

# The Hales–Jewett Theorem

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## Abstract

This article is a formalisation of a proof of the Hales–Jewett theorem presented in the textbook *Ramsey Theory* by Graham et al. [1].

The Hales–Jewett theorem is a result in Ramsey Theory which states that, for any non-negative integers  $r$  and  $t$ , there exists a minimal dimension  $N$ , such that any  $r$ -coloured  $N'$ -dimensional cube over  $t$  elements (with  $N' \geq N$ ) contains a monochromatic line. This theorem generalises Van der Waerden’s Theorem, which has already been formalised in another AFP entry [2].

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```

theory Hales-Jewett
  imports Main HOL-Library.Disjoint-Sets HOL-Library.FuncSet
begin

```

## 1 Preliminaries

The Hales–Jewett Theorem is at its core a statement about sets of tuples called the  $n$ -dimensional cube over  $t$  elements (denoted by  $C_t^n$ ); i.e. the set  $\{0, \dots, t-1\}^n$ , where  $\{0, \dots, t-1\}$  is called the base. We represent tuples by functions  $f : \{0, \dots, n-1\} \rightarrow \{0, \dots, t-1\}$  because they're easier to deal with. The set of tuples then becomes the function space  $\{0, \dots, t-1\}^{\{0, \dots, n-1\}}$ . Furthermore,  $r$ -colourings of the cube are represented by mappings from the function space to the set  $\{0, \dots, r-1\}$ .

### 1.1 The $n$ -dimensional cube over $t$ elements

Function spaces in Isabelle are supported by the library component FuncSet. In essence,  $f \in A \rightarrow_E B$  means  $a \in A \implies f a \in B$  and  $a \notin A \implies f a = \text{undefined}$

The (canonical)  $n$ -dimensional cube over  $t$  elements is defined in the following using the variables:

```

n:  nat  dimension
t:  nat  number of elements

```

**definition** *cube* :: *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  (*nat*  $\Rightarrow$  *nat*) *set*  
**where** *cube* *n* *t*  $\equiv$   $\{..<n\} \rightarrow_E \{..<t\}$

For any function  $f$  whose image under a set  $A$  is a subset of another set  $B$ , there's a unique function  $g$  in the function space  $B^A$  that equals  $f$  everywhere in  $A$ . The function  $g$  is usually written as  $f|_A$  in the mathematical literature.

**lemma** *PiE-uniqueness*:  $f \text{ ' } A \subseteq B \implies \exists! g \in A \rightarrow_E B. \forall a \in A. g a = f a$   
*<proof>*

Any prefix of length  $j$  of an  $n$ -tuple (i.e. element of  $C_t^n$ ) is a  $j$ -tuple (i.e. element of  $C_t^j$ ).

**lemma** *cube-restrict*:  
**assumes**  $j < n$   
**and**  $y \in \text{cube } n \ t$   
**shows**  $(\lambda g \in \{..<j\}. y \ g) \in \text{cube } j \ t$  *<proof>*

Narrowing down the obvious fact  $B^A \subseteq C^A$  if  $B \subseteq C$  to a specific case for cubes.

**lemma** *cube-subset*:  $\text{cube } n \ t \subseteq \text{cube } n \ (t + 1)$

*<proof>*

A simplifying definition for the 0-dimensional cube.

**lemma** *cube0-alt-def*:  $\text{cube } 0 \ t = \{\lambda x. \text{undefined}\}$   
*<proof>*

The cardinality of the  $n$ -dimensional over  $t$  elements is simply a consequence of the overarching definition of the cardinality of function spaces (over finite sets).

**lemma** *cube-card*:  $\text{card} (\{..<n::\text{nat}\} \rightarrow_E \{..<t::\text{nat}\}) = t \wedge n$   
*<proof>*

A simplifying definition for the  $n$ -dimensional cube over a single element, i.e. the single  $n$ -dimensional point  $(0, \dots, 0)$ .

**lemma** *cube1-alt-def*:  $\text{cube } n \ 1 = \{\lambda x \in \{..<n\}. 0\}$  *<proof>*

## 1.2 Lines

The property of being a line in  $C_t^n$  is defined in the following using the variables:

$L$ :  $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$  line  
 $n$ :  $\text{nat}$  dimension of cube  
 $t$ :  $\text{nat}$  the size of the cube's base

**definition** *is-line* ::  $(\text{nat} \Rightarrow (\text{nat} \Rightarrow \text{nat})) \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{bool}$

**where** *is-line*  $L \ n \ t \equiv (L \in \{..<t\} \rightarrow_E \text{cube } n \ t \wedge$   
 $((\forall j < n. (\forall x < t. \forall y < t. L \ x \ j = L \ y \ j) \vee (\forall s < t. L \ s \ j = s))$   
 $\wedge (\exists j < n. (\forall s < t. L \ s \ j = s))))$

We introduce an elimination rule to relate lines with the more general definition of a subspace (see below).

**lemma** *is-line-elim-t-1*:

**assumes** *is-line*  $L \ n \ t$  **and**  $t = 1$

**obtains**  $B_0 \ B_1$

**where**  $B_0 \cup B_1 = \{..<n\} \wedge B_0 \cap B_1 = \{\}$   $\wedge$   
 $B_0 \neq \{\} \wedge (\forall j \in B_1. (\forall x < t. \forall y < t. L \ x \ j = L \ y$   
 $j)) \wedge (\forall j \in B_0. (\forall s < t. L \ s \ j = s))$

*<proof>*

The next two lemmas are used to simplify proofs by enabling us to use the resulting facts directly. This avoids having to unfold the definition of *is-line* each time.

**lemma** *line-points-in-cube*:

**assumes** *is-line*  $L \ n \ t$

**and**  $s < t$

**shows**  $L \ s \in \text{cube } n \ t$

*<proof>*

**lemma** *line-points-in-cube-unfolded*:

**assumes** *is-line*  $L$   $n$   $t$

**and**  $s < t$

**and**  $j < n$

**shows**  $L$   $s$   $j \in \{..<t\}$

*<proof>*

The incrementation of all elements of a set is defined in the following using the variables:

$n$ : *nat* increment size

$S$ : *nat set* set

**definition** *set-incr* ::  $nat \Rightarrow nat\ set \Rightarrow nat\ set$

**where**

$set-incr\ n\ S \equiv (\lambda a. a + n) ' S$

**lemma** *set-incr-disjnt*:

**assumes** *disjnt*  $A$   $B$

**shows** *disjnt* ( $set-incr\ n\ A$ ) ( $set-incr\ n\ B$ )

*<proof>*

**lemma** *set-incr-disjoint-family*:

**assumes** *disjoint-family-on*  $B$   $\{..k\}$

**shows** *disjoint-family-on*  $(\lambda i. set-incr\ n\ (B\ i))$   $\{..k\}$

*<proof>*

**lemma** *set-incr-altdef*:  $set-incr\ n\ S = (+)\ n\ ' S$

*<proof>*

**lemma** *set-incr-image*:

**assumes**  $(\bigcup i \in \{..k\}. B\ i) = \{..<n\}$

**shows**  $(\bigcup i \in \{..k\}. set-incr\ m\ (B\ i)) = \{m..<m+n\}$

*<proof>*

Each tuple of dimension  $k + 1$  can be split into a tuple of dimension 1 (the first entry) and a tuple of dimension  $k$  (the remaining entries).

**lemma** *split-cube*:

**assumes**  $x \in cube\ (k+1)\ t$

**shows**  $(\lambda y \in \{..<1\}. x\ y) \in cube\ 1\ t$

**and**  $(\lambda y \in \{..<k\}. x\ (y + 1)) \in cube\ k\ t$

*<proof>*

### 1.3 Subspaces

The property of being a  $k$ -dimensional subspace of  $C_t^n$  is defined in the following using the variables:

$S$ :  $(nat \Rightarrow nat) \Rightarrow nat \Rightarrow nat$  the subspace  
 $k$ :  $nat$  the dimension of the subspace  
 $n$ :  $nat$  the dimension of the cube  
 $t$ :  $nat$  the size of the cube's base

**definition** *is-subspace*

**where** *is-subspace*  $S k n t \equiv (\exists B f. \text{disjoint-family-on } B \{..k\} \wedge \bigcup (B \text{ ' } \{..k\}) = \{..<n\} \wedge (\{ \} \notin B \text{ ' } \{..<k\}) \wedge f \in (B k) \rightarrow_E \{..<t\} \wedge S \in (\text{cube } k t) \rightarrow_E (\text{cube } n t) \wedge (\forall y \in \text{cube } k t. (\forall i \in B k. S y i = f i) \wedge (\forall j < k. \forall i \in B j. (S y) i = y j)))$

A  $k$ -dimensional subspace of  $C_t^n$  can be thought of as an embedding of the  $C_t^k$  into  $C_t^n$ , akin to how a  $k$ -dimensional vector subspace of  $\mathbf{R}^n$  may be thought of as an embedding of  $\mathbf{R}^k$  into  $\mathbf{R}^n$ .

**lemma** *subspace-inj-on-cube*:

**assumes** *is-subspace*  $S k n t$

**shows** *inj-on*  $S (\text{cube } k t)$

*<proof>*

The following is required to handle base cases in the key lemmas.

**lemma** *dim0-subspace-ex*:

**assumes**  $t > 0$

**shows**  $\exists S. \text{is-subspace } S 0 n t$

*<proof>*

## 1.4 Equivalence classes

Defining the equivalence classes of *cube*  $n (t + 1)$ :  $\{\text{classes } n t 0, \dots, \text{classes } n t n\}$

**definition** *classes*

**where** *classes*  $n t \equiv (\lambda i. \{x . x \in (\text{cube } n (t + 1)) \wedge (\forall u \in \{(n-i)..<n\}. x u = t) \wedge t \notin x \text{ ' } \{..<(n - i)\}\})$

**lemma** *classes-subset-cube*:  $\text{classes } n t i \subseteq \text{cube } n (t+1)$  *<proof>*

**definition** *layered-subspace*

**where** *layered-subspace*  $S k n t r \chi \equiv (\text{is-subspace } S k n (t + 1) \wedge (\forall i \in \{..k\}. \exists c < r. \forall x \in \text{classes } k t i. \chi (S x) = c) \wedge \chi \in (\text{cube } n (t + 1) \rightarrow_E \{..<r\})$

**lemma** *layered-eq-classes*:

**assumes** *layered-subspace*  $S k n t r \chi$

**shows**  $\forall i \in \{..k\}. \forall x \in \text{classes } k t i. \forall y \in \text{classes } k t i.$

$\chi (S x) = \chi (S y)$

*<proof>*

**lemma** *dim0-layered-subspace-ex*:

**assumes**  $\chi \in (\text{cube } n (t + 1)) \rightarrow_E \{..<r::nat\}$

**shows**  $\exists S. \text{layered-subspace } S (0::\text{nat}) n t r \chi$   
 $\langle \text{proof} \rangle$

**lemma** *disjoint-family-onI* [intro]:  
**assumes**  $\bigwedge m n. m \in S \implies n \in S \implies m \neq n$   
 $\implies A m \cap A n = \{\}$   
**shows** *disjoint-family-on*  $A S$   
 $\langle \text{proof} \rangle$

**lemma** *fun-ex*:  $a \in A \implies b \in B \implies \exists f \in A$   
 $\rightarrow_E B. f a = b$   
 $\langle \text{proof} \rangle$

**lemma** *ex-bij-betw-nat-finite-2*:  
**assumes**  $\text{card } A = n$   
**and**  $n > 0$   
**shows**  $\exists f. \text{bij-betw } f A \{..<n\}$   
 $\langle \text{proof} \rangle$

**lemma** *one-dim-cube-eq-nat-set*:  $\text{bij-betw } (\lambda f. f 0) (\text{cube } 1 k) \{..<k\}$   
 $\langle \text{proof} \rangle$

An alternative introduction rule for the  $\exists!x$  quantifier, which means "there exists exactly one  $x$ ".

**lemma** *ex1I-alt*:  $(\exists x. P x \wedge (\forall y. P y \longrightarrow x = y)) \implies (\exists!x. P x)$   
 $\langle \text{proof} \rangle$

**lemma** *nat-set-eq-one-dim-cube*:  $\text{bij-betw } (\lambda x. \lambda y \in \{..<1::\text{nat}\}. x) \{..<k::\text{nat}\} (\text{cube } 1 k)$   
 $\langle \text{proof} \rangle$

A bijection  $f$  between domains  $A_1$  and  $A_2$  creates a correspondence between functions in  $A_1 \rightarrow B$  and  $A_2 \rightarrow B$ .

**lemma** *bij-domain-PiE*:  
**assumes**  $\text{bij-betw } f A1 A2$   
**and**  $g \in A2 \rightarrow_E B$   
**shows**  $(\text{restrict } (g \circ f) A1) \in A1 \rightarrow_E B$   
 $\langle \text{proof} \rangle$

The following three lemmas relate lines to 1-dimensional subspaces (in the natural way). This is a direct consequence of the elimination rule *is-line-elim* introduced above.

**lemma** *line-is-dim1-subspace-t-1*:  
**assumes**  $n > 0$   
**and**  $\text{is-line } L n 1$   
**shows**  $\text{is-subspace } (\text{restrict } (\lambda y. L (y 0)) (\text{cube } 1 1)) 1 n 1$   
 $\langle \text{proof} \rangle$

**lemma** *line-is-dim1-subspace-t-ge-1*:

**assumes**  $n > 0$   
**and**  $t > 1$   
**and** *is-line*  $L n t$   
**shows** *is-subspace* (*restrict* ( $\lambda y. L (y 0)$ ) (*cube* 1  $t$ )) 1  $n t$   
 ⟨*proof*⟩

**lemma** *line-is-dim1-subspace*:

**assumes**  $n > 0$   
**and**  $t > 0$   
**and** *is-line*  $L n t$   
**shows** *is-subspace* (*restrict* ( $\lambda y. L (y 0)$ ) (*cube* 1  $t$ )) 1  $n t$   
 ⟨*proof*⟩

The key property of the existence of a minimal dimension  $N$ , such that for any  $r$ -colouring in  $C_t^{N'}$  (for  $N' \geq N$ ) there exists a monochromatic line is defined in the following using the variables:

$r$ : *nat* the number of colours  
 $t$ : *nat* the size of of the base

**definition** *hj*

**where**  $hj\ r\ t \equiv (\exists N > 0. \forall N' \geq N. \forall \chi. \chi \in (\textit{cube}\ N')$   
 $t) \rightarrow_E \{..<r::nat\} \rightarrow (\exists L. \exists c < r. \textit{is-line}\ L\ N'\ t$   
 $\wedge (\forall y \in L\ ' \{..<t\}. \chi\ y = c))$

The key property of the existence of a minimal dimension  $N$ , such that for any  $r$ -colouring in  $C_t^{N'}$  (for  $N' \geq N$ ) there exists a layered subspace of dimension  $k$  is defined in the following using the variables:

$r$ : *nat* the number of colours  
 $t$ : *nat* the size of of the base  
 $k$ : *nat* the dimension of the subspace

**definition** *lhj*

**where**  $lhj\ r\ t\ k \equiv (\exists N > 0. \forall N' \geq N. \forall \chi. \chi \in$   
 $(\textit{cube}\ N'\ (t + 1)) \rightarrow_E \{..<r::nat\} \rightarrow (\exists S.$   
 $\textit{layered-subspace}\ S\ k\ N'\ t\ r\ \chi)$

We state some useful facts about 1-dimensional subspaces.

**lemma** *dim1-subspace-elims*:

**assumes** *disjoint-family-on*  $B\ \{..1::nat\}$  **and**  $\bigcup (B\ ' \{..1::nat\}) = \{..<n\}$  **and**  
 $(\{}$   
 $\notin B\ ' \{..<1::nat\})$  **and**  $f \in (B\ 1) \rightarrow_E \{..<t\}$  **and**  $S \in (\textit{cube}\ 1$   
 $t) \rightarrow_E (\textit{cube}\ n\ t)$  **and**  $(\forall y \in \textit{cube}\ 1\ t. (\forall i \in B\ 1. S\ y\ i$   
 $= f\ i) \wedge (\forall j < 1. \forall i \in B\ j. (S\ y)\ i = y\ j))$   
**shows**  $B\ 0 \cup B\ 1 = \{..<n\}$   
**and**  $B\ 0 \cap B\ 1 = \{}$   
**and**  $(\forall y \in \textit{cube}\ 1\ t. (\forall i \in B\ 1. S\ y\ i = f\ i) \wedge (\forall i \in B\ 0. (S\ y)\ i = y\ 0))$   
**and**  $B\ 0 \neq \{}$   
 ⟨*proof*⟩

We state some properties of cubes.

**lemma** *cube-props*:

**assumes**  $s < t$

**shows**  $\exists p \in \text{cube } 1 \ t. \ p \ 0 = s$

**and**  $(\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s) \ 0 = s$

**and**  $(\lambda s \in \{..<t\}. \ S \ (\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s)) \ s =$

$(\lambda s \in \{..<t\}. \ S \ (\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s)) \ ((\text{SOME } p. \ p \in \text{cube } 1 \ t$

$\wedge p \ 0 = s) \ 0)$

**and**  $(\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s) \in \text{cube } 1 \ t$

*<proof>*

The following lemma relates 1-dimensional subspaces to lines, thus establishing a bidirectional correspondence between the two together with *line-is-dim1-subspace*.

**lemma** *dim1-subspace-is-line*:

**assumes**  $t > 0$

**and** *is-subspace*  $S \ 1 \ n \ t$

**shows** *is-line*  $(\lambda s \in \{..<t\}. \ S \ (\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s)) \ n \ t$

*<proof>*

**lemma** *bij-unique-inv*:

**assumes** *bij-betw*  $f \ A \ B$

**and**  $x \in B$

**shows**  $\exists! y \in A. \ (\text{the-inv-into } A \ f) \ x = y$

*<proof>*

**lemma** *inv-into-cube-props*:

**assumes**  $s < t$

**shows** *the-inv-into*  $(\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s \in \text{cube } 1 \ t$

**and** *the-inv-into*  $(\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s \ 0 = s$

*<proof>*

**lemma** *some-inv-into*:

**assumes**  $s < t$

**shows**  $(\text{SOME } p. \ p \in \text{cube } 1 \ t \wedge p \ 0 = s) = (\text{the-inv-into } (\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s)$

*<proof>*

**lemma** *some-inv-into-2*:

**assumes**  $s < t$

**shows**  $(\text{SOME } p. \ p \in \text{cube } 1 \ (t+1) \wedge p \ 0 = s) = (\text{the-inv-into } (\text{cube } 1 \ t) \ (\lambda f. \ f \ 0) \ s)$

*<proof>*

**lemma** *dim1-layered-subspace-as-line*:

**assumes**  $t > 0$

**and** *layered-subspace*  $S \ 1 \ n \ t \ r \ \chi$

**shows**  $\exists c1 \ c2. \ c1 < r \wedge c2 < r \wedge (\forall s < t. \ \chi \ (S \ (\text{SOME } p. \ p \in \text{cube } 1$

$(t+1) \wedge p \ 0 = s)) = c1) \wedge \chi \ (S \ (\text{SOME } p. \ p \in \text{cube } 1 \ (t+1) \wedge p \ 0 = t)) = c2$

*<proof>*

**lemma** *dim1-layered-subspace-mono-line*:

**assumes**  $t > 0$   
**and** *layered-subspace*  $S\ 1\ n\ t\ r\ \chi$   
**shows**  $\forall s < t. \forall l < t. \chi (S (SOME\ p. p \in cube\ 1\ (t+1) \wedge p\ 0 = s)) =$   
 $\chi (S (SOME\ p. p \in cube\ 1\ (t+1) \wedge p\ 0 = l)) \wedge \chi (S (SOME\ p. p \in cube\ 1$   
 $(t+1) \wedge p\ 0 = s)) < r$   
 $\langle proof \rangle$

**definition**  $join :: (nat \Rightarrow 'a) \Rightarrow (nat \Rightarrow 'a) \Rightarrow nat$   
 $\Rightarrow nat \Rightarrow (nat \Rightarrow 'a)$

**where**  
 $join\ f\ g\ n\ m \equiv (\lambda x. \text{if } x \in \{..<n\} \text{ then } f\ x \text{ else (if } x \in \{n..<n+m\} \text{ then } g$   
 $(x - n) \text{ else undefined}))$

**lemma** *join-cubes*:

**assumes**  $f \in cube\ n\ (t+1)$   
**and**  $g \in cube\ m\ (t+1)$   
**shows**  $join\ f\ g\ n\ m \in cube\ (n+m)\ (t+1)$   
 $\langle proof \rangle$

**lemma** *subspace-elems-embed*:

**assumes** *is-subspace*  $S\ k\ n\ t$   
**shows**  $S\ ' (cube\ k\ t) \subseteq cube\ n\ t$   
 $\langle proof \rangle$

## 2 Core proofs

The numbering of the theorems has been borrowed from the textbook [1].

### 2.1 Theorem 4

#### 2.1.1 Base case of Theorem 4

**lemma** *hj-imp-lhj-base*:

**fixes**  $r\ t$   
**assumes**  $t > 0$   
**and**  $\bigwedge r'. hj\ r'\ t$   
**shows**  $lhj\ r\ t\ 1$   
 $\langle proof \rangle$

#### 2.1.2 Induction step of theorem 4

The proof has four parts:

1. We obtain two layered subspaces of dimension 1 and  $k$  (respectively), whose existence is guaranteed by the assumption *lhj* (i.e. the induction hypothesis). Additionally, we prove some useful facts about these.
2. We construct a  $k+1$ -dimensional subspace with the goal of showing that it is layered.

3. We prove that our construction is a subspace in the first place.
4. We prove that it is a layered subspace.

**lemma** *hj-imp-lhj-step*:

**fixes**  $r\ k$   
**assumes**  $t > 0$   
**and**  $k \geq 1$   
**and** *True*  
**and**  $(\bigwedge r\ k'. k' \leq k \implies lhj\ r\ t\ k')$   
**and**  $r > 0$   
**shows**  $lhj\ r\ t\ (k+1)$   
 $\langle proof \rangle$

### Part 1: Obtaining the subspaces $L$ and $S$

Recall that *lhj* claims the existence of a layered subspace for any colouring (of a fixed size, where the size of a colouring refers to the number of colours). Therefore, the colourings have to be defined first, before the layered subspaces can be obtained. The colouring  $\chi L$  here is  $\chi^*$  in the book [1], an  $s$ -colouring; see the fact *s-coloured* a couple of lines below.

$\langle proof \rangle$

### Part 2: Constructing the $(k + 1)$ -dimensional subspace $T$

Below, *Tset* is the set as defined in the book [1]. It represents the  $(k + 1)$ -dimensional subspace. In this construction, subspaces (e.g.  $T$ ) are functions whose image is a set. See the fact *im-T-eq-Tset* below.

Having obtained our subspaces  $S$  and  $L$ , we define the  $(k + 1)$ -dimensional subspace very straightforwardly. Namely,  $T = L \times S$ . Since we represent tuples by function sets, we need an appropriate operator that mirrors the Cartesian product  $\times$  for these. We call this *join* and define it for elements of a function set.

$\langle proof \rangle$

### Part 3: Proving that $T$ is a subspace

To prove something is a subspace, we have to provide the  $B$  and  $f$  satisfying the subspace properties. We construct  $BT$  and  $fT$  from  $BS$ ,  $fS$  and  $BL$ ,  $fL$ , which correspond to the  $k$ -dimensional subspace  $S$  and the 1-dimensional subspace (i.e. line)  $L$ , respectively.

$\langle proof \rangle$

### Part 4: Proving $T$ is layered

The following redefinition of the classes makes proving the layered property easier.

*<proof>*

**theorem** *hj-imp-lhj*:

**fixes**  $k$

**assumes**  $\wedge r'. \text{hj } r' t$

**shows**  $\text{lhj } r t k$

*<proof>*

## 2.2 Theorem 5

We provide a way to construct a monochromatic line in  $C_{t+1}^n$  from a  $k$ -dimensional  $k$ -coloured layered subspace  $S$  in  $C_{t+1}^n$ . The idea is to rely on the fact that there are  $k + 1$  classes in  $S$ , but only  $k$  colours. It thus follows from the Pigeonhole Principle that two classes must share the same colour. The way classes are defined allows for a straightforward construction of a line with points only from those two classes. Thus we have our monochromatic line.

**theorem** *layered-subspace-to-mono-line*:

**assumes** *layered-subspace*  $S k n t k \chi$

**and**  $t > 0$

**shows**  $(\exists L. \exists c < k. \text{is-line } L n (t+1) \wedge (\forall y \in L. \{..<t+1\}. \chi y = c))$

*<proof>*

## 2.3 Corollary 6

**corollary** *lhj-imp-hj*:

**assumes**  $(\wedge r k. \text{lhj } r t k)$

**and**  $t > 0$

**shows**  $(\text{hj } r (t+1))$

*<proof>*

## 2.4 Main result

### 2.4.1 Edge cases and auxiliary lemmas

**lemma** *single-point-line*:

**assumes**  $N > 0$

**shows** *is-line*  $(\lambda s \in \{..<1\}. \lambda a \in \{..<N\}. 0) N 1$

*<proof>*

**lemma** *single-point-line-is-monochromatic*:

**assumes**  $\chi \in \text{cube } N 1 \rightarrow_E \{..<r\} N > 0$

**shows**  $(\exists c < r. \text{is-line } (\lambda s \in \{..<1\}. \lambda a \in \{..<N\}. 0) N 1 \wedge (\forall i \in$

$\{..<1\}. \lambda a \in \{..<N\}. 0) \{..<1\}. \chi i = c))$

*<proof>*

**lemma** *hj-r-nonzero-t-0*:

**assumes**  $r > 0$

**shows**  $hj\ r\ 0$   
*<proof>*

Any cube over 1 element always has a single point, which also forms the only line in the cube. Since it's a single point line, it's trivially monochromatic. We show the result for dimension 1.

**lemma** *hj-t-1*:  $hj\ r\ 1$   
*<proof>*

## 2.4.2 Main theorem

We state the main result  $hj\ r\ t$ . The explanation for the choice of assumption is offered subsequently.

**theorem** *hales-jewett*:  
**assumes**  $\neg(r = 0 \wedge t = 0)$   
**shows**  $hj\ r\ t$   
*<proof>*

We offer a justification for having excluded the special case  $r = t = 0$  from the statement of the main theorem *hales-jewett*. The exclusion is a consequence of the fact that colourings are defined as members of the function set  $cube\ n\ t \rightarrow_E \{..<r\}$ , which for  $r = t = 0$  means there's a dummy colouring  $\lambda x. undefined$ , even though  $cube\ n\ 0 = \{\}$  for  $n > 0$ . Hence, in this case, no line exists at all (let alone one monochromatic under the aforementioned colouring). This means  $hj\ 0\ 0 \implies False$ —but only because of the quirky behaviour of the  $FuncSet\ cube\ n\ t \rightarrow_E \{..<r\}$ . This could have been circumvented by letting colourings  $\chi$  be arbitrary functions constraint only by  $\chi \text{ ' } cube\ n\ t \subseteq \{..<r\}$ . We avoided this in order to have consistency with the cube's definition, for which  $FuncSets$  were crucial because the proof heavily relies on arguments about the cardinality of the cube. The constraint  $x \text{ ' } \{..<n\} \subseteq \{..<t\}$  for elements  $x$  of  $C_t^n$  would not have sufficed there, as there are infinitely many functions over the naturals satisfying it.

**end**

## References

- [1] R. L. Graham, B. L. Rothschild, and J. H. Spencer. *Ramsey Theory, 2nd Edition*. Wiley-Interscience, March 1990.
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