E″*
shows *Sum PO E E″ A B C*
⟨*proof*⟩

lemma *sum-x-axis-unit-change*:
assumes *Sum PO E E′ A B C*
and *Col PO E U*
and *U* ≠ *PO*
shows *Sum PO U E′ A B C*
⟨*proof*⟩

lemma *change-grid-sum-0*:
assumes *PO E ParStrict O′ E′*
and *Ar1 PO E A B C*
and *Ar1 O′ E′ A′ B′ C′*
and *PO O′ Pj E E′*
and *PO O′ Pj A A′*
and *PO O′ Pj B B′*
and *PO O′ Pj C C′*
and *Sum PO E E′ A B C*
and *A* = *PO*
shows *Sum O′ E′ E A′ B′ C′*
⟨*proof*⟩

lemma *change-grid-sum*:
assumes $PO\ E\ ParStrict\ O'\ E'$
and $Ar1\ PO\ E\ A\ B\ C$
and $Ar1\ O'\ E'\ A'\ B'\ C'$
and $PO\ O'\ Pj\ E\ E'$
and $PO\ O'\ Pj\ A\ A'$
and $PO\ O'\ Pj\ B\ B'$
and $PO\ O'\ Pj\ C\ C'$
and $Sum\ PO\ E\ E'\ A\ B\ C$
shows $Sum\ O'\ E'\ E\ A'\ B'\ C'$
⟨*proof*⟩

lemma *double-null-null*:
assumes $Sum\ PO\ E\ E'\ A\ A\ PO$
shows $A = PO$
⟨*proof*⟩

lemma *not-null-double-not-null*:
assumes $Sum\ PO\ E\ E'\ A\ A\ C$
and $A \neq PO$
shows $C \neq PO$
⟨*proof*⟩

lemma *double-not-null-not-nul*:
assumes $Sum\ PO\ E\ E'\ A\ A\ C$
and $C \neq PO$
shows $A \neq PO$
⟨*proof*⟩

lemma *diff-ar2*:
assumes $Diff\ PO\ E\ E'\ A\ B\ AMB$
shows $Ar2\ PO\ E\ E'\ A\ B\ AMB$
⟨*proof*⟩

lemma *diff-null*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Col\ PO\ E\ A$
shows $Diff\ PO\ E\ E'\ A\ A\ PO$
⟨*proof*⟩

lemma *diff-exists*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Col\ PO\ E\ A$
and $Col\ PO\ E\ B$
shows $\exists\ D.\ Diff\ PO\ E\ E'\ A\ B\ D$
⟨*proof*⟩

lemma *diff-uniqueness*:
assumes $Diff\ PO\ E\ E'\ A\ B\ D1$
and $Diff\ PO\ E\ E'\ A\ B\ D2$
shows $D1 = D2$
⟨*proof*⟩

lemma *sum-ar2*:
assumes $Sum\ PO\ E\ E'\ A\ B\ C$
shows $Ar2\ PO\ E\ E'\ A\ B\ C$
⟨*proof*⟩

lemma *diff-A-O*:
assumes *grid-ok*: $\neg\ Col\ PO\ E\ E'$
and $Col\ PO\ E\ A$
shows $Diff\ PO\ E\ E'\ A\ PO\ A$
⟨*proof*⟩

lemma *diff-O-A*:
assumes $Opp\ PO\ E\ E'\ A\ mA$

shows *Diff PO E E′ PO A mA*
⟨*proof*⟩

lemma *diff-O-A-opp*:
assumes *Diff PO E E′ PO A mA*
shows *Opp PO E E′ A mA*
⟨*proof*⟩

lemma *diff-uniquenessA*:
assumes *Diff PO E E′ A B C*
and *Diff PO E E′ A′ B C*
shows $A = A'$
⟨*proof*⟩

lemma *diff-uniquenessB*:
assumes *Diff PO E E′ A B C*
and *Diff PO E E′ A B′ C*
shows $B = B'$
⟨*proof*⟩

lemma *diff-null-eq*:
assumes *Diff PO E E′ A B PO*
shows $A = B$
⟨*proof*⟩

lemma *midpoint-opp*:
assumes *Ar2 PO E E′ PO A B*
and *PO Midpoint A B*
shows *Opp PO E E′ A B*
⟨*proof*⟩

lemma *sum-diff*:
assumes *Sum PO E E′ A B S*
shows *Diff PO E E′ S A B*
⟨*proof*⟩

lemma *diff-sum*:
assumes *Diff PO E E′ S A B*
shows *Sum PO E E′ A B S*
⟨*proof*⟩

lemma *diff-opp*:
assumes *Diff PO E E′ A B AmB*
and *Diff PO E E′ B A BmA*
shows *Opp PO E E′ AmB BmA*
⟨*proof*⟩

lemma *sum-stable*:
assumes $A = B$
and *Sum PO E E′ A C S1*
and *Sum PO E E′ B C S2*
shows $S1 = S2$
⟨*proof*⟩

lemma *diff-stable*:
assumes $A = B$
and *Diff PO E E′ A C D1*
and *Diff PO E E′ B C D2*
shows $D1 = D2$
⟨*proof*⟩

lemma *plg-to-sum*:
assumes *Ar2 PO E E′ A B C*
and *ParallelogramFlat PO A C B*
shows *Sum PO E E′ A B C*
⟨*proof*⟩

lemma *opp-midpoint*:
  assumes *Opp PO E E′ A MA*
  shows *PO Midpoint A MA*
⟨*proof*⟩

lemma *diff-to-plg*:
  assumes $A \neq PO \lor B \neq PO$
    and *Diff PO E E′ B A dBA*
  shows *ParallelogramFlat PO A B dBA*
  ⟨*proof*⟩

lemma *sum3-col*:
  assumes *sum3 PO E E′ A B C S*
  shows ¬ *Col PO E E′* ∧ *Col PO E A* ∧ *Col PO E B* ∧ *Col PO E C* ∧ *Col PO E S*
  ⟨*proof*⟩

lemma *sum3-permut*:
  assumes *sum3 PO E E′ A B C S*
  shows *sum3 PO E E′ C A B S*
  ⟨*proof*⟩

lemma *sum3-comm-1-2*:
  assumes *sum3 PO E E′ A B C S*
  shows *sum3 PO E E′ B A C S*
  ⟨*proof*⟩

lemma *sum3-comm-2-3*:
  assumes *sum3 PO E E′ A B C S*
  shows *sum3 PO E E′ A C B S*
  ⟨*proof*⟩

lemma *sum3-exists*:
  assumes *Ar2 PO E E′ A B C*
  shows ∃ *S*. *sum3 PO E E′ A B C S*
  ⟨*proof*⟩

lemma *sum3-uniqueness*:
  assumes *sum3 PO E E′ A B C S1*
    and *sum3 PO E E′ A B C S2*
  shows *S1* = *S2*
  ⟨*proof*⟩

lemma *sum4-col*:
  assumes *Sum4 PO E E′ A B C D S*
  shows ¬ *Col PO E E′* ∧ *Col PO E A* ∧ *Col PO E B* ∧ *Col PO E C* ∧ *Col PO E D* ∧ *Col PO E S*
  ⟨*proof*⟩

lemma *sum22-col*:
  assumes *sum22 PO E E′ A B C D S*
  shows ¬ *Col PO E E′* ∧ *Col PO E A* ∧ *Col PO E B* ∧ *Col PO E C* ∧ *Col PO E D* ∧ *Col PO E S*
  ⟨*proof*⟩

lemma *sum-to-sum3*:
  assumes *Sum PO E E′ A B AB*
    and *Sum PO E E′ AB X S*
  shows *sum3 PO E E′ A B X S*
  ⟨*proof*⟩

lemma *sum3-to-sum4*:
  assumes *sum3 PO E E′ A B C ABC*
    and *Sum PO E E′ ABC X S*
  shows *Sum4 PO E E′ A B C X S*
  ⟨*proof*⟩

lemma *sum-A-exists*:

assumes *Ar2 PO E E′ A AB PO*
shows ∃ *B*. *Sum PO E E′ A B AB*
⟨*proof*⟩

lemma *sum-B-exists*:
assumes *Ar2 PO E E′ B AB PO*
shows ∃ *A*. *Sum PO E E′ A B AB*
⟨*proof*⟩

lemma *sum4-equiv-a*:
assumes *Sum4 PO E E′ A B C D S*
shows *sum22 PO E E′ A B C D S*
⟨*proof*⟩

lemma *sum4-equiv-b*:
assumes *sum22 PO E E′ A B C D S*
shows *Sum4 PO E E′ A B C D S*
⟨*proof*⟩

lemma *sum4-equiv*:
shows *Sum4 PO E E′ A B C D S* ⟷ *sum22 PO E E′ A B C D S*
⟨*proof*⟩

lemma *sum4-permut*:
assumes *Sum4 PO E E′ A B C D S*
shows *Sum4 PO E E′ D A B C S*
⟨*proof*⟩

lemma *sum22-permut*:
assumes *sum22 PO E E′ A B C D S*
shows *sum22 PO E E′ D A B C S*
⟨*proof*⟩

lemma *sum4-comm*:
assumes *Sum4 PO E E′ A B C D S*
shows *Sum4 PO E E′ B A C D S*
⟨*proof*⟩

lemma *sum22-comm*:
assumes *sum22 PO E E′ A B C D S*
shows *sum22 PO E E′ B A C D S*
⟨*proof*⟩

lemma *sum-abcd*:
assumes *Sum PO E E′ A B AB*
and *Sum PO E E′ C D CD*
and *Sum PO E E′ B C BC*
and *Sum PO E E′ A D AD*
and *Sum PO E E′ AB CD S*
shows *Sum PO E E′ BC AD S*
⟨*proof*⟩

lemma *sum-diff-diff-a*:
assumes *Diff PO E E′ B A dBA*
and *Diff PO E E′ C B dCB*
and *Diff PO E E′ C A dCA*
shows *Sum PO E E′ dCB dBA dCA*
⟨*proof*⟩

lemma *sum-diff-diff-b*:
assumes *Diff PO E E′ B A dBA*
and *Diff PO E E′ C B dCB*

and *Sum PO E E′ dCB dBA dCA*
shows *Diff PO E E′ C A dCA*
⟨*proof*⟩

lemma *sum-diff2-diff-sum2-a*:
assumes *Sum PO E E′ A B C*
and *Sum PO E E′ X Y Z*
and *Diff PO E E′ X A dXA*
and *Diff PO E E′ Y B dYB*
and *Sum PO E E′ dXA dYB dZC*
shows *Diff PO E E′ Z C dZC*
⟨*proof*⟩

lemma *sum-diff2-diff-sum2-b*:
assumes *Sum PO E E′ A B C*
and *Sum PO E E′ X Y Z*
and *Diff PO E E′ X A dXA*
and *Diff PO E E′ Y B dYB*
and *Diff PO E E′ Z C dZC*
shows *Sum PO E E′ dXA dYB dZC*
⟨*proof*⟩

lemma *sum-opp*:
assumes *Sum PO E E′ X MX PO*
shows *Opp PO E E′ X MX*
⟨*proof*⟩

lemma *sum-diff-diff*:
assumes *Diff PO E E′ AX BX AXMBX*
and *Diff PO E E′ AX CX AXMCX*
and *Diff PO E E′ BX CX BXMCX*
shows *Sum PO E E′ AXMBX BXMCX AXMCX*
⟨*proof*⟩

lemma *prod-to-prodp*:
assumes *Prod PO E E′ A B C*
shows *Prodp PO E E′ A B C*
⟨*proof*⟩

lemma *project-pj*:
assumes *P P′ Proj A B X Y*
shows *X Y Pj P P′*
⟨*proof*⟩

lemma *prodp-to-prod*:
assumes *Prodp PO E E′ A B C*
shows *Prod PO E E′ A B C*
⟨*proof*⟩

lemma *prod-exists*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Col PO E A*
and *Col PO E B*
shows ∃ *C*. *Prod PO E E′ A B C*
⟨*proof*⟩

lemma *prod-uniqueness*:
assumes *grid-ok*: ¬ *Col PO E E′*
and *Prod PO E E′ A B C1*
and *Prod PO E E′ A B C2*
shows *C1* = *C2*
⟨*proof*⟩

lemma *prod-0-l*:

assumes ¬ *Col PO E E′*
and *Col PO E A*
shows *Prod PO E E′ PO A PO*
⟨*proof*⟩

lemma *prod-0-r*:
assumes ¬ *Col PO E E′*
and *Col PO E A*
shows *Prod PO E E′ A PO PO*
⟨*proof*⟩

lemma *prod-1-l*:
assumes ¬ *Col PO E E′*
and *Col PO E A*
shows *Prod PO E E′ E A A*
⟨*proof*⟩

lemma *prod-1-r*:
assumes ¬ *Col PO E E′*
and *Col PO E A*
shows *Prod PO E E′ A E A*
⟨*proof*⟩

lemma *inv-exists*:
assumes ¬ *Col PO E E′*
and *Col PO E A*
and $A \neq PO$
shows ∃ *IA*. *Prod PO E E′ IA A E*
⟨*proof*⟩

lemma *prod-null*:
assumes *Prod PO E E′ A B PO*
shows $A = PO \vee B = PO$
⟨*proof*⟩

lemma *prod-y-axis-change*:
assumes *Prod PO E E′ A B C*
and ¬ *Col PO E E′′*
shows *Prod PO E E′′ A B C*
⟨*proof*⟩

lemma *proj-preserves-prod*:
assumes *Prod PO E E′ A B C*
and *Ar1 PO E′ A′ B′ C′*
and *E E′ Pj A A′*
and *E E′ Pj B B′*
and *E E′ Pj C C′*
shows *Prod PO E′ E A′ B′ C′*
⟨*proof*⟩

lemma *prod-assoc1*:
assumes *Prod PO E E′ A B AB*
and *Prod PO E E′ B C BC*
and *Prod PO E E′ A BC ABC*
shows *Prod PO E E′ AB C ABC*
⟨*proof*⟩

lemma *prod-assoc2*:

assumes *Prod PO E E′ A B AB* **and**
Prod PO E E′ B C BC
and *Prod PO E E′ AB C ABC*
shows *Prod PO E E′ A BC ABC*
⟨*proof*⟩

lemma *prod-assoc*:
assumes *Prod PO E E′ A B AB*
and *Prod PO E E′ B C BC*
shows *Prod PO E E′ A BC ABC* ⟷ *Prod PO E E′ AB C ABC*
⟨*proof*⟩

lemma *prod-comm*:
assumes *Prod PO E E′ A B C*
shows *Prod PO E E′ B A C*
⟨*proof*⟩

lemma *prod-O-l-eq*:
assumes *Prod PO E E′ PO B C*
shows $C = PO$
⟨*proof*⟩

lemma *prod-O-r-eq*:
assumes *Prod PO E E′ A PO C*
shows $C = PO$
⟨*proof*⟩

lemma *prod-uniquenessA*:
assumes $B \neq PO$
and *Prod PO E E′ A B C*
and *Prod PO E E′ A′ B C*
shows $A = A'$
⟨*proof*⟩

lemma *prod-uniquenessB*:
assumes $A \neq PO$
and *Prod PO E E′ A B C*
and *Prod PO E E′ A B′ C*
shows $B = B'$
⟨*proof*⟩

lemma *distr-l*:
assumes *Sum PO E E′ B C D*
and *Prod PO E E′ A B AB*
and *Prod PO E E′ A C AC*
and *Prod PO E E′ A D AD*
shows *Sum PO E E′ AB AC AD*
⟨*proof*⟩

lemma *distr-r*:
assumes *Sum PO E E′ A B D*
and *Prod PO E E′ A C AC*
and *Prod PO E E′ B C BC*
and *Prod PO E E′ D C DC*
shows *Sum PO E E′ AC BC DC*
⟨*proof*⟩

lemma *prod-1-l-eq*:
assumes *Prod PO E E′ A B B*
shows $A = E \lor B = PO$
⟨*proof*⟩

lemma *prod-1-r-eq*:
assumes *Prod PO E E′ A B A*
shows $B = E \lor A = PO$
⟨*proof*⟩

lemma *change-grid-prod-l-O*:
assumes *PO E ParStrict O′ E′*
and *Ar1 PO E PO B C*
and *Ar1 O′ E′ A′ B′ C′*

and *PO O′ Pj PO A′*

and *PO O′ Pj C C′*
and *Prod PO E E′ PO B C*
shows *Prod O′ E′ E A′ B′ C′*
⟨*proof*⟩

lemma *change-grid-prod1*:
assumes *PO E ParStrict O′ E′*
and *Ar1 PO E E B C*
and *Ar1 O′ E′ A′ B′ C′*
and *PO O′ Pj E E′*
and *PO O′ Pj E A′*
and *PO O′ Pj B B′*
and *PO O′ Pj C C′*
and *Prod PO E E′ E B C*
shows *Prod O′ E′ E A′ B′ C′*
⟨*proof*⟩

lemma *change-grid-prod*:
assumes *PO E ParStrict O′ E′*
and *Ar1 PO E A B C*
and *Ar1 O′ E′ A′ B′ C′*
and *PO O′ Pj E E′*
and *PO O′ Pj A A′*
and *PO O′ Pj B B′*
and *PO O′ Pj C C′*
and *Prod PO E E′ A B C*
shows *Prod O′ E′ E A′ B′ C′*
⟨*proof*⟩

lemma *prod-sym*:
assumes *Prod PO E E′ A B C*
shows *Prod PO E E′ B A C*
⟨*proof*⟩

lemma *l14-31-1*:
assumes *Ar2p4 PO E E′ A B C D*
and $C \neq PO$

and ∃ *X*. *Prod PO E E′ A B X* ∧ *Prod PO E E′ C D X*
shows *Prod PO C E′ A B D*
⟨*proof*⟩

lemma *l14-31-2*:
assumes *Ar2p4 PO E E′ A B C D*
and *C* ≠ *PO*
and *Prod PO C E′ A B D*
shows ∃ *X*. *Prod PO E E′ A B X* ∧ *Prod PO E E′ C D X*
⟨*proof*⟩

lemma *prod-x-axis-unit-change*:
assumes *Ar2p4 PO E E′ A B C D*
and *Col PO E U*
and *U* ≠ *PO*
and ∃ *X*. *Prod PO E E′ A B X* ∧ *Prod PO E E′ C D X*
shows ∃ *Y*. *Prod PO U E′ A B Y* ∧ *Prod PO U E′ C D Y*
⟨*proof*⟩

lemma *opp-prod*:
assumes *Opp PO E E′ E ME*
and *Opp PO E E′ X MX*
shows *Prod PO E E′ X ME MX*
⟨*proof*⟩

lemma *distr-l-diff*:
assumes *Diff PO E E′ B C BMC*
and *Prod PO E E′ A B AB*
and *Prod PO E E′ A C AC*
and *Prod PO E E′ A BMC ABMC*
shows *Diff PO E E′ AB AC ABMC*
⟨*proof*⟩

lemma *diff-of-squares*:
assumes *Prod PO E E′ A A A2*
and *Prod PO E E′ B B B2*
and *Diff PO E E′ A2 B2 A2MB2*
and *Sum PO E E′ A B APB*
and *Diff PO E E′ A B AMB*
and *Prod PO E E′ APB AMB F*
shows *A2MB2* = *F*
⟨*proof*⟩

lemma *eq-squares-eq-or-opp*:
assumes *Prod PO E E′ A A A2*
and *Prod PO E E′ B B A2*
shows *A* = *B* ∨ *Opp PO E E′ A B*
⟨*proof*⟩

lemma *diff-2-prod*:
assumes *Opp PO E E′ E ME*
and *Diff PO E E′ A B AMB*
and *Diff PO E E′ B A BMA*
shows *Prod PO E E′ AMB ME BMA*
⟨*proof*⟩

lemma *l14-36-a*:
assumes *Sum PO E E′ A B C* **and**
PO Out A B
shows *Bet PO A C*
⟨*proof*⟩

lemma *l14-36-b*:
assumes *Sum PO E E′ A B C*
and *PO Out A B*
shows *PO* ≠ *A* ∧ *PO* ≠ *C* ∧ *A* ≠ *C*

⟨proof⟩

lemma *pos-null-neg*:
assumes *Opp PO E E′ A MA*
shows *Ps PO E A* ∨ *PO* = *A* ∨ *Ps PO E MA*
⟨proof⟩

lemma *sum-pos-pos*:
assumes *Ps PO E A*
and *Ps PO E B*
and *Sum PO E E′ A B AB*
shows *Ps PO E AB*
⟨proof⟩

lemma *prod-pos-pos*:
assumes *Ps PO E A*
and *Ps PO E B*
and *Prod PO E E′ A B AB*
shows *Ps PO E AB*
⟨proof⟩

lemma *pos-not-neg*:
assumes *Ps PO E A*
shows ¬ *Ng PO E A*
⟨proof⟩

lemma *neg-not-pos*:
assumes *Ng PO E A*
shows ¬ *Ps PO E A*
⟨proof⟩

lemma *opp-pos-neg*:
assumes *Ps PO E A*
and *Opp PO E E′ A MA*
shows *Ng PO E MA*
⟨proof⟩

lemma *opp-neg-pos*:
assumes *Ng PO E A*
and *Opp PO E E′ A MA*
shows *Ps PO E MA*
⟨proof⟩

lemma *ltP-ar2*:
assumes *LtP PO E E′ A B*
shows *Ar2 PO E E′ A B A*
⟨proof⟩

lemma *ltP-neq*:
assumes *LtP PO E E′ A B*
shows $A \neq B$
⟨proof⟩

lemma *leP-refl*:
shows *LeP PO E E′ A A*
⟨proof⟩

lemma *ltP-sum-pos*:
assumes *Ps PO E B*
and *Sum PO E E′ A B C*
shows *LtP PO E E′ A C*
⟨proof⟩

lemma *pos-opp-neg*:
assumes *Ps PO E A*
and *Opp PO E E′ A mA*

shows *Ng PO E mA*
⟨*proof*⟩

lemma *diff-pos-diff-neg*:
assumes *Diff PO E E′ A B AmB*
and *Diff PO E E′ B A BmA*
and *Ps PO E AmB*
shows *Ng PO E BmA*
⟨*proof*⟩

lemma *not-pos-and-neg*:
shows ¬(*Ps PO E A* ∧ *Ng PO E A*)
⟨*proof*⟩

lemma *leP-asym*:
assumes *LeP PO E E′ A B*
and *LeP PO E E′ B A*
shows *A* = *B*
⟨*proof*⟩

lemma *leP-trans*:
assumes *LeP PO E E′ A B*
and *LeP PO E E′ B C*
shows *LeP PO E E′ A C*
⟨*proof*⟩

lemma *leP-sum-leP*:
assumes *LeP PO E E′ A X*
and *LeP PO E E′ B Y*
and *Sum PO E E′ A B C*
and *Sum PO E E′ X Y Z*
shows *LeP PO E E′ C Z*
⟨*proof*⟩

lemma *square-pos*:
assumes *PO* ≠ *A*
and *Prod PO E E′ A A A2*
shows *Ps PO E A2*
⟨*proof*⟩

lemma *ltP-neg*:
assumes *LtP PO E E′ A PO*
shows *Ng PO E A*
⟨*proof*⟩

lemma *ps-le*:
assumes ¬ *Col PO E E′*
and *Bet PO X E* ∨ *Bet PO E X*
shows *LeP PO E E′ PO X*
⟨*proof*⟩

lemma *lt-diff-ps*:
assumes *LtP PO E E′ Y X*
and *Diff PO E E′ X Y XMY*
shows *Ps PO E XMY*
⟨*proof*⟩

lemma *col-2-le-or-ge*:
assumes ¬ *Col PO E E′*
and *Col PO E A*
and *Col PO E B*
shows *LeP PO E E′ A B* ∨ *LeP PO E E′ B A*
⟨*proof*⟩

lemma *compatibility-of-sum-with-order*:
assumes *LeP PO E E′ A B*
and *Sum PO E E′ A C APC*
and *Sum PO E E′ B C BPC*
shows *LeP PO E E′ APC BPC*
⟨*proof*⟩

lemma *compatibility-of-prod-with-order*:
assumes *LeP PO E E′ PO A*
and *LeP PO E E′ PO B*
and *Prod PO E E′ A B AB*
shows *LeP PO E E′ PO AB*
⟨*proof*⟩

lemma *pos-inv-pos*:
assumes $PO \neq A$
and *LeP PO E E′ PO A*
and *Prod PO E E′ IA A E*
shows *LeP PO E E′ PO IA*
⟨*proof*⟩

lemma *le-pos-prod-le*:
assumes *LeP PO E E′ A B*
and *LeP PO E E′ PO C*
and *Prod PO E E′ A C AC*
and *Prod PO E E′ B C BC*
shows *LeP PO E E′ AC BC*
⟨*proof*⟩

lemma *bet-lt12-le23*:
assumes *Bet A B C*
and *LtP PO E E′ A B*
shows *LeP PO E E′ B C*
⟨*proof*⟩

lemma *bet-lt12-le13*:
assumes *Bet A B C*
and *LtP PO E E′ A B*
shows *LeP PO E E′ A C*
⟨*proof*⟩

lemma *bet-lt21-le32*:
assumes *Bet A B C*
and *LtP PO E E′ B A*
shows *LeP PO E E′ C B*
⟨*proof*⟩

lemma *bet-lt21-le31*:
assumes *Bet A B C*
and *LtP PO E E′ B A*
shows *LeP PO E E′ C A*
⟨*proof*⟩

lemma *opp-2-le-le*:
assumes *Opp PO E E′ A MA*
and *Opp PO E E′ B MB*
and *LeP PO E E′ A B*
shows *LeP PO E E′ MB MA*
⟨*proof*⟩

lemma *diff-2-le-le*:
assumes *Diff PO E E′ A C AMC*
and *Diff PO E E′ B C BMC*
and *LeP PO E E′ A B*
shows *LeP PO E E′ AMC BMC*
⟨*proof*⟩

lemma *length-pos*:
assumes *Length PO E E' A B L*
shows *LeP PO E E' PO L*
⟨*proof*⟩

lemma *length-id-1*:
assumes *Length PO E E' A B PO*
shows $A = B$
⟨*proof*⟩

lemma *length-id-2*:
assumes $PO \neq E$
shows *Length PO E E' A A PO*
⟨*proof*⟩

lemma *length-id*:
shows *Length PO E E' A B PO* $\longleftrightarrow (A = B \wedge PO \neq E)$
⟨*proof*⟩

lemma *length-eq-cong-1*:
assumes *Length PO E E' A B AB*
and *Length PO E E' C D AB*
shows *Cong A B C D*
⟨*proof*⟩

lemma *length-eq-cong-2*:
assumes *Length PO E E' A B AB*
and *Cong A B C D*
shows *Length PO E E' C D AB*
⟨*proof*⟩

lemma *ltP-pos*:
assumes *LtP PO E E' PO A*
shows *Ps PO E A*
⟨*proof*⟩

lemma *bet-leP*:
assumes *Bet PO AB CD*
and *LeP PO E E' PO AB*
and *LeP PO E E' PO CD*
shows *LeP PO E E' AB CD*
⟨*proof*⟩

lemma *leP-bet*:
assumes *LeP PO E E' AB CD*
and *LeP PO E E' PO AB*
and *LeP PO E E' PO CD*
shows *Bet PO AB CD*
⟨*proof*⟩

lemma *length-Ar2*:
assumes *Length PO E E' A B AB*
shows (*Col PO E AB* $\wedge \neg$ *Col PO E E'*) $\vee$ *AB* = *PO*
⟨*proof*⟩

lemma *length-leP-le-1*:
assumes *Length PO E E' A B AB*
and *Length PO E E' C D CD*
and *LeP PO E E' AB CD*
shows *A B Le C D*
⟨*proof*⟩

lemma *length-leP-le-2*:
  assumes *Length PO E E′ A B AB*
    and *Length PO E E′ C D CD*
    and *A B Le C D*
  shows *LeP PO E E′ AB CD*
⟨*proof*⟩

lemma *l15-3*:
  assumes *Sum PO E E′ A B C*
  shows *Cong PO B A C*
  ⟨*proof*⟩

lemma *length-uniqueness*:
  assumes *Length PO E E′ A B AB*
    and *Length PO E E′ A B AB′*
  shows $AB = AB'$
  ⟨*proof*⟩

lemma *length-Ps*:
  assumes $AB \neq PO$
    and *Length PO E E′ A B AB*
  shows *Ps PO E AB*
  ⟨*proof*⟩

lemma *length-not-col-null*:
  assumes *Col PO E E′*
    and *Length PO E E′ A B AB*
  shows $AB = PO$
  ⟨*proof*⟩

lemma *not-triangular-equality1*:
  assumes $PO \neq E$
  shows ¬ (∀ *E′ B C AB BC AC*. *Bet A B C* ∧ *Length PO E E′ A B AB* ∧ *Length PO E E′ B C BC* ∧
    Length PO E E′ A C AC ⟶ *Sum PO E E′ AB BC AC*)
  ⟨*proof*⟩

lemma *triangular-equality*:
  assumes $PO \neq E$
    and *Bet A B C*
    and *IsLength PO E E′ A B AB*
    and *IsLength PO E E′ B C BC*
    and *IsLength PO E E′ A C AC*
  shows *Sumg PO E E′ AB BC AC*
⟨*proof*⟩

lemma *length-O*:
  assumes $PO \neq E$
  shows *Length PO E E′ PO PO PO*
  ⟨*proof*⟩

lemma *triangular-equality-bis*:
  assumes $A \neq B \vee A \neq C \vee B \neq C$
    and $PO \neq E$
    and *Bet A B C*
    and *Length PO E E′ A B AB*
    and *Length PO E E′ B C BC*
    and *Length PO E E′ A C AC*
  shows *Sum PO E E′ AB BC AC*
⟨*proof*⟩

lemma *length-out*:

assumes $A \neq B$
and $C \neq D$
and *Length PO E E' A B AB*
and *Length PO E E' C D CD*
shows *PO Out AB CD*
⟨*proof*⟩

lemma *image-preserves-bet1*:
assumes *Bet A B C*
and *A A' Reflect X Y*
and *B B' Reflect X Y*
and *C C' Reflect X Y*
shows *Bet A' B' C'*
⟨*proof*⟩

lemma *image-preserves-col*:
assumes *Col A B C*
and *A A' Reflect X Y*
and *B B' Reflect X Y*
and *C C' Reflect X Y*
shows *Col A' B' C'*
⟨*proof*⟩

lemma *image-preserves-out*:
assumes *A Out B C*
and *A A' Reflect X Y*
and *B B' Reflect X Y*
and *C C' Reflect X Y*
shows *A' Out B' C'*
⟨*proof*⟩

lemma *project-preserves-out*:
assumes *A Out B C*
and ¬ *A B Par X Y*
and *A A' Proj P Q X Y*
and *B B' Proj P Q X Y*
and *C C' Proj P Q X Y*
shows *A' Out B' C'*
⟨*proof*⟩

lemma *conga-bet-conga*:
assumes *A B C CongA D E F*
and $A' \neq B$
and $C' \neq B$
and $D' \neq E$
and $F' \neq E$
and *Bet A B A'*
and *Bet C B C'*
and *Bet D E D'*
and *Bet F E F'*
shows *A' B C' CongA D' E F'*
⟨*proof*⟩

lemma *thales*:
assumes $PO \neq E$
and *Col P A B*
and *Col P C D*
and ¬ *Col P A C*
and *A C Pj B D*
and *Length PO E E' P A A1*
and *Length PO E E' P B B1*
and *Length PO E E' P C C1*
and *Length PO E E' P D D1*
and *Prodg PO E E' A1 D1 AD*
shows *Prodg PO E E' C1 B1 AD*
⟨*proof*⟩

lemma *length-existence*:
  assumes $\neg$ *Col PO E E'*
  shows $\exists$ *AB*. *Length PO E E' A B AB*
⟨*proof*⟩

lemma *l15-7*:
  assumes $PO \neq E$
    and *Per A C B*
    and *H PerpAt C H A B*
    and *Length PO E E' A B AB*
    and *Length PO E E' A C AC*
    and *Length PO E E' A H AH*
  shows *Prod PO E E' AC AC AC2* $\longleftrightarrow$ *Prod PO E E' AB AH AC2*
⟨*proof*⟩

lemma *l15-7-1*:
  assumes $PO \neq E$
    and *Per A C B*
    and *H PerpAt C H A B*
    and *Length PO E E' A B AB*
    and *Length PO E E' A C AC*
    and *Length PO E E' A H AH*
    and *Prod PO E E' AC AC AC2*
  shows *Prod PO E E' AB AH AC2*
  ⟨*proof*⟩

lemma *l15-7-2*:
  assumes $PO \neq E$
    and *Per A C B*
    and *H PerpAt C H A B*
    and *Length PO E E' A B AB*
    and *Length PO E E' A C AC*
    and *Length PO E E' A H AH*
    and *Prod PO E E' AB AH AC2*
  shows *Prod PO E E' AC AC AC2*
  ⟨*proof*⟩

lemma *length-sym*:
  assumes *Length PO E E' A B AB*
  shows *Length PO E E' B A AB*
  ⟨*proof*⟩

lemma *pythagoras*:
  assumes $PO \neq E$
    and *Per A C B*
    and *Length PO E E' A B AB*
    and *Length PO E E' A C AC*
    and *Length PO E E' B C BC*
    and *Prod PO E E' AC AC AC2*
    and *Prod PO E E' BC BC BC2*
    and *Prod PO E E' AB AB AB2*
  shows *Sum PO E E' AC2 BC2 AB2*
⟨*proof*⟩

lemma *is-length-exists*:
  assumes $\neg$ *Col PO E E'*
  shows $\exists$ *XY*. *IsLength PO E E' X Y XY*
⟨*proof*⟩

lemma *lt-to-ltp*:
  assumes *Length PO E E' A B L*
    and *Length PO E E' C D M*

and *A B Lt C D*
shows *LtP PO E E′ L M*
⟨*proof*⟩

lemma *ltp-to-lep*:
assumes *LtP PO E E′ L M*
shows *LeP PO E E′ L M*
⟨*proof*⟩

lemma *ltp-to-lt*:
assumes *Length PO E E′ A B L*
and *Length PO E E′ C D M*
and *LtP PO E E′ L M*
shows *A B Lt C D*
⟨*proof*⟩

lemma *prod-col*:
assumes *Ar2 PO E E′ A B A*
and *Prod PO E E′ A B AB*
shows *Col PO E AB*
⟨*proof*⟩

lemma *square-increase-strict*:
assumes *Ar2 PO E E′ A B A*
and *Ps PO E A*
and *Ps PO E B*
and *LtP PO E E′ A B*
and *Prod PO E E′ A A A2*
and *Prod PO E E′ B B B2*
shows *LtP PO E E′ A2 B2*
⟨*proof*⟩

lemma *square-increase*:
assumes *Ar2 PO E E′ A B A*
and *Ps PO E A*
and *Ps PO E B*
and *LeP PO E E′ A B*
and *Prod PO E E′ A A A2*
and *Prod PO E E′ B B B2*
shows *LeP PO E E′ A2 B2*
⟨*proof*⟩

lemma *signeq--prod-pos*:
assumes *SignEq PO E A B*
and *Prod PO E E′ A B C*
shows *Ps PO E C*
⟨*proof*⟩

lemma *pos-neg--prod-neg*:
assumes *Ps PO E A*
and *Ng PO E B*
and *Prod PO E E′ A B C*
shows *Ng PO E C*
⟨*proof*⟩

lemma *not-signEq-prod-neg*:
assumes $A \neq PO$
and $B \neq PO$
and ¬ *SignEq PO E A B*
and *Prod PO E E′ A B C*
shows *Ng PO E C*
⟨*proof*⟩

lemma *prod-pos--signeq*:
assumes $A \neq PO$
and $B \neq PO$

and *Prod PO E E' A B C*
and *Ps PO E C*
shows *SignEq PO E A B*
⟨*proof*⟩

lemma *prod-ng---not-signeq*:
assumes *Prod PO E E' A B C*
and *Ng PO E C*
shows ¬ *SignEq PO E A B*
⟨*proof*⟩

lemma *ltp--diff-pos*:
assumes *LtP PO E E' A B*
and *Diff PO E E' B A D*
shows *Ps PO E D*
⟨*proof*⟩

lemma *diff-pos--ltp*:
assumes *Diff PO E E' B A D*
and *Ps PO E D*
shows *LtP PO E E' A B*
⟨*proof*⟩

lemma *square-increase-rev*:
assumes *Ps PO E A*
and *Ps PO E B*
and *LtP PO E E' A2 B2*
and *Prod PO E E' A A A2*
and *Prod PO E E' B B B2*
shows *LtP PO E E' A B*
⟨*proof*⟩

lemma *ltp--ltps*:
assumes *LtP PO E E' A B*
shows *LtPs PO E E' A B*
⟨*proof*⟩

lemma *ltps--ltp*:
assumes *LtPs PO E E' A B*
shows *LtP PO E E' A B*
⟨*proof*⟩

lemma *ltp--lep-neq*:
assumes *LtP PO E E' A B*
shows *LeP PO E E' A B* ∧ *A* ≠ *B*
⟨*proof*⟩

lemma *lep-neq--ltp*:
assumes *LeP PO E E' A B*
and *A* ≠ *B*
shows *LtP PO E E' A B*
⟨*proof*⟩

lemma *sum-preserves-ltp*:
assumes *LtP PO E E' A B*
and *Sum PO E E' A C AC*
and *Sum PO E E' B C BC*
shows *LtP PO E E' AC BC*
⟨*proof*⟩

lemma *sum-preserves-lep*:
assumes *LeP PO E E' A B*
and *Sum PO E E' A C AC*
and *Sum PO E E' B C BC*
shows *LeP PO E E' AC BC*
⟨*proof*⟩

lemma *sum-preserves-ltp-rev*:
  assumes *Sum PO E E′ A C AC*
    and *Sum PO E E′ B C BC*
    and *LtP PO E E′ AC BC*
  shows *LtP PO E E′ A B*
  ⟨*proof*⟩

lemma *sum-preserves-lep-rev*:
  assumes *Sum PO E E′ A C AC*
    and *Sum PO E E′ B C BC*
    and *LeP PO E E′ AC BC*
  shows *LeP PO E E′ A B*
  ⟨*proof*⟩

lemma *cong2-lea--le*:
  assumes *Cong A B D E*
    and *Cong A C D F*
    and *F D E LeA C A B*
  shows *E F Le B C*
⟨*proof*⟩

lemma *lea-out-lea*:
  assumes *B Out A A′*
    and *B Out C C′*
    and *E Out D D′*
    and *E Out F F′*
    and *A B C LeA D E F*
  shows *A′ B C′ LeA D′ E F′*
  ⟨*proof*⟩

lemma *lta-out-lta*:
  assumes *B Out A A′*
    and *B Out C C′*
    and *E Out D D′*
    and *E Out F F′*
    and *A B C LtA D E F*
  shows *A′ B C′ LtA D′ E F′*
  ⟨*proof*⟩

lemma *pythagoras-obtuse*:
  assumes $PO \neq E$
    and *Obtuse A C B*
    and *Length PO E E′ A B AB*
    and *Length PO E E′ A C AC*
    and *Length PO E E′ B C BC*
    and *Prod PO E E′ AC AC AC2*
    and *Prod PO E E′ BC BC BC2*
    and *Prod PO E E′ AB AB AB2*
    and *Sum PO E E′ AC2 BC2 S2*
  shows *LtP PO E E′ S2 AB2*
⟨*proof*⟩

lemma *pythagoras-obtuse-or-per*:
  assumes $PO \neq E$
    and *Obtuse A C B* ∨ *Per A C B*
    and *Length PO E E′ A B AB*
    and *Length PO E E′ A C AC*
    and *Length PO E E′ B C BC*
    and *Prod PO E E′ AC AC AC2*
    and *Prod PO E E′ BC BC BC2*
    and *Prod PO E E′ AB AB AB2*
    and *Sum PO E E′ AC2 BC2 S2*
  shows *LeP PO E E′ S2 AB2*
⟨*proof*⟩

lemma *pythagoras-acute*:
  assumes $PO \neq E$
    and *Acute A C B*
    and *Length PO E E' A B AB*
    and *Length PO E E' A C AC*
    and *Length PO E E' B C BC*
    and *Prod PO E E' AC AC AC2*
    and *Prod PO E E' BC BC BC2*
    and *Prod PO E E' AB AB AB2*
    and *Sum PO E E' AC2 BC2 S2*
  shows *LtP PO E E' AB2 S2*
⟨*proof*⟩

lemma *pyth-context*:
  assumes ¬ *Col PO E E'*
  shows ∃ *AB BC AC AB2 BC2 AC2 SS. Col PO E AB* ∧ *Col PO E BC* ∧ *Col PO E AC* ∧
    Col PO E AB2 ∧ *Col PO E BC2* ∧ *Col PO E AC2* ∧
    Length PO E E' A B AB ∧ *Length PO E E' B C BC* ∧
    Length PO E E' A C AC ∧ *Prod PO E E' AB AB AB2* ∧
    Prod PO E E' BC BC BC2 ∧ *Prod PO E E' AC AC AC2* ∧
    Sum PO E E' AB2 BC2 SS
⟨*proof*⟩

lemma *length-pos-or-null*:
  assumes *Length PO E E' A B AB*
  shows *Ps PO E AB* ∨ $A = B$
  ⟨*proof*⟩

lemma *sum-pos-null*:
  assumes ¬ *Ng PO E A*
    and ¬ *Ng PO E B*
    and *Sum PO E E' A B PO*
  shows $A = PO \wedge B = PO$
  ⟨*proof*⟩

lemma *length-not-neg*:
  assumes *Length PO E E' A B AB*
  shows ¬ *Ng PO E AB*
  ⟨*proof*⟩

lemma *signEq-refl*:
  assumes $PO \neq E$
    and *Col PO E A*
  shows $A = PO$ ∨ *SignEq PO E A A*
  ⟨*proof*⟩

lemma *square-not-neg*:
  assumes *Prod PO E E' A A A2*
  shows ¬ *Ng PO E A2*
  ⟨*proof*⟩

lemma *root-uniqueness*:
  assumes ¬ *Ng PO E A*
    and ¬ *Ng PO E B*
    and *Prod PO E E' A A C*
    and *Prod PO E E' B B C*
  shows $A = B$
  ⟨*proof*⟩

lemma *inter-tangent-circle*:
  assumes $P \neq Q$
    and *Cong P PO Q PO*
    and *Col P PO Q*
    and *P M Le P PO*
    and *Q M Le Q PO*
  shows $M = PO$

⟨*proof*⟩

lemma *inter-circle-per*:
assumes *Cong P A Q A*
and *P M Le P A*
and *Q M Le Q A*
and *A T Projp P Q*
and *Per P T M*
shows *T M Le T A*
⟨*proof*⟩

lemma *inter-circle-obtuse*:
assumes *Cong P A Q A*
and *P M Le P A*
and *Q M Le Q A*
and *A T Projp P Q*
and *Obtuse P T M* ∨ *Per P T M*
shows *T M Le T A*
⟨*proof*⟩

lemma *circle-projp-between*:
assumes *Cong P A Q A*
and *A T Projp P Q*
shows *Bet P T Q*
⟨*proof*⟩

lemma *inter-circle*:
assumes *Cong P A Q A*
and *P M Le P A*
and *Q M Le Q A*
and *A T Projp P Q*
shows *T M Le T A*
⟨*proof*⟩

lemma *projp-lt*:
assumes *Cong P A Q A*
and *A T Projp P Q*
shows *T A Lt P A*
⟨*proof*⟩

lemma *Ps-Col*:
assumes *Ps PO E A*
shows *Col PO E A*
⟨*proof*⟩

lemma *PythRel-exists*:
assumes ¬ *Col PO E E'*
shows ∀ *A B*. *Col PO E A* ∧ *Col PO E B* ⟶ (∃ *C*. *PythRel PO E E' A B C*)
⟨*proof*⟩

lemma *opp-same-square*:
assumes *Opp PO E E' A B*
and *Prod PO E E' A A A2*
shows *Prod PO E E' B B A2*
⟨*proof*⟩

lemma *PythOK*:
assumes *PythRel PO E E' A B C*
and *Prod PO E E' A A A2*
and *Prod PO E E' B B B2*
and *Prod PO E E' C C C2*
shows *Sum PO E E' A2 B2 C2*
⟨*proof*⟩

lemma *PythRel-uniqueness*:
assumes *PythRel PO E E′ A B C1*
and *PythRel PO E E′ A B C2*
and (*Ps PO E C1* ∧ *Ps PO E C2*) ∨ *C1* = *PO*
shows *C1* = *C2*
⟨*proof*⟩

end
end

16 Tarski Euclidean 2D Continuous

theory *Tarski-Euclidean-2D-Continuous*

imports
Tarski-Euclidean-2D
Tarski-Neutral-Continuous
begin

locale *Tarski-Euclidean-2D-Continuous* = *Tarski-Euclidean-2D* + *Tarski-Neutral-Continuous*

end

theory *Highschool-Neutral*

imports
Tarski-Neutral
begin

17 Highschool Neutral

context *Tarski-neutral-dimensionless*

begin

17.1 Definitions

17.2 Propositions

lemma *bisector-existence*:
assumes *A* ≠ *B* **and** *B* ≠ *C*
shows ∃ *E*. *E InAngle A B C* ∧ *A B E CongA E B C*
⟨*proof*⟩

lemma *not-col-bfoot-not-equality*:
assumes ¬ *Col A B C* **and**
Coplanar A B C I **and**
Col A B H1 **and**

A B I CongA I B C **and**
A B Perp I H1 **and**
B C Perp I H2
shows *H1* ≠ *B* ∧ *H2* ≠ *B*
⟨*proof*⟩

lemma *bisector-foot-out-out*:
assumes ¬ *Col A B C* **and**
Coplanar A B C I **and**
Col B C H2 **and**
A B I CongA I B C **and**
A B Perp I H1 **and**
B C Perp I H2 **and**
B Out H1 A

shows *B Out H2 C*
⟨*proof*⟩

lemma *not-col-efoot-not-equality*:
assumes ¬ *Col A B C* **and**
Coplanar A B C I **and**
Col A B H1 **and**
Col B C H2 **and**
Cong I H1 I H2 **and**
A B Perp I H1 **and**
B C Perp I H2
shows *H1* ≠ *B* ∧ *H2* ≠ *B*
⟨*proof*⟩

lemma *equality-foot-out-out*:
assumes ¬ *Col A B C* **and**
I InAngle A B C **and**
Col B C H2 **and**
Cong I H1 I H2 **and**
A B Perp I H1 **and**
B C Perp I H2 **and**
B Out H1 A
shows *B Out H2 C*
⟨*proof*⟩

lemma *bisector-perp-equality*:
assumes *Coplanar A B C I* **and**
Col B H1 A **and**
Col B C H2 **and**
A B Perp I H1 **and**
B C Perp I H2 **and**
A B I CongA I B C
shows *Cong I H1 I H2*
⟨*proof*⟩

lemma *perp-equality-bisector*:
assumes ¬ *Col A B C* **and**
I InAngle A B C **and**
Col B H1 A **and**
Col B H2 C **and**
A B Perp I H1 **and**
B C Perp I H2 **and**
Cong I H1 I H2
shows *A B I CongA I B C*
⟨*proof*⟩

lemma *col-permut132*:
assumes *Col A B C*
shows *Col A C B*
⟨*proof*⟩

lemma *col-permut213*:
assumes *Col A B C*
shows *Col B A C*
⟨*proof*⟩

lemma *col-permut231*:
assumes *Col A B C*
shows *Col B C A*
⟨*proof*⟩

lemma *col-permut312*:
assumes *Col A B C*
shows *Col C A B*
⟨*proof*⟩

lemma *col-permut321*:
assumes *Col A B C*
shows *Col C B A*
⟨*proof*⟩

lemma *col123-124--col134*:
assumes $P \neq Q$ **and**
Col P Q A **and**
Col P Q B
shows *Col P A B*
⟨*proof*⟩

lemma *col123-124--col234*:
assumes $P \neq Q$ **and**
Col P Q A **and**
Col P Q B
shows *Col Q A B*
⟨*proof*⟩

lemma *triangle-mid-par-strict*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C
shows *A B ParStrict Q P*
⟨*proof*⟩

end
end

theory *Highschool-Euclidean*

imports
Tarski-Neutral-Continuity
Highschool-Neutral
Tarski-Euclidean

begin

18 Highschool Euclidean dimensionless

context *Tarski-Euclidean*

begin

lemma *triangle-mid-par-strict-cong-aux*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B ParStrict Q P* ∧ *Cong A R P Q* ∧ *Cong B R P Q*
⟨*proof*⟩

lemma *triangle-par-mid*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
A B Par Q P **and**
Col Q A C
shows *Q Midpoint A C*
⟨*proof*⟩

lemma *triangle-mid-par-strict-cong-1*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B ParStrict Q P* ∧ *Cong A R P Q*
⟨*proof*⟩

lemma *triangle-mid-par-strict-cong-2*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B ParStrict Q P* ∧ *Cong B R P Q*
⟨*proof*⟩

lemma *triangle-mid-par-strict-cong-simp*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C
shows *A B ParStrict Q P*
⟨*proof*⟩

lemma *triangle-mid-par-strict-cong*:
assumes ¬ *Col A B C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B ParStrict Q P* ∧ *A C ParStrict P R* ∧
B C ParStrict Q R ∧ *Cong A R P Q* ∧
Cong B R P Q ∧ *Cong A Q P R* ∧
Cong C Q P R ∧ *Cong B P Q R* ∧ *Cong C P Q R*
⟨*proof*⟩

lemma *triangle-mid-par-flat-cong-aux*:
assumes *A* ≠ *B* **and**
Col A B C **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B Par Q P* ∧ *Cong A R P Q* ∧ *Cong B R P Q*
⟨*proof*⟩

lemma *triangle-mid-par-flat-cong-1*:
assumes *A* ≠ *B* **and**
Col A B C **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B Par Q P* ∧ *Cong A R P Q*
⟨*proof*⟩

lemma *triangle-mid-par-flat-cong-2*:
assumes *A* ≠ *B* **and**
Col A B C **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B Par Q P* ∧ *Cong B R P Q*
⟨*proof*⟩

lemma *triangle-mid-par-flat-cong*:
assumes *A* ≠ *B* **and**
A ≠ *C* **and**
B ≠ *C* **and**

Col A B C **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B Par Q P* ∧ *A C Par P R* ∧ *B C Par Q R* ∧
Cong A R P Q ∧ *Cong B R P Q* ∧ *Cong A Q P R* ∧
Cong C Q P R ∧ *Cong B P Q R* ∧ *Cong C P Q R*
⟨*proof*⟩

lemma *triangle-mid-par-flat*:
assumes *A* ≠ *B* **and**
Col A B C **and**
P Midpoint B C **and**
Q Midpoint A C
shows *A B Par Q P*
⟨*proof*⟩

lemma *triangle-mid-par*:
assumes *A* ≠ *B* **and**
P Midpoint B C **and**
Q Midpoint A C
shows *A B Par Q P*
⟨*proof*⟩

lemma *triangle-mid-par-cong*:
assumes *A* ≠ *B* **and**
A ≠ *C* **and**
B ≠ *C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *A B Par Q P* ∧ *A C Par P R* ∧ *B C Par Q R* ∧
Cong A R P Q ∧ *Cong B R P Q* ∧ *Cong A Q P R* ∧
Cong C Q P R ∧ *Cong B P Q R* ∧ *Cong C P Q R*
⟨*proof*⟩

lemma *triangle-mid-par-cong-1*:
assumes *B* ≠ *C* **and**
P Midpoint B C **and**
Q Midpoint A C **and**
R Midpoint A B
shows *B C Par Q R* ∧ *Cong B P Q R*
⟨*proof*⟩

lemma *midpoint-thales*:
assumes ¬ *Col A B C* **and**
P Midpoint A B **and**
Cong P A P C
shows *Per A C B*
⟨*proof*⟩

lemma *midpoint-thales-reci*:
assumes *Per A C B* **and**
P Midpoint A B
shows *Cong P A P B* ∧ *Cong P B P C*
⟨*proof*⟩

lemma *midpoint-thales-reci-1*:
assumes *Per A C B* **and**
P Midpoint A B
shows *Cong P A P B*
⟨*proof*⟩

lemma *midpoint-thales-reci-2*:
assumes *Per A C B* **and**
P Midpoint A B

shows *Cong P B P C*
⟨*proof*⟩

lemma *Per-mid-rectangle*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per B A C **and**
I Midpoint B C **and**
J Midpoint A C **and**
K Midpoint A B
shows *Rectangle A J I K*
⟨*proof*⟩

lemma *varignon*:
assumes $A \neq C$ **and**
$B \neq D$ **and**
¬ *Col I J K* **and**
I Midpoint A B **and**
J Midpoint B C **and**
K Midpoint C D **and**
L Midpoint A D
shows *Parallelogram I J K L*
⟨*proof*⟩

lemma *varignon-aux-aux*:
assumes $A \neq C$ **and**
$J \neq L$ **and**
I Midpoint A B **and**
J Midpoint B C **and**
K Midpoint C D **and**
L Midpoint A D
shows *Parallelogram I J K L*
⟨*proof*⟩

lemma *varignon-aux*:
assumes
$J \neq L$ **and**
I Midpoint A B **and**
J Midpoint B C **and**
K Midpoint C D **and**
L Midpoint A D
shows *Parallelogram I J K L*
⟨*proof*⟩

lemma *varignon-bis*:
assumes $A \neq C \lor B \neq D$ **and**
I Midpoint A B **and**
J Midpoint B C **and**
K Midpoint C D **and**
L Midpoint A D
shows *Parallelogram I J K L*
⟨*proof*⟩

lemma *quadrileral-midpoints*:
assumes ¬ *Col I J K* **and**
I Midpoint A B **and**
J Midpoint B C **and**
K Midpoint C D **and**
L Midpoint A D **and**
X Midpoint I K **and**
Y Midpoint J L
shows $X = Y$
⟨*proof*⟩

lemma *is-circumcenter-coplanar*:

assumes *G IsCircumcenter A B C*
shows *Coplanar G A B C*
⟨*proof*⟩

lemma *circumcenter-cong-1*:
assumes *G IsCircumcenter A B C*
shows *Cong A G B G*
⟨*proof*⟩

lemma *circumcenter-cong-2*:
assumes *G IsCircumcenter A B C*
shows *Cong B G C G*
⟨*proof*⟩

lemma *circumcenter-cong-3*:
assumes *G IsCircumcenter A B C*
shows *Cong C G A G*
⟨*proof*⟩

lemma *circumcenter-cong*:
assumes *G IsCircumcenter A B C*
shows *Cong A G B G* ∧ *Cong B G C G* ∧ *Cong C G A G*
⟨*proof*⟩

lemma *is-circumcenter-perm-1*:
assumes *G IsCircumcenter A B C*
shows*G IsCircumcenter A C B*
⟨*proof*⟩

lemma *is-circumcenter-perm-2*:
assumes *G IsCircumcenter A B C*
shows*G IsCircumcenter B A C*
⟨*proof*⟩

lemma *is-circumcenter-perm-3*:
assumes *G IsCircumcenter A B C*
shows*G IsCircumcenter B C A*
⟨*proof*⟩

lemma *is-circumcenter-perm-4*:
assumes *G IsCircumcenter A B C*
shows*G IsCircumcenter C A B*
⟨*proof*⟩

lemma *is-circumcenter-perm-5*:
assumes *G IsCircumcenter A B C*
shows*G IsCircumcenter C B A*
⟨*proof*⟩

lemma *is-circumcenter-perm*:
assumes *G IsCircumcenter A B C*
shows*G IsCircumcenter A B C* ∧ *G IsCircumcenter A C B* ∧
G IsCircumcenter B A C ∧ *G IsCircumcenter B C A* ∧
G IsCircumcenter C A B ∧ *G IsCircumcenter C B A*
⟨*proof*⟩

lemma *is-circumcenter-cases*:
assumes *G IsCircumcenter A B C* ∨ *G IsCircumcenter A C B* ∨
G IsCircumcenter B A C ∨ *G IsCircumcenter B C A* ∨
G IsCircumcenter C A B ∨ *G IsCircumcenter C B A*
shows *G IsCircumcenter A B C*
⟨*proof*⟩

lemma *circumcenter-perp*:
assumes
$B \neq C$ **and**

$G \neq A'$ **and**
G *IsCircumcenter* A B C **and**
A' *Midpoint* B C
shows G A' *PerpBisect* B C
⟨*proof*⟩

lemma *exists-circumcenter*:
assumes $\neg$ *Col* A B C
shows $\exists$ G. G *IsCircumcenter* A B C
⟨*proof*⟩

lemma *circumcenter-perp-all*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
$G \neq A'$ **and**
$G \neq B'$ **and**
$G \neq C'$ **and**
G *IsCircumcenter* A B C **and**
A' *Midpoint* B C **and**
B' *Midpoint* A C **and**
C' *Midpoint* A B
shows G A' *PerpBisect* B C $\wedge$ G B' *PerpBisect* A C $\wedge$ G C' *PerpBisect* A B
⟨*proof*⟩

lemma *circumcenter-intersect*:
assumes $A \neq B$ **and**
$G \neq C'$ **and**
C' *Midpoint* A B **and**
G A' *PerpBisect* B C **and**
G B' *PerpBisect* A C
shows G C' *Perp* A B
⟨*proof*⟩

lemma *is-circumcenter-uniqueness*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
$A \neq C$ **and**
G1 IsCircumcenter A B C **and**
G2 IsCircumcenter A B C
shows *G1* = *G2*
⟨*proof*⟩

lemma *midpoint-thales-reci-circum*:
assumes *Per* A C B **and**
P *Midpoint* A B
shows P *IsCircumcenter* A B C
⟨*proof*⟩

lemma *circumcenter-per*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Per A B C **and**
P *IsCircumcenter* A B C
shows P *Midpoint* A C
⟨*proof*⟩

lemma *is-orthocenter-coplanar*:
assumes H *IsOrthocenter* A B C
shows *Coplanar* H A B C
⟨*proof*⟩

lemma *construct-intersection*:
assumes $\neg$ *Col* A B C **and**
A C *Par* B *X1* **and**

A B Par C X2 **and**
B C Par A X3
shows ∃ *E. Col E A X3* ∧ *Col E B X1*
⟨*proof*⟩

lemma *not-col-par-col-diff*:
assumes ¬ *Col A B C* **and**
A B Par C D **and**
Col C D E
shows *A* ≠ *E*
⟨*proof*⟩

lemma *construct-triangle*:
assumes ¬ *Col A B C*
shows ∃ *D E F*.
Col B D F ∧ *Col A E F* ∧ *Col C D E* ∧
A B Par C D ∧ *A C Par B D* ∧ *B C Par A E* ∧
A B Par C E ∧ *A C Par B F* ∧ *B C Par A F* ∧
D ≠ *E* ∧ *D* ≠ *F* ∧ *E* ≠ *F*
⟨*proof*⟩

lemma *diff-not-col-col-par4-mid*:
assumes *D* ≠ *E* **and**
¬ *Col A B C* **and**
Col C D E **and**
A B Par C D **and**
A B Par C E **and**
A E Par B C **and**
A C Par B D
shows *C Midpoint D E*
⟨*proof*⟩

lemma *altitude-is-perp-bisect*:
assumes *A* ≠ *P* **and**
E ≠ *F* **and**
A A1 Perp B C **and**
Col P A1 A **and**
Col A E F **and**
B C Par A E **and**
A Midpoint E F
shows *A P PerpBisect E F*
⟨*proof*⟩

lemma *altitude-intersect*:
assumes ¬ *Col A B C* **and**
A A1 Perp B C **and**
B B1 Perp A C **and**
C C1 Perp A B **and**
Col P A A1 **and**
Col P B B1
shows *Col P C C1*
⟨*proof*⟩

lemma *IsOrthocenter-cases*:
assumes *G IsOrthocenter A B C* ∨ *G IsOrthocenter A C B* ∨
G IsOrthocenter B A C ∨ *G IsOrthocenter B C A* ∨
G IsOrthocenter C A B ∨ *G IsOrthocenter C B A*
shows *G IsOrthocenter A B C*
⟨*proof*⟩

lemma *IsOrthocenter-perm*:
assumes *G IsOrthocenter A B C*
shows *G IsOrthocenter A B C* ∧ *G IsOrthocenter A C B* ∧
G IsOrthocenter B A C ∧ *G IsOrthocenter B C A* ∧
G IsOrthocenter C A B ∧ *G IsOrthocenter C B A*

⟨*proof*⟩

lemma *IsOrthocenter-perm-1*:
assumes *G IsOrthocenter A B C*
shows *G IsOrthocenter A C B*
⟨*proof*⟩

lemma *IsOrthocenter-perm-2*:
assumes *G IsOrthocenter B A C*
shows *G IsOrthocenter A C B*
⟨*proof*⟩

lemma *IsOrthocenter-perm-3*:
assumes *G IsOrthocenter B C A*
shows *G IsOrthocenter A C B*
⟨*proof*⟩

lemma *IsOrthocenter-perm-4*:
assumes *G IsOrthocenter C A B*
shows *G IsOrthocenter A C B*
⟨*proof*⟩

lemma *IsOrthocenter-perm-5*:
assumes *G IsOrthocenter C B A*
shows *G IsOrthocenter A C B*
⟨*proof*⟩

lemma *orthocenter-per*:
assumes *Per A B C* **and**
H IsOrthocenter A B C
shows *H* = *B*
⟨*proof*⟩

lemma *orthocenter-col*:
assumes *Col H B C* **and**
H IsOrthocenter A B C
shows *H* = *B* ∨ *H* = *C*
⟨*proof*⟩

lemma *intersection-two-medians-exist*:
assumes ¬ *Col A B C* **and**
I Midpoint B C **and**
J Midpoint A C
shows ∃ *G*. *Col G A I* ∧ *Col G B J*
⟨*proof*⟩

lemma *intersection-two-medians-exist-unique*:
assumes ¬ *Col A B C* **and**
I Midpoint B C **and**
J Midpoint A C
shows ∃! *G*. *Col G A I* ∧ *Col G B J*
⟨*proof*⟩

lemma *intersection-two-medians-unique-R1*:
assumes ¬ *Col A B C* **and**
I Midpoint B C **and**
J Midpoint A C **and**
Col G1 A I **and**
Col G1 B J **and**
Col G2 A I **and**
Col G2 B J
shows *G1* = *G2*
⟨*proof*⟩

lemma *is-gravity-center-coplanar*:
assumes *G IsGravityCenter A B C*

shows *Coplanar G A B C*
⟨*proof*⟩

lemma *is-gravity-center-exist-unique*:
assumes ¬ *Col A B C*
shows ∃! *G*. *G IsGravityCenter A B C*
⟨*proof*⟩

lemma *three-medians-intersect*:
assumes ¬ *Col A B C* **and**
I Midpoint B C **and**
J Midpoint A C **and**
K Midpoint A B
shows ∃ *G*. *Col G A I* ∧ *Col G B J* ∧ *Col G C K*
⟨*proof*⟩

lemma *is-gravity-center-col*:
assumes *G IsGravityCenter A B C* **and**
I Midpoint A B
shows *Col G I C*
⟨*proof*⟩

lemma *is-gravity-center-diff-1*:
assumes *G IsGravityCenter A B C*
shows $G \neq A$
⟨*proof*⟩

lemma *is-gravity-center-diff-2*:
assumes *G IsGravityCenter A B C*
shows $G \neq B$
⟨*proof*⟩

lemma *is-gravity-center-diff-3*:
assumes *G IsGravityCenter A B C*
shows $G \neq C$
⟨*proof*⟩

lemma *is-gravity-center-third-aux-lem*:
assumes *G IsGravityCenter A B C* **and**
I Midpoint B C **and** *J Midpoint A C* **and**
Col G A I **and** *Col G B J*
shows $G \neq I \land G \neq J$
⟨*proof*⟩

lemma *is-gravity-center-third-aux-1*:
assumes *G IsGravityCenter A B C*
shows ¬ *Col B G C*
⟨*proof*⟩

lemma *is-gravity-center-third-aux-2*:
assumes *G IsGravityCenter A B C*
shows ¬ *Col C G A*
⟨*proof*⟩

lemma *is-gravity-center-third-aux-3*:
assumes *G IsGravityCenter A B C*
shows ¬ *Col A G B*
⟨*proof*⟩

lemma *is-gravity-center-third*:
assumes *G IsGravityCenter A B C* **and**
G' Midpoint A G **and**
A' Midpoint B C
shows *G Midpoint A' G'*

⟨proof⟩

lemma *is-gravity-center-third-reci*:
assumes A' *Midpoint B C* **and**
A'' *Midpoint A G* **and**
G Midpoint A' A'' **and**
¬ *Col A B C*
shows *G IsGravityCenter A B C*
⟨proof⟩

lemma *is-gravity-center-perm*:
assumes *G IsGravityCenter A B C*
shows *G IsGravityCenter A B C* ∧ *G IsGravityCenter A C B* ∧
G IsGravityCenter B A C ∧ *G IsGravityCenter B C A* ∧
G IsGravityCenter C A B ∧ *G IsGravityCenter C B A*
⟨proof⟩

lemma *is-gravity-center-perm-1*:
assumes *G IsGravityCenter A B C*
shows *G IsGravityCenter A C B*
⟨proof⟩

lemma *is-gravity-center-perm-2*:
assumes *G IsGravityCenter A B C*
shows *G IsGravityCenter B A C*
⟨proof⟩

lemma *is-gravity-center-perm-3*:
assumes *G IsGravityCenter A B C*
shows *G IsGravityCenter B C A*
⟨proof⟩

lemma *is-gravity-center-perm-4*:
assumes *G IsGravityCenter A B C*
shows *G IsGravityCenter C A B*
⟨proof⟩

lemma *is-gravity-center-perm-5*:
assumes *G IsGravityCenter A B C*
shows *G IsGravityCenter C B A*
⟨proof⟩

lemma *is-gravity-center-cases*:
assumes *G IsGravityCenter A B C* ∨ *G IsGravityCenter A C B* ∨
G IsGravityCenter B A C ∨ *G IsGravityCenter B C A* ∨
G IsGravityCenter C A B ∨ *G IsGravityCenter C B A*
shows *G IsGravityCenter A B C*
⟨proof⟩

lemma *concyclic-aux*:
assumes *Concyclic2 A B C D*
shows ∃ *P*. *Cong P A P B* ∧ *Cong P A P C* ∧ *Cong P A P D* ∧ *Coplanar A B C P*
⟨proof⟩

lemma *concyclic-trans*:
assumes ¬ *Col A B C* **and**
Concyclic2 A B C D **and**
Concyclic2 A B C E
shows *Concyclic2 A B D E*
⟨proof⟩

lemma *concyclic-perm-1*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 A B D C*
⟨proof⟩

lemma *concyclic-perm-2*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 A C B D*
⟨*proof*⟩

lemma *concyclic-perm-3*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 A C D B*
⟨*proof*⟩

lemma *concyclic-perm-4*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 A D B C*
⟨*proof*⟩

lemma *concyclic-perm-5*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 A D C B*
⟨*proof*⟩

lemma *concyclic-perm-6*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 B A C D*
⟨*proof*⟩

lemma *concyclic-perm-7*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 B A D C*
⟨*proof*⟩

lemma *concyclic-perm-8*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 B C A D*
⟨*proof*⟩

lemma *concyclic-perm-9*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 B C D A*
⟨*proof*⟩

lemma *concyclic-perm-10*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 B D A C*
⟨*proof*⟩

lemma *concyclic-perm-11*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 B D C A*
⟨*proof*⟩

lemma *concyclic-perm-12*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 C A B D*
⟨*proof*⟩

lemma *concyclic-perm-13*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 C A D B*
⟨*proof*⟩

lemma *concyclic-perm-14*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 C B A D*
⟨*proof*⟩

lemma *concyclic-perm-15*:

assumes *Concyclic2 A B C D*
shows *Concyclic2 C B D A*
⟨*proof*⟩

lemma *concyclic-perm-16*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 C D A B*
⟨*proof*⟩

lemma *concyclic-perm-17*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 C D B A*
⟨*proof*⟩

lemma *concyclic-perm-18*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 D A B C*
⟨*proof*⟩

lemma *concyclic-perm-19*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 D A C B*
⟨*proof*⟩

lemma *concyclic-perm-20*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 D B A C*
⟨*proof*⟩

lemma *concyclic-perm-21*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 D B C A*
⟨*proof*⟩

lemma *concyclic-perm-22*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 D C A B*
⟨*proof*⟩

lemma *concyclic-perm-23*:
assumes *Concyclic2 A B C D*
shows *Concyclic2 D C B A*
⟨*proof*⟩

lemma *concyclic-1123*:
assumes $\neg$ *Col A B C*
shows *Concyclic2 A A B C*
⟨*proof*⟩

lemma *concyclic-not-col-or-eq-aux*:
assumes *Concyclic2 A B C D*
shows $A = B \vee A = C \vee B = C \vee \neg$ *Col A B C*
⟨*proof*⟩

lemma *concyclic-not-col-or-eq*:
assumes *Concyclic2 A B C A'*
shows $A' = C \vee A' = B \vee A = B \vee A = C \vee A = A' \vee (\neg\ Col\ A\ B\ A' \wedge \neg\ Col\ A\ C\ A')$
⟨*proof*⟩

lemma *Euler-line-special-case*:
assumes *Per A B C* **and**
G IsGravityCenter A B C **and**
H IsOrthocenter A B C **and**
P IsCircumcenter A B C
shows *Col G H P*
⟨*proof*⟩

lemma *gravity-center-change-triangle*:
  assumes *G IsGravityCenter A B C* **and**
    I Midpoint B C **and**
    I Midpoint B′ C′ **and**
    ¬ *Col A B′ C′*
  shows *G IsGravityCenter A B′ C′*
⟨*proof*⟩

lemma *Euler-line*:
  assumes ¬ *Col A B C* **and**
    G IsGravityCenter A B C **and**
    H IsOrthocenter A B C **and**
    P IsCircumcenter A B C
  shows *Col G H P*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex35*:
  assumes ¬ *Col G A Z* **and**
    Per G A Z **and**
    F Midpoint A Z **and**
    E Midpoint G Z **and**
    R Midpoint G A
  shows *Rectangle F E R A*
  ⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex36-aux*:
  assumes ¬ *Col A B C* **and**
    I Midpoint A B **and**
    J Midpoint A C **and**
    K Midpoint B C
  shows *Plg I J K B*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex36*:
  assumes *B A C isosceles* **and**
    A H Perp B C **and**
    Col B H C **and**
    I Midpoint A B **and**
    J Midpoint A C
  shows *Rhombus A I H J*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex37*:
  assumes ¬ *Col S E L* **and**
    I Midpoint L S **and**
    M Midpoint S E **and**
    A Midpoint E L **and**
    A P PerpBisect L E **and**
    Coplanar S E L P
  shows *A P Perp I M*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex38* :
assumes ¬ *Col E A U* **and**
N Midpoint E A **and**
M Midpoint E U **and**
L Midpoint U A
shows ∃ *P*. *Col P E L* ∧ *P Midpoint M N*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex39*:
assumes ¬ *Col A C T* **and**
S IsCircumcenter C T A **and**
S Midpoint C T **and**
Col C H A **and**
S H Perp A C
shows *H Midpoint A C*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex40*:
assumes ¬ *Col A B C* **and**
Bet C R B **and**
M Midpoint A B **and**
N Midpoint A C **and**
S Midpoint B R **and**
T Midpoint R C
shows *M S Par N T* ∧ *Parallelogram M S T N*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex41* :
assumes ¬ *Col T L H* **and**
E Midpoint T L **and**
P Midpoint T H **and**
A ≠ *T* **and**
A ≠ *P* **and**
A ≠ *E* **and**
A ≠ *S* **and**
Bet T A S **and**
Bet P A E **and**
Bet H S L **and**
S ≠ *H* **and**
S ≠ *L*
shows *S A E CongA T S H* ∧ *A Midpoint T S*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex42*:
assumes ¬ *Col A B C* **and**
G IsGravityCenter A B C **and**
I Midpoint A C **and**
J Midpoint A B **and**
K Midpoint B G **and**
L Midpoint C G
shows *Parallelogram I J K L*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex44-1*:
  assumes *ParallelogramStrict A B C D* **and**
    I Midpoint A D **and**
    J Midpoint B C
  shows *Parallelogram B J D I*
  ⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex45*:
  assumes ¬ *Col A B C* **and**
    I Midpoint B C **and**
    Col I A M **and**
    A Midpoint I M **and**
    J Midpoint A I **and**
    J K Par A C **and**
    Col K I C **and**
    Col G C A **and**
    Col G M K
  shows *K Midpoint I C* ∧ *A K Par M C* ∧ *G IsGravityCenter C M I*
⟨*proof*⟩

lemma *sesamath-4ieme-G2-ex47* :
  assumes ¬ *Col A C C′* **and**
    ¬ *Col A A′ C′* **and**
    Col B′ A′ C′ **and**
    B Midpoint A C **and**
    A A′ Par B B′ **and**
    A A′ Par C C′
  shows *B′ Midpoint A′ C′*
⟨*proof*⟩

end
end

theory *Highschool-Euclidean-2D*

imports
  Highschool-Euclidean
  Tarski-Euclidean-2D

begin

19 Highschool Euclidean 2D

context *Tarski-Euclidean-2D*

begin

19.1 Definitions

19.2 Propositions

lemma *NewEX1*:
  assumes *Square A B C D* **and**
    Cong K A K C **and**
    Cong K A A C
  shows *Col B K D*
⟨*proof*⟩

end
end

theory *Tarski-Non-Euclidean*

imports
Tarski-Postulate-Parallels

begin

20 Tarski Non Euclidean

20.1 Definitions

locale *Tarski-Non-Euclidean* = *Tarski-neutral-dimensionless* +
fixes *A0 B0 C0 D0 T0*
assumes *NERule1*: *Bet A0 D0 T0*
and *NERule2*: *Bet B0 D0 C0*
and *NERule3*: $A0 \neq D0$
and *not-tarski-s-parallel-postulate-aux*:
$\forall\ X\ Y.\ ((Bet\ A0\ B0\ X \wedge Bet\ A0\ C0\ Y) \longrightarrow \neg\ Bet\ X\ T0\ Y)$

context *Tarski-Non-Euclidean*

begin

20.2 Propositions

lemma *Tarski-Pre-Non-Euclidean-aux*:
shows $\exists\ A\ B\ C\ D\ T.\ \neg\ ((Bet\ A\ D\ T \wedge Bet\ B\ D\ C \wedge A \neq D) \longrightarrow (\exists\ X\ Y.\ Bet\ A\ B\ X \wedge Bet\ A\ C\ Y \wedge Bet\ X\ T\ Y))$
⟨*proof*⟩

lemma *NPost01*:
shows ¬ *Postulate01*
⟨*proof*⟩

lemma *NPost02*:
shows ¬ *Postulate02*
⟨*proof*⟩

lemma *NPost05*:
shows ¬ *Postulate05*
⟨*proof*⟩

lemma *NPost06*:
shows ¬ *Postulate06*
⟨*proof*⟩

lemma *NPost07*:
shows ¬ *Postulate07*
⟨*proof*⟩

lemma *NPost08*:
shows ¬ *Postulate08*
⟨*proof*⟩

lemma *NPost09*:
shows ¬ *Postulate09*
⟨*proof*⟩

lemma *NPost10*:
shows ¬ *Postulate10*
⟨*proof*⟩

lemma *NPost11*:
shows ¬ *Postulate11*
⟨*proof*⟩

lemma *NPost12*:
shows ¬ *Postulate12*
⟨*proof*⟩

lemma *NPost13*:
shows ¬ *Postulate13*
⟨*proof*⟩

lemma *NPost14*:
shows ¬ *Postulate14*
⟨*proof*⟩

lemma *NPost15*:
shows ¬ *Postulate15*
⟨*proof*⟩

lemma *NPost16*:
shows ¬ *Postulate16*
⟨*proof*⟩

lemma *NPost17*:
shows ¬ *Postulate17*
⟨*proof*⟩

lemma *NPost18*:
shows ¬ *Postulate18*
⟨*proof*⟩

lemma *NPost19*:
shows ¬ *Postulate19*
⟨*proof*⟩

lemma *NPost20*:
shows ¬ *Postulate20*
⟨*proof*⟩

theorem *not-tarski-s-parallel-postulate-thm*:
shows ∃ *A B C D T*. (*Bet A D T* ∧ *Bet B D C* ∧ *A* ≠ *D* ∧
(∀ *X Y*. (*Bet A B X* ∧ *Bet A C Y*) ⟶ ¬ *Bet X T Y*))
⟨*proof*⟩

theorem *not-playfair-s-postulate-thm*:
shows ∃ *A1 A2 B1 B2 C1 C2 P*. *A1 A2 Par B1 B2* ∧
Col P B1 B2 ∧ *A1 A2 Par C1 C2* ∧ *Col P C1 C2* ∧
(¬ *Col C1 B1 B2* ∨ ¬ *Col C2 B1 B2*)
⟨*proof*⟩

theorem *not-postulate-of-transitivity-of-parallelism-thm*:
shows ∃ *A1 A2 B1 B2 C1 C2*. *A1 A2 Par B1 B2* ∧ *B1 B2 Par C1 C2* ∧ ¬ (*A1 A2 Par C1 C2*)
⟨*proof*⟩

theorem *not-midpoint-converse-postulate-thm*:
shows ∃ *A B C P Q*. ¬ *Col A B C* ∧ *P Midpoint B C* ∧ *A B Par Q P* ∧ *Col A C Q* ∧ ¬ *Q Midpoint A C*
⟨*proof*⟩

theorem *not-alternate-interior-angles-postulate-thm*:
shows ∃ *A B C D*. *A C TS B D* ∧ *A B Par C D* ∧ ¬ *B A C CongA D C A*
⟨*proof*⟩

theorem *not-consecutive-interior-angles-postulate-thm*:
shows ∃ *A B C D*. *B C OS A D* ∧ *A B Par C D* ∧ ¬ *A B C SuppA B C D*

⟨proof⟩

theorem *not-perpendicular-transversal-postulate-thm*:
shows ∃ *A B C D P Q*. *A B Par C D* ∧ *A B Perp P Q* ∧ *Coplanar C D P Q* ∧ ¬ *C D Perp P Q*
⟨proof⟩

theorem *not-postulate-of-parallelism-of-perpendicular-transversals-thm*:
shows ∃ *A1 A2 B1 B2 C1 C2 D1 D2*.
A1 A2 Par B1 B2 ∧ *A1 A2 Perp C1 C2* ∧ *B1 B2 Perp D1 D2* ∧
Coplanar A1 A2 C1 D1 ∧ *Coplanar A1 A2 C1 D2* ∧
Coplanar A1 A2 C2 D1 ∧ *Coplanar A1 A2 C2 D2* ∧ ¬ *C1 C2 Par D1 D2*
⟨proof⟩

theorem *not-universal-posidonius-postulate*:
shows ∃ *A1 A2 A3 A4 B1 B2 B3 B4*.
A1 A2 Par B1 B2 ∧ *Col A1 A2 A3* ∧ *Col B1 B2 B3* ∧ *A1 A2 Perp A3 B3* ∧
Col A1 A2 A4 ∧ *Col B1 B2 B4* ∧ *A1 A2 Perp A4 B4* ∧ ¬ *Cong A3 B3 A4 B4*
⟨proof⟩

theorem *not-alternative-playfair-s-postulate-thm*:
shows ∃ *A1 A2 B1 B2 C1 C2 P*.
P Perp2 A1 A2 B1 B2 ∧ ¬ *Col A1 A2 P* ∧ *Col P B1 B2* ∧
Coplanar A1 A2 B1 B2 ∧ *A1 A2 Par C1 C2* ∧ *Col P C1 C2* ∧
(¬ *Col C1 B1 B2* ∨ ¬ *Col C2 B1 B2*)
⟨proof⟩

theorem *not-proclus-postulate-thm*:
shows ∃ *A B C D P Q*. (∀ *Y*. *A B Par C D* ∧ *Col A B P* ∧
¬ *Col A B Q* ∧ *Coplanar C D P Q* ∧ (¬ *Col P Q Y* ∨ ¬ *Col C D Y*))
⟨proof⟩

theorem *not-alternative-proclus-postulate-thm*:
shows ∃ *A B C D P Q*. (∀ *Y*. *P Perp2 A B C D* ∧ ¬ *Col C D P* ∧ *Coplanar A B C D* ∧ *Col A B P* ∧
¬ *Col A B Q* ∧ *Coplanar C D P Q* ∧ (¬ *Col P Q Y* ∨ ¬ *Col C D Y*))
⟨proof⟩

theorem *not-triangle-circumscription-principle-thm*:
shows ∃ *A B C*. ∀ *D*. (¬ *Col A B C* ∧ (¬ *Cong A D B D* ∨ ¬ *Cong A D C D* ∨ ¬ *Coplanar A B C D*))
⟨proof⟩

theorem *not-inverse-projection-postulate-thm*:
shows ∃ *A B C P Q*. ∀ *Y*.
Acute A B C ∧ *B Out A P* ∧ *P* ≠ *Q* ∧ *Per B P Q* ∧ *Coplanar A B C Q* ∧
(¬ *B Out C Y* ∨ ¬ *Col P Q Y*)
⟨proof⟩

theorem *not-euclid-5-thm*:
shows ∃ *P Q R S T U*. ∀ *I*.
BetS P T Q ∧ *BetS R T S* ∧ *BetS Q U R* ∧ ¬ *Col P Q S* ∧
Cong P T Q T ∧ *Cong R T S T* ∧ (¬ *BetS S Q I* ∨ ¬ *BetS P U I*)
⟨proof⟩

theorem *not-strong-parallel-postulate-thm*:
shows ∃ *P Q R S T U*. ∀ *I*.
BetS P T Q ∧ *BetS R T S* ∧ ¬ *Col P R U* ∧ *Coplanar P Q R U* ∧
Cong P T Q T ∧ *Cong R T S T* ∧ (¬ *Col S Q I* ∨ ¬ *Col P U I*)
⟨proof⟩

theorem *not-alternative-strong-parallel-postulate-thm*:
shows ∃ *A B C D P Q R*. ∀ *Y*.
B C OS A D ∧ *A B C B C D SumA P Q R* ∧ ¬ *Bet P Q R* ∧ (¬ *Col B A Y* ∨ ¬ *Col C D Y*)
⟨proof⟩

theorem *not-euclid-s-parallel-postulate-thm*:
shows ∃ *A B C D P Q R*. ∀ *Y*.
B C OS A D ∧ *SAMS A B C B C D* ∧ *A B C B C D SumA P Q R* ∧

¬ *Bet P Q R* ∧ (¬ *B Out A Y* ∨ ¬ *C Out D Y*)
⟨*proof*⟩

end
end

theory *Tarski-Non-Euclidean-Aristotle*

imports
Tarski-Non-Euclidean

begin

21 Tarski Non Euclidean Aristotle

21.1 Definitions

locale *Tarski-Non-Euclidean-Aristotle* = *Tarski-Non-Euclidean* +
assumes *aristotle*: *aristotle-s-axiom*

context *Tarski-Non-Euclidean-Aristotle*

begin

21.2 Propositions

lemma *NPost03*:
shows ¬ *Postulate03*
⟨*proof*⟩

lemma *NPost21*:
shows ¬ *Postulate21*
⟨*proof*⟩

lemma *NPost22*:
shows ¬ *Postulate22*
⟨*proof*⟩

lemma *NPost23*:
shows ¬ *Postulate23*
⟨*proof*⟩

lemma *NPost24*:
shows ¬ *Postulate24*
⟨*proof*⟩

lemma *NPost25*:
shows ¬ *Postulate25*
⟨*proof*⟩

lemma *NPost26*:
shows ¬ *Postulate26*
⟨*proof*⟩

lemma *NPost27*:
shows ¬ *Postulate27*
⟨*proof*⟩

lemma *NPost28*:
shows ¬ *Postulate28*
⟨*proof*⟩

lemma *NPost29*:
shows ¬ *Postulate29*
⟨*proof*⟩

lemma *NPost30*:
shows ¬ *Postulate30*
⟨*proof*⟩

theorem *not-triangle-postulate-thm*:
shows ∃ *A B C D E F*. *A B C TriSumA D E F* ∧ ¬ *Bet D E F*
⟨*proof*⟩

theorem *not-postulate-of-existence-of-a-triangle-whose-angles-sum-to-two-rights-thm*:
shows ∀ *A B C D E F*. *A B C TriSumA D E F* ∧ *Bet D E F* ⟶ *Col A B C*
⟨*proof*⟩

theorem *not-posidonius-postulate-thm*:
shows ∀ *A1 A2 B1 B2*. *Col A1 A2 B1* ∨ *B1* = *B2* ∨ ¬ *Coplanar A1 A2 B1 B2* ∨
(∃ *A3 A4 B3 B4*. *Col A1 A2 A3* ∧ *Col B1 B2 B3* ∧ *A1 A2 Perp A3 B3* ∧
Col A1 A2 A4 ∧ *Col B1 B2 B4* ∧ *A1 A2 Perp A4 B4* ∧
¬ *Cong A3 B3 A4 B4*)
⟨*proof*⟩

theorem *not-posidonius-postulate-thm-1*:
shows ∀ *A1 A2 B*. *Col A1 A2 B* ∨ (∃ *A3 A4 B3 B4*. *Col A1 A2 A3* ∧ *Col B B B3* ∧ *A1 A2 Perp A3 B3* ∧
Col A1 A2 A4 ∧ *Col B B B4* ∧ *A1 A2 Perp A4 B4* ∧
¬ *Cong A3 B3 A4 B4*)
⟨*proof*⟩

theorem *not-postulate-of-existence-of-similar-triangles-thm*:
shows ∀ *A B C D E F*. (*Col A B C* ∨ *Cong A B D E* ∨ ¬ *A B C CongA D E F* ∨
¬ *B C A CongA E F D* ∨ ¬ *C A B CongA F D E*)
⟨*proof*⟩

theorem *not-thales-postulate-thm*:
shows ∃ *A B C M*. *M Midpoint A B* ∧ *Cong M A M C* ∧ ¬ *Per A C B*
⟨*proof*⟩

theorem *not-thales-converse-postulate-thm*:
shows ∃ *A B C M*. *M Midpoint A B* ∧ *Per A C B* ∧ ¬ *Cong M A M C*
⟨*proof*⟩

theorem *not-existential-thales-postulate-thm*:
shows ∀ *A B C M*. (*M Midpoint A B* ∧ *Cong M A M C* ∧ *Per A C B*) ⟶ *Col A B C*
⟨*proof*⟩

theorem *not-postulate-of-right-saccheri-quadrilaterals-thm*:
shows ∃ *A B C D*. *Saccheri A B C D* ∧ ¬ *Per A B C*
⟨*proof*⟩

theorem *not-postulate-of-existence-of-a-right-saccheri-quadrilateral-thm*:
shows ∀ *A B C D*. ¬ (*Saccheri A B C D* ∧ *Per A B C*)
⟨*proof*⟩

theorem *not-postulate-of-right-lambert-quadrilaterals-thm*:
shows ∃ *A B C D*. *Lambert A B C D* ∧ ¬ *Per B C D*
⟨*proof*⟩

theorem *not-postulate-of-existence-of-a-right-lambert-quadrilateral-thm*:
shows ∀ *A B C D*. ¬ (*Lambert A B C D* ∧ *Per B C D*)
⟨*proof*⟩

end
end

theory *Tarski-Non-Euclidean-Archimedean*

imports
Tarski-Non-Euclidean-Aristotle

begin

22 Tarski Non Euclidean

22.1 Definitions

locale *Tarski-Non-Euclidean-Archimedean* = *Tarski-Non-Euclidean-Aristotle* +
assumes *archimedes*: *archimedes-axiom*

context *Tarski-Non-Euclidean-Archimedean*

begin

22.2 Propositions

lemma *NPost04*:
shows ¬ *Postulate04*
⟨*proof*⟩

lemma *NPost31*:
shows ¬ *Postulate31*
⟨*proof*⟩

lemma *NPost32*:
shows ¬ *Postulate32*
⟨*proof*⟩

lemma *NPost33*:
shows ¬ *Postulate33*
⟨*proof*⟩

lemma *NPost34*:
shows ¬ *Postulate34*
⟨*proof*⟩

lemma *NPost35*:
shows ¬ *Postulate35*
⟨*proof*⟩

theorem *not-bachmann-s-lotschnittaxiom*:
shows ∃ *P Q R P1 R1*. *P* ≠ *Q* ∧ *Q* ≠ *R* ∧ *Per P Q R* ∧ *Per Q P P1* ∧ *Per Q R R1* ∧
Coplanar P Q R P1 ∧ *Coplanar P Q R R1* ∧
(∀ *S*. ¬ *Col P P1 S* ∨ ¬ *Col R R1 S*)
⟨*proof*⟩

theorem *not-weak-inverse-projection-postulate*:
shows ∃ *A B C D E F P Q Y*. *Acute A B C* ∧ *Per D E F* ∧ *A B C A B C SumA D E F* ∧
B Out A P ∧ *P* ≠ *Q* ∧ *Per B P Q* ∧ *Coplanar A B C Q* ∧
(¬ *B Out C Y* ∨ ¬ *Col P Q Y*)
⟨*proof*⟩

theorem *not-weak-tarski-s-parallel-postulate*:
shows ∃ *A B C T*. *Per A B C* ∧ *T InAngle A B C* ∧
(∀ *X Y*. ¬ *B Out A X* ∨ ¬ *B Out C Y* ∨ ¬ *Bet X T Y*)
⟨*proof*⟩

theorem *not-weak-triangle-circumscription-principle*:
shows ∃ *A B C A1 A2 B1 B2*. ¬ *Col A B C* ∧ *Per A C B* ∧ *A1 A2 PerpBisect B C* ∧
B1 B2 PerpBisect A C ∧ *Coplanar A B C A1* ∧ *Coplanar A B C A2* ∧
Coplanar A B C B1 ∧ *Coplanar A B C B2* ∧
(∀ *I*. ¬ *Col A1 A2 I* ∨ ¬ *Col B1 B2 I*)
⟨*proof*⟩

theorem *not-legendre-s-parallel-postulate*:
  fixes $A\ B\ C$
  assumes $\neg$ *Col* $A\ B\ C$
    and *Acute* $A\ B\ C$
  shows $\exists\ T.\ T$ *InAngle* $A\ B\ C \wedge (\forall\ X\ Y.\ \neg\ B$ *Out* $A\ X \vee \neg\ B$ *Out* $C\ Y \vee \neg$ *Bet* $X\ T\ Y)$
  ⟨*proof*⟩

theorem *not-existential-playfair-s-postulate*:
  assumes $\neg$ *Col A1 A2 P*
  shows $\exists$ *B1 B2 C1 C2*. *A1 A2 Par B1 B2* $\wedge$ *Col P B1 B2* $\wedge$ *A1 A2 Par C1 C2* $\wedge$ *Col P C1 C2* $\wedge$
      ($\neg$ *Col C1 B1 B2* $\vee$ $\neg$ *Col C2 B1 B2*)
  ⟨*proof*⟩

end
end

theory *Gupta-Neutral*

imports
  Tarski-Neutral

begin

23 Gupta inspired variant of Tarski - Geometry

23.1 Axioms - neutral dimension less

locale *Gupta-neutral-dimensionless* =
  fixes *GPA GPB GPC* :: $'p$
  and *BetG* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
  and *CongG* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$

assumes *cong-pseudo-reflexivityG*: $\forall\ a\ b.$

CongG $a\ b\ b\ a$

and *cong-inner-transitivityG*: $\forall\ a\ b\ p\ q\ r\ s.$

CongG $a\ b\ p\ q\ \wedge$
CongG $a\ b\ r\ s$
$\longrightarrow$
CongG $p\ q\ r\ s$

and *cong-identityG*: $\forall\ a\ b\ c.$

CongG $a\ b\ c\ c$
$\longrightarrow$
$a = b$

and *segment-constructionG*: $\forall\ a\ b\ c\ q.$

$\exists x.$ (*BetG* $q\ a\ x\ \wedge$ *CongG* $a\ x\ b\ c$)

and *five-segmentG*: $\forall\ a\ b\ c\ d\ a'\ b'\ c'\ d'.$

$a \neq b\ \wedge$
BetG $a\ b\ c\ \wedge$
BetG $a'\ b'\ c' \wedge$
CongG $a\ b\ a'\ b'\ \wedge$
CongG $b\ c\ b'\ c'\ \wedge$
CongG $a\ d\ a'\ d'\ \wedge$
CongG $b\ d\ b'\ d'$
$\longrightarrow$
CongG $c\ d\ c'\ d'$

and *between-symmetryG*: $\forall$ *a b c*.

BetG a b c
$\longrightarrow$
BetG c b a

and *between-inner-transitivityG*: $\forall$ *a b c d*.

BetG a b d $\wedge$
BetG b c d
$\longrightarrow$
BetG a b c

and *inner-paschG*: $\forall$ *a b c p q*.

BetG a p c $\wedge$
BetG b q c $\wedge$
$a \neq p \wedge$
$c \neq p \wedge$
$b \neq q \wedge$
$c \neq q \wedge$
$\neg$ (*BetG a b c* $\vee$ *BetG b c a* $\vee$ *BetG c a b*)
$\longrightarrow$
($\exists$ *x*. *BetG p x b* $\wedge$ *BetG q x a*)

and *lower-dimG*: $\neg$ (*BetG GPA GPB GPC* $\vee$ *BetG GPB GPC GPA* $\vee$ *BetG GPC GPA GPB*)

context *Gupta-neutral-dimensionless*

begin

23.2 Definitions

definition *ColG* :: *'p⇒'p⇒'p⇒bool* **where**
ColG A B C $\equiv$ *BetG A B C* $\vee$ *BetG B C A* $\vee$ *BetG C A B*

23.3 Propositions

lemma *g2-1*:
shows *CongG A B A B*
⟨*proof*⟩

lemma *g2-2*:
assumes *CongG A B C D*
shows *CongG C D A B*
⟨*proof*⟩

lemma *cong-inner-transitivityT*:
assumes *CongG A B C D* **and**
CongG A B E F
shows *CongG C D E F*
⟨*proof*⟩

lemma *g2-3*:
assumes *CongG A B C D*
shows *CongG B A C D*
⟨*proof*⟩

lemma *g2-4*:
assumes *CongG A B C D*
shows *CongG A B D C*
⟨*proof*⟩

lemma *g2-5-a*:

assumes *CongG A B C D* **and**
A = *B*
shows *C* = *D*
⟨*proof*⟩

lemma *g2-5-b*:
assumes *CongG A B C D* **and**
C = *D*
shows *A* = *B*
⟨*proof*⟩

lemma *g2-5*:
assumes *CongG A B C D*
shows $A = B \longleftrightarrow C = D$
⟨*proof*⟩

lemma *g2-6*:
shows *BetG A B B* ∧ *CongG B B A A*
⟨*proof*⟩

lemma *g2-7*:
assumes *CongG A B A′ B′* **and**
CongG B C B′ C′ **and**
BetG A B C **and**
BetG A′ B′ C′
shows *CongG A C A′ C′*
⟨*proof*⟩

lemma *g2-8*:
assumes *BetG A B C* **and**
BetG A B D **and**
CongG B C B D **and**
$A \neq B$
shows *C* = *D*
⟨*proof*⟩

lemma *g2-9*:
assumes *BetG A B A*
shows *BetG C A B*
⟨*proof*⟩

lemma *g2-10*:
assumes *BetG A B A*
shows *BetG C B A*
⟨*proof*⟩

lemma *g2-11*:
assumes *BetG A B A* **and**
$A \neq B$
shows ∃ *D*. *BetG C D C* ∧ *BetG D C D* ∧ $C \neq D$
⟨*proof*⟩

lemma *g2-12*:
assumes *BetG A B A*
shows *BetG A B C*
⟨*proof*⟩

lemma *g2-13*:
assumes *BetG A B A* **and**
$A \neq B$
shows *BetG C B C*
⟨*proof*⟩

lemma *g2-14-1*:
assumes *BetG A B A* **and**
$A \neq B$

shows *BetG C D C*
⟨*proof*⟩

lemma *g2-14-2*:
assumes *BetG A B A* **and**
$A \neq B$
shows *BetG D C D*
⟨*proof*⟩

lemma *g2-14*:
assumes *BetG A B A* **and**
$A \neq B$
shows *BetG C D C* ∧ *BetG D C D*
⟨*proof*⟩

lemma *g2-15*:
assumes *BetG A B A* **and**
$A \neq B$
shows *BetG C D E*
⟨*proof*⟩

lemma *g2-16*:
assumes ¬ *BetG C D E* **and**
BetG A B A
shows $A = B$
⟨*proof*⟩

lemma *between-identityT*:
assumes *BetG A B A*
shows $A = B$
⟨*proof*⟩

lemma *cong-trivial-identityT*:
shows *CongG A A B B*
⟨*proof*⟩

lemma *l2-11T*:
assumes *BetG A B C* **and**
BetG A′ B′ C′ **and**
CongG A B A′ B′ **and**
CongG B C B′ C′
shows *CongG A C A′ C′*
⟨*proof*⟩

lemma *construction-uniquenessT*:
assumes $Q \neq A$ **and**
BetG Q A X **and**
CongG A X B C **and**
BetG Q A Y **and**
CongG A Y B C
shows $X = Y$
⟨*proof*⟩

lemma *between-trivialT*:
shows *BetG A B B*
⟨*proof*⟩

lemma *bet-decG*:
shows *BetG A B C* ∨ ¬ *BetG A B C*
⟨*proof*⟩

lemma *col-decG*:
shows *ColG A B C* ∨ ¬ *ColG A B C*
⟨*proof*⟩

lemma *inner-paschT*:

assumes *BetG A P C* **and**
BetG B Q C
shows ∃ *x*. *BetG P x B* ∧ *BetG Q x A*
⟨*proof*⟩

end
end

theory *Gupta-Neutral-2D*

imports
Gupta-Neutral

begin

24 Gupta 2D

24.1 Axioms Gupta 2D

locale *Gupta-neutral-2D* = *Gupta-neutral-dimensionless GPA GPB GPC BetG CongG*
for *GPA GPB GPC* :: ′*p*
and *BetG* :: ′*p* ⇒ ′*p* ⇒ ′*p* ⇒ *bool*
and *CongG* :: ′*p* ⇒ ′*p* ⇒ ′*p* ⇒ ′*p* ⇒ *bool* +
assumes *upper-dimG*: ∀ *a b c p q*.
$p \neq q$ ∧
$a \neq b$ ∧
$a \neq c$ ∧
$c \neq b$ ∧
CongG a p a q ∧
CongG b p b q ∧
CongG c p c q
⟶
(*BetG a b c* ∨ *BetG b c a* ∨ *BetG c a b*)

context *Gupta-neutral-2D*

begin

24.2 Definitions

24.3 Propositions

lemma *upper-dimT* :
assumes $P \neq Q$ **and**
CongG A P A Q **and**
CongG B P B Q **and**
CongG C P C Q
shows *BetG A B C* ∨ *BetG B C A* ∨ *BetG C A B*
⟨*proof*⟩

end

end

theory *Gupta-Euclidean*

imports
Gupta-Neutral

begin

25 Gupta Euclidean

25.1 Axioms Gupta Euclidean

locale *Gupta-Euclidean* = *Gupta-neutral-dimensionless GPA GPB GPC BetG CongG*
for *GPA GPB GPC* :: $'p$
and *BetG* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$
and *CongG* :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ +
assumes *euclidG*: $\forall\ A\ B\ C\ D\ T.$
$BetG\ A\ D\ T \wedge BetG\ B\ D\ C \wedge B \neq D \wedge D \neq C \wedge$
$\neg\ (BetG\ A\ B\ C \vee BetG\ B\ C\ A \vee BetG\ C\ A\ B) \longrightarrow$
$(\exists\ x\ y.\ BetG\ A\ B\ x \wedge BetG\ A\ C\ y \wedge BetG\ x\ T\ y)$

25.2 Definitions

25.3 Propositions

end

theory *Hilbert-Neutral-2D*

imports *Hilbert-Neutral*

begin

26 Hilbert - Geometry - Neutral 2D

26.1 Axioms: Hilbert neutral 2D

locale *Hilbert-neutral-2D* = *Hilbert-neutral-dimensionless IncidL IncidP EqL EqP IsL IsP BetH CongH CongaH*
for
IncidL :: $'p \Rightarrow 'b \Rightarrow bool$ **and**
IncidP :: $'p \Rightarrow 'c \Rightarrow bool$ **and**
EqL :: $'b \Rightarrow 'b \Rightarrow bool$ **and**
EqP :: $'c \Rightarrow 'c \Rightarrow bool$ **and**
IsL :: $'b \Rightarrow bool$ **and**
IsP :: $'c \Rightarrow bool$ **and**
BetH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongaH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ +
assumes *pasch-2D* :
$IsL\ l \wedge\ \neg\ ColH\ A\ B\ C \wedge \neg\ IncidL\ C\ l \wedge cut\ l\ A\ B \longrightarrow (cut\ l\ A\ C \vee cut\ l\ B\ C)$

context *Hilbert-neutral-2D*

begin

26.2 Definitions

26.3 Propositions

lemma *plane-separation-2D*:
assumes $\neg\ ColH\ A\ X\ Y$ **and**
$\neg\ ColH\ B\ X\ Y$
shows $cut'\ A\ B\ X\ Y \vee same\text{-}side'\ A\ B\ X\ Y$
⟨*proof*⟩

lemma *col-upper-dim*:
assumes *ColH A P Q* **and**
$P \neq Q$ **and**
$A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
$A \neq P$ **and**
$A \neq Q$ **and**
$B \neq P$ **and**

$B \neq Q$ **and**
$C \neq P$ **and**
$C \neq Q$ **and**
CongH A P A Q **and**
CongH B P B Q **and**
CongH C P C Q
shows *Bet* A B C $\vee$ *Bet* B C A $\vee$ *Bet* C A B
⟨*proof*⟩

lemma *TS-upper-dim*:
assumes *cut'* A B P Q **and**
$P \neq Q$ **and**
$A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
$A \neq P$ **and**
$A \neq Q$ **and**
$B \neq P$ **and**
$B \neq Q$ **and**
$C \neq P$ **and**
$C \neq Q$ **and**
CongH A P A Q **and**
CongH B P B Q **and**
CongH C P C Q
shows *Bet* A B C $\vee$ *Bet* B C A $\vee$ *Bet* C A B
⟨*proof*⟩

lemma *cut'-comm*:
assumes *cut'* A B X Y
shows *cut'* B A X Y
⟨*proof*⟩

lemma *TS-upper-dim-bis*:
assumes *BetH* P I Q **and**
BetH I B A **and**
$P \neq Q$ **and**
$A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
$A \neq P$ **and**
$A \neq Q$ **and**
$B \neq P$ **and**
$B \neq Q$ **and**
$C \neq P$ **and**
$C \neq Q$ **and**
CongH A P A Q **and**
CongH B P B Q **and**
CongH C P C Q
shows *Bet* A B C $\vee$ *Bet* B C A $\vee$ *Bet* C A B
⟨*proof*⟩

lemma *upper-dim*:
assumes $P \neq Q$ **and**
$A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
Cong A P A Q **and**
Cong B P B Q **and**
Cong C P C Q
shows *Bet* A B C $\vee$ *Bet* B C A $\vee$ *Bet* C A B
⟨*proof*⟩

end
end

theory *Hilbert-Neutral-3D*

imports *Hilbert-Neutral*

begin

27 Hilbert - Geometry - Neutral 3D

27.1 Axioms: neutral 3D

locale *Hilbert-neutral-3D* = *Hilbert-neutral-dimensionless IncidL IncidP EqL EqP IsL IsP BetH CongH CongaH*
for
IncidL :: $'p \Rightarrow 'b \Rightarrow bool$ **and**
IncidP :: $'p \Rightarrow 'c \Rightarrow bool$ **and**
EqL :: $'b \Rightarrow 'b \Rightarrow bool$ **and**
EqP :: $'c \Rightarrow 'c \Rightarrow bool$ **and**
IsL :: $'b \Rightarrow bool$ **and**
IsP :: $'c \Rightarrow bool$ **and**
BetH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongaH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ +
fixes *HS1 HS2 HS3 HS4* :: $'p$
assumes *plane-intersection*:
$IsP\ p \wedge IsP\ q \wedge IncidP\ A\ p \wedge IncidP\ A\ q$
$\longrightarrow$
$(\exists\ B.\ IncidP\ B\ p \wedge IncidP\ B\ q \wedge A \neq B)$
and *lower-dim-3*:
$\neg (\exists\ p.\ IsP\ p \wedge IncidP\ HS1\ p \wedge IncidP\ HS2\ p \wedge IncidP\ HS3\ p \wedge IncidP\ HS4\ p)$

end

theory *Hilbert-Euclidean*

imports
Hilbert-Neutral
Tarski-Euclidean
Tarski-Neutral-Hilbert

begin

28 Hilbert - Geometry - Euclidean

28.1 Axioms: Euclidean

locale *Hilbert-Euclidean* = *Hilbert-neutral-dimensionless IncidL IncidP EqL EqP IsL IsP BetH CongH CongaH*
for
IncidL :: $'p \Rightarrow 'b \Rightarrow bool$ **and**
IncidP :: $'p \Rightarrow 'c \Rightarrow bool$ **and**
EqL :: $'b \Rightarrow 'b \Rightarrow bool$ **and**
EqP :: $'c \Rightarrow 'c \Rightarrow bool$ **and**
IsL :: $'b \Rightarrow bool$ **and**
IsP :: $'c \Rightarrow bool$ **and**
BetH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ **and**
CongaH :: $'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow 'p \Rightarrow bool$ +
assumes *euclid-uniqueness*: $\forall\ l\ P\ m1\ m2.\ IsL\ l \wedge IsL\ m1 \wedge IsL\ m2 \wedge$
$\neg\ IncidL\ P\ l \wedge Para\ l\ m1 \wedge IncidL\ P\ m1 \wedge Para\ l\ m2 \wedge IncidL\ P\ m2$
$\longrightarrow$
$EqL\ m1\ m2$

context *Tarski-Euclidean*

begin

28.2 Definitions

28.3 Propositions

lemma *Para-Par*:
  assumes $A \neq B$ **and**
    $C \neq D$ **and**
    Para-H (*Line A B*) (*Line C D*)
  shows *A B Par C D*
⟨*proof*⟩

lemma *axiom-euclid-uniqueness*:
  assumes ¬ *IncidentL P l* **and**
    Para-H l m1 **and**
    IncidentL P m1 **and**
    Para-H l m2 **and**
    IncidentL P m2
  shows *m1 =l= m2*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-Model-Gupta-Neutral*

imports *Gupta-Neutral*

begin

context *Gupta-neutral-dimensionless*

begin

interpretation *Interpretation-Tarski-neutral-dimensionless*: *Tarski-neutral-dimensionless*
where *TPA* = *GPA* **and** *TPB* = *GPB* **and** *TPC* = *GPC* **and** *Bet* = *BetG* **and** *Cong* = *CongG*
⟨*proof*⟩

28.4 Transport theorem from Tarski Neutral for Gupta Euclidean Model

lemma *g-l5-2*:
assumes $A \neq B$ **and**
BetG A B C **and**
BetG A B D
shows *BetG B C D* ∨ *BetG B D C*
⟨*proof*⟩

lemma *g-l5-3*:
assumes *BetG A B D* **and**
BetG A C D
shows *BetG A B C* ∨ *BetG A C B*
⟨*proof*⟩

lemma *g-between-exchange4*:
assumes *BetG A B C* **and**
BetG A C D
shows *BetG A B D*
⟨*proof*⟩

lemma *g-between-inner-transitivity*:
assumes *BetG A B D* **and**
BetG B C D
shows *BetG A B C*
⟨*proof*⟩

lemma *g-not-bet-distincts*:
assumes ¬ *BetG A B C*

shows $A \neq B \wedge B \neq C$
⟨*proof*⟩

lemma *g-between-symmetry*:
assumes *BetG A B C*
shows *BetG C B A*
⟨*proof*⟩

lemma *g-between-trivial*:
shows *BetG A B B*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-2D-Model-Gupta-Neutral-2D*

imports
Gupta-Neutral-2D
Tarski-Neutral-2D

begin

context *Gupta-neutral-2D*

begin

interpretation *Interpretation-Tarski-neutral-2D* : *Tarski-neutral-2D*
where $TPA = GPA$ **and** $TPB = GPB$ **and** $TPC = GPC$ **and** $Bet = BetG$ **and** $Cong = CongG$
⟨*proof*⟩

end
end

theory *Tarski-Euclidean-Model-Gupta-Euclidean*

imports
Tarski-Neutral-Model-Gupta-Neutral
Gupta-Euclidean
Tarski-Euclidean

begin

context *Gupta-Euclidean*

begin

28.5 Interpretation Tarski Gupta

lemma *euclidT*:
assumes *BetG A D T* **and**
BetG B D C **and**
$A \neq D$ **and**
$\forall\ A\ B\ C\ D\ T.$
$BetG\ A\ D\ T \wedge BetG\ B\ D\ C \wedge A \neq D$
$\longrightarrow$
$(\exists\ X\ Y.\ BetG\ A\ B\ X \wedge BetG\ A\ C\ Y \wedge BetG\ X\ T\ Y)$
shows $\exists\ X\ Y.\ BetG\ A\ B\ X \wedge BetG\ A\ C\ Y \wedge BetG\ X\ T\ Y$
⟨*proof*⟩

lemma *lem-euclidG*:
assumes *BetG A D T* **and**
BetG B D C **and**
$A \neq D$

shows ∃ *X Y*. *BetG A B X* ∧ *BetG A C Y* ∧ *BetG X T Y*
⟨*proof*⟩

interpretation *Interpretation-Tarski-euclidean* : *Tarski-Euclidean*
where *TPA* = *GPA* **and** *TPB* = *GPB* **and** *TPC* = *GPC* **and** *Bet* = *BetG* **and** *Cong* = *CongG*
⟨*proof*⟩

end
end

theory *Gupta-Neutral-Model-Tarski-Neutral*

imports
Tarski-Neutral
Gupta-Neutral

begin

context *Tarski-neutral-dimensionless*

begin

interpretation *Interpretation-Gupta-neutral-dimensionless* : *Gupta-neutral-dimensionless*
where *GPA* = *TPA* **and** *GPB* = *TPB* **and** *GPC* = *TPC* **and** *BetG* = *Bet* **and** *CongG* = *Cong*
⟨*proof*⟩

end
end

theory *Gupta-Neutral-2D-Model-Tarski-Neutral-2D*

imports
Tarski-Neutral-2D
Gupta-Neutral-2D

begin

context *Tarski-neutral-2D*

begin

interpretation *Interpretation-Gupta-neutral-2D* : *Gupta-neutral-2D*
where *GPA* = *TPA* **and** *GPB* = *TPB* **and** *GPC* = *TPC* **and** *BetG* = *Bet* **and** *CongG* = *Cong*
⟨*proof*⟩

end
end

theory *Gupta-Euclidean-Model-Tarski-Euclidean*

imports
Tarski-Euclidean
Gupta-Euclidean

begin

context *Tarski-Euclidean*

begin

interpretation *Interpretation-Gupta-euclidean* : *Gupta-Euclidean*
where *GPA* = *TPA* **and** *GPB* = *TPB* **and** *GPC* = *TPC* **and** *BetG* = *Bet* **and** *CongG* = *Cong*
⟨*proof*⟩

end
end

theory *Hilbert-Neutral-Model-Tarski-Neutral*

imports *Tarski-Neutral-Hilbert*

begin

29 Hilbert Neutral - Tarski Neutral Model

29.1 Interpretation

context *Tarski-neutral-dimensionless*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre*: *Hilbert-neutral-dimensionless-pre*
where *IncidL* = *IncidentL* **and**
IncidP = *IncidentP* **and**
EqL = *EqTL* **and**
EqP = *EqTP* **and**
IsL = *isLine* **and**
IsP = *isPlane* **and**
BetH = *Between-H* **and**
CongH = *Cong* **and**
CongaH = *CongA-H*
⟨*proof*⟩

interpretation *Intrepretation-Hilbert-neutral-dimensionless*: *Hilbert-neutral-dimensionless*
where *IncidL* = *IncidentL* **and**
IncidP = *IncidentP* **and**
EqL = *EqTL* **and**
EqP = *EqTP* **and**
IsL = *isLine* **and**
IsP = *isPlane* **and**
BetH = *Between-H* **and**
CongH = *Cong* **and**
CongaH = *CongA-H* **and**
PP = *TPA* **and**
PQ = *TPB* **and**
PR = *TPC*
⟨*proof*⟩

end
end

theory *Hilbert-Neutral-2D-Model-Tarski-Neutral-2D*

imports
Tarski-Neutral-Hilbert
Tarski-Neutral-2D
Hilbert-Neutral-2D

begin

30 Hilbert Neutral 2D - Tarski Neutral 2D Model

30.1 Interpretation

context *Tarski-neutral-2D*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre*: *Hilbert-neutral-dimensionless-pre*
  where *IncidL* = *IncidentL* **and**
    IncidP = *IncidentP* **and**
    EqL = *EqTL* **and**
    EqP = *EqTP* **and**
    IsL = *isLine* **and**
    IsP = *isPlane* **and**
    BetH = *Between-H* **and**
    CongH = *Cong* **and**
    CongaH = *CongA-H*
⟨*proof*⟩

interpretation *Intrepretation-Hilbert-neutral-dimensionless*: *Hilbert-neutral-dimensionless*
  where *IncidL* = *IncidentL* **and**
    IncidP = *IncidentP* **and**
    EqL = *EqTL* **and**
    EqP = *EqTP* **and**
    IsL = *isLine* **and**
    IsP = *isPlane* **and**
    BetH = *Between-H* **and**
    CongH = *Cong* **and**
    CongaH = *CongA-H* **and**
    PP = *TPA* **and**
    PQ = *TPB* **and**
    PR = *TPC*
⟨*proof*⟩

interpretation *Intrepretation-Hilbert-neutral-2D*: *Hilbert-neutral-2D*
  where *IncidL* = *IncidentL* **and**
    IncidP = *IncidentP* **and**
    EqL = *EqTL* **and**
    EqP = *EqTP* **and**
    IsL = *isLine* **and**
    IsP = *isPlane* **and**
    BetH = *Between-H* **and**
    CongH = *Cong* **and**
    CongaH = *CongA-H* **and**
    PP = *TPA* **and**
    PQ = *TPB* **and**
    PR = *TPC*
⟨*proof*⟩

end
end

theory *Hilbert-Neutral-3D-Model-Tarski-Neutral-3D*

imports
  Tarski-Neutral-3D-Hilbert
  Hilbert-Neutral-3D

begin

31 Hilbert Neutral 3D - Tarski Neutral 3D Model

31.1 Interpretation

context *Tarski-neutral-3D*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre*: *Hilbert-neutral-dimensionless-pre*
  where *IncidL* = *IncidentL* **and** *IncidP* = *IncidentP* **and** *EqL* = *EqTL*

and *EqP* = *EqTP* **and** *IsL* = *isLine* **and** *IsP* = *isPlane*
and *BetH* = *Between-H* **and** *CongH* = *Cong* **and** *CongaH* = *CongA-H*
⟨*proof*⟩

interpretation *Intrepretation-Hilbert-neutral-dimensionless*: *Hilbert-neutral-dimensionless*
where *IncidL* = *IncidentL* **and** *IncidP* = *IncidentP* **and** *EqL* = *EqTL*
and *EqP* = *EqTP* **and** *IsL* = *isLine* **and** *IsP* = *isPlane*
and *BetH* = *Between-H* **and** *CongH* = *Cong* **and** *CongaH* = *CongA-H*
and *PP* = *TPA* **and** *PQ* = *TPB* **and** *PR* = *TPC*
⟨*proof*⟩

interpretation *Intrepretation-Hilbert-neutral-3D*: *Hilbert-neutral-3D*
where *IncidL* = *IncidentL* **and** *IncidP* = *IncidentP* **and** *EqL* = *EqTL*
and *EqP* = *EqTP* **and** *IsL* = *isLine* **and** *IsP* = *isPlane*
and *BetH* = *Between-H* **and** *CongH* = *Cong* **and** *CongaH* = *CongA-H*
and *PP* = *TPA* **and** *PQ* = *TPB* **and** *PR* = *TPC*
and *HS1* = *TS1* **and** *HS2* = *TS2* **and** *HS3* = *TS3*
and *HS4* = *TS4*
⟨*proof*⟩

end
end

theory *Hilbert-Euclidean-Model-Tarski-Euclidean*

imports *Hilbert-Euclidean*

begin

32 Hilbert Euclidean - Tarski Euclidean Model

32.1 Interpretation

context *Tarski-Euclidean*

begin

interpretation *Interpretation-Hilbert-neutral-dimensionless-pre*: *Hilbert-neutral-dimensionless-pre*
where *IncidL* = *IncidentL* **and** *IncidP* = *IncidentP* **and** *EqL* = *EqTL*
and *EqP* = *EqTP* **and** *IsL* = *isLine* **and** *IsP* = *isPlane*
and *BetH* = *Between-H* **and** *CongH* = *Cong* **and** *CongaH* = *CongA-H*
⟨*proof*⟩

interpretation *Intrepretation-Hilbert-neutral-dimensionless*: *Hilbert-neutral-dimensionless*
where *IncidL* = *IncidentL* **and** *IncidP* = *IncidentP* **and** *EqL* = *EqTL*
and *EqP* = *EqTP* **and** *IsL* = *isLine* **and** *IsP* = *isPlane*
and *BetH* = *Between-H* **and** *CongH* = *Cong* **and** *CongaH* = *CongA-H*
and *PP* = *TPA* **and** *PQ* = *TPB* **and** *PR* = *TPC*
⟨*proof*⟩

interpretation *Intrepretation-Hilbert-euclidean*: *Hilbert-Euclidean*
where *IncidL* = *IncidentL* **and** *IncidP* = *IncidentP* **and** *EqL* = *EqTL*
and *EqP* = *EqTP* **and** *IsL* = *isLine* **and** *IsP* = *isPlane*
and *BetH* = *Between-H* **and** *CongH* = *Cong* **and** *CongaH* = *CongA-H*
and *PP* = *TPA* **and** *PQ* = *TPB* **and** *PR* = *TPC*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-Model-Hilbert-Neutral*

imports
Tarski-Postulate-Parallels

Hilbert-Neutral
Gupta-Neutral

begin

33 Tarski Neutral Model Hilbert Neutral

context *Hilbert-neutral-dimensionless*

begin

interpretation *Interp-Gupta-of-Tarski-neutral-dimensionless*:
Gupta-neutral-dimensionless
where *BetG* = *Bet* **and**
CongG = *Cong* **and**
GPA = *PP* **and**
GPB = *PQ* **and**
GPC = *PR*
⟨*proof*⟩

interpretation *H-to-T* : *Tarski-neutral-dimensionless*
where *Bet* = *Bet* **and**
Cong = *Cong* **and**
TPA = *PP* **and**
TPB = *PQ* **and**
TPC = *PR*
⟨*proof*⟩

lemma *MidH--Mid*:
assumes *M Midpoint A B*
shows *H-to-T.Midpoint M A B*
⟨*proof*⟩

lemma *Mid--MidH*:
assumes *H-to-T.Midpoint M A B*
shows *M Midpoint A B*
⟨*proof*⟩

lemma *col-colh-1*:
assumes *H-to-T.Col A B C*
shows *ColH A B C*
⟨*proof*⟩

lemma *col-colh-2*:
assumes *ColH A B C*
shows *H-to-T.Col A B C*
⟨*proof*⟩

lemma *col-colh*:
shows *H-to-T.Col A B C* ⟷ *ColH A B C*
⟨*proof*⟩

lemma *line-col*:
assumes *IsL l* **and**
IncidL A l **and**
IncidL B l **and**
IncidL C l
shows *H-to-T.Col A B C*
⟨*proof*⟩

lemma *bet--beth*:
assumes $A \neq B$ **and**
$B \neq C$ **and**
Bet A B C
shows *BetH A B C*

⟨proof⟩

lemma *coplanar-plane0*:
assumes *ColH A B X* **and**
ColH C D X
shows ∃ *p*. *IncidP A p* ∧ *IncidP B p* ∧ *IncidP C p* ∧ *IncidP D p*
⟨proof⟩

lemma *coplanar-plane1*:
assumes *Bet A B X* ∨ *Bet B X A* ∨ *Bet X A B* **and**
Bet C D X ∨ *Bet D X C* ∨ *Bet X C D*
shows ∃ *p*. *IsP p* ∧ *IncidP A p* ∧ *IncidP B p* ∧ *IncidP C p* ∧ *IncidP D p*
⟨proof⟩

lemma *coplanar-plane*:
assumes *H-to-T.Coplanar A B C D*
shows ∃ *p*. *IsP p* ∧ *IncidP A p* ∧ *IncidP B p* ∧ *IncidP C p* ∧ *IncidP D p*
⟨proof⟩

lemma *plane-coplanar*:
assumes *IncidP A p* **and**
IncidP B p **and**
IncidP C p **and**
IncidP D p
shows *H-to-T.Coplanar A B C D*
⟨proof⟩

lemma *pars--para*:
assumes *IncidL A l* **and**
IncidL B l **and**
IncidL C m **and**
IncidL D m **and**
H-to-T.ParStrict A B C D
shows *Para l m*
⟨proof⟩

lemma *par--or-eq-para*:
assumes *IncidL A l* **and**
IncidL B l **and**
IncidL C m **and**
IncidL D m **and**
H-to-T.Par A B C D
shows *Para l m* ∨ *EqL l m*
⟨proof⟩

lemma *tarski-upper-dim*:
assumes
plane-intersection-assms: ∀ *A p q*. *IsP p* ∧ *IsP q* ∧ *IncidP A p* ∧ *IncidP A q*
⟶ (∃ *B*. *IncidP B p* ∧ *IncidP B q* ∧ *A* ≠ *B*) **and**
lower-dim-3-assms: ¬ (∃ *p*. *IsP p* ∧ *IncidP HS1 p* ∧ *IncidP HS2 p* ∧
IncidP HS3 p ∧ *IncidP HS4 p*) **and**
P ≠ *Q* **and**
Q ≠ *R* **and**
P ≠ *R* **and**
Cong A P A Q **and**
Cong B P B Q **and**
Cong C P C Q **and**
Cong A P A R **and**
Cong B P B R **and**
Cong C P C R
shows *Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*
⟨proof⟩

lemma *Col--ColH*:
assumes *H-to-T.Col A B C*
shows *ColH A B C*

⟨proof⟩

lemma *ColH--Col*:
assumes *ColH A B C*
shows *H-to-T.Col A B C*
⟨proof⟩

lemma *playfair-s-postulateH*:
assumes *euclid-uniqueness*: $\forall$ *l P m1 m2*. *IsL l* $\wedge$ *IsL m1* $\wedge$ *IsL m2* $\wedge$
$\neg$ *IncidL P l* $\wedge$ *Para l m1* $\wedge$ *IncidL P m1* $\wedge$ *Para l m2* $\wedge$ *IncidL P m2* $\longrightarrow$
EqL m1 m2
shows *H-to-T.playfair-s-postulate*
⟨proof⟩

lemma *tarski-s-euclid-aux*:
assumes *euclid-uniqueness*: $\forall$ *l P m1 m2*. *IsL l* $\wedge$ *IsL m1* $\wedge$ *IsL m2* $\wedge$
$\neg$ *IncidL P l* $\wedge$ *Para l m1* $\wedge$ *IncidL P m1* $\wedge$ *Para l m2* $\wedge$ *IncidL P m2* $\longrightarrow$
EqL m1 m2
shows *H-to-T.tarski-s-parallel-postulate*
⟨proof⟩

lemma *tarski-s-euclid*:
assumes *euclid-uniqueness*: $\forall$ *l P m1 m2*. *IsL l* $\wedge$ *IsL m1* $\wedge$ *IsL m2* $\wedge$
$\neg$ *IncidL P l* $\wedge$ *Para l m1* $\wedge$ *IncidL P m1* $\wedge$ *Para l m2* $\wedge$ *IncidL P m2* $\longrightarrow$
EqL m1 m2
shows
$\forall$ *A B C D T*. *Bet A D T* $\wedge$ *Bet B D C* $\wedge$ $A \neq D \longrightarrow$ ($\exists$ *X Y*. *Bet A B X* $\wedge$ *Bet A C Y* $\wedge$ *Bet X T Y*)
⟨proof⟩

end
end

theory *Tarski-Neutral-2D-Model-Hilbert-Neutral-2D*

imports
Tarski-Neutral-2D
Hilbert-Neutral-2D

begin

34 Tarski Neutral 2D - Hilbert Neutral 2D Model

context *Hilbert-neutral-2D*

begin

lemma *plane-separation-2D*:
assumes $\neg$ *ColH A X Y* **and**
$\neg$ *ColH B X Y*
shows *cut′ A B X Y* $\vee$ *same-side′ A B X Y*
⟨proof⟩

lemma *col-upper-dim*:
assumes *ColH A P Q* **and**
$P \neq Q$ **and**
$A \neq B$ **and**
$A \neq C$ **and**
$B \neq C$ **and**
$A \neq P$ **and**
$A \neq Q$ **and**
$B \neq P$ **and**
$B \neq Q$ **and**
$C \neq P$ **and**
$C \neq Q$ **and**

CongH A P A Q **and**
CongH B P B Q **and**
CongH C P C Q
shows *Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*
⟨*proof*⟩

lemma *TS-upper-dim*:
assumes *cut′ A B P Q* **and**
P ≠ *Q* **and**
A ≠ *B* **and**
A ≠ *C* **and**
B ≠ *C* **and**
A ≠ *P* **and**
A ≠ *Q* **and**
B ≠ *P* **and**
B ≠ *Q* **and**
C ≠ *P* **and**
C ≠ *Q* **and**
CongH A P A Q **and**
CongH B P B Q **and**
CongH C P C Q
shows *Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*
⟨*proof*⟩

lemma *cut′-comm*:
assumes *cut′ A B X Y*
shows *cut′ B A X Y*
⟨*proof*⟩

lemma *TS-upper-dim-bis*:
assumes *BetH P I Q* **and**
BetH I B A **and**
P ≠ *Q* **and**
A ≠ *B* **and**
A ≠ *C* **and**
B ≠ *C* **and**
A ≠ *P* **and**
A ≠ *Q* **and**
B ≠ *P* **and**
B ≠ *Q* **and**
C ≠ *P* **and**
C ≠ *Q* **and**
CongH A P A Q **and**
CongH B P B Q **and**
CongH C P C Q
shows *Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*
⟨*proof*⟩

lemma *upper-dim*:
assumes *P* ≠ *Q* **and**
A ≠ *B* **and**
A ≠ *C* **and**
B ≠ *C* **and**
Cong A P A Q **and**
Cong B P B Q **and**
Cong C P C Q
shows *Bet A B C* ∨ *Bet B C A* ∨ *Bet C A B*
⟨*proof*⟩

interpretation *H2D-to-T2D* : *Tarski-neutral-2D*
where *Bet* = *Bet* **and**
Cong = *Cong* **and**
TPA = *PP* **and**
TPB = *PQ* **and**
TPC = *PR*
⟨*proof*⟩

end
end

theory *Tarski-Neutral-3D-Model-Hilbert-Neutral-3D*

imports
Tarski-Neutral-3D
Hilbert-Neutral-3D
Tarski-Neutral-Model-Hilbert-Neutral

begin

35 Tarski Neutral 3D - Hilbert Neutral 3D Model

context *Hilbert-neutral-3D*

begin

interpretation *H3D-to-T3D* : *Tarski-neutral-3D*
where *Bet* = *Bet* **and**
Cong = *Cong* **and**
TPA = *PP* **and**
TPB = *PQ* **and**
TPC = *PR* **and**
TS1 = *HS1* **and** *TS2* = *HS2* **and** *TS3* = *HS3* **and** *TS4* = *HS4*
⟨*proof*⟩

end
end

theory *Tarski-Euclidean-Model-Hilbert-Euclidean*

imports
Hilbert-Euclidean
Tarski-Neutral-3D-Model-Hilbert-Neutral-3D

begin

36 Tarski Euclidean Model Hilbert Euclidean

context *Hilbert-Euclidean*

begin

interpretation *Heucl-to-Teucl* : *Tarski-Euclidean*
where *Bet* = *Bet* **and**
Cong = *Cong* **and**
TPA = *PP* **and**
TPB = *PQ* **and**
TPC = *PR*
⟨*proof*⟩

end

end

References

[1] M. Beeson, P. Boutry, G. Braun, C. Gries, and J. Narboux. GeoCoq, June 2018.

[2] M. Beeson, J. Narboux, and F. Wiedijk. Proof-checking euclid. *Annals of Mathematics and Artificial Intelligence*, 85(2):213–257, 2019.

[3] M. Beeson and L. Wos. Otter proofs in tarskian geometry. In *International Joint Conference on Automated Reasoning*, pages 495–510. Springer, 2014.

[4] M. Beeson and L. Wos. Finding proofs in tarskian geometry. *Journal of Automated Reasoning*, 58(1):181–207, 2017.

[5] P. Boutry. *On the formalization of foundations of geometry*. PhD thesis, Université de Strasbourg, 2018.

[6] P. Boutry, G. Braun, and J. Narboux. Formalization of the Arithmetization of Euclidean Plane Geometry and Applications. *Journal of Symbolic Computation*, 90:149–168, 2019.

[7] P. Boutry, C. Gries, J. Narboux, and P. Schreck. Parallel postulates and continuity axioms: a mechanized study in intuitionistic logic using Coq. *Journal of Automated Reasoning*, page 68, 2017. online first.

[8] R. Coghetto and A. Grabowski. Tarski geometry axioms - part II. *Formalized Mathematics*, 24(2):157–166, 2016.

[9] R. Coghetto and A. Grabowski. Tarski geometry axioms. part III. *Formalized Mathematics*, 25(4):289–313, 2017.

[10] R. Coghetto and A. Grabowski. Tarski geometry axioms. part IV - right angle. *Formalized Mathematics*, 27(1):75–85, 2019.

[11] M. Doré and K. Broda. Intuitive reasoning in formalized mathematics with elfe. In B. M. McLaren, R. Reilly, S. Zvacek, and J. Uhomoibhi, editors, *Computer Supported Education - 10th International Conference, CSEDU 2018, Funchal, Madeira, Portugal, March 15-17, 2018, Revised Selected Papers*, volume 1022 of *Communications in Computer and Information Science*, pages 549–571. Springer, 2018.

[12] C. Gries, P. Boutry, and J. Narboux. Somme des angles d'un triangle et unicité de la parallèle : une preuve d'équivalence formalisée en Coq. In *Les vingt-septièmes Journées Francophones des Langages Applicatifs (JFLA 2016)*, Actes des Vingt-septièmes Journées Francophones des Langages Applicatifs (JFLA 2016), page 15, Saint Malo, France, Jan. 2016. Jade Algave and Julien Signoles.

[13] T. J. M. Makarios. The independence of Tarski's Euclidean axiom. *Archive of Formal Proofs*, Oct. 2012. https://isa-afp.org/entries/Tarskis_Geometry.html, Formal proof development.

[14] G. E. Martin. *The foundations of geometry and the non-Euclidean plane*. Springer Science & Business Media, 2012.

[15] J. Narboux. Mechanical Theorem Proving in Tarski's geometry. In F. B. Eugenio Roanes Lozano, editor, *Automated Deduction in Geometry 2006*, volume 4869 of *LNCS*, pages 139–156, Pontevedra, Spain, Aug. 2006. Francisco Botana, Springer.

[16] J. Narboux, P. Janicic, and J. Fleuriot. Computer-assisted Theorem Proving in Synthetic Geometry. In M. Sitharam, A. S. John, and J. Sidman, editors, *Handbook of Geometric Constraint Systems Principles*, Discrete Mathematics and Its Applications. Chapman and Hall/CRC , 2018.

[17] J. Narboux, P. Janicic, and J. Fleuriot. Computer-assisted theorem proving in synthetic geometry. *Handbook of Geometric Constraint Systems Principles*, pages 25–73, 2018.

[18] T. Nipkow. Structured proofs in isar/hol. In *International Workshop on Types for Proofs and Programs*, pages 259–278. Springer, 2002.

[19] W. Richter, A. Grabowski, and J. Alama. Tarski geometry axioms. *Formalized Mathematics*, 22(2):167–176, 2014.

[20] W. Schwabhäuser, W. Szmielew, and A. Tarski. *Metamathematische Methoden in der Geometrie*. Springer-Verlag, Berlin, 1983.

[21] D. Simić, F. Marić, and P. Boutry. Poincaré disc model. *Archive of Formal Proofs*, Dec. 2019. https://isa-afp.org/entries/Poincare_Disc.html, Formal proof development.

[22] D. Simic, F. Maric, and P. Boutry. Formalization of the poincaré disc model of hyperbolic geometry. *J. Autom. Reasoning*, 64(4), 2020.

[23] S. Stojanović, V. Pavlović, and P. Janičić. A coherent logic based geometry theorem prover capable of producing formal and readable proofs. In *International Workshop on Automated Deduction in Geometry*, pages 201–220. Springer, 2010.

[24] G. Sutcliffe and C. Suttner. The tptp problem library. *Journal of Automated Reasoning*, 21(2):177–203, 1998.

[25] S. S. Đurđević, J. Narboux, and P. Janičić. Automated generation of machine verifiable and readable proofs: a case study of tarski's geometry. *Annals of Mathematics and Artificial Intelligence*, 74(3-4):249–269, 2015.

[26] F. Wiedijk. A synthesis of the procedural and declarative styles of interactive theorem proving. *Logical Methods in Computer Science*, 8, 2012.