The Fundamental Group of the Circle

Arthur Freitas Ramos David Barros Hulak
Ruy J. G. B. de Queiroz

September 2, 2026

Abstract

We formalise the classical theorem of algebraic topology that the fundamental group of the circle is isomorphic to the additive group of the integers, $\pi_1(S^1) \cong \mathbb{Z}$. The circle is modelled as the unit sphere in the complex plane with basepoint 1, and the carrier of the fundamental group is the set of path-homotopy classes of loops based at 1, with concatenation as the group operation. The group laws are obtained from the homotopy groupoid laws of the Isabelle analysis library. The isomorphism with $\mathbb{Z}$ is given by the degree map, sending a homotopy class to the winding number about the origin of any representative loop; that this is a bijective group homomorphism follows from the winding-number classification of loops in the punctured plane, together with a radial retraction onto the circle. The entry is self-contained on top of the Isabelle distribution.

AI assistance was used for proof engineering. The final definitions, statements, and proofs are checked by Isabelle.

Contents

1 Introduction

The fundamental group of the circle is one of the first non-trivial computations in algebraic topology: it is the infinite cyclic group, so $\pi_1(S^1) \cong \mathbb{Z}$. The result underlies many classical theorems, including the Brouwer fixed-point theorem in dimension two and the fundamental theorem of algebra. Standard textbook treatments can be found in Hatcher [1] and Munkres [2].

Rather than developing covering-space theory, this entry follows an analytic route that is directly supported by the Isabelle distribution. The circle is modelled as the unit sphere in the complex plane with basepoint 1, and the group is built from path-homotopy classes of based loops. The isomorphism with $\mathbb{Z}$ is realised by the *degree* map, which sends a homotopy class to the winding number about the origin of any representative loop. The winding-number theory of the Isabelle analysis library provides exactly the facts needed: winding numbers are integer valued on loops, additive under concatenation, and classify loops in the punctured plane up to homotopy. A radial retraction of the punctured plane onto the circle transfers homotopies back to the circle, which yields injectivity of the degree map.

2 The Fundamental Group of the Circle

theory *Fundamental-Group-Circle*
  imports
    HOL−Complex-Analysis.Complex-Analysis
    HOL−Algebra.Group
begin

We formalise the classical result of algebraic topology that the fundamental group of the circle is isomorphic to the additive group of integers, $\pi_1(S^1) \cong \mathbb{Z}$.

The development is self-contained on top of the Isabelle distribution. It models the circle as the unit sphere in the complex plane, with basepoint *1*. Based loops are paths in the circle starting and ending at *1*, and the carrier of the fundamental group is the set of path-homotopy classes of such loops, with concatenation as the group operation. The group structure is obtained directly from the homotopy "groupoid" laws in *HOL−Analysis.Homotopy*.

The isomorphism with $\mathbb{Z}$ is realised by the *degree* map, which sends a homotopy class to the winding number of any representative loop about the origin. That this is a well-defined bijective homomorphism rests on the classification of loops in the punctured plane up to homotopy by their winding number, from *HOL−Complex-Analysis.Riemann-Mapping*, together with a radial retraction bridging homotopies in the punctured plane and in the circle.

AI assistance was used for proof engineering. The final definitions, statements, and proofs are checked by Isabelle.

2.1 The circle and based loops

definition *circle* :: *complex set* **where**
circle = *sphere 0 1*

lemma *one-in-circle* [*simp*]: (*1*::*complex*) ∈ *circle*
⟨*proof*⟩

lemma *zero-notin-circle* [*simp*]: (*0*::*complex*) ∉ *circle*
⟨*proof*⟩

lemma *circle-subset-punctured*: *circle* ⊆ − {*0*}
⟨*proof*⟩

lemma *norm-of-circle*: *z* ∈ *circle* ⟹ *norm z* = *1*
⟨*proof*⟩

A (based) loop is a path in the circle starting and finishing at *1*.

definition *is-loop* :: (*real* ⇒ *complex*) ⇒ *bool* **where**
is-loop p ⟷ *path p* ∧ *path-image p* ⊆ *circle* ∧ *pathstart p* = *1* ∧ *pathfinish p* = *1*

lemma *is-loop-path*: *is-loop p* ⟹ *path p*
⟨*proof*⟩

lemma *is-loop-image*: *is-loop p* ⟹ *path-image p* ⊆ *circle*
⟨*proof*⟩

lemma *is-loop-pathstart*: *is-loop p* ⟹ *pathstart p* = *1*
⟨*proof*⟩

lemma *is-loop-pathfinish*: *is-loop p* ⟹ *pathfinish p* = *1*
⟨*proof*⟩

lemma *is-loop-selfloop*: *is-loop p* ⟹ *pathfinish p* = *pathstart p*
⟨*proof*⟩

lemma *is-loop-connect*: ⟦*is-loop p*; *is-loop q*⟧ ⟹ *pathfinish p* = *pathstart q*
⟨*proof*⟩

lemma *is-loop-zero-notin-image*: *is-loop p* ⟹ *0* ∉ *path-image p*
⟨*proof*⟩

lemma *is-loop-linepath-1* [*simp*]: *is-loop* (*linepath 1 1*)
⟨*proof*⟩

lemma *is-loop-join* [*simp*]:
assumes *is-loop p* **and** *is-loop q*
shows *is-loop* (*p* +++ *q*)
⟨*proof*⟩

lemma *is-loop-reversepath* [*simp*]:
  assumes *is-loop p*
  shows *is-loop* (*reversepath p*)
  ⟨*proof*⟩

2.2 Homotopy classes of loops

definition *loop-class* :: (*real* ⇒ *complex*) ⇒ (*real* ⇒ *complex*) *set* **where**
  loop-class p = {*q*. *is-loop q* ∧ *homotopic-paths circle p q*}

definition *loops-carrier* :: (*real* ⇒ *complex*) *set set* **where**
  loops-carrier = {*loop-class p* |*p*. *is-loop p*}

definition *rep* :: (*real* ⇒ *complex*) *set* ⇒ (*real* ⇒ *complex*) **where**
  rep A = (*SOME p*. *is-loop p* ∧ *A* = *loop-class p*)

lemma *in-loop-class*: *is-loop p* ⟹ *p* ∈ *loop-class p*
  ⟨*proof*⟩

lemma *loop-class-eqI*:
  assumes *is-loop p is-loop q homotopic-paths circle p q*
  shows *loop-class p* = *loop-class q*
⟨*proof*⟩

lemma *loop-class-eq-iff*:
  assumes *is-loop p is-loop q*
  shows *loop-class p* = *loop-class q* ⟷ *homotopic-paths circle p q*
⟨*proof*⟩

lemma *loop-class-in-carrier* [*simp*]: *is-loop p* ⟹ *loop-class p* ∈ *loops-carrier*
  ⟨*proof*⟩

lemma *rep-props*:
  assumes *A* ∈ *loops-carrier*
  shows *is-loop* (*rep A*) ∧ *loop-class* (*rep A*) = *A*
⟨*proof*⟩

lemma *is-loop-rep*: *A* ∈ *loops-carrier* ⟹ *is-loop* (*rep A*)
  ⟨*proof*⟩

lemma *loop-class-rep*: *A* ∈ *loops-carrier* ⟹ *loop-class* (*rep A*) = *A*
  ⟨*proof*⟩

2.3 The fundamental group of the circle

definition *pi1-circle* :: (*real* ⇒ *complex*) *set monoid* **where**
  pi1-circle =
    (| *carrier* = *loops-carrier*,
      mult = (λ*A B*. *loop-class* (*rep A* +++ *rep B*)),

one = *loop-class* (*linepath 1 1*) ⦈

lemma *carrier-pi1* [*simp*]: *carrier pi1-circle* = *loops-carrier*
⟨*proof*⟩

lemma *one-pi1* [*simp*]: $\mathbf{1}_{pi1\text{-}circle}$ = *loop-class* (*linepath 1 1*)
⟨*proof*⟩

Multiplication of classes is concatenation, independent of representatives.

lemma *pi1-mult-class*:
assumes *p*: *is-loop p* **and** *q*: *is-loop q*
shows *loop-class p* $\otimes_{pi1\text{-}circle}$ *loop-class q* = *loop-class* (*p* +++ *q*)
⟨*proof*⟩

lemma *group-pi1-circle*: *group pi1-circle*
⟨*proof*⟩

2.4 The integers as a group

definition *integer-group* :: *int monoid* **where**
integer-group = ⦇ *carrier* = *UNIV*, *mult* = (+), *one* = *0* ⦈

lemma *carrier-integer-group* [*simp*]: *carrier integer-group* = *UNIV*
⟨*proof*⟩

lemma *mult-integer-group* [*simp*]: *x* $\otimes_{integer\text{-}group}$ *y* = *x* + *y*
⟨*proof*⟩

lemma *one-integer-group* [*simp*]: $\mathbf{1}_{integer\text{-}group}$ = *0*
⟨*proof*⟩

lemma *group-integer-group*: *group integer-group*
⟨*proof*⟩

2.5 The winding-number degree

definition *wnd* :: (*real* ⇒ *complex*) ⇒ *int* **where**
wnd p = (*SOME n. winding-number p 0* = *of-int n*)

lemma *wnd-eq*:
assumes *is-loop p*
shows *winding-number p 0* = *of-int* (*wnd p*)
⟨*proof*⟩

lemma *wnd-join*:
assumes *is-loop p is-loop q*
shows *wnd* (*p* +++ *q*) = *wnd p* + *wnd q*
⟨*proof*⟩

lemma *wnd-reversepath*:
  assumes *is-loop p*
  shows $wnd\ (reversepath\ p) = -\ wnd\ p$
⟨*proof*⟩

lemma *wnd-linepath-1*: $wnd\ (linepath\ 1\ 1) = 0$
⟨*proof*⟩

lemma *wnd-homotopic*:
  assumes *is-loop p is-loop q homotopic-paths circle p q*
  shows $wnd\ p = wnd\ q$
⟨*proof*⟩

definition $deg :: (real \Rightarrow complex)\ set \Rightarrow int$ **where**
  $deg\ A = wnd\ (rep\ A)$

lemma *deg-class*:
  assumes *is-loop p*
  shows $deg\ (loop\text{-}class\ p) = wnd\ p$
⟨*proof*⟩

2.6 The degree is a homomorphism

lemma *deg-mult*:
  assumes $A \in loops\text{-}carrier\ B \in loops\text{-}carrier$
  shows $deg\ (A \otimes_{pi1\text{-}circle} B) = deg\ A + deg\ B$
⟨*proof*⟩

lemma *deg-hom*: $deg \in hom\ pi1\text{-}circle\ integer\text{-}group$
  ⟨*proof*⟩

2.7 The degree is injective

A radial retraction of the punctured plane onto the circle. It fixes the circle pointwise and lets us transfer a homotopy in the punctured plane back into the circle.

definition $retr :: complex \Rightarrow complex$ **where**
  $retr\ z = z\ /\ of\text{-}real\ (norm\ z)$

lemma *retr-fixes*: $z \in circle \Longrightarrow retr\ z = z$
  ⟨*proof*⟩

lemma *continuous-on-retr*: $continuous\text{-}on\ (-\ \{0\})\ retr$
  ⟨*proof*⟩

lemma *retr-into-circle*: $retr \in (-\ \{0\}) \rightarrow circle$
⟨*proof*⟩

lemma *homotopic-paths-retr-self*:

assumes *is-loop p*
shows *homotopic-paths circle p* (*retr* ∘ *p*)
⟨*proof*⟩

lemma *deg-inj*: *inj-on deg* (*carrier pi1-circle*)
⟨*proof*⟩

2.8 The degree is surjective

lemma *is-loop-circlepath*: *is-loop* (*circlepath 0 1*)
⟨*proof*⟩

lemma *wnd-circlepath*: *wnd* (*circlepath 0 1*) = *1*
⟨*proof*⟩

The *n*-fold concatenation of the standard circle loop.

fun *cpow* :: *nat* ⇒ (*real* ⇒ *complex*) **where**
cpow 0 = *linepath 1 1*
| *cpow* (*Suc n*) = *circlepath 0 1* +++ *cpow n*

lemma *is-loop-cpow*: *is-loop* (*cpow n*)
⟨*proof*⟩

lemma *wnd-cpow*: *wnd* (*cpow n*) = *int n*
⟨*proof*⟩

lemma *deg-surj*: *deg* ' *carrier pi1-circle* = *UNIV*
⟨*proof*⟩

2.9 The fundamental theorem

theorem *pi1-circle-iso-integers*: *pi1-circle* ≅ *integer-group*
⟨*proof*⟩

theorem *group-hom-deg*: *group-hom pi1-circle integer-group deg*
⟨*proof*⟩

end

References

[1] A. Hatcher. *Algebraic Topology.* Cambridge University Press, Cambridge, 2002. Available at https://pi.math.cornell.edu/~hatcher/AT/ATpage.html.

[2] J. R. Munkres. *Topology.* Prentice Hall, Upper Saddle River, NJ, 2nd edition, 2000.