Conformance Relations between Input/Output Languages

Robert Sachtleben

February 6, 2026

Abstract

This entry formalises the paper of the same name by Huang et al. [1] and presents a unifying characterisation of well-known conformance relations such as equivalence and language inclusion (reduction) on languages over input/output pairs. This characterisation simplifies comparisons between conformance relations and from it a fundamental necessary and sufficient criterion for conformance testing is developed.

Contents

1 Preliminaries **2**

2 Conformance Relations **6**

3 Unifying Characterisations **7**
3.1 $\preceq$ Conformance 7
3.2 $\leq$ Conformance 12

4 Comparing Conformance Relations **19**
4.1 Completely Specified Languages 21

5 Conformance Testing **21**

6 Reductions Between Relations **29**
6.1 Quasi-Equivalence via Quasi-Reduction and Absences 29
6.2 Quasi-Reduction via Reduction and explicit Undefined Behaviour 49
6.3 Strong Reduction via Reduction and Undefinedness Outputs 66

theory *Input-Output-Language-Conformance*
imports *HOL−Library.Sublist*
begin

1 Preliminaries

type-synonym $('a)$ *alphabet* $= 'a$ *set*
type-synonym $('x,'y)$ *word* $= ('x \times 'y)$ *list*
type-synonym $('x,'y)$ *language* $= ('x,'y)$ *word set*
type-synonym $('y)$ *output-relation* $= ('y$ *set* $\times$ $'y$ *set*$)$ *set*

fun *is-language* :: $'x$ *alphabet* $\Rightarrow$ $'y$ *alphabet* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
is-language X Y L = (
— nonempty
$(L \neq \{\}) \wedge$
$(\forall\ \pi \in L\ .$
— over X and Y
$(\forall\ xy \in set\ \pi\ .\ fst\ xy \in X \wedge snd\ xy \in Y) \wedge$
— prefix closed
$(\forall\ \pi'\ .\ prefix\ \pi'\ \pi \longrightarrow \pi' \in L)))$

lemma *language-contains-nil* :
assumes *is-language X Y L*
shows $[] \in L$
using *assms* **by** *auto*

lemma *language-intersection-is-language* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows *is-language X Y* $(L1 \cap L2)$
using *assms*
using *language-contains-nil*[*OF assms*(*1*)] *language-contains-nil*[*OF assms*(*2*)]
unfolding *is-language.simps*
by (*metis IntD1 IntD2 IntI disjoint-iff*)

fun *language-for-state* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *word* $\Rightarrow$ $('x,'y)$ *language* **where**
language-for-state L $\pi = \{\tau\ .\ \pi@\tau \in L\}$

notation *language-for-state* ($\langle\mathcal{L}[\text{-},\text{-}]\rangle$)

lemma *language-for-state-is-language* :
assumes *is-language X Y L*
and $\pi \in L$
shows *is-language X Y* $\mathcal{L}[L,\pi]$
proof −
have $\bigwedge \tau\ .\ \tau \in \mathcal{L}[L,\pi] \Longrightarrow (\forall\ xy \in set\ \tau\ .\ fst\ xy \in X \wedge snd\ xy \in Y) \wedge (\forall\ \tau'\ .$
$prefix\ \tau'\ \tau \longrightarrow \tau' \in \mathcal{L}[L,\pi])$
proof −
fix τ **assume** $\tau \in \mathcal{L}[L,\pi]$

then have $\pi@\tau \in L$ **by** *auto*
then have $\bigwedge xy\ .\ xy \in set\ (\pi@\tau) \Longrightarrow fst\ xy \in X \wedge snd\ xy \in Y$
and $\bigwedge \pi'\ .\ prefix\ \pi'\ (\pi@\tau) \Longrightarrow \pi' \in L$
using *assms*(*1*) **by** *auto*

have $\bigwedge xy\ .\ xy \in set\ \tau \Longrightarrow fst\ xy \in X \wedge snd\ xy \in Y$
using $\langle \bigwedge xy\ .\ xy \in set\ (\pi@\tau) \Longrightarrow fst\ xy \in X \wedge snd\ xy \in Y \rangle$ **by** *auto*
moreover have $\bigwedge \tau'\ .\ prefix\ \tau'\ \tau \Longrightarrow \tau' \in \mathcal{L}[L,\pi]$
by (*simp add*: $\langle \bigwedge \pi'.\ prefix\ \pi'\ (\pi\ @\ \tau) \Longrightarrow \pi' \in L \rangle$)
ultimately show $(\forall\ xy \in set\ \tau\ .\ fst\ xy \in X \wedge snd\ xy \in Y) \wedge (\forall\ \tau'\ .\ prefix\ \tau'\ \tau \longrightarrow \tau' \in \mathcal{L}[L,\pi])$
by *simp*
qed
moreover have $\mathcal{L}[L,\pi] \neq \{\}$
using *assms*(*2*)
by (*metis* (*no-types*, *lifting*) *append.right-neutral empty-Collect-eq language-for-state.simps*)

ultimately show *?thesis*
by *simp*
qed

lemma *language-of-state-empty-iff* :
assumes *is-language X Y L*
shows $(\mathcal{L}[L,\pi] = \{\}) \longleftrightarrow (\pi \notin L)$
using *assms* **unfolding** *is-language.simps language-for-state.simps*
by (*metis Collect-empty-eq append.right-neutral prefixI*)

fun *are-equivalent-for-language* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *word* $\Rightarrow$ $('x,'y)$ *word* $\Rightarrow$ *bool* **where**
are-equivalent-for-language L α β $= (\mathcal{L}[L,\alpha] = \mathcal{L}[L,\beta])$

abbreviation(*input*) *input-projection* $\pi \equiv map\ fst\ \pi$
abbreviation(*input*) *output-projection* $\pi \equiv map\ snd\ \pi$
notation *input-projection* $(\langle [\text{-}]_I \rangle)$
notation *output-projection* $(\langle [\text{-}]_O \rangle)$

fun *is-executable* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *word* $\Rightarrow$ $'x$ *list* $\Rightarrow$ *bool* **where**
is-executable L π *xs* $= (\exists\ \tau \in \mathcal{L}[L,\pi]\ .\ [\tau]_I = xs)$

fun *executable-sequences* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *word* $\Rightarrow$ $'x$ *list set* **where**
executable-sequences L $\pi = \{xs\ .\ is\text{-}executable\ L\ \pi\ xs\}$

fun *executable-inputs* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *word* $\Rightarrow$ $'x$ *set* **where**
executable-inputs L $\pi = \{x\ .\ is\text{-}executable\ L\ \pi\ [x]\}$

notation *executable-inputs* (‹*exec*[-,-]›)

lemma *executable-sequences-alt-def* : *executable-sequences L π* = {*xs* . ∃ *ys* . *length ys* = *length xs* ∧ *zip xs ys* ∈ $\mathcal{L}$[*L*,*π*]}
proof −
have ∗:⋀ *A xs* . (∃ *τ*∈*A*. *map fst τ* = *xs*) = (∃ *ys*. *length ys* = *length xs* ∧ *zip xs ys* ∈ *A*)
by (*metis length-map map-fst-zip zip-map-fst-snd*)

show *?thesis*
unfolding *executable-sequences.simps is-executable.simps*
unfolding ∗
by *simp*
qed

lemma *executable-inputs-alt-def* : *executable-inputs L π* = {*x* . ∃ *y* . [(*x*,*y*)] ∈ $\mathcal{L}$[*L*,*π*]}
proof −
have ∗:⋀ *A xs* . (∃ *τ*∈*A*. *map fst τ* = *xs*) = (∃ *ys*. *length ys* = *length xs* ∧ *zip xs ys* ∈ *A*)
by (*metis length-map map-fst-zip zip-map-fst-snd*)

have ∗∗: ⋀ *A x* . (∃ *ys*. *length ys* = *length* [*x*] ∧ *zip* [*x*] *ys* ∈ *A*) = (∃ *y* . [(*x*,*y*)] ∈ *A*)
by (*metis length-Suc-conv length-map zip-Cons-Cons zip-Nil*)

show *?thesis*
unfolding *executable-inputs.simps is-executable.simps*
unfolding ∗
unfolding ∗∗
by *fastforce*
qed

lemma *executable-inputs-in-alphabet* :
assumes *is-language X Y L*
and *x* ∈ *exec*[*L*,*π*]
shows *x* ∈ *X*
using *assms* **unfolding** *executable-inputs-alt-def* **by** *auto*

fun *output-sequences* :: ('*x*,'*y*) *language* ⇒ ('*x*,'*y*) *word* ⇒ '*x list* ⇒ '*y list set*
where
output-sequences L π xs = *output-projection* ‘ {*τ* ∈ $\mathcal{L}$[*L*,*π*] . [*τ*]$_I$ = *xs*}

lemma *prefix-closure-no-member* :
assumes *is-language X Y L*
and *π* ∉ *L*

shows $\pi@\tau \notin L$
by (*meson assms*(*1*) *assms*(*2*) *is-language.elims*(*2*) *prefixI*)

lemma *output-sequences-empty-iff* :
assumes *is-language X Y L*
shows (*output-sequences L* π *xs* = {}) = (($\pi \notin L$) $\vee$ ($\neg$ *is-executable L* π *xs*))
unfolding *output-sequences.simps is-executable.simps language-for-state.simps*
using *Collect-empty-eq assms image-empty mem-Collect-eq prefix-closure-no-member*
by *auto*

fun *outputs* :: ('*x*,'*y*) *language* $\Rightarrow$ ('*x*,'*y*) *word* $\Rightarrow$ '*x* $\Rightarrow$ '*y set* **where**
outputs L π *x* = {*y* . [(*x*,*y*)] $\in \mathcal{L}$[*L*,π]}

notation *outputs* (‹*out*[-,-,-]›)

lemma *outputs-in-alphabet* :
assumes *is-language X Y L*
shows *out*[*L*,π,*x*] $\subseteq$ *Y*
using *assms* **by** *auto*

lemma *outputs-executable* : (*out*[*L*,π,*x*] = {}) $\longleftrightarrow$ (*x* $\notin$ *exec*[*L*,π])
by *auto*

fun *is-completely-specified-for* :: '*x set* $\Rightarrow$ ('*x*,'*y*) *language* $\Rightarrow$ *bool* **where**
is-completely-specified-for X L = ($\forall$ $\pi \in L$. $\forall$ *x* $\in$ *X* . *out*[*L*,π,*x*] $\neq$ {})

lemma *prefix-executable* :
assumes *is-language X Y L*
and $\pi \in L$
and *i* < *length* π
shows *fst* (π ! *i*) $\in$ *exec*[*L*,*take i* π]
proof −
define π' **where** π' = *take i* π
moreover define π'' **where** π'' = *drop* (*Suc i*) π
moreover define *xy* **where** *xy* = π ! *i*
ultimately have $\pi = \pi'@[xy]@\pi''$
by (*simp add*: *Cons-nth-drop-Suc assms*(*3*))
then have $\pi'@[xy] \in L$
using *assms*(*1*,*2*) **by** *auto*
then show *?thesis*
unfolding π'*-def xy-def*
unfolding *executable-inputs-alt-def language-for-state.simps*
by (*metis* (*mono-tags*, *lifting*) *CollectI eq-fst-iff*)

qed

2 Conformance Relations

definition *language-equivalence* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
language-equivalence L1 L2 = $(L1 = L2)$

definition *language-inclusion* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**

language-inclusion L1 L2 = $(L1 \subseteq L2)$

abbreviation(*input*) *reduction L1 L2* $\equiv$ *language-inclusion L1 L2*

definition *quasi-equivalence* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
quasi-equivalence L1 L2 = $(\forall\ \pi \in L1 \cap L2\ .\ \forall\ x \in exec[L2,\pi]\ .\ out[L1,\pi,x] = out[L2,\pi,x])$

definition *quasi-reduction* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
quasi-reduction L1 L2 = $(\forall\ \pi \in L1 \cap L2\ .\ \forall\ x \in exec[L2,\pi]\ .\ (out[L1,\pi,x] \neq \{\} \wedge out[L1,\pi,x] \subseteq out[L2,\pi,x]))$

definition *strong-reduction* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
strong-reduction L1 L2 = $(quasi\text{-}reduction\ L1\ L2 \wedge (\forall\ \pi \in L1 \cap L2\ .\ \forall\ x\ .\ out[L2,\pi,x] = \{\} \longrightarrow out[L1,\pi,x] = \{\}))$

definition *semi-equivalence* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
semi-equivalence L1 L2 = $(\forall\ \pi \in L1 \cap L2\ .\ \forall\ x \in exec[L2,\pi]\ .$
$(out[L1,\pi,x] = \{\} \vee out[L1,\pi,x] = out[L2,\pi,x]) \wedge$
$(\exists\ x'\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}))$

definition *semi-reduction* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
semi-reduction L1 L2 = $(\forall\ \pi \in L1 \cap L2\ .\ \forall\ x \in exec[L2,\pi]\ .$
$(out[L1,\pi,x] \subseteq out[L2,\pi,x]) \wedge$
$(\exists\ x'\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}))$

definition *strong-semi-equivalence* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
strong-semi-equivalence L1 L2 = $(\forall\ \pi \in L1 \cap L2\ .\ \forall\ x\ .$
$(x \in exec[L2,\pi] \longrightarrow ((out[L1,\pi,x] = \{\} \vee out[L1,\pi,x] = out[L2,\pi,x]) \wedge (\exists\ x'\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}))) \wedge$
$(x \notin exec[L2,\pi] \longrightarrow out[L1,\pi,x] = \{\}))$

definition *strong-semi-reduction* :: $('x,'y)$ *language* $\Rightarrow$ $('x,'y)$ *language* $\Rightarrow$ *bool* **where**
strong-semi-reduction L1 L2 = $(\forall\ \pi \in L1 \cap L2\ .\ \forall\ x\ .$
$(x \in exec[L2,\pi] \longrightarrow (out[L1,\pi,x] \subseteq out[L2,\pi,x] \wedge (\exists\ x'\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}))) \wedge$
$(x \notin exec[L2,\pi] \longrightarrow out[L1,\pi,x] = \{\}))$

3 Unifying Characterisations

3.1 $\preceq$ Conformance

fun *type-1-conforms* :: ('*x*,'*y*) *language* ⇒ '*x alphabet* ⇒ '*y output-relation* ⇒ ('*x*,'*y*) *language* ⇒ *bool* **where**
type-1-conforms L1 X H L2 = (∀ π ∈ *L1* ∩ *L2* . ∀ *x* ∈ *X* . (*out*[*L1*,π,*x*],*out*[*L2*,π,*x*]) ∈ *H*)

notation *type-1-conforms* (‹- ⪯[-,-] -›)

fun *equiv* :: '*y alphabet* ⇒ '*y output-relation* **where**
equiv Y = {(*A*,*A*) | *A* . *A* ⊆ *Y*}

fun *red* :: '*y alphabet* ⇒ '*y output-relation* **where**
red Y = {(*A*,*B*) | *A B* . *A* ⊆ *B* ∧ *B* ⊆ *Y*}

fun *quasieq* :: '*y alphabet* ⇒ '*y output-relation* **where**
quasieq Y = {(*A*,*A*) | *A* . *A* ⊆ *Y*} ∪ {(*A*,{}) | *A* . *A* ⊆ *Y*}

fun *quasired* :: '*y alphabet* ⇒ '*y output-relation* **where**
quasired Y = {(*A*,*B*) | *A B* . *A* ≠ {} ∧ *A* ⊆ *B* ∧ *B* ⊆ *Y*} ∪ {(*C*,{}) | *C* . *C* ⊆ *Y*}

fun *strongred* :: '*y alphabet* ⇒ '*y output-relation* **where**
strongred Y = {(*A*,*B*) | *A B* . *A* ≠ {} ∧ *A* ⊆ *B* ∧ *B* ⊆ *Y*} ∪ {({},{})}

lemma *red-type-1* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows *reduction L1 L2* ⟷ (*L1* ⪯[*X*,*red Y*] *L2*)
unfolding *language-inclusion-def* **proof**
show *L1* ⊆ *L2* ⟹ *L1* ⪯[*X*,*red Y*] *L2*
using *outputs-in-alphabet*[*OF assms*(*2*)]
unfolding *type-1-conforms.simps red.simps*
by *auto*

show *L1* ⪯[*X*,*red Y*] *L2* ⟹ *L1* ⊆ *L2*
proof
fix π **assume** π ∈ *L1* **and** *L1* ⪯[*X*,*red Y*] *L2*

then show π ∈ *L2* **proof** (*induction* π *rule*: *rev-induct*)
case *Nil*
then show *?case* **using** *assms*(*2*) **by** *auto*
next
case (*snoc xy* π)
then have π ∈ *L1* **and** π ∈ *L1* ∩ *L2*
using *assms*(*1*) **by** *auto*

```
    obtain x y where xy = (x,y)
      by fastforce
    then have y ∈ out[L1,π,x]
      using snoc.prems(1)
      by simp
    moreover have x ∈ X and y ∈ Y
      using snoc.prems(1) assms(1) unfolding ‹xy = (x,y)› by auto
    ultimately have y ∈ out[L2,π,x]
      using snoc.prems(2) ‹π ∈ L1 ∩ L2›
      unfolding type-1-conforms.simps
      by fastforce
    then show ?case
      using ‹xy = (x, y)› by auto
  qed
 qed
qed


lemma equiv-by-reduction : (L1 ≼[X,equiv Y] L2) ⟷ ((L1 ≼[X,red Y] L2) ∧
(L2 ≼[X,red Y] L1))
 by fastforce

lemma equiv-type-1 :
 assumes is-language X Y L1
 and     is-language X Y L2
shows (L1 = L2) ⟷ (L1 ≼[X,equiv Y] L2)
 unfolding equiv-by-reduction
 unfolding red-type-1[OF assms(1,2), symmetric]
 unfolding red-type-1[OF assms(2,1), symmetric]
 unfolding language-inclusion-def
 by blast


lemma quasired-type-1 :
 assumes is-language X Y L1
 and     is-language X Y L2
shows quasi-reduction L1 L2 ⟷ (L1 ≼[X,quasired Y] L2)
proof
 have ⋀ π x . quasi-reduction L1 L2 ⟹ π ∈ L1 ∩ L2 ⟹ x ∈ X ⟹ (out[L1,π,x],
out[L2,π,x]) ∈ quasired Y
 proof −
   fix π x assume quasi-reduction L1 L2 and π ∈ L1 ∩ L2 and x ∈ X

   show (out[L1,π,x], out[L2,π,x]) ∈ quasired Y
   proof (cases x ∈ exec[L2,π])
     case False
     then show ?thesis
       by (metis (mono-tags, lifting) CollectI UnCI assms(1) outputs-executable
```

outputs-in-alphabet quasired.elims)
next
case *True*
then obtain *y* **where** $y \in out[L2,\pi,x]$ **by** *auto*
then have $out[L1,\pi,x] \subseteq out[L2,\pi,x]$ **and** $out[L1,\pi,x] \neq \{\}$
using ‹$\pi \in L1 \cap L2$› ‹$x \in X$› ‹*quasi-reduction L1 L2*›
unfolding *quasi-reduction-def* **by** *force+*
moreover have $out[L2,\pi,x] \subseteq Y$
by (*meson assms*(2) *outputs-in-alphabet*)
ultimately show *?thesis*
unfolding *quasired.simps* **by** *blast*
qed
qed
then show *quasi-reduction L1 L2* $\Longrightarrow$ $(L1 \preceq[X,quasired\ Y]\ L2)$
by *auto*

have $\bigwedge \pi\ x\ .\ L1 \preceq[X,quasired\ Y]\ L2 \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow out[L1,\pi,x] \subseteq out[L2,\pi,x]$
and $\bigwedge \pi\ x\ .\ L1 \preceq[X,quasired\ Y]\ L2 \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow out[L1,\pi,x] \neq \{\}$
proof −
fix $\pi\ x$ **assume** $L1 \preceq[X,quasired\ Y]\ L2$ **and** $\pi \in L1 \cap L2$ **and** $x \in exec[L2,\pi]$
then have $x \in X$
using *executable-inputs-in-alphabet*[*OF assms*(2)] **by** *auto*

have $out[L2,\pi,x] \neq \{\}$
using ‹$x \in exec[L2,\pi]$›
by (*meson outputs-executable*)
moreover have $(out[L1,\pi,x],out[L2,\pi,x]) \in quasired\ Y$
by (*meson* ‹$L1 \preceq[X,quasired\ Y]\ L2$› ‹$\pi \in L1 \cap L2$› ‹$x \in X$› *type-1-conforms.elims*(2))
ultimately show $out[L1,\pi,x] \subseteq out[L2,\pi,x]$
and $out[L1,\pi,x] \neq \{\}$
unfolding *quasired.simps*
by *blast+*
qed
then show $L1 \preceq[X,quasired\ Y]\ L2 \Longrightarrow$ *quasi-reduction L1 L2*
by (*meson quasi-reduction-def*)
qed

lemma *quasieq-type-1* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows *quasi-equivalence L1 L2* $\longleftrightarrow$ $(L1 \preceq[X,quasieq\ Y]\ L2)$
proof
have $\bigwedge \pi\ x\ .\ quasi\text{-}equivalence\ L1\ L2 \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in quasieq\ Y$
proof −

fix π x **assume** *quasi-equivalence L1 L2* **and** $\pi \in L1 \cap L2$ **and** $x \in X$

show (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in$ *quasieq Y*
proof (*cases* $x \in$ *exec*[*L2*,π])
case *False*
then show *?thesis*
by (*metis* (*mono-tags*, *lifting*) *CollectI UnCI assms*(*1*) *outputs-executable outputs-in-alphabet quasieq.simps*)
next
case *True*
then show *?thesis*
by (*metis* (*mono-tags*, *lifting*) *CollectI UnCI* ‹$\pi \in L1 \cap L2$› ‹*quasi-equivalence L1 L2*› *assms*(*1*) *outputs-in-alphabet quasi-equivalence-def quasieq.simps*)
qed
qed
then show *quasi-equivalence L1 L2* $\Longrightarrow$ (*L1* $\preceq$[*X*,*quasieq Y*] *L2*)
by *auto*

have $\bigwedge \pi\ x$. *L1* $\preceq$[*X*,*quasieq Y*] *L2* $\Longrightarrow$ $\pi \in L1 \cap L2 \Longrightarrow x \in$ *exec*[*L2*,π] $\Longrightarrow$ *out*[*L1*,π,*x*] = *out*[*L2*,π,*x*]
proof −
fix π x **assume** *L1* $\preceq$[*X*,*quasieq Y*] *L2* **and** $\pi \in L1 \cap L2$ **and** $x \in$ *exec*[*L2*,π]
then have $x \in X$
using *executable-inputs-in-alphabet*[*OF assms*(*2*)] **by** *auto*

have *out*[*L2*,π,*x*] $\neq$ {}
using ‹$x \in$ *exec*[*L2*,π]›
by (*meson outputs-executable*)
moreover have (*out*[*L1*,π,*x*],*out*[*L2*,π,*x*]) $\in$ *quasieq Y*
by (*meson* ‹*L1* $\preceq$[*X*,*quasieq Y*] *L2*› ‹$\pi \in L1 \cap L2$› ‹$x \in X$› *type-1-conforms.elims*(*2*))
ultimately show *out*[*L1*,π,*x*] = *out*[*L2*,π,*x*]
unfolding *quasieq.simps*
by *blast*
qed
then show *L1* $\preceq$[*X*,*quasieq Y*] *L2* $\Longrightarrow$ *quasi-equivalence L1 L2*
by (*meson quasi-equivalence-def*)
qed

lemma *strongred-type-1* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows *strong-reduction L1 L2* $\longleftrightarrow$ (*L1* $\preceq$[*X*,*strongred Y*] *L2*)
proof
have $\bigwedge \pi\ x$. *strong-reduction L1 L2* $\Longrightarrow$ $\pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow$ (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in$ *strongred Y*
proof −
fix π x **assume** *strong-reduction L1 L2* **and** $\pi \in L1 \cap L2$ **and** $x \in X$

```
    have out[L2,π,x] ⊆ Y
      using outputs-in-alphabet[OF assms(2)] .

    show (out[L1,π,x], out[L2,π,x]) ∈ strongred Y
    proof (cases x ∈ exec[L2,π])
      case False
      then have out[L2,π,x] = {}
        using outputs-executable by force
      then have out[L1,π,x] = {}
        using ‹strong-reduction L1 L2› ‹π ∈ L1 ∩ L2›
        unfolding strong-reduction-def by blast
      then show ?thesis
        using ‹out[L2,π,x] = {}› by auto
    next
      case True
      then have out[L1,π,x] ≠ {}
        using ‹strong-reduction L1 L2› ‹π ∈ L1 ∩ L2›
        unfolding strong-reduction-def
        by (meson quasi-reduction-def)
      moreover have out[L1,π,x] ⊆ out[L2,π,x]
       by (meson True ‹π ∈ L1 ∩ L2› ‹strong-reduction L1 L2› quasi-reduction-def
strong-reduction-def)
      ultimately show ?thesis
        unfolding strongred.simps
        using outputs-executable outputs-in-alphabet[OF assms(2)]
        by force
    qed
  qed
  then show strong-reduction L1 L2 ⟹ (L1 ≼[X,strongred Y] L2)
    by auto


  have ⋀ π x . L1 ≼[X,strongred Y] L2 ⟹ π ∈ L1 ∩ L2 ⟹ x ∈ exec[L2,π] ⟹
out[L1,π,x] ≠ {}
   and ⋀ π x . L1 ≼[X,strongred Y] L2 ⟹ π ∈ L1 ∩ L2 ⟹ x ∈ exec[L2,π] ⟹
out[L1,π,x] ⊆ out[L2,π,x]
  proof −
   fix π x y assume L1 ≼[X,strongred Y] L2 and π ∈ L1 ∩ L2 and x ∈ exec[L2,π]
    then have x ∈ X
      using executable-inputs-in-alphabet[OF assms(2)] by auto

    have out[L2,π,x] ≠ {}
      using ‹x ∈ exec[L2,π]›
      by (meson outputs-executable)
    moreover have (out[L1,π,x],out[L2,π,x]) ∈ strongred Y
    by (meson ‹L1 ≼[X,strongred Y] L2› ‹π ∈ L1 ∩ L2› ‹x ∈ X› type-1-conforms.elims(2))
    ultimately show out[L1,π,x] ≠ {} and out[L1,π,x] ⊆ out[L2,π,x]
      unfolding strongred.simps
```

by *blast+*
qed
moreover have $\bigwedge \pi\ x$. *L1* $\preceq$[*X*,*strongred Y*] *L2* $\Longrightarrow$ $\pi \in$ *L1* $\cap$ *L2* $\Longrightarrow$ *out*[*L2*,π,*x*] = {} $\Longrightarrow$ *out*[*L1*,π,*x*] = {}
proof −
fix π *x* **assume** *L1* $\preceq$[*X*,*strongred Y*] *L2* **and** $\pi \in$ *L1* $\cap$ *L2* **and** *out*[*L2*,π,*x*] = {}

show *out*[*L1*,π,*x*] = {}
proof (*rule ccontr*)
assume *out*[*L1*,π,*x*] $\neq$ {}
then have $x \in X$
by (*meson assms(1) executable-inputs-in-alphabet outputs-executable*)
then have *out*[*L2*,π,*x*] $\neq$ {}
using ‹*L1* $\preceq$[*X*,*strongred Y*] *L2*› ‹$\pi \in$ *L1* $\cap$ *L2*› ‹*out*[*L1*,π,*x*] $\neq$ {}› **by** *fastforce*
then show *False*
using ‹*out*[*L2*,π,*x*] = {}› **by** *simp*
qed
qed
ultimately show *L1* $\preceq$[*X*,*strongred Y*] *L2* $\Longrightarrow$ *strong-reduction L1 L2*
unfolding *strong-reduction-def quasi-reduction-def* **by** *blast*
qed

3.2 $\leq$ Conformance

fun *type-2-conforms* :: ('*x*,'*y*) *language* $\Rightarrow$ '*x alphabet* $\Rightarrow$ '*y output-relation* $\Rightarrow$ ('*x*,'*y*) *language* $\Rightarrow$ *bool* **where**
type-2-conforms L1 X H L2 = (
(∀ $\pi \in$ *L1* $\cap$ *L2* . ∀ $x \in X$. (*out*[*L1*,π,*x*],*out*[*L2*,π,*x*]) $\in$ *H*) ∧
(∀ $\pi \in$ *L1* $\cap$ *L2* . *exec*[*L2*,π] $\neq$ {} ⟶ (∃ *x* . *out*[*L1*,π,*x*] $\cap$ *out*[*L2*,π,*x*] $\neq$ {})))

notation *type-2-conforms* (‹- ≤[-,-] -›)

fun *semieq* :: '*y alphabet* $\Rightarrow$ '*y output-relation* **where**
semieq Y = {(*A*,*A*) | *A* . *A* $\subseteq$ *Y*} $\cup$ {({},*A*) | *A* . *A* $\subseteq$ *Y*} $\cup$ {(*A*,{}) | *A* . *A* $\subseteq$ *Y*}

fun *semired* :: '*y alphabet* $\Rightarrow$ '*y output-relation* **where**
semired Y = {(*A*,*B*) | *A B* . *A* $\subseteq$ *B* ∧ *B* $\subseteq$ *Y*} $\cup$ {(*C*,{}) | *C* . *C* $\subseteq$ *Y*}

fun *strongsemieq* :: '*y alphabet* $\Rightarrow$ '*y output-relation* **where**
strongsemieq Y = {(*A*,*A*) | *A* . *A* $\subseteq$ *Y*} $\cup$ {({},*A*) | *A* . *A* $\subseteq$ *Y*}

fun *strongsemired* :: '*y alphabet* $\Rightarrow$ '*y output-relation* **where**
strongsemired Y = {(*A*,*B*) | *A B* . *A* $\subseteq$ *B* ∧ *B* $\subseteq$ *Y*}

lemma *strongsemieq-alt-def* : *strongsemieq Y* = *semieq Y* $\cap$ *red Y*

by *auto*

lemma *strongsemired-alt-def* : *strongsemired Y = red Y*
by *auto*

lemma *semired-type-2* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows (*semi-reduction L1 L2*) $\longleftrightarrow$ (*L1* $\leq$[*X*, *semired Y*] *L2*)
proof
show *semi-reduction L1 L2* $\Longrightarrow$ *L1* $\leq$[*X*, *semired Y*] *L2*
proof −
assume *semi-reduction L1 L2*
then have *p1*: $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow (out[L1,\pi,x] \subseteq out[L2,\pi,x])$
and *p2*: $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow \exists\ x'\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}$
unfolding *semi-reduction-def* **by** *blast+*

have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in semired\ Y$
by (*metis* (*mono-tags*, *lifting*) *CollectI UnCI assms*(*1*) *assms*(*2*) *outputs-executable outputs-in-alphabet p1 semired.simps*)
moreover have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow exec[L2,\pi] \neq \{\} \Longrightarrow \exists x.\ out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
using *p2* **by** *fastforce*
ultimately show *L1* $\leq$[*X*, *semired Y*] *L2*
by *auto*
qed

show *L1* $\leq$[*X*,*semired Y*] *L2* $\Longrightarrow$ *semi-reduction L1 L2*
proof −
assume *L1* $\leq$[*X*,*semired Y*] *L2*
then have *p1* : $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow (out[L1,\pi,x],out[L2,\pi,x]) \in semired\ Y$
and *p2* : $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow exec[L2,\pi] \neq \{\} \Longrightarrow \exists\ x\ .\ out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
by *auto*

have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow (out[L1,\pi,x] \subseteq out[L2,\pi,x])$
proof −
fix π *x* **assume** $\pi \in L1 \cap L2$ **and** $x \in exec[L2,\pi]$
then have (*out*[*L1*,π,*x*],*out*[*L2*,π,*x*]) $\in$ *semired Y*
using *p1 executable-inputs-in-alphabet*[*OF assms*(*2*)] **by** *auto*
moreover have *out*[*L2*,π,*x*] $\neq$ {}
using ‹*x* $\in$ *exec*[*L2*,π]› **by** *auto*
ultimately show (*out*[*L1*,π,*x*] $\subseteq$ *out*[*L2*,π,*x*])
unfolding *semired.simps* **by** *blast*

```
    qed
   moreover have ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ exec[L2,π] ⟹ ∃ x′ . out[L1,π,x′]
∩ out[L2,π,x′] ≠ {}
      using p2 by blast
    ultimately show ?thesis
      unfolding semi-reduction-def by blast
  qed
qed


lemma semieq-type-2 :
  assumes is-language X Y L1
  and     is-language X Y L2
shows (semi-equivalence L1 L2) ⟷ (L1 ≤[X, semieq Y] L2)
proof
  show semi-equivalence L1 L2 ⟹ L1 ≤[X, semieq Y] L2
  proof −
    assume semi-equivalence L1 L2
    then have p1: ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ exec[L2,π] ⟹ out[L1,π,x] = {}
∨ out[L1,π,x] = out[L2,π,x]
        and  p2: ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ exec[L2,π] ⟹ ∃ x′ . out[L1,π,x′]
∩ out[L2,π,x′] ≠ {}
      unfolding semi-equivalence-def by blast+

    have ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ X ⟹ (out[L1,π,x], out[L2,π,x]) ∈ semieq
Y
    proof −
      fix π x assume π ∈ L1 ∩ L2 and x ∈ X
      show (out[L1,π,x], out[L2,π,x]) ∈ semieq Y
      proof (cases x ∈ exec[L2,π])
        case True
        then have out[L2,π,x] ≠ {} by auto
        then show ?thesis
          using p1[OF ‹π ∈ L1 ∩ L2› True]
          using outputs-in-alphabet[OF assms(2)]
          unfolding semieq.simps
          by fastforce
      next
        case False
        then show ?thesis
          by (metis (mono-tags, lifting) CollectI UnI2 assms(1) outputs-executable
outputs-in-alphabet semieq.elims)
      qed
    qed
   moreover have ⋀ π x . π ∈ L1 ∩ L2 ⟹ exec[L2,π] ≠ {} ⟹ ∃ x. out[L1,π,x]
∩ out[L2,π,x] ≠ {}
      using p2 by fastforce
    ultimately show L1 ≤[X, semieq Y] L2
      by auto
```

qed

show *L1* $\leq$[*X*,*semieq Y*] *L2* $\Longrightarrow$ *semi-equivalence L1 L2*
proof −
assume *L1* $\leq$[*X*,*semieq Y*] *L2*
then have *p1* : $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow$ (*out*[*L1*,π,*x*],*out*[*L2*,π,*x*]) $\in$ *semieq Y*
and *p2* : $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow$ *exec*[*L2*,π] $\neq \{\} \Longrightarrow \exists\ x\ .$ *out*[*L1*,π,*x*] $\cap$ *out*[*L2*,π,*x*] $\neq \{\}$
by *auto*

have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in$ *exec*[*L2*,π] $\Longrightarrow$ *out*[*L1*,π,*x*] $= \{\} \vee$ *out*[*L1*,π,*x*] $=$ *out*[*L2*,π,*x*]
proof −
fix π *x* **assume** $\pi \in L1 \cap L2$ **and** $x \in$ *exec*[*L2*,π]
then have (*out*[*L1*,π,*x*],*out*[*L2*,π,*x*]) $\in$ *semieq Y*
using *p1 executable-inputs-in-alphabet*[*OF assms*(*2*)] **by** *auto*
moreover have *out*[*L2*,π,*x*] $\neq \{\}$
using ‹*x* $\in$ *exec*[*L2*,π]› **by** *auto*
ultimately show *out*[*L1*,π,*x*] $= \{\} \vee$ *out*[*L1*,π,*x*] $=$ *out*[*L2*,π,*x*]
unfolding *semieq.simps*
by *blast*
qed
moreover have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in$ *exec*[*L2*,π] $\Longrightarrow \exists\ x'\ .$ *out*[*L1*,π,*x*′] $\cap$ *out*[*L2*,π,*x*′] $\neq \{\}$
using *p2* **by** *blast*
ultimately show *?thesis*
unfolding *semi-equivalence-def* **by** *blast*
qed
qed

lemma *strongsemired-type-2* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows (*strong-semi-reduction L1 L2*) $\longleftrightarrow$ (*L1* $\leq$[*X*, *strongsemired Y*] *L2*)
proof
show *strong-semi-reduction L1 L2* $\Longrightarrow$ *L1* $\leq$[*X*, *strongsemired Y*] *L2*
proof −
assume *strong-semi-reduction L1 L2*
then have *p1*: $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in$ *exec*[*L2*,π] $\Longrightarrow$ (*out*[*L1*,π,*x*] $\subseteq$ *out*[*L2*,π,*x*])
and *p2*: $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in$ *exec*[*L2*,π] $\Longrightarrow \exists\ x'\ .$ *out*[*L1*,π,*x*′] $\cap$ *out*[*L2*,π,*x*′] $\neq \{\}$
and *p3*: $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \notin$ *exec*[*L2*,π] $\Longrightarrow$ *out*[*L1*,π,*x*] $= \{\}$
unfolding *strong-semi-reduction-def* **by** *blast*+

have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow$ (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in$ *strongsemired Y*

unfolding *strongsemired.simps*
by (*metis* (*mono-tags*, *lifting*) *CollectI assms*(2) *outputs-executable outputs-in-alphabet p1 p3 set-eq-subset*)
moreover have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow exec[L2,\pi] \neq \{\} \Longrightarrow \exists x.\ out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
using *p2* **by** *fastforce*
ultimately show *L1* $\leq$[*X*, *strongsemired Y*] *L2*
by *auto*
qed

show *L1* $\leq$[*X*,*strongsemired Y*] *L2* $\Longrightarrow$ *strong-semi-reduction L1 L2*
proof −
assume *L1* $\leq$[*X*,*strongsemired Y*] *L2*
then have *p1* : $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow (out[L1,\pi,x],out[L2,\pi,x]) \in strongsemired\ Y$
and *p2* : $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow exec[L2,\pi] \neq \{\} \Longrightarrow \exists\ x\ .\ out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
by *auto*

have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow (out[L1,\pi,x] \subseteq out[L2,\pi,x])$
proof −
fix π *x* **assume** $\pi \in L1 \cap L2$ **and** $x \in exec[L2,\pi]$
then have $(out[L1,\pi,x],out[L2,\pi,x]) \in semired\ Y$
using *p1 executable-inputs-in-alphabet*[*OF assms*(2)] **by** *auto*
moreover have $out[L2,\pi,x] \neq \{\}$
using ‹$x \in exec[L2,\pi]$› **by** *auto*
ultimately show $(out[L1,\pi,x] \subseteq out[L2,\pi,x])$
unfolding *semired.simps* **by** *blast*
qed
moreover have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow \exists\ x'\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}$
using *p2* **by** *blast*
moreover have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \notin exec[L2,\pi] \Longrightarrow out[L1,\pi,x] = \{\}$
proof −
fix π *x* **assume** $\pi \in L1 \cap L2$ **and** $x \notin exec[L2,\pi]$

have $(out[L1,\pi,x],out[L2,\pi,x]) \in strongsemired\ Y$
proof (*cases* $x \in exec[L1,\pi]$)
case *True*
then show *?thesis*
by (*meson* ‹$\pi \in L1 \cap L2$› *assms*(1) *executable-inputs-in-alphabet p1*)
next
case *False*
then show *?thesis*
using ‹$x \notin exec[L2,\pi]$› **by** *fastforce*
qed
moreover have $out[L2,\pi,x] = \{\}$
using ‹$x \notin exec[L2,\pi]$› **by** *auto*
ultimately show $out[L1,\pi,x] = \{\}$

```
      unfolding strongsemired.simps
      by blast
    qed
    ultimately show ?thesis
      unfolding strong-semi-reduction-def by blast
  qed
qed


lemma strongsemieq-type-2 :
  assumes is-language X Y L1
  and     is-language X Y L2
shows (strong-semi-equivalence L1 L2) ⟷ (L1 ≤[X, strongsemieq Y] L2)
proof
  show strong-semi-equivalence L1 L2 ⟹ L1 ≤[X, strongsemieq Y] L2
  proof −
    assume strong-semi-equivalence L1 L2
    then have p1: ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ exec[L2,π] ⟹ out[L1,π,x] = {}
∨ out[L1,π,x] = out[L2,π,x]
        and  p2: ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ exec[L2,π] ⟹ ∃ x′ . out[L1,π,x′]
∩ out[L2,π,x′] ≠ {}
        and  p3: ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∉ exec[L2,π] ⟹ out[L1,π,x] = {}
      unfolding strong-semi-equivalence-def by blast+

     have ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ X ⟹ (out[L1,π,x], out[L2,π,x]) ∈
strongsemieq Y
    proof −
      fix π x assume π ∈ L1 ∩ L2 and x ∈ X
      show (out[L1,π,x], out[L2,π,x]) ∈ strongsemieq Y
      proof (cases x ∈ exec[L2,π])
        case True
        then have out[L2,π,x] ≠ {} by auto
        then show ?thesis
          using p1[OF ‹π ∈ L1 ∩ L2› True]
          using outputs-in-alphabet[OF assms(2)]
          by fastforce
      next
        case False
        then show ?thesis
          using ‹π ∈ L1 ∩ L2› p3 by fastforce
      qed
    qed
    moreover have ⋀ π x . π ∈ L1 ∩ L2 ⟹ exec[L2,π] ≠ {} ⟹ ∃ x. out[L1,π,x]
∩ out[L2,π,x] ≠ {}
      using p2 by fastforce
    ultimately show L1 ≤[X, strongsemieq Y] L2
      by auto
  qed
```

show $L1 \leq[X,strongsemieq\ Y]\ L2 \Longrightarrow strong\text{-}semi\text{-}equivalence\ L1\ L2$
proof −
assume $L1 \leq[X,strongsemieq\ Y]\ L2$
then have $p1 : \bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow (out[L1,\pi,x],out[L2,\pi,x]) \in strongsemieq\ Y$
and $p2 : \bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow exec[L2,\pi] \neq \{\} \Longrightarrow \exists\ x\ .\ out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
by *auto*

have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow out[L1,\pi,x] = \{\} \vee out[L1,\pi,x] = out[L2,\pi,x]$
proof −
fix $\pi\ x$ **assume** $\pi \in L1 \cap L2$ **and** $x \in exec[L2,\pi]$
then have $(out[L1,\pi,x],out[L2,\pi,x]) \in semieq\ Y$
using *p1 executable-inputs-in-alphabet*[*OF assms*(*2*)] **by** *auto*
moreover have $out[L2,\pi,x] \neq \{\}$
using ‹$x \in exec[L2,\pi]$› **by** *auto*
ultimately show $out[L1,\pi,x] = \{\} \vee out[L1,\pi,x] = out[L2,\pi,x]$
unfolding *semieq.simps*
by *blast*
qed
moreover have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow \exists\ x'\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}$
using *p2* **by** *blast*
moreover have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \notin exec[L2,\pi] \Longrightarrow out[L1,\pi,x] = \{\}$
proof −
fix $\pi\ x$ **assume** $\pi \in L1 \cap L2$ **and** $x \notin exec[L2,\pi]$

have $(out[L1,\pi,x],out[L2,\pi,x]) \in strongsemieq\ Y$
proof (*cases* $x \in exec[L1,\pi]$)
case *True*
then show *?thesis*
by (*meson* ‹$\pi \in L1 \cap L2$› *assms*(*1*) *executable-inputs-in-alphabet p1*)
next
case *False*
then show *?thesis*
using ‹$x \notin exec[L2,\pi]$› **by** *fastforce*
qed
moreover have $out[L2,\pi,x] = \{\}$
using ‹$x \notin exec[L2,\pi]$› **by** *auto*
ultimately show $out[L1,\pi,x] = \{\}$
unfolding *strongsemieq.simps*
by *blast*
qed
ultimately show *?thesis*
unfolding *strong-semi-equivalence-def* **by** *blast*
qed
qed

4 Comparing Conformance Relations

lemma *type-1-subset* :
assumes $L1 \preceq[X,H1]\ L2$
and $H1 \subseteq H2$
shows $L1 \preceq[X,H2]\ L2$
using *assms* **by** *auto*

lemma *type-1-subsets* :
shows $equiv\ Y \subseteq strongred\ Y$
and $equiv\ Y \subseteq quasieq\ Y$
and $strongred\ Y \subseteq red\ Y$
and $strongred\ Y \subseteq quasired\ Y$
and $quasieq\ Y \subseteq quasired\ Y$
by *auto*

lemma *type-1-implications* :
shows $L1 \preceq[X,\ equiv\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ strongred\ Y]\ L2$
and $L1 \preceq[X,\ equiv\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ red\ Y]\ L2$
and $L1 \preceq[X,\ equiv\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ quasired\ Y]\ L2$
and $L1 \preceq[X,\ equiv\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ quasieq\ Y]\ L2$
and $L1 \preceq[X,\ strongred\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ red\ Y]\ L2$
and $L1 \preceq[X,\ strongred\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ quasired\ Y]\ L2$
and $L1 \preceq[X,\ quasieq\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ quasired\ Y]\ L2$
using *type-1-subset*[*OF - type-1-subsets*(*4*), *of L1 X Y L2*]
using *type-1-subset*[*OF - type-1-subsets*(*5*), *of L1 X Y L2*]
by *auto*

lemma *type-2-subset* :
assumes $L1 \leq[X,H1]\ L2$
and $H1 \subseteq H2$
shows $L1 \leq[X,H2]\ L2$
using *assms* **by** *auto*

lemma *type-2-subsets* :
shows $strongsemieq\ Y \subseteq strongsemired\ Y$
and $strongsemieq\ Y \subseteq semieq\ Y$
and $semieq\ Y \subseteq semired\ Y$
and $strongsemired\ Y \subseteq semired\ Y$
and $strongsemired\ Y \subseteq red\ Y$
by *auto*

lemma *type-2-implications* :
shows $L1 \leq[X,\ strongsemieq\ Y]\ L2 \Longrightarrow L1 \leq[X,\ strongsemired\ Y]\ L2$
and $L1 \leq[X,\ strongsemieq\ Y]\ L2 \Longrightarrow L1 \leq[X,\ semieq\ Y]\ L2$
and $L1 \leq[X,\ strongsemieq\ Y]\ L2 \Longrightarrow L1 \leq[X,\ semired\ Y]\ L2$
and $L1 \leq[X,\ strongsemired\ Y]\ L2 \Longrightarrow L1 \leq[X,\ semired\ Y]\ L2$
and $L1 \leq[X,\ semieq\ Y]\ L2 \Longrightarrow L1 \leq[X,\ semired\ Y]\ L2$

by *auto*

lemma *type-1-conformance-to-type-2* :
assumes *is-language X Y L2*
and $L1 \preceq[X,H1]\ L2$
and $H1 \subseteq H2$
and $\bigwedge A\ B\ .\ (A,B) \in H1 \Longrightarrow B \neq \{\} \Longrightarrow A \cap B \neq \{\}$
shows $L1 \leq[X,H2]\ L2$
proof −
have $\bigwedge \pi\ x\ .\ \pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in H2$
using *assms*(*2*,*3*) **by** *auto*
moreover have $\bigwedge \pi\ .\ \pi \in L1 \cap L2 \Longrightarrow exec[L2,\pi] \neq \{\} \Longrightarrow \exists\, x.\ out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
proof −
fix π **assume** $\pi \in L1 \cap L2$ **and** $exec[L2,\pi] \neq \{\}$
then obtain x **where** $x \in exec[L2,\pi]$
by *blast*
then have $x \in X$
by (*meson assms*(*1*) *executable-inputs-in-alphabet*)
then have $(out[L1,\pi,x],\ out[L2,\pi,x]) \in H1$
using ‹$\pi \in L1 \cap L2$› *assms*(*2*) **by** *auto*
moreover have $out[L2,\pi,x] \neq \{\}$
by (*meson* ‹$x \in exec[L2,\pi]$› *outputs-executable*)
ultimately have $out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
using *assms*(*4*) **by** *blast*
then show $\exists\, x.\ out[L1,\pi,x] \cap out[L2,\pi,x] \neq \{\}$
by *blast*
qed
ultimately show *?thesis*
by *auto*
qed

lemma *type-1-and-2-mixed-implications* :
assumes *is-language X Y L2*
shows $L1 \leq[X,\ strongsemieq\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ red\ Y]\ L2$
and $L1 \leq[X,\ strongsemired\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ red\ Y]\ L2$
and $L1 \preceq[X,\ quasieq\ Y]\ L2 \Longrightarrow L1 \leq[X,\ semieq\ Y]\ L2$
and $L1 \preceq[X,\ quasired\ Y]\ L2 \Longrightarrow L1 \leq[X,\ semired\ Y]\ L2$
and $L1 \preceq[X,\ equiv\ Y]\ L2 \Longrightarrow L1 \leq[X,\ strongsemieq\ Y]\ L2$
and $L1 \preceq[X,\ strongred\ Y]\ L2 \Longrightarrow L1 \leq[X,\ strongsemired\ Y]\ L2$
proof −

show $L1 \leq[X,\ strongsemieq\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ red\ Y]\ L2$
and $L1 \leq[X,\ strongsemired\ Y]\ L2 \Longrightarrow L1 \preceq[X,\ red\ Y]\ L2$
by *auto*

have $\bigwedge A\ B\ .\ (A,B) \in quasieq\ Y \Longrightarrow B \neq \{\} \Longrightarrow A \cap B \neq \{\}$
by *auto*

moreover have *quasieq Y* $\subseteq$ *semieq Y*
by *auto*
ultimately show *L1* $\preceq$[*X*, *quasieq Y*] *L2* $\Longrightarrow$ *L1* $\leq$[*X*, *semieq Y*] *L2*
using *type-1-conformance-to-type-2*[*OF assms*] **by** *blast*

have $\bigwedge$ *A B* . (*A*,*B*) $\in$ *quasired Y* $\Longrightarrow$ *B* $\neq$ {} $\Longrightarrow$ *A* $\cap$ *B* $\neq$ {}
by *auto*
moreover have *quasired Y* $\subseteq$ *semired Y*
unfolding *quasired.simps semired.simps* **by** *blast*
ultimately show *L1* $\preceq$[*X*, *quasired Y*] *L2* $\Longrightarrow$ *L1* $\leq$[*X*, *semired Y*] *L2*
using *type-1-conformance-to-type-2*[*OF assms*] **by** *blast*

have $\bigwedge$ *A B* . (*A*,*B*) $\in$ *equiv Y* $\Longrightarrow$ *B* $\neq$ {} $\Longrightarrow$ *A* $\cap$ *B* $\neq$ {}
by *auto*
moreover have *equiv Y* $\subseteq$ *strongsemieq Y*
unfolding *equiv.simps strongsemieq.simps* **by** *blast*
ultimately show *L1* $\preceq$[*X*, *equiv Y*] *L2* $\Longrightarrow$ *L1* $\leq$[*X*, *strongsemieq Y*] *L2*
using *type-1-conformance-to-type-2*[*OF assms*] **by** *blast*

have $\bigwedge$ *A B* . (*A*,*B*) $\in$ *strongred Y* $\Longrightarrow$ *B* $\neq$ {} $\Longrightarrow$ *A* $\cap$ *B* $\neq$ {}
by *auto*
moreover have *strongred Y* $\subseteq$ *strongsemired Y*
unfolding *strongred.simps strongsemired.simps* **by** *blast*
ultimately show *L1* $\preceq$[*X*, *strongred Y*] *L2* $\Longrightarrow$ *L1* $\leq$[*X*, *strongsemired Y*] *L2*
using *type-1-conformance-to-type-2*[*OF assms*] **by** *blast*
qed

4.1 Completely Specified Languages

definition *partiality-component* :: *'y set* $\Rightarrow$ *'y output-relation* **where**
partiality-component Y = {(*A*,{}) | *A* . *A* $\subseteq$ *Y*} $\cup$ {({},*A*) | *A* . *A* $\subseteq$ *Y*}

abbreviation(*input*) Π *Y* $\equiv$ *partiality-component Y*

lemma *conformance-without-partiality* :
shows *strongsemieq Y* $-$ Π *Y* = *semieq Y* $-$ Π *Y*
and *semieq Y* $-$ Π *Y* = *equiv Y* $-$ Π *Y*
and *strongsemired Y* $-$ Π *Y* = *semired Y* $-$ Π *Y*
and *semired Y* $-$ Π *Y* = *red Y* $-$ Π *Y*
unfolding *partiality-component-def*
by *fastforce+*

5 Conformance Testing

type-synonym ('*x*,'*y*) *state-cover* = ('*x*,'*y*) *language*
type-synonym ('*x*,'*y*) *transition-cover* = ('*x*,'*y*) *state-cover* $\times$ '*x set*

fun *is-state-cover* :: ('*x*,'*y*) *language* $\Rightarrow$ ('*x*,'*y*) *language* $\Rightarrow$ ('*x*,'*y*) *state-cover* $\Rightarrow$

bool **where**
is-state-cover L1 L2 V = (∀ *π* ∈ *L1* ∩ *L2* . ∃ *α* ∈ *V* . $\mathcal{L}$[*L1*,*π*] = $\mathcal{L}$[*L1*,*α*] ∧ $\mathcal{L}$[*L2*,*π*] = $\mathcal{L}$[*L2*,*α*])

lemma *state-cover-subset* :
assumes *is-language X Y L1*
and *is-language X Y L2*
and *is-state-cover L1 L2 V*
and *π* ∈ *L1* ∩ *L2*
obtains *α* **where** *α* ∈ *V*
and *α* ∈ *L1* ∩ *L2*
and $\mathcal{L}$[*L1*,*π*] = $\mathcal{L}$[*L1*,*α*]
and $\mathcal{L}$[*L2*,*π*] = $\mathcal{L}$[*L2*,*α*]
proof −
obtain *α* **where** *α* ∈ *V*
and $\mathcal{L}$[*L1*,*π*] = $\mathcal{L}$[*L1*,*α*]
and $\mathcal{L}$[*L2*,*π*] = $\mathcal{L}$[*L2*,*α*]
using *assms*
by (*meson is-state-cover.elims*(*2*))
moreover have $\mathcal{L}$[*L1*,*π*] ≠ {} **and** $\mathcal{L}$[*L2*,*π*] ≠ {}
by (*metis Collect-empty-eq-bot Int-iff append.right-neutral assms*(*4*) *empty-def language-for-state.elims*)+
ultimately have *α* ∈ *L1* ∩ *L2*
using *language-of-state-empty-iff*[*OF assms*(*1*)] *language-of-state-empty-iff*[*OF assms*(*2*)]
by *blast*
then show *?thesis* **using** *that*[*OF* ‹*α* ∈ *V*› - ‹$\mathcal{L}$[*L1*,*π*] = $\mathcal{L}$[*L1*,*α*]› ‹$\mathcal{L}$[*L2*,*π*] = $\mathcal{L}$[*L2*,*α*]›]
by *blast*
qed

theorem *sufficient-condition-for-type-1-conformance* :
assumes *is-language X Y L1*
and *is-language X Y L2*
and *is-state-cover L1 L2 V*
shows (*L1* ⪯[*X*,*H*] *L2*) ⟷ (∀ *π* ∈ *V* . ∀ *x* ∈ *X* . *π* ∈ *L1* ∩ *L2* ⟶ (*out*[*L1*,*π*,*x*], *out*[*L2*,*π*,*x*]) ∈ *H*)
proof
show (*L1* ⪯[*X*,*H*] *L2*) ⟹ (∀ *π* ∈ *V* . ∀ *x* ∈ *X* . *π* ∈ *L1* ∩ *L2* ⟶ (*out*[*L1*,*π*,*x*], *out*[*L2*,*π*,*x*]) ∈ *H*)
by *auto*

have (⋀ *π x* . *π* ∈ *V* ⟹ *x* ∈ *X* ⟹ *π* ∈ *L1* ∩ *L2* ⟹ (*out*[*L1*,*π*,*x*], *out*[*L2*,*π*,*x*]) ∈ *H*) ⟹ (⋀ *π x* . *π* ∈ *L1* ∩ *L2* ⟹ *x* ∈ *X* ⟹ (*out*[*L1*,*π*,*x*], *out*[*L2*,*π*,*x*]) ∈ *H*)
proof −
fix *π x* **assume** (⋀ *π x* . *π* ∈ *V* ⟹ *x* ∈ *X* ⟹ *π* ∈ *L1* ∩ *L2* ⟹ (*out*[*L1*,*π*,*x*],

$out[L2,\pi,x]) \in H$)
and $\pi \in L1 \cap L2$
and $x \in X$

obtain α **where** $\alpha \in V$ **and** $\alpha \in L1 \cap L2$ **and** $\mathcal{L}[L1,\pi] = \mathcal{L}[L1,\alpha]$ **and** $\mathcal{L}[L2,\pi] = \mathcal{L}[L2,\alpha]$
using *state-cover-subset*[*OF assms* ‹$\pi \in L1 \cap L2$›] **by** *auto*
then have $out[L1,\pi,x] = out[L1,\alpha,x]$ **and** $out[L2,\pi,x] = out[L2,\alpha,x]$
by *force+*
moreover have $(out[L1,\alpha,x],\ out[L2,\alpha,x]) \in H$
using ‹$(\bigwedge \pi\ x\ .\ \pi \in V \Longrightarrow x \in X \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in H)$› ‹$\alpha \in V$› ‹$x \in X$› ‹$\alpha \in L1 \cap L2$›
by *blast*
ultimately show $(out[L1,\pi,x],\ out[L2,\pi,x]) \in H$
by *simp*
qed
then show $\forall \pi{\in}V.\ \forall x{\in}X.\ \pi \in L1 \cap L2 \longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in H \Longrightarrow L1 \preceq[X,H]\ L2$
by *auto*
qed

theorem *sufficient-condition-for-type-2-conformance* :
assumes *is-language X Y L1*
and *is-language X Y L2*
and *is-state-cover L1 L2 V*
shows $(L1 \leq[X,H]\ L2) \longleftrightarrow (\forall\ \pi \in V\ .\ \forall\ x \in X\ .\ \pi \in L1 \cap L2 \longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in H \wedge (out[L2,\pi,x] \neq \{\} \longrightarrow (\exists\ x' \in X\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\})))$
proof

have $\bigwedge\ \pi\ x\ .\ (L1 \leq[X,H]\ L2) \Longrightarrow \pi \in V \Longrightarrow x \in X \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in H \wedge (out[L2,\pi,x] \neq \{\} \longrightarrow (\exists\ x' \in X\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}))$
proof −
fix $\pi\ x$ **assume** $L1 \leq[X,H]\ L2$ **and** $\pi \in V$ **and** $x \in X$ **and** $\pi \in L1 \cap L2$

have $(out[L1,\pi,x],\ out[L2,\pi,x]) \in H$
using ‹$L1 \leq[X,H]\ L2$› ‹$\pi \in L1 \cap L2$› ‹$x \in X$› **by** *force*
moreover have $out[L2,\pi,x] \neq \{\} \Longrightarrow (\exists\ x' \in X\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\})$
by (*metis* (*no-types*, *lifting*) ‹$L1 \leq[X,H]\ L2$› ‹$\pi \in L1 \cap L2$› *assms*(2) *empty-iff executable-inputs-in-alphabet inf-bot-right outputs-executable type-2-conforms.elims*(2))

ultimately show $(out[L1,\pi,x],\ out[L2,\pi,x]) \in H \wedge (out[L2,\pi,x] \neq \{\} \longrightarrow (\exists\ x' \in X\ .\ out[L1,\pi,x'] \cap out[L2,\pi,x'] \neq \{\}))$
by *blast*
qed
then show $(L1 \leq[X,H]\ L2) \Longrightarrow (\forall\ \pi \in V\ .\ \forall\ x \in X\ .\ \pi \in L1 \cap L2 \longrightarrow (out[L1,\pi,x],\ out[L2,\pi,x]) \in H \wedge (out[L2,\pi,x] \neq \{\} \longrightarrow (\exists\ x' \in X\ .\ out[L1,\pi,x']$

$\cap$ *out*[*L2*,π,*x'*] $\neq$ {})))
by *auto*

have ($\bigwedge$ π *x* . $\pi \in V \Longrightarrow x \in X \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow$ (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in H \wedge$ (*out*[*L2*,π,*x*] $\neq$ {} $\longrightarrow$ ($\exists$ *x'* $\in X$. *out*[*L1*,π,*x'*] $\cap$ *out*[*L2*,π,*x'*] $\neq$ {}))) $\Longrightarrow$ ($\bigwedge$ π *x* . $\pi \in L1 \cap L2 \Longrightarrow x \in X \Longrightarrow$ (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in H$)
by (*meson assms*(*1*) *assms*(*2*) *assms*(*3*) *sufficient-condition-for-type-1-conformance type-1-conforms.elims*(*2*))
moreover have ($\bigwedge$ π *x* . $\pi \in V \Longrightarrow x \in X \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow$ (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in H \wedge$ (*out*[*L2*,π,*x*] $\neq$ {} $\longrightarrow$ ($\exists$ *x'* $\in X$. *out*[*L1*,π,*x'*] $\cap$ *out*[*L2*,π,*x'*] $\neq$ {}))) $\Longrightarrow$ ($\bigwedge$ π . $\pi \in L1 \cap L2 \Longrightarrow$ *exec*[*L2*,π] $\neq$ {} $\Longrightarrow$ ($\exists$ *x* . *out*[*L1*,π,*x*] $\cap$ *out*[*L2*,π,*x*] $\neq$ {}))
proof −
fix π **assume** $\pi \in L1 \cap L2$
and *exec*[*L2*,π] $\neq$ {}
and $*$: ($\bigwedge$ π *x* . $\pi \in V \Longrightarrow x \in X \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow$ (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in H \wedge$ (*out*[*L2*,π,*x*] $\neq$ {} $\longrightarrow$ ($\exists$ *x'* $\in X$. *out*[*L1*,π,*x'*] $\cap$ *out*[*L2*,π,*x'*] $\neq$ {})))
then obtain *x* **where** $x \in X$ **and** *out*[*L2*,π,*x*] $\neq$ {}
by (*metis all-not-in-conv assms*(*2*) *executable-inputs-in-alphabet outputs-executable*)
moreover obtain α **where** $\alpha \in V$
and $\alpha \in L1 \cap L2$
and $\mathcal{L}$[*L1*,π] = $\mathcal{L}$[*L1*,α]
and $\mathcal{L}$[*L2*,π] = $\mathcal{L}$[*L2*,α]
using *state-cover-subset*[*OF assms* ‹$\pi \in L1 \cap L2$›] **by** *blast*
ultimately show ($\exists$ *x* . *out*[*L1*,π,*x*] $\cap$ *out*[*L2*,π,*x*] $\neq$ {})
using $*$
by (*metis outputs.elims*)
qed
ultimately show ($\forall$ $\pi \in V$. $\forall$ $x \in X$. $\pi \in L1 \cap L2 \longrightarrow$ (*out*[*L1*,π,*x*], *out*[*L2*,π,*x*]) $\in H \wedge$ (*out*[*L2*,π,*x*] $\neq$ {} $\longrightarrow$ ($\exists$ *x'* $\in X$. *out*[*L1*,π,*x'*] $\cap$ *out*[*L2*,π,*x'*] $\neq$ {}))) $\Longrightarrow$ (*L1* $\leq$[*X*,*H*] *L2*)
by *auto*
qed

lemma *intersections-card-helper* :
assumes *finite X*
and *finite Y*
shows *card* {*A* $\cap$ *B* | *A B* . *A* $\in X \wedge$ *B* $\in$ *Y*} $\leq$ *card X* $*$ *card Y*
proof −
have {*A* $\cap$ *B* | *A B* . *A* $\in X \wedge$ *B* $\in$ *Y*} = (λ (*A*,*B*) . *A* $\cap$ *B*) ‘ (*X* $\times$ *Y*)
by *auto*
then have *card* {*A* $\cap$ *B* | *A B* . *A* $\in X \wedge$ *B* $\in$ *Y*} $\leq$ *card* (*X* $\times$ *Y*)
by (*metis* (*no-types*, *lifting*) *assms*(*1*) *assms*(*2*) *card-image-le finite-SigmaI*)
then show *?thesis*
by (*simp add*: *card-cartesian-product*)

qed

lemma *prefix-length-take* :
(*prefix xs ys* ∧ *length xs* ≤ *k*) ⟷ (*prefix xs* (*take k ys*))
proof
show *prefix xs ys* ∧ *length xs* ≤ *k* ⟹ *prefix xs* (*take k ys*)
using *prefix-length-prefix take-is-prefix* **by** *fastforce*
show *prefix xs* (*take k ys*) ⟹ *prefix xs ys* ∧ *length xs* ≤ *k*
by (*metis le-trans length-take min.cobounded2 prefix-length-le prefix-order.order-trans take-is-prefix*)
qed

lemma *brute-force-state-cover* :
assumes *is-language X Y L1*
and *is-language X Y L2*
and *finite* {*L*[*L1*,π] | π . π ∈ *L1*}
and *finite* {*L*[*L2*,π] | π . π ∈ *L2*}
and *card* {*L*[*L1*,π] | π . π ∈ *L1*} ≤ *n*
and *card* {*L*[*L2*,π] | π . π ∈ *L2*} ≤ *m*
shows *is-state-cover L1 L2* {α . *length* α ≤ *m* ∗ *n* − *1* ∧ (∀ *xy* ∈ *set* α . *fst xy* ∈ *X* ∧ *snd xy* ∈ *Y*)}
proof (*rule ccontr*)
let *?V* = {α. *length* α ≤ *m* ∗ *n* − *1* ∧ (∀ *xy*∈*set* α. *fst xy* ∈ *X* ∧ *snd xy* ∈ *Y*)}
assume ¬ *is-state-cover L1 L2 ?V*

define *is-covered* **where** *is-covered* = (λ π . ∃ α ∈ *?V* . *L*[*L1*,π] = *L*[*L1*,α] ∧ *L*[*L2*,π] = *L*[*L2*,α])
define *missing-traces* **where** *missing-traces* = {τ . τ ∈ *L1* ∩ *L2* ∧ ¬*is-covered* τ}
define τ **where** τ = *arg-min length* (λ π . π ∈ *missing-traces*)

have *missing-traces* ≠ {}
using ‹¬ *is-state-cover L1 L2 ?V*›
using *is-covered-def missing-traces-def* **by** *fastforce*
then have τ ∈ *missing-traces*
⋀ τ′ . τ′ ∈ *missing-traces* ⟹ *length* τ ≤ *length* τ′
using *arg-min-nat-lemma*[**where** *P*=(λ π . π ∈ *missing-traces*) **and** *m*=*length*]

unfolding τ*-def*[*symmetric*] **by** *blast*+
then have τ*-props*: τ ∈ *L1* ∩ *L2*
⋀ α . α ∈ *?V* ⟹ ¬(*L*[*L1*,τ] = *L*[*L1*,α] ∧ *L*[*L2*,τ] = *L*[*L2*,α])
unfolding *missing-traces-def is-covered-def* **by** *blast*+

have $\bigwedge xy\ .\ xy \in set\ \tau \Longrightarrow fst\ xy \in X \wedge snd\ xy \in Y$
using ‹$\tau \in L1 \cap L2$› *assms(1)* **by** *auto*
moreover have $\tau \notin ?V$
using *τ-props(2)* **by** *blast*
ultimately have *length* $\tau > m{*}n{-}1$
by *simp*

let *?L12* = $\{\mathcal{L}[L1,\pi] \mid \pi\ .\ \pi \in L1\} \times \{\mathcal{L}[L2,\pi] \mid \pi\ .\ \pi \in L2\}$

have *finite ?L12*
using *assms(3,4)*
by *blast*

have *card ?L12* $\leq m{*}n$
using *assms(3,4,5,6)*
by (*metis* (*no-types*, *lifting*) *Sigma-cong card-cartesian-product mult.commute mult-le-mono*)

let *?visited-states* = $\{(\mathcal{L}[L1,\tau'],\mathcal{L}[L2,\tau']) \mid \tau'\ .\ \tau' \in set\ (prefixes\ \tau) \wedge length\ \tau' \leq m * n - 1\}$

have $\bigwedge \tau'\ .\ \tau' \in set\ (prefixes\ \tau) \Longrightarrow \tau' \in L1 \cap L2$
by (*meson τ-props(1) assms(1) assms(2) in-set-prefixes is-language.elims(2) language-intersection-is-language*)
then have *?visited-states* $\subseteq$ *?L12*
by *blast*
then have *card ?visited-states* $\leq m * n$
using ‹*finite ?L12*› ‹*card ?L12* $\leq m * n$›
by (*meson card-mono dual-order.trans*)

have *no-index-loop* : $\bigwedge i\ j\ .\ i < j \Longrightarrow j \leq length\ \tau \Longrightarrow \mathcal{L}[L1,\ take\ i\ \tau] \neq \mathcal{L}[L1,\ take\ j\ \tau] \vee \mathcal{L}[L2,\ take\ i\ \tau] \neq \mathcal{L}[L2,\ take\ j\ \tau]$
proof (*rule ccontr*)
fix *i j*
assume $i < j$ **and** $j \leq$ *length* τ **and** $\neg$ ($\mathcal{L}[L1,take\ i\ \tau] \neq \mathcal{L}[L1,take\ j\ \tau] \vee \mathcal{L}[L2,take\ i\ \tau] \neq \mathcal{L}[L2,take\ j\ \tau]$)
then have $\mathcal{L}[L1,take\ i\ \tau] = \mathcal{L}[L1,take\ j\ \tau]$ **and** $\mathcal{L}[L2,take\ i\ \tau] = \mathcal{L}[L2,take\ j\ \tau]$
by *auto*

have $\{\tau'.\ \tau\ @\ \tau' \in L1\} = \{\tau'.\ take\ j\ \tau\ @\ drop\ j\ \tau\ @\ \tau' \in L1\}$
by (*metis append.assoc append-take-drop-id*)
have $\{\tau'.\ \tau\ @\ \tau' \in L2\} = \{\tau'.\ take\ j\ \tau\ @\ drop\ j\ \tau\ @\ \tau' \in L2\}$
by (*metis append.assoc append-take-drop-id*)

have $\mathcal{L}$[*L1*,*take i* τ @ *drop j* τ] = $\mathcal{L}$[*L1*,τ]
using ‹$\mathcal{L}$[*L1*,*take i* τ] = $\mathcal{L}$[*L1*,*take j* τ]›
unfolding *language-for-state.simps*
unfolding ‹$\{\tau'.\ \tau\ @\ \tau' \in L1\} = \{\tau'.\ take\ j\ \tau\ @\ drop\ j\ \tau\ @\ \tau' \in L1\}$› *append.assoc* **by** *blast*
moreover have $\mathcal{L}$[*L2*,*take i* τ @ *drop j* τ] = $\mathcal{L}$[*L2*,τ]
using ‹$\mathcal{L}$[*L2*,*take i* τ] = $\mathcal{L}$[*L2*,*take j* τ]›
unfolding *language-for-state.simps*
unfolding ‹$\{\tau'.\ \tau\ @\ \tau' \in L2\} = \{\tau'.\ take\ j\ \tau\ @\ drop\ j\ \tau\ @\ \tau' \in L2\}$› *append.assoc* **by** *blast*

have (*take i* τ @ *drop j* τ) ∈ *missing-traces*
proof (*rule ccontr*)
assume *take i* τ @ *drop j* τ ∉ *missing-traces*
moreover have *take i* τ @ *drop j* τ ∈ *L1* ∩ *L2*
by (*metis IntD1 IntD2 IntI* ‹$\mathcal{L}$[*L1*,*take i* τ] = $\mathcal{L}$[*L1*,*take j* τ]› ‹$\mathcal{L}$[*L2*,*take i* τ] = $\mathcal{L}$[*L2*,*take j* τ]› τ-*props*(*1*) *append-take-drop-id language-for-state.elims mem-Collect-eq*)
ultimately obtain α **where** *length* $\alpha \le m * n - 1$
($\forall$ *xy*∈*set* α. *fst xy* ∈ *X* ∧ *snd xy* ∈ *Y*)
$\mathcal{L}$[*L1*,*take i* τ @ *drop j* τ] = $\mathcal{L}$[*L1*,α]
$\mathcal{L}$[*L2*,*take i* τ @ *drop j* τ] = $\mathcal{L}$[*L2*,α]
unfolding *missing-traces-def is-covered-def*
by *blast*
then have $\tau \notin$ *missing-traces*
unfolding *missing-traces-def is-covered-def*
using ‹τ ∈ *L1* ∩ *L2*›
unfolding ‹$\mathcal{L}$[*L1*,*take i* τ @ *drop j* τ] = $\mathcal{L}$[*L1*,τ]›
unfolding ‹$\mathcal{L}$[*L2*,*take i* τ @ *drop j* τ] = $\mathcal{L}$[*L2*,τ]›
by *blast*
then show *False*
using ‹τ ∈ *missing-traces*› **by** *simp*
qed
moreover have *length* (*take i* τ @ *drop j* τ) < *length* τ
using ‹*i* < *j*› ‹*j* ≤ *length* τ›
by (*induction* τ *arbitrary*: *i j*; *auto*)
ultimately show *False*
using ‹$\bigwedge \tau'$. $\tau' \in$ *missing-traces* ⟹ *length* τ ≤ *length* τ'›
using *leD* **by** *blast*
qed

have *no-prefix-loop* : $\bigwedge \tau'\ \tau''$. $\tau' \in$ *set* (*prefixes* τ) ⟹ $\tau'' \in$ *set* (*prefixes* τ) ⟹ $\tau' \neq \tau''$ ⟹ ($\mathcal{L}$[*L1*,τ'],$\mathcal{L}$[*L2*,τ']) ≠ ($\mathcal{L}$[*L1*,τ''],$\mathcal{L}$[*L2*,τ''])
proof −

fix τ' τ'' **assume** $\tau' \in$ *set* (*prefixes* τ) **and** $\tau'' \in$ *set* (*prefixes* τ) **and** $\tau' \neq \tau''$

obtain i **where** $\tau' =$ *take* i τ **and** $i \leq$ *length* τ
using ‹$\tau' \in$ *set* (*prefixes* τ)›
by (*metis append-eq-conv-conj in-set-prefixes linorder-linear prefix-def take-all-iff*)

obtain j **where** $\tau'' =$ *take* j τ **and** $j \leq$ *length* τ
using ‹$\tau'' \in$ *set* (*prefixes* τ)›
by (*metis append-eq-conv-conj in-set-prefixes linorder-linear prefix-def take-all-iff*)

have $i \neq j$
using ‹$\tau' =$ *take* i τ› ‹$\tau' \neq \tau''$› ‹$\tau'' =$ *take* j τ› **by** *blast*
then consider (*a*) $i < j$ | (*b*) $j < i$
using *nat-neq-iff* **by** *blast*
then show ($\mathcal{L}$[*L1*,τ'],$\mathcal{L}$[*L2*,τ']) $\neq$ ($\mathcal{L}$[*L1*,τ''],$\mathcal{L}$[*L2*,τ''])
using *no-index-loop*
using ‹$j \leq$ *length* τ› ‹$i \leq$ *length* τ›
unfolding ‹$\tau' =$ *take* i τ› ‹$\tau'' =$ *take* j τ›
by (*cases*; *blast*)
qed
then have *inj-on* (λ τ' . ($\mathcal{L}$[*L1*,τ'], $\mathcal{L}$[*L2*,τ'])) {τ' . $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$}
using *inj-onI*
by (*metis* (*mono-tags*, *lifting*) *mem-Collect-eq*)
then have *card* ((λ τ' . ($\mathcal{L}$[*L1*,τ'], $\mathcal{L}$[*L2*,τ'])) ‘ {τ' . $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$}) = *card* {τ' . $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$}
using *card-image* **by** *blast*
moreover have *?visited-states* = (λ τ' . ($\mathcal{L}$[*L1*,τ'], $\mathcal{L}$[*L2*,τ'])) ‘ {τ' . $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$}
by *auto*
ultimately have *card ?visited-states* = *card* {τ' . $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$}
by *simp*
moreover have *card* {τ' . $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$} = $m*n$
proof −
have {τ' . $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$} = *set* (*prefixes* (*take* ($m*n-1$) τ))
unfolding *in-set-prefixes prefix-length-take*
by *auto*
moreover have *length* (*take* ($m*n-1$) τ) = $m*n-1$
using ‹*length* $\tau >$ $m*n-1$› **by** *auto*
ultimately show *?thesis*
using *length-prefixes distinct-prefixes*
by (*metis* ‹*card* {($\mathcal{L}$[*L1*,τ'], $\mathcal{L}$[*L2*,τ']) |τ'. $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$} = *card* {$\tau' \in$ *set* (*prefixes* τ). *length* $\tau' \leq m * n - 1$}› ‹*card* {($\mathcal{L}$[*L1*,τ'], $\mathcal{L}$[*L2*,τ']) |τ'. $\tau' \in$ *set* (*prefixes* τ) $\wedge$ *length* $\tau' \leq m * n - 1$} $\leq m * n$› *distinct-card less-diff-conv not-less-iff-gr-or-eq order-le-less*)

qed

ultimately have *card ?visited-states = m∗n*
by *simp*
then have *?visited-states = ?L12*
by (*metis* (*no-types*, *lifting*) ‹*card* ({$\mathcal{L}$[*L1*,π] |π. π ∈ *L1*} × {$\mathcal{L}$[*L2*,π] |π. π ∈ *L2*}) ≤ *m* ∗ *n*› ‹*finite* ({$\mathcal{L}$[*L1*,π] |π. π ∈ *L1*} × {$\mathcal{L}$[*L2*,π] |π. π ∈ *L2*})› ‹{($\mathcal{L}$[*L1*,τ'], $\mathcal{L}$[*L2*,τ']) |τ'. τ' ∈ *set* (*prefixes* τ) ∧ *length* τ' ≤ *m* ∗ *n* − *1*} ⊆ {$\mathcal{L}$[*L1*,π] |π. π ∈ *L1*} × {$\mathcal{L}$[*L2*,π] |π. π ∈ *L2*}› *card-seteq*)

have ($\mathcal{L}$[*L1*,τ], $\mathcal{L}$[*L2*,τ]) ∈ *?L12*
using ‹τ ∈ *L1* ∩ *L2*›
by *blast*
moreover have ($\mathcal{L}$[*L1*,τ], $\mathcal{L}$[*L2*,τ]) ∉ *?visited-states*
proof
assume ($\mathcal{L}$[*L1*,τ], $\mathcal{L}$[*L2*,τ]) ∈ *?visited-states*
then obtain τ' **where** ($\mathcal{L}$[*L1*,τ], $\mathcal{L}$[*L2*,τ]) = ($\mathcal{L}$[*L1*,τ'],$\mathcal{L}$[*L2*,τ'])
and τ' ∈ *set* (*prefixes* τ)
and *length* τ' ≤ *m* ∗ *n* − *1*
by *blast*

then have $\tau \neq \tau'$
using ‹*length* τ > *m∗n−1*› **by** *auto*

show *False*
using ‹($\mathcal{L}$[*L1*,τ], $\mathcal{L}$[*L2*,τ]) = ($\mathcal{L}$[*L1*,τ'],$\mathcal{L}$[*L2*,τ'])›
using *no-prefix-loop*[*OF* - ‹τ' ∈ *set* (*prefixes* τ)› ‹$\tau \neq \tau'$›]
by *simp*
qed
ultimately show *False*
unfolding ‹*?visited-states = ?L12*›
by *blast*
qed

6 Reductions Between Relations

6.1 Quasi-Equivalence via Quasi-Reduction and Absences

fun *absence-completion* :: *'x alphabet* ⇒ *'y alphabet* ⇒ (*'x*,*'y*) *language* ⇒ (*'x*, *'y* × *bool*) *language* **where**
absence-completion X Y L =
((λ π . *map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) π) ‘ *L*)
∪ {(*map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) π)@[(*x*,(*y*,*False*))]@τ | π *x y* τ . π ∈ *L* ∧

$out[L,\pi,x] \neq \{\} \wedge y \in Y \wedge y \notin out[L,\pi,x] \wedge (\forall\ (x,(y,a)) \in set\ \tau\ .\ x \in X \wedge y \in Y)\}$

lemma *absence-completion-is-language* :
assumes *is-language X Y L*
shows *is-language X* $(Y \times UNIV)$ *(absence-completion X Y L)*
proof −
let *?L* = *(absence-completion X Y L)*
have $[] \in ?L$
using *language-contains-nil*[*OF assms*] **by** *auto*

have $?L \neq \{\}$
using *language-contains-nil*[*OF assms*] **by** *auto*
moreover have $\bigwedge \gamma\ xy\ .\ \gamma \in ?L \Longrightarrow xy \in set\ \gamma \Longrightarrow fst\ xy \in X \wedge snd\ xy \in (Y \times UNIV)$
and $\bigwedge \gamma\ \gamma'\ .\ \gamma \in ?L \Longrightarrow prefix\ \gamma'\ \gamma \Longrightarrow \gamma' \in ?L$
proof −
fix $\gamma\ xy\ \gamma'$ **assume** $\gamma \in ?L$
then consider *(a)* $\gamma \in ((\lambda\ \pi\ .\ map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)\ `\ L)$ |
(b) $\gamma \in \{(map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)@[(x,(y,False))]@\tau \mid \pi\ x\ y\ \tau\ .\ \pi \in L \wedge out[L,\pi,x] \neq \{\} \wedge y \in Y \wedge y \notin out[L,\pi,x] \wedge (\forall\ (x,(y,a)) \in set\ \tau\ .\ x \in X \wedge y \in Y)\}$
unfolding *absence-completion.simps* **by** *blast*
then have $(xy \in set\ \gamma \longrightarrow fst\ xy \in X \wedge snd\ xy \in (Y \times UNIV)) \wedge (prefix\ \gamma'\ \gamma \longrightarrow \gamma' \in ?L)$
proof *cases*
case *a*
then obtain π **where** $*{:}\gamma = map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi$ **and** $\pi \in L$
by *auto*
then have *p1*: $\bigwedge xy\ .\ xy \in set\ \pi \Longrightarrow fst\ xy \in X \wedge snd\ xy \in Y$
and *p2*: $\bigwedge \pi'\ .\ prefix\ \pi'\ \pi \Longrightarrow \pi' \in L$
using *assms* **by** *auto*

have $xy \in set\ \gamma \Longrightarrow fst\ xy \in X \wedge snd\ xy \in (Y \times UNIV)$
proof −
assume $xy \in set\ \gamma$
then have $(fst\ xy, fst\ (snd\ xy)) \in set\ \pi$ **and** $snd\ (snd\ xy) = True$
unfolding $*$ **by** *auto*
then show $fst\ xy \in X \wedge snd\ xy \in (Y \times UNIV)$
by *(metis p1 SigmaI UNIV-I fst-conv prod.collapse snd-conv)*
qed
moreover have $prefix\ \gamma'\ \gamma \Longrightarrow \gamma' \in ?L$
proof −
assume $prefix\ \gamma'\ \gamma$
then obtain i **where** $\gamma' = take\ i\ \gamma$
by *(metis append-eq-conv-conj prefix-def)*
then have $\gamma' = map\ (\lambda(x,y)\ .\ (x,(y,True)))\ (take\ i\ \pi)$
unfolding $*$ **using** *take-map* **by** *blast*
moreover have $take\ i\ \pi \in L$

```
    using p2 ‹π ∈ L› take-is-prefix by blast
   ultimately have γ' ∈ ((λ π . map (λ(x,y) . (x,(y,True))) π) ‘ L)
    by simp
   then show γ' ∈ ?L
    by auto
  qed
  ultimately show ?thesis by blast
next
  case b
then obtain π x y τ where ∗: γ = (map (λ(x,y) . (x,(y,True))) π)@[(x,(y,False))]@τ
                      and π ∈ L
                      and out[L,π,x] ≠ {}
                      and y ∈ Y
                      and y ∉ out[L,π,x]
                      and (∀ (x,(y,a)) ∈ set τ . x ∈ X ∧ y ∈ Y)
   by blast
  then have p1: ⋀ xy . xy ∈ set π ⟹ fst xy ∈ X ∧ snd xy ∈ Y
       and p2: ⋀ π' . prefix π' π ⟹ π' ∈ L
   using assms by auto

  have x ∈ X
   using ‹out[L,π,x] ≠ {}› assms
   by (meson executable-inputs-in-alphabet outputs-executable)

  have xy ∈ set γ ⟹ fst xy ∈ X ∧ snd xy ∈ (Y × UNIV)
  proof −
   assume xy ∈ set γ
   then consider (b1) xy ∈ set (map (λ(x,y) . (x,(y,True))) π) |
                 (b2) xy = (x,(y,False)) |
                 (b3) xy ∈ set τ
    unfolding ∗ by force
   then show ?thesis proof cases
    case b1
    then have (fst xy, fst (snd xy)) ∈ set π and snd (snd xy) = True
     unfolding ∗ by auto
    then show fst xy ∈ X ∧ snd xy ∈ (Y × UNIV)
     by (metis p1 SigmaI UNIV-I fst-conv prod.collapse snd-conv)
   next
    case b2
    then show ?thesis
     using ‹x ∈ X› ‹y ∈ Y› by simp
   next
    case b3
    then show ?thesis
     using ‹(∀ (x,(y,a)) ∈ set τ . x ∈ X ∧ y ∈ Y)› by force
   qed
  qed
  moreover have prefix γ' γ ⟹ γ' ∈ ?L
  proof −
```

```
      assume prefix γ' γ
      then obtain i where γ' = take i γ
        by (metis append-eq-conv-conj prefix-def)
      then consider (b1) i ≤ length π |
                    (b2) i > length π
        by linarith
      then show γ' ∈ ?L proof cases
        case b1
        then have i ≤ length (map (λ(x, y). (x, y, True)) π)
          by auto
        then have γ' = map (λ(x,y) . (x,(y,True))) (take i π)
          unfolding ∗ ‹γ' = take i γ›
          by (simp add: take-map)
        moreover have take i π ∈ L
          using p2 ‹π ∈ L› take-is-prefix by blast
        ultimately have γ' ∈ ((λ π . map (λ(x,y) . (x,(y,True))) π) ‘ L)
          by simp
        then show γ' ∈ ?L
          by auto
      next
        case b2
        then have i > length (map (λ(x, y). (x, y, True)) π)
          by auto

       have ⋀ k xs ys . k > length xs ⟹ take k (xs@ys) = xs@(take (k − length
xs) ys)
         by simp
        have take-helper: ⋀ k xs y zs . k > length xs ⟹ take k (xs@[y]@zs) =
xs@[y]@(take (k − length xs − 1) zs)
          by (metis One-nat-def Suc-pred ‹⋀ys xs k. length xs < k ⟹ take k
(xs @ ys) = xs @ take (k − length xs) ys› append-Cons append-Nil take-Suc-Cons
zero-less-diff)

       have ∗∗: γ' = (map (λ(x,y) . (x,(y,True))) π)@[(x,(y,False))]@(take (i −
length π − 1) τ)
         unfolding ∗ ‹γ' = take i γ›
          using take-helper[OF ‹i > length (map (λ(x, y). (x, y, True)) π)›] by
simp

       have (∀ (x,(y,a)) ∈ set (take (i − length π − 1) τ) . x ∈ X ∧ y ∈ Y)
         using ‹(∀ (x,(y,a)) ∈ set τ . x ∈ X ∧ y ∈ Y)›
         by (meson in-set-takeD)
       then show ?thesis
         unfolding ∗∗ absence-completion.simps
         using ‹π ∈ L› ‹out[L,π,x] ≠ {}› ‹y ∈ Y› ‹y ∉ out[L,π,x]›
         by blast
     qed
   qed
   ultimately show ?thesis by simp
```

```
    qed
    then show xy ∈ set γ ⟹ fst xy ∈ X ∧ snd xy ∈ (Y × UNIV)
         and prefix γ' γ ⟹ γ' ∈ ?L
      by blast+
  qed
  ultimately show ?thesis
    unfolding is-language.simps by blast
qed

lemma absence-completion-inclusion-R :
  assumes is-language X Y L
  and     π ∈ absence-completion X Y L
shows (map (λ(x,y,a) . (x,y)) π ∈ L) ⟷ (∀ (x,y,a) ∈ set π . a = True)
proof −
  define L'a where L'a = ((λ π . map (λ(x,y) . (x,(y,True))) π) ‘ L)
  define L'b where L'b = {(map (λ(x,y) . (x,(y,True))) π)@[(x,(y,False))]@τ | π
x y τ . π ∈ L ∧ out[L,π,x] ≠ {} ∧ y ∈ Y ∧ y ∉ out[L,π,x] ∧ (∀ (x,(y,a)) ∈ set τ
. x ∈ X ∧ y ∈ Y)}


  have ⋀ π xya . π ∈ L'a ⟹ xya ∈ set π ⟹ snd (snd xya) = True
    unfolding L'a-def by auto
  moreover have ⋀ π . π ∈ L'b ⟹ ∃ xya ∈ set π . snd (snd xya) = False
    unfolding L'b-def by auto
  moreover have π ∈ L'a ∪ L'b
    using assms(2) unfolding absence-completion.simps L'a-def L'b-def .
  ultimately have (∀ (x,y,a) ∈ set π . a = True) = (π ∈ L'a)
    by fastforce

  show ?thesis proof (cases (∀ (x,y,a) ∈ set π . a = True))
    case True
    then obtain τ where π = map (λ(x, y). (x, y, True)) τ
                    and τ ∈ L
      unfolding ‹(∀ (x,y,a) ∈ set π . a = True) = (π ∈ L'a)› L'a-def
      by blast

    have map (λ(x, y, a). (x, y)) π = τ
      unfolding ‹π = map (λ(x, y). (x, y, True)) τ›
      by (induction τ; auto)
    show ?thesis
      using True ‹τ ∈ L›
      unfolding ‹(∀ (x,y,a) ∈ set π . a = True) = (π ∈ L'a)› L'a-def
      unfolding ‹map (λ(x, y, a). (x, y)) π = τ›
      unfolding ‹π = map (λ(x, y). (x, y, True)) τ›
      by blast
  next
    case False
    then have π ∈ L'b
      using ‹(∀ (x,y,a) ∈ set π . a = True) = (π ∈ L'a)› ‹π ∈ L'a ∪ L'b› by blast
```

then obtain $\tau\ x\ y\ \tau'$ **where** $\pi = (map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \tau)@[(x,(y,False))]@\tau'$
and $\tau \in L$
and $out[L,\tau,x] \neq \{\}$
and $y \in Y$
and $y \notin out[L,\tau,x]$
and $(\forall\ (x,(y,a)) \in set\ \tau'\ .\ x \in X \wedge y \in Y)$
unfolding *L'b-def* **by** *blast*
then have $\tau@[(x,y)] \notin L$
by *fastforce*
then have $\tau@[(x,y)]@(map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \tau') \notin L$
using *assms(1)*
by (*metis append.assoc prefix-closure-no-member*)
moreover have $map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi = \tau@[(x,y)]@(map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \tau')$
unfolding ‹$\pi = (map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \tau)@[(x,(y,False))]@\tau'$›
by (*induction* τ; *auto*)
ultimately have $map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi \notin L$
by *simp*
then show *?thesis*
using *False* **by** *blast*
qed
qed

lemma *absence-completion-inclusion-L* :
$(\pi \in L) \longleftrightarrow (map\ (\lambda(x,y)\ .\ (x,y,True))\ \pi \in absence\text{-}completion\ X\ Y\ L)$
proof −

let *?L* = *absence-completion X Y L*
define *L'a* **where** $L'a = ((\lambda\ \pi\ .\ map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)\ `\ L)$
define *L'b* **where** $L'b = \{(map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)@[(x,(y,False))]@\tau \mid \pi\ x\ y\ \tau\ .\ \pi \in L \wedge out[L,\pi,x] \neq \{\} \wedge y \in Y \wedge y \notin out[L,\pi,x] \wedge (\forall\ (x,(y,a)) \in set\ \tau\ .\ x \in X \wedge y \in Y)\}$
have *?L* = *L'a* ∪ *L'b*
unfolding *L'a-def L'b-def absence-completion.simps* **by** *blast*

have $\bigwedge\ \pi\ .\ \pi \in L'b \Longrightarrow \exists\ xya \in set\ \pi\ .\ snd\ (snd\ xya) = False$
unfolding *L'b-def* **by** *auto*
then have $(map\ (\lambda(x,y)\ .\ (x,y,True))\ \pi \in ?L) = (map\ (\lambda(x,y)\ .\ (x,y,True))\ \pi \in L'a)$
unfolding ‹*?L* = *L'a* ∪ *L'b*›
by *fastforce*

have *inj* $(\lambda\ \pi\ .\ map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)$
by (*simp add*: *inj-def*)
then show *?thesis*
unfolding ‹$(map\ (\lambda(x,y)\ .\ (x,y,True))\ \pi \in ?L) = (map\ (\lambda(x,y)\ .\ (x,y,True))\ \pi \in L'a)$›
unfolding *L'a-def*
by (*simp add*: *image-iff inj-def*)

qed

fun *is-present* :: ('*x*,'*y* × *bool*) *word* ⇒ ('*x*,'*y*) *language* ⇒ *bool* **where**
is-present π *L* = (π ∈ *map* (λ(*x*, *y*). (*x*, *y*, *True*)) ‘ *L*)

lemma *is-present-rev* :
assumes *is-present* π *L*
shows *map* (λ(*x*, *y*, *a*). (*x*, *y*)) π ∈ *L*
proof −
obtain π′ **where** π = *map* (λ(*x*, *y*). (*x*, *y*, *True*)) π′ **and** π′ ∈ *L*
using *assms* **by** *auto*
moreover have *map* (λ(*x*, *y*, *a*). (*x*, *y*)) (*map* (λ(*x*, *y*). (*x*, *y*, *True*)) π′) = π′
by (*induction* π′; *auto*)
ultimately show *?thesis*
by *force*
qed

lemma *absence-completion-out* :
assumes *is-language X Y L*
and *x* ∈ *X*
and π ∈ *absence-completion X Y L*
shows *is-present* π *L* ⟹ *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*] ≠ {} ⟹ *out*[*absence-completion X Y L*, π, *x*] = {(*y*,*True*) | *y* . *y* ∈ *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*]} ∪ {(*y*,*False*) | *y* . *y* ∈ *Y* ∧ *y* ∉ *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*]}
and *is-present* π *L* ⟹ *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*] = {} ⟹ *out*[*absence-completion X Y L*, π, *x*] = {}
and ¬ *is-present* π *L* ⟹ *out*[*absence-completion X Y L*, π, *x*] = *Y* × *UNIV*
proof −

let *?L* = *absence-completion X Y L*
define *L*′*a* **where** *L*′*a* = ((λ π . *map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) π) ‘ *L*)
define *L*′*b* **where** *L*′*b* = {(*map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) π)@[(*x*,(*y*,*False*))]@τ | π *x y* τ . π ∈ *L* ∧ *out*[*L*,π,*x*] ≠ {} ∧ *y* ∈ *Y* ∧ *y* ∉ *out*[*L*,π,*x*] ∧ (∀ (*x*,(*y*,*a*)) ∈ *set* τ . *x* ∈ *X* ∧ *y* ∈ *Y*)}
have *?L* = *L*′*a* ∪ *L*′*b*
unfolding *L*′*a-def L*′*b-def absence-completion.simps* **by** *blast*
then have *out*[*?L*, π, *x*] = {*y*. π @ [(*x*, *y*)] ∈ *L*′*a*} ∪ {*y*. π @ [(*x*, *y*)] ∈ *L*′*b*}
unfolding *outputs.simps language-for-state.simps* **by** *blast*

show *is-present* π *L* ⟹ *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*] ≠ {} ⟹ *out*[*?L*, π, *x*] = {(*y*,*True*) | *y* . *y* ∈ *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*]} ∪ {(*y*,*False*) | *y* . *y* ∈ *Y* ∧ *y* ∉ *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*]}
proof −
assume *is-present* π *L* **and** *out*[*L*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*] ≠ {}
then have *map* (λ(*x*, *y*, *a*). (*x*, *y*)) π ∈ *L*
using *assms*(*1*) **by** *auto*

have $\{y.\ \pi\ @\ [(x,\ y)] \in L'a\} = \{(y,True) \mid y\ .\ y \in out[L,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi,x]\}$
proof
show $\{y.\ \pi\ @\ [(x,\ y)] \in L'a\} \subseteq \{(y,\ True)\ |y.\ y \in out[L,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi,x]\}$
proof
fix *ya* **assume** $ya \in \{y.\ \pi\ @\ [(x,\ y)] \in L'a\}$
then have $\pi\ @\ [(x,\ ya)] \in map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ `\ L$
unfolding *L'a-def* **by** *blast*
then obtain γ **where** $\gamma \in L$ **and** $\pi\ @\ [(x,\ ya)] = map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ \gamma$
by *blast*
then have $length\ (\pi\ @\ [(x,\ ya)]) = length\ \gamma$
by *auto*
then obtain $\gamma'\ xy$ **where** $\gamma = \gamma'@[xy]$
by (*metis add.right-neutral dual-order.strict-iff-not length-append-singleton less-add-Suc2 rev-exhaust take0 take-all-iff*)
then have $(x,ya) = (\lambda(x,\ y).\ (x,\ y,\ True))\ xy$
using $\langle\pi\ @\ [(x,\ ya)] = map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ \gamma\rangle$ **unfolding** $\langle\gamma = \gamma'@[xy]\rangle$ **by** *auto*
then have $ya = (snd\ xy,\ True)$ **and** $xy = (x,snd\ xy)$
by (*simp add*: *split-beta*)+
moreover define y **where** $y = snd\ xy$
ultimately have $ya = (y,\ True)$ **and** $xy = (x,y)$
by *auto*

have $\pi = map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ \gamma'$
using $\langle\pi\ @\ [(x,\ ya)] = map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ \gamma\rangle$ **unfolding** $\langle\gamma = \gamma'@[xy]\rangle$ **by** *auto*
then have $map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi = \gamma'$
by (*induction* π *arbitrary*: γ'; *auto*)

have $[(x,\ y)] \in \{\tau.\ map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi\ @\ \tau \in L\}$
using $\langle\gamma \in L\rangle$
unfolding $\langle\gamma = \gamma'@[xy]\rangle$ $\langle ya = (y,\ True)\rangle$ $\langle xy = (x,y)\rangle$
unfolding $\langle map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi = \gamma'\rangle$
by *auto*
then show $ya \in \{(y,\ True)\ |y.\ y \in out[L,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi,x]\}$
unfolding $\langle ya = (snd\ xy,\ True)\rangle$ *outputs.simps language-for-state.simps*
unfolding $\langle ya = (y,\ True)\rangle$ $\langle xy = (x,y)\rangle$ $\langle map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi = \gamma'\rangle$
by *auto*
qed
show $\{(y,\ True)\ |y.\ y \in out[L,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi,x]\} \subseteq \{y.\ \pi\ @\ [(x,\ y)] \in L'a\}$
proof
fix *ya* **assume** $ya \in \{(y,\ True)\ |y.\ y \in out[L,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi,x]\}$
then obtain y **where** $ya = (y,True)$ **and** $y \in out[L,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ \pi,x]$

by *blast*
then have $[(x, y)] \in \{\tau.\ map\ (\lambda(x, y, a).\ (x, y))\ \pi\ @\ \tau \in L\}$
unfolding *outputs.simps language-for-state.simps* **by** *auto*
then have $(map\ (\lambda(x, y, a).\ (x, y))\ \pi)\ @\ [(x,y)] \in L$
by *auto*
moreover have $map\ (\lambda(x, y).\ (x, y, True))\ ((map\ (\lambda(x, y, a).\ (x, y))\ \pi)\ @\ [(x,y)]) = \pi\ @\ [(x, (y, True))]$
using ‹*is-present* π L› **unfolding** *is-present.simps*
by (*induction* π *arbitrary*: x y; *auto*)
ultimately have $\pi\ @\ [(x, (y, True))] \in L'a$
unfolding $L'a$*-def*
by *force*
then show $ya \in \{y.\ \pi\ @\ [(x, y)] \in L'a\}$
unfolding ‹$ya = (y,True)$›
by *blast*
qed
qed
moreover have $\{y.\ \pi\ @\ [(x, y)] \in L'b\} = \{(y,False)\ |\ y\ .\ y \in Y \wedge y \notin out[L,map\ (\lambda(x, y, a).\ (x, y))\ \pi,x]\}$
proof
show $\{y.\ \pi\ @\ [(x, y)] \in L'b\} \subseteq \{(y, False)\ |y.\ y \in Y \wedge y \notin out[L,map\ (\lambda(x, y, a).\ (x, y))\ \pi,x]\}$
proof
fix ya **assume** $ya \in \{y.\ \pi\ @\ [(x, y)] \in L'b\}$
then have $\pi\ @\ [(x,ya)] \in L'b$
by *auto*
then obtain $\pi'\ x'\ y'\ \tau$ **where** $\pi\ @\ [(x,ya)]\ = map\ (\lambda(x, y).\ (x, y, True))\ \pi'\ @\ [(x', y', False)]\ @\ \tau$
and $\pi' \in L$
and $out[L,\pi',x'] \neq \{\}$
and $y' \in Y$
and $y' \notin out[L,\pi',x']$
and $(\forall\,(x, y, a) \in set\ \tau.\ x \in X \wedge y \in Y)$
unfolding $L'b$*-def* **by** *blast*

obtain π'' **where** $\pi = map\ (\lambda(x, y).\ (x, y, True))\ \pi''$ **and** $\pi'' \in L$
using ‹*is-present* π L› **by** *auto*
then have $\bigwedge xya\ .\ xya \in set\ \pi \Longrightarrow snd\ (snd\ xya) = True$
by (*induction* π; *auto*)

have $\tau = []$
proof (*rule ccontr*)
assume $\tau \neq []$
then obtain $\tau'\ xyz$ **where** $\tau = \tau'@[xyz]$
by (*metis append-butlast-last-id*)
then have $\pi = map\ (\lambda(x, y).\ (x, y, True))\ \pi'\ @\ [(x', y', False)]\ @\ \tau'$
using ‹$\pi\ @\ [(x,ya)]\ = map\ (\lambda(x, y).\ (x, y, True))\ \pi'\ @\ [(x', y', False)]\ @\ \tau$›
by *auto*

```
      then have (x', y', False) ∈ set π
        by auto
      then show False
        using ‹⋀ xya . xya ∈ set π ⟹ snd (snd xya) = True› by force
    qed
   then have x' = x and ya = (y', False) and π = map (λ(x, y). (x, y, True)) π'
      using ‹π @ [(x,ya)]  = map (λ(x, y). (x, y, True)) π' @ [(x', y', False)] @ τ›
      by auto

    have ∗: map (λ(x, y, a). (x, y)) (map (λ(x, y). (x, y, True)) π') = π'
      by (induction π'; auto)

    have y' ∉ out[L,map (λ(x, y, a). (x, y)) π,x]
      using ‹y' ∉ out[L,π',x']›
      unfolding outputs.simps language-for-state.simps
      unfolding ‹π = map (λ(x, y). (x, y, True)) π'› ‹x' = x›
      unfolding ∗ .
    then show ya ∈ {(y, False) |y. y ∈ Y ∧ y ∉ out[L,map (λ(x, y, a). (x, y)) π,x]}
      using ‹y' ∈ Y›
      unfolding ‹ya = (y', False)› by auto
  qed

  show {(y, False) |y. y ∈ Y ∧ y ∉ out[L,map (λ(x, y, a). (x, y)) π,x]} ⊆ {y. π @ [(x, y)] ∈ L'b}
  proof
    fix ya assume ya ∈ {(y, False) |y. y ∈ Y ∧ y ∉ out[L,map (λ(x, y, a). (x, y)) π,x]}
    then obtain y where ya = (y,False)
                    and y ∈ Y
                    and y ∉ out[L,map (λ(x, y, a). (x, y)) π,x]
      by blast

    obtain π' where π = map (λ(x, y). (x, y, True)) π' and π' ∈ L
      using ‹is-present π L› by auto
    have ∗: map (λ(x, y, a). (x, y)) (map (λ(x, y). (x, y, True)) π') = π'
      by (induction π'; auto)
    have out[L,π',x] ≠ {}
      using ‹out[L,map (λ(x, y, a). (x, y)) π,x] ≠ {}›
      unfolding ‹π = map (λ(x, y). (x, y, True)) π'› ∗ .
    have y ∉ out[L,π',x]
      using ‹y ∉ out[L,map (λ(x, y, a). (x, y)) π,x]›
      unfolding ‹π = map (λ(x, y). (x, y, True)) π'› ∗ .

    have π@[(x,ya)] = map (λ(x, y). (x, y, True)) π' @ [(x, y, False)]
      unfolding ‹ya = (y,False)› ‹π = map (λ(x, y). (x, y, True)) π'›
      by auto
```

```
      then show ya ∈ {y. π @ [(x, y)] ∈ L'b}
        unfolding L'b-def
        using ‹π' ∈ L› ‹out[L,π',x] ≠ {}› ‹y ∈ Y› ‹y ∉ out[L,π',x]›
        by force
    qed
  qed
  ultimately show ?thesis
   unfolding ‹out[?L, π, x] = {y. π @ [(x, y)] ∈ L'a} ∪ {y. π @ [(x, y)] ∈ L'b}›
   by blast
 qed


 show is-present π L ⟹ out[L,map (λ(x, y, a). (x, y)) π,x] = {} ⟹ out[absence-completion
X Y L, π, x] = {}
 proof −
  assume is-present π L and out[L,map (λ(x, y, a). (x, y)) π,x] = {}

  obtain π' where π = map (λ(x, y). (x, y, True)) π' and π' ∈ L
   using ‹is-present π L› by auto
  have ∗: map (λ(x, y, a). (x, y)) (map (λ(x, y). (x, y, True)) π') = π'
   by (induction π'; auto)
  then have map (λ(x, y, a). (x, y)) π = π'
   using ‹π = map (λ(x, y). (x, y, True)) π'› by blast

  have {y. π @ [(x, y)] ∈ L'a} = {}
  proof −
   have ∄ y . π @ [(x, y)] ∈ L'a
   proof
    assume ∃ y. π @ [(x, y)] ∈ L'a
    then obtain ya where π @ [(x, ya)] ∈ L'a
     by blast
    then obtain π'' where π'' ∈ L and map (λ(x, y). (x, y, True)) π'' = π @
[(x, ya)]
     unfolding L'a-def by force
    then have (x,ya) = (λ(x, y). (x, y, True)) (last π'')
        by (metis (mono-tags, lifting) append-is-Nil-conv last-map last-snoc
list.map-disc-iff not-Cons-self2)
    then obtain y where ya = (y,True)
     by (simp add: split-beta)

    have map (λ(x, y). (x, y, True)) π'' = map (λ(x, y). (x, y, True)) (π' @
[(x, y)])
     using ‹map (λ(x, y). (x, y, True)) π'' = π @ [(x, ya)]›
     unfolding ‹π = map (λ(x, y). (x, y, True)) π'› ‹ya = (y,True)› by auto
    moreover have inj (λ(x, y). (x, y, True))
     by (simp add: inj-def)
    ultimately have π'' = π' @ [(x,y)]
     using inj-map-eq-map by blast
```

```
      show False
        using ‹π'' ∈ L› ‹out[L,map (λ(x, y, a). (x, y)) π,x] = {}›
        unfolding ‹map (λ(x, y, a). (x, y)) π = π'› ‹π'' = π' @ [(x,y)]›
        by simp
    qed
    then show ?thesis
      by blast
  qed
  moreover have {y. π @ [(x, y)] ∈ L'b} = {}
  proof −
    have ∄ y . π @ [(x, y)] ∈ L'b
    proof
      assume ∃ y. π @ [(x, y)] ∈ L'b
      then obtain ya where π @ [(x, ya)] ∈ L'b
        by blast
      then obtain π'' x' y' τ where π @ [(x,ya)] = map (λ(x, y). (x, y, True))
π'' @ [(x', y', False)] @ τ
                              and π'' ∈ L
                              and out[L,π'',x'] ≠ {}
                              and y' ∈ Y
                              and y' ∉ out[L,π'',x']
                              and (∀ (x, y, a)∈set τ. x ∈ X ∧ y ∈ Y)
        unfolding L'b-def by blast

      have ⋀ xya . xya ∈ set π ⟹ snd (snd xya) = True
        using ‹π = map (λ(x, y). (x, y, True)) π'›
        by (induction π; auto)

      have τ = []
      proof (rule ccontr)
        assume τ ≠ []
        then obtain τ' xyz where τ = τ'@[xyz]
          by (metis append-butlast-last-id)
        then have π = map (λ(x, y). (x, y, True)) π'' @ [(x', y', False)] @ τ'
          using ‹π @ [(x,ya)] = map (λ(x, y). (x, y, True)) π'' @ [(x', y', False)]
@ τ›
          by auto
        then have (x', y', False) ∈ set π
          by auto
        then show False
          using ‹⋀ xya . xya ∈ set π ⟹ snd (snd xya) = True› by force
      qed
     then have x' = x and ya = (y', False) and π = map (λ(x, y). (x, y, True))
π''
        using ‹π @ [(x,ya)] = map (λ(x, y). (x, y, True)) π'' @ [(x', y', False)]
@ τ›
        by auto
      moreover have inj (λ(x, y). (x, y, True))
        by (simp add: inj-def)
```

```
      ultimately have π'' = π'
        unfolding ‹π = map (λ(x, y). (x, y, True)) π'›
        using map-injective by blast
      then show False
        using ‹out[L,π'',x'] ≠ {}› ‹out[L,map (λ(x, y, a). (x, y)) π,x] = {}›
        unfolding ‹map (λ(x, y, a). (x, y)) π = π'› ‹x' = x›
        by blast
    qed
    then show ?thesis
      by blast
  qed
  ultimately show ?thesis
   unfolding ‹out[?L, π, x] = {y. π @ [(x, y)] ∈ L'a} ∪ {y. π @ [(x, y)] ∈ L'b}›
    by blast
qed

show ¬ is-present π L ⟹ out[absence-completion X Y L,π,x] = Y × UNIV
proof

  show out[absence-completion X Y L,π,x] ⊆ Y × UNIV
    using absence-completion-is-language[OF assms(1)]
    by (meson outputs-in-alphabet)



  assume ¬ is-present π L
  then have π ∉ L'a
    unfolding L'a-def by auto
  then have π ∈ L'b
    using ‹π ∈ ?L› ‹?L = L'a ∪ L'b› by blast
  then obtain π' x' y' τ where π = map (λ(x, y). (x, y, True)) π' @ [(x', y',
False)] @ τ
                          and π' ∈ L
                          and out[L,π',x'] ≠ {}
                          and y' ∈ Y
                          and y' ∉ out[L,π',x']
                          and (∀(x, y, a)∈set τ. x ∈ X ∧ y ∈ Y)
    unfolding L'b-def by blast

  show Y × UNIV ⊆ out[absence-completion X Y L,π,x]
  proof
    fix ya assume ya ∈ Y × (UNIV :: bool set)

    have π@[(x,ya)] = map (λ(x, y). (x, y, True)) π' @ [(x', y', False)] @ (τ @
[(x,ya)])
      using ‹π = map (λ(x, y). (x, y, True)) π' @ [(x', y', False)] @ τ›
      by auto
    moreover have ‹(∀(x, y, a)∈set (τ @ [(x,ya)]) . x ∈ X ∧ y ∈ Y)›
      using ‹(∀(x, y, a)∈set τ. x ∈ X ∧ y ∈ Y)› ‹x ∈ X› ‹ya ∈ Y × (UNIV ::
```

bool set)›
by *auto*
ultimately have $\pi@[(x,ya)] \in L'b$
unfolding *L'b-def*
using ‹$\pi' \in L$› ‹$out[L,\pi',x'] \neq \{\}$› ‹$y' \in Y$› ‹$y' \notin out[L,\pi',x']$›
by *blast*
then show $ya \in out[?L,\pi,x]$
unfolding ‹$out[?L, \pi, x] = \{y.\ \pi\ @\ [(x,\ y)] \in L'a\} \cup \{y.\ \pi\ @\ [(x,\ y)] \in L'b\}$›
by *blast*
qed
qed
qed

theorem *quasieq-via-quasired* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows ($L1 \preceq[X,quasieq\ Y]\ L2$) $\longleftrightarrow$ ((*absence-completion X Y L1*) $\preceq[X,\ quasired\ (Y \times UNIV)]$ (*absence-completion X Y L2*))
proof

define *L1'* **where** $L1' = absence\text{-}completion\ X\ Y\ L1$
define *L2'* **where** $L2' = absence\text{-}completion\ X\ Y\ L2$

define *L1'a* **where** $L1'a = ((\lambda\ \pi\ .\ map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)\ `\ L1)$
define *L1'b* **where** $L1'b = \{(map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)@[(x,(y,False))]@\tau\ |\ \pi\ x\ y\ \tau\ .\ \pi \in L1 \wedge out[L1,\pi,x] \neq \{\} \wedge y \in Y \wedge y \notin out[L1,\pi,x] \wedge (\forall\ (x,(y,a)) \in set\ \tau\ .\ x \in X \wedge y \in Y)\}$
define *L2'a* **where** $L2'a = ((\lambda\ \pi\ .\ map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)\ `\ L2)$
define *L2'b* **where** $L2'b = \{(map\ (\lambda(x,y)\ .\ (x,(y,True)))\ \pi)@[(x,(y,False))]@\tau\ |\ \pi\ x\ y\ \tau\ .\ \pi \in L2 \wedge out[L2,\pi,x] \neq \{\} \wedge y \in Y \wedge y \notin out[L2,\pi,x] \wedge (\forall\ (x,(y,a)) \in set\ \tau\ .\ x \in X \wedge y \in Y)\}$

have $\bigwedge\ \pi\ xya\ .\ \pi \in L1'a \Longrightarrow xya \in set\ \pi \Longrightarrow snd\ (snd\ xya) = True$
unfolding *L1'a-def* **by** *auto*
moreover have $\bigwedge\ \pi\ xya\ .\ \pi \in L2'a \Longrightarrow xya \in set\ \pi \Longrightarrow snd\ (snd\ xya) = True$
unfolding *L2'a-def* **by** *auto*
moreover have $\bigwedge\ \pi\ .\ \pi \in L1'b \Longrightarrow \exists\ xya \in set\ \pi\ .\ snd\ (snd\ xya) = False$
unfolding *L1'b-def* **by** *auto*
moreover have $\bigwedge\ \pi\ .\ \pi \in L2'b \Longrightarrow \exists\ xya \in set\ \pi\ .\ snd\ (snd\ xya) = False$
unfolding *L2'b-def* **by** *auto*
ultimately have $L1'a \cap L2'b = \{\}$ **and** $L1'b \cap L2'a = \{\}$
by *blast+*
moreover have $L1' = L1'a \cup L1'b$
unfolding *L1'-def L1'a-def L1'b-def* **by** *auto*

```
  moreover have L2' = L2'a ∪ L2'b
    unfolding L2'-def L2'a-def L2'b-def by auto
  ultimately have L1' ∩ L2' = (L1'a ∩ L2'a) ∪ (L1'b ∩ L2'b)
    by blast

  have inj (λ π . map (λ(x,y) . (x,(y,True))) π)
    by (simp add: inj-def)
  then have L1'a ∩ L2'a = ((λ π . map (λ(x,y) . (x,(y,True))) π) ` (L1 ∩ L2))
    unfolding L1'a-def L2'a-def
    using image-Int by blast

 have intersection-b: L1'b ∩ L2'b = {(map (λ(x,y) . (x,(y,True))) π)@[(x,(y,False))]@τ
| π x y τ . π ∈ L1 ∩ L2 ∧ out[L1,π,x] ≠ {} ∧ out[L2,π,x] ≠ {} ∧ y ∈ Y ∧ y ∉
out[L1,π,x] ∧ y ∉ out[L2,π,x] ∧ (∀ (x,(y,a)) ∈ set τ . x ∈ X ∧ y ∈ Y)}
    (is L1'b ∩ L2'b = ?L12'b)
  proof
    show ?L12'b ⊆ L1'b ∩ L2'b
      unfolding L1'b-def L2'b-def by blast
    show L1'b ∩ L2'b ⊆ ?L12'b
    proof
      fix γ assume γ ∈ L1'b ∩ L2'b

    obtain π1 x1 y1 τ1 where γ = (map (λ(x,y) . (x,(y,True))) π1)@[(x1,(y1,False))]@τ1

                         and π1 ∈ L1
                         and out[L1,π1,x1] ≠ {}
                         and y1 ∈ Y
                         and y1 ∉ out[L1,π1,x1]
                         and (∀ (x,(y,a)) ∈ set τ1 . x ∈ X ∧ y ∈ Y)
      using ‹γ ∈ L1'b ∩ L2'b› unfolding L1'b-def by blast

    obtain π2 x2 y2 τ2 where γ = (map (λ(x,y) . (x,(y,True))) π2)@[(x2,(y2,False))]@τ2

                         and π2 ∈ L2
                         and out[L2,π2,x2] ≠ {}
                         and y2 ∈ Y
                         and y2 ∉ out[L2,π2,x2]
                         and (∀ (x,(y,a)) ∈ set τ2 . x ∈ X ∧ y ∈ Y)
      using ‹γ ∈ L1'b ∩ L2'b› unfolding L2'b-def by blast

      have ⋀ i . i < length π1 ⟹ snd (snd (γ ! i)) = True
      proof −
        fix i assume i < length π1
        then have i < length (map (λ(x,y) . (x,(y,True))) π1) by auto
        then have γ ! i = (map (λ(x,y) . (x,(y,True))) π1) ! i
          unfolding ‹γ = (map (λ(x,y) . (x,(y,True))) π1)@[(x1,(y1,False))]@τ1›
          by (simp add: nth-append)
        also have ... = (λ(x,y) . (x,(y,True))) (π1 ! i)
          using ‹i < length π1› nth-map by blast
```

finally show *snd* (*snd* (*γ* ! *i*)) = *True*
by (*metis* (*no-types*, *lifting*) *case-prod-conv old.prod.exhaust snd-conv*)
qed
have *γ* ! *length π1* = (*x1*,(*y1*,*False*))
unfolding ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π1*)@[(*x1*,(*y1*,*False*))]@*τ1*›
by (*metis append-Cons length-map nth-append-length*)
have ⋀ *i* . *i* < *length π2* ⟹ *snd* (*snd* (*γ* ! *i*)) = *True*
proof −
fix *i* **assume** *i* < *length π2*
then have *i* < *length* (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π2*) **by** *auto*
then have *γ* ! *i* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π2*) ! *i*
unfolding ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π2*)@[(*x2*,(*y2*,*False*))]@*τ2*›
by (*simp add*: *nth-append*)
also have ... = (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) (*π2* ! *i*)
using ‹*i* < *length π2*› *nth-map* **by** *blast*
finally show *snd* (*snd* (*γ* ! *i*)) = *True*
by (*metis* (*no-types*, *lifting*) *case-prod-conv old.prod.exhaust snd-conv*)
qed
have *γ* ! *length π2* = (*x2*,(*y2*,*False*))
unfolding ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π2*)@[(*x2*,(*y2*,*False*))]@*τ2*›
by (*metis append-Cons length-map nth-append-length*)

have *length π1* = *length π2*
by (*metis* ‹⋀*i*. *i* < *length π1* ⟹ *snd* (*snd* (*γ* ! *i*)) = *True*› ‹⋀*i*. *i* < *length π2* ⟹ *snd* (*snd* (*γ* ! *i*)) = *True*› ‹*γ* ! *length π1* = (*x1*, *y1*, *False*)› ‹*γ* ! *length π2* = (*x2*, *y2*, *False*)› *not-less-iff-gr-or-eq snd-conv*)
then have *π1* = *π2*
using ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π1*)@[(*x1*,(*y1*,*False*))]@*τ1*› ‹*inj* (*λ π* . *map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π*)›
unfolding ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π2*)@[(*x2*,(*y2*,*False*))]@*τ2*›
using *map-injective* **by** *fastforce*
then have [(*x1*,(*y1*,*False*))]@*τ1* = [(*x2*,(*y2*,*False*))]@*τ2*
using ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π1*)@[(*x1*,(*y1*,*False*))]@*τ1*›
unfolding ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π2*)@[(*x2*,(*y2*,*False*))]@*τ2*›
by *force*
then have *x1* = *x2* **and** *y1* = *y2* **and** *τ1* = *τ2*
by *auto*

show *γ* ∈ *?L12'b*
using ‹*π1* ∈ *L1*› ‹*out*[*L1*,*π1*,*x1*] ≠ {}› ‹*y1* ∈ *Y*› ‹*y1* ∉ *out*[*L1*,*π1*,*x1*]› ‹(∀ (*x*,(*y*,*a*)) ∈ *set τ1* . *x* ∈ *X* ∧ *y* ∈ *Y*)›
using ‹*π2* ∈ *L2*› ‹*out*[*L2*,*π2*,*x2*] ≠ {}› ‹*y2* ∈ *Y*› ‹*y2* ∉ *out*[*L2*,*π2*,*x2*]› ‹(∀ (*x*,(*y*,*a*)) ∈ *set τ2* . *x* ∈ *X* ∧ *y* ∈ *Y*)›
unfolding ‹*π1* = *π2*› ‹*x1* = *x2*› ‹*y1* = *y2*› ‹*τ1* = *τ2*› ‹*γ* = (*map* (*λ*(*x*,*y*) . (*x*,(*y*,*True*))) *π2*)@[(*x2*,(*y2*,*False*))]@*τ2*›
by *blast*
qed
qed

have *is-language X* (*Y* × *UNIV*) *L1′*
using *absence-completion-is-language*[*OF assms*(*1*)] **unfolding** *L1′-def* **.**
have *is-language X* (*Y* × *UNIV*) *L2′*
using *absence-completion-is-language*[*OF assms*(*2*)] **unfolding** *L2′-def* **.**

have (*L1* ⪯[*X*,*quasieq Y*] *L2*) = *quasi-equivalence L1 L2*
using *quasieq-type-1*[*OF assms*] **by** *blast*

have (*L1′* ⪯[*X*,*quasired* (*Y* × *UNIV*)] *L2′*) = *quasi-reduction L1′ L2′*
using *quasired-type-1*[*OF* ‹*is-language X* (*Y* × *UNIV*) *L1′*› ‹*is-language X* (*Y* × *UNIV*) *L2′*›] **by** *blast*

have ⋀ π *x* . *quasi-equivalence L1 L2* ⟹ π ∈ *L1′* ∩ *L2′* ⟹ *x* ∈ *exec*[*L2′*,π] ⟹ (*out*[*L1′*,π,*x*] ≠ {} ∧ *out*[*L1′*,π,*x*] ⊆ *out*[*L2′*,π,*x*])
proof −
fix π *x* **assume** *quasi-equivalence L1 L2* **and** π ∈ *L1′* ∩ *L2′* **and** *x* ∈ *exec*[*L2′*,π]

have *x* ∈ *X*
using ‹*x* ∈ *exec*[*L2′*,π]› *absence-completion-is-language*[*OF assms*(*2*)]
by (*metis L2′-def executable-inputs-in-alphabet*)
have π ∈ *absence-completion X Y L1* **and** π ∈ *absence-completion X Y L2*
using ‹π ∈ *L1′* ∩ *L2′*› **unfolding** *L1′-def L2′-def* **by** *blast*+

consider (*a*) π ∈ *L1′a* ∩ *L2′a* | (*b*) π ∈ (*L1′b* ∩ *L2′b*) − (*L1′a* ∩ *L2′a*)
using ‹π ∈ *L1′* ∩ *L2′*› ‹*L1′* ∩ *L2′* = (*L1′a* ∩ *L2′a*) ∪ (*L1′b* ∩ *L2′b*)› **by** *blast*
then show (*out*[*L1′*,π,*x*] ≠ {} ∧ *out*[*L1′*,π,*x*] ⊆ *out*[*L2′*,π,*x*])
proof *cases*
case *a*
then obtain τ **where** τ ∈ *L1* ∩ *L2*
and π = *map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) τ
using ‹*L1′a* ∩ *L2′a* = ((λ π . *map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) π) ‘ (*L1* ∩ *L2*))› **by** *blast*

have *map* (λ(*x*, *y*, *a*). (*x*, *y*)) π = τ
unfolding ‹π = *map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) τ› **by** (*induction* τ; *auto*)

have *is-present* π *L1* **and** *is-present* π *L2*
using ‹τ ∈ *L1* ∩ *L2*› **unfolding** ‹π = *map* (λ(*x*,*y*) . (*x*,(*y*,*True*))) τ› **by** *auto*

have *out*[*L2*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π,*x*] ≠ {}
using ‹*x* ∈ *exec*[*L2′*,π]›
using *absence-completion-out*(*2*)[*OF assms*(*2*) ‹*x* ∈ *X*› ‹π ∈ *absence-completion X Y L2*› ‹*is-present* π *L2*›]
unfolding *L2′-def*[*symmetric*]
by (*meson outputs-executable*)
then have *x* ∈ *exec*[*L2*,*map* (λ(*x*, *y*, *a*). (*x*, *y*)) π]
by *auto*

then have *out[L1,map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π*,*x*] ≠ {} **and** *out[L1,map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π*,*x*] = *out[L2,map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π*,*x*]
using ‹*quasi-equivalence L1 L2*› ‹*τ* ∈ *L1* ∩ *L2*›
unfolding *quasi-equivalence-def* ‹*map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π* = *τ*› **by** *force+*

have *out[L1′,π,x]* = *out[L2′,π,x]*
unfolding *L1′-def L2′-def*
unfolding *absence-completion-out(1)*[*OF assms(2)* ‹*x* ∈ *X*› ‹*π* ∈ *absence-completion X Y L2*› ‹*is-present π L2*› ‹*out[L2,map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π*,*x*] ≠ {}›]
unfolding *absence-completion-out(1)*[*OF assms(1)* ‹*x* ∈ *X*› ‹*π* ∈ *absence-completion X Y L1*› ‹*is-present π L1*› ‹*out[L1,map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π*,*x*] ≠ {}›]
using ‹*quasi-equivalence L1 L2*› ‹*τ* ∈ *L1* ∩ *L2*› ‹*x* ∈ *exec[L2,map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π*]›
unfolding *quasi-equivalence-def*
unfolding ‹*map* (λ(*x*, *y*, *a*). (*x*, *y*)) *π* = *τ*›
by *blast*
then show *?thesis*
by (*metis* ‹*x* ∈ *exec[L2′,π]*› *dual-order.refl outputs-executable*)
next
case *b*

then obtain *π′ x′ y′ τ′* **where** *π* = *map* (λ(*x*, *y*). (*x*, *y*, *True*)) *π′* @ [(*x′*, *y′*, *False*)] @ *τ′*
and *π′* ∈ *L1* ∩ *L2*
and *out[L1,π′,x′]* ≠ {}
and *out[L2,π′,x′]* ≠ {}
and *y′* ∈ *Y*
and *y′* ∉ *out[L1,π′,x′]*
and *y′* ∉ *out[L2,π′,x′]*
and (∀(*x*, *y*, *a*)∈*set τ′*. *x* ∈ *X* ∧ *y* ∈ *Y*)
unfolding *intersection-b*
by *blast*

have ¬ *is-present π L1*
using ‹*L1′a* ≡ *map* (λ(*x*, *y*). (*x*, *y*, *True*)) ‘ *L1*› ‹*L1′a* ∩ *L2′b* = {}› *b* **by** *auto*

have ¬ *is-present π L2*
using ‹*L2′a* ≡ *map* (λ(*x*, *y*). (*x*, *y*, *True*)) ‘ *L2*› ‹*L1′b* ∩ *L2′a* = {}› *b* **by** *auto*

show *?thesis*
unfolding *L1′-def L2′-def*
unfolding *absence-completion-out(3)*[*OF assms(1)* ‹*x* ∈ *X*› ‹*π* ∈ *absence-completion X Y L1*› ‹¬ *is-present π L1*›]
unfolding *absence-completion-out(3)*[*OF assms(2)* ‹*x* ∈ *X*› ‹*π* ∈ *absence-completion X Y L2*› ‹¬ *is-present π L2*›]

using ‹$y' \in Y$›
by *blast*
qed
qed
then show $L1 \preceq[X,quasieq\ Y]\ L2 \Longrightarrow (absence\text{-}completion\ X\ Y\ L1) \preceq[X,quasired\ (Y \times UNIV)]\ (absence\text{-}completion\ X\ Y\ L2)$
unfolding $L1'$-*def*[*symmetric*] $L2'$-*def*[*symmetric*]
unfolding ‹$(L1' \preceq[X,quasired\ (Y \times UNIV)]\ L2') = quasi\text{-}reduction\ L1'\ L2'$›
unfolding ‹$(L1 \preceq[X,quasieq\ Y]\ L2) = quasi\text{-}equivalence\ L1\ L2$›
unfolding *quasi-reduction-def*
by *blast*

have $\bigwedge \pi\ x\ .\ quasi\text{-}reduction\ L1'\ L2' \Longrightarrow \pi \in L1 \cap L2 \Longrightarrow x \in exec[L2,\pi] \Longrightarrow out[L1,\pi,x] = out[L2,\pi,x]$
proof −
fix $\pi\ x$ **assume** *quasi-reduction* $L1'\ L2'$ **and** $\pi \in L1 \cap L2$ **and** $x \in exec[L2,\pi]$
then have $x \in X$
by (*meson assms*(2) *executable-inputs-in-alphabet*)

let $?\pi = map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ \pi$
have $map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ ?\pi = \pi$
by (*induction* π; *auto*)
then have $out[L2,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ ?\pi,x] \neq \{\}$
using ‹$x \in exec[L2,\pi]$› **by** *auto*

have *is-present* $?\pi$ $L1$ **and** *is-present* $?\pi$ $L2$
using ‹$\pi \in L1 \cap L2$› **by** *auto*

have $?\pi \in L1'a \cap L2'a$
using $L1'a$-*def* ‹$L2'a \equiv map\ (\lambda(x,\ y).\ (x,\ y,\ True))$ ‘ $L2$› ‹*is-present* $(map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ \pi)\ L1$› ‹*is-present* $(map\ (\lambda(x,\ y).\ (x,\ y,\ True))\ \pi)\ L2$› **by** *auto*
then have $?\pi \in absence\text{-}completion\ X\ Y\ L1$ **and** $?\pi \in absence\text{-}completion\ X\ Y\ L2$ **and** $?\pi \in L1' \cap L2'$
unfolding $L1'$-*def*[*symmetric*] $L2'$-*def*[*symmetric*]
unfolding ‹$L1' = L1'a \cup L1'b$› ‹$L2' = L2'a \cup L2'b$›
by *blast*+

have $out[L2',?\pi,x] = \{(y,\ True)\ |y.\ y \in out[L2,\pi,x]\} \cup \{(y,\ False)\ |y.\ y \in Y \wedge y \notin out[L2,\pi,x]\}$
using *absence-completion-out*(1)[*OF assms*(2) ‹$x \in X$› ‹$?\pi \in absence\text{-}completion\ X\ Y\ L2$› ‹*is-present* $?\pi\ L2$› ‹$out[L2,map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ ?\pi,x] \neq \{\}$›]
unfolding $L2'$-*def*[*symmetric*] ‹$map\ (\lambda(x,\ y,\ a).\ (x,\ y))\ ?\pi = \pi$› **.**
then have $x \in exec[L2',?\pi]$
using ‹$x \in exec[L2,\pi]$› **by** *fastforce*
then have $out[L1',?\pi,x] \neq \{\}$ **and** $out[L1',?\pi,x] \subseteq out[L2',?\pi,x]$
using ‹*quasi-reduction* $L1'\ L2'$› ‹$?\pi \in L1' \cap L2'$›
unfolding *quasi-reduction-def*

by *blast+*

have $out[L1,\pi,x] \neq \{\}$
by (*metis L1′-def* ‹*is-present* (*map* (λ(*x*, *y*). (*x*, *y*, *True*)) π) *L1*› ‹*map* (λ(*x*, *y*). (*x*, *y*, *True*)) π ∈ *absence-completion X Y L1*› ‹*map* (λ(*x*, *y*, *a*). (*x*, *y*)) (*map* (λ(*x*, *y*). (*x*, *y*, *True*)) π) = π› ‹*out*[*L1′*,*map* (λ(*x*, *y*). (*x*, *y*, *True*)) π,*x*] ≠ {}› ‹*x* ∈ *X*› *absence-completion-out*(*2*) *assms*(*1*))
then have $out[L1',?\pi,x] = \{(y,\ True)\ |y.\ y \in out[L1,\pi,x]\} \cup \{(y,\ False)\ |y.\ y \in Y \wedge y \notin out[L1,\pi,x]\}$
using *absence-completion-out*(*1*)[*OF assms*(*1*) ‹*x* ∈ *X*› ‹*?π* ∈ *absence-completion X Y L1*› ‹*is-present ?π L1*›]
unfolding *L1′-def*[*symmetric*] ‹*map* (λ(*x*, *y*, *a*). (*x*, *y*)) *?π* = π›
by *blast*

have $out[L1,\pi,x] \subseteq Y$ **and** $out[L2,\pi,x] \subseteq Y$
by (*meson assms*(*1*,*2*) *outputs-in-alphabet*)+

have $\bigwedge y\ .\ y \in out[L1,\pi,x] \Longrightarrow y \in out[L2,\pi,x]$
proof −
fix *y* **assume** $y \in out[L1,\pi,x]$
then have $(y,\ True) \in out[L1',?\pi,x]$
unfolding ‹$out[L1',?\pi,x] = \{(y,\ True)\ |y.\ y \in out[L1,\pi,x]\} \cup \{(y,\ False)\ |y.\ y \in Y \wedge y \notin out[L1,\pi,x]\}$› **by** *blast*
then have $(y,\ True) \in out[L2',?\pi,x]$
using ‹$out[L1',?\pi,x] \subseteq out[L2',?\pi,x]$› **by** *blast*
then show $y \in out[L2,\pi,x]$
unfolding ‹$out[L2',?\pi,x] = \{(y,\ True)\ |y.\ y \in out[L2,\pi,x]\} \cup \{(y,\ False)\ |y.\ y \in Y \wedge y \notin out[L2,\pi,x]\}$›
by *fastforce*
qed
moreover have $\bigwedge y\ .\ y \in out[L2,\pi,x] \Longrightarrow y \in out[L1,\pi,x]$
proof −
fix *y* **assume** $y \in out[L2,\pi,x]$
then have $(y,\ True) \in out[L2',?\pi,x]$ **and** $(y,\ False) \notin out[L2',?\pi,x]$
unfolding ‹$out[L2',?\pi,x] = \{(y,\ True)\ |y.\ y \in out[L2,\pi,x]\} \cup \{(y,\ False)\ |y.\ y \in Y \wedge y \notin out[L2,\pi,x]\}$› **by** *blast+*
moreover have $(y,\ True) \in out[L1',?\pi,x] \vee (y,\ False) \in out[L1',?\pi,x]$
unfolding ‹$out[L1',?\pi,x] = \{(y,\ True)\ |y.\ y \in out[L1,\pi,x]\} \cup \{(y,\ False)\ |y.\ y \in Y \wedge y \notin out[L1,\pi,x]\}$›
using ‹$out[L2,\pi,x] \subseteq Y$› ‹$y \in out[L2,\pi,x]$› **by** *auto*
ultimately have $(y,\ True) \in out[L1',?\pi,x]$
using ‹$out[L1',?\pi,x] \subseteq out[L2',?\pi,x]$› **by** *blast*
then show $y \in out[L1,\pi,x]$
unfolding ‹$out[L1',?\pi,x] = \{(y,\ True)\ |y.\ y \in out[L1,\pi,x]\} \cup \{(y,\ False)\ |y.\ y \in Y \wedge y \notin out[L1,\pi,x]\}$›
by *fastforce*
qed
ultimately show $out[L1,\pi,x] = out[L2,\pi,x]$
by *blast*

qed
then show (*absence-completion X Y L1*) $\preceq$[*X*,*quasired* (*Y* × *UNIV*)] (*absence-completion X Y L2*) ⟹ *L1* $\preceq$[*X*,*quasieq Y*] *L2*
unfolding *L1'-def*[*symmetric*] *L2'-def*[*symmetric*]
unfolding ‹(*L1'* $\preceq$[*X*,*quasired* (*Y* × *UNIV*)] *L2'*) = *quasi-reduction L1' L2'*›
unfolding ‹(*L1* $\preceq$[*X*,*quasieq Y*] *L2*) = *quasi-equivalence L1 L2*›
unfolding *quasi-reduction-def quasi-equivalence-def*
by *blast*
qed

6.2 Quasi-Reduction via Reduction and explicit Undefined Behaviour

fun *bottom-completion* :: *'x alphabet* ⇒ *'y alphabet* ⇒ (*'x*,*'y*) *language* ⇒ (*'x*, *'y option*) *language* **where**
bottom-completion X Y L =
((λ π . *map* (λ(*x*,*y*) . (*x*,*Some y*)) π) ‘ *L*)
∪ {(*map* (λ(*x*,*y*) . (*x*,*Some y*)) π)@[(*x*,*y*)]@τ | π *x y* τ . π ∈ *L* ∧ *out*[*L*,π,*x*] = {} ∧ *x* ∈ *X* ∧ (*y* = *None* ∨ *y* ∈ *Some* ‘ *Y*) ∧ (∀ (*x*,*y*) ∈ *set* τ . *x* ∈ *X* ∧ (*y* = *None* ∨ *y* ∈ *Some* ‘ *Y*))}

lemma *bottom-completion-is-language* :
assumes *is-language X Y L*
shows *is-language X* ({*None*} ∪ *Some* ‘ *Y*) (*bottom-completion X Y L*)
proof −
let *?L* = *bottom-completion X Y L*

have *?L* ≠ {}
using *language-contains-nil*[*OF assms*] **by** *auto*
moreover have ⋀ π . π ∈ *?L* ⟹ (∀ *xy* ∈ *set* π . *fst xy* ∈ *X* ∧ *snd xy* ∈ ({*None*} ∪ *Some* ‘ *Y*)) ∧ (∀ π' . *prefix* π' π ⟶ π' ∈ *?L*)
proof −
fix π **assume** π ∈ *?L*
then consider (*a*) π ∈ ((λ π . *map* (λ(*x*,*y*) . (*x*,*Some y*)) π) ‘ *L*) |
(*b*) π ∈ {(*map* (λ(*x*,*y*) . (*x*,*Some y*)) π)@[(*x*,*y*)]@τ | π *x y* τ . π ∈ *L* ∧ *out*[*L*,π,*x*] = {} ∧ *x* ∈ *X* ∧ (*y* = *None* ∨ *y* ∈ *Some* ‘ *Y*) ∧ (∀ (*x*,*y*) ∈ *set* τ . *x* ∈ *X* ∧ (*y* = *None* ∨ *y* ∈ *Some* ‘ *Y*))}
unfolding *bottom-completion.simps* **by** *blast*
then show (∀ *xy* ∈ *set* π . *fst xy* ∈ *X* ∧ *snd xy* ∈ ({*None*} ∪ *Some* ‘ *Y*)) ∧ (∀ π' . *prefix* π' π ⟶ π' ∈ *?L*)
proof *cases*
case *a*
then obtain π' **where** π = *map* (λ(*x*, *y*). (*x*, *Some y*)) π' **and** π' ∈ *L*
by *auto*
then have (∀ *xy* ∈ *set* π' . *fst xy* ∈ *X* ∧ *snd xy* ∈ *Y*)
and (∀ π'' . *prefix* π'' π' ⟶ π'' ∈ *L*)
using *assms* **by** *auto*

have ($\forall\ \pi'$. *prefix* $\pi'\ \pi \longrightarrow \pi' \in ((\lambda\ \pi$. *map* $(\lambda(x,y)$. $(x,$*Some* $y))\ \pi)$ ‘ $L))$
using ‹$(\forall\ \pi''$. *prefix* $\pi''\ \pi' \longrightarrow \pi'' \in L)$› **unfolding** ‹$\pi =$ *map* $(\lambda(x,\ y).$ $(x,$ *Some* $y))\ \pi'$›
using *prefix-map-rightE* **by** *force*
then have ($\forall\ \pi'$. *prefix* $\pi'\ \pi \longrightarrow \pi' \in$ *?L*)
by *auto*
moreover have ($\forall\ xy \in$ *set* π . *fst* $xy \in X \wedge$ *snd* $xy \in (\{$*None*$\} \cup$ *Some* ‘ $Y))$
using ‹$(\forall\ xy \in$ *set* π' . *fst* $xy \in X \wedge$ *snd* $xy \in Y)$› **unfolding** ‹$\pi =$ *map* $(\lambda(x,\ y).$ $(x,$ *Some* $y))\ \pi'$›
by (*induction* π'; *auto*)
ultimately show *?thesis*
by *blast*
next
case *b*
then obtain $\pi'\ x\ y\ \tau$ **where** $\pi =$ (*map* $(\lambda(x,y)$. $(x,$*Some* $y))\ \pi')@[(x,y)]@\tau$

and $\pi' \in L$
and *out*$[L,\pi',x] = \{\}$
and $x \in X$
and $(y =$ *None* $\vee\ y \in$ *Some* ‘ $Y)$
and $(\forall\ (x,y) \in$ *set* τ . $x \in X \wedge (y =$ *None* $\vee\ y \in$ *Some* ‘ $Y))$
by *blast*
then have ($\forall\ xy \in$ *set* π' . *fst* $xy \in X \wedge$ *snd* $xy \in Y$)
and ($\forall\ \pi''$. *prefix* $\pi''\ \pi' \longrightarrow \pi'' \in L$)
using *assms* **by** *auto*

have ($\forall\ xy \in$ *set* (*map* $(\lambda(x,y)$. $(x,$*Some* $y))\ \pi')$. *fst* $xy \in X \wedge$ *snd* $xy \in (\{$*None*$\} \cup$ *Some* ‘ $Y))$
using ‹$(\forall\ xy \in$ *set* π' . *fst* $xy \in X \wedge$ *snd* $xy \in Y)$›
by (*induction* π'; *auto*)
moreover have *set* $\pi =$ *set* (*map* $(\lambda(x,y)$. $(x,$*Some* $y))\ \pi') \cup \{(x,y)\} \cup$ *set* τ
unfolding ‹$\pi =$ (*map* $(\lambda(x,y)$. $(x,$*Some* $y))\ \pi')@[(x,y)]@\tau$›
by *simp*
ultimately have ($\forall\ xy \in$ *set* π . *fst* $xy \in X \wedge$ *snd* $xy \in (\{$*None*$\} \cup$ *Some* ‘ $Y))$
using ‹$x \in X$› ‹$(y =$ *None* $\vee\ y \in$ *Some* ‘ $Y)$› ‹$(\forall\ (x,y) \in$ *set* τ . $x \in X \wedge (y =$ *None* $\vee\ y \in$ *Some* ‘ $Y))$›
by *auto*
moreover have $\bigwedge\ \pi''$. *prefix* $\pi''\ \pi \Longrightarrow \pi'' \in$ *?L*
proof −
fix π'' **assume** *prefix* $\pi''\ \pi$
then obtain i **where** $\pi'' =$ *take* $i\ \pi$
by (*metis append-eq-conv-conj prefix-def*)
then consider (*b1*) $i \leq$ *length* π' |
(*b2*) $i >$ *length* π'
by *linarith*
then show $\pi'' \in$ *?L* **proof** *cases*
case *b1*

```
        then have i ≤ length (map (λ(x,y) . (x,Some y)) π')
          by auto
        then have π'' = map (λ(x,y) . (x,Some y)) (take i π')
          unfolding ‹π'' = take i π›
          using ‹π = map (λ(x, y). (x, Some y)) π' @ [(x, y)] @ τ› take-map by
fastforce
        moreover have take i π' ∈ L
          using ‹π' ∈ L› take-is-prefix
          using ‹∀π''. prefix π'' π' ⟶ π'' ∈ L› by blast
        ultimately have π'' ∈ ((λ π . map (λ(x,y) . (x,Some y)) π) ‘ L)
          by simp
        then show π'' ∈ ?L
          by auto
      next
        case b2
        then have i > length (map (λ(x,y) . (x,Some y)) π')
          by auto

       have ⋀ k xs ys . k > length xs ⟹ take k (xs@ys) = xs@(take (k − length
xs) ys)
          by simp
        have take-helper: ⋀ k xs y zs . k > length xs ⟹ take k (xs@[y]@zs) =
xs@[y]@(take (k − length xs − 1) zs)
          by (metis One-nat-def Suc-pred ‹⋀ys xs k. length xs < k ⟹ take k
(xs @ ys) = xs @ take (k − length xs) ys› append-Cons append-Nil take-Suc-Cons
zero-less-diff)

        have **: π'' = (map (λ(x,y) . (x,Some y)) π')@[(x,y)]@(take (i − length
π' − 1) τ)
          unfolding ‹π = map (λ(x, y). (x, Some y)) π' @ [(x, y)] @ τ›  ‹π'' =
take i π›
          using take-helper[OF ‹i > length (map (λ(x,y) . (x,Some y)) π')›] by
simp

        have (∀ (x,y) ∈ set (take (i − length π' − 1) τ) . x ∈ X ∧ (y = None ∨
y ∈ Some ‘ Y))
          using ‹(∀ (x,y) ∈ set τ . x ∈ X ∧ (y = None ∨ y ∈ Some ‘ Y))›
          by (meson in-set-takeD)
        then show ?thesis
          unfolding ** bottom-completion.simps
         using ‹π' ∈ L› ‹out[L,π',x] = {}› ‹x ∈ X› ‹(y = None ∨ y ∈ Some ‘ Y)›
          by blast
       qed
     qed
     ultimately show ?thesis by auto
   qed
  qed
  ultimately show ?thesis
    unfolding is-language.simps by blast
```

qed

fun *is-not-undefined* :: ('*x*,'*y option*) *word* ⇒ ('*x*,'*y*) *language* ⇒ *bool* **where**
is-not-undefined π *L* = (π ∈ *map* (λ(*x*, *y*). (*x*, *Some y*)) ‘ *L*)

lemma *bottom-id* : *map* (λ(*x*,*y*) . (*x*, *the y*)) (*map* (λ(*x*, *y*). (*x*, *Some y*)) π) = π
by (*induction* π; *auto*)

fun *maximum-prefix-with-property* :: ('*a list* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* **where**
maximum-prefix-with-property P xs = (*last* (*filter P* (*prefixes xs*)))

lemma *maximum-prefix-with-property-props* :
assumes ∃ *ys* ∈ *set* (*prefixes xs*) . *P ys*
shows *P* (*maximum-prefix-with-property P xs*)
and (*maximum-prefix-with-property P xs*) ∈ *set* (*prefixes xs*)
and ⋀ *ys* . *prefix ys xs* ⟹ *P ys* ⟹ *length ys* ≤ *length* (*maximum-prefix-with-property P xs*)
proof −

have *P* (*maximum-prefix-with-property P xs*) ∧
(*maximum-prefix-with-property P xs*) ∈ *set* (*prefixes xs*) ∧
(∀ *ys* . *prefix ys xs* ⟶ *P ys* ⟶ *length ys* ≤ *length* (*maximum-prefix-with-property P xs*))
using *assms*
proof (*induction xs rule*: *rev-induct*)
case *Nil*
then show *?case* **by** *auto*
next
case (*snoc x xs*)
have *prefixes* (*xs* @ [*x*]) = (*prefixes xs*)@[*xs* @ [*x*]]
by *simp*

show *?case* **proof** (*cases P* (*xs*@[*x*]))
case *True*
then have *maximum-prefix-with-property P* (*xs* @ [*x*]) = (*xs* @ [*x*])
unfolding *maximum-prefix-with-property.simps* ‹*prefixes* (*xs* @ [*x*]) = (*prefixes xs*)@[*xs* @ [*x*]]›
by *auto*
show *?thesis*
using *True*
unfolding ‹*maximum-prefix-with-property P* (*xs* @ [*x*]) = (*xs*@[*x*])›
using *in-set-prefixes prefix-length-le* **by** *blast*
next
case *False*

then have *maximum-prefix-with-property P (xs@[x]) = maximum-prefix-with-property P xs*
unfolding *maximum-prefix-with-property.simps ‹prefixes (xs @ [x]) = (prefixes xs)@[xs @ [x]]›*
by *auto*

have *∃ a∈set (prefixes xs). P a*
using *snoc.prems False* **unfolding** *‹prefixes (xs @ [x]) = (prefixes xs)@[xs @ [x]]›* **by** *auto*

show *?thesis*
using *snoc.IH[OF ‹∃ a∈set (prefixes xs). P a›] False*
unfolding *‹maximum-prefix-with-property P (xs@[x]) = maximum-prefix-with-property P xs›*
unfolding *‹prefixes (xs @ [x]) = (prefixes xs)@[xs @ [x]]›* **by** *auto*
qed
qed
then show *P (maximum-prefix-with-property P xs)*
and *(maximum-prefix-with-property P xs) ∈ set (prefixes xs)*
and *⋀ ys . prefix ys xs ⟹ P ys ⟹ length ys ≤ length (maximum-prefix-with-property P xs)*
by *blast+*
qed

lemma *bottom-completion-out* :
assumes *is-language X Y L*
and *x ∈ X*
and *π ∈ bottom-completion X Y L*
shows *is-not-undefined π L ⟹ out[L,map (λ(x,y) . (x, the y)) π,x] ≠ {} ⟹ out[bottom-completion X Y L, π, x] = Some ‘ out[L, map (λ(x,y) . (x, the y)) π, x]*
and *is-not-undefined π L ⟹ out[L,map (λ(x,y) . (x, the y)) π,x] = {} ⟹ out[bottom-completion X Y L, π, x] = {None} ∪ Some ‘ Y*
and *¬ is-not-undefined π L ⟹ out[bottom-completion X Y L, π, x] = {None} ∪ Some ‘ Y*
proof −

let *?L = bottom-completion X Y L*
define *L′a* **where** *L′a = ((λ π . map (λ(x,y) . (x,Some y)) π) ‘ L)*
define *L′b* **where** *L′b = {(map (λ(x,y) . (x,Some y)) π)@[(x,y)]@τ | π x y τ . π ∈ L ∧ out[L,π,x] = {} ∧ x ∈ X ∧ (y = None ∨ y ∈ Some ‘ Y) ∧ (∀ (x,y) ∈ set τ . x ∈ X ∧ (y = None ∨ y ∈ Some ‘ Y))}*
have *?L = L′a ∪ L′b*
unfolding *L′a-def L′b-def bottom-completion.simps* **by** *blast*
then have *out[?L, π, x] = {y. π @ [(x, y)] ∈ L′a} ∪ {y. π @ [(x, y)] ∈ L′b}*
unfolding *outputs.simps language-for-state.simps* **by** *blast*

have *is-language X ({None} ∪ Some ‘ Y) ?L*
using *bottom-completion-is-language[OF assms(1)]* **.**

show *is-not-undefined* π $L \Longrightarrow$ *out*[*L*,*map* (λ(*x*,*y*) . (*x*, *the y*)) π,*x*] $\neq$ {} $\Longrightarrow$ *out*[*bottom-completion X Y L*, π, *x*] = *Some* ‘ *out*[*L*, *map* (λ(*x*,*y*) . (*x*, *the y*)) π, *x*]
and *is-not-undefined* π $L \Longrightarrow$ *out*[*L*,*map* (λ(*x*,*y*) . (*x*, *the y*)) π,*x*] = {} $\Longrightarrow$ *out*[*bottom-completion X Y L*, π, *x*] = {*None*} $\cup$ *Some* ‘ *Y*
proof −
assume *is-not-undefined* π *L*
then obtain π' **where** π = *map* (λ(*x*, *y*). (*x*, *Some y*)) π' **and** $\pi' \in L$
by *auto*
then have *map* (λ(*x*, *y*). (*x*, *the y*)) π = π'
using *bottom-id* **by** *auto*

have {*y*. π @ [(*x*, *y*)] $\in$ *L*′*a*} = *Some* ‘ *out*[*L*,*map* (λ(*x*,*y*) . (*x*, *the y*)) π,*x*]
proof
show {*y*. π @ [(*x*, *y*)] $\in$ *L*′*a*} $\subseteq$ *Some* ‘ *out*[*L*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*]
proof
fix *y* **assume** *y* $\in$ {*y*. π @ [(*x*, *y*)] $\in$ *L*′*a*}
then have π @ [(*x*, *y*)] $\in$ *L*′*a* **by** *auto*
then obtain π' **where** π @ [(*x*, *y*)] = *map* (λ(*x*,*y*) . (*x*,*Some y*)) π' **and** π' $\in$ *L*
unfolding *L*′*a-def* **by** *blast*
then have *length* (π @ [(*x*, *y*)]) = *length* π'
by *auto*
then obtain γ' *xy* **where** $\pi' = \gamma'$@[*xy*]
by (*metis add.right-neutral dual-order.strict-iff-not length-append-singleton less-add-Suc2 rev-exhaust take0 take-all-iff*)
then have (*x*,*y*) = (λ(*x*, *y*). (*x*, *Some y*)) *xy*
using ‹π @ [(*x*, *y*)] = *map* (λ(*x*, *y*). (*x*, *Some y*)) π'› **unfolding** ‹$\pi' = \gamma'$@[*xy*]› **by** *auto*
then have *y* = *Some* (*snd xy*) **and** *xy* = (*x*,*snd xy*)
by (*simp add*: *split-beta*)+
moreover define *y*′ **where** *y*′ = *snd xy*
ultimately have *y* = *Some y*′ **and** *xy* = (*x*,*y*′)
by *auto*

have *map* (λ(*x*, *y*). (*x*, *the y*)) π = γ'
using ‹π @ [(*x*, *y*)] = *map* (λ(*x*,*y*) . (*x*,*Some y*)) π'› **unfolding** ‹$\pi' = \gamma'$@[*xy*]›
using *bottom-id* **by** *auto*

have *y*′ $\in$ *out*[*L*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*]
using ‹$\pi' \in L$›
unfolding ‹*map* (λ(*x*, *y*). (*x*, *the y*)) π = γ'› ‹$\pi' = \gamma'$@[*xy*]› ‹*xy* = (*x*,*y*′)›
by *auto*
then show *y* $\in$ *Some* ‘ *out*[*L*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*]
unfolding ‹*y* = *Some y*′› **by** *blast*
qed
show *Some* ‘ *out*[*L*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*] $\subseteq$ {*y*. π @ [(*x*, *y*)] $\in$ *L*′*a*}

```
    proof
      fix y assume y ∈ Some ‘ out[L,map (λ(x, y). (x, the y)) π,x]
       then obtain y' where y = Some y' and y' ∈ out[L,map (λ(x, y). (x, the
y)) π,x]
        by blast
      then have π'@[(x,y')] ∈ L
        unfolding ‹map (λ(x, y). (x, the y)) π = π'› by auto
      then show y ∈ {y. π @ [(x, y)] ∈ L'a}
        unfolding L'a-def ‹π = map (λ(x, y). (x, Some y)) π'›
        using ‹y = Some y'› image-iff by fastforce
    qed
  qed


  show out[L,map (λ(x,y) . (x, the y)) π,x] ≠ {} ⟹ out[bottom-completion X Y
L, π, x] = Some ‘ out[L, map (λ(x,y) . (x, the y)) π, x]
  proof −
    assume out[L,map (λ(x,y) . (x, the y)) π,x] ≠ {}
    then obtain ya where π'@[(x,ya)] ∈ L
      using ‹π' ∈ L› unfolding ‹map (λ(x,y) . (x, the y)) π = π'› by auto


    have {y. π @ [(x, y)] ∈ L'b} = {}
    proof (rule ccontr)
      assume {y. π @ [(x, y)] ∈ L'b} ≠ {}
      then obtain y where π @ [(x, y)] ∈ L'b by blast
       then obtain π'' x' y' τ where π @ [(x, y)] = (map (λ(x,y) . (x,Some y))
π'')@[(x',y')]@τ
                                and π'' ∈ L
                                and out[L,π'',x'] = {}
                                and x' ∈ X
                                and (y' = None ∨ y' ∈ Some ‘ Y)
                               and (∀ (x,y) ∈ set τ . x ∈ X ∧ (y = None ∨ y ∈ Some
‘ Y))
       unfolding L'b-def
       by blast

      have ⋀ y'' . π''@[(x',y'')] ∉ L
        using ‹π'' ∈ L› ‹out[L,π'',x'] = {}›
        unfolding outputs.simps language-for-state.simps by force

      have length π' = length π''
      proof −
        have length π' = length π
          using ‹map (λ(x, y). (x, the y)) π = π'› length-map by blast

        have ¬ length π' < length π''
        proof
```

assume *length* $\pi' < length\ \pi''$
then have *length* $\pi'' = Suc\ (length\ \pi')$
by (*metis* (*no-types*, *lifting*) *One-nat-def* ‹π @ [(x, y)] = *map* ($\lambda(x,$ $y).$ (x, *Some* y)) π'' @ [(x', y')] @ τ› ‹*length* π' = *length* π› *add-diff-cancel-left' length-append length-append-singleton length-map list.size*(*3*) *not-less-eq plus-1-eq-Suc zero-less-Suc zero-less-diff*)
then have *length* $\pi'' >$ *length* π
by (*simp add*: ‹π = *map* ($\lambda(x,$ $y).$ (x, *Some* y)) π'›)
then show *False*
by (*metis* (*no-types*, *lifting*) *One-nat-def* ‹π @ [(x, y)] = *map* ($\lambda(x,$ $y).$ (x, *Some* y)) π'' @ [(x', y')] @ τ› *length-Cons length-append length-append-singleton length-map less-add-same-cancel1 list.size*(*3*) *not-less-eq plus-1-eq-Suc zero-less-Suc*)

qed
moreover have ¬ *length* $\pi'' <$ *length* π'
proof
assume *length* $\pi'' <$ *length* π'
then have *prefix* ((*map* ($\lambda(x,y)$. (x,*Some* y)) π'')@[(x',y')]) (*map* ($\lambda(x,y)$. (x,*Some* y)) π')
by (*metis* (*no-types*, *lifting*) ‹π = *map* ($\lambda(x,$ $y).$ (x, *Some* y)) π'› ‹π @ [(x, y)] = *map* ($\lambda(x,$ $y).$ (x, *Some* y)) π'' @ [(x', y')] @ τ› *append.assoc length-append-singleton length-map linorder-not-le not-less-eq prefixI prefix-length-prefix*)

then have *prefix* π'' π'
by (*metis append-prefixD bottom-id map-mono-prefix*)
then have *take* (*length* π'') π' = π''
by (*metis append-eq-conv-conj prefix-def*)

have (x',y') = (((*map* ($\lambda(x,y)$. (x,*Some* y)) π'')@[(x',y')])) ! (*length* π'')
by (*induction* π'' *arbitrary*: x' y'; *auto*)
then have (x',y') = (*map* ($\lambda(x,y)$. (x,*Some* y)) π') ! (*length* π'')
by (*metis* (*no-types*, *lifting*) ‹π = *map* ($\lambda(x,$ $y).$ (x, *Some* y)) π'› ‹π @ [(x, y)] = *map* ($\lambda(x,$ $y).$ (x, *Some* y)) π'' @ [(x', y')] @ τ› ‹*length* $\pi'' <$ *length* π'› *append-Cons length-map nth-append nth-append-length*)
then have *fst* (π' ! (*length* π'')) = x'
by (*simp add*: ‹*length* $\pi'' <$ *length* π'› *split-beta*)

have *out*[L, *take* (*length* π'') π', *fst* (π' ! (*length* π''))] = {}
unfolding ‹*take* (*length* π'') π' = π''› ‹*fst* (π' ! (*length* π'')) = x'›
using ‹*out*[L,π'',x'] = {}› .
moreover have $\bigwedge$ i . $i <$ *length* $\pi' \Longrightarrow$ *out*[L, *take* i π', *fst* (π' ! i)] ≠ {}
using *prefix-executable*[*OF assms*(*1*) ‹π' ∈ L›]
by (*meson outputs-executable*)
ultimately show *False*
using ‹*length* $\pi'' <$ *length* π'› **by** *blast*
qed
ultimately show *?thesis*
by *simp*
qed

then have $\pi'' = \pi'$
by (*metis* ‹π @ [(x, y)] = *map* ($\lambda(x, y).$ (x, *Some* y)) π'' @ [(x', y')] @ τ› ‹*map* ($\lambda(x, y).$ (x, *the* y)) π = π'› *append-eq-append-conv bottom-id length-map*)

show *False*
using ‹π @ [(x, y)] = *map* ($\lambda(x, y).$ (x, *Some* y)) π'' @ [(x', y')] @ τ› ‹π'' = π'› ‹*map* ($\lambda(x, y).$ (x, *the* y)) π = π'› ‹*out*[L,π'',x'] = {}› ‹*out*[L,*map* ($\lambda(x, y).$ (x, *the* y)) π,x] ≠ {}›
by *force*
qed
then show *?thesis*
using ‹*out*[*bottom-completion* X Y L,π,x] = {y. π @ [(x, y)] ∈ $L'a$} ∪ {y. π @ [(x, y)] ∈ $L'b$}›
using ‹{y. π @ [(x, y)] ∈ $L'a$} = *Some* ‘ *out*[L,*map* ($\lambda(x,y)$. (x, *the* y)) π,x]›
by *force*
qed

show *out*[L,*map* ($\lambda(x,y)$. (x, *the* y)) π,x] = {} ⟹ *out*[*bottom-completion* X Y L, π, x] = {*None*} ∪ *Some* ‘ Y
proof −
assume *out*[L,*map* ($\lambda(x,y)$. (x, *the* y)) π,x] = {}
then have {y. π @ [(x, y)] ∈ $L'a$} = {}
unfolding ‹{y. π @ [(x, y)] ∈ $L'a$} = *Some* ‘ *out*[L,*map* ($\lambda(x,y)$. (x, *the* y)) π,x]› **by** *blast*
moreover have {y. π @ [(x, y)] ∈ $L'b$} = {*None*} ∪ *Some* ‘ Y
proof
show {y. π @ [(x, y)] ∈ $L'b$} ⊆ {*None*} ∪ *Some* ‘ Y
proof
fix y **assume** y ∈ {y. π @ [(x, y)] ∈ $L'b$}
then have π @ [(x, y)] ∈ $L'b$ **by** *blast*
then obtain π'' x' y' τ **where** π @ [(x, y)] = (*map* ($\lambda(x,y)$. (x,*Some* y)) π'')@[(x',y')]@τ
and $\pi'' \in L$
and *out*[L,π'',x'] = {}
and $x' \in X$
and (y' = *None* ∨ $y' \in$ *Some* ‘ Y)
and (∀ (x,y) ∈ *set* τ . $x \in X \wedge$ (y = *None* ∨ $y \in$ *Some* ‘ Y))
unfolding $L'b$-*def*
by *blast*

show $y \in$ {*None*} ∪ *Some* ‘ Y
by (*metis* (*no-types*, *lifting*) *Un-insert-right* ‹*out*[*bottom-completion* X Y L,π,x] = {y. π @ [(x, y)] ∈ $L'a$} ∪ {y. π @ [(x, y)] ∈ $L'b$}› ‹$y \in$ {y. π @ [(x, y)] ∈ $L'b$}› *assms*(*1*) *bottom-completion-is-language insert-subset mk-disjoint-insert outputs-in-alphabet*)
qed

```
      show {None} ∪ Some ‘ Y ⊆ {y. π @ [(x, y)] ∈ L′b}
      proof
        fix y assume y ∈ {None} ∪ Some ‘ Y

        have π @ [(x, y)] = map (λ(x, y). (x, Some y)) π′ @ [(x, y)] @ []
          by (simp add: ‹π = map (λ(x, y). (x, Some y)) π′›)
        moreover note ‹π′ ∈ L›
        moreover have out[L,π′,x] = {}
         using ‹out[L,map (λ(x,y) . (x, the y)) π,x] = {}› unfolding ‹map (λ(x,y)
. (x, the y)) π = π′› .
        moreover note ‹x ∈ X›
        moreover have (y = None ∨ y ∈ Some ‘ Y)
          using ‹y ∈ {None} ∪ Some ‘ Y› by blast
        moreover have (∀(x, y)∈set []. x ∈ X ∧ (y = None ∨ y ∈ Some ‘ Y))
          by simp
        ultimately show y ∈ {y. π @ [(x, y)] ∈ L′b}
          unfolding L′b-def by blast
      qed
    qed
    ultimately show ?thesis
      using ‹out[bottom-completion X Y L,π,x] = {y. π @ [(x, y)] ∈ L′a} ∪ {y. π
@ [(x, y)] ∈ L′b}›
       using ‹{y. π @ [(x, y)] ∈ L′a} = Some ‘ out[L,map (λ(x,y) . (x, the y))
π,x]›
      by force
  qed
qed

show ¬ is-not-undefined π L ⟹ out[bottom-completion X Y L,π,x] = {None}
∪ Some ‘ Y
proof −
  assume ¬ is-not-undefined π L
  then have π ∉ L′a
    unfolding L′a-def by auto

  have {y. π @ [(x, y)] ∈ L′a} = {}
  proof (rule ccontr)
    assume {y. π @ [(x, y)] ∈ L′a} ≠ {}
    then obtain y where π @ [(x, y)] ∈ L′a by blast
    then obtain γ where π @ [(x, y)] = map (λ(x, y). (x, Some y)) γ and γ ∈
L
      unfolding L′a-def by blast
    then have π = map (λ(x, y). (x, Some y)) (butlast γ)
      by (metis (mono-tags, lifting) butlast-snoc map-butlast)
    moreover have butlast γ ∈ L
      using ‹γ ∈ L› assms(1)
      by (simp add: prefixeq-butlast)
    ultimately show False
      using ‹π ∉ L′a›
```

```
      using L'a-def by blast
    qed
    then have out[?L, π, x] = {y. π @ [(x, y)] ∈ L'b}
      using ‹out[bottom-completion X Y L,π,x] = {y. π @ [(x, y)] ∈ L'a} ∪ {y. π
@ [(x, y)] ∈ L'b}› by blast
    also have ... = {None} ∪ Some ‘ Y
    proof
      show {y. π @ [(x, y)] ∈ L'b} ⊆ {None} ∪ Some ‘ Y
      proof
        fix y assume y ∈ {y. π @ [(x, y)] ∈ L'b}
        then obtain π' x' y' τ where π @ [(x, y)] = (map (λ(x,y) . (x,Some y))
π')@[(x',y')]@τ
                                and π' ∈ L
                                and out[L,π',x'] = {}
                                and x' ∈ X
                                and (y' = None ∨ y' ∈ Some ‘ Y)
                               and (∀ (x,y) ∈ set τ . x ∈ X ∧ (y = None ∨ y ∈ Some
‘ Y))
          unfolding L'b-def
          by blast

        have (x,y) ∈ set ([(x',y')]@τ)
            by (metis ‹π @ [(x, y)] = map (λ(x, y). (x, Some y)) π' @ [(x', y')]
@ τ› append-is-Nil-conv last-appendR last-in-set last-snoc length-Cons list.size(3)
nat.simps(3))
        then show y ∈ {None} ∪ Some ‘ Y
          using ‹(y' = None ∨ y' ∈ Some ‘ Y)› ‹(∀ (x,y) ∈ set τ . x ∈ X ∧ (y =
None ∨ y ∈ Some ‘ Y))› by auto
      qed
      show {None} ∪ Some ‘ Y ⊆ {y. π @ [(x, y)] ∈ L'b}
      proof
        fix y assume y ∈ {None} ∪ Some ‘ Y

        have π ∈ L'b
          using ‹π ∉ L'a› ‹?L = L'a ∪ L'b› assms(3) by fastforce
      then obtain π' x' y' τ where π = (map (λ(x,y) . (x,Some y)) π')@[(x',y')]@τ

                                and π' ∈ L
                                and out[L,π',x'] = {}
                                and x' ∈ X
                                and (y' = None ∨ y' ∈ Some ‘ Y)
                               and (∀ (x,y) ∈ set τ . x ∈ X ∧ (y = None ∨ y ∈ Some
‘ Y))
          unfolding L'b-def
          by blast

        have π @ [(x,y)] = (map (λ(x,y) . (x,Some y)) π')@[(x',y')]@(τ@[(x,y)])
          unfolding ‹π = (map (λ(x,y) . (x,Some y)) π')@[(x',y')]@τ› by auto
         moreover note ‹π' ∈ L› and ‹out[L,π',x'] = {}› and ‹x' ∈ X› and ‹(y'
```

$= None \vee y' \in Some\ `\ Y\rangle$
moreover have $(\forall\ (x,y) \in set\ (\tau@[(x,y)])\ .\ x \in X \wedge (y = None \vee y \in Some\ `\ Y))$
using $\langle \forall\ (x,y) \in set\ \tau\ .\ x \in X \wedge (y = None \vee y \in Some\ `\ Y)\rangle\ \langle y \in \{None\} \cup Some\ `\ Y\rangle\ \langle x \in X\rangle$
by *auto*
ultimately show $y \in \{y.\ \pi\ @\ [(x,\ y)] \in L'b\}$
unfolding *L'b-def* **by** *blast*
qed
qed
finally show $out[?L,\pi,x] = \{None\} \cup Some\ `\ Y$ **.**
qed
qed

theorem *quasired-via-red* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows $(L1 \preceq[X,quasired\ Y]\ L2) \longleftrightarrow ((bottom\text{-}completion\ X\ Y\ L1) \preceq[X,\ red\ (\{None\} \cup Some\ `\ Y)]\ (bottom\text{-}completion\ X\ Y\ L2))$
proof −

define $L1'$ **where** $L1' = bottom\text{-}completion\ X\ Y\ L1$
define $L2'$ **where** $L2' = bottom\text{-}completion\ X\ Y\ L2$

define $L1'a$ **where** $L1'a = ((\lambda\ \pi\ .\ map\ (\lambda(x,y)\ .\ (x,Some\ y))\ \pi)\ `\ L1)$
define $L1'b$ **where** $L1'b = \{(map\ (\lambda(x,y)\ .\ (x,Some\ y))\ \pi)@[(x,y)]@\tau \mid \pi\ x\ y\ \tau\ .\ \pi \in L1 \wedge out[L1,\pi,x] = \{\} \wedge x \in X \wedge (y = None \vee y \in Some\ `\ Y) \wedge (\forall\ (x,y) \in set\ \tau\ .\ x \in X \wedge (y = None \vee y \in Some\ `\ Y))\}$
define $L2'a$ **where** $L2'a = ((\lambda\ \pi\ .\ map\ (\lambda(x,y)\ .\ (x,Some\ y))\ \pi)\ `\ L2)$
define $L2'b$ **where** $L2'b = \{(map\ (\lambda(x,y)\ .\ (x,Some\ y))\ \pi)@[(x,y)]@\tau \mid \pi\ x\ y\ \tau\ .\ \pi \in L2 \wedge out[L2,\pi,x] = \{\} \wedge x \in X \wedge (y = None \vee y \in Some\ `\ Y) \wedge (\forall\ (x,y) \in set\ \tau\ .\ x \in X \wedge (y = None \vee y \in Some\ `\ Y))\}$

let $?L1 = bottom\text{-}completion\ X\ Y\ L1$

have $?L1 = L1'a \cup L1'b$
unfolding *L1'a-def L1'b-def bottom-completion.simps* **by** *blast*
then have $\bigwedge \pi\ x\ .\ out[?L1,\ \pi,\ x] = \{y.\ \pi\ @\ [(x,\ y)] \in L1'a\} \cup \{y.\ \pi\ @\ [(x,\ y)] \in L1'b\}$
unfolding *outputs.simps language-for-state.simps* **by** *blast*

let $?L2 = bottom\text{-}completion\ X\ Y\ L2$

have $?L2 = L2'a \cup L2'b$
unfolding *L2'a-def L2'b-def bottom-completion.simps* **by** *blast*
then have $\bigwedge \pi\ x\ .\ out[?L2,\ \pi,\ x] = \{y.\ \pi\ @\ [(x,\ y)] \in L2'a\} \cup \{y.\ \pi\ @\ [(x,\ y)] \in L2'b\}$

unfolding *outputs.simps language-for-state.simps* **by** *blast*

have *is-language X* ({*None*} ∪ *Some* ‘ *Y*) *?L1*
using *bottom-completion-is-language*[*OF assms*(*1*)] **.**
have *is-language X* ({*None*} ∪ *Some* ‘ *Y*) *?L2*
using *bottom-completion-is-language*[*OF assms*(*2*)] **.**
then have ⋀ *π x* . *out*[*bottom-completion X Y L2*,*π*,*x*] ⊆ {*None*} ∪ *Some* ‘ *Y*
by (*meson outputs-in-alphabet*)

have (*?L1* ⪯[*X*, *red* ({*None*} ∪ *Some* ‘ *Y*)] *?L2*) = (∀ *π* ∈ *?L1* ∩ *?L2* . ∀ *x* ∈ *X* . *out*[*?L1*,*π*,*x*] ⊆ *out*[*?L2*,*π*,*x*])
unfolding *type-1-conforms.simps red.simps*
using ‹⋀ *π x* . *out*[*bottom-completion X Y L2*,*π*,*x*] ⊆ {*None*} ∪ *Some* ‘ *Y*› **by** *force*
also have ... = (∀ *π* ∈ *?L1* ∩ *?L2* . ∀ *x* ∈ *X* . (*out*[*?L2*,*π*,*x*] = {*None*} ∪ *Some* ‘ *Y* ∨ (*out*[*?L1*,*π*,*x*] ≠ {} ∧ *out*[*?L1*,*π*,*x*] ⊆ *out*[*?L2*,*π*,*x*])))
by (*metis* (*no-types*, *lifting*) *IntD1* ‹*is-language X* ({*None*} ∪ *Some* ‘ *Y*) (*bottom-completion X Y L1*)› ‹*is-language X* ({*None*} ∪ *Some* ‘ *Y*) (*bottom-completion X Y L2*)› *assms*(*1*) *bottom-completion-out*(*1*) *bottom-completion-out*(*2*) *bottom-completion-out*(*3*) *image-is-empty outputs-in-alphabet subset-antisym*)
also have ... = (∀ *π* ∈ *?L1* ∩ *?L2* . ∀ *x* ∈ *X* . (*out*[*?L2*,*π*,*x*] = {*None*} ∪ *Some* ‘ *Y* ∨ (*is-not-undefined π L1* ∧ *is-not-undefined π L2* ∧ *out*[*L1*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*] ≠ {} ∧ *out*[*L1*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*] ⊆ *out*[*L2*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*])))
proof −
have ⋀ *π x* . *π* ∈ *?L1* ∩ *?L2* ⟹ *x* ∈ *X* ⟹ *out*[*?L2*,*π*,*x*] ≠ {*None*} ∪ *Some* ‘ *Y* ⟹
(*out*[*?L1*,*π*,*x*] ≠ {} ∧ *out*[*?L1*,*π*,*x*] ⊆ *out*[*?L2*,*π*,*x*]) = (*is-not-undefined π L1* ∧ *is-not-undefined π L2* ∧ *out*[*L1*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*] ≠ {} ∧ *out*[*L1*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*] ⊆ *out*[*L2*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*])
proof −
fix *π x* **assume** *π* ∈ *?L1* ∩ *?L2* **and** *x* ∈ *X* **and** *out*[*?L2*,*π*,*x*] ≠ {*None*} ∪ *Some* ‘ *Y*
then have *π* ∈ *?L1* **and** *π* ∈ *?L2* **by** *blast*+

have *is-not-undefined π L2*
using *bottom-completion-out*[*OF assms*(*2*) ‹*x* ∈ *X*› ‹*π* ∈ *?L2*›]
using ‹*out*[*bottom-completion X Y L2*,*π*,*x*] ≠ {*None*} ∪ *Some* ‘ *Y*› **by** *fastforce*
have *out*[*L2*,*map* (λ(*x*, *y*). (*x*, *the y*)) *π*,*x*] ≠ {}
using *bottom-completion-out*(*1*,*2*)[*OF assms*(*2*) ‹*x* ∈ *X*› ‹*π* ∈ *?L2*›]
using ‹*is-not-undefined π L2*› ‹*out*[*bottom-completion X Y L2*,*π*,*x*] ≠ {*None*} ∪ *Some* ‘ *Y*› **by** *blast*

show (*out*[*?L1*,*π*,*x*] ≠ {} ∧ *out*[*?L1*,*π*,*x*] ⊆ *out*[*?L2*,*π*,*x*]) = (*is-not-undefined π L1* ∧ *is-not-undefined π L2* ∧ *out*[*L1*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*] ≠ {} ∧ *out*[*L1*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*] ⊆ *out*[*L2*,*map* (λ(*x*,*y*) . (*x*, *the y*)) *π*,*x*])
proof (*cases is-not-undefined π L1*)
case *False*

then have *out[?L1,π,x] = {None} ∪ Some ‘ Y*
by (*meson IntD1 ‹π ∈ bottom-completion X Y L1 ∩ bottom-completion X Y L2› ‹x ∈ X› assms(1) bottom-completion-out(3)*)
then have ¬ (*out[?L1,π,x] ⊆ out[?L2,π,x]*)
by (*metis ‹is-language X ({None} ∪ Some ‘ Y) (bottom-completion X Y L2)› ‹out[bottom-completion X Y L2,π,x] ≠ {None} ∪ Some ‘ Y› outputs-in-alphabet subset-antisym*)
then show *?thesis*
using *False* **by** *presburger*
next
case *True*

have (*out[?L1,π,x] ≠ {} ∧ out[?L1,π,x] ⊆ out[?L2,π,x]*) = (*out[L1,map (λ(x,y) . (x, the y)) π,x] ≠ {} ∧ out[L1,map (λ(x,y) . (x, the y)) π,x] ⊆ out[L2,map (λ(x,y) . (x, the y)) π,x]*)
proof (*cases out[L1,map (λ(x, y). (x, the y)) π,x] = {}*)
case *True*

have ¬ (*out[?L1,π,x] ≠ {} ∧ out[?L1,π,x] ⊆ out[?L2,π,x]*)
unfolding *bottom-completion-out(2)[OF assms(1) ‹x ∈ X› ‹π ∈ ?L1› ‹is-not-undefined π L1› True]*
by (*meson ‹⋀x π. out[bottom-completion X Y L2,π,x] ⊆ {None} ∪ Some ‘ Y› ‹out[bottom-completion X Y L2,π,x] ≠ {None} ∪ Some ‘ Y› subset-antisym*)
moreover have ¬ (*out[L1,map (λ(x,y) . (x, the y)) π,x] ≠ {} ∧ out[L1,map (λ(x,y) . (x, the y)) π,x] ⊆ out[L2,map (λ(x,y) . (x, the y)) π,x]*)
using *True* **by** *simp*
ultimately show *?thesis* **by** *blast*
next
case *False*
show *?thesis*
unfolding *bottom-completion-out(1)[OF assms(1) ‹x ∈ X› ‹π ∈ ?L1› ‹is-not-undefined π L1› False]*
unfolding *bottom-completion-out(1)[OF assms(2) ‹x ∈ X› ‹π ∈ ?L2› ‹is-not-undefined π L2› ‹out[L2,map (λ(x, y). (x, the y)) π,x] ≠ {}›]*
by *blast*
qed
then show *?thesis*
using *‹is-not-undefined π L1› ‹is-not-undefined π L2›*
by *blast*
qed
qed
then show *?thesis*
by *meson*
qed
also have *... = ((∀ π ∈ ?L1 ∩ ?L2 . ∀ x ∈ X . ¬ is-not-undefined π L1 ⟶ is-not-undefined π L2 ⟶ out[?L2,π,x] = {None} ∪ Some ‘ Y)*
∧ (∀ π ∈ L1 ∩ L2 . ∀ x ∈ X . out[L2,π,x] = {} ∨ (out[L1,π,x] ≠ {} ∧ out[L1,π,x] ⊆ out[L2,π,x])))
(**is** *?A = ?B*)

```
proof
  show ?A ⟹ ?B
  proof −
    assume ?A

    have ⋀ π x . π ∈ ?L1 ∩ ?L2 ⟹ x ∈ X ⟹ ¬ is-not-undefined π L1 ⟹
is-not-undefined π L2 ⟹ out[?L2,π,x] = {None} ∪ Some ‘ Y
      using ‹?A› by blast
    moreover have ⋀ π x . π ∈ L1 ∩ L2 ⟹ x ∈ X ⟹ out[L2,π,x] = {} ∨
(out[L1,π,x] ≠ {} ∧ out[L1,π,x] ⊆ out[L2,π,x])
    proof −
      fix π x assume π ∈ L1 ∩ L2 and x ∈ X

      let ?π = map (λ(x, y). (x, Some y)) π

      have is-not-undefined ?π L1 and is-not-undefined ?π L2
        using ‹π ∈ L1 ∩ L2› by auto
      then have ?π ∈ ?L1 and ?π ∈ ?L2
        by auto

      show out[L2,π,x] = {} ∨ (out[L1,π,x] ≠ {} ∧ out[L1,π,x] ⊆ out[L2,π,x])
      proof (cases out[L2,π,x] = {})
        case True
        then show ?thesis by auto
      next
        case False
        then have out[bottom-completion X Y L2,?π,x] ≠ {None} ∪ Some ‘ Y
            using bottom-completion-out(1)[OF assms(2) ‹x ∈ X› ‹?π ∈ ?L2›
‹is-not-undefined ?π L2›]
          unfolding bottom-id
          by force
       then have out[L1,map (λ(x, y). (x, the y)) ?π,x] ≠ {} ∧ out[L1,map (λ(x,
y). (x, the y)) ?π,x] ⊆ out[L2,map (λ(x, y). (x, the y)) ?π,x]
          using ‹?A›
          using ‹?π ∈ ?L1› ‹?π ∈ ?L2› ‹x ∈ X›
          by blast
            then show out[L2,π,x] = {} ∨ (out[L1,π,x] ≠ {} ∧ out[L1,π,x] ⊆
out[L2,π,x])
          unfolding bottom-id by blast
      qed
    qed
    ultimately show ?B
      by meson
  qed
  show ?B ⟹ ?A
  proof −
    assume ?B

    have ⋀ π x . π ∈ ?L1 ∩ ?L2 ⟹ x ∈ X ⟹ out[?L2,π,x] = {None} ∪ Some
```

‘ $Y \vee$ *is-not-undefined* π *L1* $\wedge$ *is-not-undefined* π *L2* $\wedge$ *out*[*L1*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*] $\subseteq$ *out*[*L2*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*]
proof −
fix π *x* **assume** $\pi \in$ *?L1* $\cap$ *?L2* **and** $x \in X$
then have $\pi \in$ *?L1* **and** $\pi \in$ *?L2* **by** *auto*

show *out*[*?L2*,π,*x*] = {*None*} $\cup$ *Some* ‘ *Y* $\vee$ *is-not-undefined* π *L1* $\wedge$ *is-not-undefined* π *L2* $\wedge$ *out*[*L1*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*] $\subseteq$ *out*[*L2*,*map* (λ(*x*, *y*). (*x*, *the y*)) π,*x*]
proof (*cases out*[*?L2*,π,*x*] = {*None*} $\cup$ *Some* ‘ *Y*)
case *True*
then show *?thesis* **by** *blast*
next
case *False*

let *?*π = *map* (λ(*x*, *y*). (*x*, *the y*)) π

have *is-not-undefined* π *L2*
using *False* ‹($\forall \pi \in$*bottom-completion X Y L1* $\cap$ *bottom-completion X Y L2*. $\forall x \in X$. $\neg$ *is-not-undefined* π *L1* $\longrightarrow$ *is-not-undefined* π *L2* $\longrightarrow$ *out*[*bottom-completion X Y L2*,π,*x*] = {*None*} $\cup$ *Some* ‘ *Y*) $\wedge$ ($\forall \pi \in$*L1* $\cap$ *L2*. $\forall x \in X$. *out*[*L2*,π,*x*] = {} $\vee$ *out*[*L1*,π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,π,*x*] $\subseteq$ *out*[*L2*,π,*x*])› ‹$\pi \in$ *bottom-completion X Y L1* $\cap$ *bottom-completion X Y L2*› ‹*x* $\in$ *X*›
by (*meson* ‹$\pi \in$ *bottom-completion X Y L2*› *assms*(*2*) *bottom-completion-out*(*3*))
then have *?*$\pi \in$ *L2*
using *bottom-id*
by (*metis* (*mono-tags*, *lifting*) *imageE is-not-undefined.elims*(*2*))

have *is-not-undefined* π *L1*
using *False* ‹($\forall \pi \in$*bottom-completion X Y L1* $\cap$ *bottom-completion X Y L2*. $\forall x \in X$. $\neg$ *is-not-undefined* π *L1* $\longrightarrow$ *is-not-undefined* π *L2* $\longrightarrow$ *out*[*bottom-completion X Y L2*,π,*x*] = {*None*} $\cup$ *Some* ‘ *Y*) $\wedge$ ($\forall \pi \in$*L1* $\cap$ *L2*. $\forall x \in X$. *out*[*L2*,π,*x*] = {} $\vee$ *out*[*L1*,π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,π,*x*] $\subseteq$ *out*[*L2*,π,*x*])› ‹$\pi \in$ *bottom-completion X Y L1* $\cap$ *bottom-completion X Y L2*› ‹*x* $\in$ *X*›
using ‹*is-not-undefined* π *L2*› **by** *blast*
then have *?*$\pi \in$ *L1*
using *bottom-id*
by (*metis* (*mono-tags*, *lifting*) *imageE is-not-undefined.elims*(*2*))

have *out*[*L2*,*?*π,*x*] $\neq$ {}
using *False bottom-completion-out*(*2*)[*OF assms*(*2*) ‹*x* $\in$ *X*› ‹$\pi \in$ *?L2*› ‹*is-not-undefined* π *L2*›]
by *blast*
then have *out*[*L1*,*?*π,*x*] $\neq$ {} **and** *out*[*L1*,*?*π,*x*] $\subseteq$ *out*[*L2*,*?*π,*x*]
using ‹*?B*› ‹*?*$\pi \in$ *L1*› ‹*?*$\pi \in$ *L2*› ‹*x* $\in$ *X*›
by (*meson IntI*)+
then show *?thesis*
using ‹*is-not-undefined* π *L1*› ‹*is-not-undefined* π *L2*›

by *blast*
qed
qed
then show *?A*
by *blast*
qed
qed
also have ... = ((∀ π ∈ *?L1* ∩ *?L2* . ∀ x ∈ X . ¬ *is-not-undefined* π *L1* ⟶ *is-not-undefined* π *L2* ⟶ *out*[*L2*,*map* (λ(x, y). (x, *the* y)) π,x] = {})
∧ (∀ π ∈ *L1* ∩ *L2* . ∀ x ∈ X . *out*[*L2*,π,x] = {} ∨ (*out*[*L1*,π,x] ≠ {} ∧ *out*[*L1*,π,x] ⊆ *out*[*L2*,π,x])))
(**is** (*?A* ∧ *?B*) = (*?C* ∧ *?B*))
proof −
have *?A* = *?C*
by (*metis IntD2 None-notin-image-Some UnCI assms*(2) *bottom-completion-out*(1) *bottom-completion-out*(2) *insertCI*)
then show *?thesis* **by** *meson*
qed
also have ... = (∀ π ∈ *L1* ∩ *L2* . ∀ x ∈ X . *out*[*L2*,π,x] = {} ∨ (*out*[*L1*,π,x] ≠ {} ∧ *out*[*L1*,π,x] ⊆ *out*[*L2*,π,x]))
(**is** (*?A* ∧ *?B*) = *?B*)
proof −
have *?B* ⟹ *?A*
proof −
assume *?B*

have ⋀ π x . π ∈ *?L1* ∩ *?L2* ⟹ x ∈ X ⟹ ¬ *is-not-undefined* π *L1* ⟹ *is-not-undefined* π *L2* ⟹ *out*[*L2*,*map* (λ(x, y). (x, *the* y)) π,x] = {}
proof (*rule ccontr*)
fix π x **assume** π ∈ *?L1* ∩ *?L2* **and** x ∈ X **and** ¬ *is-not-undefined* π *L1* **and** *is-not-undefined* π *L2*
and *out*[*L2*,*map* (λ(x, y). (x, *the* y)) π,x] ≠ {}

let *?π* = *map* (λ(x, y). (x, *the* y)) π
have *?π* ∈ *L2*
by (*metis* (*mono-tags*, *lifting*) ‹*is-not-undefined* π *L2*› *bottom-id image-iff is-not-undefined.elims*(2))

have π ∈ *?L1*
using ‹π ∈ *?L1* ∩ *?L2*› **by** *auto*
moreover have π ∉ *L1'a*
unfolding *L1'a-def* **using** ‹¬ *is-not-undefined* π *L1*› **by** *auto*
ultimately have π ∈ *L1'b*
unfolding ‹*?L1* = *L1'a* ∪ *L1'b*› **by** *blast*
then obtain π' x' y' τ **where** π = (*map* (λ(x,y) . (x,*Some* y)) π')@[(x',y')]@τ
and π' ∈ *L1*
and *out*[*L1*,π',x'] = {}

and $x' \in X$
and $(y' = None \vee y' \in Some\ `\ Y)$
and $(\forall\ (x,y) \in set\ \tau\ .\ x \in X \wedge (y = None \vee y \in Some\ `\ Y))$
unfolding *L1'b-def*
by *blast*

have $?\pi = (\pi'@[(x',\ the\ y')])\ @\ (map\ (\lambda(x,\ y).\ (x,\ the\ y))\ \tau)$
unfolding $\langle \pi = (map\ (\lambda(x,y)\ .\ (x,Some\ y))\ \pi')@[(x',y')]@\tau \rangle$
using *bottom-id* **by** (*induction* π' *arbitrary*: $x'\ y'\ \tau$; *auto*)
then have $\pi'@[(x',\ the\ y')] \in L2$ **and** $\pi' \in L2$
using $\langle ?\pi \in L2 \rangle$
by (*metis assms*(2) *prefix-closure-no-member*)+
then have $out[L2,\pi',x'] \neq \{\}$
by *fastforce*

show *False*
using $\langle ?B \rangle$ $\langle \pi' \in L1 \rangle$ $\langle \pi' \in L2 \rangle$ $\langle x' \in X \rangle$ $\langle out[L2,\pi',x'] \neq \{\} \rangle$ $\langle out[L1,\pi',x'] = \{\} \rangle$
by *blast*
qed
then show *?A*
by *blast*
qed
then show *?thesis* **by** *meson*
qed
also have $\ldots = (L1 \preceq[X,quasired\ Y]\ L2)$
unfolding *quasired-type-1*[*OF assms*, *symmetric*] *quasi-reduction-def*
by (*meson assms*(2) *executable-inputs-in-alphabet outputs-executable*)
finally show *?thesis*
by *meson*
qed

6.3 Strong Reduction via Reduction and Undefinedness Outputs

fun *non-bottom-shortening* :: $('x,'y\ option)\ word \Rightarrow ('x,'y\ option)\ word$ **where**
$non\text{-}bottom\text{-}shortening\ \pi = filter\ (\lambda\ (x,y)\ .\ y \neq None)\ \pi$

fun *non-bottom-projection* :: $('x,'y\ option)\ word \Rightarrow ('x,'y)\ word$ **where**
$non\text{-}bottom\text{-}projection\ \pi = map\ (\lambda(x,y)\ .\ (x,the\ y))\ (non\text{-}bottom\text{-}shortening\ \pi)$

lemma *non-bottom-projection-split*: $non\text{-}bottom\text{-}projection\ (\pi'@\pi'') = (non\text{-}bottom\text{-}projection\ \pi')@(non\text{-}bottom\text{-}projection\ \pi'')$
by (*induction* π' *arbitrary*: π''; *auto*)

lemma *non-bottom-projection-id* : $non\text{-}bottom\text{-}projection\ (map\ (\lambda(x,y)\ .\ (x,Some\ y))\ \pi) = \pi$
by (*induction* π; *auto*)

fun *undefinedness-completion* :: *'x alphabet* ⇒ (*'x*,*'y*) *language* ⇒ (*'x*, *'y option*) *language* **where**
undefinedness-completion X L =
{π . *non-bottom-projection* π ∈ *L* ∧ (∀ π′ *x* π″ . π = π′ @ [(*x*,*None*)] @ π″ ⟶ *x* ∈ *X* ∧ *out*[*L*, *non-bottom-projection* π′, *x*] = {})}

lemma *undefinedness-completion-is-language* :
assumes *is-language X Y L*
shows *is-language X* ({*None*} ∪ *Some* ‘ *Y*) (*undefinedness-completion X L*)
proof −
let *?L* = *undefinedness-completion X L*

have [] ∈ *L*
using *language-contains-nil*[*OF assms*] **.**
moreover have *non-bottom-projection* [] = []
by *auto*
ultimately have [] ∈ *?L*
by *simp*
then have *?L* ≠ {}
by *blast*
moreover have ⋀ π . π ∈ *?L* ⟹ (⋀ *xy* . *xy* ∈ *set* π ⟹ *fst xy* ∈ *X* ∧ *snd xy* ∈ ({*None*} ∪ *Some* ‘ *Y*))
and ⋀ π . π ∈ *?L* ⟹ (⋀ π′ . *prefix* π′ π ⟹ π′ ∈ *?L*)
proof −
fix π **assume** π ∈ *?L*
then have *p1*: *non-bottom-projection* π ∈ *L*
and *p2*: ⋀ π′ *x* π″ . π = π′ @ [(*x*,*None*)] @ π″ ⟹ *x* ∈ *X* ∧ *out*[*L*, *non-bottom-projection* π′, *x*] = {}
by *auto*

show ⋀ *xy* . *xy* ∈ *set* π ⟹ *fst xy* ∈ *X* ∧ *snd xy* ∈ ({*None*} ∪ *Some* ‘ *Y*)
proof −
fix *xy* **assume** *xy* ∈ *set* π
then obtain π′ *x y* π″ **where** *xy* = (*x*,*y*) **and** π = π′ @ [(*x*,*y*)] @ π″
by (*metis append-Cons append-Nil old.prod.exhaust split-list*)

show *fst xy* ∈ *X* ∧ *snd xy* ∈ ({*None*} ∪ *Some* ‘ *Y*)
proof (*cases snd xy*)
case *None*
then show *?thesis*
unfolding ‹*xy* = (*x*,*y*)› *snd-conv*
using *p2* ‹π = π′ @ [(*x*,*y*)] @ π″›
by *simp*
next
case (*Some y*′)
then have *y* = *Some y*′

```
      unfolding ‹xy = (x,y)› by auto
    have (x,y') ∈ set (non-bottom-projection π)
      unfolding ‹π = π' @ [(x,y)] @ π''› ‹y = Some y'›
      by auto
    then show ?thesis
      unfolding ‹xy = (x,y)› snd-conv ‹y = Some y'› fst-conv
      using p1 assms
      unfolding is-language.simps by fastforce
  qed
 qed

 show ⋀ π' . prefix π' π ⟹ π' ∈ ?L
 proof −
  fix π' assume prefix π' π
  then obtain π'' where π = π'@π''
    using prefixE by blast

 have non-bottom-projection π = (non-bottom-projection π')@(non-bottom-projection
π'')
    unfolding ‹π = π'@π''›
    using non-bottom-projection-split .
  then have non-bottom-projection π' ∈ L
    by (metis assms p1 prefix-closure-no-member)
  moreover have ⋀ π''' x π'''' . π' = π''' @ [(x,None)] @ π'''' ⟹ x ∈ X ∧
out[L, non-bottom-projection π''', x] = {}
    using p2 unfolding ‹π = π'@π''›
    by (metis append.assoc)
  ultimately show π' ∈ ?L
    by fastforce
 qed
qed
 ultimately show ?thesis
  by (meson is-language.elims(3))
qed


lemma undefinedness-completion-inclusion :
 assumes π ∈ L
shows map (λ(x,y) . (x,Some y)) π ∈ undefinedness-completion X L
proof −
 let ?π = map (λ(x,y) . (x,Some y)) π

 have ⋀ a . (a,None) ∉ set ?π
  by (induction π; auto)
 then have ∀ π' x π'' . ?π = π' @ [(x,None)] @ π'' ⟶ x ∈ X ∧ out[L,
non-bottom-projection π', x] = {}
  by (metis Cons-eq-appendI in-set-conv-decomp)
 moreover have non-bottom-projection ?π ∈ L
  using ‹π ∈ L› unfolding non-bottom-projection-id .
```

```
  ultimately show ?thesis
    by auto
qed


lemma undefinedness-completion-out-shortening :
  assumes is-language X Y L
  and     π ∈ undefinedness-completion X L
  and     x ∈ X
shows out[undefinedness-completion X L, π, x] = out[undefinedness-completion X
L, non-bottom-shortening π, x]
using assms(2,3) proof (induction length π arbitrary: π x rule: less-induct)
  case less

  let ?L = undefinedness-completion X L

  show ?case proof (cases π rule: rev-cases)
    case Nil
    then show ?thesis by auto
  next
    case (snoc π' xy)

    then obtain x' y' where xy = (x',y') by fastforce

    have x' ∈ X
      using snoc less.prems(1) unfolding ‹xy = (x',y')›
      using undefinedness-completion-is-language[OF assms(1)]
      by (metis fst-conv is-language.elims(2) last-in-set snoc-eq-iff-butlast)

    have π' ∈ ?L
      using snoc less.prems(1)
      using undefinedness-completion-is-language[OF assms(1)]
      using prefix-closure-no-member by blast

    show ?thesis proof (cases y')
      case None

      then have non-bottom-shortening π = non-bottom-shortening π'
        unfolding ‹xy = (x',y')› snoc by auto
    then have out[?L, non-bottom-shortening π, x] = out[?L, non-bottom-shortening
π', x]
        by simp
      also have ... = out[?L, π', x]
        using less.hyps[OF - ‹π' ∈ ?L› ‹x ∈ X›] unfolding snoc
        by (metis Suc-lessD length-append-singleton not-less-eq)
      also have ... = out[?L, π, x]
      proof

        show out[?L, π', x] ⊆ out[?L, π, x]
```

proof
fix y **assume** $y \in out[?L, \pi', x]$
then have $\pi'@[(x,y)] \in ?L$
by *auto*
then have *p1*: *non-bottom-projection* $(\pi'@[(x,y)]) \in L$
and *p2*: $\bigwedge \gamma'\ a\ \gamma''\ .\ \pi'@[(x,y)] = \gamma' @ [(a,None)] @ \gamma'' \Longrightarrow a \in X \wedge out[L,$ *non-bottom-projection* $\gamma', a] = \{\}$
by *auto*

have *non-bottom-projection* $(\pi@[(x,y)]) =$ *non-bottom-projection* $(\pi'@[(x,y)])$
unfolding *snoc* ‹$xy = (x',y')$› *None* **by** *auto*
then have *non-bottom-projection* $(\pi@[(x,y)]) \in L$
using *p1* **by** *simp*
moreover have $\bigwedge \gamma'\ a\ \gamma''\ .\ \pi@[(x,y)] = \gamma' @ [(a,None)] @ \gamma'' \Longrightarrow a \in X \wedge out[L,$ *non-bottom-projection* $\gamma', a] = \{\}$
proof −
fix $\gamma'\ a\ \gamma''$ **assume** $\pi@[(x,y)] = \gamma' @ [(a,None)] @ \gamma''$
then have $\pi'@[(x',None)]@[(x,y)] = \gamma' @ [(a,None)] @ \gamma''$
unfolding *snoc* ‹$xy = (x',y')$› *None* **by** *auto*

show $a \in X \wedge out[L,$ *non-bottom-projection* $\gamma', a] = \{\}$
proof (*cases* γ'' *rule*: *rev-cases*)
case *Nil*
then show *?thesis*
using ‹$\pi\ @\ [(x,\ y)] = \gamma' @ [(a,\ None)] @ \gamma''$› ‹*non-bottom-shortening* $\pi =$ *non-bottom-shortening* π'› *p2* **by** *auto*
next
case (*snoc* $\gamma'''\ xy'$)
then show *?thesis*
using ‹$\pi\ @\ [(x,\ y)] = \gamma' @ [(a,\ None)] @ \gamma''$› *less.prems*(*1*) **by** *force*
qed
qed
ultimately show $y \in out[?L, \pi, x]$
by *auto*
qed

show $out[?L, \pi, x] \subseteq out[?L, \pi', x]$
proof
fix y **assume** $y \in out[?L, \pi, x]$
then have $\pi'@[(x',None)]@[(x,y)] \in ?L$
unfolding *snoc* ‹$xy = (x',y')$› *None*
by *auto*
then have *p1*: *non-bottom-projection* $(\pi'@[(x',None)]@[(x,y)]) \in L$
and *p2*: $\bigwedge \gamma'\ a\ \gamma''\ .\ \pi'@[(x',None)]@[(x,y)] = \gamma' @ [(a,None)] @ \gamma'' \Longrightarrow a \in X \wedge out[L,$ *non-bottom-projection* $\gamma', a] = \{\}$
by *auto*

have *non-bottom-projection* $(\pi'@[(x',None)]@[(x,y)]) =$ *non-bottom-projection* $(\pi'@[(x,y)])$

```
          by auto
        then have non-bottom-projection (π'@[(x,y)]) ∈ L
          using p1 by auto
        moreover have ⋀ γ' a γ'' . π'@[(x,y)] = γ' @ [(a,None)] @ γ'' ⟹ a ∈
X ∧ out[L, non-bottom-projection γ', a] = {}
        proof −
          fix γ' a γ'' assume π'@[(x,y)] = γ' @ [(a,None)] @ γ''

          show a ∈ X ∧ out[L, non-bottom-projection γ', a] = {}
          proof (cases γ'' rule: rev-cases)
            case Nil
            then show ?thesis
                    by (metis None ‹π' @ [(x, y)] = γ' @ [(a, None)] @ γ''›
‹non-bottom-shortening π = non-bottom-shortening π'› ‹xy = (x', y')› append.assoc
append.right-neutral append1-eq-conv non-bottom-projection.simps p2 snoc)
          next
            case (snoc γ''' xy')
            then show ?thesis
                 using ‹π' @ [(x, y)] = γ' @ [(a, None)] @ γ''› ‹π' ∈ undefined-
ness-completion X L› by force
          qed
        qed
        ultimately show y ∈ out[?L, π', x]
          by auto
      qed
    qed
    finally show ?thesis
      by blast
  next
    case (Some y'')

    have non-bottom-shortening π = (non-bottom-shortening π')@[(x',Some y'')]
      unfolding snoc ‹xy = (x',y')› Some by auto
    then have non-bottom-projection π = (non-bottom-projection π')@[(x',y'')]
      by auto

    have π' @ [(x', Some y'')] ∈ ?L
      using less.prems(1) unfolding snoc ‹xy = (x',y')› Some .
    then have Some y'' ∈ out[?L,π',x']
      by auto
    moreover have out[?L,π',x'] = out[?L,non-bottom-shortening π',x']
      using less.hyps[OF - ‹π' ∈ ?L› ‹x' ∈ X›]
      unfolding snoc ‹xy = (x',y')› Some
      by (metis length-append-singleton lessI)
    ultimately have Some y'' ∈ out[?L,non-bottom-shortening π',x']
      by blast


    show ?thesis
```

proof
show *out*[*?L*,*π*,*x*] ⊆ *out*[*?L*,*non-bottom-shortening π*,*x*]
proof
fix *y* **assume** *y* ∈ *out*[*?L*,*π*,*x*]
then have *π′*@[(*x′*,*Some y″*)]@[(*x*,*y*)] ∈ *?L*
unfolding *snoc* ‹*xy* = (*x′*,*y′*)› *Some* **by** *auto*
then have *p1*: *non-bottom-projection* (*π′*@[(*x′*,*Some y″*)]@[(*x*,*y*)]) ∈ *L*
and *p2*: ⋀ *γ′ a γ″* . *π′*@[(*x′*,*Some y″*)]@[(*x*,*y*)] = *γ′* @ [(*a*,*None*)] @ *γ″* ⟹ *a* ∈ *X* ∧ *out*[*L*, *non-bottom-projection γ′*, *a*] = {}
by *auto*

have *non-bottom-projection* ((*non-bottom-shortening π*)@[(*x*,*y*)]) = *non-bottom-projection* (*π′*@[(*x′*,*Some y″*)]@[(*x*,*y*)])
unfolding ‹*non-bottom-shortening π* = (*non-bottom-shortening π′*)@[(*x′*,*Some y″*)]›
by *auto*
then have *non-bottom-projection* ((*non-bottom-shortening π*)@[(*x*,*y*)]) ∈ *L*
using *p1* **by** *simp*
moreover have ⋀ *γ′ a γ″* . (*non-bottom-shortening π*)@[(*x*,*y*)] = *γ′* @ [(*a*,*None*)] @ *γ″* ⟹ *a* ∈ *X* ∧ *out*[*L*, *non-bottom-projection γ′*, *a*] = {}
proof −
fix *γ′ a γ″* **assume** (*non-bottom-shortening π*)@[(*x*,*y*)] = *γ′* @ [(*a*,*None*)] @ *γ″*
moreover have (*a*, *None*) ∉ *set* (*non-bottom-shortening π*)
by (*induction π*; *auto*)
moreover have ⋀ *xs a ys b zs* . *xs*@[*a*] = *ys*@[*b*]@*zs* ⟹ *b* ∉ *set xs* ⟹ *zs* = []
by (*metis append-Cons append-Nil butlast.simps*(*2*) *butlast-snoc in-set-butlast-appendI list.distinct*(*1*) *list.sel*(*1*) *list.set-sel*(*1*))
ultimately have *γ″* = []
by *fastforce*
then have *γ′* = *non-bottom-shortening π*
and *x* = *a*
and *y* = *None*
using ‹(*non-bottom-shortening π*)@[(*x*,*y*)] = *γ′* @ [(*a*,*None*)] @ *γ″*›
by *auto*

show *a* ∈ *X* ∧ *out*[*L*, *non-bottom-projection γ′*, *a*] = {}
using ‹*x* ∈ *X*› **unfolding** ‹*x* = *a*›
unfolding ‹*γ′* = *non-bottom-shortening π*›
by (*metis* (*no-types*, *lifting*) ‹*non-bottom-projection* (*non-bottom-shortening π* @ [(*x*, *y*)]) = *non-bottom-projection* (*π′* @ [(*x′*, *Some y″*)] @ [(*x*, *y*)])› ‹*x* = *a*› ‹*y* = *None*› *append.assoc append.right-neutral append-same-eq non-bottom-projection-split p2*)
qed
ultimately show *y* ∈ *out*[*?L*,*non-bottom-shortening π*,*x*]
by *auto*
qed

```
      show out[?L,non-bottom-shortening π,x] ⊆ out[?L,π,x]
      proof
        fix y assume y ∈ out[?L,non-bottom-shortening π,x]
        then have (non-bottom-shortening π')@[(x',Some y'')]@[(x,y)] ∈ ?L
          unfolding snoc ‹xy = (x',y')› Some by auto
      then have p1: non-bottom-projection ((non-bottom-shortening π')@[(x',Some
y'')]@[(x,y)]) ∈ L
          and p2: ⋀ γ' a γ'' . (non-bottom-shortening π')@[(x',Some y'')]@[(x,y)]
= γ' @ [(a,None)] @ γ'' ⟹ a ∈ X ∧ out[L, non-bottom-projection γ', a] = {}
          by auto

            have non-bottom-projection ((non-bottom-shortening π')@[(x',Some
y'')]@[(x,y)]) = non-bottom-projection (π@[(x,y)])
          unfolding snoc ‹xy = (x',y')› Some by auto
        then have non-bottom-projection (π@[(x,y)]) ∈ L
          using p1 by presburger
        moreover have ⋀ γ' a γ'' . π@[(x,y)] = γ' @ [(a,None)] @ γ'' ⟹ a ∈
X ∧ out[L, non-bottom-projection γ', a] = {}
        proof
          fix γ' a γ'' assume π@[(x,y)] = γ' @ [(a,None)] @ γ''
          then have (a,None) ∈ set (π@[(x,y)])
            by auto
          then consider (a,None) ∈ set π | (a,None) = (x,y)
            by auto
          then show a ∈ X
                by (metis assms(1) fst-conv is-language.elims(2) less.prems(1)
less.prems(2) undefinedness-completion-is-language)

          show out[L,non-bottom-projection γ',a] = {}
          proof (cases γ'' rule: rev-cases)
            case Nil
            then have π = γ' and x = a and y = None
              using ‹π@[(x,y)] = γ' @ [(a,None)] @ γ''› by auto
            then show ?thesis
                    by (metis (no-types, opaque-lifting) ‹non-bottom-projection
(non-bottom-shortening π' @ [(x', Some y'')] @ [(x, y)]) = non-bottom-projection
(π @ [(x, y)])› ‹non-bottom-shortening π = non-bottom-shortening π' @ [(x', Some
y'')]› append.assoc append-Cons append-Nil append-same-eq non-bottom-projection-split
p2)
          next
            case (snoc γ''' xy')
            then have π = γ' @ [(a, None)] @ γ'''
              using ‹π@[(x,y)] = γ' @ [(a,None)] @ γ''› by auto

            have γ' @ [(a, None)] ∈ ?L
              using less.prems(1) unfolding ‹π = γ' @ [(a, None)] @ γ'''›
              using undefinedness-completion-is-language[OF assms(1)]
              by (metis append-assoc prefix-closure-no-member)
```

```
        then show out[L, non-bottom-projection γ', a] = {}
          by auto
      qed
    qed
    ultimately show y ∈ out[?L,π,x]
      by auto
  qed
 qed
qed
```

```
lemma undefinedness-completion-out-projection-not-empty :
  assumes is-language X Y L
  and     π ∈ undefinedness-completion X L
  and     x ∈ X
  and     out[L, non-bottom-projection π, x] ≠ {}
shows out[undefinedness-completion X L, non-bottom-shortening π, x] = Some ‘
out[L, non-bottom-projection π, x]
proof

  let ?L = undefinedness-completion X L

  have π@[(x,None)] ∉ ?L
    using assms(4) by auto
  then have None ∉ out[?L,π,x]
    by auto
  then have None ∉ out[?L,non-bottom-shortening π,x]
    using undefinedness-completion-out-shortening[OF assms(1,2,3)] by blast
  then have (non-bottom-shortening π)@[(x,None)] ∉ ?L
    by auto

  show out[?L, non-bottom-shortening π, x] ⊆ Some ‘ out[L, non-bottom-projection
π, x]
  proof
    fix y assume y ∈ out[?L, non-bottom-shortening π, x]
    then have (non-bottom-shortening π) @ [(x,y)] ∈ ?L by auto
    then have y ≠ None
      using ‹(non-bottom-shortening π)@[(x,None)] ∉ ?L›
      by meson
    then obtain y' where y = Some y'
      by auto

   have non-bottom-projection ((non-bottom-shortening π) @ [(x,y)]) = (non-bottom-projection
π) @ [(x,y')]
      unfolding ‹y = Some y'›
      by (induction π; auto)
```

```
    then have (non-bottom-projection π) @ [(x,y')] ∈ L
      using ‹(non-bottom-shortening π) @ [(x,y)] ∈ ?L› unfolding ‹y = Some y'›
      by auto
    then show y ∈ Some ‘ out[L, non-bottom-projection π, x]
      unfolding ‹y = Some y'› by auto
  qed

  show Some ‘ out[L,non-bottom-projection π,x] ⊆ out[?L,non-bottom-shortening
π,x]
  proof
    fix y assume y ∈ Some ‘ out[L,non-bottom-projection π,x]
   then obtain y' where y = Some y' and y' ∈ out[L,non-bottom-projection π,x]
      by auto
    then have (non-bottom-projection π) @ [(x,y')] ∈ L
      by auto
     moreover have non-bottom-projection ((non-bottom-shortening π) @ [(x,y)])
= (non-bottom-projection π) @ [(x,y')]
      unfolding ‹y = Some y'›
      by (induction π; auto)
    ultimately have non-bottom-projection ((non-bottom-shortening π) @ [(x,y)])
∈ L
      unfolding ‹y = Some y'›
      by auto
     moreover have ⋀ π' x' π'' . ((non-bottom-shortening π) @ [(x,y)]) = π' @
[(x',None)] @ π'' ⟹ x' ∈ X ∧ out[L, non-bottom-projection π', x'] = {}
    proof −
     fix π' x' π'' assume ((non-bottom-shortening π) @ [(x,y)]) = π' @ [(x',None)]
@ π''
      then have (x',None) ∈ set (non-bottom-shortening π)
        by (metis ‹y = Some y'› append-Cons in-set-conv-decomp old.prod.inject
option.distinct(1) rotate1.simps(2) set-ConsD set-rotate1)
      then have False
        by (induction π; auto)
      then show x' ∈ X ∧ out[L, non-bottom-projection π', x'] = {}
        by blast
    qed
    ultimately show y ∈ out[?L,non-bottom-shortening π,x]
      by auto
  qed
qed


lemma undefinedness-completion-out-projection-empty :
  assumes is-language X Y L
  and     π ∈ undefinedness-completion X L
  and     x ∈ X
  and     out[L, non-bottom-projection π, x] = {}
shows out[undefinedness-completion X L, non-bottom-shortening π, x] = {None}
proof
```

let *?L = undefinedness-completion X L*

have *p1*: *non-bottom-projection* $\pi \in L$
and *p2*: $\bigwedge \pi'\ x\ \pi''\ .\ \pi = \pi' \mathbin{@} [(x,None)] \mathbin{@} \pi'' \Longrightarrow x \in X \wedge$ *out*[*L*, *non-bottom-projection* π', *x*] = {}
using *assms*(*2*) **by** *auto*

have *non-bottom-projection* $(\pi@[(x,None)]) \in L$
using *p1* **by** *auto*
moreover have $\bigwedge \pi'\ x'\ \pi''\ .\ \pi@[(x,None)] = \pi' \mathbin{@} [(x',None)] \mathbin{@} \pi'' \Longrightarrow x' \in X \wedge$ *out*[*L*, *non-bottom-projection* π', x'] = {}
proof −
fix $\pi'\ x'\ \pi''$ **assume** $\pi@[(x,None)] = \pi' \mathbin{@} [(x',None)] \mathbin{@} \pi''$
show $x' \in X \wedge$ *out*[*L*, *non-bottom-projection* π', x'] = {}
proof (*cases* π'' *rule*: *rev-cases*)
case *Nil*
then show *?thesis*
using ‹$\pi \mathbin{@} [(x, None)] = \pi' \mathbin{@} [(x', None)] \mathbin{@} \pi''$› *assms*(*3*) *assms*(*4*) **by** *auto*
next
case (*snoc ys y*)
then show *?thesis*
using ‹$\pi \mathbin{@} [(x, None)] = \pi' \mathbin{@} [(x', None)] \mathbin{@} \pi''$› *p2* **by** *auto*
qed
qed
ultimately have $\pi@[(x,None)] \in$ *?L*
by *auto*
then show {*None*} ⊆ *out*[*?L*,*non-bottom-shortening* π,*x*]
unfolding *undefinedness-completion-out-shortening*[*OF assms*(*1*,*2*,*3*), *symmetric*]
by *auto*

show *out*[*?L*,*non-bottom-shortening* π,*x*] ⊆ {*None*}
proof (*rule ccontr*)
assume ¬ *out*[*?L*,*non-bottom-shortening* π,*x*] ⊆ {*None*}
then obtain *y* **where** $y \in$ *out*[*?L*,*non-bottom-shortening* π,*x*] **and** $y \neq$ *None*
by *blast*
then obtain y' **where** $y =$ *Some* y'
by *auto*

have $\pi@[(x,Some\ y')] \in$ *?L*
using ‹$y \in$ *out*[*?L*,*non-bottom-shortening* π,*x*]›
unfolding ‹$y =$ *Some* y'›
unfolding *undefinedness-completion-out-shortening*[*OF assms*(*1*,*2*,*3*), *symmetric*]
by *auto*
then have (*non-bottom-projection* π)@[(*x*,y')] ∈ *L*
by *auto*

then show *False*
using *assms(4)* **by** *auto*
qed
qed

theorem *strongred-via-red* :
assumes *is-language X Y L1*
and *is-language X Y L2*
shows (*L1* $\preceq$[*X*,*strongred Y*] *L2*) $\longleftrightarrow$ ((*undefinedness-completion X L1*) $\preceq$[*X*, *red* ({*None*} $\cup$ *Some* ‘ *Y*)] (*undefinedness-completion X L2*))
proof −

let *?L1* = *undefinedness-completion X L1*
let *?L2* = *undefinedness-completion X L2*

have (*L1* $\preceq$[*X*,*strongred Y*] *L2*) = ($\forall$ π $\in$ *L1* $\cap$ *L2* . $\forall$ *x* $\in$ *X* . (*out*[*L1*,π,*x*] = {} $\wedge$ *out*[*L2*,π,*x*] = {}) $\vee$ (*out*[*L1*,π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,π,*x*] $\subseteq$ *out*[*L2*,π,*x*]))
(**is** *?A* = *?B*)
proof
show *?A* $\Longrightarrow$ *?B*
unfolding *strongred-type-1*[*OF assms*, *symmetric*] *strong-reduction-def quasi-reduction-def*
by (*metis outputs-executable*)
show *?B* $\Longrightarrow$ *?A*
unfolding *strongred-type-1*[*OF assms*, *symmetric*] *strong-reduction-def quasi-reduction-def*
by (*metis assms(1) assms(2) executable-inputs-in-alphabet outputs-executable subset-empty*)
qed
also have ... = ($\forall$ π $\in$ *?L1* $\cap$ *?L2* . $\forall$ *x* $\in$ *X* . (*out*[*L1*,*non-bottom-projection* π,*x*] = {} $\wedge$ *out*[*L2*,*non-bottom-projection* π,*x*] = {}) $\vee$ (*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*non-bottom-projection* π,*x*] $\subseteq$ *out*[*L2*,*non-bottom-projection* π,*x*]))
(**is** *?A* = *?B*)
proof
have $\bigwedge$ π *x* . *?A* $\Longrightarrow$ π $\in$ *?L1* $\cap$ *?L2* $\Longrightarrow$ *x* $\in$ *X* $\Longrightarrow$ (*out*[*L1*,*non-bottom-projection* π,*x*] = {} $\wedge$ *out*[*L2*,*non-bottom-projection* π,*x*] = {}) $\vee$ (*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*non-bottom-projection* π,*x*] $\subseteq$ *out*[*L2*,*non-bottom-projection* π,*x*])
proof −
fix π *x* **assume** *?A* **and** π $\in$ *?L1* $\cap$ *?L2* **and** *x* $\in$ *X*

let *?*π = *non-bottom-projection* π

have *?*π $\in$ *L1*
and *?*π $\in$ *L2*
using ‹π $\in$ *?L1* $\cap$ *?L2*› **by** *auto*
then show (*out*[*L1*,*?*π,*x*] = {} $\wedge$ *out*[*L2*,*?*π,*x*] = {}) $\vee$ (*out*[*L1*,*?*π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*?*π,*x*] $\subseteq$ *out*[*L2*,*?*π,*x*])
using ‹*?A*› ‹*x* $\in$ *X*› **by** *blast*

qed
then show *?A* $\Longrightarrow$ *?B*
by *blast*

have $\bigwedge$ π *x* . *?B* $\Longrightarrow$ π $\in$ *L1* $\cap$ *L2* $\Longrightarrow$ *x* $\in$ *X* $\Longrightarrow$ (*out*[*L1*,π,*x*] = {} $\wedge$ *out*[*L2*,π,*x*] = {}) $\vee$ (*out*[*L1*,π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,π,*x*] $\subseteq$ *out*[*L2*,π,*x*])
proof −
fix π *x* **assume** *?B* **and** π $\in$ *L1* $\cap$ *L2* **and** *x* $\in$ *X*

let *?*π = *map* (λ(*x*,*y*) . (*x*,*Some y*)) π

have *?*π $\in$ *?L1* **and** *?*π $\in$ *?L2*
using ‹π $\in$ *L1* $\cap$ *L2*› *undefinedness-completion-inclusion* **by** *blast+*
then have (*out*[*L1*,*non-bottom-projection ?*π,*x*] = {} $\wedge$ *out*[*L2*,*non-bottom-projection ?*π,*x*] = {}) $\vee$ (*out*[*L1*,*non-bottom-projection ?*π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*non-bottom-projection ?*π,*x*] $\subseteq$ *out*[*L2*,*non-bottom-projection ?*π,*x*])
using ‹*?B*› ‹*x* $\in$ *X*› **by** *blast*
then show (*out*[*L1*,π,*x*] = {} $\wedge$ *out*[*L2*,π,*x*] = {}) $\vee$ (*out*[*L1*,π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,π,*x*] $\subseteq$ *out*[*L2*,π,*x*])
unfolding *non-bottom-projection-id* **.**
qed
then show *?B* $\Longrightarrow$ *?A*
by *blast*
qed
also have ... = ($\forall$ π $\in$ *?L1* $\cap$ *?L2* . $\forall$ *x* $\in$ *X* . (*out*[*?L1*,π,*x*] = {*None*} $\wedge$ *out*[*?L2*,π,*x*] = {*None*}) $\vee$ (*out*[*?L1*,π,*x*] $\neq$ {*None*} $\wedge$ *out*[*?L1*,π,*x*] $\subseteq$ *out*[*?L2*,π,*x*]))
proof −
have $\bigwedge$ π *x* . π $\in$ *?L1* $\cap$ *?L2* $\Longrightarrow$ *x* $\in$ *X* $\Longrightarrow$ (*out*[*L1*,*non-bottom-projection* π,*x*] = {} $\wedge$ *out*[*L2*,*non-bottom-projection* π,*x*] = {}) = (*out*[*?L1*,π,*x*] = {*None*} $\wedge$ *out*[*?L2*,π,*x*] = {*None*})
by (*metis IntD1 IntD2 None-notin-image-Some assms*(*1*) *assms*(*2*) *insertCI undefinedness-completion-out-projection-empty undefinedness-completion-out-projection-not-empty undefinedness-completion-out-shortening*)
moreover have $\bigwedge$ π *x* . π $\in$ *?L1* $\cap$ *?L2* $\Longrightarrow$ *x* $\in$ *X* $\Longrightarrow$ (*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*non-bottom-projection* π,*x*] $\subseteq$ *out*[*L2*,*non-bottom-projection* π,*x*]) = (*out*[*?L1*,π,*x*] $\neq$ {*None*} $\wedge$ *out*[*?L1*,π,*x*] $\subseteq$ *out*[*?L2*,π,*x*])
proof −
fix π *x* **assume** π $\in$ *?L1* $\cap$ *?L2* **and** *x* $\in$ *X*
then have π $\in$ *?L1* **and** π $\in$ *?L2* **by** *auto*

have (*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {}) = (*out*[*?L1*,π,*x*] $\neq$ {*None*})
by (*metis None-notin-image-Some* ‹π $\in$ *undefinedness-completion X L1*› ‹*x* $\in$ *X*› *assms*(*1*) *singletonI undefinedness-completion-out-projection-empty undefinedness-completion-out-projection-not-empty undefinedness-completion-out-shortening*)

show (*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {} $\wedge$ *out*[*L1*,*non-bottom-projection* π,*x*] $\subseteq$ *out*[*L2*,*non-bottom-projection* π,*x*]) = (*out*[*?L1*,π,*x*] $\neq$ {*None*} $\wedge$ *out*[*?L1*,π,*x*] $\subseteq$ *out*[*?L2*,π,*x*])

proof (*cases out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {})
case *False*
then show *?thesis* **using** ‹(*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {}) = (*out*[*?L1*,π,*x*] $\neq$ {*None*})› **by** *blast*
next
case *True*
have *out*[*undefinedness-completion X L1*,π,*x*] = *Some* ‘ *out*[*L1*,*non-bottom-projection* π,*x*]
using *undefinedness-completion-out-projection-not-empty*[*OF assms*(*1*) ‹π $\in$ *?L1*› ‹*x* $\in$ *X*› *True*]
unfolding *undefinedness-completion-out-shortening*[*OF assms*(*1*) ‹π $\in$ *?L1*› ‹*x* $\in$ *X*›,*symmetric*] **.**

show *?thesis* **proof** (*cases out*[*L2*,*non-bottom-projection* π,*x*] = {})
case *True*
then show *?thesis*
by (*metis* ‹(*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {}) = (*out*[*undefinedness-completion X L1*,π,*x*] $\neq$ {*None*})› ‹π $\in$ *undefinedness-completion X L2*› ‹*out*[*undefinedness-completion X L1*,π,*x*] = *Some* ‘ *out*[*L1*,*non-bottom-projection* π,*x*]› ‹*x* $\in$ *X*› *assms*(*2*) *image-is-empty subset-empty subset-singletonD undefinedness-completion-out-projection-empty undefinedness-completion-out-shortening*)
next
case *False*

have *out*[*undefinedness-completion X L2*,π,*x*] = *Some* ‘ *out*[*L2*,*non-bottom-projection* π,*x*]
using *undefinedness-completion-out-projection-not-empty*[*OF assms*(*2*) ‹π $\in$ *?L2*› ‹*x* $\in$ *X*› *False*]
unfolding *undefinedness-completion-out-shortening*[*OF assms*(*2*) ‹π $\in$ *?L2*› ‹*x* $\in$ *X*›,*symmetric*] **.**

show *?thesis*
unfolding ‹*out*[*undefinedness-completion X L1*,π,*x*] = *Some* ‘ *out*[*L1*,*non-bottom-projection* π,*x*]›
unfolding ‹*out*[*undefinedness-completion X L2*,π,*x*] = *Some* ‘ *out*[*L2*,*non-bottom-projection* π,*x*]›
by (*metis* ‹(*out*[*L1*,*non-bottom-projection* π,*x*] $\neq$ {}) = (*out*[*undefinedness-completion X L1*,π,*x*] $\neq$ {*None*})› ‹*out*[*undefinedness-completion X L1*,π,*x*] = *Some* ‘ *out*[*L1*,*non-bottom-projection* π,*x*]› *subset-image-iff these-image-Some-eq*)
qed
qed
qed
ultimately show *?thesis*
by *meson*
qed
also have ... = ($\forall$ π $\in$ *?L1* $\cap$ *?L2* . $\forall$ *x* $\in$ *X* . *out*[*?L1*,π,*x*] $\subseteq$ *out*[*?L2*,π,*x*])
(**is** *?A* = *?B*)
proof

show *?A* $\Longrightarrow$ *?B*
by *blast*
show *?B* $\Longrightarrow$ *?A*
by (*metis IntD2 None-notin-image-Some assms*(*2*) *insert-subset undefinedness-completion-out-projection-empty undefinedness-completion-out-projection-not-empty undefinedness-completion-out-shortening*)
qed
also have ... = (*?L1* $\preceq$[*X*, *red* ({*None*} $\cup$ *Some* ‘ *Y*)] *?L2*)
unfolding *type-1-conforms.simps red.simps*
using *outputs-in-alphabet*[*OF undefinedness-completion-is-language*[*OF assms*(*2*)]]
by *force*
finally show *?thesis* **.**
qed

end

References

[1] W.-l. Huang and R. Sachtleben. *Conformance Relations Between Input/Output Languages*, pages 49–67. Springer Nature Switzerland, Cham, 2023.