The Földes–Hammer Characterization of Split Graphs

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Abstract

This entry formalizes the classical Földes–Hammer theorem characterizing finite split graphs as exactly the graphs with no induced copy of $2K_2$, C_4, or C_5. It builds on the AFP session `Undirected_Graph_Theory`.

1 Overview

A finite graph is split when its vertex set can be partitioned into a clique and an independent set. The main theorem of this entry is `foldes_hammer` in `Foldes_Hammer.thy`, which proves the equivalence between split graphs and the absence of induced $2K_2$, C_4, and C_5.

The forward direction is established by analyzing how any split partition excludes each forbidden configuration. The converse direction follows an extremal argument: choose a maximum clique whose complement minimizes the number of induced edges, orient any remaining outside edge according to nested nonneighbor sets in the clique, and swap one clique vertex with an outside vertex to derive a contradiction.

2 Sources

The entry follows the classical theorem due to Földes and Hammer [2] and standard graph-class background as presented by Brandstädt, Le, and Spinrad [1].

Contents

1 Overview **1**

2 Sources [1]

theory *Foldes-Hammer*
  imports *Undirected-Graph-Theory.Undirected-Graphs-Root*
begin

context *fin-sgraph*
begin

definition *is-clique* :: $'a\ set \Rightarrow bool$ **where**
  $\textit{is-clique}\ C \longleftrightarrow C \subseteq V \wedge \textit{induced-edges}\ C = \textit{all-edges}\ C$

definition *cliques* :: $'a\ set\ set$ **where**
  $\textit{cliques} = \{C.\ \textit{is-clique}\ C\}$

definition *maximum-clique* :: $'a\ set \Rightarrow bool$ **where**
  $\textit{maximum-clique}\ K \longleftrightarrow K \in \textit{cliques} \wedge (\forall C{\in}\textit{cliques}.\ \textit{card}\ C \leq \textit{card}\ K)$

definition *outside-edge-count* :: $'a\ set \Rightarrow nat$ **where**
  $\textit{outside-edge-count}\ K = \textit{card}\ (\textit{induced-edges}\ (V - K))$

definition *is-split-partition* :: $'a\ set \Rightarrow 'a\ set \Rightarrow bool$ **where**
  $\textit{is-split-partition}\ C\ I \longleftrightarrow C \cap I = \{\} \wedge C \cup I = V \wedge \textit{is-clique}\ C \wedge \textit{is-independent-set}\ I$

definition *is-split-graph* :: $bool$ **where**
  $\textit{is-split-graph} \longleftrightarrow (\exists\, C\ I.\ \textit{is-split-partition}\ C\ I)$

definition *induces-2K2* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
  $\textit{induces-2K2}\ a\ b\ c\ d \longleftrightarrow$
    $\textit{distinct}\ [a, b, c, d] \wedge$
    $\textit{vert-adj}\ a\ b \wedge \textit{vert-adj}\ c\ d \wedge$
    $\neg\ \textit{vert-adj}\ a\ c \wedge \neg\ \textit{vert-adj}\ a\ d \wedge \neg\ \textit{vert-adj}\ b\ c \wedge \neg\ \textit{vert-adj}\ b\ d$

definition *induces-C4* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
  $\textit{induces-C4}\ a\ b\ c\ d \longleftrightarrow$
    $\textit{distinct}\ [a, b, c, d] \wedge$
    $\textit{vert-adj}\ a\ b \wedge \textit{vert-adj}\ b\ c \wedge \textit{vert-adj}\ c\ d \wedge \textit{vert-adj}\ d\ a \wedge$
    $\neg\ \textit{vert-adj}\ a\ c \wedge \neg\ \textit{vert-adj}\ b\ d$

definition *induces-C5* :: $'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow 'a \Rightarrow bool$ **where**
  $\textit{induces-C5}\ a\ b\ c\ d\ e \longleftrightarrow$
    $\textit{distinct}\ [a, b, c, d, e] \wedge$
    $\textit{vert-adj}\ a\ b \wedge \textit{vert-adj}\ b\ c \wedge \textit{vert-adj}\ c\ d \wedge \textit{vert-adj}\ d\ e \wedge \textit{vert-adj}\ e\ a \wedge$
    $\neg\ \textit{vert-adj}\ a\ c \wedge \neg\ \textit{vert-adj}\ a\ d \wedge \neg\ \textit{vert-adj}\ b\ d \wedge \neg\ \textit{vert-adj}\ b\ e \wedge \neg\ \textit{vert-adj}\ c\ e$

definition *has-induced-2K2* :: $bool$ **where**
  $\textit{has-induced-2K2} \longleftrightarrow (\exists\, a\ b\ c\ d.\ \textit{induces-2K2}\ a\ b\ c\ d)$

definition *has-induced-C4* :: *bool* **where**
has-induced-C4 $\longleftrightarrow$ ($\exists$ *a b c d*. *induces-C4 a b c d*)

definition *has-induced-C5* :: *bool* **where**
has-induced-C5 $\longleftrightarrow$ ($\exists$ *a b c d e*. *induces-C5 a b c d e*)

definition *foldes-hammer-free* :: *bool* **where**
foldes-hammer-free $\longleftrightarrow$ $\neg$ *has-induced-2K2* $\wedge$ $\neg$ *has-induced-C4* $\wedge$ $\neg$ *has-induced-C5*

lemma *is-clique-alt*:
is-clique C $\longleftrightarrow$ $C \subseteq V \wedge (\forall u{\in}C.\ \forall v{\in}C.\ u \neq v \longrightarrow$ *vert-adj u v*)
⟨*proof*⟩

lemma *clique-pair-adj*:
assumes *is-clique C* $u \in C$ $v \in C$ $u \neq v$
shows *vert-adj u v*
⟨*proof*⟩

lemma *clique-mono*:
assumes *is-clique C* $D \subseteq C$
shows *is-clique D*
⟨*proof*⟩

lemma *vert-adj-neq*:
assumes *vert-adj u v*
shows $u \neq v$
⟨*proof*⟩

lemma *empty-clique*: *is-clique* {}
⟨*proof*⟩

lemma *singleton-clique*:
assumes $v \in V$
shows *is-clique* $\{v\}$
⟨*proof*⟩

lemma *finite-cliques*: *finite cliques*
⟨*proof*⟩

lemma *cliques-nonempty*: {} $\in$ *cliques*
⟨*proof*⟩

lemma *maximum-clique-exists*:
obtains *K* **where** *maximum-clique K*
⟨*proof*⟩

lemma *maximum-clique-finite*:
assumes *maximum-clique K*

shows *finite* K
⟨*proof*⟩

lemma *finite-induced-edges-subset-V*:
assumes $S \subseteq V$
shows *finite* (*induced-edges* S)
⟨*proof*⟩

lemma *obtain-best-clique*:
obtains K **where** *maximum-clique* K
and $\forall L.$ *maximum-clique* $L \longrightarrow$ *outside-edge-count* $K \leq$ *outside-edge-count* L
⟨*proof*⟩

lemma *insert-clique-if-all-adj*:
assumes C: *is-clique* C
and *vV*: $v \in V$
and *vnot*: $v \notin C$
and *adj*: $\forall u \in C.$ *vert-adj* u v
shows *is-clique* (*insert* v C)
⟨*proof*⟩

lemma *maximum-clique-outside-has-nonneighbor*:
assumes *maxK*: *maximum-clique* K
and *vV*: $v \in V$
and *vnot*: $v \notin K$
shows $\exists u \in K. \neg$ *vert-adj* u v
⟨*proof*⟩

lemma *split-graph-alt*:
is-split-graph $\longleftrightarrow$ ($\exists C.$ *is-clique* $C \wedge$ *is-independent-set* $(V - C)$)
⟨*proof*⟩

lemma *independent-pair-not-adj*:
assumes *is-independent-set* I $u \in I$ $v \in I$
shows $\neg$ *vert-adj* u v
⟨*proof*⟩

lemma *split-partition-edge-hits-clique*:
assumes *sp*: *is-split-partition* C I
and *uv*: *vert-adj* u v
shows $u \in C \vee v \in C$
⟨*proof*⟩

lemma *split-partition-nonedge-not-both-clique*:
assumes *sp*: *is-split-partition* C I
and *uV*: $u \in V$
and *vV*: $v \in V$
and *uv*: $u \neq v$
and *not-adj*: $\neg$ *vert-adj* u v

shows $\neg (u \in C \wedge v \in C)$
$\langle proof \rangle$

lemma *induces-2K2-inV*:
assumes *induces-2K2 a b c d*
shows $a \in V$ $b \in V$ $c \in V$ $d \in V$
$\langle proof \rangle$

lemma *induces-C4-inV*:
assumes *induces-C4 a b c d*
shows $a \in V$ $b \in V$ $c \in V$ $d \in V$
$\langle proof \rangle$

lemma *induces-C5-inV*:
assumes *induces-C5 a b c d e*
shows $a \in V$ $b \in V$ $c \in V$ $d \in V$ $e \in V$
$\langle proof \rangle$

lemma *split-partition-forbids-2K2*:
assumes *sp*: *is-split-partition C I*
and *H*: *induces-2K2 a b c d*
shows *False*
$\langle proof \rangle$

lemma *split-partition-forbids-C4*:
assumes *sp*: *is-split-partition C I*
and *H*: *induces-C4 a b c d*
shows *False*
$\langle proof \rangle$

lemma *split-partition-forbids-C5*:
assumes *sp*: *is-split-partition C I*
and *H*: *induces-C5 a b c d e*
shows *False*
$\langle proof \rangle$

lemma *split-graph-imp-foldes-hammer-free*:
assumes *is-split-graph*
shows *foldes-hammer-free*
$\langle proof \rangle$

lemma *outside-adjacent-nonneighbor-sets-nested*:
assumes *fh*: *foldes-hammer-free*
and *Kmax*: *maximum-clique K*
and *uO*: $u \in V - K$
and *vO*: $v \in V - K$
and *uv*: *vert-adj u v*
shows $\{x{\in}K.\ \neg\ \textit{vert-adj}\ x\ u\} \subseteq \{x{\in}K.\ \neg\ \textit{vert-adj}\ x\ v\} \vee$
$\{x{\in}K.\ \neg\ \textit{vert-adj}\ x\ v\} \subseteq \{x{\in}K.\ \neg\ \textit{vert-adj}\ x\ u\}$

$\langle proof \rangle$

lemma *outside-adjacent-smaller-nonneighbor-set-singleton*:
  assumes *fh*: *foldes-hammer-free*
    and *Kmax*: *maximum-clique K*
    and *uO*: $u \in V - K$
    and *vO*: $v \in V - K$
    and *uv*: *vert-adj u v*
    and *AB*: $\{x{\in}K.\ \neg$ *vert-adj x u*$\} \subseteq \{x{\in}K.\ \neg$ *vert-adj x v*$\}$
  obtains a **where** $\{x{\in}K.\ \neg$ *vert-adj x u*$\} = \{a\}$ **and** $a \in K$ **and** $\neg$ *vert-adj a u*
and $\neg$ *vert-adj a v*
$\langle proof \rangle$

lemma *orient-outside-edge-for-swap*:
  assumes *fh*: *foldes-hammer-free*
    and *Kmax*: *maximum-clique K*
    and *uO*: $u \in V - K$
    and *vO*: $v \in V - K$
    and *uv*: *vert-adj u v*
  obtains $u'\ v'\ a$ **where**
      $\{u', v'\} = \{u, v\}$
    and $u' \in V - K$
    and $v' \in V - K$
    and *vert-adj* $u'\ v'$
    and $\{x{\in}K.\ \neg$ *vert-adj x* $u'\} = \{a\}$
    and $a \in K$
    and $\neg$ *vert-adj a* u'
    and $\neg$ *vert-adj a* v'
$\langle proof \rangle$

lemma *replace-unique-nonneighbor-maximum-clique*:
  assumes *Kmax*: *maximum-clique K*
    and *uO*: $u \in V - K$
    and *Au*: $\{x{\in}K.\ \neg$ *vert-adj x u*$\} = \{a\}$
  shows *maximum-clique* (*insert u* $(K - \{a\})$)
$\langle proof \rangle$

lemma *replacement-edge-transfer*:
  assumes *fh*: *foldes-hammer-free*
    and *Kmax*: *maximum-clique K*
    and *uO*: $u \in V - K$
    and *vO*: $v \in V - K$
    and *uv*: *vert-adj u v*
    and *Au*: $\{x{\in}K.\ \neg$ *vert-adj x u*$\} = \{a\}$
    and *aNv*: $\neg$ *vert-adj a v*
    and *zO*: $z \in V - K$
    and *zne*: $z \neq u$
    and *az*: *vert-adj a z*
  shows *vert-adj u z*

⟨*proof*⟩

lemma *outside-set-after-swap*:
  assumes *uO*: $u \in V - K$
    and *aV*: $a \in V$
    and *aK*: $a \in K$
  shows $V - \mathit{insert}\ u\ (K - \{a\}) = \mathit{insert}\ a\ ((V - K) - \{u\})$
  ⟨*proof*⟩

lemma *swap-outside-edges-inj*:
  assumes *uO*: $u \in V - K$
    and *aK*: $a \in K$
  shows *inj-on* (%*e*. *if* $a \in e$ *then insert u* $(e - \{a\})$ *else e*)
    (*induced-edges* ($V -$ *insert u* $(K - \{a\})$))
⟨*proof*⟩

lemma *swap-outside-edges-image-subset*:
  assumes *fh*: *foldes-hammer-free*
    and *Kmax*: *maximum-clique K*
    and *uO*: $u \in V - K$
    and *vO*: $v \in V - K$
    and *uv*: *vert-adj u v*
    and *Au*: $\{x{\in}K.\ \neg$ *vert-adj x u*$\} = \{a\}$
    and *aNv*: $\neg$ *vert-adj a v*
  shows (%*e*. *if* $a \in e$ *then insert u* $(e - \{a\})$ *else e*) ‘
    induced-edges ($V -$ *insert u* $(K - \{a\})$)
    $\subseteq$ *induced-edges* $(V - K)$
⟨*proof*⟩

lemma *swap-outside-edges-missing-uv*:
  assumes *uO*: $u \in V - K$
    and *vO*: $v \in V - K$
    and *uv*: *vert-adj u v*
    and *aK*: $a \in K$
    and *aNv*: $\neg$ *vert-adj a v*
  shows $\{u, v\} \in$ *induced-edges* $(V - K)$
    and $\{u, v\} \notin$ (%*e*. *if* $a \in e$ *then insert u* $(e - \{a\})$ *else e*) ‘
    induced-edges ($V -$ *insert u* $(K - \{a\})$)
⟨*proof*⟩

lemma *outside-edge-count-swap-strict*:
  assumes *fh*: *foldes-hammer-free*
    and *Kmax*: *maximum-clique K*
    and *uO*: $u \in V - K$
    and *vO*: $v \in V - K$
    and *uv*: *vert-adj u v*
    and *Au*: $\{x{\in}K.\ \neg$ *vert-adj x u*$\} = \{a\}$
    and *aNv*: $\neg$ *vert-adj a v*
  shows *outside-edge-count* (*insert u* $(K - \{a\})$) $<$ *outside-edge-count K*

⟨*proof*⟩

theorem *foldes-hammer-converse*:
  assumes *fh*: *foldes-hammer-free*
  shows *is-split-graph*
⟨*proof*⟩

theorem *foldes-hammer*:
  is-split-graph $\longleftrightarrow$ *foldes-hammer-free*
  ⟨*proof*⟩

end

end

References

[1] A. Brandstädt, V. B. Le, and J. P. Spinrad. *Graph Classes: A Survey*. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1999. DOI: https://doi.org/10.1137/1.9780898719796.

[2] S. Földes and P. L. Hammer. Split graphs. *Congressus Numerantium*, 19:311–315, 1977.