

The Fair Games Theorem

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Abstract

This is the optional stopping theorem (or fair games theorem) for real-valued discrete-time stochastic processes on a σ -finite filtered measure space. A theory of piecewise-constant stopping times — the function taking i on an \mathcal{F}_i -measurable set S and j on its complement — is developed, together with the corresponding decomposition of integrals over the two pieces. The central results then prove both directions of the theorem: if X is a submartingale, then for any bounded stopping times $\tau \leq \pi$ the expected stopped values satisfy $\mathbb{E}[X_\tau] \leq \mathbb{E}[X_\pi]$ (via the telescoping identity and the submartingale set-integral inequality); conversely, monotonicity of expected stopped values over all bounded stopping times characterises submartingales. It is further shown that the stopped process X^τ of a submartingale is again a submartingale. Throughout, the theorems are cross-referenced to their counterparts in Mathlib’s **OptionalStopping**, from which it was translated by Claude 4.6. It was polished manually afterwards.

The fair games theorem is #62 on the *Top 100 Mathematical Theorems*.

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1 Integrability of Stopped Values and Processes

```
theory Stopped-Value-Integration
imports Doob-Convergence.Stopping-Time
begin
```

Integrability of stopped values and stopped processes, and the telescoping identity for differences of stopped values. These results bridge the gap between the existing AFP theories (Martingales, Doob_Convergence) and the optional stopping theorem.

```
context nat-sigma-finite-filtered-measure
begin
```

1.1 Stopped value as a sum of indicators

A stopped value with a bounded stopping time can be written as a finite sum of indicators.

```
lemma stopped-value-eq-sum:
fixes X :: nat  $\Rightarrow$  'a  $\Rightarrow$  'b :: real-vector
assumes  $\tau$ -st: stopping-time  $\tau$  and  $\tau$ -bnd:  $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq N$ 
assumes  $\omega$ -in:  $\omega \in \text{space } M$ 
shows stopped-value X  $\tau \omega = (\sum i \leq N. \text{indicator } \{\omega \in \text{space } M. \tau \omega = i\} \omega *_{R}$ 
X i  $\omega)$ 
 $\langle$ proof $\rangle$ 
```

1.2 Telescoping identity for stopped values

The difference of stopped values can be expressed as a sum of indicator-weighted increments. This corresponds to `stoppedValue_sub_eq_sum` in Mathlib.

```
lemma stopped-value-sub-eq-sum:
fixes X :: nat  $\Rightarrow$  'a  $\Rightarrow$  real
assumes  $\tau$ -st: stopping-time  $\tau$  and  $\pi$ -st: stopping-time  $\pi$ 
and le:  $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq \pi \omega$ 
and bnd:  $\bigwedge \omega. \omega \in \text{space } M \implies \pi \omega \leq N$ 
and  $\omega$ -in:  $\omega \in \text{space } M$ 
shows stopped-value X  $\pi \omega - \text{stopped-value } X \tau \omega =$ 
 $(\sum i \leq N. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \omega * (X (\text{Suc } i) \omega - X$ 
i  $\omega))$ 
 $\langle$ proof $\rangle$ 
```

If each $X i$ is integrable and the stopping time is bounded, then the stopped value is integrable. This corresponds to `integrable_stoppedValue` in Mathlib.

```
lemma integrable-stopped-value:
fixes X :: nat  $\Rightarrow$  'a  $\Rightarrow$  real
assumes  $\tau$ -st: stopping-time  $\tau$  and int-X:  $\bigwedge i. \text{integrable } M (X i)$ 
```

and τ -*bound*: $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq N$
shows *integrable* M (*stopped-value* $X \tau$)
<proof>

1.3 Stopped process

The stopped process X^τ is defined as $X (\min i \tau)$.

definition *stopped-process* :: $(\text{nat} \Rightarrow 'a \Rightarrow 'b) \Rightarrow ('a \Rightarrow \text{nat}) \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'b$
where

stopped-process $X \tau i \omega \equiv X (\min i (\tau \omega)) \omega$

lemma *stopped-process-eq-stopped-value*:
stopped-process $X \tau i = \text{stopped-value } X (\lambda \omega. \min i (\tau \omega))$
<proof>

lemma *integrable-stopped-process*:
fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$
assumes *stopping-time* $\tau \bigwedge i. \text{integrable } M (X i)$
shows *integrable* M (*stopped-process* $X \tau n$)
<proof>

end

end

2 Piecewise Stopping Times

theory *Piecewise-Stopping-Time*
imports *Stopped-Value-Integration*

begin

Piecewise constant stopping times and their interaction with stopped values and integration. These are needed for the converse direction of the optional stopping theorem.

context *nat-sigma-finite-filtered-measure*
begin

2.1 Piecewise constant stopping times

Given $i \leq j$ and an F i -measurable set S , the function that returns i on S and j on its complement is a stopping time. This corresponds to `isStoppingTime_piecewise_const` in Mathlib.

lemma *stopping-time-piecewise-const*:
assumes $i \leq j$ **and** $S: S \in \text{sets } (F i)$
shows *stopping-time* $(\lambda \omega. \text{if } \omega \in S \text{ then } i \text{ else } j)$
<proof>

The stopped value at a piecewise constant stopping time decomposes into a piecewise function. This corresponds to `stoppedValue_piecewise_const` in Mathlib.

lemma *stopped-value-piecewise-const*:

assumes $S \subseteq \text{space } M$

shows *stopped-value* $X (\lambda\omega. \text{if } \omega \in S \text{ then } i \text{ else } j) = (\lambda\omega. \text{if } \omega \in S \text{ then } X i \omega \text{ else } X j \omega)$

<proof>

2.2 Integration over piecewise functions

The integral of a piecewise function splits into integrals over the two pieces. This corresponds to `integral_piecewise` in Mathlib.

lemma *piecewise-eq-indicator-sum*:

fixes $f g :: 'a \Rightarrow \text{real}$

assumes $S \in \text{sets } M \ \omega \in \text{space } M$

shows *(if* $\omega \in S$ *then* $f \ \omega$ *else* $g \ \omega$) *=* *indicat-real* $S \ \omega * f \ \omega + \text{indicat-real } (\text{space } M - S) \ \omega * g \ \omega$

<proof>

lemma *integral-piecewise*:

fixes $f g :: 'a \Rightarrow \text{real}$

assumes *S-meas*: $S \in \text{sets } M$ **and** *int-f*: *integrable* $M \ f$ **and** *int-g*: *integrable* $M \ g$

shows $(\int \omega. (\text{if } \omega \in S \text{ then } f \ \omega \text{ else } g \ \omega) \ \partial M) =$

set-lebesgue-integral $M \ S \ f + \text{set-lebesgue-integral } M \ (\text{space } M - S) \ g$

<proof>

end

end

3 Fair Games Theorem

theory *Optional-Stopping*

imports

Piecewise-Stopping-Time

Martingales.Martingale

begin

The optional stopping theorem (fair games theorem): an adapted integrable process is a submartingale if and only if for all bounded stopping times τ and π with $\tau \leq \pi$, the expected stopped value at τ is at most that at π . We also prove that the stopped process of a submartingale is a submartingale.

context *nat-sigma-finite-filtered-measure*

begin

3.1 Helper lemmas

lemma *indicator-set-in-F*:

assumes τ -st: *stopping-time* τ **and** π -st: *stopping-time* π

shows $\{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \in \text{sets } (F i)$

<proof>

3.2 Forward direction

If X is a submartingale and $\tau \leq \pi$ are bounded stopping times, then $\text{integral}^L M (\text{stopped-value } X \tau) \leq \text{integral}^L M (\text{stopped-value } X \pi)$. This corresponds to `Submartingale.expected_stoppedValue_mono` in Mathlib.

theorem *expected-stopped-value-mono*:

fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$

assumes *sub: submartingale-linorder* $M F 0 X$

and τ -st: *stopping-time* τ **and** π -st: *stopping-time* π

and *le*: $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq \pi \omega$

and *bnd*: $\bigwedge \omega. \omega \in \text{space } M \implies \pi \omega \leq N$

shows $(\int \omega. \text{stopped-value } X \tau \omega \partial M) \leq (\int \omega. \text{stopped-value } X \pi \omega \partial M)$

<proof>

3.3 Converse direction

If an adapted integrable process satisfies the monotonicity of expected stopped values for all bounded stopping times, then it is a submartingale. This corresponds to `submartingale_of_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-of-expected-stopped-value-mono*:

fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$

assumes *adapted-process* $M F 0 X$

and *integrable*: $\bigwedge i. \text{integrable } M (X i)$

and *mono*: $\bigwedge \tau \pi N. \text{stopping-time } \tau \implies \text{stopping-time } \pi \implies$

$(\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq \pi \omega) \implies (\bigwedge \omega. \omega \in \text{space } M \implies \pi \omega \leq N) \implies$

$(\int \omega. \text{stopped-value } X \tau \omega \partial M) \leq (\int \omega. \text{stopped-value } X \pi \omega \partial M)$

shows *submartingale* $M F 0 X$

<proof>

3.4 The optional stopping theorem (iff)

The full characterization: an adapted integrable process is a submartingale iff expected stopped values are monotone for all bounded stopping times. This corresponds to `submartingale_iff_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-iff-expected-stopped-value-mono*:

fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$

assumes *adapted-process* $M F 0 X$ $\bigwedge i. \text{integrable } M (X i)$

shows *submartingale* $M F 0 X \longleftrightarrow$

$(\forall \tau \pi. \text{stopping-time } \tau \longrightarrow \text{stopping-time } \pi \longrightarrow$

$(\forall \omega \in \text{space } M. \tau \omega \leq \pi \omega) \longrightarrow (\exists N. \forall \omega \in \text{space } M. \pi \omega \leq N) \longrightarrow$

$(\int \omega. \text{stopped-value } X \tau \omega \partial M) \leq (\int \omega. \text{stopped-value } X \pi \omega \partial M)$ (is ?L = ?R)
 <proof>

3.5 Stopped process of a submartingale

The stopped process of a submartingale with respect to a stopping time is a submartingale. This corresponds to `Submartingale.stoppedProcess` in Mathlib.

We first show the stopped process is adapted. The proof proceeds by induction: $X^\tau 0 = X 0$ is trivially $F 0$ -measurable, and $X^\tau (\text{Suc } n) = \text{if } \tau \leq n \text{ then } X^\tau n \text{ else } X (\text{Suc } n)$ is $F (\text{Suc } n)$ -measurable by the induction hypothesis, adaptedness of X , and the stopping time property.

lemma *adapted-stopped-process:*

fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$
assumes *adapted: adapted-process M F 0 X* **and** *τ -st: stopping-time τ*
shows *adapted-process M F 0 (stopped-process X τ)*
 <proof>

theorem *stopped-process-submartingale:*

fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$
assumes *sub: submartingale-linorder M F 0 X*
and *τ -st: stopping-time τ*
shows *submartingale M F 0 (stopped-process X τ)*
 <proof>

end

end