(Extended) Interval Analysis

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Abstract

Interval analysis (also called interval arithmetic) is a well known mathematical technique to analyse or mitigate rounding errors or measurement errors. Thus, it is promising to integrate interval analysis into program verification environments. Such an integration is not only useful for the verification of numerical algorithms: the need to ensure that computations stay within certain bounds is common. For example to show that computations stay within the hardware bounds of a given number representation.

Another application is the verification of cyber-physical systems, where a discretised implementation approximates a system described in physical quantities expressed using perfect mathematical reals, and perfect ordinary differential equations.

In this AFP entry, we formalise extended interval analysis, including the concept of inclusion isotone (or inclusion isotonic) (extended) interval analysis. The main result is the formal proof that interval-splitting converges for Lipschitz-continuous interval isotone functions. From pragmatic perspective, we provide the datatypes and theory required for integrating interval analysis into other formalisations and applications.

Keywords: Extended Interval Analysis, Formalising Mathematics, Isabelle/HOL

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1 Introduction

Interval analysis [4] in general and, in particular, inclusion isotone extended interval arithmetic [5] are well known mathematical techniques to analyse or mitigate rounding errors or measurement errors. Thus, it is promising to integrate interval analysis into program verification environments. Such an integration is not only useful for the verification of numerical algorithms: the need to ensure that computations stay within certain bounds is common. For example to show that computations stay within the hardware bounds of a given number representation.

Another application is the verification of cyber-physical systems, were a discretised implementation approximates a system described in physical quantities expressed using perfect mathematical reals, and perfect ordinary differential equations. Moreover, first applications of interval analysis to the security analysis of neural networks are promising [6, 3] and, when combined with existing works of verifying neural networks in Isabelle [2] provide a verification approach using a "correct-by-construction" verification tool for neural networks.

In this AFP entry, we formalise extended interval analysis, including the concept of inclusion isotone (extended) interval analysis. The main result is the formal proof that interval-splitting converges for Lipschitz-continuous inclusion isotone functions. From pragmatic perspective, we provide the datatypes and theory required for integrating interval analysis into other formalisations and applications. In more detail, our contributions are:

1. a conservative formalisation of (extended) interval arithmetic in Isabelle/HOL, including inclusion isotonicity;
2. we formally prove that interval-splitting converges for Lipschitz-continuous inclusion isotone functions;

From an end-user's perspective, the main entry points into this session are the following three theories:

- `Interval_Analysis` (Chapter 10): This theory provides interval analysis over standard types such as real or integer. All operations work over (closed) intervals.
- `Extended_Interval_Analysis` (Chapter 12): This theory provides extended interval analysis over the type extended reals. All operations work over (closed) intervals.
- `Extended_Multi_Interval_Analysis` (Chapter 14): This theory provides extended multi-interval analysis over the type extended reals. All operations work over multi-intervals, i.e., lists of (closed) intervals.

The following publication [1] gives a high-level overview of this AFP entry:

> A. D. Brucker, T. Cameron-Burke, and A. Stell. Formally verified interval arithmetic and its application to program verification. In 13th IEEE/ACM International Conference on Formal Methods in Software Engineering (FormaliSE 2024). IEEE, 2024.

The rest of this document is automatically generated from the formalisation in Isabelle/HOL, i.e., all content is checked by Isabelle. Overall, the structure of this document follows the theory dependencies (see Figure 1.1).

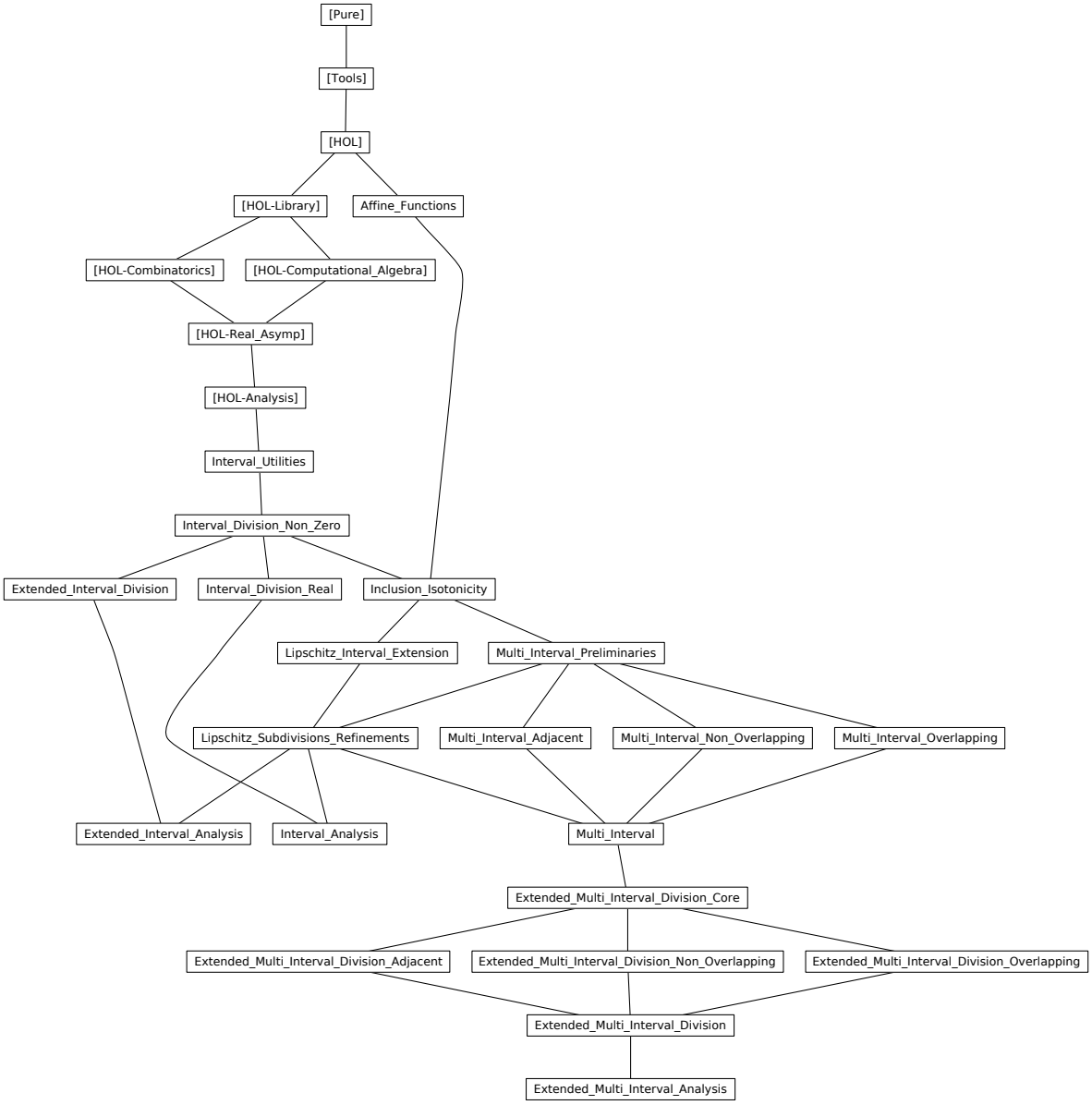

Figure 1.1: The Dependency Graph of the Isabelle Theories.

Generated Sessions

2 Interval Utilities (Interval_Utilities)

theory
Interval_Utilities
imports
HOL−Library.Interval
HOL−Analysis.Analysis
HOL−Library.Interval_Float
begin

2.1 Preliminaries

lemma *compact_set_of*:
fixes *X*::‹'*a*::{*preorder*,*topological_space*, *ordered_euclidean_space*} *interval*›
shows ‹*compact* (*set_of X*)›
by(*simp add*:*set_of_eq compact_interval*[*of lower X upper X*])

lemma *bounded_set_of*:
fixes *X*::‹'*a*::{*preorder*,*topological_space*, *ordered_euclidean_space*} *interval*›
shows ‹*bounded* (*set_of X*)›
by(*simp add*:*set_of_eq compact_interval*[*of lower X upper X*])

lemma *compact_img_set_of*:
fixes *X* :: ‹*real interval*› **and** *f*::‹*real* ⇒ *real*›
assumes ‹*continuous_on* (*set_of X*) *f*›
shows ‹*compact* (*f* ' *set_of X*)›
using *compact_continuous_image*[*of set_of X f*, *simplified assms*] *compact_interval*
unfolding *set_of_eq*
by *simp*

lemma *sup_inf_dist_bounded*:
fixes *X*::‹*real set*›
shows ‹*bdd_below X* ⟹ *bdd_above X* ==> ∀ *x* ∈ *X*. ∀ *x'* ∈ *X*. *dist x x'* ≤ *Sup X* − *Inf X*›
using *cInf_lower*[*of _ X*] *cSup_upper*[*of _ X*]
apply(*auto simp add*:*dist_real_def*)[1]
by (*smt* (*z3*))

lemma *set_of_nonempty*[*simp*]: ‹*set_of X* ≠ {}›
by(*simp add*:*set_of_eq*)

lemma *lower_in_interval*[*simp*]:‹*lower X* ∈$_i$ *X*›
by(*simp add*: *in_intervalI*)

lemma *upper_in_interval*[*simp*]:‹*upper X* ∈$_i$ *X*›
by(*simp add*: *in_intervalI*)

lemma *bdd_below_set_of*: ‹*bdd_below* (*set_of X*)›
by (*metis atLeastAtMost_iff bdd_below.unfold set_of_eq*)

```
lemma bdd_above_set_of: ‹bdd_above (set_of X)›
 by (metis atLeastAtMost_iff bdd_above.unfold set_of_eq)

lemma closed_set_of: ‹closed (set_of (X::real interval))›
 by (metis  closed_real_atLeastAtMost set_of_eq)

lemma set_f_nonempty: ‹f ' set_of X ≠ {}›
 apply(simp add:image_def set_of_eq)
 by fastforce

lemma interval_linorder_case_split[case_names LeftOf Including  RightOf]:
assumes ‹(upper x < c ⟹ P (x::('a::linorder interval)))›
  ‹( c ∈ᵢ x ⟹ P x)›
  ‹( c < lower x ⟹ P x)›
 shows‹ P x›
proof(insert assms, cases  upper x < c)
 case True
 then show ?thesis
  using assms(1) by blast
next
 case False
 then show ?thesis
 proof(cases c ∈ᵢ x)
  case True
  then show ?thesis
   using assms(2) by blast
 next
  case False
  then show ?thesis
   by (meson assms(1) assms(3) in_intervalI not_le_imp_less)
 qed
qed

lemma foldl_conj_True:
‹foldl (∧) x xs = list_all (λ e. e = True)  (x#xs)›
 by(induction xs rule:rev_induct, auto)

lemma foldl_conj_set_True:
‹foldl (∧) x xs = (∀  e ∈ set (x#xs) . e = True)›
 by(induction xs rule:rev_induct, auto)
```

2.2 Interval Bounds and Set Conversion

```
lemma sup_set_of:
 fixes X :: 'a::{conditionally_complete_lattice} interval
 shows Sup (set_of X) = upper X
 unfolding set_of_def upper_def
 using cSup_atLeastAtMost lower_le_upper[of X]
 by (simp add: upper_def lower_def)

lemma inf_set_of:
 fixes X :: 'a::{conditionally_complete_lattice} interval
```

shows *Inf* $(set_of\ X) = lower\ X$
unfolding *set_of_def lower_def*
using *cInf_atLeastAtMost lower_le_upper*[*of X*]
by (*simp add*: *upper_def lower_def*)

lemma *inf_le_sup_set_of*:
fixes *X* :: *'a*::{*conditionally_complete_lattice*} *interval*
shows*Inf* $(set_of\ X) \leq Sup\ (set_of\ X)$
using *sup_set_of inf_set_of lower_le_upper*
by *metis*

lemma *in_bounds*: ‹$x \in_i X \Longrightarrow lower\ X \leq x \wedge x \leq upper\ X$›
by (*simp add*: *set_of_eq*)

lemma *lower_bounds*[*simp*]:
assumes ‹$L \leq U$›
shows ‹$lower\ (Interval(L,U)) = L$ ›
using *assms*
apply (*simp add*: *lower.rep_eq*)
by (*simp add*: *bounds_of_interval_eq_lower_upper lower_Interval upper_Interval*)

lemma *upper_bounds* [*simp*]:
assumes ‹$L \leq U$›
shows ‹$upper\ (Interval(L,U)) = U$›
using *assms*
apply (*simp add*: *upper.rep_eq*)
by (*simp add*: *bounds_of_interval_eq_lower_upper lower_Interval upper_Interval*)

lemma *lower_point_interval*[*simp*]: ‹$lower\ (Interval\ (x,x)) = x$›
by (*simp*)
lemma *upper_point_interval*[*simp*]: ‹$upper\ (Interval\ (x,x)) = x$›
by (*simp*)

lemma *map2_nth*:
assumes ‹$length\ xs = length\ ys$›
and ‹$n < length\ xs$›
shows ‹$(map2\ f\ xs\ ys)!n = f\ (xs!n)\ (ys!n)$›
using *assms* **by** *simp*

lemma *map_set*: ‹$a \in set\ (map\ f\ X) \Longrightarrow\ (\exists\ x \in set\ X\ .\ f\ x = a)$›
by *auto*
lemma *map_pair_set_left*: ‹$(a,b) \in set\ (zip\ (map\ f\ X)\ (map\ f\ Y)) \Longrightarrow (\exists\ x \in set\ X\ .\ f\ x = a)$›
by (*meson map_set set_zip_leftD*)
lemma *map_pair_set_right*: ‹$(a,b) \in set\ (zip\ (map\ f\ X)\ (map\ f\ Y)) \Longrightarrow\ (\exists\ y \in set\ Y\ .\ f\ y = b)$›
by (*meson map_set set_zip_rightD*)
lemma *map_pair_set*: ‹$(a,b) \in set\ (zip\ (map\ f\ X)\ (map\ f\ Y)) \Longrightarrow\ (\exists\ x \in set\ X\ .\ f\ x = a) \wedge\ (\exists\ y \in set\ Y\ .\ f\ y = b)$›
by (*meson map_pair_set_left set_zip_rightD zip_same*)

lemma *map_pair_f_all*:
assumes ‹$length\ X = length\ Y$›
shows ‹$(\forall\ (x, y) \in set\ (zip\ (map\ f\ X)\ (map\ f\ Y)).\ x \leq y) = (\forall\ (x, y) \in set\ (zip\ X\ Y\ ).\ f\ x \leq f\ y)$›
by(*insert assms*(1), *induction X Y rule*:*list_induct2*, *auto*)

definition *map_interval_swap* :: ‹$('a::linorder \times 'a)\ list \Rightarrow 'a\ interval\ list$› **where**

‹map_interval_swap = map (λ (x,y). Interval (if x ≤ y then (x,y) else (y,x)))›

definition *mk_interval* :: ‹('a::linorder × 'a) ⇒ 'a interval › **where**
‹mk_interval = (λ (x,y). Interval (if x ≤ y then (x,y) else (y,x)))›

definition *mk_interval'* :: ‹('a::linorder × 'a) ⇒ 'a interval › **where**
‹mk_interval' = (λ (x,y). (if x ≤ y then Interval(x,y) else Interval(y,x)))›

lemma *map_interval_swap_code*[*code*]:
‹ map_interval_swap = map (λ (x,y). the (if x ≤ y then Interval' x y else Interval' y x))›
unfolding *map_interval_swap_def* **by**(*rule ext*, *rule arg_cong*, *auto simp add*: *Interval'.abs_eq*)

lemma *mk_interval_code*[*code*]:
‹mk_interval = (λ (x,y). the (if x ≤ y then Interval' x y else Interval' y x))›
unfolding *mk_interval_def* **by**(*rule ext*, *rule arg_cong*, *auto simp add*: *Interval'.abs_eq*)

lemma *mk_interval'*:
‹mk_interval = (λ (x,y). (if x ≤ y then Interval(x, y) else Interval(y, x)))›
unfolding *mk_interval_def* **by**(*rule ext*, *rule arg_cong*, *auto*)

lemma *mk_interval_lower*[*simp*]: ‹ lower (mk_interval (x,y)) = (if x ≤ y then x else y)›
by(*simp add*: *lower_def Interval_inverse mk_interval_def*)

lemma *mk_interval_upper*[*simp*]: ‹ upper (mk_interval (x,y)) = (if x ≤ y then y else x)›
by(*simp add*: *upper_def Interval_inverse mk_interval_def*)

2.3 Linear Order on List of Intervals

definition
le_interval_list :: ‹('a::linorder) interval list ⇒ 'a interval list ⇒ bool› (‹(_/ ≤$_I$ _)› [51, 51] 50)
where
‹le_interval_list Xs Ys ≡ (length Xs = length Ys) ∧ (foldl (∧) True (map2 (≤) Xs Ys))›

lemma *le_interval_single*: ‹(x ≤ y) = ([x] ≤$_I$ [y])›
unfolding *le_interval_list_def* **by** *simp*

lemma *le_intervall_empty*[*simp*]: ‹[] ≤$_I$ []›
unfolding *le_interval_list_def* **by** *simp*

lemma *le_interval_list_rev*: ‹(is ≤$_I$ js) = (rev is ≤$_I$ rev js)›
unfolding *le_interval_list_def*
by(*safe*, *simp_all add*: *foldl_conj_set_True zip_rev*)

lemma *le_interval_list_imp_length*:
assumes ‹Xs ≤$_I$ Ys› **shows** ‹length Xs = length Ys›
using *assms* **unfolding** *le_interval_list_def*
by *simp*

lemma *lsplit_left*: **assumes** ‹length (xs) = length (ys)›
and ‹(∀ n<length (x # xs). (x # xs) ! n ≤ (y # ys) ! n)› **shows** ‹
((∀ n < length xs. xs!n ≤ ys!n) ∧ x ≤ y)›
using *assms* **by** *auto*

```
lemma lsplit_right: assumes ‹length (xs) = length (ys)›
 and ‹((∀ n < length xs. xs!n ≤ ys!n) ∧ x ≤ y)›
shows ‹(n<length (x # xs) ⟶ (x # xs) ! n ≤ (y # ys) ! n)›
proof(cases n=0)
 case True
 then show ?thesis using assms by(simp)
next
 case False
 then show ?thesis using assms by simp
qed

lemma lsplit: assumes ‹length (xs) = length (ys)›
 shows ‹(∀ n<length (x # xs). (x # xs) ! n ≤ (y # ys) ! n) =
    ((∀ n < length xs. xs!n ≤ ys!n) ∧ x ≤ y)›
 using assms lsplit_left lsplit_right by metis

lemma le_interval_list_all':
 assumes ‹length Xs = length Ys› and ‹Xs ≤I Ys› shows ‹∀ n < length Xs. Xs!n ≤ Ys!n›
proof(insert assms, induction rule:list_induct2)
 case Nil
 then show ?case by simp
next
 case (Cons x xs y ys)
 then show ?case
  using lsplit[of xs ys x y] le_interval_list_def
  unfolding le_interval_list_def
  by(auto simp add: foldl_conj_True)
qed

lemma le_interval_list_all2:
 assumes ‹length Xs = length Ys›
  and ‹∀ n<length Xs . (Xs!n ≤ Ys!n)›
 shows ‹Xs ≤I Ys›
proof(insert assms, induction rule:list_induct2)
 case Nil
 then show ?case by simp
next
 case (Cons x xs y ys)
 then show ?case
  using lsplit[of xs ys x y] le_interval_list_def
  unfolding le_interval_list_def
  by(auto simp add: foldl_conj_True)
qed

lemma le_interval_list_all:
 assumes ‹Xs ≤I Ys› shows ‹∀ n < length Xs. Xs!n ≤ Ys!n›
 using assms le_interval_list_all' le_interval_list_imp_length
 by auto

lemma le_interval_list_imp:
 assumes ‹Xs ≤I Ys› shows ‹n < length Xs ⟶ Xs!n ≤ Ys!n›
 using assms le_interval_list_all' le_interval_list_imp_length
 by auto
```

```
lemma interval_set_leq_eq: ‹(X ≤ Y) = (set_of X ⊆ set_of Y)›
 for X :: ‹'a::linordered_ring interval›
 by (simp add: less_eq_interval_def set_of_eq)

lemma times_interval_right:
 fixes X Y C ::‹'a::linordered_ring interval›
 assumes ‹X ≤ Y›
 shows ‹C * X ≤ C * Y›
 using assms[simplified interval_set_leq_eq]
 apply(subst interval_set_leq_eq)
 by (simp add: set_of_mul_inc_right)


lemma times_interval_left:
 fixes X Y C ::‹'a::{real_normed_algebra,linordered_ring,linear_continuum_topology} interval›
 assumes ‹X ≤ Y›
 shows ‹X * C ≤ Y * C›
 using assms[simplified interval_set_leq_eq]
 apply(subst interval_set_leq_eq)
 by (simp add: set_of_mul_inc_left)
```

2.4 Support for Lists of Intervals

```
abbreviation in_interval_list::‹('a::preorder) list ⇒ 'a interval list ⇒ bool› (‹(_/ ∈I _)› [51, 51] 50)
 where ‹in_interval_list xs Xs ≡ foldl (∧) True (map2 (in_interval) xs Xs)›

lemma interval_of_in_interval_list[simp]: ‹xs ∈I map interval_of xs›
proof(induction xs)
 case Nil
 then show ?case by simp
next
 case (Cons a xs)
 then show ?case
  by (simp add: in_interval_to_interval)
qed

lemma interval_of_in_eq: ‹interval_of x ≤ X = (x ∈i X)›
 by (simp add: less_eq_interval_def set_of_eq)

lemma interval_of_list_in:
 assumes ‹length inputs = length Inputs›
shows ‹(map interval_of inputs ≤I Inputs) = (inputs ∈I Inputs)›
unfolding le_interval_list_def
proof(insert assms, induction inputs Inputs rule:list_induct2)
 case Nil
 then show ?case by simp
next
 case (Cons x xs y ys)
 then show ?case
 proof(cases interval_of x ≤ y)
  case True
  then show ?thesis
```

```
    using Cons by(simp add: interval_of_in_eq)
  next
    case False
    then show ?thesis using foldl_conj_True interval_of_in_eq by auto
  qed
qed
```

2.5 Interval Width and Arithmetic Operations

```
lemma interval_width_addition:
  fixes A:: 'a::{linordered_ring} interval
  shows ‹width (A + B) = width A + width B›
  by (simp add: width_def)

lemma interval_width_times:
  fixes a :: 'a::{linordered_ring} and A :: 'a interval
  shows width (interval_of a * A) = |a| * width A
proof −
  have width (interval_of a * A) = upper (interval_of a * A) − lower (interval_of a * A)
    by (simp add: width_def)
  also have ... = (if a > 0 then a * upper A − a * lower A else a * lower A − a * upper A)
    proof −
      have upper (interval_of a * A) = max (a * lower A) (a * upper A)
        by (simp add: upper_times)
      moreover have lower (interval_of a * A) = min (a * lower A) (a * upper A)
        by (simp add: lower_times)
      ultimately show ?thesis
        by (simp add: mult_left_mono mult_left_mono_neg)
    qed
  also have ... = |a| * (upper A − lower A)
    by (simp add: right_diff_distrib)
  finally show ?thesis
    by (simp add: width_def)
qed

lemma interval_sup_width:
  fixes X Y :: 'a::{linordered_ring, lattice} interval
  shows width (sup X Y) = max (upper X) (upper Y) − min (lower X) (lower Y)
proof −
  have a0: ∀ i ia. lower (sup i ia) = min (lower i::real) (lower ia)
    by (simp add: inf_min)
  have a1: ∀ i ia. upper (sup i ia) = max (upper i::real) (upper ia)
    by (simp add: sup_max)
  show ?thesis
    using a0 a1 by (simp add: inf_min sup_max width_def)
qed

lemma width_expanded: ‹interval_of (width Y) = Interval(upper Y − lower Y, upper Y − lower Y)›
  unfolding width_def interval_of_def by simp

lemma interval_width_positive:
  fixes X :: ‹'a::{linordered_ring} interval›
  shows ‹0 ≤ width X›
```

```
 using lower_le_upper
 by (simp add: width_def)
```

2.6 Interval Multiplication

```
lemma interval_interval_times:
‹X * Y = Interval(Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)},
            Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)})›
 by (simp add: times_interval_def bounds_of_interval_eq_lower_upper Let_def min_def)

lemma interval_times_scalar: ‹interval_of a * A = mk_interval(a * lower A, a * upper A)›
 proof −
   have upper (interval_of a * A) = max (a * lower A) (a * upper A)
    by (simp add: upper_times )
   moreover have lower (interval_of a * A) = min (a * lower A) (a * upper A)
    by (simp add: lower_times)
   ultimately show ?thesis
    by (simp add: interval_eq_iff)
 qed

lemma interval_times_scalar_pos_l:
 fixes a :: 'a::{ordered_semiring,times,linorder}
 assumes ‹0 ≤ a›
 shows ‹interval_of a * A = Interval(a * lower A, a * upper A)›
 proof −
  have upper (interval_of a * A) = a * upper A
   using assms
   by (simp add: upper_times mult_left_mono)
  moreover have lower (interval_of a * A) = a * lower A
   using assms
   by (simp add: lower_times mult_left_mono)
  ultimately show ?thesis
   using assms
   by (metis bounds_of_interval_eq_lower_upper bounds_of_interval_inverse lower_le_upper)
 qed

lemma interval_times_scalar_pos_r:
 fixes a :: 'a::{linordered_idom}
 assumes ‹0 ≤ a›
 shows ‹A * interval_of a = Interval(a * lower A, a * upper A)›
 proof −
  have upper (A * interval_of a) = a * upper A
   using assms
   by (simp add: mult.commute mult_left_mono upper_times)
  moreover have lower (A * interval_of a ) = a * lower A
   using assms
   by (simp add: lower_times mult_right_mono)
  ultimately show ?thesis
   using assms
   by (metis bounds_of_interval_eq_lower_upper bounds_of_interval_inverse lower_le_upper)
 qed
```

2.7 Distance-based Properties of Intervals

Given two real intervals *X* and *Y*, and two real numbers *a* and *b*, the width of the sum of the scaled intervals is equivalent to the width of the two individual intervals.

lemma *width_of_scaled_interval_sum*:
fixes *X* :: *'a*::{*linordered_ring*} *interval*
shows ‹*width* (*interval_of a* ∗ *X* + *interval_of b* ∗ *Y*) = |*a*| ∗ *width X* + |*b*| ∗ *width Y*›
using *interval_width_addition interval_width_times* **by** *metis*

lemma *width_of_product_interval_bound_real*:
fixes *X* :: *real interval*
shows ‹*interval_of* (*width* (*X* ∗ *Y*)) ≤ *abs_interval*(*X*) ∗ *interval_of* (*width Y*) + *abs_interval*(*Y*) ∗ *interval_of* (*width X*)›
proof −
have *a0*: *lower X* ≤ *upper X* ‹*lower Y* ≤ *upper Y*›
using *lower_le_upper* **by** *simp_all*
have *a1*: ‹*width Y* ≥ *0*› **and** *a1'*: ‹*width X* ≥ *0*›
using *a0 interval_width_positive* **by** *simp_all*
have *a2*: ‹*abs_interval*(*X*) ∗ *interval_of* (*width Y*) = *Interval* (*width Y* ∗ *lower* (*abs_interval X*), *width Y* ∗ *upper* (*abs_interval X*))›
using *interval_times_scalar_pos_r a0 a1* **by** *blast*
have *a3*: ‹*abs_interval*(*Y*) ∗ *interval_of* (*width X*) = *Interval* (*width X* ∗ *lower* (*abs_interval Y*), *width X* ∗ *upper* (*abs_interval Y*))›
using *interval_times_scalar_pos_r a0 interval_width_positive* **by** *blast*
have *a4*: ‹*abs_interval*(*X*) ∗ *interval_of* (*width Y*) + *abs_interval*(*Y*) ∗ *interval_of* (*width X*) = *Interval* (*width Y* ∗ *lower* (*abs_interval X*) + *width X* ∗ *lower* (*abs_interval Y*), *width Y* ∗ *upper* (*abs_interval X*) + *width X* ∗ *upper* (*abs_interval Y*))›
using *a0 a2 a3* **unfolding** *plus_interval_def*
apply (*simp add*:*bounds_of_interval_eq_lower_upper mult_left_mono interval_width_positive*)
by (*auto simp add*:*bounds_of_interval_eq_lower_upper mult_left_mono interval_width_positive*)
have *a5*: ‹*width*(*X* ∗ *Y*) = *Max* {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)} − *Min* {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)}›
using *lower_times upper_times* **unfolding** *width_def* **by** *metis*
have *a6*: ‹*Max* {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)} − *Min* {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)} ≤ *width Y* ∗ *upper* (*abs_interval X*) + *width X* ∗ *upper* (*abs_interval Y*)›
using *a0 a1 a1'* **unfolding** *width_def*
by (*simp*, *smt* (*verit*) *a0 mult.commute mult_less_cancel_left_pos mult_minus_left right_diff_distrib*)
have *a7*: ‹*width Y* ∗ *lower* (*abs_interval X*) + *width X* ∗ *lower* (*abs_interval Y*) ≤ *Max* {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)} − *Min* {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)}›
using *a0 a1 a1'* **unfolding** *width_def*
by (*simp*, *smt* (*verit*) *a0 mult.commute mult_less_cancel_left_pos mult_minus_left right_diff_distrib*)
have ‹*interval_of* (*Max* {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)} −
Min {(*lower X* ∗ *lower Y*), (*lower X* ∗ *upper Y*), (*upper X* ∗ *lower Y*), (*upper X* ∗ *upper Y*)}) ≤
Interval (*width Y* ∗ *lower* (*abs_interval X*) + *width X* ∗ *lower* (*abs_interval Y*),
width Y ∗ *upper* (*abs_interval X*) + *width X* ∗ *upper* (*abs_interval Y*))›
using *a6 a7* **unfolding** *less_eq_interval_def*
by (*smt* (*verit*, *ccfv_threshold*) *lower_bounds lower_interval_of upper_bounds upper_interval_of*)
then show *?thesis* **using** *a4 a5 a6* **by** *presburger*
qed

lemma *width_of_product_interval_bound_int*:

```
 fixes X :: int interval
 shows ‹interval_of (width (X * Y)) ≤ abs_interval(X) * interval_of (width Y) + abs_interval(Y) * interval_of (width X)›
proof −
 have a0: lower X ≤ upper X ‹lower Y ≤ upper Y›
  using lower_le_upper by simp_all
 have a1: ‹width Y ≥ 0› and ‹width X ≥ 0›
  using a0 interval_width_positive by simp_all
  have a2: ‹abs_interval(X) * interval_of (width Y) = Interval (width Y * lower (abs_interval X), width Y * upper (abs_interval X))›
  using interval_times_scalar_pos_r a0 a1 by blast
  have a3: ‹abs_interval(Y) * interval_of (width X) = Interval (width X * lower (abs_interval Y), width X * upper (abs_interval Y))›
  using interval_times_scalar_pos_r a0 interval_width_positive by blast
 have a4: ‹abs_interval(X) * interval_of (width Y) + abs_interval(Y) * interval_of (width X) = Interval (width Y * lower (abs_interval X) + width X * lower (abs_interval Y), width Y * upper (abs_interval X) + width X * upper (abs_interval Y))›
  using a0 a2 a3 unfolding plus_interval_def
  by (auto simp add:bounds_of_interval_eq_lower_upper mult_left_mono interval_width_positive)
 have a5: ‹width(X * Y) = Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} − Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)}›
  using lower_times upper_times unfolding width_def by metis
 have a6: ‹Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} − Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} ≤ width Y * upper (abs_interval X) + width X * upper (abs_interval Y)›
  using a0 unfolding width_def
  by (simp, smt (verit) a0 mult.commute mult_less_cancel_left_pos mult_minus_left right_diff_distrib)
 have a7: ‹width Y * lower (abs_interval X) + width X * lower (abs_interval Y) ≤ Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} − Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)}›
  using a0 unfolding width_def
  apply(auto)[1]
  by (smt (verit) a0(1) a0(2) left_diff_distrib mult_less_cancel_left_pos int_distrib(3)
     mult_less_cancel_left mult_minus_right mult.commute mult_le_0_iff right_diff_distrib'
     mult_left_mono mult_le_cancel_right right_diff_distrib zero_less_mult_iff )+
 have ‹interval_of (Max {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)} −
           Min {(lower X * lower Y), (lower X * upper Y), (upper X * lower Y), (upper X * upper Y)}) ≤
     Interval (width Y * lower (abs_interval X) + width X * lower (abs_interval Y),
          width Y * upper (abs_interval X) + width X * upper (abs_interval Y))›
  using a6 a7 unfolding less_eq_interval_def
  by (smt (verit, ccfv_threshold) lower_bounds lower_interval_of upper_bounds upper_interval_of)
 then show ?thesis using a4 a5 a6 by presburger
qed

end
```

3 Basic Properties of Interval Division (Interval_Division_Non_Zero)

theory
Interval_Division_Non_Zero
imports
Interval_Utilities
begin

The theory *HOL−Library.Interval* does not define a division operation on intervals. In the following we define a locale capturing the core properties of division by an interval that does not contain zero.

3.1 Preliminaries

lemma *division_leq_neg*:
fixes $x :: 'a::\{linordered_field\}$
assumes $0 < x$ **and** $y < 0$ **and** $z < 0$ **and** $y \leq z$
shows $x / z \leq x / y$
proof −
have $x * y \leq x * z$ **using** *assms* **by** *simp*
hence $(x * y) / (y * z) \leq (x * z) / (y * z)$
using *assms*
by (*simp add*: *divide_right_mono zero_less_mult_iff neg_divide_le_eq*)
thus *?thesis* **using** *assms* **by** *auto*[*1*]
qed

lemma *division_leq*:
fixes $x :: 'a::\{linordered_field\}$
assumes $0 < x$ **and** $y \leq z$ **and** ‹$y \neq 0 \wedge z \neq 0$› **and** ‹$(y < 0 \wedge z < 0) \vee (0 < y \wedge 0 < z)$›
shows $x / z \leq x / y$
proof (*cases* ‹$(y < 0 \wedge z < 0)$›)
case *True*
then show *?thesis* **using** *assms division_leq_neg* **by** *blast*
next
case *False*
have ‹$(0 < y \wedge 0 < z)$› **using** *assms False* **by** *blast*
then show *?thesis*
using *assms*
by (*simp add*: *frac_le*)
qed

lemma *upper_leq_lower_div*:
fixes $Y :: 'a::\{linordered_field\}$ *interval*
assumes ‹*lower Y* $\leq$ *upper Y*› **and** ‹$\neg$ $0 \in_i Y$›
shows ‹$1 / $ *upper Y* $\leq 1 /$ *lower Y*›
using *assms division_leq frac_le*
by (*metis atLeastAtMost_iff inverse_eq_divide le_imp_inverse_le*

le_imp_inverse_le_neg linorder_not_less set_of_eq)

3.2 A Locale for Interval Division Where the Quotient-Interval does not Contain Zero

locale *interval_division* = *inverse* +
 constrains *inverse* :: ‹'*a*::{*linordered_field*, *real_normed_algebra*,*linear_continuum_topology*} *interval* ⇒ '*a interval*›
 and *divide* :: ‹'*a*::{*linordered_field*, *real_normed_algebra*,*linear_continuum_topology*} *interval* ⇒ '*a interval* ⇒ '*a interval*›
 assumes *inverse_left*: ‹¬ *0* ∈$_i$ *x* ⟹ *1* ≤ (*inverse x*) ∗ *x*›
 and *divide*: ‹¬ *0* ∈$_i$ *y* ⟹ *x* ≤ (*divide x y*) ∗ *y*›
begin
end

lemma *interval_non_zero_eq*:
 ‹¬ *0* ∈$_i$ (*i*::'*a*::{*linorder*, *zero*} *interval*) = (*lower i* < *0* ∧ *upper i* < *0*) ∨ (*lower i* > *0* ∧ *upper i* > *0*)›
 by (*metis in_bounds in_intervalI linorder_not_less lower_le_upper order_le_less_trans*)

lemma *inverse_includes_one*:
 assumes ‹¬ *0* ∈$_i$ (*i*::'*a*::{*division_ring*,*linordered_ring*} *interval*)›
 shows ‹*1* ∈$_i$ (*mk_interval* (*1* / *upper i*, *1* / *lower i*)) ∗ *i*›
 using *assms interval_non_zero_eq*[*of i*]
 apply(*simp add*: *set_of_eq*)
 apply(*safe*, *simp_all*)
 by (*metis in_bounds lower_in_interval mk_interval_upper nonzero_eq_divide_eq times_in_intervalI upper_in_interval*)+

lemma *inverse_includes_one'*:
 assumes ‹¬ *0* ∈$_i$ (*i*::'*a*::{*division_ring*,*linordered_ring*} *interval*)›
 shows ‹*1* ≤ (*mk_interval* (*1* / *upper i*, *1* / *lower i*)) ∗ *i*›
 by (*simp add*: *assms in_bounds inverse_includes_one less_eq_interval_def*)

locale *interval_division_inverse* = *inverse* +
 constrains *inverse* :: ‹'*a*::{*linordered_field*, *real_normed_algebra*,*linear_continuum_topology*} *interval* ⇒ '*a interval*›
 and *divide* :: ‹'*a*::{*linordered_field*, *real_normed_algebra*,*linear_continuum_topology*} *interval* ⇒ '*a interval* ⇒ '*a interval*›
 assumes *inverse_non_zero_def*: ‹¬ *0* ∈$_i$ *x* ⟹ (*inverse x*) = *mk_interval*(*1* / (*upper x*), *1* / (*lower x*))›
 and *divide_non_zero_def*: ‹¬ *0* ∈$_i$ *y* ⟹ (*divide x y*) = *inverse y* ∗ *x*›
begin

sublocale *interval_division divide inverse*
 apply(*standard*)
 subgoal
 by (*simp add*: *inverse_includes_one' inverse_non_zero_def*)
 subgoal
 by (*metis* (*no_types*, *opaque_lifting*) *divide_non_zero_def interval_mul_commute inverse_includes_one' inverse_non_zero_def mult.assoc one_times_ivl_right times_interval_right*)
 done

lemma *inverse_left_ge_one*:
 assumes ‹¬ *0* ∈$_i$ *x*›
 shows ‹*1* ≤ (*inverse x*) ∗ *x*›

```
proof −
 have lower_ne_zero: ‹lower x ≠ 0›
  using assms lower_in_interval by metis
 have upper_ne_zero: ‹lower x ≠ 0›
  using assms lower_in_interval by metis
 have ‹1 ≤ (mk_interval ( 1 / (upper x), 1 / (lower x))) ∗ x›
 proof(cases 1 / upper x ≤ 1 / lower x)
  case True note ∗ = this
  then show ?thesis
  proof(cases upper x = lower x)
   case True
   then show ?thesis
    using upper_times[of mk_interval (1 / upper x, 1 / lower x) x]
        lower_times[of mk_interval (1 / upper x, 1 / lower x) x]
        interval_eq_iff[of mk_interval (1 / upper x, 1 / lower x) ∗ x 1]
        lower_ne_zero upper_ne_zero
    unfolding mk_interval'
    by simp
  next
   case False
   then show ?thesis
    using interval_eq_iff[of mk_interval (1 / upper x, 1 / lower x) ∗ x 1]
        upper_times[of mk_interval (1 / upper x, 1 / lower x) x]
        lower_times[of mk_interval (1 / upper x, 1 / lower x) x]
   proof −
    have 1 / lower x = upper (mk_interval (1 / upper x, 1 / lower x))
     by (simp add: ∗)
    then show ?thesis
     by (metis (no_types) in_bounds less_eq_interval_def lower_in_interval lower_one
        nonzero_divide_eq_eq times_in_intervalI upper_in_interval upper_ne_zero upper_one)
   qed
  qed
 next
  case False
  then show ?thesis
    using interval_eq_iff[of mk_interval (1 / upper x, 1 / lower x) ∗ x 1]
        upper_times[of mk_interval (1 / upper x, 1 / lower x) x]
        lower_times[of mk_interval (1 / upper x, 1 / lower x) x]
    using assms lower_le_upper upper_leq_lower_div by blast
  qed
  then show ?thesis
 by (simp add: assms inverse_non_zero_def)
 qed

lemma division_right_ge_refl:
 assumes ‹¬ 0 ∈ᵢ y›
 shows ‹x ≤ x ∗ ((inverse y) ∗ y)›
proof −
 have a1: ‹set_of 1 ⊆ set_of ((inverse y) ∗ y)›
  using inverse_left_ge_one[of y, simplified assms, simplified]
  by (simp add: interval_set_leq_eq)
 show ?thesis
  using set_of_mul_inc_right[of 1 mk_interval (1 / upper y, 1 / lower y) ∗ y x,
        simplified one_times_ivl_right[of x] a1, simplified]
```

```
  by (metis a1 interval_set_leq_eq one_times_ivl_right times_interval_right)
qed

lemma division_right_ge_refl':
 assumes ‹¬ o ∈ᵢ y›
 shows ‹x ≤ x * inverse y * y›
 by (simp add: assms division_right_ge_refl mult.assoc)

lemma interval_div_constant:
 assumes ‹o ∉ set_of Y› and ‹o ≤ x›
 shows ‹divide (interval_of x) Y = Interval( x / upper Y, x / lower Y)›
proof −
 have l: ‹lower Y ≤ upper Y› using lower_le_upper by simp
 have ‹1 / upper Y ≤ 1 / lower Y› using assms l
  by (metis divide_left_mono frac_le in_intervalI linorder_not_less mult_neg_neg order_less_le
     zero_less_one_class.zero_le_one)
 then show ?thesis
  using
  interval_of.abs_eq[of x]
  assms divide_non_zero_def[of Y interval_of x, simplified assms, simplified]
  inverse_non_zero_def[of Y, simplified assms, simplified]
  interval_times_scalar_pos_l interval_times_scalar_pos_r by fastforce
qed

lemma interval_of_width:
 assumes ‹¬ o ∈ᵢ Y›
 shows ‹interval_of(width (divide (interval_of 1) Y)) = Interval( 1 / lower Y − 1 / upper Y, 1 / lower Y − 1 / upper Y) ›
proof(cases Y rule:interval_linorder_case_split[of _ o λ Y. interval_of(width (divide (interval_of 1) Y))
                                                = Interval( 1 / lower Y − 1 / upper Y, 1 / lower Y − 1 / upper Y) ])
 case LeftOf
 have ‹1 / upper Y ≤ 1 / lower Y›
  using assms division_leq_neg LeftOf
  by (simp add: le_divide_eq)
 then show ?case
  using interval_div_constant upper_bounds lower_bounds assms
  unfolding width_def interval_of_def by fastforce
next
 case Including
 then show ?case using assms by simp
next
 case RightOf
 have ‹1 / upper Y ≤ 1 / lower Y›
  by (simp add: assms upper_leq_lower_div)
 then show ?case
  using interval_div_constant upper_bounds lower_bounds assms
  unfolding width_def interval_of_def by fastforce
qed

lemma abs_pos:
 assumes ‹o < lower Y › and ‹¬ o ∈ᵢ Y›
 shows ‹abs_interval(divide (interval_of 1) Y) = Interval( 1 / upper Y, 1 / lower Y)›
proof −
 have l: ‹lower Y ≤ upper Y› using lower_le_upper by simp
 have ‹o < 1 / upper Y ›
```

```
  by (metis assms(1) l dual_order.strict_trans1 zero_less_divide_1_iff)
 moreover have ‹ 0 < 1 / lower Y›
  by (metis assms(1) zero_less_divide_1_iff)
 moreover have ‹1 / upper Y ≤ 1 / lower Y›
  using assms by (simp add: frac_le)
 moreover have ‹divide (interval_of 1) Y = Interval( 1 / upper Y, 1 / lower Y)›
  using assms interval_div_constant[of Y 1] by simp
 ultimately show ?thesis
  unfolding abs_interval_def by (simp add: bounds_of_interval_eq_lower_upper)
qed

lemma abs_neg:
 assumes ‹upper Y < 0 › and ‹¬ 0 ∈ᵢ Y›
 shows ‹abs_interval(divide (interval_of 1) Y) = Interval(1 / |lower Y|, 1 / |upper Y|)›
proof −
 have l:  ‹lower Y ≤ upper Y› using lower_le_upper by simp
 have i0: ‹ 1 / upper Y < 0 › and i1: ‹ 1 / lower Y < 0›
  using assms by (simp, meson assms(1) divide_less_0_1_iff lower_le_upper order_le_less_trans)
 moreover have i2: ‹|upper Y| ≤ |lower Y|›
  using assms l by linarith
 then have i3: ‹|1 / lower Y| ≤ |1 / upper Y|›
  using assms division_leq_neg i1
  by (simp add: division_leq)
 moreover have ‹divide (interval_of 1) Y = Interval( 1 / upper Y, 1 / lower Y)›
  using assms interval_div_constant[of Y 1] by simp
 moreover have ‹abs_interval(Interval( 1 / upper Y, 1 / lower Y)) = Interval(|1 / lower Y|, |1 / upper Y|)›
  using assms i0 i1 i2 i3 unfolding abs_interval_def min_def max_def
  by (simp add: bounds_of_interval_eq_lower_upper)
 moreover have ‹... = Interval(1 / |lower Y|, 1 / |upper Y|)›
  by auto[1]
 ultimately show ?thesis
  using assms interval_div_constant by force
qed


end


end
```

4 A Naive Interval Division for Real Intervals (Interval_Division_Real)

theory
Interval_Division_Real
imports
Interval_Division_Non_Zero
begin

The theory *HOL−Library.Interval* does not define a division operation on intervals. Actually, In the following we define division in a straight forward way. This is possible, as in HOL, the property *?a / 0 = 0* holds. Therefore, we do not need to use, in the first instance, extended interval analysis (e.g., based on the type *ereal*. As a consequence, results obtained using this definition might differ from results obtained using definitions of divisions using extended reals (e.g., [4]).

instantiation *interval* :: ({*linordered_field*, *real_normed_algebra*,*linear_continuum_topology*}) *inverse*
begin
definition *inverse_interval* :: *'a interval* ⇒ *'a interval*
where *inverse_interval a* = *mk_interval* (*1* / (*upper a*), *1* / (*lower a*))
definition *divide_interval* :: *'a interval* ⇒ *'a interval* ⇒ *'a interval*
where *divide_interval a b* = *inverse b* ∗ *a*
instance ..
end

interpretation *interval_division_inverse divide inverse*
apply (*unfold_locales*)
subgoal by (*simp add*: *inverse_interval_def*)
subgoal by(*simp add*: *divide_interval_def*)
done

lemma *width_of_reciprocal_interval_bound_real*:
fixes *Y* :: *real interval*
assumes ‹¬ *0* ∈*i* *Y*›
shows ‹*interval_of*(*width* ((*interval_of 1*) / *Y*)) ≤
(*abs_interval*((*interval_of 1*) / *Y*) ∗ *abs_interval*((*interval_of 1*) / *Y*)) ∗ *interval_of*(*width Y*)›
proof(*cases Y rule*:*interval_linorder_case_split*[*of _ 0* λ *Y*. *interval_of*(*width* ((*interval_of 1*) / *Y*)) ≤
(*abs_interval*((*interval_of 1*) / *Y*)) ∗ (*abs_interval*((*interval_of 1*) / *Y*)) ∗ *interval_of*(*width Y*)])
case *LeftOf*
have *a0*: ‹*lower Y* ≤ *upper Y*› **using** *lower_le_upper* **by** *simp*
have *a1*: ‹*interval_of*(*width* ((*interval_of 1*) / *Y*)) = *Interval*(*1* / *lower Y* − *1* / *upper Y*, *1* / *lower Y* − *1* / *upper Y*) ›
using *assms interval_of_width* **by** *blast*
have *a2*: ‹(*abs_interval*((*interval_of 1*) / *Y*) ∗ *abs_interval*((*interval_of 1*) / *Y*)) ∗ *interval_of*(*width Y*) = *Interval*((*upper Y* − *lower Y*) ∗ *1* / |*lower Y*| ∗ *1* / |*lower Y*|, (*upper Y* − *lower Y*) ∗ *1* / |*upper Y*| ∗ *1* / |*upper Y*|)›
proof −
have *b0*: ‹*1* / |*lower Y*| ≤ *1* / |*upper Y*|› **using** *assms a0 LeftOf*
by (*smt* (*verit*, *best*) *frac_le*)
have *b1*: ‹*0* ≤ *upper Y* − *lower Y*›
using *assms LeftOf* **by** *simp*

```
  moreover have b2: ‹(abs_interval((interval_of 1) / Y) * abs_interval((interval_of 1) / Y)) * interval_of(width Y) =
Interval(1/|lower Y|, 1/|upper Y|) * Interval(1/|lower Y|, 1/|upper Y|) * Interval(upper Y − lower Y, upper Y − lower Y)›
    using assms LeftOf abs_neg[of Y] width_expanded[of Y] by simp
  moreover have b3: ‹... = Interval(Min { (1 /|lower Y| * 1 /|lower Y|), (1 /|lower Y| * 1 /|upper Y|), (1 /|upper Y| * 1
/|lower Y|), (1 /|upper Y| * 1 /|upper Y|)}, Max { (1 /|lower Y| * 1 /|lower Y|), (1 /|lower Y| * 1 /|upper Y|), (1 /|upper
Y| * 1 /|lower Y|), (1 /|upper Y| * 1 /|upper Y|)}) * Interval(upper Y − lower Y, upper Y − lower Y) ›
    using assms ao LeftOf interval_interval_times[of Interval(1/|lower Y|, 1/|upper Y|) Interval(1/|lower Y|, 1/|upper Y|)]
    using lower_bounds upper_bounds bo by simp
  have b4: ‹Min {(1 /|lower Y| * 1 /|lower Y|), (1 /|lower Y| * 1 /|upper Y|), (1 /|upper Y| * 1 /|lower Y|), (1 /|upper Y|
* 1 /|upper Y|)} = (1 /|lower Y| * 1 /|lower Y|)›
    using assms ao LeftOf upper_leq_lower_div
    apply(simp add: min_def, safe)
    apply (smt (z3) divide_divide_eq_left' divide_less_cancel divide_minus_right nonzero_minus_divide_divide)
     apply argo+
   by (smt (z3) ao divide_divide_eq_left' divide_le_cancel divide_nonneg_neg frac_le minus_divide_right mult.commute
mult_left_mono mult_minus_left)
  have b5: ‹Max {(1 /|lower Y| * 1 /|lower Y|), (1 /|lower Y| * 1 /|upper Y|), (1 /|upper Y| * 1 /|lower Y|), (1 /|upper Y|
* 1 /|upper Y|)} = (1 /|upper Y| * 1 /|upper Y|)›
    using assms ao LeftOf upper_leq_lower_div
    apply (simp add: max_def, safe)
    apply (smt (verit, del_insts) frac_le mult_mono_nonpos_nonpos mult_neg_neg)
    apply argo
      apply argo
    apply (smt (z3) bo divide_divide_eq_left' divide_less_cancel minus_divide_right nonzero_minus_divide_divide)
    apply (smt (verit, best) division_leq mult_mono_nonpos_nonpos not_real_square_gt_zero)
    by argo+
  moreover have ‹(abs_interval((interval_of 1) / Y) * abs_interval((interval_of 1) / Y)) * interval_of(width Y) = Inter-
val(1 /|lower Y| * 1 /|lower Y|, 1 /|upper Y| * 1 /|upper Y|) * Interval(upper Y − lower Y, upper Y − lower Y)›
    using b2 b3 b4 b5 by presburger
  moreover have ‹... = Interval((upper Y − lower Y) * 1 /|lower Y| * 1 /|lower Y|, (upper Y − lower Y) * 1 /|upper Y| *
1 /|upper Y|)›
    using assms ao LeftOf b4 b5 interval_times_scalar_pos_r[of upper Y − lower Y Interval(1 /|lower Y| * 1 /|lower Y|, 1
/|upper Y| * 1 /|upper Y|)]
   unfolding interval_of_def by simp
  ultimately show ?thesis by metis
 qed
 have a3: ‹(upper Y − lower Y) * 1 /|lower Y| * 1 /|lower Y| ≤ 1 / lower Y − 1 / upper Y ›
 proof −
  have (1 / lower Y) − (1 / upper Y) = (upper Y − lower Y) / (lower Y * upper Y)
   using assms ao LeftOf
   by (smt (verit) diff_divide_distrib nonzero_divide_mult_cancel_left nonzero_divide_mult_cancel_right)
  moreover have ‹(upper Y − lower Y) * 1 /|lower Y| * 1 /|lower Y| = (upper Y − lower Y) / |lower Y|^2›
   using assms LeftOf by (simp add: power2_eq_square)
  moreover have ‹((upper Y − lower Y) / |lower Y|^2) ≤ (upper Y − lower Y) / (lower Y * upper Y)›
   using assms ao LeftOf
   by (smt (verit) frac_le minus_mult_minus mult_left_mono_neg mult_neg_neg power2_eq_square)
  ultimately show ?thesis by metis
 qed
 have a4: ‹1 / lower Y − 1 / upper Y ≤ (upper Y − lower Y) * 1 /|upper Y| * 1 /|upper Y|›
 proof −
  have (1 / lower Y) − (1 / upper Y) = (upper Y − lower Y) / (lower Y * upper Y)
   using assms ao LeftOf
   by (smt (verit) diff_divide_distrib nonzero_divide_mult_cancel_left nonzero_divide_mult_cancel_right)
  moreover have ‹(upper Y − lower Y) * 1 /|upper Y| * 1 /|upper Y| = (upper Y − lower Y) / |upper Y|^2›
```

```
    using assms LeftOf by (simp add: power2_eq_square)
   moreover have ‹(upper Y − lower Y) / (lower Y * upper Y) ≤ ((upper Y − lower Y) / |upper Y|^2)›
    using assms LeftOf
    by (simp add: frac_le power2_eq_square zero_less_mult_iff)
   ultimately show ?thesis by metis
  qed
  have ‹Interval( 1 / lower Y − 1 / upper Y, 1 / lower Y − 1 / upper Y) ≤ Interval((upper Y − lower Y) * 1 /|lower Y| *
1 /|lower Y|, (upper Y − lower Y) * 1 /|upper Y| * 1 /|upper Y|)›
   using a3 a4 unfolding less_eq_interval_def by simp
  then show ?case using a1 a2 a3 a4 by simp
next
  case Including
  then show ?case using assms by simp
next
  case RightOf
  have a0: ‹lower Y ≤ upper Y› using lower_le_upper by simp
  have a1: ‹interval_of(width ((interval_of 1) / Y)) = Interval( 1 / lower Y − 1 / upper Y, 1 / lower Y − 1 / upper Y)›
   using assms interval_of_width by blast
  have a2: ‹(abs_interval((interval_of 1) / Y) * abs_interval((interval_of 1) / Y)) * interval_of(width Y) = Interval((upper
Y − lower Y) * 1 /upper Y * 1 /upper Y, (upper Y − lower Y) * 1 /lower Y * 1 /lower Y)›
  proof −
   have b0: ‹0 ≤ upper Y − lower Y›
    using assms RightOf by simp
   moreover have b1: ‹(abs_interval((interval_of 1) / Y) * abs_interval((interval_of 1) / Y)) * interval_of(width Y) =
Interval(1/upper Y, 1/lower Y) * Interval(1/upper Y, 1/lower Y) * Interval(upper Y − lower Y, upper Y − lower Y)›
    using assms RightOf abs_pos[of Y] width_expanded[of Y] by simp
   moreover have b2: ‹... = Interval(Min { (1 /upper Y * 1 /upper Y), (1 /upper Y * 1 /lower Y), (1 /lower Y * 1 /upper
Y), (1 /lower Y * 1 /lower Y)}, Max { (1 /upper Y * 1 /upper Y), (1 /upper Y * 1 /lower Y), (1 /lower Y * 1 /upper Y), (1
/lower Y * 1 /lower Y)}) * Interval(upper Y − lower Y, upper Y − lower Y)›
    using assms RightOf upper_leq_lower_div interval_interval_times[of Interval(1 /upper Y, 1 / lower Y) Interval(1 /upper
Y, 1 / lower Y)]
    by fastforce
   have b3: ‹Min {(1 /upper Y * 1 /upper Y), (1 /upper Y * 1 /lower Y), (1 /lower Y * 1 /upper Y), (1 /lower Y * 1 /lower
Y)} = (1 /upper Y * 1 /upper Y)›
    using assms a0 RightOf upper_leq_lower_div
    apply(simp add: min_def, safe)
    apply (smt (verit, ccfv_SIG) frac_le mult_less_cancel_left_pos zero_less_mult_iff)
    apply argo+
    by (simp add: frac_le)
   have b4: ‹Max { (1 /upper Y * 1 /upper Y), (1 /upper Y * 1 /lower Y), (1 /lower Y * 1 /upper Y), (1 /lower Y * 1 /lower
Y)} = (1 /lower Y * 1 /lower Y)›
    using assms a0 RightOf upper_leq_lower_div
    apply (simp add: max_def, safe)
    subgoal
     using frac_le le_numeral_extra(4) less_numeral_extra(3) zero_le_one
      mult_mono' not_real_square_gt_zero order_less_imp_le by metis
    apply argo+
    apply (simp add: frac_le)
    subgoal
     using frac_le zero_le_one mult_mono' mult_pos_pos
      order.refl order_less_imp_le by meson
    by argo+
   moreover have ‹(abs_interval((interval_of 1) / Y) * abs_interval((interval_of 1) / Y)) * interval_of(width Y) = Inter-
val(1 /upper Y * 1 /upper Y, 1 /lower Y * 1 /lower Y) * Interval(upper Y − lower Y, upper Y − lower Y)›
```

```
    using b1 b2 b3 b4 by presburger
   moreover have ‹... = Interval((upper Y − lower Y) ∗ 1 /upper Y ∗ 1 /upper Y, (upper Y − lower Y) ∗ 1 /lower Y ∗ 1
/lower Y)›
    using assms RightOf bo interval_times_scalar_pos_r[of upper Y − lower Y Interval(1 /upper Y ∗ 1 /upper Y, 1 /lower
Y ∗ 1 /lower Y)]
    unfolding interval_of_def
   by (smt (verit, del_insts) divide_cancel_left divide_nonneg_nonneg frac_le interval_times_scalar_pos_r lower_bounds
nonzero_divide_mult_cancel_right times_divide_eq_right upper_bounds width_def width_expanded)
  ultimately show ?thesis by simp
 qed
 have a3: ‹1 /upper Y ∗ 1 /upper Y ∗ (upper Y − lower Y) ≤ 1 / lower Y − 1 / upper Y›
  proof −
   have bo: ‹0 < upper Y› using assms ao RightOf by argo
   then have (1 / lower Y) − (1 / upper Y) = (upper Y − lower Y) / (lower Y ∗ upper Y)
    using assms RightOf bo by (simp add: diff_divide_distrib)
   then have b1: ‹((1/ upper Y) ∗ (1/ upper Y) ∗ (upper Y − lower Y) ∗ (lower Y ∗ upper Y) ≤ (upper Y − lower Y)) =
((1/ upper Y) ∗ (1/ upper Y) ∗ (upper Y − lower Y) ≤ 1 / lower Y − 1 / upper Y)›
    using assms RightOf bo by (simp add: pos_le_divide_eq)
   then have b2: ‹((upper Y − lower Y) ∗ lower Y / upper Y) ≤ upper Y − lower Y›
    using assms ao RightOf bo by (smt (verit) mult_left_less_imp_less pos_less_divide_eq)
   moreover have ‹(upper Y − lower Y) ∗ lower Y ≤ (upper Y − lower Y) ∗ upper Y›
    using assms RightOf bo calculation pos_divide_le_eq by blast
   show ?thesis using assms bo b1 b2 by force
  qed
  have a4: ‹1 / lower Y − 1 / upper Y ≤ 1 /lower Y ∗ 1 /lower Y ∗ (upper Y − lower Y)›
  proof −
   have bo: ‹0 < upper Y› using assms ao RightOf by argo
   then have (1 / lower Y) − (1 / upper Y) = (upper Y − lower Y) / (lower Y ∗ upper Y)
    using assms RightOf bo by (simp add: diff_divide_distrib)
   then have b1: ‹(upper Y − lower Y ≤ (1/ lower Y) ∗ (1/ lower Y) ∗ (upper Y − lower Y) ∗ (lower Y ∗ upper Y)) = (1 /
lower Y − 1 / upper Y ≤ ((1/ lower Y) ∗ (1/ lower Y) ∗ (upper Y − lower Y) ))›
    using assms RightOf bo by (simp add: pos_divide_le_eq)
   then have b2: ‹upper Y − lower Y ≤ ((upper Y − lower Y) ∗ upper Y / lower Y)›
    using assms ao RightOf bo by (smt (verit) division_leq mult_pos_pos nonzero_mult_div_cancel_right)
   moreover have ‹(upper Y − lower Y) ∗ lower Y ≤ (upper Y − lower Y) ∗ upper Y›
    using calculation pos_divide_le_eq by (simp add: RightOf pos_le_divide_eq)
   show ?thesis using assms bo b1 b2 RightOf by simp
  qed
  have ‹Interval( 1 / lower Y − 1 / upper Y, 1 / lower Y − 1 / upper Y) ≤ Interval((upper Y − lower Y) ∗ 1 /upper Y ∗
1 /upper Y, (upper Y − lower Y) ∗ 1 /lower Y ∗ 1 /lower Y)›
    using a3 a4 unfolding less_eq_interval_def by simp
 then show ?case using a1 a2 a3 a4 by simp
qed

end
```

5 Affine Functions (Affine_Functions)

In this theory, we provide formalisation of affine functions (sometimes also called linear polynomial functions).

theory
Affine_Functions
imports
Complex_Main
begin

5.1 Definition of Affine Functions, Alternative Definitions, and Special Cases

locale *affine_fun* =
fixes *f*
assumes ‹∃ *b*. *linear* (λ *x*. *f x* − *b*)›

lemma *affine_fun_alt*:
‹*affine_fun f* = (∃ *c g*. (*f* = (λ *x*. *g x* + *c*)) ∧ *linear g*)›
unfolding *affine_fun_def*
by(*safe*, *force*, *simp add*: *Real_Vector_Spaces.linear_iff*)

lemma *affine_fun_real_linfun*:
‹*affine_fun* (*f*::(*real* ⇒ *real*)) = (∃ *a b* . *f* = (λ *x*. *a* ∗ *x* + *b*))›
by(*simp add*: *affine_fun_alt*, *metis real_linearD linear_times*)

lemma *linear_is_affine_fun*: ‹*linear f* ⟹ *affine_fun f*›
by (*standard*, *simp add*: *Real_Vector_Spaces.linear_iff*)

lemma *affine_zero_is_linear*: **assumes** ‹*affine_fun f*› **and** ‹*f o* = *o*› **shows** ‹*linear f*›
apply(*insert assms*)
unfolding *affine_fun_def*
using *linear_o* **by** *fastforce*

lemma *affine_add*:
assumes ‹*affine_fun f*› **and** ‹*affine_fun g*›
shows ‹*affine_fun* (λ *x*. *f x* + *g x*)›
using *assms linear_compose_add*
by(*auto simp add*: *affine_fun_alt*, *force*)

lemma *scaleR*:
assumes ‹*affine_fun f* › **shows** ‹*affine_fun* (λ *x*. *a* ∗$_R$ (*f x*))›
using *assms*
apply(*simp add*: *affine_fun_alt*)
by (*metis linear_compose_scale_right scaleR_right_distrib*)

lemma *real_affine_funD*:
fixes *f* :: *real* ⇒ *real*
assumes *affine_fun f* **obtains** *c b* **where** *f* = (λ *x*. *c* ∗ *x* + *b*)
using *assms*

apply(*simp add*: *affine_fun_alt*)
by (*metis real_linearD*)

5.2 Common Linear Polynomial Functions

lemma *affine_fun_const*[*simp*]: ‹*affine_fun* (λ *x*. *c*)›
apply(*simp add*: *affine_fun_alt*)
using *module_hom_zero* **by** *force*

lemma *affine_fun_id*[*simp*]: ‹*affine_fun* (λ *x*. *x*)›
by (*simp add*: *linear_is_affine_fun module_hom_ident*)

lemma *affine_fun_mult*[*simp*]: ‹*affine_fun* (λ *x*. (*c*::'*a*::*real_algebra*) ∗ *x*)›
by(*simp add*: *linear_is_affine_fun*)

lemma *affine_fun_scaled*[*simp*]: ‹*affine_fun* (λ *x*. *x* /*c*)›
for *c* :: '*a*::*real_normed_field*
using *bounded_linear_divide*[*of c*] *linear_is_affine_fun*[*of* (λ *x*. *x* /*c*)] *bounded_linear.linear*
by *blast*

lemma *affine_fun_add*[*simp*]: ‹*affine_fun* (λ *x*. *x* + *c*)›
apply(*simp add*: *affine_fun_alt*)
using *module_hom_ident* **by** *force*

lemma *affine_fun_diff*[*simp*]: ‹*affine_fun* (λ *x*. *x* − *c*)›
apply(*simp add*: *affine_fun_alt*)
using *module_hom_ident* **by** *force*

lemma *affine_fun_triv*[*simp*]: ‹*affine_fun* (λ *x*. *a* ∗$_R$ *x* + *c*)›
apply(*simp add*: *affine_fun_alt*)
by *fastforce*

lemma *affine_fun_add_const*[*simp*]: **assumes** ‹*affine_fun f*› **shows** ‹*affine_fun* (λ *x*. (*f x*) + *c*)›
using *assms* **apply**(*simp add*:*affine_fun_alt*)
by (*metis add.commute add.left_commute*)

lemma *affine_fun_diff_const*[*simp*]: **assumes** ‹*affine_fun f*› **shows** ‹*affine_fun* (λ *x*. (*f x*) − *c*)›
using *assms* **apply**(*simp add*:*affine_fun_alt*) **by** *force*

lemma *affine_fun_comp*[*simp*]: **assumes** ‹*affine_fun* (*f*)›
and ‹*affine_fun* (*g*)› **shows** ‹*affine_fun* (*f* ∘ *g*)›
using *assms*
unfolding *affine_fun_alt*
apply(*simp add*:*o_def*)
using *linear_add linear_compose*[*simplified o_def*] *add.assoc*
by *metis*

lemma *affine_fun_linear*[*simp*]: **assumes** ‹*affine_fun f*› **shows** ‹*affine_fun* (λ *x*. *a* ∗$_R$ (*f x*) + *c*)›
by(*rule affine_fun_comp*[*of* λ *x*. *a* ∗$_R$ *x* + *c f*, *simplified o_def*], *simp_all add*: *assms*)

5.3 Linear Polynomial Functions and Orderings

lemma *affine_fun_leq_pos*:

assumes ‹*affine_fun* (*f*::*real* ⇒ *real*)› **and** ‹*affine_fun g*›
and ‹*x*∈{*o*..*u*}› **and** ‹(*f o* ≤ *g o*)› **and** ‹(*f u* ≤ *g u*)›
shows ‹*f x* ≤ *g x*›
using *assms*
apply(*auto simp add*: *linear_o affine_fun_real_linfun*)[1]
by (*smt* (*verit*) *left_diff_distrib mult_left_mono_neg mult_right_mono*)

lemma *affine_fun_leq_neg*:
assumes ‹*affine_fun* (*f*::*real* ⇒ *real*)› **and** ‹*affine_fun g*›
and ‹*x*∈{*l*..*o*}› **and** ‹(*f l* ≤ *g l*)› **and** ‹(*f o* ≤ *g o*)›
shows ‹*f x* ≤ *g x*›
using *assms*
apply(*auto simp add*: *linear_o affine_fun_real_linfun*)[1]
using *Groups.mult_ac*(2) *left_diff_distrib mult_right_mono*
by (*smt* (*verit*, *ccfv_SIG*))

lemma *affine_fun_leq*:
assumes ‹*affine_fun* (*f*::*real* ⇒ *real*)› **and** ‹*affine_fun g*›
and ‹*x*∈{*l*..*u*}› **and** ‹(*f l* ≤ *g l*)› **and** ‹(*f u* ≤ *g u*)›
shows ‹*f x* ≤ *g* (*x*::*real*)›
using *assms affine_fun_leq_neg*[*of f g x l*] *affine_fun_leq_pos*[*of f g x u*]
apply(*simp add*: *linear_o affine_fun_real_linfun*)
by (*smt* (*verit*, *ccfv_SIG*) *left_diff_distrib mult_left_mono_neg*)

lemma *affine_fun_le_pos*:
assumes ‹*affine_fun* (*f*::*real* ⇒ *real*)› **and** ‹*affine_fun g*›
and ‹*x*∈{*o*..*u*}› **and** ‹(*f o* < *g o*)› **and** ‹(*f u* < *g u*)›
shows ‹*f x* < *g* (*x*::*real*)›
using *assms*
apply(*auto simp add*: *linear_o affine_fun_real_linfun*)[1]
using *Groups.mult_ac*(2) *left_diff_distrib mult_right_mono*
by (*smt* (*verit*, *best*))

lemma *affine_fun_le_neg*:
assumes ‹*affine_fun* (*f*::*real* ⇒ *real*)› **and** ‹*affine_fun g*›
and ‹*x*∈{*l*..*o*}› **and** ‹(*f l* < *g l*)› **and** ‹(*f o* < *g o*)›
shows ‹*f x* < *g* (*x*::*real*)›
using *assms*
apply(*auto simp add*: *linear_o affine_fun_real_linfun*)[1]
using *Groups.mult_ac*(2) *left_diff_distrib mult_right_mono*
by (*smt* (*verit*, *ccfv_SIG*))

lemma *affine_fun_le*:
assumes ‹*affine_fun* (*f*::*real* ⇒ *real*)› **and** ‹*affine_fun g*›
and ‹*x*∈{*l*..*u*}› **and** ‹(*f l* < *g l*)› **and** ‹(*f u* < *g u*)›
shows ‹*f x* < *g* (*x*::*real*)›
using *assms affine_fun_le_neg*[*of f g x l*] *affine_fun_le_pos*[*of f g x u*]
apply(*simp add*: *linear_o affine_fun_real_linfun*)
by (*smt* (*verit*, *ccfv_SIG*) *left_diff_distrib mult_left_mono_neg*)

end

6 Interval Inclusion Isotonicity (Inclusion_Isotonicity)

theory
Inclusion_Isotonicity
imports
Interval_Utilities
Affine_Functions
Interval_Division_Non_Zero
begin

6.1 Interval Extension

6.1.1 Textbook Definition of Interval Extension

definition
is_interval_extension_of :: ‹$('a$::*preorder interval* $\Rightarrow$ $'b$::*preorder interval*$) \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool*›
($\mathbf{infix}$ ‹(*is'_interval'_extension'_of*)› 50)
where
‹$((F$ *is_interval_extension_of* $f)) = (\forall\ x.\ (F\ (interval_of\ x)) = interval_of\ (f\ x))$›

definition‹*extend_natural* $f = (\lambda\ X.$ *mk_interval* $(f\ (lower\ X), f(upper\ X)))$›

lemma *interval_extension_comp*:
assumes *interval_ext_F*: ‹*F is_interval_extension_of f*›
and *interval_ext_G*: ‹*G is_interval_extension_of g*›
shows ‹$(F$ *o* $G)$ *is_interval_extension_of* $(f$ *o* $g)$›
using *assms* **unfolding** *is_interval_extension_of_def*
by *simp*

lemma *interval_extension_const*: ‹$(\lambda\ x.$ *interval_of c*$)$ *is_interval_extension_of* $(\lambda\ x.\ c)$›
unfolding *is_interval_extension_of_def*
by $(simp\ add$: *interval_eq_iff*$)$

lemma *interval_extension_id*: ‹$(\lambda\ x.\ x)$ *is_interval_extension_of* $(\lambda\ x.\ x)$›
unfolding *is_interval_extension_of_def*
by $(simp\ add$: *interval_eq_iff*$)$

6.1.2 A Stronger Definition of Interval Extension

definition
is_natural_interval_extension_of :: ‹$('a$::*linorder interval* $\Rightarrow$ $'b$::*linorder interval*$) \Rightarrow ('a \Rightarrow 'b) \Rightarrow$ *bool*›
($\mathbf{infix}$ ‹(*is'_natural'_interval'_extension'_of*)› 50)
where
‹$((F$ *is_natural_interval_extension_of* $f)) = (\forall\ l\ u.\ (F\ (mk_interval\ (l,u))) = mk_interval\ (f\ l, f\ u))$›

lemma ‹$(extend_natural\ f)$ *is_interval_extension_of f*›
unfolding *is_interval_extension_of_def extend_natural_def*
by$(auto\ simp\ add$: *mk_interval'_interval_of_def*$)$

lemma ‹(*extend_natural f*) *is_natural_interval_extension_of f*›
unfolding *is_natural_interval_extension_of_def extend_natural_def*
by(*auto simp add*: *mk_interval' interval_of_def*)

lemma *natural_interval_extension_implies_interval_extension*:
assumes ‹*F is_natural_interval_extension_of f*› **shows** ‹*F is_interval_extension_of f*›
using *assms* **unfolding** *is_natural_interval_extension_of_def is_interval_extension_of_def*
mk_interval_def interval_of_def
by(*auto split*:*if_splits*)

lemma *const_add_is_natural_interval_extensions*:
‹(λ *x*. (*interval_of c*) + *x*) *is_natural_interval_extension_of* (λ *x*. *c* + *x*)›
unfolding *is_natural_interval_extension_of_def mk_interval_def*
by (*simp add*: *add_mono_thms_linordered_semiring*(1) *antisym interval_eq_iff add_mono*
split:*if_splits*)

lemma *const_times_is_natural_interval_extensions*:
‹(λ *x*. (*interval_of c*) ∗ *x*) *is_natural_interval_extension_of* (λ *x*. *c* ∗ *x*)›
unfolding *is_natural_interval_extension_of_def mk_interval_def*
unfolding *times_interval_def Let_def*
by(*simp add*: *interval_eq_iff Interval_inverse interval_of .rep_eq add_mono*
split:*if_splits*)

lemma *const_is_interval_extension*: ‹*F is_natural_interval_extension_of* (λ *x*. *b*) ⟹ *F* = (λ *x*.(*interval_of b*))›
unfolding *is_natural_interval_extension_of_def*
apply(*auto simp add*:*mk_interval' interval_of_def split*:*if_splits*)[1]
by (*metis bounds_of_interval_eq_lower_upper bounds_of_interval_inverse lower_le_upper*)

lemma *id_is_interval_extension*: ‹*F is_natural_interval_extension_of* (λ *x*. *x*) ⟹ *F* = (λ *x*. *x*)›
unfolding *is_natural_interval_extension_of_def*
apply(*auto simp add*:*mk_interval' interval_of_def split*:*if_splits*)[1]
by (*metis bounds_of_interval_eq_lower_upper bounds_of_interval_inverse lower_le_upper*)

lemma *extend_natural_is_interval_extension*:
‹(*extend_natural f*) *is_natural_interval_extension_of f*›
unfolding *extend_natural_def is_natural_interval_extension_of_def mk_interval'_def*
by (*smt* (*z3*) *case_prod_conv mk_interval_def lower_bounds nle_le upper_bounds*)

lemma *is_natural_interval_extension_implies_bounds*:
fixes *F* :: ‹*real interval* ⇒ *real interval*›
assumes ‹*F is_natural_interval_extension_of f*› **and** ‹*F x* ≤ *F x'*›
shows
‹((*f* (*lower x'*)) ≤ (*f* (*lower x*))) ∨ ((*f* (*upper x'*)) ≤ (*f* (*upper x*)))›
by (*smt* (*verit*) *assms interval_eqI is_natural_interval_extension_of_def less_eq_interval_def*
lower_le_upper mk_interval_lower mk_interval_upper)

lemma *interval_extension_lower*:
‹((*F is_interval_extension_of f*)) ⟹ *lower* (*F* (*interval_of x*)) = (*f x*)›
unfolding *is_interval_extension_of_def* **by** *simp*

lemma *interval_extension_upper*:
‹((*F is_interval_extension_of f*)) ⟹ *upper* (*F* (*interval_of x*)) = (*f x*)›
unfolding *is_interval_extension_of_def* **by** *simp*

lemma *is_interval_extension_eq_upper_and_lower*:
 ‹((*F is_interval_extension_of f*))
 = (∀ *x*. *lower* (*F* (*interval_of x*)) = (*f x*) ∧ *upper* (*F* (*interval_of x*)) = (*f x*))›
 unfolding *is_interval_extension_of_def*
 by (*simp add*: *interval_eq_iff*)

lemma *interval_extension_lower_simp*[*simp*]:
 assumes ‹*F is_interval_extension_of f*› **and** ‹*X* = *Interval*(*x*,*x*)›
 shows ‹*lower* (*F X*) = *f x* ›
 by (*metis assms interval_extension_lower interval_of*.*abs_eq*)

lemma *interval_extension_upper_simp*[*simp*]:
 assumes ‹*F is_interval_extension_of f*› **and** ‹*X* = *Interval*(*x*,*x*)›
 shows ‹*upper* (*F X*) = *f x* ›
 by (*metis assms interval_extension_upper interval_of*.*abs_eq*)

6.2 Interval Inclusion Isotonicity

In this section, we introduce the concept of inclusion isotonicity. The formalization in this theory generalises the definitions from [5]:

definition
 inclusion_isotonic :: ‹('*a*::*preorder interval* ⇒ '*b*::*preorder interval*) ⇒ *bool*›
 where
‹*inclusion_isotonic f* = (∀ *x x'*. *x* ≤ *x'* ⟶ (*f x*) ≤ (*f x'*))›

We can immediately show that any natural extension of an affine function of from type *real* to *real* is interval inclusion isotonic:

lemma *real_affine_fun_is_inclusion_isotonicM*:
 assumes ‹*affine_fun* (*f*::*real* => *real*)›
 shows ‹*inclusion_isotonic* (*extend_natural f*)›
 using *assms*
 unfolding *inclusion_isotonic_def extend_natural_def Interval.less_eq_interval_def affine_fun_real_linfun*
 by(*auto*, (*metis lower_le_upper mult_le_cancel_left nle_le*)+)

definition
 inclusion_isotonic_on :: ‹('*a interval* ⇒ *bool*) ⇒ ('*a*::*preorder interval* ⇒ '*b*::*preorder interval*) ⇒ *bool*›
 where
‹*inclusion_isotonic_on P f* = (∀ *x x'*. *P x* ∧ *P x'* ∧ *x* ≤ *x'* ⟶ (*f x*) ≤ (*f x'*))›

lemma *inclusion_isotonic_eq*: ‹*inclusion_isotonic_on* (λ *x* . *True*) = *inclusion_isotonic*›
 unfolding *inclusion_isotonic_on_def inclusion_isotonic_def*
 by *simp*

definition *inclusion_isotonic_binary* :: ‹('*a*::*preorder interval* ⇒ '*a interval* ⇒ '*b*::*preorder interval*) ⇒ *bool*›
 where
‹*inclusion_isotonic_binary f* = (∀ *x x' y y'*. *x* ≤ *x'* ∧ *y* ≤ *y'* ⟶ (*f x y*) ≤ (*f x' y'*))›

definition *inclusion_isotonic_binary_on* :: ‹('*a interval* ⇒ *bool*) ⇒ ('*a*::*preorder interval* ⇒ '*a interval* ⇒ '*b*::*preorder interval*) ⇒ *bool*›
 where
‹*inclusion_isotonic_binary_on P f* = (∀ *x x' y y'*. *P x* ∧ *P x'* ∧ *P y* ∧ *P y'* ∧ *x* ≤ *x'* ∧ *y* ≤ *y'* ⟶ (*f x y*) ≤ (*f x' y'*))›

lemma *inclusion_isotonic_binary_eq*: ‹*inclusion_isotonic_binary_on* $(\lambda x\ .\ True) = inclusion_isotonic_binary$›
unfolding *inclusion_isotonic_binary_on_def inclusion_isotonic_binary_def*
by *simp*

definition *inclusion_isotonicM_n* :: ‹*nat* ⇒ (*'a*::*linorder interval list* ⇒ *'a*::*linorder interval*) ⇒ *bool*› **where**
‹*inclusion_isotonicM_n n f* = (∀ *is js*. (*length is* = *n* ∧ *length js* = *n* ∧ (*is* $\leq_I$ *js*)) ⟶ *f is* ≤ *f js*)›

definition *inclusion_isotonicM_n_on* :: ‹(*'a interval* ⇒ *bool*) ⇒ *nat* ⇒ (*'a*::*linorder interval list* ⇒ *'a*::*linorder interval*) ⇒ *bool*› **where**
‹*inclusion_isotonicM_n_on P n f* = (∀ *is js*. (∀ *i* ∈ *set is*. *P i*) ∧ (∀ *j* ∈ *set js*. *P j*) ∧ (*length is* = *n* ∧ *length js* = *n* ∧ (*is* $\leq_I$ *js*)) ⟶ *f is* ≤ *f js*)›

lemma *inclusion_isotonicM_n_eq*: ‹*inclusion_isotonicM_n_on* $(\lambda x\ .\ True) = inclusion_isotonicM_n$›
unfolding *inclusion_isotonicM_n_on_def inclusion_isotonicM_n_def*
by *simp*

Finally, we extend the definition to functions over lists and show that the three definitions of interval inclusion isotonicity are, for their corresponding types, equivalent:

locale *inclusion_isotonicM* =
fixes *n* :: *nat*
and *f* :: ‹(*'a*::*linorder*) *interval list* ⇒ *'a interval list*›
assumes *inclusion_isotonicM* :
‹(∀ *is js*. (*length is* = *n*) ∧ (*length js* = *n*) ∧ (*is* $\leq_I$ *js*) ⟶ *f is* $\leq_I$ *f js*)›
begin
end

locale *inclusion_isotonicM_on* =
fixes *P* :: ‹*'a*::*linorder interval* ⇒ *bool*›
and *n* :: *nat*
and *f* :: ‹(*'a*::*linorder*) *interval list* ⇒ *'a interval list*›
assumes *inclusion_isotonicM* :
‹(∀ *is js*. (∀ *i* ∈ *set is*. *P i*) ∧ (∀ *j* ∈ *set js*. *P j*) ∧ (*length is* = *n*) ∧ (*length js* = *n*) ∧ (*is* $\leq_I$ *js*) ⟶ *f is* $\leq_I$ *f js*)›
begin
end

lemma *inclusion_isotonicM_on_eq*: ‹*inclusion_isotonicM_on* $(\lambda x.\ True) = inclusion_isotonicM$›
unfolding *inclusion_isotonicM_on_def inclusion_isotonicM_def*
by *simp*

lemma *inclusion_isotonic_conv*:
‹*inclusion_isotonic g* = *inclusion_isotonicM 1* $(\lambda xs\ .$ *case xs of* $[x] \Rightarrow [g\ x])$›
unfolding *inclusion_isotonic_def inclusion_isotonicM_def*
by(*auto simp add*: *le_interval_single split*:*list.split*)

lemma *inclusion_isotonicM_n_conv1*:
‹*inclusion_isotonicM_n n f* = *inclusion_isotonicM n* $(\lambda xs.\ [f\ xs])$›
unfolding *inclusion_isotonicM_n_def inclusion_isotonicM_def le_interval_list_def*
by(*auto*)

lemma *inclusion_isotonicM_conv2*:
assumes ‹*inclusion_isotonicM n f*›
and ‹∀ *xs*. (*length xs* = *n*) ⟶ *N* = (*length* (*f xs*))›
shows ‹*inclusion_isotonicM n* $(\lambda xs.$ *if* $n' < N$ *then* $[(f\ xs)!n']$ *else* $[])$›

using *assms* **unfolding** *inclusion_isotonicM_def*
apply(*simp split*:*if_splits*, *safe*)
apply(*subst le_interval_single*[*symmetric*])
apply(*subst le_interval_list_all*)
subgoal by *blast*
subgoal by *blast*
subgoal by(*rule TrueI*)
done

lemma *inclusion_isotonicM_n_on_conv1*:
‹*inclusion_isotonicM_n_on P n f* = *inclusion_isotonicM_on P n* (λ *xs*. [*f xs*])›
unfolding *inclusion_isotonicM_n_on_def inclusion_isotonicM_on_def le_interval_list_def*
by(*auto*)

lemma *inclusion_isotonicM_on_conv2*:
assumes ‹*inclusion_isotonicM_on P n f*›
and ‹∀ *xs*. (*length xs* = *n*) ⟶ *N* = (*length* (*f xs*))›
shows ‹*inclusion_isotonicM_on P n* (λ *xs*. *if n'* < *N then* [(*f xs*)!*n'*] *else* [])›
using *assms* **unfolding** *inclusion_isotonicM_on_def*
apply(*simp split*:*if_splits*, *safe*)
apply(*subst le_interval_single*[*symmetric*])
apply(*subst le_interval_list_all*)
subgoal by *blast*
subgoal by *blast*
subgoal by(*rule TrueI*)
done

lemma *inclusion_isotonic_binary_conv1*:
‹*inclusion_isotonic_binary f* = *inclusion_isotonicM_n 2* (λ *xs* . *case xs of* [*x*,*y*] ⇒ *f x y*)›
unfolding *inclusion_isotonic_binary_def inclusion_isotonicM_n_def le_interval_list_def*
by(*auto simp add*: *le_interval_list_def split*:*list.split*)

lemma *inclusion_isotonic_binary_conv2*:
‹*inclusion_isotonic_binary f* = *inclusion_isotonicM 2* (λ *xs* . *case xs of* [*x*,*y*] ⇒ [*f x y*])›
apply(*simp add*: *inclusion_isotonic_binary_conv1*)
apply(*simp add*: *inclusion_isotonicM_n_conv1*)
unfolding *inclusion_isotonicM_def*
by(*auto split*:*list.split*)

lemma *inclusion_isotonic_binary_on_conv1*:
‹*inclusion_isotonic_binary_on P f* = *inclusion_isotonicM_n_on P 2* (λ *xs* . *case xs of* [*x*,*y*] ⇒ *f x y*)›
unfolding *inclusion_isotonic_binary_on_def inclusion_isotonicM_n_on_def le_interval_list_def*
by(*auto simp add*: *le_interval_list_def split*:*list.split*)

lemma *inclusion_isotonic_binary_on_conv2*:
‹*inclusion_isotonic_binary_on P f* = *inclusion_isotonicM_on P 2* (λ *xs* . *case xs of* [*x*,*y*] ⇒ [*f x y*])›
apply(*simp add*: *inclusion_isotonic_binary_on_conv1*)
apply(*simp add*: *inclusion_isotonicM_n_on_conv1*)
unfolding *inclusion_isotonicM_on_def*
by(*auto split*:*list.split*)

6.2.1 Compositionality of Interval Inclusion Isotonicy

lemma *inclusion_isotonicM_comp*:

assumes ‹*inclusion_isotonicM n f*›
and ‹*inclusion_isotonicM m g*›
and ‹∀ *xs*. *length xs* = *n* ⟶ *length* (*f xs*) = *m*›
shows ‹*inclusion_isotonicM n* (*g o f*)›
using *assms* **unfolding** *inclusion_isotonicM_def*
by(*simp*)

lemma *inclusion_isotonicM_on_comp*:
assumes ‹*inclusion_isotonicM_on P n f*›
and ‹*inclusion_isotonicM_on Q m g*›
and ‹∀ *xs*. *length xs* = *n* ⟶ *length* (*f xs*) = *m*›
and ‹∀ *is*. (∀ *i* ∈ *set is*. *P i*) ⟶ (∀ *x*∈*set* (*f is*). *Q x*)›
shows ‹*inclusion_isotonicM_on P n* (*g o f*)›
using *assms* **unfolding** *inclusion_isotonicM_on_def o_def*
by(*auto*)

lemma *inclusion_isotonic_comp*:
assumes ‹*inclusion_isotonic f*›
and ‹*inclusion_isotonic g*›
shows ‹*inclusion_isotonic* (*g o f*)›
using *assms* **unfolding** *inclusion_isotonic_def*
by(*simp*)

lemma *inclusion_isotonic_on_comp*:
assumes ‹*inclusion_isotonic_on P f*›
and ‹*inclusion_isotonic_on Q g*›
and ‹∀ *x*. *P x* ⟶ *Q* (*f x*)›
shows ‹*inclusion_isotonic_on P* (*g o f*)›
using *assms* **unfolding** *inclusion_isotonic_on_def*
by(*simp*)

6.2.2 Interval Inclusion Isontonicity of the Core Operator

Unary Minus (Negation)

lemma *inclusion_isotonicM_uminus*[*simp*]: ‹*inclusion_isotonic uminus*›
by (*simp add*: *inclusion_isotonic_def less_eq_interval_def*)

Addition

lemma *inclusion_isotonicM_plus*[*simp*]: ‹*inclusion_isotonic_binary* (+)›
by (*simp add*: *inclusion_isotonic_binary_def less_eq_interval_def add_mono*)

Substraction

lemma *inclusion_isotonicM_minus*[*simp*]: ‹*inclusion_isotonic_binary* (−)›
by (*simp add*: *inclusion_isotonic_binary_def less_eq_interval_def diff_mono*)

Multiplication

lemma *inclusion_isotonicM_times*[*simp*]:
‹*inclusion_isotonic_binary* (λ *x y*. (*x*::'*a*::{*linordered_ring*, *real_normed_algebra*, *linear_continuum_topology*} *interval*) ∗ *y*)›
unfolding *inclusion_isotonic_binary_def* **using** *set_of_mul_inc interval_set_leq_eq*
by *metis*

6.2.3 Interval Inclusion Isotonicity of Various Functions

lemma *inclusion_isotonicM_n_empty*[*simp*]: ‹*inclusion_isotonicM n* (λ *xs*. [])›
unfolding *inclusion_isotonicM_def* **by**(*simp*)

lemma *inclusion_isotonic_id*[*simp*]: ‹*inclusion_isotonic id*›
unfolding *inclusion_isotonic_def le_interval_list_def*
by(*simp*)

lemma *inclusion_isotonicM_id*[*simp*]: ‹*inclusion_isotonicM n id*›
unfolding *inclusion_isotonicM_def le_interval_list_def*
by(*simp*)

lemma *inclusion_isotonicM_hd*[*simp*]:
assumes ‹*0* < *n*›
shows ‹*inclusion_isotonicM_n n hd*›
unfolding *inclusion_isotonicM_n_def le_interval_list_def*
proof(*insert assms*, *induction n rule*:*nat_induct_non_zero*)
case *1*
then show *?case*
by (*auto*, *metis* (*no_types*, *lifting*) *Nil_eq_zip_iff case_prodD foldl_conj_set_True hd_zip insert_iff*
length_0_conv list.map_disc_iff list.map_sel(*1*) *list.set*(*2*) *list.set_sel*(*1*) *nat.distinct*(*1*))
next
case (*Suc n*)
then show *?case*
by (*auto*, *metis* (*no_types*, *lifting*) *Nil_eq_zip_iff Zero_not_Suc case_prodD foldl_conj_set_True hd_zip*
insert_iff length_0_conv list.map_disc_iff list.map_sel(*1*) *list.set*(*2*) *list.set_sel*(*1*))
qed

lemma *inclusion_isotonic_add_const1*[*simp*]:
‹*inclusion_isotonic* (λ *x*. *x* + *c*)›
unfolding *inclusion_isotonic_def*
by (*simp add*: *add_mono_thms_linordered_semiring*(*3*) *less_eq_interval_def*)

lemma *inclusion_isotonicM_1_add_const2*[*simp*]:
‹*inclusion_isotonic* (λ *x*. *c* + *x*)›
unfolding *inclusion_isotonic_def*
by (*simp add*: *add_mono_thms_linordered_semiring*(*2*) *less_eq_interval_def*)

lemma *inclusion_isotonic_times_right*[*simp*]:
‹*inclusion_isotonic* (λ *x*. *C*∗ (*x*::'*a*::*linordered_ring interval*))›
unfolding *inclusion_isotonic_def*
using *times_interval_right* **by** *auto*

lemma *inclusion_isotonic_times_left*[*simp*]:
‹*inclusion_isotonic* (λ *x*. (*x*::'*a*::{*real_normed_algebra*,*linordered_ring*,*linear_continuum_topology*} *interval*) ∗ *C*)›
unfolding *inclusion_isotonic_def*
using *times_interval_left* **by** *auto*

lemma *map_inclusion_isotonicM*:
assumes ‹*inclusion_isotonic f*›
shows ‹*inclusion_isotonicM n* (*map f*)›
using *assms map_set map_pair_set_left map_pair_set_right map_pair_set*
unfolding *inclusion_isotonicM_def le_interval_list_def inclusion_isotonic_def*

by(*simp add*: *foldl_conj_set_True map_pair_f_all*, *blast*)

lemma *inclusion_isotonicM_fun_plus*:
assumes ‹*inclusion_isotonic f*› **and** ‹*inclusion_isotonic g*›
shows ‹*inclusion_isotonic* (λ *x*. *f x* + *g x*)›
using *assms* **unfolding** *inclusion_isotonic_def*
by (*simp add*: *add_mono_thms_linordered_semiring*(*1*) *less_eq_interval_def*)

lemma *inclusion_isotonic_plus_const*:
assumes ‹*inclusion_isotonic f*› **and** ‹*inclusion_isotonic g*›
shows ‹*inclusion_isotonic* (λ *x*. *g x* + *c*)›
using *assms* **unfolding** *inclusion_isotonic_def*
by (*simp add*: *add_mono_thms_linordered_semiring*(*1*) *less_eq_interval_def*)

lemma *inclusion_isotonic_times_const_real*:
assumes ‹*inclusion_isotonic f*›
shows ‹*inclusion_isotonic* (λ *x*. (*c*::*real interval*) ∗ (*f x*))›
using *inclusion_isotonic_comp assms*
unfolding *inclusion_isotonic_def*
by (*simp add*: *times_interval_right*)

lemma *intervall_le_foldr*:
assumes ‹*inclusion_isotonic_binary f*›
shows ‹*length js* = *length is* ⟹ *is* ≤$_I$ *js* ⟹ (*foldr f is e*) ≤ (*foldr f js e*)›
by (*induction rule*: *list_induct2*)
(*use assms* **in** ‹*simp_all add*: *less_eq_interval_def le_interval_list_def inclusion_isotonic_binary_def foldl_conj_True*›)

lemma *intervall_le_foldr_map*:
assumes ‹*inclusion_isotonic_binary f*›
and ‹*inclusion_isotonic g*›
shows ‹*length js* = *length is* ⟹ *is* ≤$_I$ *js* ⟹ (*foldr f* (*map g is*) *e*) ≤ (*foldr f* (*map g js*) *e*)›
proof (*induction rule*:*list_induct2*)
case *Nil*
then show *?case*
by (*simp add*: *less_eq_interval_def*)
next
case (*Cons x xs y ys*)
then show *?case*
using *assms* **unfolding** *inclusion_isotonicM_n_def le_interval_list_def*
by(*simp add*: *foldl_conj_True inclusion_isotonic_def inclusion_isotonic_binary_def*)
qed

lemma *intervall_le_foldr_e*:
assumes ‹*inclusion_isotonic_binary f*›
and ‹*is* ≤ *js*›
shows ‹(*foldr f xs is*) ≤ (*foldr f xs js*)›
proof (*induction xs*)
case *Nil*
then show *?case* **using** *assms* **by**(*simp*)
next
case (*Cons x*)
then show *?case*
using *assms* **unfolding** *le_interval_list_def inclusion_isotonic_binary_def*
by (*simp add*: *less_eq_interval_def*)

qed

lemma *foldr_inclusion_isotonicM_e*:
assumes ‹*inclusion_isotonic_binary f*›
shows ‹∀ *x*. *inclusion_isotonic* (*foldr f x*)›
using *assms*
unfolding *inclusion_isotonic_def*
by(*simp add*: *intervall_le_foldr_e*)

lemma *foldr_inclusion_isotonicM*:
assumes ‹*inclusion_isotonic_binary f*›
shows ‹*inclusion_isotonicM_n n* (λ *x*. *foldr f x e*)›
using *assms*
unfolding *inclusion_isotonicM_n_def* **using** *intervall_le_foldr*
by *auto*

lemma *foldr_inclusion_isotonicM_g*:
assumes ‹*inclusion_isotonic_binary f*›
and ‹*inclusion_isotonicM n g*›
shows ‹*inclusion_isotonicM_n n* (λ *x*. *foldr f* ((*g x*)) *e*)›
using *assms*(*2*)
unfolding *inclusion_isotonicM_n_def inclusion_isotonicM_def*
by (*metis assms*(*1*) *intervall_le_foldr le_interval_list_imp_length*)

lemma *foldr_map_inclusion_isotonicM_g*:
assumes ‹*inclusion_isotonic_binary f*›
and ‹*inclusion_isotonic g* ›
and ‹*inclusion_isotonicM n P*›
shows ‹*inclusion_isotonicM_n n* (λ *x*. *foldr f* (*map g* (*P x*)) *e*)›
using *intervall_le_foldr_map assms*(*3*)
unfolding *inclusion_isotonicM_n_def inclusion_isotonicM_def*
by (*metis* (*no_types*, *lifting*) *assms*(*1*) *assms*(*2*) *intervall_le_foldr_map le_interval_list_imp_length*)

lemma *foldl_inclusion_isotonicM*:
assumes ‹*inclusion_isotonic_binary f*›
shows ‹*inclusion_isotonicM_n n* (*foldl f e*)›
unfolding *inclusion_isotonicM_n_def*
apply(*subst foldl_conv_foldr*)
apply(*subst foldl_conv_foldr*)
using *assms foldr_inclusion_isotonicM*[*simplified inclusion_isotonicM_def*]
le_interval_list_rev length_rev
unfolding *inclusion_isotonicM_n_def inclusion_isotonic_binary_def*
by (*metis*)

lemma *fold_inclusion_isotonicM*:
assumes ‹*inclusion_isotonic_binary f*›
shows ‹*inclusion_isotonicM_n n* (λ *x*. *fold f x e*)›
unfolding *inclusion_isotonicM_n_def*
apply(*subst foldl_conv_fold*[*symmetric*])
apply(*subst foldl_conv_fold*[*symmetric*])
using *assms foldl_inclusion_isotonicM*[*simplified inclusion_isotonicM_def*]
unfolding *inclusion_isotonic_binary_def inclusion_isotonicM_n_def*
by (*metis*)

lemma *map2_inclusion_isotonicM_left*: **assumes** ‹*inclusion_isotonic_binary f*›
shows ‹*inclusion_isotonicM n* (*map2 f xs*)›
unfolding *inclusion_isotonicM_def inclusion_isotonic_binary_def*
apply(*safe*,*subst le_interval_list_all2*, *simp_all*)
using *le_interval_list_imp le_interval_list_all*
by (*metis assms dual_order.refl inclusion_isotonic_binary_def less_eq_interval_def*)

lemma *map2_inclusion_isotonicM_right*: **assumes** ‹*inclusion_isotonic_binary f*›
shows ‹*inclusion_isotonicM n* (λ *ys*. *map2 f ys xs*)›
unfolding *inclusion_isotonicM_def inclusion_isotonic_binary_def*
apply(*safe*,*subst le_interval_list_all2*, *simp_all*)
using *le_interval_list_imp le_interval_list_all*
by (*metis assms dual_order.refl inclusion_isotonic_binary_def less_eq_interval_def*)

lemma *map2_inclusion_isotonicM_right_g*:
assumes ‹*inclusion_isotonic_binary f*›
and ‹∀ *xs*. *length* (*g xs*) = *length xs*›
and ‹*inclusion_isotonicM n g*›
and ‹*length xs* = *n*›
and ‹*length is* = *n*›
and ‹*length js* = *n*›
and ‹*is* ≤$_I$ *js*›
shows‹*map2 f* (*g is*) (*h xs*) ≤$_I$ *map2 f* (*g js*) (*h xs*)›
using *assms* **unfolding** *inclusion_isotonic_binary_def*
apply(*subst le_interval_list_all2*, *simp_all*)
using *assms* **unfolding** *inclusion_isotonic_binary_def*
by (*metis dual_order.refl inclusion_isotonicM_def*
le_interval_list_imp less_eq_interval_def)

lemma *inclusion_isotonicM_map*:
assumes ‹∀ *x* ∈ *set xs* . *g x* ≤ *h x*›
shows ‹(*map g xs*) ≤$_I$ (*map h xs*)›
using *assms* **by**(*subst le_interval_list_all2*, *simp_all*)

6.3 Interval Extension and Inclusion Properties

lemma *interval_extension_inclusion*:
assumes ‹*F is_interval_extension_of f*›
shows ‹∀ *X* . *interval_of* (*f X*) ≤ *F* (*interval_of X*)›
using *assms*
unfolding *is_interval_extension_of_def*
by (*simp add*: *in_interval_to_interval interval_of_in_eq*)

lemma *interval_extension_subset_const*:
assumes *interval_ext_F*: ‹*F is_interval_extension_of f*›
shows ‹∀ *X* . *set_of* (*interval_of* (*f X*)) ⊆ *set_of* (*F* (*interval_of X*))›
using *assms*
unfolding *is_interval_extension_of_def set_of_def* **by** *auto*

lemma *fundamental_theorem_of_interval_analysis*:
fixes *F* :: ‹*real interval* ⇒ *real interval*›

```
  assumes interval_ext_F: ‹F is_interval_extension_of f›
   and   inc_isontonic_F: ‹inclusion_isotonic F›
shows ‹∀ X . f ‘ (set_of X) ⊆ set_of (F X)›
proof
  fix X
  show ‹ f ‘ (set_of X) ⊆ set_of (F X)›
  proof
    fix y
    assume ‹y ∈ f ‘ (set_of X)›
    then obtain x where i: ‹x ∈ set_of X› and ‹y = f x› by auto
    then have ‹interval_of (f x) = F (interval_of x)› using assms unfolding is_interval_extension_of_def by simp
    then have ‹y ∈ set_of (F (interval_of x))› using ‹y = f x› by (metis in_interval_to_interval)
    then have ‹interval_of x ≤ X› using ‹x ∈ set_of X› using interval_of_in_eq by blast
    then have ‹F (interval_of x) ≤ F X› using inc_isontonic_F unfolding inclusion_isotonic_def by simp
    then show ‹y ∈ set_of (F X)› using ‹y ∈ set_of(F (interval_of x))› using set_of_subset_iff ' by auto
  qed
qed

lemma interval_extension_bounds:
  assumes inclusion_isotonic F
  and F is_interval_extension_of f
  shows (∀ x ∈ set_of X. lower (F X) ≤ f x) ∨ (∀ x ∈ set_of X. f x ≤ lower (F X))
  using assms
  by (metis (no_types, lifting) in_bounds inclusion_isotonic_def interval_of_in_eq
                    is_interval_extension_of_def)

lemma inclusion_isotonic_extension_subset:
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows ‹(f ‘ set_of X) ⊆ set_of (F X)›
  using assms interval_of_in_eq
  unfolding inclusion_isotonic_def set_of_eq is_interval_extension_of_def
  by (metis (mono_tags, lifting) image_subsetI)

lemma inclusion_isotonic_extension_includes:
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows ‹∀ x ∈ set_of X. f x ∈ set_of (F X)›
  using assms inclusion_isotonic_extension_subset
  by blast

lemma inclusion_isotonic_extension_lower_bound:
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows ‹∀ x ∈ set_of X. lower (F X) ≤ f x›
  using assms inclusion_isotonic_extension_includes
  using in_bounds by blast

lemma inclusion_isotonic_extension_upper_bound:
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows ‹∀ x ∈ set_of X. f x ≤ upper (F X)›
  using assms inclusion_isotonic_extension_includes
  using in_bounds by blast
```

```
lemma inclusion_isotonic_inf:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows ‹lower (F (X::real interval)) ≤ Inf (f ‘ set_of X)›
  using inclusion_isotonic_extension_subset[of F f X, simplified assms]
  cInf_superset_mono[of f ‘ set_of X  set_of (F X)]
  by (simp add: bdd_below_set_of  inf_set_of)

lemma inclusion_isotonic_sup:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows ‹Sup (f ‘ set_of X) ≤ upper (F X)›
  using inclusion_isotonic_extension_subset[of F f X, simplified assms]
      cSup_subset_mono[of f ‘ set_of X set_of (F X)]
  by (simp add: bdd_above_set_of sup_set_of)

lemma lower_inf:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows Inf (f ‘ set_of X) ≤ f (lower X)
  using cInf_superset_mono[of {f (lower X)} f ‘ set_of X]
  by (metis assms(1) assms(2) bdd_below_mono bdd_below_set_of bot.extremum
            cInf_singleton imageI insert_not_empty insert_subsetI
            fundamental_theorem_of_interval_analysis lower_in_interval)
lemma upper_sup:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows f (upper X) ≤ Sup (f ‘ set_of X)
  using cSup_subset_mono[of {f (upper X)} f ‘ set_of X]
  by (meson assms(1) assms(2) bdd_above_mono bdd_above_set_of cSup_upper
        imageI fundamental_theorem_of_interval_analysis upper_in_interval)

lemma lower_F_f:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows lower (F X) ≤ f (lower X)
  by (simp add: assms(1) assms(2) inclusion_isotonic_extension_lower_bound)

lemma upper_F_f:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F
  and F is_interval_extension_of f
  shows f (upper X) ≤ upper (F X)
  by (simp add: assms(1) assms(2) inclusion_isotonic_extension_upper_bound)

lemma inclusion_isotonic_interval_extension_le:
   assumes inclusion_isotonic: ‹inclusion_isotonic F›
```

and *interval_extension*: ‹*F is_interval_extension_of f*›
and *adjacent*: ‹*upper a* = *lower b*›
shows ‹*lower* (*F b*) ≤ *upper* (*F a*)›
using *inclusion_isotonic_extension_lower_bound*[*of F f*, *simplified assms*, *simplified*]
inclusion_isotonic_extension_upper_bound[*of F f*, *simplified assms*, *simplified*]
le_left_mono lower_in_interval upper_in_interval
by(*metis adjacent*)

6.4 Division

context *interval_division_inverse*
begin
lemma *inclusion_isotonic_on_inverse*[*simp*]:
‹*inclusion_isotonic_on* (λ *x* . ¬ *o* ∈$_i$ *x*) ((*inverse*::('*a interval* ⇒ '*a interval*)))›
using *inverse_non_zero_def*
unfolding *inclusion_isotonic_on_def less_eq_interval_def*
by (*smt* (*z3*) *dual_order.trans in_bounds in_intervalI lower_le_upper mk_interval_lower*
mk_interval_upper upper_leq_lower_div)

lemma *inclusion_isotonic_on_division*[*simp*]:
‹*inclusion_isotonic_binary_on* (λ *x* . ¬ *o* ∈$_i$ *x*) (λ *x y*. *divide x y*)›
using *inclusion_isotonicM_times divide_non_zero_def inclusion_isotonic_on_inverse*
unfolding *o_def inclusion_isotonic_binary_on_def*
inclusion_isotonic_on_def inclusion_isotonic_binary_def
by *metis*

end

end

7 Lipschitz Continuity of Intervals (Lipschitz_Interval_Extension)

An extension of of Lipschitz Continuity, developed for the Lipschitz Continuity of intervals.

theory
Lipschitz_Interval_Extension
imports
Inclusion_Isotonicity
HOL−Analysis.Lipschitz
Interval_Utilities
begin

7.1 Definition of Lipschitz Continuity on Intervals

An interval extension F is said to be lipschitz in X if there is a constant L such that $w\ (F\ X) \leq L w\ X$ for every $X \subseteq X$. Hence the width of $F\ X$ approaches 0 at least linearly with the width of X.

definition *lipschitzI_on* :: *'a*::{*zero*,*minus*,*preorder*,*times*} ⇒ *'a interval set* ⇒ (*'a interval* ⇒ *'a interval*) ⇒ *bool*
where *lipschitzI_on C U F* ⟷ (*0* ≤ *C* ∧ (∀ *x* ∈ *U*. *width* (*F x*) ≤ *C* ∗ *width x*))

The definition *lipschitzI_on* refers to definition 6.1 in[4]

open_bundle *lipschitzI_syntax*
begin
notation *lipschitzI_on* (‹_ −*lipschitzI'_on*› [*1000*])
end

lemma *lipschitzI_onI*: *C*−*lipschitzI_on U F*
if ⋀*x* . *x* ∈ *U* ⟹ *width* (*F x*) ≤ *C* ∗ *width x* *0* ≤ *C*
using *that* **by** (*auto simp*: *lipschitzI_on_def*)

lemma *lipschitzI_onD*:
width (*F x*) ≤ *C* ∗ *width x*
if *C*−*lipschitzI_on U F x* ∈ *U*
using *that* **by** (*auto simp*: *lipschitzI_on_def*)

lemma *lipschitzI_on_nonneg*:
0 ≤ *C* **if** *C*−*lipschitzI_on U F*
using *that* **by** (*auto simp*: *lipschitzI_on_def*)

lemma *lipschitz_on_normD*:
norm (*width* (*F x*)) ≤ *C* ∗ *norm* (*width x*)
if *C*−*lipschitzI_on U F x* ∈ *U*
using *lipschitzI_onD*[*OF that*]
by (*simp add*: *width_def dist_norm*)

lemma *lipschitzI_on_mono*: *L*−*lipschitzI_on D* (*F*:: *'a*::{*linordered_ring*} *interval* ⇒ *'a interval*)
if *M*−*lipschitzI_on E F M* ≤ *L D* ⊆ *E*

```
 using that  diff_ge_0_iff_ge lower_le_upper
  order_trans[OF _ mult_right_mono]
 unfolding lipschitzI_on_def width_def
 by (metis (no_types, lifting) order_trans subsetD width_def)

lemmas lipschitzI_on_subset
 = lipschitzI_on_mono[OF _ _ order_refl]
 and lipschitzI_on_le = lipschitzI_on_mono[OF _ order_refl]

lemma lipschitzI_on_leI:
 C−lipschitzI_on U F
 if ⋀x. x ∈ U ⟹ width (F x) ≤ C * width x
    0 ≤ C
 for F::'a::{minus,preorder,times,zero} interval ⇒ 'a interval
proof (rule lipschitzI_onI)
 fix x assume x: x ∈ U
 consider upper x ≤ lower x | lower x ≤ upper x
  by simp
 then show width (F x) ≤ C * width x
 proof cases
  case 1
  then have width (F x) ≤ C * width x
   by (auto intro!: that x)
  then show ?thesis
   by (simp add: dist_commute)
 qed (auto intro!: that x)
qed fact
```

7.1.1 Lipschitz Continuity of Operations

Let F and G be inclusion isotonic interval extensions with F Lipschitz in Y and G Lipschitz in X and $G\,X \subseteq Y$. Then the composition $H\,X = F\,(G\,X)$ is Lipschitz in X and is inclusion isotonic

```
lemma lipschitzI_on_compose:
 fixes U :: ‹'a::{linordered_ring} interval set›
 assumes f: C−lipschitzI_on U F and g: D−lipschitzI_on (F‘U) G
 shows (D * C)−lipschitzI_on U (G ∘ F)
proof (rule lipschitzI_onI)
 show D * C ≥ 0 using lipschitzI_on_nonneg[OF f] lipschitzI_on_nonneg[OF g] by simp
 fix x assume H: x ∈ U
 have width (G(F x)) ≤ D * C * width x
  using lipschitzI_onD[OF g] H assms
  unfolding width_def lipschitzI_on_def apply simp
  by (smt (verit, ccfv_SIG) mult.assoc mult_left_mono order_trans)
 then show width ((G ∘ F) x) ≤ D * C * width x
  unfolding comp_def by simp
qed
```

The definition $[\![?C−lipschitzI_on\ ?U\ ?F;\ ?D−lipschitzI_on\ (?F\ `\ ?U)\ ?G]\!] \Longrightarrow (?D * ?C)−lipschitzI_on\ ?U\ (?G \circ ?F)$ refers to lemma 6.3 in [4]

```
lemma lipschitzI_on_compose2:
 fixes U :: ‹'a::{linordered_ring} interval set›
 assumes F: C−lipschitzI_on U F and G: D−lipschitzI_on (F‘U) G
 shows (D * C)−lipschitzI_on U (λx. G (F x))
```

using *lipschitzI_on_compose F G* **unfolding** *o_def* **by** *blast*

lemma *lipschitzI_on_cong*:
C−*lipschitzI_on U G* ⟷ *D*−*lipschitzI_on V F*
if *C* = *D U* = *V* ⋀*x*. *x* ∈ *V* ⟹ *G x* = *F x*
using *that* **by** (*auto simp*: *lipschitzI_on_def*)

lemma *lipschitzI_on_transform*:
C−*lipschitzI_on U G*
if *C*−*lipschitzI_on U F*
⋀*x*. *x* ∈ *U* ⟹ *G x* = *F x*
using *that*
unfolding *lipschitzI_on_def width_def*
by *simp*

lemma *lipschitz_on_empty_iff*: *C*−*lipschitzI_on* {} *f* ⟷ *C* ≥ *0*
by (*auto simp*: *lipschitzI_on_def*)

lemma *lipschitz_on_id*: (*1*::*real*)−*lipschitzI_on U* (λ*x*. *x*)
by (*auto simp*: *lipschitzI_on_def*)

lemma *lipschitz_on_constant*:
assumes ‹*lower c* = *upper c*›
shows (*0*::*real*)−*lipschitzI_on U* (λ*x*. *c*)
using *assms* **by** (*auto simp*: *lipschitzI_on_def width_def*)

lemma *lipschitzI_on_mult*:
fixes *X* :: *'a*::{*linordered_idom*}
assumes *lipschitzI_on C U f*
and *1* ≤ *X*
shows *lipschitzI_on* (*X*∗*C*) *U f*
using *assms interval_width_positive lower_le_upper mult_le_cancel_right1*
unfolding *lipschitzI_on_def width_def*
by (*smt* (*verit*) *diff_ge_0_iff_ge linorder_not_le mult.assoc order_trans*)

7.1.2 Interval bounds on reals

lemma *bounded_image_real*:
fixes *X* :: *real interval* **and** *f* :: *real* ⇒ *real*
assumes ∀ *x*∈{*lower X*..*upper X*}.
f (*lower X*) − *L* ∗ (*upper X* − *lower X*) ≤ *f x* ∧ *f x* ≤ *f* (*lower X*) + *L* ∗ (*upper X* − *lower X*)
shows ∃ *x e*. ∀ *y*∈*f* ‘ {*lower X*..*upper X*}. *dist x y* ≤ *e*
proof −
let *?x* = *f* (*lower X*)
let *?e* = *L* ∗ (*upper X* − *lower X*)
have ∀ *y*∈*f* ‘ {*lower X*..*upper X*}. *dist ?x y* ≤ *?e*
proof
fix *y* **assume** *y* ∈ *f* ‘ {*lower X*..*upper X*}
then obtain *x* **where** *x* ∈ {*lower X*..*upper X*} **and** *y* = *f x* **by** *auto*
then have *?x* − *?e* ≤ *y* ∧ *y* ≤ *?x* + *?e* **using** *assms* **by** *auto*
then show *dist ?x y* ≤ *?e*
unfolding *dist_real_def* **by** *force*
qed
then show *?thesis* **by** *auto*

```
qed

lemma lipschitz_bounded_image_real:
 fixes X :: real set and f :: real ⇒ real
 assumes bounded X and L−lipschitz_on X f
 shows bounded (f ‘ X)
 using assms(1) assms(2) bounded_uniformly_continuous_image lipschitz_on_uniformly_continuous by blast

lemma inf_le_sup_image_real:
 fixes X :: real interval and f :: real ⇒ real
 assumes L−lipschitz_on (set_of X) f
 shows  Inf (f ‘ set_of X) ≤ Sup (f ‘ set_of X)
proof −
 have bounded (f ‘ set_of X)
  using assms bounded_set_of lipschitz_bounded_image_real by blast
 then have bdd_below (f ‘ set_of X) and bdd_above (f ‘ set_of X)
  using bounded_imp_bdd_below bounded_imp_bdd_above by auto
 then have Inf (f ‘ set_of X) ≤ Sup (f ‘ set_of X)
  by (metis set_of_nonempty cInf_le_cSup empty_is_image)
 then show ?thesis by simp
qed

lemma sup_image_le_real:
 fixes f :: real ⇒ real and F :: real interval ⇒ real interval and X :: real interval
 assumes f ‘ set_of X ⊆ set_of (F X)
  and L−lipschitz_on (set_of X) f
 shows Sup (f ‘ set_of X) ≤ Sup (set_of (F X))
proof −
 have a0: bounded (f ‘ set_of X)
  using lipschitz_bounded_image_real[of set_of X] assms bounded_set_of[of X] by simp
 have a1: bdd_above (f ‘ set_of X)
  using assms
  using a0 bounded_imp_bdd_above by presburger
 then have a2: ∀ y∈f ‘ set_of X. y ≤ Sup (f ‘ set_of X) using bounded_has_Sup(1) a0
  by blast
 then have a3: Sup (f ‘ set_of X) ≤ Sup (set_of (F X))
    using set_of_nonempty assms a0 a1 a2 atLeastAtMost_iff closed_real_atLeastAtMost closed_subset_contains_Sup
empty_is_image set_of_eq sup_set_of
  by metis
 then show ?thesis by simp
qed

lemma inf_image_le_real:
 fixes f :: real ⇒ real and F :: real interval ⇒ real interval and X :: real interval
 assumes f ‘ set_of X ⊆ set_of (F X)
  and L−lipschitz_on (set_of X) f
 shows Inf (set_of (F X)) ≤ Inf (f ‘ (set_of X))
proof −
 have a0: bounded (f ‘ set_of X)
  using lipschitz_bounded_image_real[of set_of X] assms bounded_set_of[of X] by simp
 have a1: bdd_above (f ‘ set_of X)
  using assms
  by (metis atLeastAtMost_iff bdd_above.unfold set_of_eq subset_eq)
 then have a2: ∀ y∈f ‘ set_of X. y ≥ Inf (f ‘ set_of X)
```

```
  using bounded_has_Inf(1) a0
  by blast
 then have a3: Inf (set_of (F X)) ≤ Inf (f ' (set_of X)) using assms
    by (metis set_of_nonempty a0 bounded_imp_bdd_below closed_real_atLeastAtMost closed_subset_contains_Inf
empty_is_image in_bounds inf_set_of set_of_eq)
 then show ?thesis by simp
qed

end
```

8 Multi-Intervals (Multi_Interval_Preliminaries)

8.1 Preliminaries

theory
Multi_Interval_Preliminaries
imports
HOL−Library.Interval
HOL−Analysis.Analysis
Inclusion_Isotonicity
begin

8.1.1 A Class for Capturing Monotonicity of Minus

We try to keep our formalisation of interval arithmetic as generic as possible. In particular, we want to support intervals of type *nat*, *int*, *real*, and *ereal*. For all these types, minus (substraction) is monotonous. Sadly, Isabelle lacks a type class capturing this. Luckily, it is very easy to define our own:

class *minus_mono* = *minus* + *linorder* +
assumes *minus_mono*: $A \leq B \Longrightarrow D \leq C \Longrightarrow A - C \leq B - D$
begin end

instance *nat*::*minus_mono*
by(*standard*, *simp*)
instance *int*::*minus_mono*
by(*standard*, *simp*)
instance *real*::*minus_mono*
by(*standard*, *simp*)
instance *integer*::*minus_mono*
by(*standard*, *simp*)
instance *ereal*::*minus_mono*
by(*standard*, *simp add*:*ereal_minus_mono*)

8.1.2 Infrastructure for Lifting Interval Operations to Lists of Intervals

definition *un_op_interval_list*::‹('*a interval* ⇒ '*a interval*)
⇒ '*a interval list* ⇒ '*a interval list*›
where
‹*un_op_interval_list op xs* = *map op xs*›

definition *bin_op_interval_list*::‹('*a interval* ⇒ '*a interval* ⇒ '*a interval*)
⇒ '*a interval list* ⇒ '*a interval list* ⇒ '*a interval list*›
where
‹*bin_op_interval_list op xs ys* = *concat* (*map* (λ *x* . *map* (*op x*) *xs*) *ys*)›

lemma *bin_op_interval_list_non_empty*: ‹(*xs* ≠ [] ∧ *ys* ≠ []) = (*bin_op_interval_list op xs ys* ≠ [])›
unfolding *bin_op_interval_list_def*
by(*auto simp add*: *ex_in_conv*)

```
lemma bin_op_interval_list_empty: ‹(xs = [] ∨ ys = []) = (bin_op_interval_list op xs ys = [])›
 unfolding bin_op_interval_list_def
 by(auto simp add: ex_in_conv)

lemma bin_op_interval_unroll: ‹bin_op_interval_list op (xs) (y#ys) = (map (op y) xs)@( bin_op_interval_list op xs ys)›
 unfolding bin_op_interval_list_def by(simp)

lemma bin_op_interval_list_commute:
 assumes op_commute: ‹⋀ x y. op x y = op y x›
 shows ‹set (bin_op_interval_list (op) xs ys) = set (bin_op_interval_list (op) ys xs)›
 unfolding bin_op_interval_list_def
proof(induction xs ys rule:list_induct2')
 case 1
 then show ?case by simp
next
 case (2 x xs)
 then show ?case by simp
next
 case (3 y ys)
 then show ?case by simp
next
 case (4 x xs y ys)
 then show ?case
  by(auto simp add: op_commute, blast)
 qed

lemma bin_op_interval_list_assoc:
 assumes op_assoc: ‹⋀ x y z. op (op x y) z = op x (op y z)›
  shows ‹set (bin_op_interval_list (op) ((bin_op_interval_list (op) xs ys)) zs) = set (bin_op_interval_list (op) xs
((bin_op_interval_list (op) ys zs)))›
 unfolding bin_op_interval_list_def
proof(induction xs ys rule:list_induct2')
 case 1
 then show ?case by simp
next
 case (2 x xs)
 then show ?case by simp
next
 case (3 y ys)
 then show ?case by simp
next
 case (4 x xs y ys)
 then show ?case
 proof(induction zs)
  case Nil
  then show ?case by simp
 next
  case (Cons a zs)
  then show ?case
   apply(auto simp add: 4 list.set_map op_assoc)[1]
   subgoal
    by (meson image1)
   subgoal
```

```
    by blast
   subgoal
    by (meson UnionI imageI insertI1)
   subgoal
    by (meson UN_I imageI insertCI)
   done
  qed
 qed
```

Lifting Unary Minus, Addition, and Multiplication

```
definition ‹iList_uminus = un_op_interval_list (λ x. − x)›
definition ‹iList_plus  = bin_op_interval_list (+)›
definition ‹iList_times  = bin_op_interval_list (∗)›

lemma iList_plus_lower:
 assumes ‹A ≠ []› and ‹B ≠ []›
shows ‹lower (hd (iList_plus A B)) = lower (hd A) + lower (hd B)›
proof(insert assms, induction B)
 case Nil
 then show ?case by simp
next
 case (Cons a B)
 then show ?case
  unfolding iList_plus_def
  apply(simp add: hd_map bin_op_interval_unroll[of (+) A a B])
  using add.commute by blast
qed

lemma iList_plus_upper:
 assumes ‹A ≠ []› and ‹B ≠ []›
shows ‹upper (hd (iList_plus A B)) = upper (hd A) + upper (hd B)›
proof(insert assms, induction B)
 case Nil
 then show ?case by simp
next
 case (Cons a B)
 then show ?case
  unfolding iList_plus_def
  apply(simp add: hd_map bin_op_interval_unroll[of (+) A a B])
  using add.commute by blast
qed

lemma iList_plus_unroll:
 ‹iList_plus ys (x # xs) = map ((+) x) ys @ iList_plus ys xs›
 by (simp add: iList_plus_def bin_op_interval_unroll[of (+) ys x xs])

lemma a ≠ [] ⟹ (iList_plus [Interval (o, o)] a) = (a::'a::{ordered_ab_group_add,zero} interval list)
proof(induction a rule:list_nonempty_induct)
 case (single x)
 then show ?case
  by (metis add.right_neutral append.right_neutral
  bin_op_interval_list_non_empty iList_plus_def iList_plus_unroll list.map(2) zero_interval_def)
next
```

```
  case (cons x xs)
  then show ?case
    by (metis add.right_neutral append_Cons append_Nil iList_plus_unroll list.map(1) list.map(2) zero_interval_def)
qed


lemma iList_plus_commute:
  ‹set (iList_plus xs ys) = set (iList_plus ys xs)›
  using bin_op_interval_list_commute[of (+) xs ys] add.commute
  unfolding iList_plus_def
  by auto

lemma iList_plus_assoc:
  ‹set (iList_plus xs (iList_plus ys zs)) = set (iList_plus (iList_plus xs ys) zs)›
  using bin_op_interval_list_assoc[of (+) xs ys zs] add.assoc
  unfolding iList_plus_def
  by auto

lemma remdups_append1:
  remdups (remdups xs @ ys) = remdups (xs @ ys)
by(induction xs) auto


lemma bin_op_interval_remdups_left:
  ‹remdups (bin_op_interval_list op (remdups x) y) = remdups (bin_op_interval_list op x y)›
proof(induction y)
  case Nil
  then show ?case
    by (simp add: bin_op_interval_list_def)
next
  case (Cons y' ys')
  then show ?case
    by (metis (no_types, opaque_lifting) bin_op_interval_list_non_empty bin_op_interval_unroll
          remdups_append1 remdups_append2 remdups_map_remdups)
qed

lemma iList_plus_remdups_left:
remdups (iList_plus (remdups a) b) = remdups (iList_plus a b)
  for a::'a::{minus_mono,ordered_ab_semigroup_add,linorder,linordered_field} interval list
proof(induction b)
  case Nil
  then show ?case
    by (metis bin_op_interval_list_empty iList_plus_def)
next
  case (Cons a b)
  then show ?case
    by (metis bin_op_interval_remdups_left iList_plus_def list.discI remdups.simps(1))
qed

lemma interval_add_eq: ‹a + b = Interval(lower a + lower b, upper a + upper b)›
  apply(subst interval_eq_iff [of a+b Interval (lower a + lower b, upper a + upper b)])
  using upper_plus[of a b] lower_plus[of a b]
  by (simp add: add_mono)
```

```
lemma iList_plus_lower_upper_eq:
 ‹iList_plus = bin_op_interval_list (λ a b. Interval(lower a + lower b, upper a + upper b))›
 unfolding iList_plus_def using interval_add_eq
 by metis
```

A Locale for Multi-Interval Division Where the Quotient does not Contain 0

```
context interval_division
begin

 definition ‹iList_inverse = un_op_interval_list inverse›
 definition ‹iList_divide  = bin_op_interval_list divide›

end
```

8.1.3 Utilities for (Sorted) Lists of Intervals

```
definition ‹cmp_lower_width = (λ x y.  if lower x = lower y then width x ≤ width y else lower x < lower y)›

definition ‹sorted_wrt_lower = sorted_wrt cmp_lower_width›

lemma interval_lower_width_eq:
‹(lower x = lower y ∧ width x = width y) = (x = (y::'a::{minus_mono, linordered_field} interval))›
 by (metis interval_eq1 le_add_diff_inverse lower_le_upper width_def)

lemma sorted_wrt_lower_distinct_lists_eq:
 assumes ‹set xs = set (ys::'a::{minus_mono, linordered_field} interval list)›
    and ‹distinct xs› and ‹distinct ys›
    and ‹sorted_wrt_lower xs› and ‹sorted_wrt_lower ys›
  shows‹ xs = ys›
proof(insert assms, unfold sorted_wrt_lower_def cmp_lower_width_def, induct xs ys rule:list_induct2')
 case 1
 then show ?case by simp
next
 case (2 x xs)
 then show ?case by simp
next
 case (3 y ys)
 then show ?case by simp
next
 case (4 x xs y ys)
 then show ?case
  using interval_lower_width_eq
  apply(auto)[1]
  subgoal by (smt (verit) insert_iff interval_lower_width_eq order.asym order_le_imp_less_or_eq)
  subgoal by (smt (verit, ccfv_SIG) antisym insertI1 insert_eq_iff interval_lower_width_eq not_less_iff_gr_or_eq)
  done
qed

definition ‹sorted_wrt_upper = sorted_wrt (λ x y. upper x ≤ upper y)›

definition ‹cmp_non_overlapping = (λ x y.  upper x ≤ lower y)›

lemma cmp_non_overlapping_lower: ‹cmp_non_overlapping x y ⟹ lower x ≤ lower y›
```

```
 using  lower_le_upper unfolding cmp_non_overlapping_def
 by(metis dual_order.trans)

lemma cmp_non_overlapping_upper: ‹cmp_non_overlapping x y ⟹ upper x ≤ upper y›
 using  lower_le_upper unfolding cmp_non_overlapping_def
 by(metis dual_order.trans)

definition ‹non_overlapping_sorted = sorted_wrt cmp_non_overlapping›
definition ‹contiguous xs = (∀ i < length xs − 1 . upper (xs ! i) = lower (xs ! (i + 1)))›

lemma non_overlapping_sorted_empty: ‹non_overlapping_sorted []›
 by (simp add: non_overlapping_sorted_def cmp_non_overlapping_def)


lemma non_overlapping_sorted_unroll:
 assumes xs ≠ []  shows non_overlapping_sorted (x # xs) = (upper x ≤ lower (hd xs) ∧  non_overlapping_sorted xs)
proof(cases xs)
 case Nil
 then show ?thesis using assms by(simp)
next
 case (Cons y ys)
 then show ?thesis
  apply(simp add:  non_overlapping_sorted_def cmp_non_overlapping_def  cmp_lower_width_def)
  using lower_le_upper
  by (metis dual_order.trans)
qed

lemma contiguous_non_overlapping: ‹contiguous (is::'a::{preorder} interval list) ⟹ non_overlapping_sorted  is›
proof(induction isrule:induct_list012)
 case 1
 then show ?case
 unfolding contiguous_def non_overlapping_sorted_def cmp_non_overlapping_def
 using lower_le_upper
 by simp
next
 case (2 x)
 then show ?case
 unfolding contiguous_def non_overlapping_sorted_def cmp_non_overlapping_def
 using lower_le_upper
 by simp
next
 case (3 x y zs)
 then show ?case
  apply(subst non_overlapping_sorted_unroll[of (y#zs) x])
  subgoal by(simp)
  subgoal apply(simp add: 3) using 3
   unfolding contiguous_def non_overlapping_sorted_def cmp_non_overlapping_def
   using lower_le_upper
   by fastforce
  done
qed

definition ‹cmp_non_adjacent = (λ x y.  upper x < lower y)›
```

```
lemma cmp_non_adjacent_lower: ‹cmp_non_adjacent x y ⟹ lower x < lower y›
 using  lower_le_upper unfolding cmp_non_adjacent_def
 by(metis dual_order.strict_trans2)

lemma cmp_non_adjacent_upper: ‹cmp_non_adjacent x y ⟹ upper x < upper y›
 using  lower_le_upper unfolding cmp_non_adjacent_def
 by(metis dual_order.strict_trans1)

definition ‹non_adjacent_sorted_wrt_lower = sorted_wrt cmp_non_adjacent›

lemma non_adjacent_sorted_wrt_lower_unroll:
 assumes xs ≠ []
 shows non_adjacent_sorted_wrt_lower (x # xs) =
     ((upper x < lower (hd xs)) ∧ non_adjacent_sorted_wrt_lower xs)
proof(cases xs)
 case Nil
 then show ?thesis using assms by(simp)
next
 case (Cons y ys)
 then show ?thesis
  apply(simp add:non_adjacent_sorted_wrt_lower_def  cmp_non_adjacent_def )
  by (meson cmp_non_adjacent_def cmp_non_adjacent_lower dual_order.strict_trans)
qed


lemma non_adjacent_implies_non_overlapping:
 assumes ‹non_adjacent_sorted_wrt_lower is› shows ‹non_overlapping_sorted is›
proof(insert assms, induction is)
 case Nil
 then show ?case
  unfolding non_overlapping_sorted_def cmp_non_overlapping_def
       non_adjacent_sorted_wrt_lower_def cmp_non_adjacent_def
  by(simp)
next
 case (Cons a is)
 then show ?case
  unfolding non_overlapping_sorted_def cmp_non_overlapping_def
       non_adjacent_sorted_wrt_lower_def cmp_non_adjacent_def
  using order.strict_implies_order by auto
qed

lemma non_overlapping_implies_sorted_wrt_lower:
 assumes ‹non_overlapping_sorted (is::'a::{minus_mono} interval list)›
 shows ‹sorted_wrt_lower is›
proof(insert assms, induction is)
 case Nil
 then show ?case unfolding sorted_wrt_lower_def by simp
next
 case (Cons a is)
 then show ?case
  unfolding non_overlapping_sorted_def sorted_wrt_lower_def
       cmp_lower_width_def cmp_non_overlapping_def
  by (simp, metis (no_types, lifting) cmp_non_adjacent_def cmp_non_adjacent_lower minus_mono
                dual_order.strict_iff_order lower_le_upper width_def )
```

```
qed

lemma non_overlapping_implies_sorted_wrt_upper:
 assumes ‹non_overlapping_sorted is›
 shows ‹sorted_wrt_upper is›
proof(insert assms, induction is rule:induct_list012)
 case 1
 then show ?case by(simp add: leD sorted_wrt_upper_def)
next
 case (2 x)
 then show ?case by(simp add: leD sorted_wrt_upper_def)
next
 case (3 x y zs)
 then show ?case
  by(auto simp add: cmp_non_overlapping_upper non_overlapping_sorted_def sorted_wrt_upper_def leD)[1]
qed

lemma non_adjacent_implies_sorted_wrt_lower:
 assumes ‹non_adjacent_sorted_wrt_lower is›
 shows ‹sorted_wrt_lower is›
proof(insert assms, induction is)
 case Nil
 then show ?case
  unfolding sorted_wrt_lower_def
  by simp
next
 case (Cons a is)
 then show ?case
  unfolding sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def
       cmp_non_adjacent_def cmp_lower_width_def width_def
  by(simp split:if_splits, metis dual_order.strict_trans2 lower_le_upper order_less_irrefl)
qed

lemma non_adjacent_implies_distinct:
 assumes ‹non_adjacent_sorted_wrt_lower is›
 shows ‹distinct is›
proof(insert assms, induction is)
 case Nil
 then show ?case
  unfolding sorted_wrt_lower_def
  by simp
next
 case (Cons a is)
 then show ?case
  unfolding sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def
       cmp_non_adjacent_def cmp_lower_width_def width_def
  by(simp split:if_splits, metis dual_order.strict_trans2 lower_le_upper order_less_irrefl)
qed

fun merge_overlapping_intervals_sorted_wrt_lower :: 'a::linorder interval list ⇒ 'a interval list  where
merge_overlapping_intervals_sorted_wrt_lower [] = [] |
merge_overlapping_intervals_sorted_wrt_lower [x] = [x] |
merge_overlapping_intervals_sorted_wrt_lower (x#y#ys) =
   ( if upper x ≤ lower y
```

```
    then x#(merge_overlapping_intervals_sorted_wrt_lower (y#ys))
    else merge_overlapping_intervals_sorted_wrt_lower ((mk_interval(lower x, max (upper x) (upper y)))#ys)
   )


lemma sorted_wrt_lower_unroll:
 assumes xs ≠ []
 shows sorted_wrt_lower (x # xs) = ((if lower x ≠ lower (hd xs)
                    then lower x < lower (hd xs)
                    else width x ≤ width (hd xs) ) ∧ sorted_wrt_lower (xs))
proof(cases xs)
 case Nil
 then show ?thesis using assms by(simp)
next
 case (Cons y ys)
 then show ?thesis
  apply(simp add: sorted_wrt_lower_def cmp_lower_width_def)
  by (smt (verit) dual_order.irrefl dual_order.strict_trans2 dual_order.trans order.strict_implies_order)
qed

lemma sorted_wrt_upper_unroll:
 assumes xs ≠ []
 shows sorted_wrt_upper (x # xs) = ((upper x ≤ upper (hd xs))∧ sorted_wrt_upper (xs))
proof(cases xs)
 case Nil
 then show ?thesis using assms by(simp)
next
 case (Cons y ys)
 then show ?thesis
  apply(simp add: sorted_wrt_upper_def cmp_lower_width_def)
  using dual_order.trans by blast
qed

lemma sorted_wrt_lower_tail: sorted_wrt_lower (x # xs) ⟹ sorted_wrt_lower (xs)
 unfolding sorted_wrt_lower_def
 by simp

lemma sorted_wrt_lower_tail':sorted_wrt_lower (x # y # ys) ⟹ sorted_wrt_lower (x # ys)
 unfolding sorted_wrt_lower_def by simp

lemma iList_plus_leq_B:
 assumes sorted_wrt_lower A and sorted_wrt_lower B and 1 < length B
 shows hd (map lower (iList_plus A B)) ≤ hd (map lower (iList_plus A (tl B)))
 proof(insert assms, induction B)
  case Nil
  then show ?case by simp
 next
  case (Cons b bs) note * = this
  then show ?case
  proof(cases bs)
   case Nil
   then show ?thesis
    using * by simp
```

```
  next
   case (Cons b' bs') note ** = this
   then have a: ‹hd (map lower (iList_plus A (b#bs))) ≤ hd (map lower (iList_plus A bs))›
   proof(induction bs)
    case Nil
    then show ?case by simp
   next
    case (Cons b'' bs'')
    then show ?case
     using * **
     by (smt (z3) add.commute add_right_mono dual_order.eq_iff hd_append iList_plus_lower
        iList_plus_unroll list.map_disc_iff list.map_sel(1) list.sel(1) list.simps(3)
        map_append order_less_imp_le sorted_wrt_lower_unroll)
   qed
   then show ?thesis
    using * ** a by simp
  qed
 qed

lemma iList_plus_leq_A:
 assumes sorted_wrt_lower A and sorted_wrt_lower B and 1 < length A
 shows hd (map lower (iList_plus A B)) ≤ hd (map lower (iList_plus (tl A) B))
 proof(insert assms, induction A)
  case Nil
  then show ?case by simp
 next
  case (Cons a as) note * = this
  then show ?case
  proof(cases as)
   case Nil
   then show ?thesis
    using * by simp
  next
   case (Cons a' as') note ** = this
   then have a: ‹hd (map lower (iList_plus (a#as) B)) ≤ hd (map lower (iList_plus as B))›
   proof(induction as)
    case Nil
    then show ?case by simp
   next
    case (Cons a'' as'')
    then show ?case
     using * **
     by (smt (verit, best) add_right_mono bin_op_interval_list_empty dual_order.eq_iff
          iList_plus_def iList_plus_lower list.map_sel(1) list.sel(1) list.simps(3)
          order_less_imp_le sorted_wrt_lower_unroll)
   qed
   then show ?thesis
    using * ** a by simp
  qed
 qed

value ‹merge_overlapping_intervals_sorted_wrt_lower [mk_interval(1::int,2), mk_interval(2,3), mk_interval(5,7),
mk_interval(6,10)]›
```

```
lemma merge_overlapping_intervals_sorted_wrt_lower_non_nil:
 assumes ‹xs ≠ []›
 shows ‹(merge_overlapping_intervals_sorted_wrt_lower xs) ≠ []›
 using assms
 by(induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct, simp_all)

lemma merge_overlapping_intervals_sorted_hd_lower:
 assumes ‹xs ≠ []›
 shows lower (hd (merge_overlapping_intervals_sorted_wrt_lower (xs))) = lower (hd xs)
proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case by(simp)
next
 case (2 x)
 then show ?case by(simp)
next
 case (3 x y ys)
 then show ?case
  using 3.IH(2) list.sel(1) lower_le_upper max.coboundedI2 mk_interval_lower
  by (metis list.discI max.commute merge_overlapping_intervals_sorted_wrt_lower.simps(3))
qed

lemma merge_overlapping_intervals_sorted_hd_upper:
 assumes ‹xs ≠ []›
 shows upper (hd xs) ≤ upper (hd (merge_overlapping_intervals_sorted_wrt_lower xs))
proof(insert assms(1), induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case by(simp)
next
 case (2 x)
 then show ?case by(simp)
next
 case (3 x y ys)
 then show ?case
  proof(cases upper x ≤ lower y)
  case True note non_overlapping = this
  then show ?thesis by simp
 next
  case False note overlapping_or_included = this
  then show ?thesis proof(cases upper x ≤ upper y)
   case True note overlapping = this
   then show ?thesis
    proof(cases lower x ≤ upper y)
     case True
     then show ?thesis
      using 3.IH(2)[simplified overlapping_or_included]
      apply(simp add: overlapping_or_included overlapping True)
      using overlapping by simp
    next
     case False
     then show ?thesis
      using  overlapping_or_included overlapping False
      by (metis lower_le_upper max.coboundedI1 max_def)
    qed
```

```
  next
   case False note included = this
   then show ?thesis
    proof(cases lower x ≤ upper y)
     case True
     then show ?thesis
      using 3.IH(2)[simplified overlapping_or_included]
      using included by simp
    next
     case False
     then show ?thesis
      using  overlapping_or_included included False
          lower_le_upper[of x] lower_le_upper[of y]
      using 3.IH(2) by fastforce
    qed
  qed
 qed
qed

lemma Interval_id[simp]: ‹Interval (lower x, upper x) = x ›
 by (simp add: interval_eq_iff)

lemma mk_interval_id[simp]: ‹(mk_interval (lower x, upper x)) = x›
 using lower_le_upper[of x] unfolding mk_interval' by(auto)

lemma merge_overlapping_intervals_sorted_hd_width:
 assumes ‹xs ≠ []›
  shows width (hd xs) ≤ width (hd (merge_overlapping_intervals_sorted_wrt_lower (xs::'a::{minus_mono} interval
list)))
proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case by(simp)
next
 case (2 x)
 then show ?case by(simp)
next
 case (3 x y ys)
 then show ?case
  proof(cases upper x ≤ lower y)
   case True note non_overlapping = this
   then show ?thesis by simp
 next
   case False note overlapping_or_included = this
   then show ?thesis proof(cases upper x ≤ upper y)
    case True note overlapping = this
    then show ?thesis
     proof(cases lower x ≤ upper y)
      case True
      then show ?thesis
       using 3.IH(2)[simplified overlapping_or_included]
       apply(simp add: width_def overlapping_or_included overlapping True)
       using True diff_right_mono list.sel(1) list.simps(3)
             merge_overlapping_intervals_sorted_hd_lower merge_overlapping_intervals_sorted_hd_upper
             mk_interval_lower mk_interval_upper order.trans overlapping minus_mono
```

```
      by (smt (verit, best) dual_order.refl)
    next
     case False
     then show ?thesis
      using  overlapping_or_included overlapping False
      by (metis lower_le_upper max.coboundedI1 max_def)
    qed
  next
   case False note included = this
   then show ?thesis
    proof(cases lower x ≤ upper y)
     case True
     then show ?thesis
      using 3.IH(2)[simplified overlapping_or_included] included
      by (metis list.discI list.sel(1) max_def
            merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id
            overlapping_or_included)
    next
     case False
     then show ?thesis
      using  overlapping_or_included included False
          lower_le_upper[of x] lower_le_upper[of y]
      using 3.IH(2)[simplified overlapping_or_included]
      by (metis list.discI list.sel(1) max_def merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id)
    qed
  qed
 qed
qed



lemma merge_overlapping_intervals_sorted_wrt_lower_sorted_lower:
 assumes ‹sorted_wrt_lower (xs::'a::{minus_mono} interval list)›
 shows ‹sorted_wrt_lower (merge_overlapping_intervals_sorted_wrt_lower xs)›
proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case
  unfolding sorted_wrt_lower_def
  by(simp)
next
 case (2 x)
 then show ?case
  unfolding sorted_wrt_lower_def
  by(simp)
next
 case (3 x y ys)
 then show ?case
  proof(cases upper x ≤ lower y)
   case True note non_overlapping = this
   then show ?thesis
    by (smt (verit) 3.IH(1) 3.prems dual_order.trans list.discI merge_overlapping_intervals_sorted_hd_lower
          merge_overlapping_intervals_sorted_hd_width merge_overlapping_intervals_sorted_wrt_lower.simps(3)
          merge_overlapping_intervals_sorted_wrt_lower_non_nil sorted_wrt_lower_unroll)
 next
```

```
    case False note overlapping_or_included = this
    then show ?thesis proof(cases upper x ≤ upper y)
     case True note overlapping = this
     then show ?thesis
      proof(cases lower x ≤ upper y)
       case True
       then show ?thesis
        using 3.IH(2)[simplified overlapping_or_included]
        by (smt (verit, ccfv_threshold) 3.prems list.sel(1) max.absorb2
            merge_overlapping_intervals_sorted_wrt_lower.simps(3)
            mk_interval_id mk_interval_lower overlapping overlapping_or_included sorted_wrt1
            sorted_wrt_lower_def sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
      next
       case False
       then show ?thesis
        using  overlapping_or_included overlapping False
        using lower_le_upper order_trans by blast
      qed
    next
     case False note included = this
     then show ?thesis
      proof(cases lower x ≤ upper y)
       case True
       then show ?thesis
        using 3.IH(2)[simplified overlapping_or_included] included
          3.prems overlapping_or_included sorted_wrt_lower_tail'
        by (metis max_def merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id)
      next
       case False
       then show ?thesis
        using  overlapping_or_included included False
           lower_le_upper[of x] lower_le_upper[of y]
        using 3.IH(2)[simplified overlapping_or_included]
          3.prems sorted_wrt_lower_tail'
        by (metis max_def merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id)
      qed
    qed
   qed
  qed

lemma merge_overlapping_intervals_sorted_sorted_non_non_overlapping:
  assumes ‹sorted_wrt_lower (xs::'a::{minus_mono} interval list)›
  shows ‹non_overlapping_sorted (merge_overlapping_intervals_sorted_wrt_lower xs)›
  proof(insert assms, induction xs rule:merge_overlapping_intervals_sorted_wrt_lower.induct)
    case 1
    then show ?case unfolding non_overlapping_sorted_def
      by (metis merge_overlapping_intervals_sorted_wrt_lower.simps(1) sorted_wrt.simps(1))
  next
    case (2 x)
    then show ?case unfolding non_overlapping_sorted_def
      by (metis merge_overlapping_intervals_sorted_wrt_lower.simps(2) sorted_wrt1)
  next
    case (3 x y ys)
    then show ?case
```

```
  proof(cases upper x ≤ lower y)
   case True note non_overlapping = this
   then show ?thesis
    using 3dual_order.trans list.discI merge_overlapping_intervals_sorted_hd_lower
      merge_overlapping_intervals_sorted_hd_width merge_overlapping_intervals_sorted_wrt_lower.simps
      merge_overlapping_intervals_sorted_wrt_lower_non_nil non_overlapping_sorted_unroll
    by (smt (verit, ccfv_threshold) list.sel(1) sorted_wrt_lower_tail)
  next
   case False note overlapping_or_included = this
   then show ?thesis proof(cases upper x ≤ upper y)
    case True note overlapping = this
    then show ?thesis
     proof(cases lower x ≤ upper y)
      case True
      then show ?thesis
       using 3.IH(2)[simplified overlapping_or_included]
       by (smt (verit) 3.prems list.discI list.sel(1)
           max.absorb2 merge_overlapping_intervals_sorted_wrt_lower.simps(3)
           mk_interval_id mk_interval_lower overlapping overlapping_or_included sorted_wrt_lower_tail'
           sorted_wrt_lower_unroll)
     next
      case False
      then show ?thesis
       using  overlapping_or_included overlapping False
       using lower_le_upper order_trans by blast
     qed
   next
    case False note included = this
    then show ?thesis
     proof(cases lower x ≤ upper y)
      case True
      then show ?thesis
       using 3.IH(2)[simplified overlapping_or_included] included
       using 3.prems overlapping_or_included sorted_wrt_lower_tail'
       by (metis max.orderE merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id nle_le)
     next
      case False
      then show ?thesis
       using  overlapping_or_included included False
           lower_le_upper[of x] lower_le_upper[of y]
       using 3.IH(2)[simplified overlapping_or_included]
       using 3.prems sorted_wrt_lower_tail'
       by (metis max.orderE merge_overlapping_intervals_sorted_wrt_lower.simps(3) mk_interval_id nle_le)
     qed
   qed
  qed
qed
```

```
fun merge_adjacent_intervals_sorted_wrt_lower :: 'a::linorder interval list ⇒ 'a interval list  where
merge_adjacent_intervals_sorted_wrt_lower [] = [] |
merge_adjacent_intervals_sorted_wrt_lower [x] = [x] |
merge_adjacent_intervals_sorted_wrt_lower (x#y#ys) =
```

```
  ( if upper x < lower y
    then x#(merge_adjacent_intervals_sorted_wrt_lower (y#ys))
    else merge_adjacent_intervals_sorted_wrt_lower ((mk_interval(lower x, max (upper y) (upper x)))#ys)
  )

value ‹merge_adjacent_intervals_sorted_wrt_lower [mk_interval(1::int,2), mk_interval(2,3), mk_interval(5,7),
mk_interval(6,10)]›

lemma merge_adjacent_intervals_sorted_wrt_lower_non_nil:
 assumes ‹xs ≠ []›
 shows ‹(merge_adjacent_intervals_sorted_wrt_lower xs) ≠ []›
 using assms by(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct, simp_all)

lemma merge_adjacent_intervals_sorted_wrt_lower_non_nil':
 shows ‹(merge_adjacent_intervals_sorted_wrt_lower (x#xs)) ≠ []›
 by(simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil)

lemma merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd:
 assumes ‹sorted_wrt_lower xs›
 shows ‹lower (hd (merge_adjacent_intervals_sorted_wrt_lower xs)) = lower (hd xs)›
 using assms
proof(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case
  by (metis merge_adjacent_intervals_sorted_wrt_lower.simps(1))
next
 case (2 x)
 then show ?case by simp
next
 case (3 x y ys)
 then show ?case
 proof(cases (sorted_wrt_lower (mk_interval (lower x, max (upper y) (upper x)) # ys)))
  case True
  then show ?thesis
   by (simp, metis 3.IH(2) list.sel(1) lower_le_upper max.coboundedI2 mk_interval_lower)
 next
  case False
  then show ?thesis
   apply(simp, subst 3 (2))
    apply (smt (verit) 3 dual_order.strict_trans leD list.discI list.sel(1) lower_le_upper max.coboundedI2
                mk_interval_lower sorted_wrt1 sorted_wrt_lower_def sorted_wrt_lower_unroll)
   apply (smt (verit) 3.prems dual_order.trans interval_eq1 list.discI list.sel(1)
               lower_le_upper max.strict_order_iff max_def mk_interval_lower mk_interval_upper
               sorted_wrt1 sorted_wrt_lower_def sorted_wrt_lower_unroll)
   by (simp add: 3 max.coboundedI2 sorted_wrt_lower_def)
  qed
 qed

lemma merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_subset:
 ‹set (map lower (merge_adjacent_intervals_sorted_wrt_lower xs)) ⊆ set (map lower xs)›
 apply(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
by(auto simp add: max.coboundedI2 mk_interval' sorted_wrt_lower_def image_def)

lemma merge_adjacent_intervals_sorted_wrt_lower_set_eq:
```

```
 assumes ‹set (xs::real interval list) = set ys›
    and ‹distinct xs› and ‹distinct ys›
    and ‹sorted_wrt_lower xs› and ‹sorted_wrt_lower ys›
 shows ‹merge_adjacent_intervals_sorted_wrt_lower xs = merge_adjacent_intervals_sorted_wrt_lower ys›
 using sorted_wrt_lower_distinct_lists_eq[of xs ys, simplified assms, simplified]
 by(auto)


lemma merge_adjacent_intervals_sorted_wrt_lower_lower_upper:
 assumes sorted_wrt_lower xs
 shows x ∈ set (merge_adjacent_intervals_sorted_wrt_lower xs) ⟹ ∃ l ∈ set xs. ∃ u ∈ set xs. lower l = lower x ∧ upper
u = upper x
proof(insert assms, induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case
  by simp
next
 case (2 x)
 then show ?case
  by simp
next
 case (3 x y ys)
 then show ?case
  proof(cases ys)
   case Nil
   then show ?thesis
    using 3
    by (simp, smt (z3) empty_iff empty_set list.sel(1) list.sel(3) list.set_cases
        max_def merge_adjacent_intervals_sorted_wrt_lower.simps(2)
        merge_adjacent_intervals_sorted_wrt_lower.simps(3)
        merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd mk_interval_lower
        mk_interval_upper)
  next
   case (Cons a list)
   then show ?thesis
    apply(auto simp add: mk_interval' max_def)[1]
    subgoal
     using 3
     by (smt (z3) insert_iff interval_eq_iff list.sel(1) list.simps(15) list.simps(3) lower_le_upper
         max.coboundedI1 max_def_raw merge_adjacent_intervals_sorted_wrt_lower.simps(3)
         mk_interval_lower mk_interval_upper sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
    subgoal
     using 3
     by (smt (verit) dual_order.trans list.discI list.inject list.sel(1) lower_le_upper max_def
         merge_adjacent_intervals_sorted_wrt_lower.elims mk_interval_id mk_interval_lower
         mk_interval_upper order.asym set_ConsD sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
    done
   qed
  qed


primrec interval_insert_sort_lower_width :: ('a::{linorder,minus}) interval ⇒ 'a interval list ⇒ 'a interval list where
 interval_insert_sort_lower_width x [] = [x] |
```

```
 interval_insert_sort_lower_width x (y#ys) =
 (if cmp_lower_width x y then (x#y#ys) else y#(interval_insert_sort_lower_width x ys))

lemma interval_insert_sort_lower_width_length:
‹length (interval_insert_sort_lower_width x xs) = 1 + length xs ›
 by(induction xs, auto)

lemma interval_insert_sort_lower_width_nonempty:
‹interval_insert_sort_lower_width x xs ≠ []›
  by(induction xs, auto)



lemma interval_insert_sort_wrt_lower:
‹sorted_wrt_lower xs ⟹ sorted_wrt_lower (interval_insert_sort_lower_width x xs)›
proof(induction xs)
 case Nil
 then show ?case by(simp add: sorted_wrt_lower_def cmp_lower_width_def)
next
 case (Cons a xs)
 then show ?case
  apply(auto)[1]
  subgoal
   by (metis cmp_lower_width_def list.discI list.sel(1) sorted_wrt_lower_unroll)
  subgoal
   apply(subst sorted_wrt_lower_unroll)
   apply(simp add: interval_insert_sort_lower_width_nonempty)
   apply(simp add: cmp_lower_width_def sorted_wrt_lower_def split:if_split)
   using interval_insert_sort_lower_width.simps(1)[of a]
       interval_insert_sort_lower_width.simps(2)
            list.sel(1) list.set_cases list.set_sel(1)
   by (smt (verit) interval_insert_sort_lower_width.simps(1) less_le_not_le nle_le)
   done
qed

lemma interval_isort_elements: set (interval_insert_sort_lower_width x xs) = {x} ∪ set xs
proof(induction xs)
 case Nil
 then show ?case by simp
next
 case (Cons a xs)
 then show ?case by(auto)
qed

lemma foldr_isort_elements: set (foldr interval_insert_sort_lower_width xs []) = set xs
proof(induction xs)
 case Nil
 then show ?case by simp
next
 case (Cons a xs)
 then show ?case
  using interval_isort_elements by(auto)
qed
```

```
definition interval_sort_lower_width :: ('a::{linorder,minus}) interval list ⇒ 'a interval list where
 interval_sort_lower_width xs = foldr interval_insert_sort_lower_width xs []

lemma interval_sort_lower_width_length: ‹length (interval_sort_lower_width xs) = (length xs)›
proof(induction xs)
 case Nil
 then show ?case unfolding interval_sort_lower_width_def by simp
next
 case (Cons a xs)
 then show ?case
  unfolding interval_sort_lower_width_def
  by (simp add: interval_insert_sort_lower_width_length)
qed

lemma interval_sort_lower_width_sorted: ‹sorted_wrt_lower  (interval_sort_lower_width xs)›
proof(induction xs)
 case Nil
 then show ?case
  unfolding interval_sort_lower_width_def sorted_wrt_lower_def
  by simp
next
 case (Cons a xs)
 then show ?case
 unfolding interval_sort_lower_width_def
 using interval_insert_sort_wrt_lower
 by auto
qed


lemma interval_sort_lower_width_set_eq:
 ‹set (interval_sort_lower_width x) = set x›
 by (simp add: foldr_isort_elements interval_sort_lower_width_def)

lemma interval_sort_lower_width_remdups:
 ‹remdups (interval_sort_lower_width (remdups xs)) =  interval_sort_lower_width (remdups xs)›
 unfolding interval_sort_lower_width_def
proof(induction xs)
 case Nil
 then show ?case by simp
next
 case (Cons a xs)
 then show ?case
  by (metis (no_types, opaque_lifting) foldr_isort_elements interval_sort_lower_width_def
        interval_sort_lower_width_length length_remdups_card_conv length_remdups_eq set_remdups)
qed

lemma interval_sort_lower_width_distinct:
 assumes ‹distinct xs› shows
 ‹distinct (interval_sort_lower_width (remdups xs))›
 using interval_sort_lower_width_remdups by auto

lemma foldr_interval_insert_sort_lower_width_distinct:
 assumes ‹distinct zs›
 shows ‹distinct (foldr interval_insert_sort_lower_width zs [])›
```

```
proof(insert assms, induction zs)
 case Nil
 then show ?case
  by simp
next
 case (Cons a zs)
 then show ?case
  by (simp, metis (no_types, opaque_lifting) Cons.prems distinct_remdups_id foldr.simps(2)
        interval_sort_lower_width_def interval_sort_lower_width_distinct o_apply)
qed


lemma non_overlapping_sorted_remdups:
non_overlapping_sorted xs ⟹ non_overlapping_sorted (remdups xs)
 proof(induction xs)
  case Nil
  then show ?case
   by(simp)
 next
  case (Cons a xs)
  then show ?case
   unfolding non_overlapping_sorted_def
   by(auto)
 qed

lemma insert_in_lower_width: x ∈ set (interval_insert_sort_lower_width a list) = (x = a ∨ x ∈ set list)
proof(induction list)
 case Nil
 then show ?case by(simp)
next
 case (Cons a list)
 then show ?case
  unfolding interval_insert_sort_lower_width_def cmp_lower_width_def
  by auto
qed

lemma remdups_set_eq:
 assumes ‹set xs = set ys›
 shows ‹set (remdups xs) = set (remdups ys)›
 using assms by simp


lemma remdups_lower_hd:
 assumes xs ≠ [] and sorted_wrt_lower xs
 shows (lower ∘ hd) (remdups xs) = (lower ∘ hd) xs
 using assms proof(induction xs rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:interval_sort_lower_width_def)
 next
  case (cons x xs)
  then show ?case
   by (simp, metis cmp_lower_width_def list.collapse order.asym set_ConsD sorted_wrt.simps(2)
       sorted_wrt_lower_def sorted_wrt_lower_unroll)
 qed
```

8.1.4 Various Notions of Validity of Sorted Lists of Intervals

Validity Tests

definition ‹*valid_mInterval_ovl is* = (*sorted_wrt_lower is* ∧ *distinct is* ∧ *is* ≠ [])›

The predicate *valid_mInterval_ovl* requires that a list of intervals is distinct and sorted with respect to the lower bound of each interval.

definition *valid_mInterval_adj* :: *'a::minus_mono interval list* ⇒ *bool*
where ‹*valid_mInterval_adj is* = (*non_overlapping_sorted is* ∧ *distinct is* ∧ *is* ≠ [])›

The predicate *valid_mInterval_adj* is strictly stronger than *valid_mInterval_ovl*:

lemma *valid_adj_imp_ovl*: ‹*valid_mInterval_adj x* ⟹ *valid_mInterval_ovl x*›
by (*simp add*: *non_overlapping_implies_sorted_wrt_lower valid_mInterval_adj_def valid_mInterval_ovl_def*)

Informally,*valid_mInterval_ovl* further limits the list of intervals to be non-overlapping. Note that adjacent intervals (i.e., intervals that share the same bounds) are allowed. For example:

lemma *valid_mInterval_adj* [*Interval*(*1::int*,*2*), *Interval*(*2*,*3*)]
apply(*simp add*: *valid_mInterval_adj_def non_overlapping_sorted_def cmp_non_overlapping_def*)
by (*smt* (*verit*) *lower_bounds*)

definition ‹*valid_mInterval_non_ovl is* = (*valid_mInterval_ovl is* ∧ *non_adjacent_sorted_wrt_lower is*)›

Informally,*valid_mInterval_non_ovl* further limits the list of intervals to also forbid adjacent intervals (i.e., intervals that share the same bounds) are allowed. It is strictly stronger than the other two predicates:

lemma *valid_non_ovl_imp_ovl*: ‹*valid_mInterval_non_ovl x* ⟹ *valid_mInterval_ovl x*›
using *valid_mInterval_non_ovl_def* **by** *blast*
lemma *valid_non_ovl_imp_adj*: ‹*valid_mInterval_non_ovl x* ⟹ *valid_mInterval_adj x*›
by (*simp add*: *non_adjacent_implies_non_overlapping valid_mInterval_adj_def valid_mInterval_non_ovl_def*
valid_mInterval_ovl_def)

lemma *valid_mInterval_non_ovl_sorted*: *valid_mInterval_non_ovl xs* ⟹ *sorted_wrt_lower xs*
by (*metis* (*no_types*, *lifting*) *valid_mInterval_non_ovl_def valid_mInterval_ovl_def*)

lemma *valid_mInterval_non_ovl_unroll*:
‹*ys* ≠ [] ⟹ *valid_mInterval_non_ovl* (*y* # *ys*) ⟹ *valid_mInterval_non_ovl ys*›
unfolding *valid_mInterval_non_ovl_def valid_mInterval_ovl_def non_adjacent_sorted_wrt_lower_def*
by(*auto simp add*: *sorted_wrt_lower_tail*)

lemma *valid_mInterval_non_ovl_eq1*:
assumes ‹*valid_mInterval_non_ovl xs*›
and ‹*valid_mInterval_non_ovl ys*›
and ‹*set xs* = *set ys*›
shows ‹*xs* = *ys*›
proof(*insert assms*, *induction xs ys rule*:*list_induct2'*)
case *1*
then show *?case*
by(*simp*)
next
case (*2 x xs*)
then show *?case*
by *simp*
next

```
  case (3 y ys)
  then show ?case
    by simp
next
  case (4 x xs y ys)
  then show ?case
  proof(cases xs)
    case Nil note xsNil = this
    then show ?thesis
    proof(cases ys)
      case Nil
      then show ?thesis
        using xsNil 4 by simp
    next
      case (Cons a list)
      then show ?thesis
        using xsNil 4
        by (simp add: valid_mInterval_non_ovl_def valid_mInterval_ovl_def)
    qed
  next
    case (Cons x' xs') note xCons = this
    then show ?thesis
    proof(cases ys)
      case Nil
      then show ?thesis
        using xCons 4
        by (simp add: valid_mInterval_non_ovl_def valid_mInterval_ovl_def)
    next
      case (Cons y' ys')
      then show ?thesis
        using xCons 4 valid_mInterval_non_ovl_unroll[of xs x] valid_mInterval_non_ovl_unroll[of ys y]
        unfolding cmp_non_adjacent_def non_adjacent_sorted_wrt_lower_def valid_mInterval_non_ovl_def
            valid_mInterval_ovl_def sorted_wrt_lower_def cmp_lower_width_def
        by (metis (no_types, lifting) 4.prems(1) 4.prems(2) cmp_non_adjacent_lower insert_eq_iff
            list.distinct(1) list.set_intros(1) list.simps(15) non_adjacent_sorted_wrt_lower_def
            order.asym set_ConsD sorted_wrt.simps(2) valid_mInterval_non_ovl_def)
    qed
  qed
qed
```

Constructors

Overlapping Intervals definition ‹ *mk_mInterval_ovl = remdups o interval_sort_lower_width*›

```
lemma mk_mInterval_ovl_non_empty: ‹is ≠ [] ⟹ (mk_mInterval_ovl is) ≠ []›
  unfolding mk_mInterval_ovl_def o_def interval_sort_lower_width_def
proof(induction is rule:list_nonempty_induct)
  case (single x)
  then show ?case by simp
next
  case (cons x xs)
  then show ?case
    apply(simp)
    by (metis (no_types, lifting) interval_insert_sort_lower_width.simps(2) list.collapse list.simps(3))
```

```
qed

lemma mk_mInterval_ovl_empty[simp]:
 mk_mInterval_ovl [] = []
 by(simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)

lemma mk_mInterval_ovl_distinct: ‹distinct (mk_mInterval_ovl is)›
 unfolding mk_mInterval_ovl_def by simp

lemma sorted_wrt_lower_remdups:
sorted_wrt_lower xs ⟹ sorted_wrt_lower (remdups xs)
 proof(induction xs)
  case Nil
  then show ?case
   by(simp)
 next
  case (Cons a xs)
  then show ?case
   unfolding sorted_wrt_lower_def
   by(auto)
 qed

lemma interval_sort_lower_width_swap_remdups:
 ‹remdups (interval_sort_lower_width xs) = interval_sort_lower_width (remdups xs)›
 for xs::'a::{minus_mono, linordered_field} interval list
proof(induction xs)
 case Nil
 then show ?case
  by (metis interval_sort_lower_width_remdups remdups.simps(1))
next
 case (Cons a xs)
 then show ?case
     by (metis (mono_tags, lifting) distinct_remdups interval_sort_lower_width_remdups
          interval_sort_lower_width_set_eq interval_sort_lower_width_sorted set_remdups
          sorted_wrt_lower_distinct_lists_eq sorted_wrt_lower_remdups)
qed


lemma mk_mInterval_ovl_sorted: ‹sorted_wrt_lower (mk_mInterval_ovl is)›
 proof(induction is)
  case Nil
  then show ?case
   unfolding mk_mInterval_ovl_def sorted_wrt_lower_def interval_sort_lower_width_def
   by simp
 next
  case (Cons a is)
  then show ?case
   using interval_sort_lower_width_sorted sorted_wrt_lower_remdups
   unfolding mk_mInterval_ovl_def sorted_wrt_lower_def interval_sort_lower_width_def o_def
   by blast
 qed

theorem mk_mInterval_ovl_valid: ‹is ≠ [] ⟹ valid_mInterval_ovl (mk_mInterval_ovl is)›
 unfolding valid_mInterval_ovl_def
```

```
 by (simp add: mk_mInterval_ovl_distinct mk_mInterval_ovl_non_empty mk_mInterval_ovl_sorted)

lemma valid_mk_mInterval_ovl_id:
 assumes ‹valid_mInterval_ovl xs›
 shows ‹mk_mInterval_ovl xs = xs›
proof(insert assms, induction xs)
 case Nil
 then show ?case
  by(simp add: valid_mInterval_ovl_def)
next
 case (Cons a xs)
 then show ?case
  apply(simp add: mk_mInterval_ovl_def interval_sort_lower_width_def valid_mInterval_ovl_def
            sorted_wrt_lower_def cmp_lower_width_def)
  by (metis (no_types, opaque_lifting) cmp_lower_width_def distinct_singleton foldr_isort_elements
        insertI1 interval_insert_sort_lower_width.simps(1) interval_insert_sort_lower_width.simps(2)
        interval_sort_lower_width_def list.distinct(1) list.simps(15) min_list.cases remdups.simps(2)
        remdups_id_iff_distinct)
qed

lemma mk_mInterval_ovl_eq:
 assumes ‹set xs = set (ys::'a::{minus_mono, linordered_field} interval list)›
 shows ‹mk_mInterval_ovl xs = mk_mInterval_ovl ys ›
 by (metis (no_types, lifting) assms comp_def interval_sort_lower_width_set_eq mk_mInterval_ovl_def
       mk_mInterval_ovl_def mk_mInterval_ovl_distinct mk_mInterval_ovl_sorted set_remdups
       sorted_wrt_lower_distinct_lists_eq)
```

Adjacent Intervals **definition** mk_mInterval_adj :: ('a::{minus, linorder,linorder}) interval list ⇒ 'a interval list

```
 where ‹mk_mInterval_adj = remdups o merge_overlapping_intervals_sorted_wrt_lower o mk_mInterval_ovl›

lemma mk_mInterval_adj_non_overlapping_sorted: ‹non_overlapping_sorted (mk_mInterval_adj
(is::'a::{minus_mono} interval list))›
 using interval_sort_lower_width_sorted sorted_wrt_lower_remdups mk_mInterval_ovl_sorted[of is]
    merge_overlapping_intervals_sorted_wrt_lower_sorted_lower[of (mk_mInterval_ovl is)]
    merge_overlapping_intervals_sorted_sorted_non_non_overlapping non_overlapping_sorted_remdups
 unfolding mk_mInterval_adj_def o_def
 by blast

lemma mk_mInterval_adj_sorted: ‹sorted_wrt_lower (mk_mInterval_adj (is::'a::minus_mono interval list))›
 using interval_sort_lower_width_sorted sorted_wrt_lower_remdups mk_mInterval_ovl_sorted[of is]
    merge_overlapping_intervals_sorted_wrt_lower_sorted_lower[of (mk_mInterval_ovl is)]
 unfolding mk_mInterval_adj_def o_def
 by auto

lemma mk_mInterval_adj_non_empty: ‹is ≠ [] ⟹ (mk_mInterval_adj is) ≠ []›
 unfolding mk_mInterval_adj_def o_def
 using mk_mInterval_ovl_non_empty merge_overlapping_intervals_sorted_wrt_lower_non_nil
 by (metis remdups_eq_nil_right_iff)

lemma mk_mInterval_adj_empty[simp]:
 mk_mInterval_adj [] = []
 by(simp add: mk_mInterval_adj_def)

lemma mk_mInterval_adj_distinct: ‹distinct (mk_mInterval_adj is)›
```

unfolding *mk_mInterval_adj_def* **by** *simp*

theorem *mk_mInterval_adj_valid*: ‹*is* ≠ [] ⟹ *valid_mInterval_adj* (*mk_mInterval_adj is*)›
unfolding *valid_mInterval_adj_def*
using *mk_mInterval_adj_non_overlapping_sorted mk_mInterval_adj_distinct mk_mInterval_adj_non_empty*
by *auto*

lemma *valid_mk_mInterval_adj_id*:
assumes ‹*valid_mInterval_adj xs*›
shows ‹*mk_mInterval_adj xs* = *xs*›
proof(*insert assms*, *induction xs*)
case *Nil*
then show *?case*
by(*simp add*: *valid_mInterval_adj_def*)
next
case (*Cons a xs*)
then show *?case*
using *valid_mk_mInterval_ovl_id*[*of a*#*xs*, *simplified valid_adj_imp_ovl*[*of a*#*xs*, *simplified Cons*, *simplified*], *simplified*]
valid_mk_mInterval_ovl_id[*of xs*, *simplified valid_adj_imp_ovl*[*of xs*, *simplified Cons*, *simplified*], *simplified*]
unfolding *valid_mInterval_adj_def mk_mInterval_adj_def o_def*
apply(*simp add*: *mk_mInterval_ovl_def interval_sort_lower_width_def valid_mInterval_ovl_def*
sorted_wrt_lower_def cmp_lower_width_def non_overlapping_sorted_def cmp_non_overlapping_def)
by (*smt* (*verit*) *One_nat_def add.commute foldr_interval_insert_sort_lower_width_distinct foldr_isort_elements*
interval_insert_sort_lower_width.simps(2) *interval_insert_sort_lower_width_length length_remdups_card_conv*
length_remdups_eq list.sel(1) *list.sel*(3) *list.set_intros*(1) *list.size*(4)
merge_overlapping_intervals_sorted_wrt_lower.elims
merge_overlapping_intervals_sorted_wrt_lower.simps(2) *remdups.simps*(2) *remdups_id_iff_distinct set_remdups*)

qed

lemma *mk_mInterval_adj_eq*:
assumes ‹*set xs* = *set* (*ys*::'*a*::{*minus_mono*, *linordered_field*} *interval list*)›
shows ‹*mk_mInterval_adj xs* = *mk_mInterval_adj ys* ›
by (*metis* (*no_types*, *lifting*) *assms comp_def interval_sort_lower_width_set_eq mk_mInterval_adj_def*
mk_mInterval_ovl_def mk_mInterval_ovl_distinct mk_mInterval_ovl_sorted set_remdups
sorted_wrt_lower_distinct_lists_eq)

lemma *mk_mInterval_ovl_id*:
mk_mInterval_ovl (*mk_mInterval_ovl x*) = *mk_mInterval_ovl x*
unfolding *mk_mInterval_ovl_def o_def*
by (*metis comp_def mk_mInterval_ovl_def mk_mInterval_ovl_empty mk_mInterval_ovl_valid valid_mk_mInterval_ovl_id*)

value *valid_mInterval_adj* (*mk_mInterval_adj* ([*Ivl* (1::*int*) 2, *Ivl 1 3*, *Ivl 1 1*]))
value *valid_mInterval_adj* (*mk_mInterval_adj* ([*Ivl* (1::*int*) 1, *Ivl 1 2*, *Ivl 2 3*]))

Non-Overlapping Intervals definition ‹*mk_mInterval_non_ovl* = *remdups o merge_adjacent_intervals_sorted_wrt_lower o mk_mInterval_ovl*›

lemma *mk_mInterval_non_ovl_distinct*:
distinct (*mk_mInterval_non_ovl is*)
by (*simp add*: *mk_mInterval_non_ovl_def*)

```
lemma mk_mInterval_non_ovl_non_empty:
 is ≠ [] ⟹ mk_mInterval_non_ovl is ≠ []
        by (simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil mk_mInterval_ovl_non_empty
mk_mInterval_non_ovl_def)

lemma mk_mInterval_non_ovl_empty[simp]:
 mk_mInterval_non_ovl [] = []
 by (simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil mk_mInterval_non_ovl_def)

lemma mk_mInterval_non_ovl_eq:
 assumes ‹set xs = set (ys::'a::{minus_mono, linordered_field} interval list)›
 shows ‹mk_mInterval_non_ovl xs = mk_mInterval_non_ovl ys ›
 unfolding mk_mInterval_non_ovl_def o_def
 by (metis (no_types, opaque_lifting) assms comp_apply foldr_isort_elements
       interval_sort_lower_width_def mk_mInterval_ovl_def mk_mInterval_ovl_distinct
       mk_mInterval_ovl_sorted set_remdups sorted_wrt_lower_distinct_lists_eq)

lemma sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower:
 sorted_wrt_lower xs ⟹ sorted_wrt_lower (merge_adjacent_intervals_sorted_wrt_lower xs)
proof(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case by simp
next
 case (2 x)
 then show ?case by simp
next
 case (3 x y ys)
 then show ?case
 proof(cases upper x < lower y)
  case True
  then show ?thesis
   using 3
   by (smt (verit, del_insts) leD list.discI list.sel(1) lower_le_upper
       merge_adjacent_intervals_sorted_wrt_lower.simps(3)
       merge_adjacent_intervals_sorted_wrt_lower_non_nil
       merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd
       sorted_wrt_lower_unroll)
 next
  case False
  then show ?thesis
   using 3
   by (smt (verit, del_insts) list.distinct(1) list.sel(1) max.absorb3 max_absorb2
       merge_adjacent_intervals_sorted_wrt_lower.simps(3) mk_interval_id mk_interval_lower
       not_le_imp_less sorted_wrt_lower_tail' sorted_wrt_lower_unroll)
 qed
qed



lemma mk_mInterval_non_ovl_sorted_wrt_lower:
 is ≠ [] ⟹ sorted_wrt_lower (mk_mInterval_non_ovl (is::int interval list))
 unfolding mk_mInterval_non_ovl_def o_def
          using mk_mInterval_ovl_sorted sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower
```

```
sorted_wrt_lower_remdups by blast

lemma valid_ovl_mkInterval_non_ovl: is ≠ [] ⟹ valid_mInterval_ovl (mk_mInterval_non_ovl is)
 unfolding valid_mInterval_ovl_def
 by (simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil mk_mInterval_non_ovl_def
                                                                  mk_mInterval_ovl_non_empty     mk_mInterval_ovl_sorted
sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower
          sorted_wrt_lower_remdups)

lemma non_adj_sorted_mkInterval_non_ovl:
sorted_wrt_lower xs
⟹ non_adjacent_sorted_wrt_lower (merge_adjacent_intervals_sorted_wrt_lower xs)
proof(induction xs rule:merge_adjacent_intervals_sorted_wrt_lower.induct)
 case 1
 then show ?case
  unfolding non_adjacent_sorted_wrt_lower_def by(simp)
next
 case (2 x)
 then show ?case
  unfolding non_adjacent_sorted_wrt_lower_def by(simp)
next
 case (3 x y ys)
 then show ?case
 proof(cases upper x < lower y)
  case True
  then show ?thesis
   apply(simp)
   apply(subst non_adjacent_sorted_wrt_lower_unroll)
   subgoal by(simp add: merge_adjacent_intervals_sorted_wrt_lower_non_nil)
   subgoal using 3 True
    by (metis list.sel(1) merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd sorted_wrt_lower_tail)
   done
 next
  case False
  then show ?thesis
  proof(cases ys)
   case Nil
   then show ?thesis
    unfolding non_adjacent_sorted_wrt_lower_def cmp_non_adjacent_def
    by(simp add: max_def 3 False split:if_splits)
  next
   case (Cons a list)
   then have a: sorted_wrt_lower (mk_interval (lower x, max (upper y) (upper x)) # ys)
   proof(cases upper y ≤ upper x)
    case True note * = this
    then show ?thesis
    proof (cases upper y = upper x)
     case True
     then show ?thesis
      using False True apply(simp)
      using 3 * False sorted_wrt_lower_unroll[of y#ys x, simplified]
          sorted_wrt_lower_tail[of y ys]
          sorted_wrt_lower_tail'[of x y ys]
      by auto
```

```
      next
       case False
       then show ?thesis
        using False True apply(simp)
        using 3 * False sorted_wrt_lower_unroll[of y#ys x, simplified]
            sorted_wrt_lower_tail[of y ys]
            sorted_wrt_lower_tail'[of x y ys]
        by auto
      qed
     next
      case False
      then show ?thesis
        using Cons False  apply(simp)
        using 3 False sorted_wrt_lower_unroll[of y#ys x, simplified]
            sorted_wrt_lower_tail[of y ys]
            sorted_wrt_lower_tail'[of x y ys]
        apply(simp)
        by (smt (verit, ccfv_threshold) less_le_not_le list.distinct(1) list.sel(1) max.absorb1
             max.assoc max.cobounded2 mk_interval_id mk_interval_lower sorted_wrt_lower_unroll)
    qed
    then show ?thesis
     using False 3[simplified a False , simplified]
     by simp
   qed
  qed
qed


lemma bin_op_mInterval_commute:
  assumes op_commute: ‹⋀ x y. op x y = op y x›
    shows ‹mk_mInterval_non_ovl (bin_op_interval_list op x y) = mk_mInterval_non_ovl (bin_op_interval_list op y
(x::'a::{minus_mono, linordered_field} interval list))›
  using bin_op_interval_list_commute  mk_mInterval_non_ovl_eq
  by (metis op_commute)

lemma iList_plus_mInterval_ovl_assoc:
  ‹mk_mInterval_ovl (iList_plus x (iList_plus y z)) = mk_mInterval_ovl (iList_plus (iList_plus x (y::'a::{minus_mono,
linordered_field} interval list)) z)›
  by (meson iList_plus_assoc mk_mInterval_ovl_eq)

lemma iList_plus_mInterval_adj_commute:
  ‹mk_mInterval_adj (iList_plus x y) = mk_mInterval_adj (iList_plus y (x::'a::{minus_mono, linordered_field} interval
list))›
  by (meson iList_plus_commute mk_mInterval_adj_eq)


lemma iList_plus_mInterval_non_ovl_assoc:
     ‹mk_mInterval_non_ovl (iList_plus x (iList_plus y z)) = mk_mInterval_non_ovl (iList_plus (iList_plus x
(y::'a::{minus_mono, linordered_field} interval list)) z)›
  by (meson iList_plus_assoc mk_mInterval_non_ovl_eq)

lemma iList_plus_mInterval_non_ovl_commute:
  ‹mk_mInterval_non_ovl (iList_plus x y) = mk_mInterval_non_ovl (iList_plus y (x::'a::{minus_mono, linordered_field}
interval list))›
```

by (*meson iList_plus_commute mk_mInterval_non_ovl_eq*)

lemma *iList_plus_mInterval_adj_assoc*:
‹*mk_mInterval_non_ovl* (*iList_plus x* (*iList_plus y z*)) = *mk_mInterval_non_ovl* (*iList_plus* (*iList_plus x* (*y*::'*a*::{*minus_mono*, *linordered_field*} *interval list*)) *z*)›
by (*meson iList_plus_assoc mk_mInterval_non_ovl_eq*)

lemma *sorted_wrt_lower_mk_mInterval_non_ovl*: *sorted_wrt_lower* (*mk_mInterval_non_ovl xs*)
unfolding *mk_mInterval_non_ovl_def*
by (*simp add*: *mk_mInterval_ovl_sorted sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower*
sorted_wrt_lower_remdups)

theorem *mk_mInterval_non_ovl_valid*: ‹*sorted_wrt_lower is* ⟹ *is* ≠ [] ⟹ *valid_mInterval_non_ovl* (*mk_mInterval_non_ovl is*)›
using *non_adj_sorted_mkInterval_non_ovl*[*of is*] *valid_ovl_mkInterval_non_ovl*[*of is*]
unfolding *valid_mInterval_non_ovl_def mk_mInterval_non_ovl_def o_def*
by (*simp add*: *distinct_remdups_id mk_mInterval_ovl_sorted non_adj_sorted_mkInterval_non_ovl non_adjacent_implies_distinct*)

lemma *valid_mk_mInterval_non_ovl_id*:
assumes ‹*valid_mInterval_non_ovl xs*›
shows ‹*mk_mInterval_non_ovl xs* = *xs*›
proof(*insert assms*, *induction xs*)
case *Nil*
then show *?case*
by(*simp add*: *valid_mInterval_non_ovl_def valid_mInterval_ovl_def*)
next
case (*Cons a xs*)
then show *?case*
using *valid_mk_mInterval_ovl_id*[*of a*#*xs*, *simplified valid_non_ovl_imp_ovl*[*of a*#*xs*, *simplified Cons*, *simplified*], *simplified*]
valid_mk_mInterval_ovl_id[*of xs*, *simplified valid_non_ovl_imp_ovl*[*of xs*, *simplified Cons*, *simplified*], *simplified*]
unfolding *valid_mInterval_non_ovl_def mk_mInterval_non_ovl_def o_def*
apply(*simp*)
apply(*simp add*: *mk_mInterval_ovl_def interval_sort_lower_width_def valid_mInterval_ovl_def*
sorted_wrt_lower_def cmp_lower_width_def non_overlapping_sorted_def cmp_non_overlapping_def)
by (*smt* (*verit*) *Cons.prems list.sel*(1) *list.sel*(3) *merge_adjacent_intervals_sorted_wrt_lower.elims*
merge_adjacent_intervals_sorted_wrt_lower.simps(2) *non_adj_sorted_mkInterval_non_ovl non_adjacent_implies_distinct*
non_adjacent_sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_unroll remdups_id_iff_distinct sorted_wrt.simps(2)
valid_mInterval_non_ovl_sorted)
qed

lemma *mk_mInterval_non_ovl_single*:
mk_mInterval_non_ovl [*x*] = [*x*]
by (*simp add*: *non_adjacent_implies_sorted_wrt_lower non_adjacent_sorted_wrt_lower_def*
valid_mInterval_non_ovl_def valid_mInterval_ovl_def valid_mk_mInterval_non_ovl_id)

lemma *mk_mInterval_non_ovl_id*:
mk_mInterval_non_ovl (*mk_mInterval_non_ovl x*) = *mk_mInterval_non_ovl x*
unfolding *mk_mInterval_non_ovl_def o_def*
by (*metis* (*no_types*, *opaque_lifting*) *comp_apply distinct_remdups_id mk_mInterval_non_ovl_def mk_mInterval_non_ovl_empty*

mk_mInterval_ovl_sorted non_adj_sorted_mkInterval_non_ovl non_adjacent_implies_distinct valid_mInterval_non_ovl_def valid_mk_mInterval_non_ovl_id valid_ovl_mkInterval_non_ovl)

value *valid_mInterval_non_ovl* (*mk_mInterval_non_ovl* ([*Ivl* (1::*int*) 2, *Ivl 1 3*, *Ivl 1 1*]))
value *valid_mInterval_non_ovl* (*mk_mInterval_non_ovl* ([*Ivl* (1::*int*) 1, *Ivl 1 2*, *Ivl 2 3*]))

value‹*mk_mInterval_ovl* [*mk_interval* ((1::*int*), 4), *mk_interval* (0,2), *mk_interval* (3,5), *mk_interval* (5,7), *mk_interval* (7,7), *mk_interval* (8,8)]›
value‹*mk_mInterval_adj* [*mk_interval* ((1::*int*), 4), *mk_interval* (0,2), *mk_interval* (3,5), *mk_interval* (5,7), *mk_interval* (7,7), *mk_interval* (8,8)]›
value‹*mk_mInterval_non_ovl* [*mk_interval* ((1::*int*), 4), *mk_interval* (0,2), *mk_interval* (3,5), *mk_interval* (5,7), *mk_interval* (7,7), *mk_interval* (8,8)]›

8.1.5 Union over a List of Intervals

definition ‹*set_of_interval_list XS* = *foldr* (λ*x a*. *set_of x* ∪ *a*) *XS* {}›

lemma *set_of_interval_list_nonempty*:
assumes *non_empty*: ‹*XS* ≠ ([]::*real interval list*)›
shows ‹(*set_of_interval_list XS*) ≠ {}›
using *assms* **proof**(*induction XS rule*:*list_nonempty_induct*)
case (*single x*)
then show *?case* **by**(*simp add*:*set_of_interval_list_def set_of_eq*)
next
case (*cons x xs*)
then show *?case*
by(*simp add*:*set_of_interval_list_def*)
qed

lemma *set_of_interval_list_bdd_below*:
assumes *non_empty*: ‹*XS* ≠ ([]::*real interval list*)›
shows ‹*bdd_below* (*set_of_interval_list XS*)›
using *assms* **proof**(*induction XS rule*:*list_nonempty_induct*)
case (*single x*)
then show *?case* **by**(*simp add*:*set_of_interval_list_def set_of_eq*)
next
case (*cons x xs*)
then show *?case*
by(*simp add*:*set_of_eq set_of_interval_list_def sorted_wrt_lower_def*)
qed

lemma *set_of_interval_list_bdd_above*:
assumes *non_empty*: ‹*XS* ≠ ([]::*real interval list*)›
shows ‹*bdd_above* (*set_of_interval_list XS*)›
using *assms* **proof**(*induction XS rule*:*list_nonempty_induct*)
case (*single x*)
then show *?case* **by**(*simp add*:*set_of_interval_list_def set_of_eq*)
next
case (*cons x xs*)
then show *?case*
by(*simp add*:*set_of_eq set_of_interval_list_def contiguous_def*)

```
qed


lemma inf_set_of_interval_list_lower:
 assumes non_empty: ‹XS ≠ ([]::real interval list)›
 and     sorted: ‹sorted_wrt_lower XS›
shows ‹Inf (set_of_interval_list XS) = lower (hd XS)›
using assms proof(induction XS rule:list_nonempty_induct)
 case (single x)
 then show ?case by(simp add:set_of_interval_list_def set_of_eq)
next
 case (cons x xs)
 then show ?case
  apply(simp add: set_of_interval_list_def)
  apply(subst cInf_union_distrib)
  subgoal by simp
  subgoal by(simp add:set_of_eq)
  subgoal by(fold set_of_interval_list_def , simp add: set_of_interval_list_nonempty)
  subgoal
   apply (fold set_of_interval_list_def)
   using sorted_wrt_lower_unroll[of xs x, simplified cons, simplified]
       set_of_interval_list_bdd_below
   by(simp)
  subgoal
   using sorted_wrt_lower_unroll[of xs x, simplified cons, simplified]
   by (metis inf.orderE interval_bounds_real(2) lower_le_upper nle_le order_less_le set_of_eq)
  done
qed

lemma contiguous_sorted_wrt_upper:
 assumes contiguous (xs:: real interval list)
 shows sorted_wrt_upper xs
 unfolding sorted_wrt_upper_def
 using assms unfolding contiguous_def lower_le_upper
 by (metis assms contiguous_non_overlapping non_overlapping_implies_sorted_wrt_upper sorted_wrt_upper_def)

lemma contiguous_sorted_wrt_lower:
 assumes contiguous (XS:: real interval list)
 shows sorted_wrt_lower XS
 unfolding sorted_wrt_lower_def
 using assms contiguous_non_overlapping[of XS] non_overlapping_implies_sorted_wrt_lower
  sorted_wrt_lower_def by metis

lemma max_last_sorted_wrt_upper:
 assumes XS ≠ [] sorted_wrt_upper (XS:: 'a::{linorder} interval list)
 shows Max (set(map upper XS)) = upper (last XS)
 using assms
proof (induction XS rule: list_nonempty_induct)
 case (single x)
 then show ?case by simp
next
 case (cons x xs)
 then have a0: sorted_wrt (λx y. upper x ≤ upper y) (x # xs) by (simp add: sorted_wrt_upper_def)
 then have a1: Max (set (map upper (x # xs))) = upper (last (x # xs)) using cons unfolding sorted_wrt_upper_def by
```

```
simp
 then show ?case using a0 a1 cons unfolding sorted_wrt_upper_def by simp
qed

lemma min_hd_sorted_wrt_lower:
 assumes XS ≠ [] sorted_wrt_lower (XS:: 'a::{linorder,minus,preorder} interval list)
 shows Min (set(map lower XS)) = lower (hd XS)
 using assms
proof (induction XS rule: list_nonempty_induct)
 case (single x)
 then show ?case by simp
next
 case (cons x xs)
 then have a0: sorted_wrt (λ x y. if lower x = lower y then width x ≤ width y else lower x < lower y) (x # xs)
  unfolding sorted_wrt_lower_def cmp_lower_width_def by simp
  then have a1: Min (set(map lower (x # xs))) = lower (hd (x # xs)) using cons unfolding sorted_wrt_lower_def
cmp_lower_width_def
     by (smt (verit, del_insts) arg_min_list.simps(2) cons.IH cons.prems f_arg_min_list_f image_set list.distinct(1)
list.exhaust_sel list.sel(1) order_less_imp_le sorted_wrt_lower_unroll)
 then show ?case using a0 a1 cons unfolding sorted_wrt_upper_def by simp
qed

lemma lower_isort:
 assumes ‹xs ≠ []›and ‹(lower o hd) xs = Min (lower ' (set xs))›
 shows ‹(lower ∘ hd) (interval_sort_lower_width xs) = (lower ∘ hd) xs›
 proof(insert assms, induction xs rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:interval_sort_lower_width_def)
 next
  case (cons x xs)
  then show ?case
   apply(simp add:interval_sort_lower_width_def)
   by (metis (no_types, opaque_lifting) comp_eq_dest_lhs cons.prems foldr.simps(2) foldr_isort_elements
        interval_insert_sort_lower_width_nonempty interval_sort_lower_width_def
        interval_sort_lower_width_sorted list.sel(1) list.set_map min_hd_sorted_wrt_lower)
 qed

lemma min_sort:
 Min (set (map lower (foldr interval_insert_sort_lower_width xs []))) = Min (set (map lower xs))
 using foldr_isort_elements
 by (metis list.set_map)

lemma mk_mInterval_lower:
 assumes xs ≠ []
 shows Min (set (map lower (mk_mInterval_non_ovl xs))) = Min (set (map lower xs))
proof (induction xs)
 case Nil
 then show ?case
  unfolding mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def
  by simp
next
 case (Cons a xs)
 have a: Min (set (map lower (foldr interval_insert_sort_lower_width xs []))) = Min (set (map lower xs))
  using foldr_isort_elements
```

```
  by (metis list.set_map)
 then show ?case
  unfolding mk_mInterval_non_ovl_def mk_mInterval_ovl_def o_def
  using Cons
  by (smt (verit, ccfv_SIG) foldr_isort_elements interval_sort_lower_width_def interval_sort_lower_width_sorted
       list.discI list.set_map merge_adjacent_intervals_sorted_wrt_lower_non_nil set_empty
        merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd  min_hd_sorted_wrt_lower
        set_remdups sorted_wrt_lower_merge_adjacent_intervals_sorted_wrt_lower sorted_wrt_lower_remdups)
qed

lemma sup_set_of_interval_list_upper:
 assumes non_empty: ‹XS ≠ ([]::real interval list)›
 and     sorted: ‹sorted_wrt_upper XS›
shows ‹Sup (set_of_interval_list XS) = upper (last XS)›
using assms proof(induction XS rule:list_nonempty_induct)
 case (single x)
 then show ?case by(simp add:set_of_interval_list_def set_of_eq)
next
 case (cons x xs)
 have Max (set (map upper (x # xs))) = upper (last (x # xs)) using max_last_sorted_wrt_upper assms cons by blast
 then show ?case
  apply(simp add: set_of_interval_list_def)
  apply(subst cSup_union_distrib)
    apply(fold set_of_interval_list_def)
  subgoal by simp
  subgoal using bdd_above_set_of by metis
  subgoal using cons.hyps set_of_interval_list_nonempty by presburger
  subgoal using cons.hyps set_of_interval_list_bdd_above by presburger
  subgoal
  proof −
   assume a1: Max (insert (upper x) (upper ‘ set xs)) = upper (if xs = [] then x else last xs)
   have f2: ∀ i. Sup (set_of i) = (upper i::real)
    by (simp add: set_of_eq)
   have [] = xs ∨ sorted_wrt_upper xs
    by (metis cons.prems sorted_wrt_upper_unroll)
   then show ?thesis
    using f2 a1 cons.IH sup_real_def by force
  qed
  done
qed

lemma compact_set_of_interval_list:
  ‹compact (set_of_interval_list (XS::('a::{preorder,ordered_euclidean_space,topological_space} interval list)))›
proof(induction XS)
 case Nil
 then show ?case by(simp add:set_of_interval_list_def)
next
 case (Cons a XS)
 then show ?case
  by(simp add:set_of_interval_list_def, subst compact_Un, simp_all add: compact_set_of Cons set_of_eq)
qed

lemma lower_le_upper_aux: ‹xs ≠ [] ⟹ non_overlapping_sorted xs ⟹ lower (hd xs) ≤ upper (last xs)›
proof(induction xs rule:induct_list012)
```

```
 case 1
 then show ?case by(simp)
next
 case (2 x)
 then show ?case by(simp)
next
 case (3 x y zs)
 then show ?case
  proof(cases zs)
   case Nil
   then show ?thesis
    using 3 dual_order.trans
    unfolding non_overlapping_sorted_def cmp_non_overlapping_def
    by (simp, smt (verit, best) dual_order.trans lower_le_upper)
  next
   case (Cons a list)
   then show ?thesis
    using 3 dual_order.trans
    unfolding non_overlapping_sorted_def cmp_non_overlapping_def
    by (simp, smt (verit, best) dual_order.trans lower_le_upper)
  qed
 qed

lemma contiguous_lower_le_upper:
 assumes non_empty: ‹XS ≠ ([]::real interval list)›
 and     contiguous: ‹contiguous XS›
shows ‹(lower (hd XS)) ≤ (upper (last XS))›
 by (simp add: contiguous contiguous_non_overlapping lower_le_upper_aux non_empty)

lemma diameter_Sup_Inf:
 assumes ‹compact X› ‹X ≠ {}›
 shows ‹diameter X ≤ Sup X − Inf X›
 using assms diameter_compact_attained[of X]
 by (metis bounded_imp_bdd_above bounded_imp_bdd_below compact_imp_bounded sup_inf_dist_bounded)


lemma diameter_width_compact:
 assumes ‹compact X› ‹bdd_below X› ‹bdd_above X› ‹X ≠ {}›
 shows ‹diameter X = Sup X − Inf X›
 using assms diameter_Sup_Inf[of X, simplified assms, simplified]
     closed_contains_Inf[of X, simplified assms, simplified]
     closed_contains_Sup[of X, simplified assms, simplified]
 by (smt (verit, best) compact_imp_bounded compact_imp_closed diameter_bounded_bound dist_real_def)

lemma diameter_contiguous:
 assumes non_empty: ‹XS ≠ ([]::real interval list)›
 and     contiguous: ‹contiguous XS›
shows ‹diameter (set_of_interval_list XS) = dist (lower (hd XS)) (upper (last XS))›
 apply(subst diameter_width_compact)
 subgoal
  by (simp add: compact_set_of_interval_list contiguous non_empty)
 subgoal
  by (simp add: ‹compact (set_of_interval_list XS)› bounded_imp_bdd_below compact_imp_bounded)
 subgoal
```

```
  by (simp add: contiguous set_of_interval_list_bdd_above non_empty)
 subgoal
  by (simp add: set_of_interval_list_nonempty non_empty)
 subgoal
  using sup_set_of_interval_list_upper[of XS, simplified assms, simplified]
      inf_set_of_interval_list_lower[of XS, simplified assms, simplified]
      contiguous_lower_le_upper[of XS, simplified assms, simplified]
  unfolding dist_real_def abs_real_def
                by (simp add: contiguous contiguous_non_overlapping contiguous_sorted_wrt_upper
non_overlapping_implies_sorted_wrt_lower)
 done


lemma interval_list_union_contiguous_lower:
 assumes non_empty: ‹XS ≠ []›
 and     sorted: ‹sorted_wrt_lower XS›
shows ‹lower (interval_list_union XS) = lower (hd XS)›
using assms proof(induction XS rule:interval_list_union.induct)
 case 1
 then show ?case by simp
next
 case (2 I)
 then show ?case by simp
next
 case (3 I v va)
 then show ?case
  unfolding sorted_wrt_lower_def cmp_lower_width_def
  apply (simp)
  by (metis inf.idem inf.strict_order_iff)
qed

lemma interval_list_union_contiguous_upper:
 assumes non_empty: ‹XS ≠ []›
 and     sorted: ‹sorted_wrt_upper XS›
shows ‹upper (interval_list_union XS) = upper (last XS)›
using assms proof(induction XS rule:interval_list_union.induct)
 case 1
 then show ?case by simp
next
 case (2 I)
 then show ?case by simp
next
 case (3 I v va)
 then show ?case
  unfolding sorted_wrt_upper_def cmp_lower_width_def
  apply (simp)
  by (metis last_in_set sup.absorb2)
qed


lemma interval_list_union_contiguous:
 assumes non_empty: ‹XS ≠ []›
 and     sorted_lower: ‹sorted_wrt_lower XS›
 and     sorted_upper: ‹sorted_wrt_upper XS›
```

shows ‹*interval_list_union XS* = *Interval* (*lower* (*hd XS*), *upper* (*last XS*))›
by (*metis Interval_id assms interval_list_union_contiguous_lower interval_list_union_contiguous_upper non_empty*)

lemma *contiguous_bounds_lower*:
assumes *non_empty*: ‹*XS* ≠ []›
and *contiguous*: ‹*contiguous* (*XS*::*real interval list*)›
shows *lower* (*hd XS*) = *Min* (*set* (*map lower XS*))
using *min_hd_sorted_wrt_lower*[*of XS*, *simplified assms contiguous_sorted_wrt_lower*[*of XS*, *simplified assms*]]
by *auto*[1]

lemma *contiguous_bounds_upper*:
assumes *non_empty*: ‹*XS* ≠ []›
and *contiguous*: ‹*contiguous* (*XS*::*real interval list*)›
shows *upper* (*last XS*) = *Max* (*set* (*map upper XS*))
using *max_last_sorted_wrt_upper*[*of XS*, *simplified assms contiguous_sorted_wrt_upper*[*of XS*, *simplified assms*]]
by *auto*[1]

lemma *set_of_interval_list_contiguous*:
assumes *non_empty*: ‹*XS* ≠ ([]::*real interval list*)›
and *contiguous*: ‹*contiguous XS*›
shows ‹*set_of_interval_list XS* = {*lower* (*hd XS*)..*upper* (*last XS*)}›
using *assms*
proof(*induction XS rule*:*list_nonempty_induct*)
case (*single x*)
then show *?case*
using *set_of_eq*
unfolding *contiguous_def set_of_interval_list_def* **by** *auto*[1]
next
case (*cons x xs*)
then show *?case*
using *set_of_eq*
sup_set_of_interval_list_upper[*of* (*x* # *xs*), *simplified cons contiguous_sorted_wrt_upper*[*of* (*x* # *xs*), *simplified assms*], *simplified*]
inf_set_of_interval_list_lower[*of* (*x* # *xs*), *simplified cons contiguous_sorted_wrt_lower*[*of* (*x* # *xs*), *simplified assms*], *simplified*]
unfolding *set_of_interval_list_def contiguous_def set_of_eq*
apply(*auto*)[1]
subgoal
using *cons.prems contiguous_non_overlapping lower_le_upper_aux non_overlapping_sorted_unroll*
by *fastforce*
subgoal
by (*smt* (*verit*, *ccfv_threshold*) *Suc_less_eq Suc_pred atLeastAtMost_iff length_greater_0_conv*
list.*sel*(1) *lower_le_upper neq_Nil_conv nth_Cons_0 nth_Cons_Suc*)
subgoal
using *less_diff_conv* **by** *force*
subgoal
by (*smt* (*verit*, *del_insts*) *One_nat_def Suc_eq_plus1 atLeastAtMost_iff hd_conv_nth*
length_greater_0_conv less_diff_conv nth_Cons_0 nth_Cons_Suc)
done
qed

lemma *set_of_interval_list_set_eq_interval_list_union_contiguous*:
assumes *non_empty*: ‹*XS* ≠ ([]::*real interval list*)›
and *contiguous*: ‹*contiguous XS*›

```
shows ‹set_of_interval_list XS = set_of (interval_list_union XS)›
 using interval_list_union_contiguous[of XS, simplified assms, simplified]
     set_of_interval_list_contiguous[of XS,simplified assms, simplified]
     contiguous_non_overlapping[of XS, simplified assms, simplified]
     non_overlapping_implies_sorted_wrt_upper[of XS]
     non_overlapping_implies_sorted_wrt_lower[of XS]
 interval_list_union_contiguous_lower[of XS]
 interval_list_union_contiguous_upper[of XS]
 apply(simp add:set_of_eq)
 by (metis non_empty)


lemma mInterval_ovl_lower_hd_min:
 ‹valid_mInterval_ovl x ⟹ Min (set (map lower x)) = (lower o hd) x›
proof(induction x rule:induct_list012)
 case 1
 then show ?case unfolding valid_mInterval_ovl_def by simp
next
 case (2 x)
 then show ?case unfolding valid_mInterval_ovl_def by simp
next
 case (3 x y zs)
 then show ?case
  apply(simp add: valid_mInterval_ovl_def non_overlapping_sorted_def cmp_non_overlapping_def image_def)
  by (metis (no_types, lifting) list.sel(1) list.simps(3) min.absorb3 min_def sorted_wrt_lower_unroll)
qed

lemma mInterval_adj_lower_hd_min:
 ‹valid_mInterval_adj x ⟹ Min (set (map lower x)) = (lower o hd) x›
proof(induction x rule:induct_list012)
 case 1
 then show ?case unfolding valid_mInterval_adj_def by simp
next
 case (2 x)
 then show ?case by simp
next
 case (3 x y zs)
 then show ?case
  apply(simp add: valid_mInterval_adj_def non_overlapping_sorted_def cmp_non_overlapping_def image_def)
  by (metis lower_le_upper min.absorb1 min.bounded_iff)
qed

lemma mInterval_non_ovl_lower_hd_min:
 ‹valid_mInterval_non_ovl x ⟹ Min (set (map lower x)) = (lower o hd) x›
 unfolding valid_mInterval_non_ovl_def
 using mInterval_ovl_lower_hd_min
 by(auto)


lemma mInterval_ovl_lower_last_max:
 ‹valid_mInterval_ovl x  ⟹ (Max (set (map lower x))) = (lower o last) x›
proof(induction x rule:induct_list012)
 case 1
 then show ?case unfolding valid_mInterval_ovl_def by simp
```

```
next
 case (2 x)
 then show ?case unfolding valid_mInterval_ovl_def by simp
next
 case (3 x y zs)
 then show ?case
  by(auto simp add: sorted_wrt_lower_def cmp_lower_width_def valid_mInterval_ovl_def
                    non_overlapping_sorted_def max_def cmp_non_overlapping_def image_def
           split: if_splits)
qed

lemma mInterval_adj_upper_hd_min:
 ‹valid_mInterval_adj x ⟹ Min (set (map upper x)) = (upper o hd) x›
proof(induction x rule:induct_list012)
 case 1
 then show ?case unfolding valid_mInterval_adj_def by simp
next
 case (2 x)
 then show ?case unfolding valid_mInterval_adj_def by simp
next
 case (3 x y zs)
 then show ?case
  apply(simp add: sorted_wrt_lower_def cmp_lower_width_def valid_mInterval_adj_def
                    non_overlapping_sorted_def max_def cmp_non_overlapping_def image_def
           split: if_splits)
  by (metis (no_types, opaque_lifting) dual_order.trans lower_le_upper min.absorb2 min.commute)
 qed

lemma mInterval_adj_upper_last_max:
 ‹valid_mInterval_adj x ⟹ Max (set (map upper x)) = (upper o last) x›
proof(induction x rule:induct_list012)
 case 1
 then show ?case unfolding valid_mInterval_adj_def by simp
next
 case (2 x)
 then show ?case unfolding valid_mInterval_adj_def by simp
next
 case (3 x y zs)
 then show ?case
  apply(simp add: sorted_wrt_lower_def cmp_lower_width_def valid_mInterval_adj_def
                    non_overlapping_sorted_def max_def cmp_non_overlapping_def image_def
           split: if_splits)
  subgoal using le_left_mono lower_le_upper by blast
  subgoal using last_in_set lower_le_upper order_trans by blast
  done
qed


lemma set_of_subeq_aux:
 ‹(⋃x∈set is. {lower x..upper x}) ⊆ {Min (lower ' (set is)) .. Max (upper ' (set is))}›
proof(induction is)
 case Nil
 then show ?case by simp
next
```

```
 case (Cons a is)
 then show ?case
  apply(auto)[1]
  apply (meson List.finite_set Min.coboundedI finite_imageI finite_insert imageI insertCI order.trans)
  by (meson List.finite_set Max.coboundedI finite_imageI finite_insert imageI insert_iff order_trans)+
qed

lemma lower_merge_adjacent_intervals:
 assumes xs ≠ []
  and ‹sorted_wrt_lower xs›
 shows (lower ∘ hd) (merge_adjacent_intervals_sorted_wrt_lower xs) = (lower ∘ hd) xs
 using assms proof(induction xs rule:list_nonempty_induct)
  case (single x)
  then show ?case by(simp add:interval_sort_lower_width_def)
 next
  case (cons x xs)
  then show ?case
   apply(simp)
   by (metis list.sel(1) merge_adjacent_intervals_sorted_wrt_lower_sorted_lower_lower_hd)
 qed

lemma sorted_wrt_lower_hd_min:
 ‹x ≠ [] ⟹ sorted_wrt_lower x ⟹ Min (set (map lower x)) = (lower o hd) x›
proof(induction x rule:induct_list012)
 case 1
 then show ?case by simp
next
 case (2 x)
 then show ?case by simp
next
 case (3 x y zs)
 then show ?case
  apply(simp add: valid_mInterval_ovl_def non_overlapping_sorted_def cmp_non_overlapping_def image_def)
  by (metis (no_types, lifting) list.sel(1) list.simps(3) min.absorb3 min_def sorted_wrt_lower_unroll)
qed

lemma lower_hd_min_over_mk_mInterval_non_ovl:
 xs ≠ [] ⟹ (lower o hd) xs = Min (lower ‘ (set xs)) ⟹ (lower ∘ hd) (mk_mInterval_non_ovl xs) = (lower ∘ hd) xs
 unfolding mk_mInterval_non_ovl_def mk_mInterval_ovl_def
      by (metis list.set_map mInterval_ovl_lower_hd_min mk_mInterval_lower mk_mInterval_non_ovl_def
mk_mInterval_ovl_def valid_ovl_mkInterval_non_ovl)


theorem valid_mInterval_non_ovl_nempty: valid_mInterval_non_ovl x ⟹ x ≠ []
 unfolding valid_mInterval_non_ovl_def valid_mInterval_ovl_def
 by simp


end
```

9 Subdivisions and Refinements (Lipschitz_Subdivisions_Refinements)

theory
Lipschitz_Subdivisions_Refinements
imports
Lipschitz_Interval_Extension
Multi_Interval_Preliminaries
begin

9.1 Subdivisions

A uniform subdivision of an interval X splits X into a vector of equal length, contiguous intervals.

definition *uniform_subdivision* :: *'a::linordered_field interval* ⇒ *nat* ⇒ *'a interval list* **where**
uniform_subdivision A n = *map* (λ*i*. *let i'* = *of_nat i in*
mk_interval (*lower A* + (*upper A* − *lower A*) ∗ *i'* / *of_nat n*,
lower A + (*upper A* − *lower A*) ∗ (*i'* + 1) / *of_nat n*)) [0..<*n*]

The definition *uniform_subdivision* refers to definition 6.2 in [4]

definition *overlapping_ordered* :: *'a::*{*preorder*} *interval list* ⇒ *bool* **where**
overlapping_ordered xs = (∀ *i*. *i* < *length xs* − 1 ⟶ *lower* (*xs* ! (*i* + 1)) ≤ *upper* (*xs* ! *i*))

definition *overlapping_non_zero_width* :: *'a::*{*preorder*, *minus*, *zero*, *ord*} *interval list* ⇒ *bool* **where**
overlapping_non_zero_width xs = (∀ *i* < *length xs* − 1 . ∃ *e*. *e* ∈$_i$ (*xs* ! (*i* + 1)) ∧ *e* ∈$_i$ (*xs* ! *i*) ∧ 0 < *width* (*xs* ! (*i* + 1))
∧ 0 < *width* (*xs* ! *i*))

definition *overlapping* :: *'a::*{*preorder*} *interval list* ⇒ *bool* **where**
overlapping xs = (∀ *i* < *length xs* − 1 . ∃ *e*. *e* ∈$_i$ (*xs* ! (*i* + 1)) ∧ *e* ∈$_i$ (*xs* ! *i*))

definition *check_is_uniform_subdivision* :: *'a::linordered_field interval* ⇒ *'a interval list* ⇒ *bool* **where**
check_is_uniform_subdivision A xs = (*let n* = *length xs in*
if n = 0 *then True*
else
let d = *width A* / *of_nat n in*
list_all (λ *x*. *width x* = *d*) *xs* ∧
contiguous xs ∧
lower (*hd xs*) = *lower A* ∧
upper (*last xs*) = *upper A*)

lemma *non_empty_subdivision*:
assumes 0 < *n*
shows *uniform_subdivision A n* ≠ []
unfolding *uniform_subdivision_def* **using** *assms* **by** *simp*

lemma *uniform_subdivision_id*: *uniform_subdivision X* 1 = [*X*]
unfolding *uniform_subdivision_def* **by** *simp*

```
lemma subdivision_length_n:
 assumes 0 < n
 shows length(uniform_subdivision A n) = n
 using assms
proof(induction n rule:nat_induct_non_zero)
 case 1
 then show ?case unfolding uniform_subdivision_def by simp
next
 case (Suc n)
 then show ?case unfolding uniform_subdivision_def by simp
qed

lemma contiguous_uniform_subdivision: contiguous (uniform_subdivision A n)
proof −
 have a0: ∀ i<length (uniform_subdivision A n) − 1.
       upper (uniform_subdivision A n ! i) = lower A + (upper A − lower A) ∗ of_nat (i + 1) / of_nat n
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
 have a1: ∀ i<length (uniform_subdivision A n) − 1.
       lower A + (upper A − lower A) ∗ of_nat (i + 1) / of_nat n = lower (uniform_subdivision A n ! (i + 1))
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
 have a2: ∀ i<length (uniform_subdivision A n) − 1.
       upper (uniform_subdivision A n ! i) = lower (uniform_subdivision A n ! (i + 1))
  using a0 a1 by simp
 have a3: contiguous (uniform_subdivision A n) =
       (∀ i<length (uniform_subdivision A n) − 1.
        upper (uniform_subdivision A n ! i) = lower (uniform_subdivision A n ! (i + 1)))
  unfolding contiguous_def by simp
 show ?thesis using a0 a1 a2 a3 by simp
qed



lemma overlapping_ordered_uniform_subdivision:
 assumes 0 < n
 shows overlapping_ordered (uniform_subdivision A n)
proof −
 have a0: ∀ i<length (uniform_subdivision A n) − 1.
       upper (uniform_subdivision A n ! i) ≥ lower A + (upper A − lower A) ∗ of_nat (i + 1) / of_nat n
  using assms
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
 have a1: ∀ i<length (uniform_subdivision A n) − 1.
       lower A + (upper A − lower A) ∗ of_nat (i + 1) / of_nat n ≥ lower (uniform_subdivision A n ! (i + 1))
  using assms
  by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
 have a2: ∀ i<length (uniform_subdivision A n) − 1.
       upper (uniform_subdivision A n ! i) ≥ lower (uniform_subdivision A n ! (i + 1))
  using a0 a1 by force
 have a3: overlapping_ordered (uniform_subdivision A n) =
       (∀ i<length (uniform_subdivision A n) − 1.
        upper (uniform_subdivision A n ! i) ≥ lower (uniform_subdivision A n ! (i + 1)))
  unfolding overlapping_ordered_def by simp
 show ?thesis using a0 a1 a2 a3 by simp
qed
```

```
lemma overlapping_uniform_subdivision:
 assumes 0 < N
 shows overlapping (uniform_subdivision X N)
 using assms
proof −
 let ?n = length (uniform_subdivision X N) − 1
 have a0: ∀ i < ?n . lower (uniform_subdivision X N ! (i + 1)) = upper (uniform_subdivision X N ! i)
  using assms contiguous_uniform_subdivision unfolding contiguous_def by metis
 have a1: ∀ i < ?n. upper (uniform_subdivision X N ! i) ∈i uniform_subdivision X N ! i
          ∧ upper (uniform_subdivision X N ! i) ∈i (uniform_subdivision X N ! (i + 1))
  using a0 in_intervalI lower_le_upper order.refl
  by metis
 have a2: ∀ i< ?n. ∃ e. e ∈i uniform_subdivision X N ! (i + 1) ∧ e ∈i uniform_subdivision X N ! i
  using a1 by auto[1]
 show ?thesis using a2 unfolding overlapping_def by simp
qed

lemma hd_lower_uniform_subdivision:
 assumes 0 < n
 shows lower ( hd (uniform_subdivision A n)) = lower A
proof −
 have lower ( hd (uniform_subdivision A n)) = lower A + (upper A − lower A) ∗ of_nat 0 / of_nat n
  using assms
  by (simp add: uniform_subdivision_def mk_interval' hd_map)
 also have ... = lower A
  by simp
 finally show ?thesis .
qed

lemma last_upper_uniform_subdivision:
 assumes 0 < n
 shows upper ( last (uniform_subdivision A n)) = upper A
 using assms
 by (simp add: uniform_subdivision_def mk_interval' last_map Let_def field_simps)

lemma uniform_subdivisions_width_single:
 fixes A ::'a::linordered_field interval
 shows ‹width (Interval (lower A + (upper A − lower A) ∗ x / of_nat n,
          lower A + (upper A − lower A) ∗ (x + 1) / of_nat n)) = width A / of_nat n›
proof −
 have lower A ≤ upper A using lower_le_upper by simp
 then have leq: lower A + (upper A − lower A) ∗ x / of_nat n ≤
       lower A + (upper A − lower A) ∗ ( x + 1) / of_nat n
  by (simp add: divide_le_cancel linorder_not_less mult_le_cancel_left)
 have U: ‹upper (Interval (lower A + (upper A − lower A) ∗ x / of_nat n,
                lower A + (upper A − lower A) ∗ ( x + 1) / of_nat n)) =
       lower A + (upper A − lower A) ∗ ( x + 1) / of_nat n›
  using upper_bounds leq by blast
 have L: ‹lower (Interval (lower A + (upper A − lower A) ∗ x / of_nat n,
                lower A + (upper A − lower A) ∗ ( x + 1) / of_nat n)) =
       lower A + (upper A − lower A) ∗ x / of_nat n›
  using lower_bounds leq by blast
 then show ?thesis using U L add_diff_cancel_left add_diff_cancel_left' diff_divide_distrib
    mult.comm_neutral vector_space_over_itself.scale_right_diff_distrib unfolding width_def
```

```
  by metis
qed

lemma uniform_subdivisions_width:
 assumes 0 < n
 shows ‹∀ A. A ∈ set (uniform_subdivision X n) ⟶ width A = width X / of_nat n›
 apply (simp add: uniform_subdivision_def mk_interval' 0_def image_def width_def Let_def split: if_split)
 apply auto[1]
 apply (metis add_diff_cancel_left' diff_divide_distrib mult.right_neutral right_diff_distrib)
 using assms uniform_subdivisions_width_single[simplified width_def]
 by (simp add: divide_right_mono mult_left_mono)

lemma uniform_subdivision_sum_width:
 assumes ‹0 < n›
 shows ‹sum_list (map width (uniform_subdivision X n)) = width X›
proof −
 have ‹∀ a . a ∈ set (uniform_subdivision X n) ⟶ width a = width X / of_nat n ›
  using uniform_subdivisions_width using assms by blast
 then have width: ∀ a . a ∈ set (map width (uniform_subdivision X n)) ⟶ a = width X / of_nat n
  unfolding width_def by auto[1]
 then have width_list: list_all (λ a . a = width X / of_nat n) (map width (uniform_subdivision X n))
  unfolding width_def using list_all_iff by blast
 then have length: length (map width (uniform_subdivision X n)) = n
  unfolding uniform_subdivision_def by simp
 then have sum_list (map width (uniform_subdivision X n)) = (width X / of_nat n) * of_nat n
  using width_list by (metis list.map_ident_strong mult_of_nat_commute sum_list_triv width)
 then show ?thesis by (simp add: assms)
qed

lemma uniform_subdivisions_distinct:
 assumes 0 < n 0 < width A
 shows distinct (uniform_subdivision A n)
proof −
 have ∀ i< n. ∀ j< n. i ≠ j ⟶ (uniform_subdivision A n) ! i ≠ (uniform_subdivision A n) ! j
 proof −
   have f1: ∀ i< n. ∀ j< n. i ≠ j ⟶ (upper A − lower A) * of_nat i / of_nat n ≠ (upper A − lower A) * of_nat j / of_nat n
    using assms(1) assms(2) divide_cancel_right less_numeral_extra(3) mult_cancel_left of_nat_eq_0_iff of_nat_eq_iff
width_def
    by metis
   have f2: ∀ i< n. ∀ j< n. i ≠ j ⟶ lower A + (upper A − lower A) * of_nat (i + 1) / of_nat n ≠ lower A + (upper A −
lower A) * of_nat (j + 1) / of_nat n
    using assms(1) assms(2) unfolding width_def by simp
   have f3: ∀ i< n. lower ((uniform_subdivision A n) ! i) = lower A + (upper A − lower A) * of_nat i / of_nat n
    using assms by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono Let_def)
   have f5: ∀ i<n − 1. (upper ((uniform_subdivision A n) ! i)) = lower ((uniform_subdivision A n) ! Suc i)
    using assms by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono)
   have f6: ∀ j<n − 1. (upper ((uniform_subdivision A n) ! j)) = lower ((uniform_subdivision A n) ! Suc j)
    using assms(1) f5 contiguous_uniform_subdivision unfolding contiguous_def subdivision_length_n by blast
   have ∀ i<n. ∀ j<n. i ≠ j ⟶ lower ((uniform_subdivision A n) ! i) ≠ lower ((uniform_subdivision A n) ! j)
    using f1 f2 f3 by auto[1]
  then show ?thesis by metis
 qed
 then show ?thesis using assms(1) distinct_conv_nth subdivision_length_n by metis
qed
```

```
lemma uniform_subdivisions_non_overlapping:
 assumes 0 < n
 shows non_overlapping_sorted (uniform_subdivision A n)
proof −
 have ∀ i<n. ∀ j<n. i < j ⟶ upper ((uniform_subdivision A n) ! i) ≤ lower ((uniform_subdivision A n) ! j)
 proof −
  have f0: ‹0 ≤ width A› unfolding width_def lower_le_upper by simp
   have f2: ∀ i<n. upper ((uniform_subdivision A n) ! i) = lower A + (upper A − lower A) * of_nat (i + 1) / of_nat n
    using assms by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono Let_def)
   have f3: ∀ j<n. lower ((uniform_subdivision A n) ! j) = lower A + (upper A − lower A) * of_nat j / of_nat n
    using assms by (simp add: uniform_subdivision_def divide_right_mono mult_left_mono Let_def)
   have f4: ∀ i<n. ∀ j<n. i < j ⟶ lower A + (upper A − lower A) * of_nat (i + 1) / of_nat n ≤ lower A + (upper A −
lower A) * of_nat j / of_nat n
    using assms divide_right_mono mult_left_mono Suc_eq_plus1 Suc_leI add_le_cancel_left
     interval_width_positive of_nat_0_le_iff of_nat_le_iff width_def
    by metis
   have ∀ i<n. ∀ j<n. i < j ⟶ upper ((uniform_subdivision A n) ! i) ≤ lower ((uniform_subdivision A n) ! j)
    using f0 f2 f3 f4 by simp
  then show ?thesis by auto[1]
 qed
 then show ?thesis
  unfolding non_overlapping_sorted_def cmp_non_overlapping_def
  by (simp add: assms(1) sorted_wrt_iff_nth_less subdivision_length_n)
qed
```

We prove that our uniform subdivision meets the multi-interval type

```
lemma uniform_subdivisions_valid_ainterval:
 assumes 0 < n 0 < width A
 shows valid_mInterval_adj(uniform_subdivision A n)
 using assms
 unfolding valid_mInterval_adj_def
 apply safe
 subgoal using uniform_subdivisions_non_overlapping by blast
 subgoal using uniform_subdivisions_distinct by blast
 subgoal using non_empty_subdivision by blast
 done

lemma uniform_subdivisions_valid:
 assumes 0 < n
 shows check_is_uniform_subdivision A (uniform_subdivision A n)
 unfolding check_is_uniform_subdivision_def Let_def
 apply (simp split: if_split)
 apply safe
 subgoal using assms uniform_subdivisions_width subdivision_length_n
  by (metis (mono_tags, lifting) Ball_set)
 subgoal using assms contiguous_uniform_subdivision by blast
 subgoal using assms hd_lower_uniform_subdivision by blast
 subgoal using assms last_upper_uniform_subdivision by blast
 done
```

9.2 Refinement

Let $F\ X$ be an inclusion isotonic, Lipschitz, interval extension for $X \subseteq Y$. A refinement $F_N\ X$ of $F\ X$ is the union of interval values of X over the elements of a uniform subdivision of X

definition *refinement* :: $('a::\{linordered_field,lattice\}\ interval \Rightarrow 'a\ interval) \Rightarrow nat \Rightarrow 'a\ interval \Rightarrow 'a\ interval$ **where**
‹*refinement F N X* = (*interval_list_union* (*map F* (*uniform_subdivision X N*)))›

definition *check_is_refinement* **where**
‹*check_is_refinement F n As B* = (*let I* = *refinement F n As in lower B* ≤ *lower I* ∧ *upper I* ≤ *upper B*)›

definition *refinement_set* :: $('a::\{linordered_field,lattice\}\ interval \Rightarrow 'a\ interval) \Rightarrow nat \Rightarrow 'a\ interval \Rightarrow 'a\ set$ **where**
‹*refinement_set F N X* = (*set_of_interval_list* (*map F* (*uniform_subdivision X N*)))›

The definition *refinement* refers to definition 6.3 in [4].

The excess width of $F\ X$ is $w\ (E\ X) = w\ (F\ X) - w\ (f\ X)$. The united extension $f\ x$ for $x \in X$ has zero excess width and we can compute $f\ x$ as closely as desired by computing refinements of an extension $F\ X$.

definition *width_set s* = *Sup s* − *Inf s*

lemma *width_set_bounded*:
fixes *X* :: ‹*real set*›
assumes ‹*bdd_below X*› ‹*bdd_above X*›
shows ‹∀ *x* ∈ *X*. ∀ *x'* ∈ *X*. *dist x x'* ≤ *width_set X*›
using *assms sup_inf_dist_bounded*
unfolding *width_set_def*
by(*simp*)

lemma *width_inclusion_isotonic_approx*:
fixes *F*::‹*real interval* ⇒ *real interval*›
assumes *inclusion_isotonic F F is_interval_extension_of f*
shows ‹*0* ≤ *width* (*F X*) − *width_set* (*f* ' *set_of X*)›
by (*smt* (*verit*, *del_insts*) *assms*(*1*) *assms*(*2*) *inclusion_isotonic_inf*
inclusion_isotonic_sup width_def width_set_def)

lemma *diameter_width*:
assumes ‹*a* ≤ *b*›
shows ‹*diameter* {*a*..*b*} = *width_set* {*a*..*b*}›
by (*simp add*: *assms linorder_not_less diameter_Sup_Inf width_set_def*)

lemma *lipschitz_dist_diameter_limit*:
fixes *S*::‹'*a*::{*metric_space*, *heine_borel*} *set*›
and *f*::‹'*a*::{*metric_space*, *heine_borel*} ⇒ '*b*::{*metric_space*, *heine_borel*}›
assumes ‹*C*−*lipschitz_on S f*› **and** ‹*bounded S*›
shows ‹*x* ∈ (*f* ' *S*) ⟹ *y* ∈ (*f* ' *S*) ⟹ *dist x y* ≤ *diameter* (*f* ' *S*)›
using *lipschitz_on_uniformly_continuous*[*of C S f*, *simplified assms*]
bounded_uniformly_continuous_image[*of S f*, *simplified assms*]
diameter_bounded_bound[*of f* ' *S x y*]
by *simp*

definition *excess_width_diameter* :: $('a::preorder\ interval \Rightarrow real\ interval) \Rightarrow ('a \Rightarrow 'b::metric_space) \Rightarrow 'a\ interval \Rightarrow real$ **where**
‹*excess_width_diameter F f X* = *width*(*F X*) − *diameter* (*f* ' *set_of X*)›

definition *excess_width_set* :: $('a::\{minus,linorder,Inf,Sup\}\ interval \Rightarrow 'a\ set) \Rightarrow ('a \Rightarrow 'a) \Rightarrow 'a\ interval \Rightarrow 'a$ **where**

‹*excess_width_set F f X* = *width_set*(*F X*) − *width_set* (*f* ‘ *set_of X*)›

definition *excess_width* :: ('*a*::{*minus*,*linorder*,*Inf*,*Sup*} *interval* ⇒ '*a interval*) ⇒ ('*a* ⇒ '*a*) ⇒ '*a interval* ⇒ '*a* **where**
‹*excess_width F f X* = *width*(*F X*) − *width_set* (*f* ‘ *set_of X*)›

The definition *excess_width* refers to definition 6.4 in [4]

lemma *width_set_of*: **fixes** *X* :: *real interval*
shows *width_set_upper_lower*: ‹*width_set* (*set_of X*) = |(*lower X*) − (*upper X*)|›
by (*simp add*: *width_set_def set_of_eq*)

lemma *width_set_dist*:
fixes *f* :: *real* ⇒ *real*
shows *width_set* (*set_of X*) = (*dist* (*lower X*) (*upper X*))
by(*simp add*:*set_of_eq width_set_def dist_real_def*)

lemma *diameter_of*: **fixes** *X* :: *real interval*
shows *diameter_upper_lower*: ‹*diameter* (*set_of X*) = |(*lower X*) − (*upper X*)|›
by (*simp add*: *linorder_not_less set_of_eq*)

lemma *diameter_dist*:
fixes *X* :: *real interval*
shows *diameter* (*set_of X*) = (*dist* (*lower X*) (*upper X*))
unfolding *set_of_eq dist_real_def abs_real_def*
using *lower_le_upper*[*of X*] *diameter_closed_interval*[*of lower X upper X*]
by *argo*

lemma *bdd_below_f_set_of*:
fixes *f* :: *real* ⇒ *real*
assumes *C*−*lipschitz_on X f*
and ‹*bounded X*› **and** ‹*X* ≠ {}›
shows ‹*bdd_below* (*f* ‘ *X*)›
using *assms atLeastAtMost_iff bdd_below.unfold bounded_imp_bdd_below image_def*
lipschitz_bounded_image_real set_of_eq set_of_nonempty
by *simp*

lemma *bdd_above_f_set_of*:
fixes *f* :: *real* ⇒ *real*
assumes *C*−*lipschitz_on* (*X*) *f*
and ‹*bounded X*› **and** ‹*X* ≠ {}›
shows ‹*bdd_above* (*f* ‘ *X*)›
using *assms atLeastAtMost_iff bdd_above.unfold bounded_imp_bdd_above image_def*
lipschitz_bounded_image_real set_of_eq set_of_nonempty
by *simp*

lemma *diameter_image_dist*:
fixes *f*::‹*real* ⇒ *real*›
assumes ‹*continuous_on* (*set_of X*) *f*›
shows ‹(∃ *x*∈ *set_of X*. ∃ *x*′∈ *set_of X* . *diameter* (*f* ‘ *set_of X*) = *dist* (*f x*) (*f x*′))›
using *assms compact_continuous_image*[*of set_of X f*, *simplified assms compact_set_of*[*of X*]]
lower_le_upper[*of X*] *diameter_closed_interval*[*of f* (*lower X*) *f*(*upper X*), *symmetric*]
diameter_compact_attained[*of f* ‘ *set_of X*] *set_f_nonempty*[*of f X*]
by *fastforce*

lemma *excess_width_inf_diameter*:

```
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F  F is_interval_extension_of f ‹ C−lipschitz_on (set_of X) f›
  shows ‹dist (Inf (f ‘ set_of X)) (lower (F X)) ≤ excess_width_diameter F f X›
    unfolding dist_real_def abs_real_def excess_width_diameter_def  width_def
    using inclusion_isotonic_inf[of F f X, simplified assms]
        inclusion_isotonic_sup[of F f X, simplified assms]
        diameter_Sup_Inf[of f ‘ set_of X, simplified assms lipschitz_on_continuous_on[of C (set_of X) f]
                                compact_img_set_of[of X f], simplified]
    by simp

lemma excess_width_inf:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F  F is_interval_extension_of f ‹ C−lipschitz_on (set_of X) f›
  shows ‹dist (Inf (f ‘ set_of X)) (lower (F X)) ≤ excess_width F f X›
    unfolding dist_real_def abs_real_def excess_width_def  width_def
    using inclusion_isotonic_inf[of F f X, simplified assms]
        inclusion_isotonic_sup[of F f X, simplified assms]
    by (simp add: width_set_def)


lemma excess_width_sup_diameter:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F  F is_interval_extension_of f ‹ C−lipschitz_on (set_of X) f›
  shows ‹dist (Sup (f ‘ set_of X)) (upper (F X)) ≤ excess_width F f X›
    unfolding dist_real_def abs_real_def excess_width_diameter_def  width_def
    using inclusion_isotonic_inf[of F f X, simplified assms]
        inclusion_isotonic_sup[of F f X, simplified assms]
        diameter_Sup_Inf[of f ‘ set_of X, simplified assms lipschitz_on_continuous_on[of C (set_of X) f]
                                compact_img_set_of[of X f], simplified]
    by (simp add: excess_width_def width_def width_set_def)

lemma excess_width_sup:
  fixes F::‹real interval ⇒ real interval›
  assumes  inclusion_isotonic F  F is_interval_extension_of f ‹ C−lipschitz_on (set_of X) f›
  shows ‹dist (Sup (f ‘ set_of X)) (upper (F X)) ≤ excess_width F f X›
    unfolding dist_real_def abs_real_def excess_width_def  width_def
    using inclusion_isotonic_inf[of F f X, simplified assms]
        inclusion_isotonic_sup[of F f X, simplified assms]
    by (simp add: width_set_def)
```

If $F\ X$ is an inclusion isotonic, Lipschitz, interval extension then the excess width of a refinement is of order $1 \mathbin{/} N$

If X and X are intervals such that $X \subseteq Y$, then there is an interval E with $\mathit{lower}\ E \leq o \wedge o \leq \mathit{upper}\ E$ such that $Y = X + E$ and $w\ Y = w\ X + w\ E$.

```
lemma interval_subset_width:
  fixes X Y :: 'a::{linordered_ring, lattice} interval
  assumes X ≤ Y
  and X_def: X = Interval(a, b) and x_valid: a ≤ b
  and Y_def: Y = Interval(c, d) and y_valid: c ≤ d
  shows ∃ E. o ∈ᵢ E ∧ Y = X + E ∧ width Y = width X + width E
proof −
  have c ≤ a b ≤ d
  proof −
```

```
    have leq: lower Y ≤ lower X ∧ upper X ≤ upper Y using assms(1) unfolding less_eq_interval_def by simp
   have X_bounds: lower X = a upper X = b unfolding X_def by (simp add: x_valid bounds_of_interval_eq_lower_upper)+
   have Y_bounds: lower Y = c upper Y = d unfolding Y_def by (simp add: y_valid bounds_of_interval_eq_lower_upper)+
    show c ≤ a b ≤ d using assms(1) X_bounds Y_bounds unfolding less_eq_interval_def by simp_all
  qed
  define e where e = c − a
  define f where f = d − b
  define E where E = Interval(e, f)
  have 0 ∈ᵢ E unfolding E_def e_def f_def using ‹c ≤ a› ‹b ≤ d› set_of_subset_iff
   by (smt (verit, ccfv_SIG) diff_ge_0_iff_ge in_intervalI le_iff_diff_le_0 lower_bounds order.trans upper_bounds)
  have Y = X + E
  proof −
   have X_bounds: lower X = a upper X = b
    unfolding X_def by (simp add: x_valid bounds_of_interval_eq_lower_upper)+
   have Y_bounds: lower Y = c upper Y = d
    unfolding Y_def by (simp add: y_valid bounds_of_interval_eq_lower_upper)+
   have E_bound_1: lower E = c − a
    unfolding E_def e_def f_def
    using ‹b ≤ d› ‹c ≤ a› diff_left_mono diff_self lower_bounds order_trans
    by metis
   have E_bound_2:upper E = d − b
    unfolding E_def e_def f_def
    using ‹b ≤ d› ‹c ≤ a›  E_bound_1 ‹0 ∈ᵢ E› diff_ge_0_iff_ge dual_order.trans in_bounds upper_bounds
    by metis
   have add: Interval(a,b) + Interval(c−a,d−b) = Interval(c,d)
    using X_bounds Y_bounds E_bound_1 E_bound_2
    by (simp add: E_def X_def Y_def e_def f_def interval_eq)
   show ?thesis unfolding E_def X_def Y_def e_def f_def using add by simp
  qed
  have width Y = width X + width E unfolding width_def using E_def X_def Y_def ‹Y = X + E› e_def f_def by force
  from ‹0 ∈ᵢ E› ‹Y = X + E› ‹width Y = width X + width E› show ?thesis by auto[1]
qed

lemma excess_width_incl:
  fixes F :: real interval ⇒ real interval and X :: real interval
  assumes int: ‹F is_interval_extension_of f›
  and iso: inclusion_isotonic F
  and L−lipschitz_on (set_of X) f
shows ‹ ∃ E . F X = Interval(Inf (f ‘ set_of X), Sup (f ‘ set_of X)) + E ›
proof −
  have a0: ‹f ‘ (set_of X) ≠ {}› using in_intervalI by fastforce
  have a1: Inf (f ‘ set_of X) ≤ Sup (f ‘ set_of X)
   using assms a0 inf_le_sup_image_real by simp
  have a2: ‹f ‘ (set_of X) ⊆ set_of (F X)›
   using assms fundamental_theorem_of_interval_analysis by simp
  have max: ‹Sup (f ‘ (set_of X)) ≤ Sup (set_of (F X))›
   using assms(3) a0 a2 sup_image_le_real[of f X F] by blast
  have max_interval: Sup (f ‘ (set_of X)) ≤ upper (F X)
   using max sup_set_of by metis
  have min: ‹Inf (set_of (F X)) ≤ Inf (f ‘ (set_of X))›
   using assms(3) a0 a2 inf_image_le_real[of f X F] by blast
  have min_interval: lower (F X) ≤ Inf (f ‘ (set_of X))
   using min inf_set_of by metis
  have lower_min: lower (Interval (Inf (f ‘ set_of X), Sup (f ‘ set_of X))) = Inf (f ‘ set_of X)
```

```
  using lower_bounds a1 by simp
 have upper_max: upper (Interval (Inf (f ' set_of X), Sup (f ' set_of X))) = Sup (f ' set_of X)
  using upper_bounds a1 by simp
 have a3: Interval(Inf (f ' set_of X), Sup (f ' set_of X)) ≤ F X
  using min_interval max_interval lower_min upper_max unfolding less_eq_interval_def by simp
 have a4: ∃ E . Interval(Inf (f ' set_of X), Sup (f ' set_of X)) + E = F X using a3
  by (metis (no_types, opaque_lifting) bounds_of_interval_eq_lower_upper
     bounds_of_interval_inverse interval_subset_width lower_le_upper)
 then show ?thesis by metis
qed

lemma excess_interval_superset_interval:
 fixes F :: real interval ⇒ real interval and X :: real interval
 assumes int: ‹F is_interval_extension_of f›
 and iso: inclusion_isotonic F
 and L−lipschitz_on (set_of X) f
 and ex: ‹ ∃ E . F X = Interval(Inf (f ' set_of X), Sup (f ' set_of X)) + E ›
shows ‹Interval(Inf (f ' set_of X), Sup (f ' set_of X)) ≤ F X›
proof −
 have lhs: lower (F X) ≤ lower (Interval(Inf (f ' set_of X), Sup (f ' set_of X)))
  using assms(1,2,3) inf_image_le_real inf_le_sup_image_real inf_set_of
      fundamental_theorem_of_interval_analysis lower_bounds
  by metis
 have rhs: upper (Interval(Inf (f ' set_of X), Sup (f ' set_of X))) ≤ upper (F X)
  using assms(1,2,3) inf_le_sup_image_real sup_image_le_real sup_set_of
      fundamental_theorem_of_interval_analysis upper_bounds
  by metis
 show ?thesis using lhs rhs unfolding less_eq_interval_def by simp
qed

lemma each_subdivision_width_order:
 fixes X :: 'a::{linordered_field,lattice,metric_space} interval
 assumes inclusion_isotonic F lipschitzI_on C U F F is_interval_extension_of f
 and set (uniform_subdivision X N) ⊆ U  0 < N Xs ∈ set (uniform_subdivision X N)
shows width(F Xs) ≤ C * width (X) / of_nat N
proof −
 have a0: ∀ Xs ∈ set (uniform_subdivision X N). width (F Xs) ≤ C * width Xs
  using assms(2) assms(4) lipschitzI_onD by blast
 have a1: ∀ Xs ∈ set (uniform_subdivision X N). width(F Xs) ≤ C * width (X) / of_nat N
  using a0 assms(5) uniform_subdivisions_width[of N X] by simp
 show ?thesis using a1 assms by simp
qed

lemma each_subdivision_excess_width_order:
 fixes X :: real interval
 assumes inclusion_isotonic F lipschitzI_on C U F F is_interval_extension_of f
 and set (uniform_subdivision X N) ⊆ U  0 < N
 and  L−lipschitz_on (set_of (interval_list_union (uniform_subdivision X N))) f
shows ∀ Xs ∈ set (uniform_subdivision X N) . excess_width F f Xs ≤ C * width (X) / of_nat N
proof −
 have a0: ∀ Xs ∈ set (uniform_subdivision X N). width (F Xs) ≤ C * width Xs
  using assms lipschitzI_onD by blast
 have a1: ∀ Xs ∈ set (uniform_subdivision X N). width(F Xs) ≤ C * width (X) / of_nat N
  using a0 assms uniform_subdivisions_width[of N X] by simp
```

```
 have a2: ∀ Xs ∈ set (uniform_subdivision X N). excess_width F f Xs ≤ C * width (X) / of_nat N
 proof −
  have b0: set (uniform_subdivision X N) ≠ {}
   using assms non_empty_subdivision by simp
  have b1: ∀ Xs ∈ set (uniform_subdivision X N) . 0 ≤ width_set (f ' set_of Xs)
  proof −
    have c0: ∀ Xs ∈ set (uniform_subdivision X N) . set_of Xs ⊆ set_of (interval_list_union (uniform_subdivision X N))
     using assms interval_list_union_correct in_set_conv_nth non_empty_subdivision
     by metis
    then have c1: ∀ Xs ∈ set (uniform_subdivision X N) . set_of Xs ≠ {} using in_intervalI by fastforce
    then have c2: ∀ Xs ∈ set (uniform_subdivision X N) . Inf (f ' set_of Xs) ≤ Sup (f ' set_of Xs)
     using assms c0 c1 inf_le_sup_image_real lipschitz_on_subset
     by (metis inf_le_sup_image_real)
    then have c3: ∀ Xs ∈ set (uniform_subdivision X N) . 0 ≤ width_set (f ' set_of Xs)
     by (simp add: width_set_def)
    then show ?thesis by simp
  qed
  have b2: ∀ Xs ∈ set (uniform_subdivision X N) . width(F Xs) − width_set (f ' set_of Xs) ≤ width(F Xs)
   using b0 b1 assms inf_set_of lower_le_upper sup_set_of inf_le_sup_image_real image_is_empty
   by (simp add: width_set_def)
  then show ?thesis
   using a1 unfolding excess_width_def width_set_def by fastforce
  qed
 show ?thesis using assms a0 a1 a2 by simp
qed
```

The theorem ⟦*inclusion_isotonic ?F*; *?C−lipschitzI_on ?U ?F*; *?F is_interval_extension_of ?f*; *set* (*uniform_subdivision ?X ?N*) ⊆ *?U*; *0* < *?N*; *?Xs* ∈ *set* (*uniform_subdivision ?X ?N*)⟧ ⟹ *width* (*?F ?Xs*) ≤ *?C* * *width ?X* / *of_nat ?N* refers to Theorem 6.1 in [4].

```
lemma sup_interval_max:
 fixes X Y :: 'a::{linordered_ring, lattice} interval
 shows sup X Y = Interval(min (lower X) (lower Y), max (upper X) (upper Y))
 using Interval.lower_sup Interval.upper_sup Interval_id inf_real_def sup_max
 by (metis inf_min)

lemma interval_inf_sup_lower: inf (lower I1) (lower I2) = lower (sup I1 I2)
 unfolding sup_interval_def
 by (metis (mono_tags, lifting) Interval.lower_sup sup_interval_def)

lemma interval_sup_sup_upper: sup (upper I1) (upper I2) = upper (sup I1 I2)
 unfolding sup_interval_def
 by (metis (mono_tags, lifting) Interval.upper_sup sup_interval_def)

lemma interval_union_lower:
 assumes contiguous Xs Xs ≠ []
 shows lower (interval_list_union Xs) = lower (Xs!0)
 using assms
proof(induction Xs rule:interval_list_union.induct)
 case 1
 then show ?case by simp
next
 case (2 I)
 then show ?case by simp
next
```

```
 case (3 I v va)
 have a0: contiguous (v # va) using 3 unfolding contiguous_def by auto
 have a1: (v # va) ≠ [] by simp
 then show ?case
  unfolding sorted_wrt_lower_def 3
 proof −
  have b0: lower ((I # v # va) ! 0) = lower I by simp
  have b1: lower I ≤ lower v
   using 3.prems a1
   unfolding contiguous_def
   by (metis Suc_eq_plus1 add_diff_cancel_left' length_Cons less_Suc_eq_0_disj
     lower_le_upper nth_Cons_0 nth_Cons_Suc plus_1_eq_Suc)
  show lower (interval_list_union (I # v # va)) = lower ((I # v # va) ! 0)
   using b0 b1 3(2) interval_list_union.simps(3) 3.IH Interval.lower_sup a0 a1 inf.orderE
    nth_Cons_0
   unfolding sorted_wrt_lower_def
   by metis
 qed
qed

lemma interval_union_upper:
 assumes contiguous Xs Xs ≠ []
 shows upper (interval_list_union Xs) = upper (last Xs)
 using assms
proof(induction Xs rule:interval_list_union.induct)
 case 1
 then show ?case by simp
next
 case (2 I)
 then show ?case by simp
next
 case (3 I v va)
 have a0: contiguous (v # va) using 3 unfolding contiguous_def by auto
 have a1: (v # va) ≠ [] by simp
 then show ?case
 proof −
  have b0: upper ((I # v # va) ! (length (I # v # va) − 1)) = upper (last (I # v # va))
   using last_conv_nth by fastforce
  have b1: upper I ≤ upper v using 3.prems unfolding contiguous_def by auto
  show upper (interval_list_union (I # v # va)) = upper (last (I # v # va))
   using b0 b1 interval_list_union.simps(3) 3.IH a0 a1 interval_list_union.simps(3)
    Interval.upper_sup last.simps
   by (metis (no_types, opaque_lifting) dual_order.trans min_list.cases sup_absorb2 sup_ge1)
 qed
qed

lemma union_set:
 assumes 0 < n
 shows interval_list_union (uniform_subdivision X n) = X
 using assms
proof (induction n rule:nat_induct_non_zero)
 case 1
 then show ?case using uniform_subdivision_id  interval_list_union.simps(2) by metis
next
```

```
 case (Suc n)
 then show ?case
 proof (induction uniform_subdivision X (Suc n) rule:interval_list_union.induct)
  case 1
  then show ?case by (metis less_Suc_eq non_empty_subdivision)
 next
  case (2 I)
    then show ?case using One_nat_def interval_list_union.simps(2) subdivision_length_n uniform_subdivision_id
zero_less_Suc
    by metis
 next
  case (3 I v va)
  then have a0: lower I = lower X
   using hd_lower_uniform_subdivision assms
   by (metis list.collapse list.simps(3) nth_Cons_0 zero_less_Suc)
  then have a1: upper (last ( I # v # va)) = upper X
   using last_upper_uniform_subdivision[of Suc n X] assms 3.hyps(2) by simp
  then have a2: contiguous ( I # v # va)
   using contiguous_uniform_subdivision assms zero_less_Suc 3.hyps(2) by metis
  then have a3: overlapping_ordered ( I # v # va)
   using overlapping_uniform_subdivision  assms zero_less_Suc 3.hyps(2)
   by (simp add: overlapping_ordered_uniform_subdivision)
  then have a4: ∀ i<length ( I # v # va) − 1. upper ((( I # v # va) ! i) = lower ((( I # v # va) ! (i + 1))
    using a0 a2 unfolding contiguous_def by simp
   have a5: ∀ i<length ( I # v # va) − 1. lower ((( I # v # va) ! i) ≤ lower ((( I # v # va) ! (i + 1))
    using a0 a2 contiguous_def lower_le_upper by metis
   have a6: ∀ i<length ( I # v # va) − 1. upper ((( I # v # va) ! i) ≤ upper ((( I # v # va) ! (i + 1))
    using a0 a2 contiguous_def lower_le_upper by metis
   then show ?case
   proof −
   have lower (interval_list_union ( I # v # va)) = lower X
   proof −
    have c0: lower (interval_list_union ( I # v # va)) = lower (sup I (interval_list_union (v # va)))
     using interval_list_union.simps(3) by metis
    have c1: lower (sup I (interval_list_union ( v # va))) = min (lower I) (lower (interval_list_union ( v # va)))
     using inf_real_def sup_interval_def sup_interval_max sup_real_def
     by (simp add: inf_min)
    have c2: min (lower I) (lower (interval_list_union (v # va))) = lower I
     using 3 hd_lower_uniform_subdivision[of Suc n X] a0
         contiguous_uniform_subdivision[of  X, simplified contiguous_def 3(2)[symmetric]]
         interval_union_lower[of ( I # v # va)]
     by (metis a2 c0 c1 list.discI nth_Cons_0)
    show ?thesis using a0 c0 c1 c2 by simp
    qed
    moreover have upper (interval_list_union ( I # v # va)) = upper X
    proof −
     have c0: upper (interval_list_union ( I # v # va)) = upper (sup I (interval_list_union ( v # va)))
      using interval_list_union.simps(3) by metis
     have c1: upper (sup I (interval_list_union (v # va))) = max (upper I) (upper (interval_list_union ( v # va)))
      using inf_real_def sup_interval_def sup_interval_max sup_real_def
      by (simp add: sup_max)
     have c2: max (upper I) (upper (interval_list_union ( v # va))) = upper (last ( I # v # va))
       using contiguous_uniform_subdivision[of  X, simplified contiguous_def 3(2)[symmetric]]
          interval_union_upper[of ( I # v # va)]
```

```
        by (metis a2 c0 c1 list.discI )
      show ?thesis using c0 c1 c2 3.hyps(2) last_upper_uniform_subdivision by auto[1]
     qed
    ultimately show ?thesis
     using interval_eqI 3 by auto[1]
   qed
  qed
qed

lemma sum_list_less:
  assumes list_all (λn. n ≤ (y::real)) xs
  shows sum_list xs ≤ y * length xs
proof −
  have ∀ x∈set xs. x ≤ y
    using assms list_all_iff by metis
  hence sum_list xs ≤ sum_list (replicate (length xs) y)
    using sum_list_mono
    by (simp add: order_less_imp_le sum_list_mono2)
  also have ... = y * length xs
    by (simp add: sum_list_replicate)
  finally show ?thesis by simp
qed

lemma in_bounds2:
  fixes X Y :: 'a::{linordered_ring} interval
  shows x ∈i X ∧ x ∈i Y ⟹
     (lower Y ≤ lower X ∧ upper Y ≤ upper X ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ∨
     (lower X ≤ lower Y ∧ upper X ≤ upper Y ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ∨
     (lower Y ≤ lower X ∧ upper X ≤ upper Y ∧ lower Y ≤ lower X ∧ lower Y ≤ upper X) ∨
     (lower X ≤ lower Y ∧ upper Y ≤ upper X ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X)
  apply(clarify)
  by (metis (full_types) in_bounds nle_le order.trans)

lemma overlapping_width_sum:
  fixes X Y :: 'a::{linordered_ring, lattice} interval
  assumes overlapping [X,Y]
  shows width (sup X Y) ≤ width X + width Y
proof −
  have a0: ∀ i<length [X, Y] − 1. ∃ e. e ∈i [X, Y] ! (i + 1) ∧ e ∈i [X, Y] ! i
    using assms unfolding overlapping_def by blast
  have a1: (lower Y ≤ lower X ∧ upper Y ≤ upper X ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ∨
     (lower X ≤ lower Y ∧ upper X ≤ upper Y ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ∨
     (lower Y ≤ lower X ∧ upper X ≤ upper Y ∧ lower Y ≤ lower X ∧ lower Y ≤ upper X) ∨
     (lower X ≤ lower Y ∧ upper Y ≤ upper X ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X)
    using assms in_bounds2[of _ X Y] unfolding overlapping_def by auto
  have a2: (lower Y ≤ lower X ∧ upper Y ≤ upper X ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ⟹ width (sup X Y) ≤
width X + width Y
    by (simp add: inf_min width_def)
  have a3: (lower X ≤ lower Y ∧ upper X ≤ upper Y ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ⟹ width (sup X Y) ≤
width X + width Y
    by (simp add: inf_min width_def)
  have a4: (lower Y ≤ lower X ∧ upper X ≤ upper Y ∧ lower Y ≤ lower X ∧ lower Y ≤ upper X) ⟹ width (sup X Y) ≤
width X + width Y
    by (simp add: inf_min sup_max width_def)
```

```
  have a5: (lower X ≤ lower Y ∧ upper Y ≤ upper X ∧ lower X ≤ upper Y ∧ lower Y ≤ upper X) ⟹ width (sup X Y) ≤
width X + width Y
    by (metis interval_width_positive le_add_same_cancel1 less_eq_interval_def sup.order_iff)
  show ?thesis using assms a0 a1 a2 a3 a4 a5 by blast
qed

lemma interval_list_union_width:
  fixes xs :: 'a::{linordered_ring, lattice} interval list
  assumes overlapping xs xs ≠ []
  shows overlapping xs ⟹ width (interval_list_union xs) ≤ sum_list (map width xs)
  using assms unfolding contiguous_def width_def
proof (induction xs rule: interval_list_union.induct)
  case 1
  then show ?case by simp
next
  case (2 I)
  then show ?case by simp
next
  case (3 I1 I2 Is)
  then show ?case unfolding overlapping_def
  proof −
    have a0: width (interval_list_union (I1 # I2 # Is)) = width (sup I1 (interval_list_union (I2 # Is)))
      using interval_list_union.simps by simp
    have a1: ... ≤ width I1 + width (interval_list_union (I2 # Is))
    proof −
      have b0: overlapping [I1, interval_list_union (I2 # Is)]
        using 3.prems list.simps(3) list.size(3) list.size(4) interval_list_union_correct
        unfolding overlapping_def
        by (smt (verit, ccfv_threshold) One_nat_def Suc_eq_plus1 diff_add_inverse2
          length_greater_0_conv less_one nth_Cons' subsetD)
      show ?thesis
        using overlapping_width_sum[of I1 (interval_list_union (I2 # Is)), simplified b0]
        by blast
    qed
    have a2: ... ≤ width I1 + sum_list (map width (I2 # Is))
    proof −
      have b0: (width I1 + width (interval_list_union (I2 # Is)) ≤ width I1 + sum_list (map width (I2 # Is))) =
            (width (interval_list_union (I2 # Is)) ≤ sum_list (map width (I2 # Is))) by simp
      have b1: overlapping (I2 # Is) using 3.prems unfolding overlapping_def by force
      have b2: I2 # Is ≠ [] by simp
      have b3: width (interval_list_union (I2 # Is)) ≤ sum_list (map width (I2 # Is))
        using 3.IH b1 b2 unfolding width_def by simp
      show ?thesis using b0 b3 by simp
    qed
    have a3: ... = sum_list (map width (I1 # I2 # Is)) by auto[1]
    show ?thesis using a0 a1 a2 a3 map_eq_conv width_def
      by (smt (verit, ccfv_SIG) dual_order.trans)
  qed
qed

lemma map_non_zero_width:
  fixes U :: 'a::{linordered_idom} interval set
  assumes C−lipschitzI_on U F inclusion_isotonic F set xs ⊆ U
  shows ∀ x ∈ set xs. 0 ≤ width x  ⟶ 0 ≤ width (F x)
```

```
proof
 fix x
 assume a0: x ∈ set xs
 show 0 ≤ width x ⟶ 0 ≤ width (F x)
 proof
  assume a1: 0 ≤ width x
  have a2: width (F x) ≤ C * width x using assms(1,3) a0 unfolding lipschitzI_on_def by blast
  have a4: width (F x) ≤ C * width x + 1 using assms(1,3) a0 unfolding lipschitzI_on_def by fastforce
  have 0 ≤ C * width x using assms(1) a1 unfolding lipschitzI_on_def by simp
  show 0 ≤ width (F x) using assms(1) a0 interval_width_positive unfolding lipschitzI_on_def interval_width_positive by
blast
 qed
qed


lemma inclusion_isotonic_preserves_overlapping:
 assumes inclusion_isotonic F xs ≠ [] F is_interval_extension_of f
 shows contiguous xs ⟹ overlapping (map F xs)
proof (induct xs rule: induct_list012)
 case 1
 then show ?case
  unfolding overlapping_def by simp
next
 case (2 x)
 then show ?case
  unfolding overlapping_def by simp
next
 case (3 x y zs)
 then show ?case
 proof −
  have a0: ∀ i<length (map F (x # y # zs)) − 1. ∃ e. e ∈ᵢ map F (x # y # zs) ! (i + 1) ∧ e ∈ᵢ map F (x # y # zs) ! i
  proof −
   have b0: length (map F (x # y # zs)) ≥ 1 using 3.prems by simp
   have b2: ∀ i<length (x # y # zs) − 1. upper ((x # y # zs) ! i) = lower ((x # y # zs) ! (i + 1))
    using 3.prems unfolding contiguous_def by auto[1]
   have b3: ∀ i<length (x # y # zs) − 1. upper ((x # y # zs) ! i) ≤ upper ((x # y # zs) ! (i + 1))
    using 3.prems unfolding contiguous_def by auto[1]
   have b4: ∀ i<length (x # y # zs) − 1. lower ((x # y # zs) ! i) ≤ lower ((x # y # zs) ! (i + 1))
    using 3.prems lower_le_upper b2 unfolding contiguous_def by metis
   have b5: ∀ i<length (x # y # zs) − 1. f (upper ((x # y # zs) ! i)) = f (lower ((x # y # zs) ! (i + 1)))
    using b2 by simp
   have b6: ∀ i<length (x # y # zs) − 1. f (upper ((x # y # zs) ! i)) ∈ᵢ F ((x # y # zs) ! i)
    using assms b2 b4 in_intervalI interval_of_in_eq
    unfolding is_interval_extension_of_def inclusion_isotonic_def
    by (metis lower_interval_of lower_le_upper upper_interval_of)
   have b7: ∀ i<length (x # y # zs) − 1. f (upper ((x # y # zs) ! i)) ∈ᵢ F ((x # y # zs) ! (i+1))
    using assms b2 b3 in_intervalI interval_of_in_eq
    unfolding is_interval_extension_of_def inclusion_isotonic_def
    by (metis lower_interval_of lower_le_upper upper_interval_of)
   have b8: ∀ i<length (x # y # zs) − 1. f (upper ((x # y # zs) ! i)) ∈ᵢ F ((x # y # zs) ! (i+1)) ∧ f (upper ((x # y #
zs) ! i)) ∈ᵢ F ((x # y # zs) ! i)
    using b6 b7 by blast
   show ?thesis using b8 apply simp
    by (metis One_nat_def Suc_eq_plus1 le_simps(2) list.map(2) list.size(4) nth_map order_less_le)
```

```
  qed
  show ?thesis using ao unfolding overlapping_def by simp
 qed
qed

lemma bounded_image_of:
 fixes f::‹real ⇒ real›
 assumes ‹L−lipschitz_on (set_of X) f›
 shows ‹bounded (f ‘ set_of X)›
 using lipschitz_bounded_image_real[of set_of X L f] assms
 using bounded_set_of set_of_nonempty by blast

lemma dist_le_diameter:
 fixes f::‹real ⇒ real ›
 assumes ‹ C−lipschitz_on (set_of X) f›
 shows ‹dist (f (upper X)) (f (lower X)) ≤ diameter (f ‘ set_of X)›
 using diameter_bounded_bound[of f‘ set_of X f (upper X) f (lower X),
     simplified
     bounded_image_of[of C X f, simplified assms]]
 by simp

lemma excess_width_inf_sup:
 fixes X :: real interval and f::‹real ⇒ real›
 assumes ‹continuous_on (set_of X) f›
 shows Inf (f ‘ set_of X) − lower (F X) + upper (F X) − Sup (f ‘ set_of X) ≤ excess_width F f X
 using diameter_Sup_Inf[of f ‘ set_of X, simplified compact_img_set_of set_f_nonempty assms]
 unfolding excess_width_def  width_def width_set_def set_of_eq
 by simp

lemma excess_width_lower_bound:
 fixes X :: real interval
 assumes  inclusion_isotonic F  F is_interval_extension_of f ‹continuous_on (set_of X) f›
 shows  Inf (f ‘ set_of X) − lower (F X) ≤ excess_width F f X
proof −
 have ao: o ≤ upper (F X) − Sup (f ‘ set_of X)
  using assms(1) assms(2) diff_ge_o_iff_ge inclusion_isotonic_sup
  by metis
 show ?thesis using ao excess_width_inf_sup assms
  by (smt (verit, ccfv_threshold))
qed

lemma excess_width_upper_bound:
 fixes X :: real interval
 assumes  inclusion_isotonic F  F is_interval_extension_of f ‹continuous_on (set_of X) f›
 shows  upper (F X) − Sup (f ‘ set_of X) ≤ excess_width F f X
proof −
 have ao: o ≤ Inf (f ‘ {lower X..upper X}) − lower (F X)
  using assms(1) assms(2) diff_ge_o_iff_ge inclusion_isotonic_sup set_of_eq
   inclusion_isotonic_inf
  by metis
 show ?thesis using ao excess_width_inf_sup assms unfolding set_of_eq
  by (smt (verit, ccfv_threshold))
qed
```

```
lemma lipschitz_excess_width_lower_bound:
 fixes X :: real interval
 assumes inclusion_isotonic F lipschitzI_on C U F F is_interval_extension_of f
 and set (uniform_subdivision X N) ⊆ U  N = 1
 and L−lipschitz_on (set_of (interval_list_union (uniform_subdivision X N))) f
 shows  Inf (f ‘ set_of X) − lower (F X) ≤ C ∗ width X
proof−
 have a0: excess_width F f X ≤ C ∗ width X
  using each_subdivision_excess_width_order[of F C U f X N L, simplified assms]
   assms(4) assms(5) assms(6) div_by_1 insert_iff less_numeral_extra(1) list.set(2) of_nat_1
   uniform_subdivision_id
  by metis
 have a1: Inf (f ‘ set_of X) − lower (F X) ≤ excess_width F f X
  using excess_width_lower_bound[of F f X, simplified assms]
  by (metis assms(5) assms(6) lipschitz_on_continuous_on union_set zero_less_one)
 show ?thesis using a0 a1 by simp
qed

lemma lipschitz_excess_width_upper_bound:
 fixes X :: real interval
 assumes inclusion_isotonic F lipschitzI_on C U F F is_interval_extension_of f
 and set (uniform_subdivision X N) ⊆ U  N = 1
 and L−lipschitz_on (set_of (interval_list_union (uniform_subdivision X N))) f
 shows  upper (F X) − Sup (f ‘ set_of X) ≤ C ∗ width X
proof−
 have a0: excess_width F f X ≤ C ∗ width X
  using each_subdivision_excess_width_order[of F C U f X N L, simplified assms]
   assms(4) assms(5) assms(6) div_by_1 insert_iff less_numeral_extra(1) list.set(2) of_nat_1
   uniform_subdivision_id
  by metis
 have a1: upper (F X) − Sup (f ‘ set_of X) ≤ excess_width F f X
  using excess_width_upper_bound[of F f X, simplified assms]
  by (metis assms(5) assms(6) lipschitz_on_continuous_on union_set zero_less_one)
 show ?thesis using a0 a1 by simp
qed

lemma excess_width_bound_inf:
 fixes X :: real interval
 assumes excess_width_bound: ‹excess_width F f X ≤ k›
 and    inclusion_isotonic: ‹inclusion_isotonic F›
 and    interval_extension: ‹F is_interval_extension_of f›
 shows ‹Inf (f ‘ set_of X) − k ≤ lower (F X)›
 using excess_width_bound[simplified excess_width_def width_def width_set_def]
     inclusion_isotonic interval_extension
 by (smt (verit, best) inclusion_isotonic_sup)

lemma excess_width_bound_sup:
 fixes X :: real interval
 assumes excess_width_bound: ‹excess_width F f X ≤ k›
 and    inclusion_isotonic: ‹inclusion_isotonic F›
 and    interval_extension: ‹F is_interval_extension_of f›
 shows ‹upper (F X) ≤ Sup (f ‘ set_of X) + k›
 using excess_width_bound[simplified excess_width_def width_def width_set_def]
     inclusion_isotonic interval_extension
```

```
 by (smt (verit, best) inclusion_isotonic_inf)

lemma set_of_interval_list_subset_inf_sup:
 assumes non_empty: ‹XS ≠ ([]::real interval list)›
shows ‹set_of_interval_list XS ⊆ {Min(set (map lower XS))..Max(set (map upper XS))}›
 using assms unfolding set_of_interval_list_def
proof(induction XS rule:list_nonempty_induct)
 case (single x)
 then show ?case by(simp add:set_of_eq)
next
 case (cons x xs)
 then show ?case
  apply(simp add:set_of_eq)
  by fastforce
qed

lemma lower_bound_F_inf:
 assumes non_empty: ‹XS ≠ ([]::real interval list)›
 and     inclusion_isotonic: ‹inclusion_isotonic F›
 and     interval_extension: ‹F is_interval_extension_of f›
 and     sorted_lower: ‹sorted_wrt_lower XS›
 and     lipschitz:  ‹0 ≤ C› ‹C−lipschitz_on ((set_of_interval_list XS)) f›
 and     excess_width_bounded: ‹(Max (set ((map (excess_width F f)) XS))) ≤ k›
shows ‹(Inf (f ‘ (set_of_interval_list XS))) − k ≤ Inf (set_of_interval_list (map F XS))›
 using assms proof(induction XS rule:list_nonempty_induct)
 case (single x)
 then show ?case
  using excess_width_bound_inf[of F f x k]
  unfolding set_of_interval_list_def
  by (simp add: inf_set_of)
next
 case (cons x xs)
 then show ?case
  using excess_width_bound_inf[of F f x]
  unfolding set_of_interval_list_def
  apply(simp)
  apply(fold set_of_interval_list_def)
  apply(subst image_Un)
  apply(subst cInf_union_distrib)
  subgoal
   by simp
  subgoal
   apply(simp add: set_of_eq)
   by (meson bounded_imp_bdd_below compact_Icc compact_continuous_image compact_imp_bounded
         lipschitz_on_continuous_on lipschitz_on_subset sup_ge1)
  subgoal
   unfolding set_of_interval_list_def
   by (metis image_is_empty set_of_interval_list_def set_of_interval_list_nonempty)
  subgoal
   using compact_set_of_interval_list[of XS]
      compact_imp_bounded[of (set_of_interval_list XS)]
   by (meson bounded_imp_bdd_below compact_continuous_image compact_imp_bounded
         compact_set_of_interval_list lipschitz_on_continuous_on lipschitz_on_mono sup_ge2)
  subgoal
```

```
    apply(subst cInf_union_distrib)
    subgoal
     by simp
    subgoal
     by (meson bdd_below_set_of)
    subgoal
     by(simp add: set_of_interval_list_nonempty)
    subgoal
     using set_of_interval_list_bdd_below by simp
    subgoal
    proof −
     assume a1: xs ≠ []
     assume a2: ⟦sorted_wrt_lower xs; C−lipschitz_on (set_of_interval_list xs) f⟧ ⟹ Inf (f ‘ set_of_interval_list xs) − k
≤ Inf (set_of_interval_list (map F xs))
     assume a3: sorted_wrt_lower (x # xs)
     assume a4: C−lipschitz_on (set_of x ∪ set_of_interval_list xs) f
     assume a5: excess_width F f x ≤ k ∧ (∀ a∈set xs. excess_width F f a ≤ k)
     assume a6: ⋀k. excess_width F f x ≤ k ⟹ Inf (f ‘ set_of x) − k ≤ lower (F x)
     have f7: sorted_wrt_lower xs
      using a3 a1 sorted_wrt_lower_unroll by blast
     have f8: Inf (set_of (F x)) = lower (F x)
      using inf_set_of by blast
     have f9: C−lipschitz_on (set_of_interval_list xs) f
      using a4 lipschitz_on_subset by blast
     have f10: inf (Inf (f ‘ set_of x)) (Inf (f ‘ set_of_interval_list xs)) ≤ Inf (f ‘ set_of x)
      using inf_le1 by blast
     have f11: inf (Inf (f ‘ set_of x)) (Inf (f ‘ set_of_interval_list xs)) ≤ Inf (f ‘ set_of_interval_list xs)
      using inf_le2 by blast
     have Inf (f ‘ set_of x) − excess_width F f x ≤ lower (F x)
      using a6 by blast
     have inf (Inf (f ‘ set_of x)) (Inf (f ‘ set_of_interval_list xs)) − k ≤ Inf (set_of (F x))
       ∧ inf (Inf (f ‘ set_of x)) (Inf (f ‘ set_of_interval_list xs)) − k ≤ Inf (set_of_interval_list (map F xs))
      using f11 f10 f9 f8 f7 a5 a2
      using ‹Inf (f ‘ set_of x) − excess_width F f x ≤ lower (F x)› by linarith
     then show ?thesis by simp
    qed
    done
   done
qed

lemma upper_bound_F_sup:
  assumes non_empty: ‹XS ≠ ([]::real interval list)›
  and     inclusion_isotonic: ‹inclusion_isotonic F›
  and     interval_extension: ‹F is_interval_extension_of f›
  and     sorted_upper: ‹sorted_wrt_upper XS›
  and     lipschitz:  ‹0 ≤ C› ‹C−lipschitz_on ((set_of_interval_list XS)) f›
  and     excess_width_bounded: ‹(Max (set ((map (excess_width F f)) XS))) ≤ k›
 shows ‹Sup (set_of_interval_list (map F XS)) ≤ (Sup (f ‘ (set_of_interval_list XS))) + k ›
 using assms proof(induction XS rule:list_nonempty_induct)
  case (single x)
  then show ?case
   using excess_width_bound_sup[of F f x k]
   unfolding set_of_interval_list_def
   by (simp add: sup_set_of)
```

```
next
  case (cons x xs)
  then show ?case
    using excess_width_bound_inf[of F f x]
    unfolding set_of_interval_list_def
    apply(simp)
    apply(fold set_of_interval_list_def)
    apply(subst image_Un)
    apply(subst cSup_union_distrib)
    subgoal
      by simp
    subgoal
      by(simp add: set_of_eq)
    subgoal
      by(simp add: set_of_interval_list_nonempty)
    subgoal
      using set_of_interval_list_bdd_above by simp
    subgoal
      apply(subst cSup_union_distrib)
      subgoal
        by simp
      subgoal
        by (meson bdd_above_mono bdd_above_set_of fundamental_theorem_of_interval_analysis)
      subgoal
        by(simp add: set_of_interval_list_nonempty)
      subgoal
        using set_of_interval_list_bdd_above
        by (meson bounded_imp_bdd_above compact_continuous_image compact_imp_bounded
          compact_set_of_interval_list lipschitz_on_continuous_on lipschitz_on_subset sup_ge2)
      subgoal
        apply(auto)[1]
        subgoal
          by (smt (verit) excess_width_def inclusion_isotonic_inf sup_ge1 sup_set_of width_def width_set_def)
        subgoal
          by (smt (verit, ccfv_threshold) lipschitz_on_subset sorted_wrt_upper_unroll sup_ge2)
        done
      done
    done
qed

lemma Inf_interval_list_approx: assumes non_empty: ‹XS ≠ ([]::real interval list)›
  and   inclusion_isotonic: ‹inclusion_isotonic F›
  and   interval_extension: ‹F is_interval_extension_of f›
  and   sorted_upper: ‹sorted_wrt_upper XS›
  and   lipschitz: ‹0 ≤ C› ‹C−lipschitz_on ((set_of_interval_list XS)) f›
  and   excess_width_bounded: ‹(Max (set ((map (excess_width F f)) XS))) ≤ k›
shows Inf (set_of_interval_list (map F XS)) ≤ Inf (f ‘ set_of_interval_list XS)
 using assms proof(induction XS rule:list_nonempty_induct)
  case (single x)
  then show ?case
    apply(simp add: set_of_interval_list_def set_of_eq)
    by (metis inclusion_isotonic_inf set_of_eq)
next
  case (cons x xs)
```

```
 then show ?case
  apply(simp add: set_of_interval_list_def set_of_eq)
  apply(subst image_Un)
  apply(subst cInf_union_distrib)
  subgoal
   by simp
  subgoal
   by simp
  subgoal
   proof −
    assume xs ≠ []
    then have set_of_interval_list (map F xs) ≠ {}
     by (simp add: set_of_interval_list_nonempty)
    then show ?thesis
     by (simp add: set_of_eq set_of_interval_list_def)
   qed
  subgoal
  proof −
   have ∀ is. bdd_below (set_of_interval_list is::real set)
    by (simp add: bounded_imp_bdd_below compact_imp_bounded compact_set_of_interval_list)
   then show ?thesis
    by (simp add: set_of_eq set_of_interval_list_def)
  qed
  subgoal
   apply(subst cInf_union_distrib)
   subgoal
    by simp
   subgoal
    by (meson bounded_imp_bdd_below compact_Icc compact_continuous_image compact_imp_bounded
          lipschitz_on_continuous_on lipschitz_on_subset sup_ge1)
   subgoal
   proof −
    assume xs ≠ []
    then have set_of_interval_list xs ≠ {}
     by (simp add: set_of_interval_list_nonempty)
    then show ?thesis
     by (simp add: set_of_eq set_of_interval_list_def)
   qed
   subgoal
   proof −
    assume C−lipschitz_on ({lower x..upper x} ∪ foldr (λx. (∪) {lower x..upper x}) xs {}) f
    then have C−lipschitz_on ({lower x..upper x} ∪ set_of_interval_list xs) f
     by (simp add: set_of_eq set_of_interval_list_def)
    then have bounded (f ‘ set_of_interval_list xs)
     by (metis compact_imp_bounded compact_set_of_interval_list lipschitz_bounded_image_real lipschitz_on_subset
sup_ge2)
    then show ?thesis
     by (simp add: bounded_imp_bdd_below set_of_eq set_of_interval_list_def)
   qed
   subgoal
    apply(simp, rule conjI)
    subgoal
     using inclusion_isotonic_inf[of F f x, simplified assms set_of_eq, simplified]
     by (smt (verit, ccfv_threshold) inclusion_isotonic_inf le_infI1 set_of_eq)
```

```
      subgoal
      proof −
       assume a1: xs ≠ []
       assume a2: ⟦sorted_wrt_upper xs; C−lipschitz_on (foldr (λx. (∪) {lower x..upper x}) xs {}) f⟧ ⟹ Inf (foldr (λx. (∪) {lower x..upper x}) (map F xs) {}) ≤ Inf (f ‘ foldr (λx. (∪) {lower x..upper x}) xs {})
       assume a3: sorted_wrt_upper (x # xs)
       assume C−lipschitz_on ({lower x..upper x} ∪ foldr (λx. (∪) {lower x..upper x}) xs {}) f
       then have C−lipschitz_on (foldr (λi. (∪) {lower i..upper i}) xs {}) f
        using lipschitz_on_mono by blast
       then show ?thesis
        using a3 a2 a1 by (meson le_infI2 sorted_wrt_upper_unroll)
      qed
      done
    done
  done
qed

lemma Sup_interval_list_approx: assumes non_empty: ‹XS ≠ ([]::real interval list)›
  and   inclusion_isotonic: ‹inclusion_isotonic F›
  and   interval_extension: ‹F is_interval_extension_of f›
  and   sorted_lower: ‹sorted_wrt_lower XS›
  and   lipschitz: ‹0 ≤ C› ‹C−lipschitz_on ((set_of_interval_list XS)) f›
  and   excess_width_bounded: ‹(Max (set ((map (excess_width F f)) XS))) ≤ k›
shows Sup (f ‘ set_of_interval_list XS) ≤ Sup (set_of_interval_list (map F XS))
 using assms proof(induction XS rule:list_nonempty_induct)
 case (single x)
 then show ?case
  apply(simp add: set_of_interval_list_def set_of_eq)
  by (metis inclusion_isotonic_sup set_of_eq)
next
 case (cons x xs)
 then show ?case
  apply(simp add: set_of_interval_list_def set_of_eq)
  apply(subst image_Un)
  apply(subst cSup_union_distrib)
  subgoal
   by simp
  subgoal
       by (meson bounded_imp_bdd_above compact_Icc compact_continuous_image compact_imp_bounded lipschitz_on_continuous_on lipschitz_on_subset sup_ge1)
  subgoal
  proof −
   assume a1: xs ≠ []
   have f2: ∀ R Ra is isa f fa. ((R::real set) ≠ Ra ∨ is ≠ isa ∨ (∃ R i. (i::real interval) ∈ set is ∧ f i R ≠ fa i R)) ∨ foldr f is R = foldr fa isa Ra
    by (meson foldr_cong)
   obtain ii :: (real interval ⇒ real set ⇒ real set) ⇒ (real interval ⇒ real set ⇒ real set) ⇒ real interval list ⇒ real interval and RR :: (real interval ⇒ real set ⇒ real set) ⇒ (real interval ⇒ real set ⇒ real set) ⇒ real interval list ⇒ real set where
    ∀ x0 x1 x3. (∃ v6 v7. v7 ∈ set x3 ∧ x1 v7 v6 ≠ x0 v7 v6) = (ii x0 x1 x3 ∈ set x3 ∧ x1 (ii x0 x1 x3) (RR x0 x1 x3) ≠ x0 (ii x0 x1 x3) (RR x0 x1 x3))
    by moura
   then have ∀ R Ra is isa f fa. (R ≠ Ra ∨ is ≠ isa ∨ ii fa f is ∈ set is ∧ f (ii fa f is) (RR fa f is) ≠ fa (ii fa f is) (RR fa f is)) ∨ foldr f is R = foldr fa isa Ra
```

```
    using f2 by presburger
   then show ?thesis
    using a1 by (smt (z3) image_is_empty set_of_eq set_of_interval_list_def set_of_interval_list_nonempty)
  qed
  subgoal
  proof −
   assume C−lipschitz_on ({lower x..upper x} ∪ foldr (λx. (∪) {lower x..upper x}) xs {}) f
   then have C−lipschitz_on ({lower x..upper x} ∪ set_of_interval_list xs) f
    by (simp add: set_of_eq set_of_interval_list_def)
   then have bounded (f ' set_of_interval_list xs)
       by (metis (no_types) compact_imp_bounded compact_set_of_interval_list lipschitz_bounded_image_real lips-
chitz_on_subset sup_ge2)
   then show ?thesis
    by (simp add: bounded_imp_bdd_above set_of_eq set_of_interval_list_def)
  qed
  subgoal
   apply(subst cSup_union_distrib)
   subgoal
   by simp
  subgoal
   by (meson bdd_above_Icc)
   subgoal
   proof −
    assume xs ≠ []
   then have set_of_interval_list (map F xs) ≠ {}
     by (simp add: set_of_interval_list_nonempty)
    then show ?thesis
     by (simp add: set_of_eq set_of_interval_list_def)
   qed
   subgoal
   proof −
    have ∀ is. bdd_above (set_of_interval_list is::real set)
     by (simp add: bounded_imp_bdd_above compact_imp_bounded compact_set_of_interval_list)
    then show ?thesis
     by (simp add: set_of_eq set_of_interval_list_def)
   qed
   subgoal
    apply(simp, rule conjI)
    subgoal
     using inclusion_isotonic_sup[of F f x, simplified assms set_of_eq, simplified] le_supI1
     by auto
    subgoal
    proof −
     assume a1: xs ≠ []
      assume a2: ⟦sorted_wrt_lower xs; C−lipschitz_on (foldr (λx. (∪) {lower x..upper x}) xs {}) f⟧ ⟹ Sup (f ' foldr
(λx. (∪) {lower x..upper x}) xs {}) ≤ Sup (foldr (λx. (∪) {lower x..upper x}) (map F xs) {})
     assume a3: sorted_wrt_lower (x # xs)
     assume a4: C−lipschitz_on ({lower x..upper x} ∪ foldr (λx. (∪) {lower x..upper x}) xs {}) f
     have f5: sorted_wrt_lower xs
      using a3 a1 sorted_wrt_lower_unroll by blast
      have f6: Sup (foldr (λi. (∪) {lower i..upper i}) (map F xs) {}) ≤ sup (upper (F x)) (Sup (foldr (λi. (∪) {lower
i..upper i}) (map F xs) {}))
       using sup_ge2 by blast
     have C−lipschitz_on (foldr (λi. (∪) {lower i..upper i}) xs {}) f
```

```
        using a4 lipschitz_on_mono by blast
       then show ?thesis
        using f6 f5 a2 by linarith
      qed
     done
    done
  done
qed


lemma map_inclusion_isotonic_excess_width_bounded:
 assumes non_empty: ‹XS ≠ ([]::real interval list)›
 and     inclusion_isotonic: ‹inclusion_isotonic F›
 and     interval_extension: ‹F is_interval_extension_of f›
 and     sorted_lower: ‹sorted_wrt_lower XS›
 and     sorted_upper: ‹sorted_wrt_upper XS›
 and     lipschitz:  ‹C−lipschitz_on ((set_of_interval_list XS)) f›
 and     excess_width_bounded: ‹(Max (set ((map (excess_width F f)) XS))) ≤ k›
shows ‹width_set (set_of_interval_list (map F XS)) − width_set (f ' (set_of_interval_list XS)) ≤ 2 ∗ k›
 and  ‹width_set (set_of_interval_list (map F XS)) − width_set (f ' (set_of_interval_list XS)) ≥ 0›
 subgoal
  unfolding width_set_def
  using lipschitz_on_nonneg[of C ((set_of_interval_list XS)) f, simplified assms, simplified]
      lower_bound_F_inf[of XS F f C k, simplified assms, simplified]
      upper_bound_F_sup[of XS F f C k, simplified assms, simplified]
  by(simp)
 subgoal
  unfolding width_set_def
  using lipschitz_on_nonneg[of C ((set_of_interval_list XS)) f, simplified assms, simplified]
      Inf_interval_list_approx[of XS F f C k, simplified assms, simplified]
      Sup_interval_list_approx[of XS F f C k, simplified assms, simplified]
  by simp
 done

lemma max_subdivision_excess_width_order:
 fixes X :: real interval
 assumes inclusion_isotonic F lipschitzI_on C U F F is_interval_extension_of f
 and set (uniform_subdivision X N) ⊆ U  0 < N
 and  L−lipschitz_on (set_of_interval_list (uniform_subdivision X N)) f
shows ‹Max (set (map (excess_width F f) (uniform_subdivision X N))) ≤ C ∗ width X / real N›
proof(cases  (set (map (excess_width F f) (uniform_subdivision X N))) = {})
 case True
 then show ?thesis
  using non_empty_subdivision[of N X, simplified assms, simplified]
  by simp
next
 case False
 then show ?thesis
  apply(subst set_map)
  using each_subdivision_excess_width_order[of F C U f X N L, simplified assms, simplified]
        set_of_interval_list_set_eq_interval_list_union_contiguous[of  (uniform_subdivision X N), simplified contiguous_uniform_subdivision[of X N] non_empty_subdivision[of N X, simplified assms, simplified], simplified]
  using assms(6) by auto[1]
qed
```

lemma *set_of_interval_list_set_eq_interval_list_union_contiguous*:
assumes *non_empty*: ‹*XS* ≠ ([]::*real interval list*)›
and *contiguous*: ‹*contiguous XS*›
shows ‹*set_of_interval_list XS* = *set_of* (*interval_list_union XS*)›
using *interval_list_union_contiguous*[*of XS*, *simplified assms*, *simplified*]
set_of_interval_list_contiguous[*of XS*,*simplified assms*, *simplified*]
contiguous_non_overlapping[*of XS*, *simplified assms*, *simplified*]
non_overlapping_implies_sorted_wrt_upper[*of XS*]
non_overlapping_implies_sorted_wrt_lower[*of XS*]
interval_list_union_contiguous_lower[*of XS*]
interval_list_union_contiguous_upper[*of XS*]
using *set_of_eq non_empty* **by** *metis*

lemma *width_eq_wdith_set*:
fixes *X* :: ‹('*a*::{*conditionally_complete_lattice*, *minus*, *preorder*}) *interval*›
shows ‹*width X* = *width_set* (*set_of X*)›
unfolding *width_def set_of_eq width_set_def* **by**(*simp*)

lemma *width_zero_lower_upper*:
fixes *X* :: *real interval*
assumes ‹*width X* = *o*›
shows ‹*lower X* = *upper X*›
using *assms width_def*[*of X*]
by *simp*

lemma *width_zero_mk_interval*:
fixes *X* :: *real interval*
assumes ‹*width X* = *o*›
shows ‹∃ *x*. *X* = *mk_interval*(*x*,*x*)›
using *assms width_def*[*of X*]
unfolding *mk_interval'*
by (*auto*, *metis Interval_id*)

lemma *width_zero_interval_of*:
fixes *X* :: *real interval*
assumes ‹*width X* = *o*›
shows ‹∃ *x*. *X* = *interval_of x*›
using *assms width_def*[*of X*]
by (*metis eq_iff_diff_eq_o interval_eqI lower_interval_of upper_interval_of*)

lemma *width_zero_interval_extension*:
fixes *F* :: *real interval* ⇒ *real interval*
assumes ‹*F is_interval_extension_of f*›
and ‹*width X* = *o*›
shows ‹*width* (*F X*) = *o*›
using *assms width_zero_interval_of*[*of X*, *simplified assms*, *simplified*]
unfolding *is_interval_extension_of_def*
by (*metis add_o add_diff_cancel lower_interval_of upper_interval_of width_def*)

9.3 Lipschitz Interval Inclusive

If F is a natural interval extension of a real valued rational function with $F\ X$ defined for $X \subseteq Y$ where X and Y are intervals or n-dimentional interval vectors then F is Lipschitz in Y

```
lemma interval_extension_bounded:
  fixes F :: real interval ⇒ real interval
  assumes ‹F is_interval_extension_of f›
  and     ‹(width (F X)) / (width X) ≤ L›
shows width (F X) ≤ L * width X
proof(cases  width X = 0)
  case True
  then show ?thesis
    using width_zero_interval_extension[of F f X, simplified True assms, simplified]
    by auto
next
  case False
  then show ?thesis
    using assms interval_width_positive[of X]
    by (metis mult.commute mult_right_mono nonzero_mult_div_cancel_left times_divide_eq_right)
qed

lemma lipschitz_on_implies_lipschitzI_on:
  fixes F :: real interval ⇒ real interval
  assumes ‹F is_interval_extension_of f›
  and ‹C−lipschitz_on X f›
  and ‹⋃ (set_of ‘ Y) ⊆ X›
  and ‹0 ≤ L›
  and ‹∀ y ∈ Y. (width (F y)) / (width y) ≤ L›
shows L−lipschitzI_on Y F
  unfolding lipschitzI_on_def
  using assms interval_extension_bounded
  by(auto)

lemma lipschitz_on_implies_lipschitzI_on2:
  fixes f :: ‹real ⇒ real›
  assumes ‹S ≠ []› and ‹0 ≤ C›
   and ‹F is_interval_extension_of f›
   and ‹0 ≤ L›
   and ‹∀ y ∈ (set S). (width (F y)) / (width y) ≤ L›
   and ‹C−lipschitz_on (set_of (interval_list_union (S))) f ›
  shows ‹L−lipschitzI_on (set (S)) F›
  apply(rule lipschitzI_onI)
  subgoal using assms interval_extension_bounded by blast
  subgoal using assms by blast
  done


lemma width_img_Max:
  assumes ‹finite S›
  shows ‹∀ x∈S. width (F x) ≤ Max (width ‘ F ‘ S)›
  using assms by auto
lemma width_Min:
  assumes ‹finite S›
```

shows ‹$\forall x{\in}S.\ Min\ (width\ `\ S) \leq width\ x$›
using *assms* **by** *auto*

lemma *lipschitzI_on_le_interval*:
fixes *F* :: ‹*real interval* ⇒ *real interval*›
assumes *inc_isontonic_F*: ‹*inclusion_isotonic F*›
and *lipschitzI_F*: ‹$C-lipschitzI_on\ \{X\}\ F$›
and *interval_inc*: ‹$x \leq X$›
shows ‹$width\ (F\ x) \leq C * width\ X$›
using *assms*
unfolding *inclusion_isotonic_def lipschitzI_on_def*
width_def less_eq_interval_def
by *fastforce*

lemma *lipschitzI_on_le_lipschitzI_on*:
fixes *F* :: ‹*real interval* ⇒ *real interval*›
assumes *inc_isontonic_F*: ‹*inclusion_isotonic F*›
and *lipschitzI_F*: ‹$C-lipschitzI_on\ \{X\}\ F$›
and *interval_inc*: ‹$x \leq X$›
and *interval_ext*: ‹*F is_interval_extension_of f*›
shows ‹$\exists\ L.\ L-lipschitzI_on\ \{x\}\ F$›
apply$(rule\ exI[of\ _\ C * (width\ X)/(width\ x)])$
apply$(subst\ lipschitzI_onI, simp_all\ add: assms)$
subgoal
apply$(rule\ conjI)$
subgoal
using *width_zero_interval_extension*$[of\ F\ f\ x, simplified\ assms, simplified]$
by *simp*
subgoal
using *inc_isontonic_F interval_inc lipschitzI_F lipschitzI_on_le_interval* **by** *blast*
done
using *lipschitzI_on_le_interval*$[of\ F\ C\ X\ x, simplified\ assms\ , simplified]$
subgoal
using *lipschitzI_on_nonneg assms*
by $(metis\ divide_nonneg_nonneg\ interval_width_positive\ mult_nonneg_nonneg)$
done

lemma *uniform_subdivision_le*:
fixes *X* :: ‹*real interval*›
assumes ‹$N{>}0$›
shows ‹$\forall\ x \in set\ (uniform_subdivision\ X\ N).\ x \leq X$›
using *last_upper_uniform_subdivision*$[of\ N\ X, simplified\ assms, simplified]$
hd_lower_uniform_subdivision$[of\ N\ X, simplified\ assms, simplified]$
non_overlapping_implies_sorted_wrt_upper$[of\ (uniform_subdivision\ X\ N), simplified\ uniform_subdivisions_non_overlapping\ assms, simplified]$
non_overlapping_implies_sorted_wrt_lower$[of\ (uniform_subdivision\ X\ N), simplified\ uniform_subdivisions_non_overlapping\ assms, simplified]$
unfolding *sorted_wrt_upper_def sorted_wrt_lower_def cmp_lower_width_def less_eq_interval_def*
by $(metis\ (no_types, lifting)\ assms\ in_set_conv_nth\ interval_list_union_correct\ non_empty_subdivision\ set_of_subset_iff\ union_set)$

lemma *lipschitzI_on_uniform_subdivision*:
fixes *F* :: ‹*real interval* ⇒ *real interval*›
assumes *inc_isontonic_F*: ‹*inclusion_isotonic F*›

```
    and lipschitzI_F:   ‹C−lipschitzI_on ({X}) F›
    and ‹N>0›
  shows ‹∀ x∈(set (uniform_subdivision X N)). width (F x) ≤ C * width X›
 using lipschitzI_on_le_interval[of F C X, simplified assms, simplified]
     uniform_subdivision_le[of N X, simplified assms, simplified ]
 by simp

lemma division_leq_pos:
 fixes x :: 'a::{linordered_field}
 assumes x > 0 and y > 0 and z > 0 and y ≤ z
 shows x / z ≤ x / y
proof −
 have x * y ≤ x * z using assms by simp
 hence (x * y) / (y * z) ≤ (x * z) / (y * z)
  using assms
  by (simp add: frac_le)
 thus ?thesis using assms by auto[1]
qed

lemma each_subdivision_width_order':
 fixes X :: real interval
 assumes F is_interval_extension_of f
 and 0 < N
 and Xs ∈ set (uniform_subdivision X N)
shows ∃ L. width(F Xs) ≤ L * width (X) / of_nat N
 by (metis assms less_numeral_extra(3) mult_eq_0_iff nonzero_divide_eq_eq
    of_nat_le_0_iff order_refl uniform_subdivisions_width width_zero_interval_extension)

lemma uniform_subdivision_min_nonzero:
 assumes ‹N > 0›
 and ‹width X > 0›
 shows ‹0 < Min (width ' set (uniform_subdivision X N))›
 using assms unfolding image_set uniform_subdivision_def Let_def width_def
 by (simp, simp add: order_le_less)

lemma uniform_subdivision_width_zero_replicate_eq:
 fixes X::‹real interval›
 assumes positive_N:  ‹0 < N›
 and zero_width_X: ‹0 = width X ›
 shows‹replicate N X = (uniform_subdivision X N)›
 using assms
proof(induction N)
 case 0
 then show ?case
  by simp
next
 case (Suc N)
 then show ?case
  using width_zero_lower_upper[of X, simplified assms, simplified]
  unfolding uniform_subdivision_def
  by (simp, metis append.right_neutral map_replicate_trivial mk_interval_id replicate_app_Cons_same)
qed

lemma set_of_interval_list_zero_width:
```

```
 fixes X::‹real interval›
 assumes positive_N: ‹0 < N›
 and zero_width_X: ‹0 = width X ›
 shows‹set_of_interval_list (uniform_subdivision X N) = {lower X..upper X}›
proof(insert assms, simp add: uniform_subdivision_width_zero_replicate_eq[of N X, simplified assms, simplified, sym-
metric], induction N)
 case 0
 then show ?case
  by(simp)
next
 case (Suc N)
 then show ?case
  unfolding set_of_interval_list_def set_of_eq
  by(simp, fastforce)
qed

lemma  width_zero_subdivision: width X = (0::real) ==> N > 0 ⟹ set (uniform_subdivision X N) = {X}
 using width_zero_lower_upper[of X]
 unfolding uniform_subdivision_def Let_def mk_interval'
 apply(auto simp add: image_def)[1]
 apply (metis Interval_id)
 apply (metis Interval_id)
 done

lemma lipschitz_on_implies_lipschitzI_on_pre:
 fixes f :: ‹real ⇒ real›
 and F  :: ‹real interval ⇒ real interval›
 assumes ‹finite S›
  and ‹0 < Min (width ' S)›
 shows ‹let max_F_width = Max (width ' (F ' S));
         min_f_width = Min (width ' S)
      in ∀ x ∈ S. width (F x) ≤ (max_F_width/min_f_width) *  width x›
 unfolding Let_def
 by (simp, smt (verit) assms division_leq_pos interval_width_positive mult_eq_0_iff
       nonzero_mult_div_cancel_right width_Min width_img_Max zero_le_divide_iff)

lemma lipschitz_on_implies_lipschitzI_on':
 fixes f :: ‹real ⇒ real›
   and F  :: ‹real interval ⇒ real interval›
 assumes  non_empty: ‹S ≠ {}›
  and       finite: ‹finite S›
  and non_zero_width: ‹0 < Min (width ' S)›
  and interval_ext_F: ‹F is_interval_extension_of f›
 shows ‹∃  L. L−lipschitzI_on S F›
 unfolding lipschitzI_on_def
 apply(rule exI[of _ (Max (width ' (F ' S)))/(Min (width ' S))])
 apply(rule conjI)
 subgoal
  apply(rule divide_nonneg_nonneg)
  subgoal
   using assms interval_width_positive
   by (metis (mono_tags, lifting) Max_in finite_imageI imageE image_is_empty)
  subgoal
   using assms interval_width_positive
```

```
    by(auto)[1]
  done
 subgoal
  by (meson assms lipschitz_on_implies_lipschitzI_on_pre)
 done


lemma natural_extension_transfer_lipschitz:
 assumes  positive_N: ‹0 < N›
 and inc_isontonic_F: ‹inclusion_isotonic F›
 and  interval_ext_F: ‹F is_natural_interval_extension_of f›
 and     lipschitz_f: ‹C−lipschitz_on (set_of X) f›
shows    ‹C−lipschitzI_on (set (uniform_subdivision X N)) F›
proof(cases 0 < width X)
 case True
 then show ?thesis
 apply(subst lipschitzI_onI)
 subgoal
  using assms
   each_subdivision_width_order'[of F f N _ X ]
   each_subdivision_width_order[of F C set (uniform_subdivision X N) f X N ]
   uniform_subdivision_le[of N X ]
  unfolding lipschitz_on_def is_natural_interval_extension_of_def
        inclusion_isotonic_def
  dist_real_def abs_real_def width_def mk_interval'
  apply(auto split:if_splits)[1]
  by (smt (z3) Interval_id interval_of.abs_eq interval_of_in_eq less_eq_interval_def lower_bounds lower_in_interval up-
per_bounds)
 subgoal
  using assms lipschitz_on_nonneg by auto
 subgoal by simp
 done
next
 case False
  have width_zero: ‹width X = 0›
  using False
  by (meson interval_width_positive linorder_not_le nle_le)
 have us_X: ‹set (uniform_subdivision X N) = {X}›
  using width_zero width_zero_subdivision
  using positive_N by blast
 then show ?thesis
  using width_zero
  apply(simp add:assms)
   apply(subst lipschitzI_onI)
  subgoal   using assms
   each_subdivision_width_order'[of F f N _ X ]
   each_subdivision_width_order[of F C set (uniform_subdivision X N) f X N ]
   uniform_subdivision_le[of N X ]
  unfolding lipschitz_on_def is_natural_interval_extension_of_def
        inclusion_isotonic_def
  dist_real_def abs_real_def width_def mk_interval'
  apply(auto split:if_splits)[1]
  by (meson interval_ext_F natural_interval_extension_implies_interval_extension)
 subgoal
```

```
  using assms lipschitz_on_nonneg by auto
 subgoal by simp
 done
qed


lemma lipschitz_on_division_lipschitz_on:
 assumes lipschitz_f: C−lipschitz_on (set_of X) f
   and non_empty: uniform_subdivision X N ≠ []
   and subdivision: Xs ∈ set(uniform_subdivision (X::real interval) N)
 shows ∃ L . L−lipschitz_on (set_of Xs) f
proof −
 fix L
 have subset: Xs ≤ X
   using assms(3) uniform_subdivision_le[of N X]
   by (metis (no_types, lifting) bot_nat_0.extremum_strict
     bot_nat_0.not_eq_extremum in_set_conv_nth length_map list.size(3) map_nth
     uniform_subdivision_def)
 show ?thesis
   by (meson assms(1) interval_set_leq_eq lipschitz_on_subset subset)
qed

lemma lipschitz_on_lischitz_on_subdivisions:
 assumes lipschitz_f: C−lipschitz_on (set_of X) f
   and non_empty: uniform_subdivision X N ≠ []
   and non_zero: 0 < N
 shows ∃ L . ∀ Xs ∈ set(uniform_subdivision (X::real interval) N). L−lipschitz_on (set_of Xs) f
proof −
 have subset: ∀ Xs ∈ set(uniform_subdivision (X::real interval) N) . Xs ≤ X
   using assms uniform_subdivision_le[of N X] by blast
 show ∃ L. ∀Xs∈set (uniform_subdivision X N). L−lipschitz_on (set_of Xs) f
 proof −
   obtain L where L: L = C using lipschitz_f by auto
   have L−lipschitz_on (set_of Xs) f if Xs ∈ set (uniform_subdivision X N) for Xs
   proof −
     show ?thesis using lipschitz_on_division_lipschitz_on[of C X f N Xs, simplified assms, simplified] L
       by (meson interval_set_leq_eq lipschitz_f lipschitz_on_subset subset that)
   qed
   thus ?thesis by auto
 qed
qed

lemma lipschitz_on_lischitz_on_subdivisions_n:
 assumes lipschitz_f: C−lipschitz_on (set_of X) f
   and non_empty: uniform_subdivision X N ≠ []
   and non_zero: 0 < N
 shows ∃ L . ∀ N > 0 . ∀ Xs ∈ set(uniform_subdivision (X::real interval) N). L−lipschitz_on (set_of Xs) f
proof −
 have subset: ∀ Xs ∈ set(uniform_subdivision (X::real interval) N) . Xs ≤ X
   using assms uniform_subdivision_le[of N X] by blast
 show ∃ L. ∀ N > 0 . ∀Xs∈set (uniform_subdivision X N). L−lipschitz_on (set_of Xs) f
 proof −
   obtain L where L: L = C using lipschitz_f by auto
   have L−lipschitz_on (set_of Xs) f if Xs ∈ set (uniform_subdivision X N) for Xs
```

```
    proof −
     show ?thesis using lipschitz_on_division_lipschitz_on[of C X f N Xs, simplified assms, simplified] L
       by (meson interval_set_leq_eq lipschitz_f lipschitz_on_subset subset that)
    qed
    thus ?thesis
     by (meson interval_set_leq_eq lipschitz_f lipschitz_on_subset uniform_subdivision_le)
  qed
qed

lemma lipschitzI_on_division_lipschitzI_on:
  assumes lipschitz_f: C−lipschitzI_on (set(uniform_subdivision X N)) F
    and non_empty: uniform_subdivision X N ≠ []
    and subdivision: Xs ∈ set(uniform_subdivision (X::real interval) N)
  shows ∃ L . L−lipschitzI_on {Xs} F
proof −
  fix L
  have subset: Xs ≤ X
    using assms(3) uniform_subdivision_le[of N X]
    by (smt (verit, del_insts) gr_zeroI list.map_disc_iff list.size(3) map_nth non_empty uniform_subdivision_def)
  show ?thesis
    by (meson empty_subsetI insert_subset lipschitzI_on_le lipschitz_f subdivision)
qed

lemma lipschitzI_on_lipschitzI_on_subdivisions:
  fixes X :: real interval
  assumes lipschitz_f: C−lipschitzI_on (set(uniform_subdivision X N)) F
    and non_zero: 0 < N
  shows ∃ L . ∀ Xs ∈ set(uniform_subdivision X N). L−lipschitzI_on {Xs} F
proof −
  have subset: ∀ Xs ∈ set(uniform_subdivision (X::real interval) N) . Xs ≤ X
    using assms uniform_subdivision_le[of N X] by blast
  show ∃L. ∀Xs∈set (uniform_subdivision X N). L−lipschitzI_on {Xs} F
  proof −
    obtain L where L: L = C using lipschitz_f by auto
    have L−lipschitzI_on {Xs} F if Xs ∈ set (uniform_subdivision X N) for Xs
    proof −
      show ?thesis using assms
        by (simp add: L lipschitzI_on_def that)
    qed
    thus ?thesis by auto
  qed
qed
```

9.4 Lipschitz Convergence

```
lemma isotonic_lipschitz_refinement':
  assumes positive_N: ‹0 < N›
  and inc_isontonic_F: ‹inclusion_isotonic F›
  and interval_ext_F: ‹F is_interval_extension_of f›
  and lipschitz_f:    ‹C−lipschitz_on (set_of X) f›
shows ‹∃ L. width_set (set_of_interval_list (map F (uniform_subdivision X N))) − width_set (f ‘ (set_of X)) ≤ 2 * (L *
width X / real N)›
proof (cases 0 < width X)
```

```
 case True
 have us_eq_set_of_X: ‹(set_of_interval_list (uniform_subdivision X N)) = set_of X›
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
        contiguous_uniform_subdivision non_empty_subdivision positive_N union_set)
 have lipschitz_f': ‹C−lipschitz_on (set_of_interval_list (uniform_subdivision X N)) f›
  by (simp add: lipschitz_f us_eq_set_of_X)
 have us_nonempty: ‹uniform_subdivision X N ≠ [] ›
  by (simp add: assms(1) non_empty_subdivision)
 have us_nonempty_set: ‹set (uniform_subdivision X N) ≠ {}›
  by (simp add: us_nonempty)
 have us_finite: ‹finite (set (uniform_subdivision X N))›
  by simp
 have sorted_lower: ‹sorted_wrt_lower (uniform_subdivision X N)›
  by (simp add: contiguous_sorted_wrt_lower contiguous_uniform_subdivision)
 have sorted_upper: ‹sorted_wrt_upper (uniform_subdivision X N)›
  by (simp add: contiguous_sorted_wrt_upper contiguous_uniform_subdivision)
 have lipschitzI: ‹∃ L. L−lipschitzI_on (set (uniform_subdivision X N)) F›
  using lipschitz_on_implies_lipschitzI_on'[of set (uniform_subdivision X N) F f, simplified
                                 us_nonempty_set us_finite interval_ext_F]
     uniform_subdivision_min_nonzero[of N X, simplified positive_N True, simplified]
  by(simp)
    have set_of_interval_eq: ‹(set_of_interval_list (uniform_subdivision X N)) = (set_of (interval_list_union
(uniform_subdivision X N)))›
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
        contiguous_uniform_subdivision us_nonempty)
 have width_bounded: ‹∃ L. Max (excess_width F f ' set (uniform_subdivision X N)) ≤ 2 * (L * width X / real N)›
  using
     each_subdivision_excess_width_order[of F _ (set (uniform_subdivision X N)) f X N C,
      simplified inc_isontonic_F interval_ext_F positive_N lipschitz_f'[simplified set_of_interval_eq], simplified]
    max_subdivision_excess_width_order[of F _ (set (uniform_subdivision X N)) f X N C,
      simplified inc_isontonic_F interval_ext_F positive_N lipschitz_f', simplified]
 proof −
  assume a1: ⋀C. C−lipschitzI_on (set (uniform_subdivision X N)) F ⟹
   Max (excess_width F f ' set (uniform_subdivision X N)) ≤ C * width X / real N
  have ∀ r ra rb. (r::real) / rb * ra = r * ra / rb
   by fastforce
  then show ?thesis
   using a1 by (metis ‹∃ L. L−lipschitzI_on (set (uniform_subdivision X N)) F›
     divide_numeral_1 le_divide_eq_numeral1(1) mult_numeral_1 order_antisym_conv order_refl)
 qed
 have width_limit:‹∃ L.  width_set (set_of_interval_list (map F (uniform_subdivision X N)))
              − width_set (f ' (set_of X))
             ≤ 2 * (L * width X / real N)›
  using width_bounded
     map_inclusion_isotonic_excess_width_bounded(1)[of uniform_subdivision X N F f C,
      simplified us_nonempty interval_ext_F inc_isontonic_F sorted_lower sorted_upper lipschitz_f', simplified]
  by (metis (no_types, lifting) us_eq_set_of_X inc_isontonic_F interval_ext_F lipschitzI lipschitz_f' list.set_map
       max_subdivision_excess_width_order order.refl positive_N)
 then show ?thesis
  by simp
next
 case False
 have width_zero: ‹width X = 0›
  using False
```

```
  by (meson interval_width_positive linorder_not_le nle_le)
 have us_nonempty: ‹uniform_subdivision X N ≠ [] ›
  by (simp add: assms(1) non_empty_subdivision)
   have set_of_interval_eq: ‹(set_of (interval_list_union (uniform_subdivision X N))) = (set_of_interval_list
(uniform_subdivision X N))›
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
        contiguous_uniform_subdivision us_nonempty)
 have w_zero_1: ‹width_set (set_of_interval_list (map F (uniform_subdivision X N))) = 0›
  by (metis (full_types) Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous

    contiguous_uniform_subdivision interval_ext_F map_replicate non_empty_subdivision positive_N
                    uniform_subdivision_width_zero_replicate_eq union_set width_eq_wdith_set width_zero
width_zero_interval_extension)
 have w_zero_2: ‹ width_set (f ' (set_of X)) = 0›
  using set_of_interval_list_zero_width[of N X, simplified positive_N width_zero, simplified]
    width_zero width_zero_lower_upper[of X, simplified width_zero, simplified]
  by (metis diff_ge_0_iff_ge inc_isontonic_F inf_le_sup_image_real interval_ext_F lipschitz_f nle_le
      width_inclusion_isotonic_approx width_set_def width_zero_interval_extension)
 then show ?thesis
  using width_zero w_zero_1 w_zero_2
  by simp
qed

lemma isotonic_lipschitz_refinementI:
 assumes positive_N: ‹0 < N›
 and inc_isontonic_F: ‹inclusion_isotonic F›
 and interval_ext_F: ‹F is_interval_extension_of f›
 and lipschitz_f: ‹L−lipschitz_on (set_of X) f›
 and lipschitz_F: ‹C−lipschitzI_on (set (uniform_subdivision X N)) F›
shows ‹width_set (set_of_interval_list (map F (uniform_subdivision X N))) − width_set (f ' (set_of X)) ≤ 2 * (C * width
X / real N)›
proof (cases 0 < width X)
 case True
 have us_eq_set_of_X: ‹(set_of_interval_list (uniform_subdivision X N)) = set_of X›
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
        contiguous_uniform_subdivision non_empty_subdivision positive_N union_set)
 have lipschitz_f ': ‹L−lipschitz_on (set_of_interval_list (uniform_subdivision X N)) f›
  by (simp add: lipschitz_f us_eq_set_of_X)
 have us_nonempty: ‹uniform_subdivision X N ≠ [] ›
  by (simp add: assms(1) non_empty_subdivision)
 have us_nonempty_set: ‹set (uniform_subdivision X N) ≠ {}›
  by (simp add: us_nonempty)
 have us_finite: ‹finite (set (uniform_subdivision X N))›
  by simp
 have sorted_lower: ‹sorted_wrt_lower (uniform_subdivision X N)›
  by (simp add: contiguous_sorted_wrt_lower contiguous_uniform_subdivision)
 have sorted_upper: ‹sorted_wrt_upper (uniform_subdivision X N)›
  by (simp add: contiguous_sorted_wrt_upper contiguous_uniform_subdivision)
 have lipschitzI: ‹C−lipschitzI_on (set (uniform_subdivision X N)) F›
  using lipschitz_F by blast
   have set_of_interval_eq: ‹(set_of_interval_list (uniform_subdivision X N)) = (set_of (interval_list_union
(uniform_subdivision X N)))›
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
        contiguous_uniform_subdivision us_nonempty)
```

```
 have width_bounded: ‹Max (excess_width F f ' set (uniform_subdivision X N)) ≤ 2 * (C * width X / real N)›
  using
      each_subdivision_excess_width_order[of F C (set (uniform_subdivision X N)) f X N _,
        simplified inc_isontonic_F interval_ext_F positive_N lipschitz_f'[simplified set_of_interval_eq], simplified]
      max_subdivision_excess_width_order[of F C (set (uniform_subdivision X N)) f X N _,
        simplified inc_isontonic_F interval_ext_F positive_N lipschitz_f', simplified]
       by (smt (verit) divide_nonneg_nonneg interval_width_positive lipschitzI_on_def lipschitz_F lipschitz_f'
of_nat_0_less_iff positive_N split_mult_pos_le)
 have width_limit:‹width_set (set_of_interval_list (map F (uniform_subdivision X N)))
                  − width_set (f ' (set_of X))
                 ≤ 2 * (C * width X / real N)›
  using width_bounded
     by (metis assms(3) inc_isontonic_F lipschitz_F lipschitz_f' map_inclusion_isotonic_excess_width_bounded(1)
max_subdivision_excess_width_order order_le_less positive_N sorted_lower sorted_upper us_eq_set_of_X us_nonempty)

  then show ?thesis
  by simp
next
 case False
 have width_zero: ‹width X = 0›
  using False
  by (meson interval_width_positive linorder_not_le nle_le)
 have us_nonempty: ‹uniform_subdivision X N ≠ [] ›
  by (simp add: assms(1) non_empty_subdivision)
   have set_of_interval_eq: ‹(set_of (interval_list_union (uniform_subdivision X N))) = (set_of_interval_list
(uniform_subdivision X N))›
  by (simp add: Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous
        contiguous_uniform_subdivision us_nonempty)
 have w_zero_1: ‹width_set (set_of_interval_list (map F (uniform_subdivision X N))) = 0›
  by (metis (full_types) Lipschitz_Subdivisions_Refinements.set_of_interval_list_set_eq_interval_list_union_contiguous

     contiguous_uniform_subdivision interval_ext_F map_replicate non_empty_subdivision positive_N
                        uniform_subdivision_width_zero_replicate_eq union_set width_eq_wdith_set width_zero
width_zero_interval_extension)
 have w_zero_2: ‹ width_set (f ' (set_of X)) = 0›
  using set_of_interval_list_zero_width[of N X, simplified positive_N width_zero, simplified]
      width_zero width_zero_lower_upper[of X, simplified width_zero, simplified]
  by (simp add: set_of_eq width_set_def)
 then show ?thesis
  using width_zero w_zero_1 w_zero_2
  by simp
qed

lemma isotonic_lipschitz_refinement:
 assumes positive_N: ‹0 < N›
 and inc_isontonic_F: ‹inclusion_isotonic F›
 and interval_ext_F: ‹F is_interval_extension_of f›
 and lipschitz_f:   ‹L−lipschitz_on (set_of X) f›
 and lipschitz_F:   ‹C−lipschitzI_on (set (uniform_subdivision X N)) F›
shows ‹excess_width_set (refinement_set F N) f X ≤ 2 * (C * width X / real N)›
 using isotonic_lipschitz_refinementI[of N F f L X C, simplified assms, simplified]
 unfolding excess_width_set_def refinement_set_def by simp

end
```

10 Interval Analysis (Interval_Analysis)

theory
Interval_Analysis
imports
Interval_Division_Real
Lipschitz_Subdivisions_Refinements
begin

This theory provides interval analysis over standard types such as real or integer. All operations work over (closed) intervals.

end

11 Extended Division on Intervals (Extended_Interval_Division)

theory
Extended_Interval_Division
imports
Interval_Division_Non_Zero
begin

In this theory, we define an extended division operation on intervals. This definition is inspired by the definition given in [4], but we use an over-approximation for the case in which zero is an element of the divisor interval. By this, we avoid the need for multi-intervals.

instantiation *interval* :: $(\{infinity, linordered_field, real_normed_algebra, linear_continuum_topology\})$ *inverse*
begin
definition *inverse_interval* :: *'a interval* $\Rightarrow$ *'a interval*
where *inverse_interval a* $=$ (
if $(\neg 0 \in_i a)$ *then mk_interval* $(1 / (upper\ a), 1 / (lower\ a))$
else if lower a $= 0 \wedge 0 <$ *upper a then mk_interval* $(1/\ upper\ a, \infty)$
else if lower a $< 0 \wedge 0 <$ *upper a then mk_interval* $(-\infty, \infty)$
else if lower a $<$ *upper a* $\wedge$ *upper a* $= 0$ *then mk_interval*$(-\infty, 1 / lower\ a)$
else undefined
)

definition *divide_interval* :: *'a interval* $\Rightarrow$ *'a interval* $\Rightarrow$ *'a interval*
where *divide_interval a b* $=$ *inverse b* $*$ *a*
instance ..
end

interpretation *interval_division_inverse divide inverse*
apply(*unfold_locales*)
subgoal
by (*simp add*: *inverse_interval_def*)
subgoal
by(*simp add*: *divide_interval_def*)
done

end

12 Extended Interval Analysis (Extended_Interval_Analysis)

theory
Extended_Interval_Analysis
imports
Extended_Interval_Division
Lipschitz_Subdivisions_Refinements
begin

This theory provides extended interval analysis over the type extended reals. All operations work over (closed) intervals.

end

12.1 Overlapping Multi-Intervals (Multi_Interval_Overlapping)

theory
Multi_Interval_Overlapping
imports
Multi_Interval_Preliminaries
begin

12.1.1 Type Definition

typedef (**overloaded**) *'a minterval_ovl* =
$\{is::'a::\{minus_mono\}\ interval\ list.\ valid_mInterval_ovl\ is\}$
morphisms *bounds_of_minterval_ovl mInterval_ovl*
unfolding *valid_mInterval_ovl_def*
apply(*intro exI*[**where** *x*=[*Interval* (*l*,*l*)] **for** *l*])
by(*auto simp add*: *sorted_wrt_lower_def non_overlapping_sorted_def*)

setup_lifting *type_definition_minterval_ovl*

lift_definition *mlower_ovl*::($'a::\{minus_mono\}$) *minterval_ovl* ⇒ *'a* **is** ‹*lower o hd*› .
lift_definition *mupper_ovl*::($'a::\{minus_mono\}$) *minterval_ovl* ⇒ *'a* **is** ‹*upper o last*› .
lift_definition *mlist_ovl*::($'a::\{minus_mono\}$) *minterval_ovl* ⇒ *'a interval list* **is** ‹*id*› .

12.1.2 Equality and Orderings

lemma *minterval_ovl_eq_iff*: *a* = *b* ⟷ *mlist_ovl a* = *mlist_ovl b*
by *transfer auto*

lemma *ainterval_eqI*: *mlist_ovl a* = *mlist_ovl b* ⟹ *a* = *b*
by (*auto simp*: *minterval_ovl_eq_iff*)

lemma *minterval_ovl_imp_upper_lower_eq* :
a = *b* ⟶ *mlower_ovl a* = *mlower_ovl b* ∧ *mupper_ovl a* = *mupper_ovl b*

```
 by transfer auto

lemma valid_mInterval_ovl_lower_le_upper:
 valid_mInterval_ovl i ⟹ (lower ∘ hd) i ≤ (upper ∘ last) i
proof(induction i)
 case Nil
 then show ?case
  unfolding valid_mInterval_ovl_def  o_def sorted_wrt_lower_def cmp_lower_width_def
  by simp
next
 case (Cons a i)
 then show ?case
  unfolding valid_mInterval_ovl_def  o_def sorted_wrt_lower_def cmp_lower_width_def
  by (metis (no_types, lifting) Cons.IH Cons.prems comp_apply distinct.simps(2)
     in_bounds last.simps list.sel(1) lower_in_interval order_less_imp_le order_less_le_trans
     sorted_wrt_lower_unroll valid_mInterval_ovl_def)
qed

lemma mlower_non_ovl_le_mupper_non_ovl[simp]: mlower_ovl i ≤ mupper_ovl i
 by(transfer, metis valid_mInterval_ovl_lower_le_upper)

lift_definition set_of_ovl :: 'a::{minus_mono} minterval_ovl ⇒ 'a set
 is λ is. ⋃x∈set is. {lower x..upper x} .

lemma not_in_ovl_eq:
 ‹(¬ e ∈ set_of_ovl xs) = (∀ x ∈ set (mlist_ovl xs). ¬ e ∈ set_of x)›
proof(induction (mlist_ovl xs))
 case Nil
 then show ?case
  by (metis UN_iff empty_iff empty_set mlist_ovl.rep_eq set_of_ovl.rep_eq)
next
 case (Cons a x)
 then show ?case
  by (simp add: mlist_ovl.rep_eq set_of_eq set_of_ovl.rep_eq)
qed

lemma in_ovl_eq:
 ‹(e ∈ set_of_ovl xs) = (∃ x ∈ set (mlist_ovl xs). e ∈ set_of x)›
proof(induction (mlist_ovl xs))
 case Nil
 then show ?case
  by (metis UN_iff empty_iff empty_set mlist_ovl.rep_eq set_of_ovl.rep_eq)
next
 case (Cons a x)
 then show ?case
  by (simp add: mlist_ovl.rep_eq set_of_eq set_of_ovl.rep_eq)
qed

context notes [[typedef_overloaded]] begin

lift_definition(code_dt) mInterval_ovl'::'a::minus_mono interval list ⇒ 'a minterval_ovl option
 is λ is. if  valid_mInterval_ovl is then Some is else None
 by (auto simp add: valid_mInterval_ovl_def)
```

lemma *mInterval_ovl'_split*:
 $P\ (mInterval_ovl'\ is) \longleftrightarrow$
  $(\forall\ ivl.\ \ valid_mInterval_ovl\ is \longrightarrow mlist_ovl\ ivl = is \longrightarrow P\ (Some\ ivl)) \wedge (\neg\ valid_mInterval_ovl\ is \longrightarrow P\ None)$
 by *transfer auto*

lemma *mInterval_ovl'_split_asm*:
 $P\ (mInterval_ovl'\ is) \longleftrightarrow$
  $\neg((\exists\ ivl.\ valid_mInterval_ovl\ is \wedge mlist_ovl\ ivl = is \wedge \neg P\ (Some\ ivl)) \vee (\neg\ valid_mInterval_ovl\ is \wedge \neg P\ None))$
 unfolding *mInterval_ovl'_split*
 by *auto*

lemmas *mInterval_ovl'_splits* = *mInterval_ovl'_split mInterval_ovl'_split_asm*

lemma *mInterval'_eq_Some*: $mInterval_ovl'\ is = Some\ i \Longrightarrow mlist_ovl\ i = is$
 by (*simp split*: *mInterval_ovl'_splits*)

end

lemma *set_of_ovl_non_zero_list_all*:
 ‹$o \notin set_of_ovl\ xs \Longrightarrow \forall\ x \in set\ (mlist_ovl\ xs).\ \neg\ o \in_i x$›
proof(*induction mlist_ovl xs*)
 case *Nil*
 then show *?case*
  by *simp*
next
 case (*Cons a x*)
 then show *?case*
  using *in_ovl_eq* **by** *blast*
qed

instantiation *minterval_ovl* :: ({*minus_mono*}) *equal*
begin

definition *equal_class.equal a b* ≡ (*mlist_ovl a* = *mlist_ovl b*)

instance proof qed (*simp add*: *equal_minterval_ovl_def minterval_ovl_eq_iff*)
end

instantiation *minterval_ovl* :: ({*minus_mono*}) *ord* **begin**

definition *less_eq_minterval_ovl* :: *'a minterval_ovl* ⇒ *'a minterval_ovl* ⇒ *bool*
 where *less_eq_minterval_ovl a b* ⟷ *mlower_ovl b* ≤ *mlower_ovl a* ∧ *mupper_ovl a* ≤ *mupper_ovl b*

definition *less_minterval_ovl* :: *'a minterval_ovl* ⇒ *'a minterval_ovl* ⇒ *bool*
 where *less_minterval_ovl x y* = (*x* ≤ *y* ∧ ¬ *y* ≤ *x*)

instance proof qed
end

instantiation *minterval_ovl* :: ({*minus_mono*,*lattice*}) *sup*
begin

lift_definition *sup_minterval_non_ovl* :: *'a minterval_ovl* ⇒ *'a minterval_ovl* ⇒ *'a minterval_ovl*
 is λ *a b*. [*Interval* (*inf* (*lower* (*hd a*)) (*lower* (*hd b*)), *sup* (*upper* (*last a*)) (*upper* (*last b*)))]

```
 by(auto simp: valid_mInterval_ovl_def sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def
          non_overlapping_sorted_def le_infI1 le_supI1 valid_mInterval_non_ovl_def mupper_ovl_def
          mlower_ovl_def)
instance
 by(standard)
end

instantiation minterval_ovl :: ({lattice,minus_mono}) preorder
begin
instance
 apply(standard)
 subgoal
  using less_minterval_ovl_def by auto
 subgoal
  by (simp add: less_eq_minterval_ovl_def)
 subgoal
  by (meson less_eq_minterval_ovl_def order.trans)
 done
end

lift_definition minterval_ovl_of :: 'a::{minus_mono} ⇒ 'a minterval_ovl is λx. [Interval(x, x)]
 unfolding valid_mInterval_ovl_def valid_mInterval_ovl_def non_adjacent_sorted_wrt_lower_def
       cmp_non_adjacent_def sorted_wrt_lower_def
 by simp

lemma mlower_ovl_minterval_ovl_of[simp]: mlower_ovl (minterval_ovl_of a) = a
 by transfer auto

lemma mupper_ovl_minterval_ovl_of[simp]: mupper_ovl (minterval_ovl_of a) = a
 by transfer auto

definition width_ovl :: 'a::{minus_mono} minterval_ovl ⇒ 'a
 where width_ovl i = mupper_ovl i − mlower_ovl i
```

12.1.3 Zero and One

```
instantiation minterval_ovl :: ({minus_mono,zero}) zero
begin

lift_definition zero_minterval_ovl::'a minterval_ovl is mk_mInterval_ovl [Interval (0, 0)]
 by (simp add: mk_mInterval_ovl_valid)

lemma mlower_ovl_zero[simp]: mlower_ovl 0 = 0
 by(transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)

lemma mupper_ovl_zero[simp]: mupper_ovl 0 = 0
 by(transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)

instance proof qed
end

instantiation minterval_ovl :: ({minus_mono,one}) one
begin
```

```
lift_definition one_minterval_ovl::'a minterval_ovl is mk_mInterval_ovl [Interval (1, 1)]
 by (simp add: mk_mInterval_ovl_valid)

lemma mlower_ovl_one[simp]: mlower_ovl 1 = 1
 by(transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)

lemma mupper_ovl_one[simp]: mupper_ovl 1 = 1
 by(transfer, simp add: mk_mInterval_ovl_def interval_sort_lower_width_def)

instance proof qed
end
```

12.1.4 Addition

```
instantiation minterval_ovl :: ({minus_mono,ordered_ab_semigroup_add,linordered_field}) plus
begin

lift_definition plus_minterval_ovl::'a minterval_ovl ⇒ 'a minterval_ovl ⇒ 'a minterval_ovl
 is λ a b . mk_mInterval_ovl (iList_plus a b)
 by (metis bin_op_interval_list_non_empty iList_plus_def mk_mInterval_ovl_valid valid_mInterval_ovl_def)

lemma valid_mk_interval_iList_plus:
 assumes valid_mInterval_ovl a and valid_mInterval_ovl b
 shows valid_mInterval_ovl (mk_mInterval_ovl (iList_plus a b))
 by (metis (no_types, lifting) assms(1) assms(2) bin_op_interval_list_empty iList_plus_lower_upper_eq
       mk_mInterval_ovl_id mk_mInterval_ovl_valid )

lift_definition plus_minterval_non_ovl::'a minterval_ovl ⇒ 'a minterval_ovl ⇒ 'a minterval_ovl
  is λ a b . mk_mInterval_ovl (iList_plus a b)
  by (simp add: valid_mk_interval_iList_plus)

lemma interval_plus_com:
 ‹a + b = b + a› for a::'a::{minus_mono,ordered_ab_semigroup_add,linordered_field} minterval_ovl
 apply(transfer)
 using plus_minterval_ovl_def
 by (metis (no_types, opaque_lifting) foldr_isort_elements iList_plus_commute
       interval_sort_lower_width_def mk_mInterval_ovl_def mk_mInterval_ovl_distinct
       mk_mInterval_ovl_sorted o_def set_remdups sorted_wrt_lower_distinct_lists_eq)

instance proof qed
end
```

12.1.5 Unary Minus

```
lemma a: (x::'a::ordered_ab_group_add interval) ≠ y ⟹ −x ≠ −y
    apply(simp add:uminus_interval_def)
   by (smt (z3) Pair_inject bounds_of_interval_inverse case_prod_Pair_iden case_prod_unfold neg_equal_iff_equal umi-
nus_interval.rep_eq)

lemma b: distinct (is::'a::ordered_ab_group_add interval list) ⟹ distinct (map (λ i. −i) is)
proof(induction is)
 case Nil
 then show ?case by simp
next
```

case (*Cons a is*)
then show *?case* **using** *a* **by** *force*
qed

instantiation *minterval_ovl* :: ({*minus_mono*, *ordered_ab_group_add*}) *uminus*
begin

lift_definition *uminus_minterval_ovl*::*'a minterval_ovl* ⇒ *'a minterval_ovl*
is λ *is* . *mk_mInterval_ovl* (*rev* (*map* (λ *i*. −*i*) *is*))
by (*metis list.map_disc_iff mk_mInterval_ovl_valid rev_is_Nil_conv valid_mInterval_ovl_def*)
instance ..
end

12.1.6 Subtraction

instantiation *minterval_ovl* :: ({*minus_mono*, *linordered_field*, *ordered_ab_group_add*}) *minus*
begin

definition *minus_minterval_ovl*::*'a minterval_ovl* ⇒ *'a minterval_ovl* ⇒ *'a minterval_ovl*
where *minus_minterval_ovl a b* = *a* + − *b*

instance ..
end

12.1.7 Multiplication

instantiation *minterval_ovl* :: ({*minus_mono*,*linordered_semiring*}) *times*
begin

lift_definition *times_minterval_ovl* :: *'a minterval_ovl* ⇒ *'a minterval_ovl* ⇒ *'a minterval_ovl*
is λ *a b* . *mk_mInterval_ovl* (*iList_times a b*)
by (*metis bin_op_interval_list_non_empty iList_times_def mk_mInterval_ovl_empty*
mk_mInterval_ovl_valid)

instance ..
end

12.1.8 Multiplicative Inverse and Division

locale *minterval_ovl_division* = *inverse* +
constrains *inverse* :: ‹*'a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_ovl ⇒ *'a minterval_ovl*›
and *divide* :: ‹*'a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_ovl ⇒ *'a minterval_ovl* ⇒ *'a minterval_ovl*›
assumes *inverse_left*: ‹¬ *o* ∈ *set_of_ovl x* ⟹ *1* ≤ (*inverse x*) ∗ *x*›
and *divide*: ‹¬ *o* ∈ *set_of_ovl y* ⟹ *x* ≤ (*divide x y*) ∗ *y*›
begin
end

locale *minterval_ovl_division_inverse* = *inverse* +
constrains *inverse* :: ‹*'a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_ovl ⇒ *'a minterval_ovl*›

and *divide* :: ‹'a::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_ovl ⇒ 'a *minterval_ovl* ⇒ 'a *minterval_ovl*›
assumes *inverse_non_zero_def*: ‹¬ *o* ∈ *set_of_ovl x* ⟹ (*inverse x*)
= *mInterval_ovl* (*mk_mInterval_ovl*(*un_op_interval_list* (λ *i*. *mk_interval* (*1* / (*upper i*), *1* / (*lower i*)))
(*mlist_ovl x*)))›
and *divide_non_zero_def*: ‹¬ *o* ∈ *set_of_ovl y* ⟹ (*divide x y*) = *inverse y* ∗ *x*›
begin

end

12.1.9 Membership

abbreviation (**in** *preorder*) *in_minterval_ovl* (‹(_/ ∈$_{no}$ _)› [*51*, *51*] *50*)
where *in_minterval_ovl x X* ≡ *x* ∈ *set_of_ovl X*

lemma *in_minterval_ovl_to_minterval_ovl*[*intro*!]: *a* ∈$_{no}$ *minterval_ovl_of a*
by (*metis* (*mono_tags*, *lifting*) *UN_iff list.set_intros*(*1*) *lower_in_interval lower_point_interval*
minterval_ovl_of.rep_eq set_of_eq set_of_ovl.rep_eq)

instance *minterval_ovl* :: ({*one*, *preorder*, *minus_mono*, *linordered_semiring*}) *power*
proof qed

lemma *set_of_one_ovl*[*simp*]: *set_of_ovl* (*1*::'a::{*one*, *order*, *minus_mono*} *minterval_ovl*) = {*1*}
apply (*auto simp*: *set_of_ovl_def*)[*1*]
subgoal
by (*simp add*: *interval_sort_lower_width_set_eq mk_mInterval_ovl_def one_minterval_ovl.rep_eq*)
subgoal
by (*simp add*: *interval_sort_lower_width_set_eq mk_mInterval_ovl_def one_minterval_ovl.rep_eq*)
done

lifting_update *minterval_ovl.lifting*
lifting_forget *minterval_ovl.lifting*

end

12.2 Non-Overlapping Multi-Intervals (Multi_Interval_Non_Overlapping)

theory
Multi_Interval_Non_Overlapping
imports
Multi_Interval_Preliminaries
begin

12.2.1 Type Definition

typedef (**overloaded**) 'a *minterval_non_ovl* =
{*is*::'a::{*minus_mono*} *interval list*. *valid_mInterval_non_ovl is*}
morphisms *bounds_of_minterval_non_ovl mInterval_non_ovl*
unfolding *valid_mInterval_non_ovl_def*
apply(*intro exI*[**where** *x*=[*Interval* (*l*,*l*)] **for** *l*])
by(*auto simp add*: *valid_mInterval_ovl_def sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def*)

setup_lifting *type_definition_minterval_non_ovl*

lift_definition *mlower_non_ovl*::('*a*::{*minus_mono*}) *minterval_non_ovl* ⇒ '*a* **is** ‹*lower o hd*› .
lift_definition *mupper_non_ovl*::('*a*::{*minus_mono*}) *minterval_non_ovl* ⇒ '*a* **is** ‹*upper o last*› .
lift_definition *mlist_non_ovl*::('*a*::{*minus_mono*}) *minterval_non_ovl* ⇒ '*a interval list* **is** ‹*id*› .

12.2.2 Equality and Orderings

lemma *minterval_non_ovl_eq_iff*: *a* = *b* ⟷ *mlist_non_ovl a* = *mlist_non_ovl b*
by *transfer auto*

lemma *ainterval_eqI*: *mlist_non_ovl a* = *mlist_non_ovl b* ⟹ *a* = *b*
by (*auto simp*: *minterval_non_ovl_eq_iff*)

lemma *minterval_non_ovl_imp_upper_lower_eq* :
a = *b* ⟶ *mlower_non_ovl a* = *mlower_non_ovl b* ∧ *mupper_non_ovl a* = *mupper_non_ovl b*
by *transfer auto*

lemma *mlower_non_ovl_le_mupper_non_ovl*[*simp*]: *mlower_non_ovl i* ≤ *mupper_non_ovl i*
by (*transfer*, *metis comp_def hd_Nil_eq_last lower_le_upper lower_le_upper_aux*
non_adjacent_implies_non_overlapping valid_mInterval_non_ovl_def)

lift_definition *set_of_non_ovl* :: '*a*::{*minus_mono*} *minterval_non_ovl* ⇒ '*a set*
is λ *is*. ⋃*x*∈*set is*. {*lower x*..*upper x*} .

lemma *set_non_ovl_of_subset*: *set_of_non_ovl* (*x*::'*a*::*minus_mono minterval_non_ovl*) ⊆ {*mlower_non_ovl x* .. *mupper_non_ovl x*}
apply(*transfer*, *simp*)
using *set_of_subeq_aux*
mInterval_ovl_lower_hd_min[*symmetric*, *simplified o_def*]
mInterval_adj_upper_last_max[*symmetric*, *simplified o_def*]
valid_non_ovl_imp_adj valid_non_ovl_imp_ovl
list.set_map
by (*metis* (*mono_tags*, *lifting*))

lemma *not_in_non_ovl_eq*:
‹(¬ *e* ∈ *set_of_non_ovl xs*) = (∀ *x* ∈ *set* (*mlist_non_ovl xs*). ¬ *e* ∈ *set_of x*)›
proof(*induction* (*mlist_non_ovl xs*))
case *Nil*
then show *?case*
by (*metis UN_iff empty_iff empty_set mlist_non_ovl.rep_eq set_of_non_ovl.rep_eq*)
next
case (*Cons a x*)
then show *?case*
by (*simp add*: *mlist_non_ovl.rep_eq set_of_eq set_of_non_ovl.rep_eq*)
qed

lemma *in_non_ovl_eq*:
‹(*e* ∈ *set_of_non_ovl xs*) = (∃ *x* ∈ *set* (*mlist_non_ovl xs*). *e* ∈ *set_of x*)›
proof(*induction* (*mlist_non_ovl xs*))
case *Nil*

```
 then show ?case
  by (metis UN_iff empty_iff empty_set mlist_non_ovl.rep_eq set_of_non_ovl.rep_eq)
next
 case (Cons a x)
 then show ?case
  by (simp add: mlist_non_ovl.rep_eq set_of_eq set_of_non_ovl.rep_eq)
qed

lemma set_of_non_ovl_non_zero_list_all:
 ‹o ∉ set_of_non_ovl xs ⟹ ∀ x ∈ set (mlist_non_ovl xs). ¬ o ∈ᵢ x›
proof(induction mlist_non_ovl xs)
 case Nil
 then show ?case
  by simp
next
 case (Cons a x)
 then show ?case
  using in_non_ovl_eq by blast
qed

context notes [[typedef_overloaded]] begin


lift_definition(code_dt) mInterval_non_ovl'::'a::minus_mono interval list ⇒ 'a minterval_non_ovl option
 is λ is. if  valid_mInterval_non_ovl is then Some is else None
 by (auto simp add: valid_mInterval_non_ovl_def)

lemma mInterval_non_ovl'_split:
 P (mInterval_non_ovl' is) ⟷
  (∀ ivl.  valid_mInterval_non_ovl is ⟶ mlist_non_ovl ivl = is ⟶ P (Some ivl)) ∧ (¬ valid_mInterval_non_ovl is ⟶
P None)
 by transfer auto

lemma mInterval_non_ovl'_split_asm:
 P (mInterval_non_ovl' is) ⟷
  ¬((∃ ivl. valid_mInterval_non_ovl is ∧ mlist_non_ovl ivl = is ∧ ¬P (Some ivl)) ∨ (¬ valid_mInterval_non_ovl is ∧ ¬P
None))
 unfolding mInterval_non_ovl'_split
 by auto

lemmas mInterval_non_ovl'_splits = mInterval_non_ovl'_split mInterval_non_ovl'_split_asm

lemma mInterval'_eq_Some: mInterval_non_ovl' is = Some i ⟹ mlist_non_ovl i = is
 by (simp split: mInterval_non_ovl'_splits)

end

instantiation minterval_non_ovl :: ({minus_mono}) equal
begin

definition equal_class.equal a b ≡ (mlist_non_ovl a = mlist_non_ovl b)

instance proof qed (simp add: equal_minterval_non_ovl_def minterval_non_ovl_eq_iff)
end
```

```
instantiation minterval_non_ovl :: ({minus_mono}) ord begin

definition less_eq_minterval_non_ovl :: 'a minterval_non_ovl ⇒ 'a minterval_non_ovl ⇒ bool
  where less_eq_minterval_non_ovl a b ⟷ mlower_non_ovl b ≤ mlower_non_ovl a ∧ mupper_non_ovl a ≤ mupper_non_ovl b

definition less_minterval_non_ovl :: 'a minterval_non_ovl ⇒ 'a minterval_non_ovl ⇒ bool
 where  less_minterval_non_ovl x y = (x ≤ y ∧ ¬ y ≤ x)

instance proof qed
end

instantiation minterval_non_ovl :: ({minus_mono,lattice}) sup
begin

lift_definition sup_minterval_non_ovl :: 'a minterval_non_ovl ⇒ 'a minterval_non_ovl ⇒ 'a minterval_non_ovl
 is λ a b. [Interval (inf (lower (hd a)) (lower (hd b)), sup (upper (last a)) (upper (last b)))]
 by(auto simp: valid_mInterval_ovl_def sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def
          non_overlapping_sorted_def le_infI1 le_supI1 valid_mInterval_non_ovl_def mupper_non_ovl_def
          mlower_non_ovl_def )

lemma mlower_non_ovl_sup[simp]: mlower_non_ovl (sup A B) = inf (mlower_non_ovl A) (mlower_non_ovl B)
 apply(transfer)
 by (metis comp_apply le_supE le_supI1 list.sel(1) lower_bounds lower_le_upper_aux sup_inf_absorb
       valid_mInterval_adj_def valid_non_ovl_imp_adj)

lemma mupper_non_ovl_sup[simp]: mupper_non_ovl (sup A B) = sup (mupper_non_ovl A) (mupper_non_ovl B)
 apply(transfer)
 by (metis (no_types, lifting) comp_def inf_sup_absorb last.simps le_infI1 le_inf_iff lower_le_upper_aux
       upper_bounds valid_mInterval_adj_def valid_non_ovl_imp_adj)
instance
 by(standard)
end

instantiation minterval_non_ovl :: ({lattice,minus_mono}) preorder
begin
instance
 apply(standard)
 subgoal
  using less_minterval_non_ovl_def by auto
 subgoal
  by (simp add: less_eq_minterval_non_ovl_def)
 subgoal
  by (meson less_eq_minterval_non_ovl_def order.trans)
 done
end

lemma set_of_minterval_non_ovl_union: set_of_non_ovl A ∪ set_of_non_ovl B ⊆ set_of_non_ovl (sup A B)
 for A::'a::{lattice, minus_mono} minterval_non_ovl
 apply(transfer, simp)
 using set_of_subeq_aux
     mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
     mInterval_adj_upper_last_max[symmetric, simplified o_def]
```

valid_non_ovl_imp_adj valid_non_ovl_imp_ovl
list.set_map
by (smt (verit) le_sup_iff lower_bounds lower_le_upper_aux lower_sup set_of_eq set_of_subset_iff
sup.commute sup.commute sup.order_iff sup_ge1 sup_ge1 sup_inf_absorb upper_bounds
valid_mInterval_adj_def)

lemma minterval_non_ovl_union_commute: sup A B = sup B A **for** A::'a::{minus_mono,lattice} minterval_non_ovl
apply (auto simp add: minterval_non_ovl_eq_iff inf.commute sup.commute)[1]
by (simp add: mlist_non_ovl.rep_eq inf_commute sup_minterval_non_ovl.rep_eq sup_commute)

lemma minterval_non_ovl_union_mono1: set_of_non_ovl a ⊆ set_of_non_ovl (sup a A)
for A :: 'a::{minus_mono,lattice} minterval_non_ovl
apply(transfer, simp)
using set_of_subeq_aux
mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
mInterval_adj_upper_last_max[symmetric, simplified o_def]
valid_non_ovl_imp_adj valid_non_ovl_imp_ovl
list.set_map
by (smt (verit, del_insts) inf.absorb_iff2 inf_le1 le_infI1 lower_bounds lower_le_upper_aux
set_of_eq set_of_subset_iff sup_ge1 upper_bounds valid_mInterval_adj_def)

lemma minterval_non_ovl_union_mono2: set_of_non_ovl A ⊆ set_of_non_ovl (sup a A) **for** A :: 'a::{lattice, minus_mono}
minterval_non_ovl
apply(transfer, simp)
using set_of_subeq_aux
mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
mInterval_adj_upper_last_max[symmetric, simplified o_def]
valid_non_ovl_imp_adj valid_non_ovl_imp_ovl
list.set_map
by (smt (verit, del_insts) inf.absorb_iff2 le_sup_iff lower_bounds lower_le_upper_aux nle_le
set_of_eq set_of_subset_iff sup_inf_absorb upper_bounds valid_mInterval_adj_def)

lift_definition minterval_non_ovl_of :: 'a::{minus_mono} ⇒ 'a minterval_non_ovl **is** λx. [Interval(x, x)]
unfolding valid_mInterval_non_ovl_def valid_mInterval_ovl_def non_adjacent_sorted_wrt_lower_def
cmp_non_adjacent_def sorted_wrt_lower_def
by simp

lemma mlower_non_ovl_minterval_non_ovl_of[simp]: mlower_non_ovl (minterval_non_ovl_of a) = a
by transfer auto

lemma mupper_non_ovl_minterval_non_ovl_of[simp]: mupper_non_ovl (minterval_non_ovl_of a) = a
by transfer auto

definition width_non_ovl :: 'a::{minus_mono} minterval_non_ovl ⇒ 'a
where width_non_ovl i = mupper_non_ovl i − mlower_non_ovl i

12.2.3 Zero and One

instantiation minterval_non_ovl :: ({minus_mono,zero}) zero
begin

lift_definition zero_minterval_non_ovl::'a minterval_non_ovl **is** mk_mInterval_non_ovl [Interval (0, 0)]
by (simp add: mk_mInterval_non_ovl_valid sorted_wrt_lower_def)

lemma *mlower_non_ovl_zero*[*simp*]: *mlower_non_ovl* 0 = 0
by(*transfer*, *simp add*: *mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

lemma *mupper_non_ovl_zero*[*simp*]: *mupper_non_ovl* 0 = 0
by(*transfer*, *simp add*: *mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

instance proof qed
end

instantiation *minterval_non_ovl* :: ({*minus_mono*,*one*}) *one*
begin

lift_definition *one_minterval_non_ovl*::'*a minterval_non_ovl* **is** *mk_mInterval_non_ovl* [*Interval* (1, 1)]
by (*metis list.simps*(3) *mk_mInterval_non_ovl_single mk_mInterval_non_ovl_valid sorted_wrt_lower_mk_mInterval_non_ovl*)

lemma *mlower_non_ovl_one*[*simp*]: *mlower_non_ovl* 1 = 1
by(*transfer*, *simp add*: *mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

lemma *mupper_non_ovl_one*[*simp*]: *mupper_non_ovl* 1 = 1
by(*transfer*, *simp add*: *mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

instance proof qed
end

12.2.4 Addition

instantiation *minterval_non_ovl* :: ({*minus_mono*,*ordered_ab_semigroup_add*,*linordered_field*}) *plus*
begin

lemma *valid_mk_interval_iList_plus*:
assumes *valid_mInterval_non_ovl a* **and** *valid_mInterval_non_ovl b*
shows *valid_mInterval_non_ovl* (*mk_mInterval_non_ovl* (*iList_plus a b*))
by (*metis* (*no_types*, *lifting*) *assms*(1) *assms*(2) *bin_op_interval_list_empty iList_plus_lower_upper_eq mk_mInterval_non_ovl_id mk_mInterval_non_ovl_non_empty mk_mInterval_non_ovl_valid sorted_wrt_lower_mk_mInterval_non_ovl*)

lift_definition *plus_minterval_non_ovl*::'*a minterval_non_ovl* ⇒ '*a minterval_non_ovl* ⇒ '*a minterval_non_ovl*
is λ *a b* . *mk_mInterval_non_ovl* (*iList_plus a b*)
by (*simp add*: *valid_mk_interval_iList_plus*)

lemma *interval_plus_com*:
‹*a* + *b* = *b* + *a*› **for** *a*::'*a*::{*minus_mono*,*ordered_ab_semigroup_add*,*linordered_field*} *minterval_non_ovl*
apply(*transfer*)
using *iList_plus_mInterval_non_ovl_commute plus_minterval_non_ovl_def*
by *auto*

instance proof qed

end

12.2.5 Unary Minus

lemma *a*: (*x*::'*a*::*ordered_ab_group_add interval*) ≠ *y* ⟹ −*x* ≠ −*y*

apply(*simp add*:*uminus_interval_def*)
by (*smt* (*z3*) *Pair_inject bounds_of_interval_inverse case_prod_Pair_iden case_prod_unfold neg_equal_iff_equal uminus_interval.rep_eq*)

lemma *b*: *distinct* (*is*::'*a*::*ordered_ab_group_add interval list*) ⟹ *distinct* (*map* (λ *i*. −*i*) *is*)
proof(*induction is*)
case *Nil*
then show *?case* **by** *simp*
next
case (*Cons a is*)
then show *?case* **using** *a* **by** *force*
qed

instantiation *minterval_non_ovl* :: ({*minus_mono*, *ordered_ab_group_add*}) *uminus*
begin

lift_definition *uminus_minterval_non_ovl*::'*a minterval_non_ovl* ⇒ '*a minterval_non_ovl*
is λ *is* . *mk_mInterval_non_ovl* (*rev* (*map* (λ *i*. −*i*) *is*))
by (*metis* (*no_types*, *lifting*) *distinct_remdups_id list.map_disc_iff mk_mInterval_non_ovl_def mk_mInterval_ovl_sorted non_adj_sorted_mkInterval_non_ovl non_adjacent_implies_distinct o_def rev_is_Nil_conv valid_mInterval_non_ovl_def valid_mInterval_ovl_def valid_ovl_mkInterval_non_ovl*)

instance ..
end

12.2.6 Subtraction

instantiation *minterval_non_ovl* :: ({*minus_mono*, *linordered_field*, *ordered_ab_group_add*}) *minus*
begin

definition *minus_minterval_non_ovl*::'*a minterval_non_ovl* ⇒ '*a minterval_non_ovl* ⇒ '*a minterval_non_ovl*
where *minus_minterval_non_ovl a b* = *a* + − *b*

instance ..
end

12.2.7 Multiplication

instantiation *minterval_non_ovl* :: ({*minus_mono*, *linordered_field*}) *times*
begin

lift_definition *times_minterval_non_ovl*::'*a minterval_non_ovl* ⇒ '*a minterval_non_ovl* ⇒ '*a minterval_non_ovl*
is λ *a b* . *mk_mInterval_non_ovl* (*iList_times a b*)
by (*metis* (*no_types*, *lifting*) *bin_op_interval_list_empty distinct_remdups_id iList_times_def mk_mInterval_non_ovl_def mk_mInterval_ovl_sorted non_adj_sorted_mkInterval_non_ovl non_adjacent_implies_distinct o_def valid_mInterval_non_ovl_def valid_mInterval_ovl_def valid_ovl_mkInterval_non_ovl*)

instance ..
end

12.2.8 Multiplicative Inverse and Division

locale *minterval_non_ovl_division* = *inverse* +
 constrains *inverse* :: ‹'*a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_non_ovl ⇒ '*a minterval_non_ovl*›
 and *divide* :: ‹'*a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_non_ovl ⇒ '*a minterval_non_ovl* ⇒ '*a minterval_non_ovl*›
 assumes *inverse_left*: ‹¬ *o* ∈ *set_of_non_ovl x* ⟹ *1* ≤ (*inverse x*) ∗ *x*›
 and *divide*: ‹¬ *o* ∈ *set_of_non_ovl y* ⟹ *x* ≤ (*divide x y*) ∗ *y*›
begin
end

locale *minterval_non_ovl_division_inverse* = *inverse* +
 constrains *inverse* :: ‹'*a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_non_ovl ⇒ '*a minterval_non_ovl*›
 and *divide* :: ‹'*a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*}
minterval_non_ovl ⇒ '*a minterval_non_ovl* ⇒ '*a minterval_non_ovl*›
 assumes *inverse_non_zero_def*: ‹¬ *o* ∈ *set_of_non_ovl x* ⟹ (*inverse x*)
 = *mInterval_non_ovl* (*mk_mInterval_non_ovl*(*un_op_interval_list* (λ *i*. *mk_interval* (*1* / (*upper i*), *1* /
(*lower i*))) (*mlist_non_ovl x*)))›
 and *divide_non_zero_def*: ‹¬ *o* ∈ *set_of_non_ovl y* ⟹ (*divide x y*) = *inverse y* ∗ *x*›
begin

end

12.2.9 Membership

abbreviation (**in** *preorder*) *in_minterval_non_ovl* (‹(_/ ∈$_{no}$ _)›) [*51*, *51*] *50*)
 where *in_minterval_non_ovl x X* ≡ *x* ∈ *set_of_non_ovl X*

lemma *in_minterval_non_ovl_to_minterval_non_ovl*[*intro*!]: *a* ∈$_{no}$ *minterval_non_ovl_of a*
 by (*simp add*: *minterval_non_ovl_of .rep_eq set_of_non_ovl_def*)

instance *minterval_non_ovl* :: ({*one*, *preorder*, *minus_mono*, *linordered_semiring*}) *power*
proof qed

lemma *set_of_one_non_ovl*[*simp*]: *set_of_non_ovl* (*1*::'*a*::{*one*, *minus_mono*, *order*} *minterval_non_ovl*) = {*1*}
 apply(*transfer*)
 by(*auto simp*: *mk_mInterval_non_ovl_def mk_mInterval_ovl_def interval_sort_lower_width_def set_of_non_ovl_def*)

lifting_update *minterval_non_ovl.lifting*
lifting_forget *minterval_non_ovl.lifting*

end

12.3 Adjacent Multi-Intervals (Multi_Interval_Adjacent)

theory
 Multi_Interval_Adjacent
imports
 Multi_Interval_Preliminaries
begin

12.3.1 A Type For Non Overlapping Multi Intervals

```
typedef (overloaded) 'a minterval_adj =
  {is::'a::{minus_mono,linorder} interval list. valid_mInterval_adj is}
  morphisms bounds_of_minterval_adj mInterval_adj
  unfolding valid_mInterval_adj_def
  apply(intro exI[where x=[Interval (l,l) ] for l])
  by(auto simp add: sorted_wrt_lower_def non_overlapping_sorted_def)

setup_lifting type_definition_minterval_adj

lift_definition mlower_adj::('a::{minus_mono}) minterval_adj ⇒ 'a is ‹lower o hd› .
lift_definition mupper_adj::('a::{minus_mono}) minterval_adj ⇒ 'a is ‹upper o last› .
lift_definition mlist_adj::('a::{minus_mono}) minterval_adj ⇒ 'a interval list is ‹id› .
```

12.3.2 Equality and Orderings

```
lemma minterval_adj_eq_iff: a = b ⟷ mlist_adj a = mlist_adj b
  by transfer auto

lemma ainterval_eqI: mlist_adj a = mlist_adj b ⟹ a = b
  by (auto simp: minterval_adj_eq_iff)

lemma minterval_adj_imp_upper_lower_eq :
  a = b ⟶ mlower_adj a = mlower_adj b ∧ mupper_adj a = mupper_adj b
  by transfer auto


lemma mlower_adj_le_mupper_adj[simp]: mlower_adj i ≤ mupper_adj i
  by (transfer, metis comp_def   lower_le_upper_aux valid_mInterval_adj_def)

lift_definition set_of_adj :: 'a::{minus_mono} minterval_adj ⇒ 'a set
  is λ is. ⋃x∈set is. {lower x..upper x} .

lemma set_adj_of_subset: set_of_adj (x::'a::minus_mono minterval_adj) ⊆ {mlower_adj x .. mupper_adj x}
  apply(transfer, simp)
  using set_of_subeq_aux
      mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
      mInterval_adj_upper_last_max[symmetric, simplified o_def]
      valid_adj_imp_ovl
      list.set_map
  by (smt (verit, best))

lemma not_in_adj_eq:
  ‹(¬ e ∈ set_of_adj xs) = (∀ x ∈ set (mlist_adj xs). ¬ e ∈ set_of x)›
proof(induction (mlist_adj xs))
  case Nil
  then show ?case
    by (metis UN_iff empty_iff empty_set mlist_adj.rep_eq set_of_adj.rep_eq)
next
  case (Cons a x)
  then show ?case
    by (simp add: mlist_adj.rep_eq set_of_eq set_of_adj.rep_eq)
qed
```

```
lemma in_adj_eq:
 ‹(e ∈ set_of_adj xs) = (∃ x ∈ set (mlist_adj xs). e ∈ set_of x)›
proof(induction (mlist_adj xs))
 case Nil
 then show ?case
  by (metis UN_iff empty_iff empty_set mlist_adj.rep_eq set_of_adj.rep_eq)
next
 case (Cons a x)
 then show ?case
  by (simp add: mlist_adj.rep_eq set_of_eq set_of_adj.rep_eq)
qed

lemma set_of_adj_non_zero_list_all:
 ‹o ∉ set_of_adj xs ⟹ ∀ x ∈ set (mlist_adj xs). ¬ o ∈ᵢ x›
proof(induction mlist_adj xs)
 case Nil
 then show ?case
  by simp
next
 case (Cons a x)
 then show ?case
  using in_adj_eq by blast
qed

context notes [[typedef_overloaded]] begin


lift_definition(code_dt) mInterval_adj'::'a::minus_mono interval list ⇒ 'a minterval_adj option
 is λ is. if  valid_mInterval_adj is then Some is else None
 by (auto simp add: valid_mInterval_adj_def)

lemma mInterval_adj'_split:
 P (mInterval_adj' is) ⟷
  (∀ ivl.  valid_mInterval_adj is ⟶ mlist_adj ivl = is ⟶ P (Some ivl)) ∧ (¬ valid_mInterval_adj is ⟶ P None)
 by transfer auto

lemma mInterval_adj'_split_asm:
 P (mInterval_adj' is) ⟷
  ¬((∃ ivl. valid_mInterval_adj is ∧ mlist_adj ivl = is ∧ ¬P (Some ivl)) ∨ (¬ valid_mInterval_adj is ∧ ¬P None))
 unfolding mInterval_adj'_split
 by auto

lemmas mInterval_adj'_splits = mInterval_adj'_split mInterval_adj'_split_asm

lemma mInterval'_eq_Some: mInterval_adj' is = Some i ⟹ mlist_adj i = is
 by (simp split: mInterval_adj'_splits)

end

instantiation minterval_adj :: ({minus_mono}) equal
begin

definition equal_class.equal a b ≡ (mlist_adj a = mlist_adj b)
```

```
instance proof qed (simp add: equal_minterval_adj_def minterval_adj_eq_iff)
end

instantiation minterval_adj :: ({minus_mono}) ord begin

definition less_eq_minterval_adj :: 'a minterval_adj ⇒ 'a minterval_adj ⇒ bool
 where less_eq_minterval_adj a b ⟷ mlower_adj b ≤ mlower_adj a ∧ mupper_adj a ≤ mupper_adj b

definition less_minterval_adj :: 'a minterval_adj ⇒ 'a minterval_adj ⇒ bool
 where  less_minterval_adj x y = (x ≤ y ∧ ¬ y ≤ x)

instance proof qed
end

instantiation minterval_adj :: ({minus_mono,lattice}) sup
begin

lift_definition sup_minterval_adj :: 'a minterval_adj ⇒ 'a minterval_adj ⇒ 'a minterval_adj
 is λ a b. [Interval (inf (lower (hd a)) (lower (hd b)), sup (upper (last a)) (upper (last b)))]
 by(auto simp: valid_mInterval_ovl_def sorted_wrt_lower_def non_adjacent_sorted_wrt_lower_def
          non_overlapping_sorted_def le_infI1 le_supI1 valid_mInterval_adj_def mupper_adj_def
          mlower_adj_def )

lemma mlower_adj_sup[simp]: mlower_adj (sup A B) = inf (mlower_adj A) (mlower_adj B)
 apply(transfer)
 by (metis comp_apply le_supE le_supI1 list.sel(1) lower_bounds lower_le_upper_aux sup_inf_absorb
       valid_mInterval_adj_def)

lemma mupper_adj_sup[simp]: mupper_adj (sup A B) = sup (mupper_adj A) (mupper_adj B)
 apply(transfer)
 by (metis (no_types, lifting) comp_def inf_sup_absorb last.simps le_infI1 le_inf_iff lower_le_upper_aux
       upper_bounds valid_mInterval_adj_def)
instance
 by(standard)
end

instantiation minterval_adj :: ({lattice,minus_mono}) preorder
begin
instance
 apply(standard)
 subgoal
  using less_minterval_adj_def by auto
 subgoal
  by (simp add: less_eq_minterval_adj_def)
 subgoal
  by (meson less_eq_minterval_adj_def order.trans)
 done
end

lemma set_of_minterval_adj_union: set_of_adj A ∪ set_of_adj B ⊆ set_of_adj (sup A B)
 for A::'a::{lattice, minus_mono} minterval_adj
 apply(transfer, simp)
 using set_of_subeq_aux
```

```
    mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
    mInterval_adj_upper_last_max[symmetric, simplified o_def]
    valid_adj_imp_ovl
    list.set_map
 by (smt (verit) le_sup_iff lower_bounds lower_le_upper_aux lower_sup set_of_eq set_of_subset_iff
       sup.commute sup.commute sup.order_iff sup_ge1 sup_ge1 sup_inf_absorb upper_bounds
       valid_mInterval_adj_def)

lemma minterval_adj_union_commute: sup A B = sup B A for A::'a::{minus_mono,lattice} minterval_adj
 apply (auto simp add: minterval_adj_eq_iff inf.commute sup.commute)[1]
 by (simp add: mlist_adj.rep_eq inf_commute sup_minterval_adj.rep_eq sup_commute)

lemma minterval_adj_union_mono1: set_of_adj a ⊆ set_of_adj (sup a A)
 for A :: 'a::{minus_mono,lattice} minterval_adj
 apply(transfer, simp)
 using set_of_subeq_aux
    mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
    mInterval_adj_upper_last_max[symmetric, simplified o_def]
    valid_adj_imp_ovl
    list.set_map
 by (smt (verit, del_insts) inf.absorb_iff2 inf_le1 le_infI1 lower_bounds lower_le_upper_aux
     set_of_eq set_of_subset_iff sup_ge1 upper_bounds valid_mInterval_adj_def)

lemma minterval_adj_union_mono2: set_of_adj A ⊆ set_of_adj (sup a A) for A :: 'a::{lattice, minus_mono} minterval_adj
 apply(transfer, simp)
 using set_of_subeq_aux
    mInterval_ovl_lower_hd_min[symmetric, simplified o_def]
    mInterval_adj_upper_last_max[symmetric, simplified o_def]
    valid_adj_imp_ovl
    list.set_map
 by (smt (verit, del_insts) inf.absorb_iff2 le_sup_iff lower_bounds lower_le_upper_aux nle_le
     set_of_eq set_of_subset_iff sup_inf_absorb upper_bounds valid_mInterval_adj_def)

lift_definition minterval_adj_of :: 'a::{minus_mono} ⇒ 'a minterval_adj is λx. [Interval(x, x)]
 unfolding valid_mInterval_adj_def valid_mInterval_ovl_def non_adjacent_sorted_wrt_lower_def
     cmp_non_adjacent_def sorted_wrt_lower_def non_overlapping_sorted_def
 by simp

lemma mlower_adj_minterval_adj_of[simp]: mlower_adj (minterval_adj_of a) = a
 by transfer auto

lemma mupper_adj_minterval_adj_of[simp]: mupper_adj (minterval_adj_of a) = a
 by transfer auto

definition width_adj :: 'a::{minus_mono} minterval_adj ⇒ 'a
 where width_adj i = mupper_adj i − mlower_adj i
```

12.3.3 Zero and One

```
instantiation minterval_adj :: ({minus_mono,zero}) zero
begin

lift_definition zero_minterval_adj::'a minterval_adj is mk_mInterval_adj [Interval (0, 0)]
 by (simp add: mk_mInterval_adj_valid)
```

lemma *mlower_adj_zero*[*simp*]: *mlower_adj* 0 = 0
by(*transfer*, *simp add*: *mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

lemma *mupper_adj_zero*[*simp*]: *mupper_adj* 0 = 0
by(*transfer*, *simp add*: *mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

instance proof qed
end

instantiation *minterval_adj* :: ({*minus_mono*,*one*}) *one*
begin

lift_definition *one_minterval_adj*::'*a minterval_adj* **is** *mk_mInterval_adj* [*Interval* (1, 1)]
by (*simp add*: *mk_mInterval_adj_valid*)

lemma *mlower_adj_one*[*simp*]: *mlower_adj* 1 = 1
by(*transfer*, *simp add*: *mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

lemma *mupper_adj_one*[*simp*]: *mupper_adj* 1 = 1
by(*transfer*, *simp add*: *mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def*)

instance proof qed
end

12.3.4 Addition

instantiation *minterval_adj* :: ({*minus_mono*,*ordered_ab_semigroup_add*,*linordered_field*}) *plus*
begin

lift_definition *plus_minterval_adj*::'*a minterval_adj* ⇒ '*a minterval_adj* ⇒ '*a minterval_adj*
is λ *a b* . *mk_mInterval_adj* (*iList_plus a b*)
by (*metis bin_op_interval_list_empty iList_plus_def mk_mInterval_adj_valid valid_mInterval_adj_def*)

instance proof qed

lemma *interval_plus_com*:
‹*a* + *b* = *b* + *a*› **for** *a*::'*a minterval_adj*
apply(*transfer*)
using *iList_plus_mInterval_adj_commute plus_minterval_adj_def*
by(*auto*)

end

12.3.5 Unary Minus

lemma *a*: (*x*::'*a*::*ordered_ab_group_add interval*) ≠ *y* ⟹ −*x* ≠ −*y*
apply(*simp add*:*uminus_interval_def*)
by (*smt* (*z3*) *Pair_inject bounds_of_interval_inverse case_prod_Pair_iden case_prod_unfold neg_equal_iff_equal uminus_interval.rep_eq*)

lemma *b*: *distinct* (*is*::'*a*::*ordered_ab_group_add interval list*) ⟹ *distinct* (*map* (λ *i*. −*i*) *is*)
proof(*induction is*)
case *Nil*

```
 then show ?case by simp
next
 case (Cons a is)
 then show ?case using a by force
qed


instantiation minterval_adj :: ({minus_mono, ordered_ab_group_add}) uminus
begin

lift_definition uminus_minterval_non_ovl::'a minterval_adj ⇒ 'a minterval_adj
        is λ is . mk_mInterval_non_ovl (rev (map (λ i. −i) is))
 by (metis (no_types, opaque_lifting) list.map_disc_iff mk_mInterval_non_ovl_id mk_mInterval_non_ovl_non_empty
    mk_mInterval_non_ovl_valid rev_is_Nil_conv sorted_wrt_lower_mk_mInterval_non_ovl
    valid_non_ovl_imp_adj)

instance ..
end
```

12.3.6 Subtraction

```
instantiation minterval_adj :: ({minus_mono, linordered_field, ordered_ab_group_add}) minus
begin

definition minus_minterval_non_ovl::'a minterval_adj ⇒ 'a minterval_adj ⇒ 'a minterval_adj
 where minus_minterval_non_ovl a b = a + − b

instance ..
end
```

12.3.7 Multiplication

```
instantiation minterval_adj :: ({minus_mono, linordered_field}) times
begin

lift_definition times_minterval_non_ovl::'a minterval_adj ⇒ 'a minterval_adj ⇒ 'a minterval_adj
 is λ a b . mk_mInterval_non_ovl (iList_times a b)
 by (metis bin_op_interval_list_empty iList_times_def mk_mInterval_non_ovl_id
        mk_mInterval_non_ovl_non_empty mk_mInterval_non_ovl_valid sorted_wrt_lower_mk_mInterval_non_ovl
        valid_non_ovl_imp_adj)

instance ..
end
```

12.3.8 Multiplicative Inverse and Division

```
locale minterval_adj_division = inverse +
 constrains inverse :: ‹'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra,linear_continuum_topology}
minterval_adj ⇒ 'a minterval_adj›
       and divide  :: ‹'a::{linordered_field, zero, minus, minus_mono, real_normed_algebra,linear_continuum_topology}
minterval_adj ⇒ 'a minterval_adj ⇒ 'a minterval_adj›
     assumes inverse_left: ‹¬ 0 ∈ set_of_adj x ⟹ 1 ≤ (inverse x) ∗ x›
       and divide:     ‹¬ 0 ∈ set_of_adj y ⟹ x ≤ (divide x y) ∗ y›
begin
```

end

locale *minterval_adj_division_inverse* = *inverse* +
constrains *inverse* :: ‹'*a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*} *minterval_adj* ⇒ '*a minterval_adj*›
and *divide* :: ‹'*a*::{*linordered_field*, *zero*, *minus*, *minus_mono*, *real_normed_algebra*,*linear_continuum_topology*} *minterval_adj* ⇒ '*a minterval_adj* ⇒ '*a minterval_adj*›
assumes *inverse_non_zero_def*: ‹¬ *0* ∈ *set_of_adj x* ⟹ (*inverse x*) = *mInterval_adj* (*mk_mInterval_adj*(*un_op_interval_list* (λ *i*. *mk_interval* (*1* / (*upper i*), *1* / (*lower i*))) (*mlist_adj x*)))›
and *divide_non_zero_def*: ‹¬ *0* ∈ *set_of_adj y* ⟹ (*divide x y*) = *inverse y* ∗ *x*›
begin
end

12.3.9 Membership

abbreviation (**in** *preorder*) *in_minterval_adj* (‹(_/ $\in_{adj}$ _)› [*51*, *51*] *50*)
where *in_minterval_adj x X* ≡ *x* ∈ *set_of_adj X*

lemma *in_minterval_adj_to_minterval_adj*[*intro*!]: *a* $\in_{adj}$ *minterval_adj_of a*
by (*metis* (*mono_tags*, *lifting*) *UN_iff list.set_intros*(*1*) *lower_in_interval lower_point_interval minterval_adj_of.rep_eq set_of_eq set_of_adj.rep_eq*)

instance *minterval_adj* :: ({*one*, *preorder*, *minus_mono*, *linordered_semiring*}) *power*
proof qed

lemma *set_of_one_adj*[*simp*]: *set_of_adj* (*1*::'*a*::{*one*, *minus_mono*, *order*} *minterval_adj*) = {*1*}
apply(*transfer*)
by(*auto simp*: *mk_mInterval_adj_def mk_mInterval_ovl_def interval_sort_lower_width_def set_of_adj_def*)

lifting_update *minterval_adj.lifting*
lifting_forget *minterval_adj.lifting*

end

12.4 Bringing Everything Together (Multi_Interval)

theory
Multi_Interval
imports
Multi_Interval_Overlapping
Multi_Interval_Non_Overlapping
Multi_Interval_Adjacent
Lipschitz_Subdivisions_Refinements
begin

For convince, we provide a theory that provides all three variants of multi-intervals. Note that the order in which these theories are imported is important: importing the theory Multi_Interval_Adjacent last ensures that it provides the default definitions and lemmas.

end

13 Extended Division on Multi-Intervals (Extended_Multi_Interval_Division_Core)

theory
Extended_Multi_Interval_Division_Core
imports
Interval_Division_Non_Zero
Multi_Interval
begin

13.1 Division over List of Intervals

In this theory, we define an extended division operation on intervals. This is a formalization of the interval division given in [4].

definition *inverse_interval* :: $('a::\{linorder,minus_mono,zero,one,inverse,infinity,uminus\})$ *interval* $\Rightarrow$ $('a$ *interval*$)$ *list*
where *inverse_interval a* $=$ (
if $(\neg\ o \in_i a)$ *then* $[mk_interval\ (1\ /\ (upper\ a), 1\ /\ (lower\ a))]$
else if lower a $= o \wedge o <$ *upper a then* $[mk_interval\ (1/\ upper\ a, \infty)]$
else if lower a $< o \wedge o <$ *upper a then* $[mk_interval\ (-\infty, 1/lower\ a), mk_interval\ (1/upper\ a, \infty)]$
else if lower a $<$ *upper a* $\wedge$ *upper a* $= o$ *then* $[mk_interval(-\infty, 1\ /\ lower\ a)]$
else undefined
)
definition ‹*minverse* $=$ *concat o* $(map\ inverse_interval)$›

13.2 Multi-Interval Division

end

13.2.1 Overlapping Multi-Intervals (Extended_Multi_Interval_Division_Overlapping)

theory
Extended_Multi_Interval_Division_Overlapping
imports
Extended_Multi_Interval_Division_Core
begin

definition ‹*minterval_ovl_inverse x* $=$ *mInterval_ovl* $(mk_mInterval_ovl(minverse\ (mlist_ovl\ x)))$›
definition ‹*minterval_ovl_divide x y* $= (minterval_ovl_inverse\ y) * x$›

lemma *set_of_ovl_non_zero_map_inverse*:
assumes ‹*o* $\notin$ *set_of_ovl xs*›
shows ‹*concat* $(map\ inverse_interval\ (mlist_ovl\ xs)) = map\ (\lambda i.\ mk_interval\ (1\ /\ upper\ i, 1\ /\ lower\ i))\ (mlist_ovl\ xs)$›
proof$(insert\ assms, induction\ mlist_ovl\ xs)$
case *Nil*
then show *?case*
by *simp*

next
case (*Cons a x*)
then show *?case*
using *set_of_ovl_non_zero_list_all*[*of xs*, *simplified Cons*, *simplified*]
by (*metis* (*no_types*, *lifting*) *concat_map_singleton inverse_interval_def map_eq_conv*)
qed

interpretation *minterval_ovl_division_inverse minterval_ovl_divide minterval_ovl_inverse*
apply(*unfold_locales*)
subgoal
using *set_of_ovl_non_zero_map_inverse*
unfolding *minterval_ovl_inverse_def minverse_def o_def un_op_interval_list_def*
by *fastforce*
subgoal by(*metis minterval_ovl_divide_def*)
done

end

13.2.2 Non Overlapping Multi-Intervals (Extended_Multi_Interval_Division_Non_Overlapping)

theory
Extended_Multi_Interval_Division_Non_Overlapping
imports
Extended_Multi_Interval_Division_Core
begin

definition ‹*minterval_non_ovl_inverse x* = *mInterval_non_ovl* (*mk_mInterval_non_ovl*(*minverse* (*mlist_non_ovl x*)))›
definition ‹*minterval_non_ovl_divide x y* = (*minterval_non_ovl_inverse y*) ∗ *x*›

lemma *set_of_non_ovl_non_zero_map_inverse*:
assumes ‹*o* ∉ *set_of_non_ovl xs*›
shows ‹*concat* (*map inverse_interval* (*mlist_non_ovl xs*)) = *map* (λ*i*. *mk_interval* (1 / *upper i*, 1 / *lower i*))
(*mlist_non_ovl xs*)›
proof(*insert assms*, *induction mlist_non_ovl xs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons a x*)
then show *?case*
using *set_of_non_ovl_non_zero_list_all*[*of xs*, *simplified Cons*, *simplified*]
by (*metis* (*no_types*, *lifting*) *concat_map_singleton inverse_interval_def map_eq_conv*)
qed

interpretation *minterval_non_ovl_division_inverse minterval_non_ovl_divide minterval_non_ovl_inverse*
apply(*unfold_locales*)
subgoal
using *set_of_non_ovl_non_zero_map_inverse*
unfolding *minterval_non_ovl_inverse_def minverse_def o_def un_op_interval_list_def*
by *fastforce*
subgoal by(*metis minterval_non_ovl_divide_def*)
done

end

13.2.3 Adjacent Multi-Intervals (Extended_Multi_Interval_Division_Adjacent)

theory
Extended_Multi_Interval_Division_Adjacent
imports
Extended_Multi_Interval_Division_Core
begin

definition ‹*minterval_adj_inverse x* = *mInterval_adj* (*mk_mInterval_adj*(*minverse* (*mlist_adj x*)))›
definition ‹*minterval_adj_divide x y* = (*minterval_adj_inverse y*) ∗ *x*›

lemma *set_of_adj_non_zero_map_inverse*:
assumes ‹*o* ∉ *set_of_adj xs*›
shows ‹*concat* (*map inverse_interval* (*mlist_adj xs*)) = *map* (λ*i*. *mk_interval* (1 / *upper i*, 1 / *lower i*)) (*mlist_adj xs*)›
proof(*insert assms*, *induction mlist_adj xs*)
case *Nil*
then show *?case*
by *simp*
next
case (*Cons a x*)
then show *?case*
using *set_of_adj_non_zero_list_all*[*of xs*, *simplified Cons*, *simplified*]
by (*metis* (*no_types*, *lifting*) *concat_map_singleton inverse_interval_def map_eq_conv*)
qed

interpretation *minterval_adj_division_inverse minterval_adj_divide minterval_adj_inverse*
apply(*unfold_locales*)
subgoal
using *set_of_adj_non_zero_map_inverse*
unfolding *minterval_adj_inverse_def minverse_def o_def un_op_interval_list_def*
by *fastforce*
subgoal by(*metis minterval_adj_divide_def*)
done

end

13.3 Bringing Everything Together (Extended_Multi_Interval_Division)

theory
Extended_Multi_Interval_Division
imports
Extended_Multi_Interval_Division_Overlapping
Extended_Multi_Interval_Division_Non_Overlapping
Extended_Multi_Interval_Division_Adjacent
begin

end

14 Extended Multi-Interval Analysis (Extended_Multi_Interval_Analysis)

theory
Extended_Multi_Interval_Analysis
imports
Extended_Multi_Interval_Division
begin

This theory provides extended multi-interval analysis over the type extended reals. All operations work over multi-intervals, i.e., lists of (closed) intervals.

end

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