

Feuerbach's Theorem

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Abstract

Feuerbach's theorem is a classical result in Euclidean geometry about certain circles associated with a triangle. It states that the nine-point circle is tangent internally to the triangle's incircle and tangent externally to each of the three excircles, where the *incircle* is the circle tangent to all three sides of the triangle; the *excircles* are circles tangent to one side and the extensions of the other two; the *nine-point circle* is the circle passing through nine special points of the triangle.

This development is an experiment in automatic translation from one proof assistant to another using AI. The HOL Light file `feuerbach.ml` was given, one proof at a time, to Claude 4.6 using the Isabelle Assistant plug-in.¹ Despite coaxing and hints, the autoformalisation process transformed a 214-line HOL Light file to nearly 1500 lines of Isabelle/HOL, though one that could be processed in five seconds. Part of the excess length comes from spelling out two transformations (translation of a point to zero, followed by rotation of a side to the vector $(1, 0)$) that in HOL Light is done silently by WLOG tactics. The current proof has been heavily simplified, in part using Claude but mostly manually, and is now under 750 lines. It is notable that Claude often relied on the **algebra** proof method, despite its relative obscurity.

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¹Part of AutoCorrode: <https://github.com/awslabs/AutoCorrode>

```

theory Feuerbach
  imports HOL-Analysis.Analysis
begin

```

1 Algebraic condition for two circles to be tangent to each other

```

lemma inner-real2:  $(x::\text{real}^2) \cdot y = x\$1 * y\$1 + x\$2 * y\$2$ 
  <proof>

```

```

lemma norm-real2-sq:  $(\text{norm } (x::\text{real}^2))^2 = (x\$1)^2 + (x\$2)^2$ 
  <proof>

```

```

lemma dist-real2-sq:  $(\text{dist } (a::\text{real}^2) b)^2 = (a\$1 - b\$1)^2 + (a\$2 - b\$2)^2$ 
  <proof>

```

```

lemma collinear-3-2d:
  collinear  $\{(a::\text{real}^2), b, c\} \longleftrightarrow$ 
   $(b\$1 - a\$1) * (c\$2 - a\$2) = (c\$1 - a\$1) * (b\$2 - a\$2)$ 
  <proof>

```

2 Geometric invariance lemmas

```

lemma collinear-linear-image:
  fixes  $f :: 'a::\text{euclidean-space} \Rightarrow 'b::\text{euclidean-space}$ 
  assumes linear  $f$  inj  $f$ 
  shows collinear  $(f ` S) = \text{collinear } S$ 
  <proof>

```

```

lemma collinear-orthogonal-transformation:
  fixes  $f :: 'a::\text{euclidean-space} \Rightarrow 'b::\text{euclidean-space}$ 
  assumes orthogonal-transformation  $f$ 
  shows collinear  $\{f a, f b, f c\} = \text{collinear } \{a, b, c\}$ 
  <proof>

```

```

lemma rotation-to-x-axis:
  fixes  $v :: \text{real}^2$ 
  assumes  $v \neq 0$ 
  obtains  $R :: \text{real}^2 \Rightarrow \text{real}^2$  where orthogonal-transformation  $R$   $(R v)\$1 =$ 
   $\text{norm } v$   $(R v)\$2 = 0$ 
  <proof>

```

```

lemma collinear-translate:
  fixes  $x :: 'a::\text{euclidean-space}$ 
  shows collinear  $\{x, y, z\} = \text{collinear } \{x - a, y - a, z - a\}$ 
  <proof>

```

```

lemma int-case-lemma:

```

fixes $c1\ c2 :: (real,2)\ vec$
assumes $0 \leq r1\ r1 < r2$ **and** $d: dist\ c1\ c2 = |r1 - r2|$ **and** $int: dist\ c1\ c2 = |r1 - r2|$
shows $c1 = c2 \wedge r1 = r2 \vee (\exists!x. dist\ c1\ x = r1 \wedge dist\ c2\ x = r2)$
<proof>

theorem *circles-tangent:*

fixes $r1\ r2 :: real$ **and** $c1\ c2 :: real^2$
assumes $0 \leq r1\ 0 \leq r2$
and $dist\ c1\ c2 = r1 + r2 \vee dist\ c1\ c2 = |r1 - r2|$
shows $c1 = c2 \wedge r1 = r2 \vee (\exists!x::real^2. dist\ c1\ x = r1 \wedge dist\ c2\ x = r2)$
<proof>

3 Feuerbach's theorem

Given a non-degenerate triangle abc , let the circle passing through the midpoints of its sides (the "9 point circle") have center $ncenter$ and radius $nradius$. Now suppose the circle with center $icenter$ and radius $iradius$ is tangent to three sides (either internally or externally to one side and two produced sides), meaning that it's either the inscribed circle or one of the three escribed circles. Then the two circles are tangent to each other, i.e. either they are identical or they touch at exactly one point.

locale *Feuerbach =*

fixes $a\ b\ c :: real^2$
and $mbc\ mac\ mab :: real^2$
and $pbc\ pac\ pab :: real^2$
and $ncenter\ icenter :: real^2$
and $nradius\ iradius :: real$
assumes $mab-def: mab = midpoint\ a\ b$
and $mbc-def: mbc = midpoint\ b\ c$
and $mac-def: mac = midpoint\ c\ a$
assumes $nr1: dist\ ncenter\ mbc = nradius$
and $nr2: dist\ ncenter\ mac = nradius$
and $nr3: dist\ ncenter\ mab = nradius$
and $ir1: dist\ icenter\ pbc = iradius$
and $ir2: dist\ icenter\ pac = iradius$
and $ir3: dist\ icenter\ pab = iradius$
and $col-ab: collinear\ \{a, b, pab\}$ **and** $orth-ab: orthogonal\ (a - b)\ (icenter - pab)$
and $col-bc: collinear\ \{b, c, pbc\}$ **and** $orth-bc: orthogonal\ (b - c)\ (icenter - pbc)$
and $col-ac: collinear\ \{a, c, pac\}$ **and** $orth-ac: orthogonal\ (a - c)\ (icenter - pac)$

begin

lemma *special:*

assumes $bx: b^2 = 0$ **and** $bne: b^1 \neq 0$ **and** $cne: c^2 \neq 0$ **and** $[simp]: a=0$
shows $ncenter = icenter \wedge nradius = iradius \vee$
 $(\exists!x::real^2. dist\ ncenter\ x = nradius \wedge dist\ icenter\ x = iradius)$
 $\langle proof \rangle$

theorem main:

assumes $\neg collinear\ \{a, b, c\}$
shows $ncenter = icenter \wedge nradius = iradius \vee$
 $(\exists!x::real^2. dist\ ncenter\ x = nradius \wedge dist\ icenter\ x = iradius)$
 $\langle proof \rangle$

end

4 Nine-point circle

As a little bonus, verify that the circle passing through the midpoints of the sides is indeed a 9-point circle, i.e. it passes through the feet of the altitudes and the midpoints of the lines joining the vertices to the orthocenter (where the altitudes intersect).

lemma nine-point-circle-1-scalar:

fixes $bv\ cx\ cy\ n1\ n2\ f1\ f2\ g1\ g2\ h1\ h2\ nr :: real$
assumes $cy \neq 0$ **and** $bv \neq 0$
— Nine-point center equidistant to midpoints of sides
— $midpoint(0,b) = (bv/2, 0)$, $midpoint(b,c) = ((bv+cx)/2, cy/2)$, $midpoint(c,0) = (cx/2, cy/2)$
and $nr1: (n1 - (bv+cx)/2)^2 + (n2 - cy/2)^2 = nr^2$
and $nr2: (n1 - cx/2)^2 + (n2 - cy/2)^2 = nr^2$
and $nr3: (n1 - bv/2)^2 + n2^2 = nr^2$
— $fab = (f1, f2)$: foot of altitude from c to ab (the x -axis)
— $collinear\ 0, (bv,0), (f1,f2): bv * f2 = f1 * 0$, so $f2 = 0$
and $fab-col: bv * f2 = 0$
— $orthogonal\ (0-(bv,0))\ ((cx,cy)-(f1,f2)): -bv*(cx-f1) + 0*(cy-f2) = 0$
and $fab-orth: bv * (cx - f1) = 0$
— $fbc = (g1, g2)$: foot of altitude from 0 to bc
— $collinear\ (bv,0), (cx,cy), (g1,g2): (cx-bv)*(g2-0) = (g1-bv)*(cy-0)$
and $fbc-col: (cx - bv) * g2 = (g1 - bv) * cy$
— $orthogonal\ ((bv,0)-(cx,cy))\ ((0,0)-(g1,g2)): (bv-cx)*(-g1) + (-cy)*(-g2) = 0$
and $fbc-orth: (bv - cx) * g1 = cy * g2$
— $fac = (h1, h2)$: foot of altitude from b to ca
— $collinear\ (cx,cy), 0, (h1,h2): (-cx)*(h2-cy) = (h1-cx)*(-cy)$
and $fac-col: cx * h2 = h1 * cy$
— $orthogonal\ ((cx,cy)-0)\ ((bv,0)-(h1,h2)): cx*(bv-h1) + cy*(0-h2) = 0$
and $fac-orth: cx * (bv - h1) = cy * h2$
shows $(n1 - f1)^2 + (n2 - f2)^2 = nr^2 \wedge (n1 - g1)^2 + (n2 - g2)^2 = nr^2 \wedge (n1 - h1)^2 + (n2 - h2)^2 = nr^2$
 $\langle proof \rangle$

theorem nine-point-circle-1:

```

fixes a b c :: real2
  and mbc mac mab :: real2
  and fbc fac fab :: real2
  and ncenter :: real2
  and nradius :: real
assumes  $\neg$  collinear {a, b, c}
  and midpoint a b = mab
  and midpoint b c = mbc
  and midpoint c a = mac
  and dist ncenter mbc = nradius
  and dist ncenter mac = nradius
  and dist ncenter mab = nradius
  and collinear {a, b, fab} orthogonal (a - b) (c - fab)
  and collinear {b, c, fbc} orthogonal (b - c) (a - fbc)
  and collinear {c, a, fac} orthogonal (c - a) (b - fac)
shows dist ncenter fab = nradius  $\wedge$  dist ncenter fbc = nradius  $\wedge$  dist ncenter fac
= nradius
<proof>

```

theorem *nine-point-circle-2*:

```

fixes a b c :: real2
  and mbc mac mab :: real2
  and fbc fac fab :: real2
  and ncenter oc :: real2
  and nradius :: real
assumes  $\neg$  collinear {a, b, c}
  and midpoint a b = mab
  and midpoint b c = mbc
  and midpoint c a = mac
  and dist ncenter mbc = nradius
  and dist ncenter mac = nradius
  and dist ncenter mab = nradius
  and collinear {a, b, fab} orthogonal (a - b) (c - fab)
  and collinear {b, c, fbc} orthogonal (b - c) (a - fbc)
  and collinear {c, a, fac} orthogonal (c - a) (b - fac)
  and collinear {oc, a, fbc} collinear {oc, b, fac} collinear {oc, c, fab}
shows dist ncenter (midpoint oc a) = nradius  $\wedge$ 
  dist ncenter (midpoint oc b) = nradius  $\wedge$ 
  dist ncenter (midpoint oc c) = nradius
<proof>

```

end