

Faithful Logic Embeddings in HOL — Deep and Shallow (Isabelle/HOL dataset)

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Abstract

A recipe for the simultaneous deployment of different forms of deep and shallow embeddings of non-classical logics in classical higher-order logic is presented, which enables interactive or even automated faithfulness proofs between the logic embeddings. The approach, which is particularly fruitful for logic education, is explained in detail in an associated CADE conference paper. This paper presents the corresponding Isabelle/HOL dataset (which is only slightly modified to meet AFP requirements).

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1 Introduction

The Isabelle/HOL dataset associated with [1] is presented. Sections 3, 4 and 5 present deep, maximally shallow, and minimally shallow embeddings of propositional modal logic (PML) in classical higher-order logic (HOL). These are connected, as a novel contribution, by automated faithfulness proofs given in Sect. 6. This connection ensures that these deep and shallow embeddings can now be used interchangeably in subsequent applications. Several experiments with the presented embeddings are presented in Sect. 7. The presented work is conceptual in nature and can be adapted to other non-classical logics. For more detailed explanations of the presented material, including a discussion of related works, see [1].

2 Preliminaries

The following preliminaries are shared between all embeddings introduced in the remainder of this paper.

```
theory PMLinHOL-preliminaries
  imports Main
begin
```

— Type declarations common for both the deep and shallow embedding

typedecl w — Type for possible worlds

typedecl \mathcal{S} — Type for propositional constant symbols

consts $p::\mathcal{S}$ $q::\mathcal{S}$ — Some propositional constant symbols

type-synonym $\mathcal{W} = w \Rightarrow bool$ — Type for sets of possible worlds

type-synonym $\mathcal{R} = w \Rightarrow w \Rightarrow bool$ — Type for accessibility relations

type-synonym $\mathcal{V} = \mathcal{S} \Rightarrow w \Rightarrow bool$ — Type for valuation functions

— Some useful predicates for accessibility relations

abbreviation(*input*) *reflexive* $\equiv \lambda R::\mathcal{R}. \forall x. R\ x\ x$

abbreviation(*input*) *symmetric* $\equiv \lambda R::\mathcal{R}. \forall x\ y. R\ x\ y \longrightarrow R\ y\ x$

abbreviation(*input*) *transitive* $\equiv \lambda R::\mathcal{R}. \forall x\ y\ z. (R\ x\ y \wedge R\ y\ z) \longrightarrow R\ x\ z$

abbreviation(*input*) *equivrel* $\equiv \lambda R::\mathcal{R}. reflexive\ R \wedge symmetric\ R \wedge transitive\ R$

abbreviation(*input*) *irreflexive* $\equiv \lambda R::\mathcal{R}. \forall x. \neg R\ x\ x$

abbreviation(*input*) *euclidean* $\equiv \lambda R::\mathcal{R}. \forall x\ y\ z. R\ x\ y \wedge R\ x\ z \longrightarrow R\ y\ z$

abbreviation(*input*) *wellfounded* $\equiv \lambda R::\mathcal{R}. \forall P::\mathcal{W}. (\forall x. (\forall y. R\ y\ x \longrightarrow P\ y) \longrightarrow P\ x) \longrightarrow (\forall x. P\ x)$

abbreviation(*input*) *converserel* $\equiv \lambda R::\mathcal{R}. \lambda y::w. \lambda x::w. R\ x\ y$

abbreviation(*input*) *conversewf* $\equiv \lambda R::\mathcal{R}. \text{wellfounded } (\text{converserel } R)$

— Bounded universal quantifier: $\forall x:W. \varphi$ stands for $\forall x. W x \longrightarrow \varphi x$

abbreviation(*input*) *BoundedAll*:: $\mathcal{W}\Rightarrow\mathcal{W}\Rightarrow\text{bool}$ **where** *BoundedAll* $W \varphi \equiv \forall x. W x \longrightarrow \varphi x$

syntax *-BoundedAll*:: $\text{pttrn}\Rightarrow\mathcal{W}\Rightarrow\text{bool}\Rightarrow\text{bool}$ $((\exists\forall(-/:-)/-)$ $[0, 0, 10]$ $10)$

translations $\forall x:W. \varphi \equiv \text{CONST } \text{BoundedAll } W (\lambda x. \varphi)$

— Backward implication; useful for aesthetic reasons

abbreviation(*input*) *Bimp* (**infixr** \longleftarrow 50) **where** $\varphi \longleftarrow \psi \equiv \psi \longrightarrow \varphi$

— Some further settings

declare $[[\text{syntax-ambiguity-warning}=\text{false}]]$

nitpick-params $[\text{user-axioms}, \text{expect}=\text{genuine}, \text{timeout}=60]$

end

3 Deep embedding of PML in HOL

theory *PMLinHOL-deep*

imports *PMLinHOL-preliminaries*

begin

— Deep embedding (of propositional modal logic in HOL)

datatype *PML* = *AtmD* \mathcal{S} $(-^d)$ | *NotD* *PML* $(-^d)$ | *ImpD* *PML* *PML* (**infixr** \supset^d 93) | *BoxD* *PML* (\Box^d)

— Further logical connectives as definitions

definition *OrD* (**infixr** \vee^d 92) **where** $\varphi \vee^d \psi \equiv \neg^d \varphi \supset^d \psi$

definition *AndD* (**infixr** \wedge^d 95) **where** $\varphi \wedge^d \psi \equiv \neg^d (\varphi \supset^d \neg^d \psi)$

definition *DiaD* (\diamond^d) **where** $\diamond^d \varphi \equiv \neg^d (\Box^d (\neg^d \varphi))$

definition *TopD* (\top^d) **where** $\top^d \equiv p^d \supset^d p^d$

definition *BotD* (\perp^d) **where** $\perp^d \equiv \neg^d \top^d$

— Definition of truth of a formula relative to a model $\langle W, R, V \rangle$ and possible world w

primrec *RelativeTruthD* :: $\mathcal{W}\Rightarrow\mathcal{R}\Rightarrow\mathcal{V}\Rightarrow w\Rightarrow\text{PML}\Rightarrow\text{bool}$ $(\langle -, -, - \rangle, - \models^d -)$ **where**

$\langle W, R, V \rangle, w \models^d a^d = (V a w)$

| $\langle W, R, V \rangle, w \models^d \neg^d \varphi = (\neg \langle W, R, V \rangle, w \models^d \varphi)$

| $\langle W, R, V \rangle, w \models^d \varphi \supset^d \psi = (\langle W, R, V \rangle, w \models^d \varphi \longrightarrow \langle W, R, V \rangle, w \models^d \psi)$

| $\langle W, R, V \rangle, w \models^d \Box^d \varphi = (\forall v:W. R w v \longrightarrow \langle W, R, V \rangle, v \models^d \varphi)$

— Definition of validity

definition *ValD* $(\models^d -)$ **where** $(\models^d \varphi) \equiv (\forall W R V. \forall w:W. \langle W, R, V \rangle, w \models^d \varphi)$

— Collection of definitions in a bag called DefD

named-theorems *DefD* **declare** *OrD-def* $[\text{DefD}, \text{simp}]$ *AndD-def* $[\text{DefD}, \text{simp}]$ *DiaD-def* $[\text{DefD}, \text{simp}]$ *TopD-def* $[\text{DefD}, \text{simp}]$ *BotD-def* $[\text{DefD}, \text{simp}]$ *RelativeTruthD-def* $[\text{DefD}, \text{simp}]$ *ValD-def* $[\text{DefD}, \text{simp}]$

end

4 Shallow embedding of PML in HOL (maximal)

theory *PMLinHOL-shallow*

imports *PMLinHOL-preliminaries*

begin

— Shallow embedding (of propositional modal logic in HOL)

type-synonym $\sigma = \mathcal{W} \Rightarrow \mathcal{R} \Rightarrow \mathcal{V} \Rightarrow w \Rightarrow bool$

definition *AtmS*:: $\mathcal{S} \Rightarrow \sigma$ ($-^s$) **where** $a^s \equiv \lambda W R V w. V a w$

definition *NegS*:: $\sigma \Rightarrow \sigma$ (\neg^s) **where** $\neg^s \varphi \equiv \lambda W R V w. \neg(\varphi W R V w)$

definition *ImpS*:: $\sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infix** \supset^s 93) **where** $\varphi \supset^s \psi \equiv \lambda W R V w. (\varphi W R V w) \longrightarrow (\psi W R V w)$

definition *BoxS*:: $\sigma \Rightarrow \sigma$ (\Box^s) **where** $\Box^s \varphi \equiv \lambda W R V w. \forall v:W. R w v \longrightarrow (\varphi W R V v)$

— Further logical connectives as definitions

definition *OrS* (**infix** \vee^s 92) **where** $\varphi \vee^s \psi \equiv \neg^s \varphi \supset^s \psi$

definition *AndS* (**infix** \wedge^s 95) **where** $\varphi \wedge^s \psi \equiv \neg^s(\varphi \supset^s \neg^s \psi)$

definition *DiaS* (\Diamond^s) **where** $\Diamond^s \varphi \equiv \neg^s(\Box^s(\neg^s \varphi))$

definition *TopS* (\top^s) **where** $\top^s \equiv p^s \supset^s p^s$

definition *BotS* (\perp^s) **where** $\perp^s \equiv \neg^s \top^s$

— Definition of truth of a formula relative to a model $\langle W, R, V \rangle$ and possible world w

definition *RelativeTruthS*:: $\mathcal{W} \Rightarrow \mathcal{R} \Rightarrow \mathcal{V} \Rightarrow w \Rightarrow \sigma \Rightarrow bool$ ($\langle -, -, - \rangle, \models^s -$) **where** $\langle W, R, V \rangle, w \models^s \varphi \equiv \varphi W R V w$

— Definition of validity

definition *ValS* ($\models^s -$) **where** $\models^s \varphi \equiv \forall W R V. \forall w:W. \langle W, R, V \rangle, w \models^s \varphi$

— Collection of definitions in a bag called DefS

named-theorems *DefS* **declare** *AtmS-def*[*DefS, simp*] *NegS-def*[*DefS, simp*] *ImpS-def*[*DefS, simp*] *BoxS-def*[*DefS, simp*] *OrS-def*[*DefS, simp*] *AndS-def*[*DefS, simp*] *DiaS-def*[*DefS, simp*] *TopS-def*[*DefS, simp*] *BotS-def*[*DefS, simp*] *RelativeTruthS-def*[*DefS, simp*] *ValS-def*[*DefS, simp*]
end

5 Shallow embedding of PML in HOL (minimal)

theory *PMLinHOL-shallow-minimal*

imports *PMLinHOL-preliminaries*

begin

— The accessibility relation R and the valuation function V are introduced as constants at the meta-level HOL

consts *R*:: \mathcal{R} *V*:: \mathcal{V}

— Shallow embedding (of propositional modal logic in HOL)

type-synonym $\sigma = w \Rightarrow bool$

definition *AtmM*:: $\mathcal{S} \Rightarrow \sigma$ ($-^m$) **where** $a^m \equiv \lambda w. V a w$

definition $NegM::\sigma\Rightarrow\sigma$ (\neg^m) **where** $\neg^m \varphi \equiv \lambda w. \neg \varphi w$
definition $ImpM::\sigma\Rightarrow\sigma\Rightarrow\sigma$ (**infixr** \supset^m 93) **where** $\varphi \supset^m \psi \equiv \lambda w. \varphi w \longrightarrow \psi w$
definition $BoxM::\sigma\Rightarrow\sigma$ (\Box^m) **where** $\Box^m \varphi \equiv \lambda w. \forall v. R w v \longrightarrow \varphi v$

— Further logical connectives as definitions

definition OrM (**infixr** \vee^m 92) **where** $\varphi \vee^m \psi \equiv \neg^m \varphi \supset^m \psi$
definition $AndM$ (**infixr** \wedge^m 95) **where** $\varphi \wedge^m \psi \equiv \neg^m (\varphi \supset^m \neg^m \psi)$
definition $DiaM$ (\diamond^m -) **where** $\diamond^m \varphi \equiv \neg^m (\Box^m (\neg^m \varphi))$
definition $TopM$ (\top^m) **where** $\top^m \equiv p^m \supset^m p^m$
definition $BotM$ (\perp^m) **where** $\perp^m \equiv \neg^m \top^m$

— Definition of truth of a formula relative to a model $\langle W, R, V \rangle$ and a possible world w

definition $RelativeTruthM::w\Rightarrow\sigma\Rightarrow bool$ (\models^m -) **where** $w \models^m \varphi \equiv \varphi w$

— Definition of validity

definition $ValM$ (\models^m -) **where** $\models^m \varphi \equiv \forall w::w. w \models^m \varphi$

— Collection of definitions in a bag called DefM

named-theorems $DefM$ **declare** $AtmM-def[DefM, simp]$ $NegM-def[DefM, simp]$
 $ImpM-def[DefM, simp]$ $BoxM-def[DefM, simp]$ $OrM-def[DefM, simp]$ $AndM-def[DefM, simp]$
 $DiaM-def[DefM, simp]$ $TopM-def[DefM, simp]$ $BotM-def[DefM, simp]$ $RelativeTruthM-def[DefM, simp]$
 $ValM-def[DefM, simp]$
end

6 Automated faithfulness proofs

theory $PMLinHOL-faithfulness$

imports $PMLinHOL-deep$ $PMLinHOL-shallow$ $PMLinHOL-shallow-minimal$
begin

— Mappings: deep to maximal shallow and deep to minimal shallow

primrec $DpToShMax$ ($\llbracket - \rrbracket$) **where** $\llbracket \varphi^d \rrbracket = \varphi^s \mid \llbracket \neg^d \varphi \rrbracket = \neg^s \llbracket \varphi \rrbracket \mid \llbracket \varphi \supset^d \psi \rrbracket = \llbracket \varphi \rrbracket \supset^s \llbracket \psi \rrbracket \mid \llbracket \Box^d \varphi \rrbracket = \Box^s \llbracket \varphi \rrbracket$
primrec $DpToShMin$ ($\llbracket - \rrbracket$) **where** $\llbracket \varphi^d \rrbracket = \varphi^m \mid \llbracket \neg^d \varphi \rrbracket = \neg^m \llbracket \varphi \rrbracket \mid \llbracket \varphi \supset^d \psi \rrbracket = \llbracket \varphi \rrbracket \supset^m \llbracket \psi \rrbracket \mid \llbracket \Box^d \varphi \rrbracket = \Box^m \llbracket \varphi \rrbracket$

— Proving faithfulness between deep and maximal shallow

theorem $Faithful1a$: $\forall W R V. \forall w:W. \langle W, R, V \rangle, w \models^d \varphi \longleftrightarrow \langle W, R, V \rangle, w \models^s \llbracket \varphi \rrbracket$ $\langle proof \rangle$

theorem $Faithful1b$: $\models^d \varphi \longleftrightarrow \models^s \llbracket \varphi \rrbracket$ $\langle proof \rangle$

theorem $Faithful2$: $\forall w. \langle (\lambda x::w. True), R, V \rangle, w \models^d \varphi \longleftrightarrow w \models^m \llbracket \varphi \rrbracket$ $\langle proof \rangle$

theorem $Faithful3$: $\forall w. \langle (\lambda x::w. True), R, V \rangle, w \models^s \llbracket \varphi \rrbracket \longleftrightarrow w \models^m \llbracket \varphi \rrbracket$ $\langle proof \rangle$

lemma $Sound1$: $\models^m \psi \longleftrightarrow (\exists \varphi. \psi = \llbracket \varphi \rrbracket \wedge \models^d \varphi)$ $\langle proof \rangle$

lemma $Sound2$: $\models^m \psi \longleftrightarrow (\exists \varphi. \psi = \llbracket \varphi \rrbracket \wedge \models^m \llbracket \varphi \rrbracket)$ $\langle proof \rangle$

end

7 Appendix: proof automation tests

7.1 Tests with the deep embedding

theory *PMLinHOL-deep-tests*

imports *PMLinHOL-deep*

begin

— Hilbert calculus: proving that the schematic axioms and rules implied by the embedding

lemma *H1*: $\models^d \varphi \supset^d (\psi \supset^d \varphi)$ *<proof>*

lemma *H2*: $\models^d (\varphi \supset^d (\psi \supset^d \gamma)) \supset^d ((\varphi \supset^d \psi) \supset^d (\varphi \supset^d \gamma))$ *<proof>*

lemma *H3*: $\models^d (\neg^d \varphi \supset^d \neg^d \psi) \supset^d (\psi \supset^d \varphi)$ *<proof>*

lemma *MP*: $\models^d \varphi \implies \models^d (\varphi \supset^d \psi) \implies \models^d \psi$ *<proof>*

lemma *HCderived1*: $\models^d (\varphi \supset^d \varphi)$ — sledgehammer(HC1 HC2 HC3 MP) returns:
by (metis HC1 HC2 MP)

<proof>

lemma *HCderived2*: $\models^d \varphi \supset^d (\neg^d \varphi \supset^d \psi)$ *<proof>*

lemma *HCderived3*: $\models^d (\neg^d \varphi \supset^d \varphi) \supset^d \varphi$ *<proof>*

lemma *HCderived4*: $\models^d (\varphi \supset^d \psi) \supset^d (\neg^d \psi \supset^d \neg^d \varphi)$ *<proof>*

lemma *Nec*: $\models^d \varphi \implies \models^d \Box^d \varphi$ *<proof>*

lemma *Dist*: $\models^d \Box^d (\varphi \supset^d \psi) \supset^d (\Box^d \varphi \supset^d \Box^d \psi)$ *<proof>*

lemma *cM*: *reflexive* $R \iff (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d \varphi \supset^d \varphi)$ —
sledgehammer: Proof found *<proof>*

lemma *cBa*: *symmetric* $R \implies (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \varphi \supset^d \Box^d (\Diamond^d \varphi))$
<proof>

lemma *cBb*: *symmetric* $R \leftarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \varphi \supset^d \Box^d (\Diamond^d \varphi))$
— sledgehammer: No proof *<proof>*

lemma *c4a*: *transitive* $R \implies (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d \varphi \supset^d \Box^d (\Box^d \varphi))$
<proof>

lemma *c4b*: *transitive* $R \leftarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d \varphi \supset^d \Box^d (\Box^d \varphi))$
— sledgehammer: No proof *<proof>*

lemma *reflexive* $R \implies (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d \varphi \supset^d \Box^d (\Box^d \varphi))$
nitpick[*card w=3*] *<proof>*

lemma $\models^d \varphi \supset^d \Box^d \varphi$ **nitpick**[*card w=2, card S=1*] *<proof>*

lemma $\models^d \Box^d (\Box^d \varphi \supset^d \Box^d \psi) \vee^d \Box^d (\Box^d \psi \supset^d \Box^d \varphi)$ **nitpick**[*card w=3*] *<proof>*

lemma $\models^d (\Diamond^d (\Box^d \varphi)) \supset^d \Box^d (\Diamond^d \varphi)$ **nitpick**[*card w=2*] *<proof>*

lemma *KB*: *symmetric* $R \implies (\forall \varphi \psi W V. \forall w: W. \langle W, R, V \rangle, w \models^d (\Diamond^d (\Box^d \varphi)) \supset^d \Box^d (\Diamond^d \varphi))$
<proof>

lemma *K4B*: *symmetric* $R \wedge$ *transitive* $R \implies (\forall \varphi \psi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d (\Box^d \varphi \supset^d \Box^d \psi) \vee^d \Box^d (\Box^d \psi \supset^d \Box^d \varphi))$ *<proof>*

end

theory *PMLinHOL-deep-further-tests*

imports *PMLinHOL-deep-tests*

begin

— Implied modal principle

lemma *K-Dia*: $\models^d (\Box^d (\varphi \supset^d \psi)) \supset^d ((\Diamond^d \varphi) \supset^d \Diamond^d \psi)$ *<proof>*

lemma *T1a*: $\models^d \Box^d p^d \supset^d ((\Diamond^d q^d) \supset^d \Diamond^d (p^d \wedge^d q^d))$ *<proof>*
lemma *T1b*: $\models^d \Box^d p^d \supset^d ((\Diamond^d q^d) \supset^d \Diamond^d (p^d \wedge^d q^d))$ — alternative interactive proof in modal object logic K
<proof>
end

theory *PMLinHOL-deep-Loeb-tests*
imports *PMLinHOL-deep*
begin

— Löb axiom: with the deep embedding automated reasoning tools are not very responsive

lemma *Loeb1*: $\forall \varphi. \models^d \Box^d (\Box^d \varphi \supset^d \varphi) \supset^d \Box^d \varphi$ **nitpick**[*card w=1, card S=1*] *<proof>*

lemma *Loeb2*: $(\text{conversewf } R \wedge \text{transitive } R) \longrightarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d (\Box^d \varphi \supset^d \varphi) \supset^d \Box^d \varphi)$ — sledgehammer: No Proof *<proof>*

lemma *Loeb3*: $(\text{conversewf } R \wedge \text{transitive } R) \longleftarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d (\Box^d \varphi \supset^d \varphi) \supset^d \Box^d \varphi)$ — sledgehammer: No Proof *<proof>*

lemma *Loeb3a*: $\text{conversewf } R \longleftarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d (\Box^d \varphi \supset^d \varphi) \supset^d \Box^d \varphi)$ — sledgehammer: No Proof *<proof>*

lemma *Loeb3b*: $\text{transitive } R \longleftarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d (\Box^d \varphi \supset^d \varphi) \supset^d \Box^d \varphi)$ — sledgehammer: No Proof *<proof>*

lemma *Loeb3c*: $\text{irreflexive } R \longleftarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^d \Box^d (\Box^d \varphi \supset^d \varphi) \supset^d \Box^d \varphi)$ — sledgehammer: No Proof *<proof>*

end

7.2 Tests with the maximal shallow embedding

theory *PMLinHOL-shallow-tests*
imports *PMLinHOL-shallow*
begin

— Hilbert calculus: proving that the schematic axioms and rules implied by the embedding

lemma *H1*: $\models^s \varphi \supset^s (\psi \supset^s \varphi)$ *<proof>*

lemma *H2*: $\models^s (\varphi \supset^s (\psi \supset^s \gamma)) \supset^s ((\varphi \supset^s \psi) \supset^s (\varphi \supset^s \gamma))$ *<proof>*

lemma *H3*: $\models^s (\neg^s \varphi \supset^s \neg^s \psi) \supset^s (\psi \supset^s \varphi)$ *<proof>*

lemma *MP*: $\models^s \varphi \implies \models^s (\varphi \supset^s \psi) \implies \models^s \psi$ *<proof>*

lemma *HCderived1*: $\models^s (\varphi \supset^s \varphi)$ — sledgehammer(HC1 HC2 HC3 MP) returns: by (metis HC1 HC2 MP) *<proof>*

lemma *HCderived2*: $\models^s \varphi \supset^s (\neg^s \varphi \supset^s \psi)$ *<proof>*

lemma *HCderived3*: $\models^s (\neg^s \varphi \supset^s \varphi) \supset^s \varphi$ *<proof>*

lemma *HCderived4*: $\models^s (\varphi \supset^s \psi) \supset^s (\neg^s \psi \supset^s \neg^s \varphi)$ *<proof>*

lemma *Nec*: $\models^s \varphi \implies \models^s \Box^s \varphi$ *<proof>*

lemma *Dist*: $\models^s \Box^s (\varphi \supset^s \psi) \supset^s (\Box^s \varphi \supset^s \Box^s \psi)$ *<proof>*

lemma *cM:reflexive* $R \longleftrightarrow (\forall \varphi W V. \forall w: W. \langle W, R, V \rangle, w \models^s \Box^s \varphi \supset^s \varphi)$ — sledgehammer: Proof found *<proof>*

lemma *cBa*: *symmetric R* \longrightarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi))$
 $\langle proof \rangle$

lemma *cBb*: *symmetric R* \longleftarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi))$
— sledgehammer: No proof $\langle proof \rangle$

lemma *c4a*: *transitive R* \longrightarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s \varphi \supset^s \Box^s(\Box^s \varphi))$
 $\langle proof \rangle$

lemma *c4b*: *transitive R* \longleftarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s \varphi \supset^s \Box^s(\Box^s \varphi))$
— sledgehammer: No proof $\langle proof \rangle$

lemma *reflexive R* \longrightarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s \varphi \supset^s \Box^s(\Box^s \varphi))$
nitpick[*card w=3*] $\langle proof \rangle$

lemma $\models^s \varphi \supset^s \Box^s \varphi$ **nitpick**[*card w=2, card S=1*] $\langle proof \rangle$

lemma $\models^s \Box^s(\Box^s \varphi \supset^s \Box^s \psi) \vee^s \Box^s(\Box^s \psi \supset^s \Box^s \varphi)$ $\langle proof \rangle$

lemma $\models^s (\Diamond^s(\Box^s \varphi)) \supset^s \Box^s(\Diamond^s \varphi)$ **nitpick**[*card w=2*] $\langle proof \rangle$

lemma *KB*: *symmetric R* \longrightarrow $(\forall \varphi \psi W V. \forall w:W. \langle W,R,V \rangle, w \models^s (\Diamond^s(\Box^s \varphi)) \supset^s \Box^s(\Diamond^s \varphi))$ $\langle proof \rangle$

lemma *K4B*: *symmetric R* \wedge *transitive R* \longrightarrow $(\forall \varphi \psi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s(\Box^s \varphi \supset^s \Box^s \psi) \vee^s \Box^s(\Box^s \psi \supset^s \Box^s \varphi))$ $\langle proof \rangle$

end

theory *PMLinHOL-shallow-further-tests*

imports *PMLinHOL-shallow-tests*

begin

— Implied modal principle

lemma *K-Dia*: $\models^s (\Box^s(\varphi \supset^s \psi)) \supset^s ((\Diamond^s \varphi) \supset^s \Diamond^s \psi)$ $\langle proof \rangle$

lemma *T1a*: $\models^s \Box^s p^s \supset^s ((\Diamond^s q^s) \supset^s \Diamond^s (p^s \wedge^s q^s))$ $\langle proof \rangle$

lemma *T1b*: $\models^s \Box^s p^s \supset^s ((\Diamond^s q^s) \supset^s \Diamond^s (p^s \wedge^s q^s))$ — alternative interactive proof in modal object logic K

$\langle proof \rangle$

end

theory *PMLinHOL-shallow-Loeb-tests*

imports *PMLinHOL-shallow*

begin

— Löb axiom: with the minimal shallow embedding automated reasoning tools are still partly responsive

lemma *Loeb1*: $\forall \varphi. \models^s \Box^s(\Box^s \varphi \supset^s \varphi) \supset^s \Box^s \varphi$ **nitpick**[*card w=1, card S=1*] $\langle proof \rangle$

lemma *Loeb2*: $(\text{conversewf } R \wedge \text{transitive } R) \longrightarrow (\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s(\Box^s \varphi \supset^s \varphi) \supset^s \Box^s \varphi)$ — sledgehammer: Proof found $\langle proof \rangle$

lemma *Loeb3*: $(\text{conversewf } R \wedge \text{transitive } R) \longleftarrow (\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s(\Box^s \varphi \supset^s \varphi) \supset^s \Box^s \varphi)$ — sledgehammer: No Proof $\langle proof \rangle$

lemma *Loeb3a*: *conversewf R* \longleftarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s(\Box^s \varphi \supset^s \varphi) \supset^s \Box^s \varphi)$ — sledgehammer: Proof found $\langle proof \rangle$

lemma *Loeb3b*: *transitive R* \longleftarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s(\Box^s \varphi \supset^s \varphi) \supset^s \Box^s \varphi)$ — sledgehammer: No Proof $\langle proof \rangle$

lemma *Loeb3c*: *irreflexive R* \longleftarrow $(\forall \varphi W V. \forall w:W. \langle W,R,V \rangle, w \models^s \Box^s(\Box^s \varphi \supset^s \varphi) \supset^s \Box^s \varphi)$ — sledgehammer: Proof found $\langle proof \rangle$

end

7.3 Tests with the minimal shallow embedding

theory *PMLinHOL-shallow-minimal-tests*

imports *PMLinHOL-shallow-minimal*

begin

— Hilbert calculus: proving that the schematic axioms and rules implied by the embedding

lemma *H1*: $\models^m \varphi \supset^m (\psi \supset^m \varphi)$ *<proof>*

lemma *H2*: $\models^m (\varphi \supset^m (\psi \supset^m \gamma)) \supset^m ((\varphi \supset^m \psi) \supset^m (\varphi \supset^m \gamma))$ *<proof>*

lemma *H3*: $\models^m (\neg^m \varphi \supset^m \neg^m \psi) \supset^m (\psi \supset^m \varphi)$ *<proof>*

lemma *MP*: $\models^m \varphi \implies \models^m (\varphi \supset^m \psi) \implies \models^m \psi$ *<proof>*

lemma *HCderived1*: $\models^m (\varphi \supset^m \varphi)$ — sledgehammer(HC1 HC2 HC3 MP) returns:
by (metis HC1 HC2 MP)
<proof>

lemma *HCderived2*: $\models^m \varphi \supset^m (\neg^m \varphi \supset^m \psi)$ *<proof>*

lemma *HCderived3*: $\models^m (\neg^m \varphi \supset^m \varphi) \supset^m \varphi$ *<proof>*

lemma *HCderived4*: $\models^m (\varphi \supset^m \psi) \supset^m (\neg^m \psi \supset^m \neg^m \varphi)$ *<proof>*

lemma *Nec*: $\models^m \varphi \implies \models^m \Box^m \varphi$ *<proof>*

lemma *Dist*: $\models^m \Box^m (\varphi \supset^m \psi) \supset^m (\Box^m \varphi \supset^m \Box^m \psi)$ *<proof>*

lemma *cM:reflexive* $R \longleftrightarrow (\forall \varphi. \models^m \Box^m \varphi \supset^m \varphi)$ *<proof>*

lemma *cBa: symmetric* $R \longrightarrow (\forall \varphi. \models^m \varphi \supset^m \Box^m \Diamond^m \varphi)$ *<proof>*

lemma *cBb: symmetric* $R \longleftarrow (\forall \varphi. \models^m \varphi \supset^m \Box^m \Diamond^m \varphi)$ *<proof>*

lemma *c4a: transitive* $R \longrightarrow (\forall \varphi. \models^m \Box^m \varphi \supset^m \Box^m (\Box^m \varphi))$ *<proof>*

lemma *c4b: transitive* $R \longleftarrow (\forall \varphi. \models^m \Box^m \varphi \supset^m \Box^m (\Box^m \varphi))$ *<proof>*

lemma *reflexive* $R \longrightarrow (\forall \varphi. \models^m \Box^m \varphi \supset^m \Box^m (\Box^m \varphi))$ **nitpick**[*card w=3, show-all*]
<proof>

lemma $\models^m \varphi \supset^m \Box^m \varphi$ **nitpick**[*card w=2, card S= 1*] *<proof>*

lemma $\models^m \Box^m (\Box^m \varphi \supset^m \Box^m \psi) \vee^m \Box^m (\Box^m \psi \supset^m \Box^m \varphi)$ **nitpick**[*card w=3*]
<proof>

lemma $\models^m (\Diamond^m (\Box^m \varphi)) \supset^m \Box^m (\Diamond^m \varphi)$ **nitpick**[*card w=2*] *<proof>*

lemma *KB: symmetric* $R \longrightarrow (\forall \varphi \psi. \models^m (\Diamond^m (\Box^m \varphi)) \supset^m \Box^m (\Diamond^m \varphi))$ *<proof>*

lemma *K4B: symmetric* $R \wedge$ *transitive* $R \longrightarrow (\forall \varphi \psi. \models^m \Box^m (\Box^m \varphi \supset^m \Box^m \psi) \vee^m \Box^m (\Box^m \psi \supset^m \Box^m \varphi))$ *<proof>*

end

theory *PMLinHOL-shallow-minimal-further-tests*

imports *PMLinHOL-shallow-minimal-tests*

begin

— Implied modal principle

lemma *K-Dia*: $\models^m (\Box^m (\varphi \supset^m \psi)) \supset^m ((\Diamond^m \varphi) \supset^m \Diamond^m \psi)$ *<proof>*

lemma *T1a*: $\models^m \Box^m p^m \supset^m ((\Diamond^m q^m) \supset^m \Diamond^m (p^m \wedge^m q^m))$ *<proof>*

lemma *T1b*: $\models^m \Box^m p^m \supset^m ((\Diamond^m q^m) \supset^m \Diamond^m (p^m \wedge^m q^m))$ — alternative inter-
active proof in modal object logic K
<proof>

end

theory *PMLinHOL-shallow-minimal-Loeb-tests*

imports *PMLinHOL-shallow-minimal*
begin
— Löb axiom: with the minimal shallow embedding automated reasoning tools are still partly responsive
lemma *Loeb1*: $\forall \varphi. \models^m \Box^m (\Box^m \varphi \supset^m \varphi) \supset^m \Box^m \varphi$ **nitpick**[*card w=1, card S=1*]
 $\langle proof \rangle$
lemma *Loeb2*: $(conversewf\ R \wedge transitive\ R) \longrightarrow (\forall \varphi. \models^m \Box^m (\Box^m \varphi \supset^m \varphi) \supset^m \Box^m \varphi)$ — sh: Proof found $\langle proof \rangle$
lemma *Loeb3*: $(conversewf\ R \wedge transitive\ R) \longleftarrow (\forall \varphi. \models^m \Box^m (\Box^m \varphi \supset^m \varphi) \supset^m \Box^m \varphi)$ — sh: No Proof $\langle proof \rangle$
lemma *Loeb3a*: $conversewf\ R \longleftarrow (\forall \varphi. \models^m \Box^m (\Box^m \varphi \supset^m \varphi) \supset^m \Box^m \varphi)$ $\langle proof \rangle$
lemma *Loeb3b*: $transitive\ R \longleftarrow (\forall \varphi. \models^m \Box^m (\Box^m \varphi \supset^m \varphi) \supset^m \Box^m \varphi)$ — sledgehammer: No Proof $\langle proof \rangle$
lemma *Loeb3c*: $irreflexive\ R \longleftarrow (\forall \varphi. \models^m \Box^m (\Box^m \varphi \supset^m \varphi) \supset^m \Box^m \varphi)$ — sledgehammer: Proof found $\langle proof \rangle$
end

8 Test Examples: Formula classification

8.1 Tests with the deep embedding: axiom schemata

theory *PMLinHOL-deep-further-tests-1*
imports *PMLinHOL-deep-tests*
begin
— What is the weakest modal logic in which the following test formulas F1-F10 are provable?
— Test with schematic axioms

consts $\varphi::PML\ \psi::PML$
abbreviation(*input*) $F1 \equiv (\Diamond^d(\Diamond^d \varphi)) \supset^d \Diamond^d \varphi$ — holds in K4
abbreviation(*input*) $F2 \equiv (\Diamond^d(\Box^d \varphi)) \supset^d \Box^d(\Diamond^d \varphi)$ — holds in KB
abbreviation(*input*) $F3 \equiv (\Diamond^d(\Box^d \varphi)) \supset^d \Box^d \varphi$ — holds in KB4
abbreviation(*input*) $F4 \equiv (\Box^d(\Diamond^d(\Box^d(\Diamond^d \varphi)))) \supset^d \Box^d(\Diamond^d \varphi)$ — holds in KB/K4
abbreviation(*input*) $F5 \equiv (\Diamond^d(\varphi \wedge^d (\Diamond^d \psi))) \supset^d ((\Diamond^d \varphi) \wedge^d (\Diamond^d \psi))$ — holds in K4
abbreviation(*input*) $F6 \equiv ((\Box^d(\varphi \supset^d \psi)) \wedge^d (\Diamond^d(\Box^d(\neg^d \psi)))) \supset^d \neg^d(\Diamond^d \psi)$ — holds in KB4
abbreviation(*input*) $F7 \equiv (\Diamond^d \varphi) \supset^d (\Box^d(\varphi \vee^d \Diamond^d \varphi))$ — holds in KB4
abbreviation(*input*) $F8 \equiv (\Diamond^d(\Box^d \varphi)) \supset^d (\varphi \vee^d \Diamond^d \varphi)$ — holds in KT/KB
abbreviation(*input*) $F9 \equiv ((\Box^d(\Diamond^d \varphi)) \wedge^d (\Box^d(\Diamond^d(\neg^d \varphi)))) \supset^d \Diamond^d(\Diamond^d \varphi)$ — holds in KT
abbreviation(*input*) $F10 \equiv ((\Box^d(\varphi \supset^d \Box^d \psi)) \wedge^d (\Box^d(\Diamond^d(\neg^d \psi)))) \supset^d \neg^d(\Box^d \psi)$ — holds in KT

declare *imp-cong*[*cong del*]

experiment begin

lemma *S5*: $\forall w:W. (\forall \varphi. \langle W, R, V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W, R, V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W, R, V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W, R, V \rangle, w \models^d F1$

— nitpick[expect=none] — sledgehammer — none — proof
 <proof>
lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F1$
 — nitpick[expect=none] — sledgehammer — none — proof
 <proof>
lemma $KB4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F1$
 — nitpick[expect=none] — sledgehammer — none — proof
 <proof>
lemma KTB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F1$
 — nitpick[expect=none] — sledgehammer — none — no prf
 <proof>
lemma KT : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F1$
 — nitpick[expect=none] — sledgehammer — none — no prf
 <proof>
lemma KB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F1$
 — nitpick[expect=none] — sledgehammer — none — no prf
 <proof>
lemma $K4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F1$
 — nitpick[expect=none] — sledgehammer — none — proof
 <proof>
lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^d F1$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 <proof>
end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F2$
 — nitpick[expect=none] — sledgehammer — none — no prf
 <proof>
lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F2$
 — nitpick[expect=none] — sledgehammer — none — no prf
 <proof>
lemma $KB4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F2$
 — nitpick[expect=none] — sledgehammer — none — no prf
 <proof>
lemma KTB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F2$
 — nitpick[expect=none] — sledgehammer — none — no prf
 <proof>

lemma *KT*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F2$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F2$
— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma *K4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F2$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *K*: $\forall w:W. \langle W,R,V \rangle, w \models^d F2$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma *S5*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *S4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KB4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KTB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KT*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *K4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *K*: $\forall w:W. \langle W,R,V \rangle, w \models^d F3$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma $KB4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma KTB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma KT : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma KB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma $K4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^d F4$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩

end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F5$

— nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩

lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F5$

— nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩

lemma $KB4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F5$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩

lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^d F6$
 — nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi))) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \Box^d(\Box^d \varphi))) \longrightarrow \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \Box^d(\Box^d \varphi))) \longrightarrow \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma $KB4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \Box^d(\Box^d \varphi))) \longrightarrow \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma KTB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi))) \longrightarrow \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma KT : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma KB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma $K4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \Box^d(\Box^d \varphi))) \longrightarrow \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^d F7$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi))) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \Box^d(\Box^d \varphi))) \longrightarrow \langle W,R,V \rangle, w \models^d F8$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi \supset^d \Box^d(\Box^d \varphi))) \longrightarrow \langle W,R,V \rangle, w \models^d F8$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KB4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F8$
— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma *KTB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F8$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KT*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F8$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F8$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *K4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F8$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *K*: $\forall w:W. \langle W,R,V \rangle, w \models^d F8$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma *S5*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F9$
— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma *S4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F9$
— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma *KB4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F9$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma *KTB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F9$
— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma *KT*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F9$
— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F9$
— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma $K4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F9$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^d F9$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma $KB4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma KTB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma KT : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \varphi) \longrightarrow \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma KB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d \varphi \supset^d \Box^d(\Diamond^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma $K4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^d (\Box^d \varphi) \supset^d \Box^d(\Box^d \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^d F10$

— nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

end

— Summary of experiments: Nitpick: ctex=16 (with simp 10, without simp 6), none=74 (with simp 0, without simp 74), unknwn=70 (with simp 70, without simp 0) Sledgehammer: proof=33 (with simp 16, without simp 17), no prf=127 (with simp 64, without simp 63)

— No conflict

end

8.2 Tests with the deep embedding: semantic frame conditions

theory *PMLinHOL-deep-further-tests-2*

imports *PMLinHOL-deep-tests*

begin

— What is the weakest modal logic in which the following test formulas F1-F10 are provable?

— Test with semantic conditions

abbreviation(*input*) *refl* (*r*) **where** $r \equiv \lambda R. \text{ reflexive } R$

abbreviation(*input*) *sym* (*s*) **where** $s \equiv \lambda R. \text{ symmetric } R$

abbreviation(*input*) *tra* (*t*) **where** $t \equiv \lambda R. \text{ transitive } R$

consts $\varphi::PML \ \psi::PML$

abbreviation(*input*) $F1 \equiv (\diamond^d(\diamond^d\varphi)) \supset^d \diamond^d\varphi$ — holds in K4

abbreviation(*input*) $F2 \equiv (\diamond^d(\Box^d\varphi)) \supset^d \Box^d(\diamond^d\varphi)$ — holds in KB

abbreviation(*input*) $F3 \equiv (\diamond^d(\Box^d\varphi)) \supset^d \Box^d\varphi$ — holds in KB4

abbreviation(*input*) $F4 \equiv (\Box^d(\diamond^d(\Box^d(\diamond^d\varphi)))) \supset^d \Box^d(\diamond^d\varphi)$ — holds in KB/K4

abbreviation(*input*) $F5 \equiv (\diamond^d(\varphi \wedge^d (\diamond^d\psi))) \supset^d ((\diamond^d\varphi) \wedge^d (\diamond^d\psi))$ — holds in K4

abbreviation(*input*) $F6 \equiv ((\Box^d(\varphi \supset^d \psi)) \wedge^d (\Box^d(\Box^d(\neg^d\psi)))) \supset^d \neg^d(\diamond^d\psi)$ — holds in KB4

abbreviation(*input*) $F7 \equiv (\diamond^d\varphi) \supset^d (\Box^d(\varphi \vee^d \diamond^d\varphi))$ — holds in KB4

abbreviation(*input*) $F8 \equiv (\diamond^d(\Box^d\varphi)) \supset^d (\varphi \vee^d \diamond^d\varphi)$ — holds in KT/KB

abbreviation(*input*) $F9 \equiv ((\Box^d(\diamond^d\varphi)) \wedge^d (\Box^d(\diamond^d(\neg^d\varphi)))) \supset^d \diamond^d(\diamond^d\varphi)$ — holds in KT

abbreviation(*input*) $F10 \equiv ((\Box^d(\varphi \supset^d \Box^d\psi)) \wedge^d (\Box^d(\diamond^d(\neg^d\psi)))) \supset^d \neg^d(\Box^d\psi)$ — holds in KT

declare *imp-cong*[*cong del*]

experiment begin

lemma $S5: \forall w:W. r \ R \wedge s \ R \wedge t \ R \longrightarrow (\langle W,R,V \rangle, w \models^d F1)$

— nitpick[expect=none] — sledgehammer — none — proof

<proof>

lemma $S4: \forall w:W. r \ R \wedge t \ R \longrightarrow (\langle W,R,V \rangle, w \models^d F1)$

— nitpick[expect=none] — sledgehammer — none — proof

<proof>

lemma $KB4: \forall w:W. s \ R \wedge t \ R \longrightarrow (\langle W,R,V \rangle, w \models^d F1)$

— nitpick[expect=none] — sledgehammer — none — proof

<proof>

lemma $KTB: \forall w:W. r \ R \wedge s \ R \longrightarrow (\langle W,R,V \rangle, w \models^d F1)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf

<proof>

lemma $KT: \forall w:W. r \ R \longrightarrow (\langle W,R,V \rangle, w \models^d F1)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf

$\langle proof \rangle$
lemma $KB: \forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^d F1)$
 — nitpick[expect=genuine] — sledgehammer — ctext — no prf
 $\langle proof \rangle$
lemma $K4: \forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^d F1)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $K: \forall w:W. (\langle W,R,V \rangle, w \models^d F1)$
 — nitpick[expect=genuine] — sledgehammer — ctext — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5: \forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $S4: \forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=genuine] — sledgehammer — ctext — no prf
 $\langle proof \rangle$
lemma $KB4: \forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $KTB: \forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $KT: \forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=genuine] — sledgehammer — ctext — no prf
 $\langle proof \rangle$
lemma $KB: \forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $K4: \forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=genuine] — sledgehammer — ctext — no prf
 $\langle proof \rangle$
lemma $K: \forall w:W. (\langle W,R,V \rangle, w \models^d F2)$
 — nitpick[expect=genuine] — sledgehammer — ctext — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5: \forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $S4: \forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=genuine] — sledgehammer — ctext — no prf
 $\langle proof \rangle$
lemma $KB4: \forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma *KTB*: $\forall w:W. \text{r } R \wedge \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma *KT*: $\forall w:W. \text{r } R \longrightarrow (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma *KB*: $\forall w:W. \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma *K4*: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^d F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
end

experiment begin

lemma *S5*: $\forall w:W. \text{r } R \wedge \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma *S4*: $\forall w:W. \text{r } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=none] — sledgehammer — none — no prf
<proof>
lemma *KB4*: $\forall w:W. \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma *KTB*: $\forall w:W. \text{r } R \wedge \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma *KT*: $\forall w:W. \text{r } R \longrightarrow (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma *KB*: $\forall w:W. \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma *K4*: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=none] — sledgehammer — none — no prf
<proof>
lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^d F4)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
end

experiment begin

lemma *S5*: $\forall w:W. \text{r } R \wedge \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F5)$
 — nitpick[expect=none] — sledgehammer — none — no prf
<proof>
lemma *S4*: $\forall w:W. \text{r } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F5)$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma KB_4 : $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F5)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^d F5)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma KT : $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^d F5)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma KB : $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^d F5)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma K_4 : $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^d F5)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^d F5)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
end

experiment begin

lemma $S5$: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma S_4 : $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma KB_4 : $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma KT : $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma KB : $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma K_4 : $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^d F6)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
end

experiment begin

lemma $S5$: $\forall w:W. \text{r } R \wedge \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma $S4$: $\forall w:W. \text{r } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma $KB4$: $\forall w:W. \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma KTB : $\forall w:W. \text{r } R \wedge \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma KT : $\forall w:W. \text{r } R \longrightarrow (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma KB : $\forall w:W. \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma $K4$: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^d F7)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

end**experiment begin**

lemma $S5$: $\forall w:W. \text{r } R \wedge \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F8)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma $S4$: $\forall w:W. \text{r } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F8)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma $KB4$: $\forall w:W. \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F8)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma KTB : $\forall w:W. \text{r } R \wedge \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F8)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma KT : $\forall w:W. \text{r } R \longrightarrow (\langle W,R,V \rangle, w \models^d F8)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma KB : $\forall w:W. \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^d F8)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma $K4$: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F8)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^d F8)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5$: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma $S4$: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma $KB4$: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma KT : $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma KB : $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma $K4$: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^d F9)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5$: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F10)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma $S4$: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F10)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma $KB4$: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^d F10)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 $\langle proof \rangle$

lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^d F10)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma KT : $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^d F10)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma KB : $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^d F10)$

— nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma $K4$: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^d F10)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^d F10)$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
end

— Summary of experiments: Nitpick: ctex=32 (with simp 8, without simp 24), none=56 (with simp 0, without simp 56), unknwn=72 (with simp 72, without simp 0) Sledgehammer: proof=70 (with simp 38, without simp 32), no prf=90 (with simp 42, without simp 48)

— No conflicts
end

8.3 Tests with the (maximal) shallow embedding: axiom schemata

theory *PMLinHOL-shallow-further-tests-1* **imports** *PMLinHOL-shallow-tests*
begin

— What is the weakest modal logic in which the following test formulas F1-F10 are provable?
 — Test with schematic axioms

consts $\varphi::\sigma$ $\psi::\sigma$

abbreviation(*input*) $F1 \equiv (\Diamond^s(\Diamond^s\varphi)) \supset^s \Diamond^s\varphi$ — holds in K4

abbreviation(*input*) $F2 \equiv (\Diamond^s(\Box^s\varphi)) \supset^s \Box^s(\Diamond^s\varphi)$ — holds in KB

abbreviation(*input*) $F3 \equiv (\Diamond^s(\Box^s\varphi)) \supset^s \Box^s\varphi$ — holds in KB4

abbreviation(*input*) $F4 \equiv (\Box^s(\Diamond^s(\Box^s(\Diamond^s\varphi)))) \supset^s \Box^s(\Diamond^s\varphi)$ — holds in KB/K4

abbreviation(*input*) $F5 \equiv (\Diamond^s(\varphi \wedge^s (\Diamond^s\psi))) \supset^s ((\Diamond^s\varphi) \wedge^s (\Diamond^s\psi))$ — holds in K4

abbreviation(*input*) $F6 \equiv ((\Box^s(\varphi \supset^s \psi)) \wedge^s (\Diamond^s(\Box^s(\neg^s\psi)))) \supset^s \neg^s(\Diamond^s\psi)$ — holds in KB4

abbreviation(*input*) $F7 \equiv (\Diamond^s\varphi) \supset^s (\Box^s(\varphi \vee^s \Diamond^s\varphi))$ — holds in KB4

abbreviation(*input*) $F8 \equiv (\Diamond^s(\Box^s\varphi)) \supset^s (\varphi \vee^s \Diamond^s\varphi)$ — holds in KT/KB

abbreviation(*input*) $F9 \equiv ((\Box^s(\Diamond^s\varphi)) \wedge^s (\Box^s(\Diamond^s(\neg^s\varphi)))) \supset^s \Diamond^s(\Diamond^s\varphi)$ — holds in KT

abbreviation(*input*) $F10 \equiv ((\Box^s(\varphi \supset^s \Box^s\psi)) \wedge^s (\Box^s(\Diamond^s(\neg^s\psi)))) \supset^s \neg^s(\Box^s\psi)$ — holds in KT

declare *imp-cong*[*cong del*]

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s\varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s\varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s\varphi) \supset^s \Box^s(\Box^s\varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F1$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma $K4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F2$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^s F2$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩

end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma $S4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma $KB4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma KTB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma KT : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \longrightarrow \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma KB : $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma $K4$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma K : $\forall w:W. \langle W,R,V \rangle, w \models^s F3$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩

end

experiment begin

lemma $S5$: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F4$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf

lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F5$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *K4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F5$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *K*: $\forall w:W. \langle W,R,V \rangle, w \models^s F5$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma *S5*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *S4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *KB4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *KTB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *KT*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \longrightarrow \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *K4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$

lemma *K*: $\forall w:W. \langle W,R,V \rangle, w \models^s F6$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma *S5*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F7$

$\langle proof \rangle$
lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F8$
— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$
lemma *K4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F8$
— nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$
lemma *K*: $\forall w:W. \langle W,R,V \rangle, w \models^s F8$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma *S5*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma *S4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma *KB4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *KTB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma *KT*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \longrightarrow \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma *KB*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi \supset^s \Box^s(\Diamond^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K4*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \Box^s(\Box^s \varphi)) \longrightarrow \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K*: $\forall w:W. \langle W,R,V \rangle, w \models^s F9$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma *S5*: $\forall w:W. (\forall \varphi. \langle W,R,V \rangle, w \models^s (\Box^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W,R,V \rangle, w \models^s \varphi$

$\supset^s \square^s(\diamond^s \varphi) \wedge (\forall \varphi. \langle W, R, V \rangle, w \models^s (\square^s \varphi) \supset^s \square^s(\square^s \varphi)) \longrightarrow \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
<proof>

lemma *S4*: $\forall w:W. (\forall \varphi. \langle W, R, V \rangle, w \models^s (\square^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W, R, V \rangle, w \models^s (\square^s \varphi) \supset^s \square^s(\square^s \varphi)) \longrightarrow \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
<proof>

lemma *KB4*: $\forall w:W. (\forall \varphi. \langle W, R, V \rangle, w \models^s \varphi \supset^s \square^s(\diamond^s \varphi)) \wedge (\forall \varphi. \langle W, R, V \rangle, w \models^s (\square^s \varphi) \supset^s \square^s(\square^s \varphi)) \longrightarrow \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma *KTB*: $\forall w:W. (\forall \varphi. \langle W, R, V \rangle, w \models^s (\square^s \varphi) \supset^s \varphi) \wedge (\forall \varphi. \langle W, R, V \rangle, w \models^s \varphi \supset^s \square^s(\diamond^s \varphi)) \longrightarrow \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
<proof>

lemma *KT*: $\forall w:W. (\forall \varphi. \langle W, R, V \rangle, w \models^s (\square^s \varphi) \supset^s \varphi) \longrightarrow \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
<proof>

lemma *KB*: $\forall w:W. (\forall \varphi. \langle W, R, V \rangle, w \models^s \varphi \supset^s \square^s(\diamond^s \varphi)) \longrightarrow \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma *K4*: $\forall w:W. (\forall \varphi. \langle W, R, V \rangle, w \models^s (\square^s \varphi) \supset^s \square^s(\square^s \varphi)) \longrightarrow \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma *K*: $\forall w:W. \langle W, R, V \rangle, w \models^s F10$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

end

— Summary of experiments: Nitpick: ctex=32 (with simp 16, without simp 16), none=0, unknwn=128 (with simp 64, without simp 64) Sledgehammer: proof=32 (with simp 24, without simp 8), no prf=128 (with simp 56, without simp 72)

— No conflict
end

8.4 Tests with the (maximal) shallow embedding: semantic frame conditions

theory *PMLinHOL-shallow-further-tests-2*

imports *PMLinHOL-shallow-tests*

begin

— What is the weakest modal logic in which the following test formulas F1-F10 are provable?

— Test with semantic conditions

abbreviation(*input*) *refl* (r) **where** r $\equiv \lambda R. \text{reflexive } R$

abbreviation(*input*) *sym* (s) **where** s $\equiv \lambda R.$ *symmetric R*
abbreviation(*input*) *tra* (t) **where** t $\equiv \lambda R.$ *transitive R*

consts $\varphi::\sigma \ \psi::\sigma$

abbreviation(*input*) *F1* $\equiv (\diamond^s(\diamond^s\varphi)) \supset^s \diamond^s\varphi$ — holds in K4

abbreviation(*input*) *F2* $\equiv (\diamond^s(\Box^s\varphi)) \supset^s \Box^s(\diamond^s\varphi)$ — holds in KB

abbreviation(*input*) *F3* $\equiv (\diamond^s(\Box^s\varphi)) \supset^s \Box^s\varphi$ — holds in KB4

abbreviation(*input*) *F4* $\equiv (\Box^s(\diamond^s(\Box^s(\diamond^s\varphi)))) \supset^s \Box^s(\diamond^s\varphi)$ — holds in KB

abbreviation(*input*) *F5* $\equiv (\diamond^s(\varphi \wedge^s (\diamond^s\psi))) \supset^s ((\diamond^s\varphi) \wedge^s (\diamond^s\psi))$ — holds in K4

abbreviation(*input*) *F6* $\equiv ((\Box^s(\varphi \supset^s \psi)) \wedge^s (\diamond^s(\Box^s(\neg^s\psi)))) \supset^s \neg^s(\diamond^s\psi)$ — holds in KB4

abbreviation(*input*) *F7* $\equiv (\diamond^s\varphi) \supset^s (\Box^s(\varphi \vee^s \diamond^s\varphi))$ — holds in KB4

abbreviation(*input*) *F8* $\equiv (\diamond^s(\Box^s\varphi)) \supset^s (\varphi \vee^s \diamond^s\varphi)$ — holds in KT and in KB

abbreviation(*input*) *F9* $\equiv ((\Box^s(\diamond^s\varphi)) \wedge^s (\Box^s(\diamond^s(\neg^s\varphi)))) \supset^s \diamond^s(\diamond^s\varphi)$ — holds in KT

abbreviation(*input*) *F10* $\equiv ((\Box^s(\varphi \supset^s \Box^s\psi)) \wedge^s (\Box^s(\diamond^s(\neg^s\psi)))) \supset^s \neg^s(\Box^s\psi)$ — holds in KT

declare *imp-cong*[*cong del*]

experiment begin

lemma *S5*: $\forall w:W. \text{r } R \wedge \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
 <proof>

lemma *S4*: $\forall w:W. \text{r } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
 <proof>

lemma *KB4*: $\forall w:W. \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
 <proof>

lemma *KTB*: $\forall w:W. \text{r } R \wedge \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 <proof>

lemma *KT*: $\forall w:W. \text{r } R \longrightarrow (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 <proof>

lemma *KB*: $\forall w:W. \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 <proof>

lemma *K4*: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=unknown] — sledgehammer — unkn — proof
 <proof>

lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^s F1)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 <proof>

end

experiment begin

lemma *S5*: $\forall w:W. \text{r } R \wedge \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle \text{proof} \rangle$

lemma *S4*: $\forall w:W. \text{r } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *KB4*: $\forall w:W. \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle \text{proof} \rangle$

lemma *KTB*: $\forall w:W. \text{r } R \wedge \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle \text{proof} \rangle$

lemma *KT*: $\forall w:W. \text{r } R \longrightarrow (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *KB*: $\forall w:W. \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle \text{proof} \rangle$

lemma *K4*: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^s F2)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

end

experiment begin

lemma *S5*: $\forall w:W. \text{r } R \wedge \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F3)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle \text{proof} \rangle$

lemma *S4*: $\forall w:W. \text{r } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *KB4*: $\forall w:W. \text{s } R \wedge \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F3)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle \text{proof} \rangle$

lemma *KTB*: $\forall w:W. \text{r } R \wedge \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^s F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *KT*: $\forall w:W. \text{r } R \longrightarrow (\langle W,R,V \rangle, w \models^s F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *KB*: $\forall w:W. \text{s } R \longrightarrow (\langle W,R,V \rangle, w \models^s F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *K4*: $\forall w:W. \text{t } R \longrightarrow (\langle W,R,V \rangle, w \models^s F3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle \text{proof} \rangle$

lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^s F3)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma $S5$: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩

lemma $S4$: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma $KB4$: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩

lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩

lemma KT : $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩

lemma KB : $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩

lemma $K4$: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^s F4)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩

end

experiment begin

lemma $S5$: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma $S4$: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma $KB4$: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩

lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩

lemma KT : $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩

lemma KB : $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf

$\langle proof \rangle$
lemma $K4$: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 $\langle proof \rangle$
lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^s F5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5$: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma $S4$: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $KB4$: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma KT : $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma KB : $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $K4$: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma K : $\forall w:W. (\langle W,R,V \rangle, w \models^s F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5$: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma $S4$: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $KB4$: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$
lemma KTB : $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

lemma *KT*: $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

lemma *KB*: $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

lemma *K4*: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^s F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma *S5*: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *S4*: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *KB4*: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *KTB*: $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *KT*: $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *KB*: $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *K4*: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^s F8)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma *S5*: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F9)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *S4*: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F9)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 $\langle proof \rangle$

lemma *KB4*: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F9)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KTB*: $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^s F9)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩
lemma *KT*: $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^s F9)$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩
lemma *KB*: $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^s F9)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *K4*: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^s F9)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^s F9)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma *S5*: $\forall w:W. r R \wedge s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩
lemma *S4*: $\forall w:W. r R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩
lemma *KB4*: $\forall w:W. s R \wedge t R \longrightarrow (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KTB*: $\forall w:W. r R \wedge s R \longrightarrow (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩
lemma *KT*: $\forall w:W. r R \longrightarrow (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=unknown] — sledgehammer — unkn — no prf
 ⟨proof⟩
lemma *KB*: $\forall w:W. s R \longrightarrow (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *K4*: $\forall w:W. t R \longrightarrow (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *K*: $\forall w:W. (\langle W,R,V \rangle, w \models^s F10)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

— Summary of experiments: Nitpick: ctex=84 (with simp 42, without simp 42),
 none=0, unknwn=76 (with simp 38, without simp 38) Sledgehammer: proof=66
 (with simp 38, without simp 28), no prf=94 (with simp 42, without simp 52)

— No conflicts
end

8.5 Tests with the (minimal) shallow embedding: axiom schemata

theory *PMLinHOL-shallow-minimal-further-tests-1*

imports *PMLinHOL-shallow-minimal* — C.B., 2025

begin

— What is the weakest modal logic in which the following test formulas F1-F10 are provable?

— Test with schematic axioms

abbreviation(*input*) $AxT \equiv \forall \varphi. \models^m (\Box^m \varphi) \supset^m \varphi$

abbreviation(*input*) $AxB \equiv \forall \varphi. \models^m \varphi \supset^m \Box^m (\Diamond^m \varphi)$

abbreviation(*input*) $Ax4 \equiv \forall \varphi. \models^m (\Box^m \varphi) \supset^m \Box^m (\Box^m \varphi)$

consts $\varphi::\sigma \ \psi::\sigma$

abbreviation(*input*) $F1 \equiv (\Diamond^m (\Diamond^m \varphi)) \supset^m \Diamond^m \varphi$ — holds in K4

abbreviation(*input*) $F2 \equiv (\Diamond^m (\Box^m \varphi)) \supset^m \Box^m (\Diamond^m \varphi)$ — holds in KB

abbreviation(*input*) $F3 \equiv (\Diamond^m (\Box^m \varphi)) \supset^m \Box^m \varphi$ — holds in KB4

abbreviation(*input*) $F4 \equiv (\Box^m (\Diamond^m (\Box^m (\Diamond^m \varphi)))) \supset^m \Box^m (\Diamond^m \varphi)$ — holds in KB/K4

abbreviation(*input*) $F5 \equiv (\Diamond^m (\varphi \wedge^m (\Diamond^m \psi))) \supset^m ((\Diamond^m \varphi) \wedge^m (\Diamond^m \psi))$ — holds in K4

abbreviation(*input*) $F6 \equiv ((\Box^m (\varphi \supset^m \psi)) \wedge^m (\Diamond^m (\Box^m (\neg^m \psi)))) \supset^m \neg^m (\Diamond^m \psi)$ — holds in KB4

abbreviation(*input*) $F7 \equiv (\Diamond^m \varphi) \supset^m (\Box^m (\varphi \vee^m \Diamond^m \varphi))$ — holds in KB4

abbreviation(*input*) $F8 \equiv (\Diamond^m (\Box^m \varphi)) \supset^m (\varphi \vee^m \Diamond^m \varphi)$ — holds in KT/KB

abbreviation(*input*) $F9 \equiv ((\Box^m (\Diamond^m \varphi)) \wedge^m (\Box^m (\Diamond^m (\neg^m \varphi)))) \supset^m \Diamond^m (\Diamond^m \varphi)$ — holds in KT

abbreviation(*input*) $F10 \equiv ((\Box^m (\varphi \supset^m \Box^m \psi)) \wedge^m (\Box^m (\Diamond^m (\neg^m \psi)))) \supset^m \neg^m (\Box^m \psi)$ — holds in KT

declare *imp-cong*[*cong del*]

nitpick-params

experiment begin

lemma $S5: AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F1$

— nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩

lemma $S4: AxT \wedge Ax4 \longrightarrow \models^m F1$

— nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩

lemma $KB4: AxB \wedge Ax4 \longrightarrow \models^m F1$

— nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩

lemma $KTB: AxT \wedge AxB \longrightarrow \models^m F1$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩

lemma *KT*: $AxT \longrightarrow \models^m F1$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
lemma *KB*: $AxB \longrightarrow \models^m F1$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
lemma *K4*: $Ax4 \longrightarrow \models^m F1$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩
lemma *K*: $\models^m F1$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
end

experiment begin

lemma *S5*: $AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F2$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩
lemma *S4*: $AxT \wedge Ax4 \longrightarrow \models^m F2$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
lemma *KB4*: $AxB \wedge Ax4 \longrightarrow \models^m F2$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩
lemma *KTB*: $AxT \wedge AxB \longrightarrow \models^m F2$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩
lemma *KT*: $AxT \longrightarrow \models^m F2$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
lemma *KB*: $AxB \longrightarrow \models^m F2$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩
lemma *K4*: $Ax4 \longrightarrow \models^m F2$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
lemma *K*: $\models^m F2$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
end

experiment begin

lemma *S5*: $AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F3$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨*proof*⟩
lemma *S4*: $AxT \wedge Ax4 \longrightarrow \models^m F3$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨*proof*⟩
lemma *KB4*: $AxB \wedge Ax4 \longrightarrow \models^m F3$

— nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *KTB*: $AxT \wedge AxB \longrightarrow \models^m F3$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KT*: $AxT \longrightarrow \models^m F3$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KB*: $AxB \longrightarrow \models^m F3$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *K4*: $Ax4 \longrightarrow \models^m F3$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *K*: $\models^m F3$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma *S5*: $AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F4$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *S4*: $AxT \wedge Ax4 \longrightarrow \models^m F4$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩
lemma *KB4*: $AxB \wedge Ax4 \longrightarrow \models^m F4$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *KTB*: $AxT \wedge AxB \longrightarrow \models^m F4$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *KT*: $AxT \longrightarrow \models^m F4$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KB*: $AxB \longrightarrow \models^m F4$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *K4*: $Ax4 \longrightarrow \models^m F4$
 — nitpick[expect=unknown] — sledgehammer — unkn — proof
 ⟨proof⟩
lemma *K*: $\models^m F4$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma *S5*: $AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F5$
 — nitpick[expect=none] — sledgehammer — none — proof

$\langle proof \rangle$
lemma $S_4: AxT \wedge Ax_4 \longrightarrow \models^m F5$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $KB_4: AxB \wedge Ax_4 \longrightarrow \models^m F5$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $KTB: AxT \wedge AxB \longrightarrow \models^m F5$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $KT: AxT \longrightarrow \models^m F5$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $KB: AxB \longrightarrow \models^m F5$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $K_4: Ax_4 \longrightarrow \models^m F5$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $K: \models^m F5$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma $S5: AxT \wedge AxB \wedge Ax_4 \longrightarrow \models^m F6$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $S_4: AxT \wedge Ax_4 \longrightarrow \models^m F6$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $KB_4: AxB \wedge Ax_4 \longrightarrow \models^m F6$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma $KTB: AxT \wedge AxB \longrightarrow \models^m F6$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $KT: AxT \longrightarrow \models^m F6$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $KB: AxB \longrightarrow \models^m F6$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $K_4: Ax_4 \longrightarrow \models^m F6$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma $K: \models^m F6$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$

end

experiment begin

lemma $S5: AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F7$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $S4: AxT \wedge Ax4 \longrightarrow \models^m F7$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
(proof)

lemma $KB4: AxB \wedge Ax4 \longrightarrow \models^m F7$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $KTB: AxT \wedge AxB \longrightarrow \models^m F7$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
(proof)

lemma $KT: AxT \longrightarrow \models^m F7$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
(proof)

lemma $KB: AxB \longrightarrow \models^m F7$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
(proof)

lemma $K4: Ax4 \longrightarrow \models^m F7$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
(proof)

lemma $K: \models^m F7$
— nitpick[expect=genuine] — sledgehammer — ctex — no prf
(proof)

end

experiment begin

lemma $S5: AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F8$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $S4: AxT \wedge Ax4 \longrightarrow \models^m F8$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $KB4: AxB \wedge Ax4 \longrightarrow \models^m F8$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $KTB: AxT \wedge AxB \longrightarrow \models^m F8$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $KT: AxT \longrightarrow \models^m F8$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $KB: AxB \longrightarrow \models^m F8$
— nitpick[expect=none] — sledgehammer — none — proof
(proof)

lemma $K4: Ax4 \longrightarrow \models^m F8$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma K : $\models^m F8$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma $S5$: $AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F9$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma $S4$: $AxT \wedge Ax4 \longrightarrow \models^m F9$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma $KB4$: $AxB \wedge Ax4 \longrightarrow \models^m F9$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma KTB : $AxT \wedge AxB \longrightarrow \models^m F9$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma KT : $AxT \longrightarrow \models^m F9$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma KB : $AxB \longrightarrow \models^m F9$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma $K4$: $Ax4 \longrightarrow \models^m F9$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma K : $\models^m F9$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma $S5$: $AxT \wedge AxB \wedge Ax4 \longrightarrow \models^m F10$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma $S4$: $AxT \wedge Ax4 \longrightarrow \models^m F10$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma $KB4$: $AxB \wedge Ax4 \longrightarrow \models^m F10$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma KTB : $AxT \wedge AxB \longrightarrow \models^m F10$
 — nitpick[expect=none] — sledgehammer — none — no prf
 ⟨proof⟩
lemma KT : $AxT \longrightarrow \models^m F10$
 — nitpick[expect=none] — sledgehammer — none — proof

$\langle proof \rangle$
lemma *KB*: $AxB \longrightarrow \models^m F10$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K4*: $Ax4 \longrightarrow \models^m F10$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K*: $\models^m F10$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

— Summary of experiments: Nitpick: ctex=84 (with simp 42, without simp 42), none=72 (with simp 36, without simp 36), unknwn=4 (with simp 2, without simp 2) Sledgehammer: proof=73 (with simp 38, without simp 35), no prf=87 (with simp 42, without simp 45)

— No conflicts
end

8.6 Tests with the (minimal) shallow embedding: semantic frame conditions

theory *PMLinHOL-shallow-minimal-further-tests-2*
imports *PMLinHOL-shallow-minimal*
begin

— What is the weakest modal logic in which the following test formulas F1-F10 are provable?

— Test with semantic conditions

abbreviation(*input*) *refl* (*r*) **where** $r \equiv \lambda R. \text{reflexive } R$

abbreviation(*input*) *sym* (*s*) **where** $s \equiv \lambda R. \text{symmetric } R$

abbreviation(*input*) *tra* (*t*) **where** $t \equiv \lambda R. \text{transitive } R$

consts $\varphi::\sigma \ \psi::\sigma$

abbreviation(*input*) *F1* $\equiv (\Diamond^m(\Diamond^m\varphi)) \supset^m \Diamond^m\varphi$ — holds in K4

abbreviation(*input*) *F2* $\equiv (\Diamond^m(\Box^m\varphi)) \supset^m \Box^m(\Diamond^m\varphi)$ — holds in KB

abbreviation(*input*) *F3* $\equiv (\Diamond^m(\Box^m\varphi)) \supset^m \Box^m\varphi$ — holds in KB4

abbreviation(*input*) *F4* $\equiv (\Box^m(\Diamond^m(\Box^m(\Diamond^m\varphi)))) \supset^m \Box^m(\Diamond^m\varphi)$ — holds in KB

abbreviation(*input*) *F5* $\equiv (\Diamond^m(\varphi \wedge^m (\Diamond^m\psi))) \supset^m ((\Diamond^m\varphi) \wedge^m (\Diamond^m\psi))$ — holds in K4

abbreviation(*input*) *F6* $\equiv ((\Box^m(\varphi \supset^m \psi)) \wedge^m (\Diamond^m(\Box^m(\neg^m\psi)))) \supset^m \neg^m(\Diamond^m\psi)$
 — holds in KB4

abbreviation(*input*) *F7* $\equiv (\Diamond^m\varphi) \supset^m (\Box^m(\varphi \vee^m \Diamond^m\varphi))$ — holds in KB4

abbreviation(*input*) *F8* $\equiv (\Diamond^m(\Box^m\varphi)) \supset^m (\varphi \vee^m \Diamond^m\varphi)$ — holds in KT and in KB

abbreviation(*input*) *F9* $\equiv ((\Box^m(\Diamond^m\varphi)) \wedge^m (\Box^m(\Diamond^m(\neg^m\varphi)))) \supset^m \Diamond^m(\Diamond^m\varphi)$
 — holds in KT

abbreviation(*input*) *F10* $\equiv ((\Box^m(\varphi \supset^m \Box^m\psi)) \wedge^m (\Box^m(\Diamond^m(\neg^m\psi)))) \supset^m \neg^m(\Box^m\psi)$
 — holds in KT

```

declare imp-cong[cong del]

experiment begin
lemma S5:  $r R \wedge s R \wedge t R \longrightarrow (\models^m F1)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩
lemma S4:  $r R \wedge t R \longrightarrow (\models^m F1)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩
lemma KB4:  $s R \wedge t R \longrightarrow (\models^m F1)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩
lemma KTB:  $r R \wedge s R \longrightarrow (\models^m F1)$ 
  — nitpick[expect=genuine] — sledgehammer — ctex — no prf
  ⟨proof⟩
lemma KT:  $r R \longrightarrow (\models^m F1)$ 
  — nitpick[expect=genuine] — sledgehammer — ctex — no prf
  ⟨proof⟩
lemma KB:  $s R \longrightarrow (\models^m F1)$ 
  — nitpick[expect=genuine] — sledgehammer — ctex — no prf
  ⟨proof⟩
lemma K4:  $t R \longrightarrow (\models^m F1)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩
lemma K:  $(\models^m F1)$ 
  — nitpick[expect=genuine] — sledgehammer — ctex — no prf
  ⟨proof⟩
end

```

```

experiment begin
lemma S5:  $r R \wedge s R \wedge t R \longrightarrow (\models^m F2)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩
lemma S4:  $r R \wedge t R \longrightarrow (\models^m F2)$ 
  — nitpick[expect=genuine] — sledgehammer — ctex — no prf
  ⟨proof⟩
lemma KB4:  $s R \wedge t R \longrightarrow (\models^m F2)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩
lemma KTB:  $r R \wedge s R \longrightarrow (\models^m F2)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩
lemma KT:  $r R \longrightarrow (\models^m F2)$ 
  — nitpick[expect=genuine] — sledgehammer — ctex — no prf
  ⟨proof⟩
lemma KB:  $s R \longrightarrow (\models^m F2)$ 
  — nitpick[expect=none] — sledgehammer — none — proof
  ⟨proof⟩

```

lemma K_4 : $t R \longrightarrow (\models^m F_2)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma K : $(\models^m F_2)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma S_5 : $r R \wedge s R \wedge t R \longrightarrow (\models^m F_3)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma S_4 : $r R \wedge t R \longrightarrow (\models^m F_3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma KB_4 : $s R \wedge t R \longrightarrow (\models^m F_3)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma KTB : $r R \wedge s R \longrightarrow (\models^m F_3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma KT : $r R \longrightarrow (\models^m F_3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma KB : $s R \longrightarrow (\models^m F_3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma K_4 : $t R \longrightarrow (\models^m F_3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma K : $(\models^m F_3)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma S_5 : $r R \wedge s R \wedge t R \longrightarrow (\models^m F_4)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma S_4 : $r R \wedge t R \longrightarrow (\models^m F_4)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma KB_4 : $s R \wedge t R \longrightarrow (\models^m F_4)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma KTB : $r R \wedge s R \longrightarrow (\models^m F_4)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma KT : $r R \longrightarrow (\models^m F_4)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KB*: $s R \longrightarrow (\models^m F_4)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *K4*: $t R \longrightarrow (\models^m F_4)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *K*: $(\models^m F_4)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma *S5*: $r R \wedge s R \wedge t R \longrightarrow (\models^m F_5)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *S4*: $r R \wedge t R \longrightarrow (\models^m F_5)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *KB4*: $s R \wedge t R \longrightarrow (\models^m F_5)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *KTB*: $r R \wedge s R \longrightarrow (\models^m F_5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KT*: $r R \longrightarrow (\models^m F_5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KB*: $s R \longrightarrow (\models^m F_5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *K4*: $t R \longrightarrow (\models^m F_5)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *K*: $(\models^m F_5)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
end

experiment begin

lemma *S5*: $r R \wedge s R \wedge t R \longrightarrow (\models^m F_6)$
 — nitpick[expect=none] — sledgehammer — none — proof
 ⟨proof⟩
lemma *S4*: $r R \wedge t R \longrightarrow (\models^m F_6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 ⟨proof⟩
lemma *KB4*: $s R \wedge t R \longrightarrow (\models^m F_6)$
 — nitpick[expect=none] — sledgehammer — none — proof

$\langle proof \rangle$
lemma *KTB*: $r R \wedge s R \longrightarrow (\models^m F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *KT*: $r R \longrightarrow (\models^m F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *KB*: $s R \longrightarrow (\models^m F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K4*: $t R \longrightarrow (\models^m F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K*: $(\models^m F6)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma *S5*: $r R \wedge s R \wedge t R \longrightarrow (\models^m F7)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma *S4*: $r R \wedge t R \longrightarrow (\models^m F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *KB4*: $s R \wedge t R \longrightarrow (\models^m F7)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$
lemma *KTB*: $r R \wedge s R \longrightarrow (\models^m F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *KT*: $r R \longrightarrow (\models^m F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *KB*: $s R \longrightarrow (\models^m F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K4*: $t R \longrightarrow (\models^m F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
lemma *K*: $(\models^m F7)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
 $\langle proof \rangle$
end

experiment begin

lemma *S5*: $r R \wedge s R \wedge t R \longrightarrow (\models^m F8)$
 — nitpick[expect=none] — sledgehammer — none — proof
 $\langle proof \rangle$

lemma S_4 : $r R \wedge t R \longrightarrow (\models^m F8)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma KB_4 : $s R \wedge t R \longrightarrow (\models^m F8)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma KTB : $r R \wedge s R \longrightarrow (\models^m F8)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma KT : $r R \longrightarrow (\models^m F8)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma KB : $s R \longrightarrow (\models^m F8)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma K_4 : $t R \longrightarrow (\models^m F8)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma K : $(\models^m F8)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
end

experiment begin

lemma $S5$: $r R \wedge s R \wedge t R \longrightarrow (\models^m F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma S_4 : $r R \wedge t R \longrightarrow (\models^m F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma KB_4 : $s R \wedge t R \longrightarrow (\models^m F9)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma KTB : $r R \wedge s R \longrightarrow (\models^m F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma KT : $r R \longrightarrow (\models^m F9)$
 — nitpick[expect=none] — sledgehammer — none — proof
<proof>
lemma KB : $s R \longrightarrow (\models^m F9)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma K_4 : $t R \longrightarrow (\models^m F9)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
lemma K : $(\models^m F9)$
 — nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>
end

experiment begin

lemma $S5$: $r R \wedge s R \wedge t R \longrightarrow (\models^m F10)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma $S4$: $r R \wedge t R \longrightarrow (\models^m F10)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma $KB4$: $s R \wedge t R \longrightarrow (\models^m F10)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma KTB : $r R \wedge s R \longrightarrow (\models^m F10)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma KT : $r R \longrightarrow (\models^m F10)$

— nitpick[expect=none] — sledgehammer — none — proof
<proof>

lemma KB : $s R \longrightarrow (\models^m F10)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma $K4$: $t R \longrightarrow (\models^m F10)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

lemma K : $(\models^m F10)$

— nitpick[expect=genuine] — sledgehammer — ctex — no prf
<proof>

end

— Summary of experiments: Nitpick: ctex=84 (with simp 42, without simp 42), none=76 (with simp 38, without simp 38), unknwn=0 Sledgehammer: proof=76 (with simp 38, without simp 38), no prf=84 (with simp 42, without simp 42)

— No conflicts

end

References

- [1] C. Benzmüller. Faithful logic embeddings in HOL — deep and shallow. In C. Barrett and U. Waldmann, editors, *Automated Deduction – CADE-30 – 30th International Conference on Automated Deduction, Stuttgart, Germany, July 28-31, 2025, Proceedings*, volume 15943 of *Lecture Notes in Computer Science*, pages 280–302. Springer, 2025. (preprint: [arXiv:2502.19311](https://arxiv.org/abs/2502.19311)).