

Irrational Rapidly Convergent Series

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Abstract

We formalize with Isabelle/HOL a proof of a theorem by J. Hančl asserting the irrationality of the sum of a series consisting of rational numbers, built up by sequences that fulfill certain properties. Even though the criterion is a number theoretic result, the proof makes use only of analytical arguments. We also formalize a corollary of the theorem for a specific series fulfilling the assumptions of the theorem.

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1 Main Theorem and Sketch of the Proof

We formalize the proof of the following theorem by J. Hančl (Theorem 3 in [1]) :

Theorem 1. (Theorem 3 in [1]) Let $A \in \mathbb{R}$ with $A > 1$. Let $\{d_n\}_{n=1}^\infty \in \mathbb{R}$ with $d_n > 1$ for all $n \in \mathbb{N}$. Let $\{a_n\}_{n=1}^\infty \in \mathbb{Z}^+$, $\{b_n\}_{n=1}^\infty \in \mathbb{Z}^+$ such that :

$$(1) \quad \lim_{n \rightarrow \infty} a_n^{\frac{1}{2^n}} = A,$$

for all sufficiently large $n \in \mathbb{N}$:

$$(2) \quad \frac{A}{a_n^{\frac{1}{2^n}}} > \prod_{j=n}^{\infty} d_j$$

and

$$(3) \lim_{n \rightarrow \infty} \frac{d_n^{2^n}}{b_n} = \infty.$$

Then the series $\alpha = \sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is an irrational number.

The first step is to show that the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ converges to some $\alpha \in \mathbb{R}$. To show that $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we argue by proof by contradiction (to this end several auxiliary lemmas are firstly shown). In particular, assuming that $\alpha \in \mathbb{Q}$, i.e. that there exist $p, q \in \mathbb{Z}^+$ such that $\alpha = \frac{p}{q}$, we show that a quantity $\mathcal{A}(n) \geq 1$ for all $n \in \mathbb{N}$. At the same time, we find $n \in \mathbb{N}$ for which $\mathcal{A}(n) < 1$, yielding a contradiction from which we deduce the irrationality of the sum of the series.

For the proof see [1]. We note that the proof involves only elementary Analysis (criteria for convergence/divergence for sequences and series and several inequalities) and not any arithmetical/number theoretic arguments. Obviously for the formal proof we had to make many intermediate arguments explicit. Proofs of length of roughly 2 A4 pages in the original paper by J. Hančl were formalized in almost 1100 lines of code.

2 Corollary

We moreover formalize the following corollary that asserts the irrationality of the sum of an instance of a series that fulfills the assumptions of the theorem :

Corollary 1. (Corollary 2 in [1]) Let $A \in \mathbb{R}$ with $A > 1$. Let $\{a_n\}_{n=1}^{\infty} \in \mathbb{Z}^+$, $\{b_n\}_{n=1}^{\infty} \in \mathbb{Z}^+$ such that :

$$\lim_{n \rightarrow \infty} a_n^{\frac{1}{2^n}} = A$$

and for all sufficiently large $n \in \mathbb{N}$ (in particular: for $n \geq 6$)

$$a_n^{\frac{1}{2^n}} (1 + 4(2/3)^n) \leq A$$

and

$$b_n \leq 2^{(4/3)^{n-1}}.$$

Then the series $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is an irrational number.

The above corollary is an immediate consequence of the theorem by setting $d_n = 1 + (2/3)^n$. For the formalized proof of the corollary one more auxiliary lemma was required.

3 Irrational Rapidly Convergent Series

theory *Irrationality-J-Hancl*

imports *HOL-Analysis.Analysis HOL-Decision-Proc.Approximation*
begin

This is the formalisation of a proof by J. Hanl, in particular of the proof of his Theorem 3 in the paper: Irrational Rapidly Convergent Series, Rend. Sem. Mat. Univ. Padova, Vol 107 (2002).

The statement asserts the irrationality of the sum of a series consisting of rational numbers defined using sequences that fulfill certain properties. Even though the statement is number-theoretic, the proof uses only arguments from introductory Analysis.

We prove the central result (theorem `Hancl3`) by contradiction, by making use of some of the auxiliary lemmas. To this end, assuming that the sum is a rational number, for a quantity $\text{ALPHA}(n)$ we show that $\text{ALPHA}(n) \geq 1$ for all $n \in \mathbb{N}$. After that we show that we can find an $n \in \mathbb{N}$ for which $\text{ALPHA}(n) < 1$ which yields a contradiction and we thus conclude that the sum of the series is rational. We finally give an immediate application of theorem `Hancl3` for a specific series (corollary `Hancl3corollary`, requiring lemma `summable_ln_plus`) which corresponds to Corollary 2 in the original paper by J. Hanl.

hide-const *floatarith.Max*

3.1 Misc

lemma *filterlim-sequentially-iff:*

filterlim f F1 sequentially \iff filterlim ($\lambda x. f (x+k)$) F1 sequentially
<proof>

lemma *filterlim-realpov-sequentially-at-top:*

($x::\text{real}$) > 1 \implies filterlim (power x) at-top sequentially
<proof>

lemma *filterlim-at-top-powr-real:*

fixes *g::'b \implies real*
assumes *filterlim f at-top F and g': (g \longrightarrow g') F g'>1*
shows *LIM x F. g x powr f x :> at-top*
<proof>

lemma *powrfinitesum:*

fixes *a::real and s::nat assumes s \leq n*
shows *($\prod_{j=s..(n::nat)}.(a \text{ powr } (2^{\hat{j}}))$) = a powr ($\sum_{j=s..(n::nat)}.(2^{\hat{j}})$)*
<proof>

lemma *summable-ratio-test-tendsto:*

fixes *f :: nat \implies 'a::banach*

assumes $c < 1$ **and** $\forall n. f\ n \neq 0$ **and** $(\lambda n. \text{norm } (f\ (\text{Suc } n)) / \text{norm } (f\ n)) \longrightarrow c$
shows *summable* f
 $\langle \text{proof} \rangle$

lemma *summable-ln-plus*:
fixes $f :: \text{nat} \Rightarrow \text{real}$
assumes *summable* f $\forall n. f\ n > 0$
shows *summable* $(\lambda n. \ln (1 + f\ n))$
 $\langle \text{proof} \rangle$

lemma *suminf-real-offset-le*:
fixes $f :: \text{nat} \Rightarrow \text{real}$
assumes $f: \bigwedge i. 0 \leq f\ i$ **and** *summable* f
shows $(\sum i. f\ (i + k)) \leq \text{suminf } f$
 $\langle \text{proof} \rangle$

lemma *factt*:
fixes $s\ n :: \text{nat}$ **assumes** $s \leq n$
shows $(\sum_{i=s..n} 2^i) < (2^{n+1}) :: \text{real}$ $\langle \text{proof} \rangle$

3.2 Auxiliary lemmas and the main proof

lemma *showpre7*:
fixes $a\ b :: \text{nat} \Rightarrow \text{int}$ **and** $q\ p :: \text{int}$
assumes $q > 0$ **and** $p > 0$ **and** $a: \forall n. a\ n > 0$ **and** $\forall n. b\ n > 0$ **and**
assumerational: $(\lambda n. b\ (n+1) / a\ (n+1)) \text{ sums } (p/q)$
shows $q * ((\prod_{j=1..n} \text{of-int } (a\ j))) * (\text{suminf } (\lambda (j::\text{nat}). (b\ (j+n+1)) / a\ (j+n+1))))$
 $= ((\prod_{j=1..n} \text{of-int } (a\ j))) * (p - q * (\sum_{j=1..n} b\ j / a\ j))$
 $\langle \text{proof} \rangle$

lemma *show7*:
fixes $d :: \text{nat} \Rightarrow \text{real}$ **and** $a\ b :: \text{nat} \Rightarrow \text{int}$ **and** $q\ p :: \text{int}$
assumes $q \geq 1$ **and** $p \geq 1$ **and** $a: \forall n. a\ n \geq 1$ **and** $\forall n. b\ n \geq 1$
and *assumerational*: $(\lambda n. b\ (n+1) / a\ (n+1)) \text{ sums } (p/q)$
shows $q * ((\prod_{j=1..n} \text{of-int } (a\ j))) * (\text{suminf } (\lambda (j::\text{nat}). (b\ (j+n+1)) / a\ (j+n+1)))) \geq 1$
(is ?L \geq -)
 $\langle \text{proof} \rangle$

lemma *show8*:
fixes $d :: \text{nat} \Rightarrow \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{int}$ **and** $s\ k :: \text{nat}$
assumes $A > 1$ **and** $d: \forall n. d\ n > 1$ **and** $a: \forall n. a\ n > 0$ **and** $s > 0$
and *convergent-prod* d
and *assu2*: $\forall n \geq s. A / \text{of-int } (a\ n) \text{ powr } (1 / \text{of-int } (2^n)) > (\prod_{j=1..n} d\ (n + j))$

shows $\forall n \geq s. (\prod j. d(j+n)) < A / (\text{MAX } j \in \{s..n\}. \text{of-int}(a j) \text{ powr}(1 / \text{of-int}(2^{\wedge} j)))$
 (proof)

lemma auxiliary1-9:

fixes $d :: \text{nat} \Rightarrow \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{int}$ **and** $s m :: \text{nat}$
assumes $d: \forall n. d n > 1$ **and** $a: \forall n. a n > 0$ **and** $s > 0$ **and** $n \geq m$ **and** $m \geq s$
and *auxifalse-assu*: $\forall n \geq m. (\text{of-int}(a(n+1)) \text{ powr}(1 / \text{of-int}(2^{\wedge}(n+1)))) <$
 $(d(n+1)) * (\text{Max}((\lambda(j::\text{nat}). (\text{of-int}(a j) \text{ powr}(1 / \text{of-int}(2^{\wedge} j)))) ' \{s..n\})))$
shows $(\text{of-int}(a(n+1)) \text{ powr}(1 / \text{of-int}(2^{\wedge}(n+1)))) <$
 $(\prod j=(m+1)..(n+1). d j) * (\text{Max}((\lambda(j::\text{nat}). (\text{of-int}(a j) \text{ powr}(1 / \text{of-int}(2^{\wedge} j)))) ' \{s..m\})))$
 (proof)

lemma show9:

fixes $d :: \text{nat} \Rightarrow \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{int}$ **and** $s :: \text{nat}$ **and** $A :: \text{real}$
assumes $A > 1$ **and** $d: \forall n. d n > 1$ **and** $a: \forall n. a n > 0$ **and** $s > 0$
and *assu1*: $((\lambda n. (\text{of-int}(a n) \text{ powr}(1 / \text{of-int}(2^{\wedge} n)))) \longrightarrow A)$ *sequentially*
and *convergent-prod d*
and $8: \forall n \geq s. \text{prodinf}(\lambda j. d(n+j))$
 $< A / (\text{Max}((\lambda(j::\text{nat}). (\text{of-int}(a j) \text{ powr}(1 / \text{of-int}(2^{\wedge} j)))) ' \{s..n\})))$
shows $\forall m \geq s. \exists n \geq m. ((\text{of-int}(a(n+1)) \text{ powr}(1 / \text{of-int}(2^{\wedge}(n+1)))) \geq$
 $(d(n+1)) * (\text{Max}((\lambda(j::\text{nat}). (\text{of-int}(a j) \text{ powr}(1 / \text{of-int}(2^{\wedge} j)))) ' \{s..n\})))$
 (proof)

lemma show10:

fixes $d :: \text{nat} \Rightarrow \text{real}$ **and** $a :: \text{nat} \Rightarrow \text{int}$ **and** $s :: \text{nat}$
assumes d [rule-format]: $\forall n. d n > 1$
and a [rule-format]: $\forall n. a n > 0$ **and** $s > 0$
and $9: \forall m \geq s. \exists n \geq m. a(n+1) \text{ powr}(1 / \text{of-int}(2^{\wedge}(n+1))) \geq$
 $d(n+1) * (\text{Max}((\lambda j. (\text{of-int}(a j) \text{ powr}(1 / \text{of-int}(2^{\wedge} j)))) ' \{s..n\})))$
shows $\forall m \geq s. \exists n \geq m. d(n+1) \text{ powr}(2^{\wedge}(n+1)) * (\prod j=1..n. \text{of-int}(a j)) *$
 $(1 / (\prod j=1..s-1. \text{of-int}(a j))) \leq a(n+1)$
 (proof)

lemma lasttoshow:

fixes $d :: \text{nat} \Rightarrow \text{real}$ **and** $a b :: \text{nat} \Rightarrow \text{int}$ **and** $q :: \text{int}$ **and** $s :: \text{nat}$
assumes $d: \forall n. d n > 1$
and $a: \forall n. a n > 0$ **and** $s > 0$ **and** $q > 0$
and $A > 1$ **and** $b: \forall n. b n > 0$ **and** $9:$
 $\forall m \geq s. \exists n \geq m. ((\text{of-int}(a(n+1)) \text{ powr}(1 / \text{of-int}(2^{\wedge}(n+1)))) \geq$
 $(d(n+1)) * (\text{Max}((\lambda(j::\text{nat}). (\text{of-int}(a j) \text{ powr}(1 / \text{of-int}(2^{\wedge} j)))) ' \{s..n\})))$
)))
and *assu3*: *filterlim*($\lambda n. (d n)^{\wedge}(2^{\wedge} n) / b n$) *at-top sequentially*
and $5: \forall_F n$ *in sequentially*. $(\sum j. (b(n+j) / (a(n+j)))) \leq 2 * b n / a n$
shows $\exists n. q * ((\prod j=1..n. \text{real-of-int}(a j)) * \text{suminf}(\lambda(j::\text{nat}). (b(j+n+1) / a$

$(j+n+1)) < 1$
 $\langle proof \rangle$

lemma

fixes $d :: nat \Rightarrow real$ **and** $a b :: nat \Rightarrow int$ **and** $A :: real$
assumes $A > 1$ **and** $d: \forall n. d n > 1$ **and** $a: \forall n. a n > 0$ **and** $b: \forall n. b n > 0$
and $assu1: ((\lambda n. (of-int (a n)) powr(1 / of-int (2^n))) \longrightarrow A)$ *sequentially*
and $assu3: filterlim (\lambda n. (d n)^(2^n) / b n)$ *at-top sequentially*
and *convergent-prod d*
shows $issummable: summable (\lambda j. b j / a j)$
and $show5: \forall_F n$ *in sequentially. $(\sum j. (b (n + j)) / (a (n + j))) \leq 2 * b n / a n$*
 $\langle proof \rangle$

theorem *Hancl3:*

fixes $d :: nat \Rightarrow real$ **and** $a b :: nat \Rightarrow int$
assumes $A > 1$ **and** $d: \forall n. d n > 1$ **and** $a: \forall n. a n > 0$ **and** $b: \forall n. b n > 0$
and $s > 0$
and $assu1: (\lambda n. (a n) powr(1 / of-int(2^n))) \longrightarrow A$
and $assu2: \forall n \geq s. A / (a n) powr (1 / of-int(2^n)) > (\prod j. d (n+j))$
and $assu3: LIM n$ *sequentially. $d n ^ 2 ^ n / b n :>$ at-top*
and *convergent-prod d*
shows $(\sum n. b n / a n) \notin \mathbb{Q}$
 $\langle proof \rangle$

corollary *Hancl3corollary:*

fixes $A :: real$ **and** $a b :: nat \Rightarrow int$
assumes $A > 1$ **and** $a: \forall n. a n > 0$ **and** $b: \forall n. b n > 0$
and $assu1: (\lambda n. (a n) powr(1 / of-int(2^n))) \longrightarrow A$
and $asscor2: \forall n \geq 6. a n powr(1 / of-int (2^n)) * (1 + 4*(2/3)^n) \leq A$
 $\wedge b n \leq 2 powr (4/3)^(n-1)$
shows $(\sum n. b n / a n) \notin \mathbb{Q}$
 $\langle proof \rangle$

end

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References

- [1] J. Hančl. Irrational rapidly convergent series. *Rendiconti del Seminario Matematico della Università di Padova*, 107:225–231, 2002.