

The Fair Games Theorem

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Abstract

This is the optional stopping theorem (or fair games theorem) for real-valued discrete-time stochastic processes on a σ -finite filtered measure space. A theory of piecewise-constant stopping times — the function taking i on an \mathcal{F}_i -measurable set S and j on its complement — is developed, together with the corresponding decomposition of integrals over the two pieces. The central results then prove both directions of the theorem: if X is a submartingale, then for any bounded stopping times $\tau \leq \pi$ the expected stopped values satisfy $\mathbb{E}[X_\tau] \leq \mathbb{E}[X_\pi]$ (via the telescoping identity and the submartingale set-integral inequality); conversely, monotonicity of expected stopped values over all bounded stopping times characterises submartingales. It is further shown that the stopped process X^τ of a submartingale is again a submartingale. Throughout, the theorems are cross-referenced to their counterparts in Mathlib’s **OptionalStopping**, from which it was translated by Claude 4.6. It was polished manually afterwards.

The fair games theorem is #62 on the *Top 100 Mathematical Theorems*.

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1 Integrability of Stopped Values and Processes

```

theory Stopped-Value-Integration
  imports Doob-Convergence.Stopping-Time
begin

```

Integrability of stopped values and stopped processes, and the telescoping identity for differences of stopped values. These results bridge the gap between the existing AFP theories (Martingales, Doob_Convergence) and the optional stopping theorem.

```

context nat-sigma-finite-filtered-measure
begin

```

1.1 Stopped value as a sum of indicators

A stopped value with a bounded stopping time can be written as a finite sum of indicators.

```

lemma stopped-value-eq-sum:
  fixes X :: nat => 'a => 'b :: real-vector
  assumes  $\tau$ -st: stopping-time  $\tau$  and  $\tau$ -bnd:  $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq N$ 
  assumes  $\omega$ -in:  $\omega \in \text{space } M$ 
  shows stopped-value X  $\tau \omega = (\sum_{i \leq N}. \text{indicator } \{\omega \in \text{space } M. \tau \omega = i\} \omega *_R X i \omega)$ 
proof -
  have stopped-value X  $\tau \omega = X (\tau \omega) \omega$ 
    by (simp add: stopped-value-def)
  also have ... = 1 *_R X ( $\tau \omega$ )  $\omega$  by (metis (full-types) scale-one)
  also have ... = indicator  $\{\omega \in \text{space } M. \tau \omega = \tau \omega\} \omega *_R X (\tau \omega) \omega +$ 
    ( $\sum_{i \in \{..N\} - \{\tau \omega\}. \text{indicator } \{\omega \in \text{space } M. \tau \omega = i\} \omega *_R X i \omega$ )
    using  $\omega$ -in by (simp add: indicator-def)
  also have ... = ( $\sum_{i \leq N}. \text{indicator } \{\omega \in \text{space } M. \tau \omega = i\} \omega *_R X i \omega$ )
    using  $\omega$ -in  $\tau$ -bnd[OF  $\omega$ -in]
    by (subst sum.remove [where x =  $\tau \omega$ ]) simp-all
  finally show ?thesis .
qed

```

1.2 Telescoping identity for stopped values

The difference of stopped values can be expressed as a sum of indicator-weighted increments. This corresponds to 'stoppedValue_sub_eq_sum' in Mathlib.

```

lemma stopped-value-sub-eq-sum:
  fixes X :: nat => 'a => real
  assumes  $\tau$ -st: stopping-time  $\tau$  and  $\pi$ -st: stopping-time  $\pi$ 
    and le:  $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq \pi \omega$ 
    and bnd:  $\bigwedge \omega. \omega \in \text{space } M \implies \pi \omega \leq N$ 
    and  $\omega$ -in:  $\omega \in \text{space } M$ 
  shows stopped-value X  $\pi \omega - \text{stopped-value } X \tau \omega =$ 

```

$(\sum_{i \leq N}. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \omega * (X (Suc\ i) \omega - X\ i \omega))$
proof –
have $(\sum_{i \leq N}. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \omega * (X (Suc\ i) \omega - X\ i \omega)) =$
 $(\sum_{i \in \{\tau \omega..<\pi \omega\}}. X (Suc\ i) \omega - X\ i \omega)$
using $\omega\text{-in bnd}[OF\ \omega\text{-in}]$
by $(\text{intro sum.mono-neutral-cong-right})$ $(\text{auto simp: indicator-def})$
also have $\dots = X (\pi \omega) \omega - X (\tau \omega) \omega$
using $\omega\text{-in le sum-Suc-diff'}$ **by** fastforce
finally show $?thesis$ **by** $(\text{simp add: stopped-value-def})$
qed

If each $X\ i$ is integrable and the stopping time is bounded, then the stopped value is integrable. This corresponds to `integrable_stoppedValue` in `Mathlib`.

lemma *integrable-stopped-value*:

fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$

assumes $\tau\text{-st: stopping-time } \tau$ **and** $\text{int-X: } \bigwedge i. \text{integrable } M (X\ i)$
and $\tau\text{-bnd: } \bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq N$

shows $\text{integrable } M (\text{stopped-value } X\ \tau)$

proof –

– Each indicator set is measurable

have $\text{meas-eq: } \bigwedge i. \{\omega \in \text{space } M. \tau \omega = i\} \in \text{sets } M$

proof –

fix $i :: \text{nat}$

have $\text{Measurable.pred } (F\ i) (\lambda \omega. \tau \omega = i)$
by $(\text{rule stopping-time-measurable-eq}[OF\ \tau\text{-st}])$ simp-all

then have $\{\omega \in \text{space } M. \tau \omega = i\} \in \text{sets } (F\ i)$
by $(\text{metis predE subalg subalgebra-def})$

then show $\{\omega \in \text{space } M. \tau \omega = i\} \in \text{sets } M$
using $\text{sets-F-subset}[of\ i]$ **by** blast

qed

– Each summand is integrable

have $\text{int-summand: } \bigwedge i. \text{integrable } M (\lambda \omega. \text{indicat-real } \{\omega \in \text{space } M. \tau \omega = i\} \omega * X\ i \omega)$
by $(\text{simp add: int-X integrable-real-mult-indicator meas-eq mult.commute})$

– The sum is integrable

have $\text{int-sum: integrable } M (\lambda \omega. \sum_{i \leq N}. \text{indicat-real } \{\omega \in \text{space } M. \tau \omega = i\} \omega * X\ i \omega)$

by $(\text{intro Bochner-Integration.integrable-sum})$ $(\text{auto intro: int-summand})$

– The stopped value agrees with the sum AE

have $\text{eq: AE } \omega \text{ in } M. (\sum_{i \leq N}. \text{indicat-real } \{\omega \in \text{space } M. \tau \omega = i\} \omega * X\ i \omega)$
 $=$
 $\text{stopped-value } X\ \tau \omega$
by (intro AE-I2) $(\text{simp add: stopped-value-eq-sum}[OF\ \tau\text{-st } \tau\text{-bnd}])$

have $\text{eq-space: } \bigwedge \omega. \omega \in \text{space } M \implies$
 $\text{stopped-value } X\ \tau \omega = (\sum_{i \leq N}. \text{indicat-real } \{\omega \in \text{space } M. \tau \omega = i\} \omega * X\ i \omega)$

qed

by (*simp add: stopped-value-eq-sum*[*OF* τ -*st* τ -*bnd*])
 — Measurability via $(\bigwedge w. w \in \text{space } ?M \implies ?f w = ?g w) \implies (?f \in ?M \rightarrow_M ?M') = (?g \in ?M \rightarrow_M ?M')$ with the sum
have *meas-sum*: $(\lambda\omega. \sum_{i \leq N}. \text{indicat-real } \{\omega \in \text{space } M. \tau \omega = i\} \omega * X i \omega)$
 \in *borel-measurable* *M*
using *int-sum* **by** (*rule borel-measurable-integrable*)
have *meas-sv*: *stopped-value* *X* $\tau \in$ *borel-measurable* *M*
using *measurable-cong*[*of* *M* *stopped-value* *X* τ
 $\lambda\omega. \sum_{i \leq N}. \text{indicat-real } \{\omega \in \text{space } M. \tau \omega = i\} \omega * X i \omega$ *borel*]
eq-space meas-sum **by** *auto*
 — Transfer integrability via AE equality
show *?thesis*
by (*rule Bochner-Integration.integrable-cong-AE-imp*[*OF* *int-sum meas-sv eq*])
qed

1.3 Stopped process

The stopped process X^τ is defined as $X (\min i \tau)$.

definition *stopped-process* :: $(\text{nat} \Rightarrow 'a \Rightarrow 'b) \Rightarrow ('a \Rightarrow \text{nat}) \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'b$
where

stopped-process *X* τ *i* $\omega \equiv X (\min i (\tau \omega)) \omega$

lemma *stopped-process-eq-stopped-value*:
stopped-process *X* τ *i* = *stopped-value* *X* $(\lambda\omega. \min i (\tau \omega))$
unfolding *stopped-process-def* *stopped-value-def* **by** *simp*

lemma *integrable-stopped-process*:
fixes *X* :: $\text{nat} \Rightarrow 'a \Rightarrow \text{real}$
assumes *stopping-time* $\tau \bigwedge i. \text{integrable } M (X i)$
shows *integrable* *M* (*stopped-process* *X* τ *n*)
proof —
have *stopping-time* $(\lambda\omega. \min n (\tau \omega))$
by (*intro stopping-time-min stopping-time-const assms*) *simp*
with *assms* **show** *?thesis*
unfolding *stopped-process-eq-stopped-value*
using *integrable-stopped-value min.cobounded1* **by** *blast*
qed

end

end

2 Piecewise Stopping Times

theory *Piecewise-Stopping-Time*
imports *Stopped-Value-Integration*

begin

Piecewise constant stopping times and their interaction with stopped values and integration. These are needed for the converse direction of the optional stopping theorem.

context *nat-sigma-finite-filtered-measure*
begin

2.1 Piecewise constant stopping times

Given $i \leq j$ and an F i -measurable set S , the function that returns i on S and j on its complement is a stopping time. This corresponds to `isStoppingTime_piecewise_const` in Mathlib.

lemma *stopping-time-piecewise-const*:
assumes $i \leq j$ **and** $S: S \in \text{sets } (F \ i)$
shows *stopping-time* $(\lambda \omega. \text{if } \omega \in S \text{ then } i \text{ else } j)$
proof (*rule stopping-timeI*)
fix ω **assume** $\omega \in \text{space } M$
show $0 \leq (\text{if } \omega \in S \text{ then } i \text{ else } j)$ **by** *simp*
next
fix $t :: \text{nat}$ **assume** $0 \leq t$
have *space-eq*: $\text{space } (F \ t) = \text{space } M$ **by** *simp*
have *top*: $\text{space } M \in \text{sets } (F \ t)$ **using** *sets.top[of F t]* **by** *simp*
show *Measurable.pred* $(F \ t)$ $(\lambda \omega. (\text{if } \omega \in S \text{ then } i \text{ else } j) \leq t)$
unfolding *Measurable.pred-def*
proof –
consider $j \leq t \mid i \leq t \wedge j \leq t \mid \neg i \leq t$
using $\langle i \leq j \rangle$ **by** *linarith*
then show $\{\omega \in \text{space } (F \ t). (\text{if } \omega \in S \text{ then } i \text{ else } j) \leq t\} \in \text{sets } (F \ t)$
proof cases
case 1
then have $\{\omega \in \text{space } (F \ t). (\text{if } \omega \in S \text{ then } i \text{ else } j) \leq t\} = \text{space } (F \ t)$
using $\langle i \leq j \rangle$ **by** (*auto simp: space-eq*)
then show *?thesis* **using** *top space-eq* **by** *simp*
next
case 2
then have $\{\omega \in \text{space } (F \ t). (\text{if } \omega \in S \text{ then } i \text{ else } j) \leq t\} = S$
using $S \text{ sets.sets-into-space subset-antisym}$ **by** *fastforce*
then show *?thesis*
using $S \ 2 \ \text{sets-F-mono}$ **by** *force*
next
case 3
then show *?thesis*
using $\langle i \leq j \rangle$ **by** (*auto simp: Measurable.pred-def space-eq split: if-splits*)
qed
qed
qed

The stopped value at a piecewise constant stopping time decomposes into a piecewise function. This corresponds to `stoppedValue_piecewise_const`

in Mathlib.

lemma *stopped-value-piecewise-const:*

assumes $S \subseteq \text{space } M$

shows *stopped-value* $X (\lambda\omega. \text{if } \omega \in S \text{ then } i \text{ else } j) = (\lambda\omega. \text{if } \omega \in S \text{ then } X i \omega \text{ else } X j \omega)$

unfolding *stopped-value-def* **by** (*simp add: if-distrib if-distribR*)

2.2 Integration over piecewise functions

The integral of a piecewise function splits into integrals over the two pieces. This corresponds to `integral_piecewise` in Mathlib.

lemma *piecewise-eq-indicator-sum:*

fixes $f g :: 'a \Rightarrow \text{real}$

assumes $S \in \text{sets } M \ \omega \in \text{space } M$

shows $(\text{if } \omega \in S \text{ then } f \ \omega \text{ else } g \ \omega) = \text{indicat-real } S \ \omega * f \ \omega + \text{indicat-real } (\text{space } M - S) \ \omega * g \ \omega$

using $\langle \omega \in \text{space } M \rangle$ **by** (*auto simp: indicator-def*)

lemma *integral-piecewise:*

fixes $f g :: 'a \Rightarrow \text{real}$

assumes $S\text{-meas}: S \in \text{sets } M$ **and** $\text{int-}f: \text{integrable } M \ f$ **and** $\text{int-}g: \text{integrable } M \ g$

shows $(\int \omega. (\text{if } \omega \in S \text{ then } f \ \omega \text{ else } g \ \omega) \ \partial M) = \text{set-lebesgue-integral } M \ S \ f + \text{set-lebesgue-integral } M \ (\text{space } M - S) \ g$

proof –

let $?h = \lambda\omega. \text{indicat-real } S \ \omega * f \ \omega + \text{indicat-real } (\text{space } M - S) \ \omega * g \ \omega$

have $\text{eq-ae}: \text{AE } \omega \text{ in } M. (\text{if } \omega \in S \text{ then } f \ \omega \text{ else } g \ \omega) = ?h \ \omega$

by (*rule AE-I2*) (*auto simp: indicator-def*)

have $\text{int-Sf}: \text{integrable } M \ (\lambda\omega. \text{indicat-real } S \ \omega * f \ \omega)$

using *integrable-mult-indicator[OF S-meas int-f]*

by (*simp add: scaleR-conv-of-real*)

have $\text{int-Sg}: \text{integrable } M \ (\lambda\omega. \text{indicat-real } (\text{space } M - S) \ \omega * g \ \omega)$

using *integrable-mult-indicator[OF - int-g]* **by** (*simp add: S-meas sets.Diff*)

have $\text{meas-if}: (\lambda\omega. \text{if } \omega \in S \text{ then } f \ \omega \text{ else } g \ \omega) \in \text{borel-measurable } M$

by (*intro measurable-If-set*) (*use int-f int-g S-meas in auto*)

have $\text{meas-h}: ?h \in \text{borel-measurable } M$

using *int-Sf int-Sg* **by** (*intro borel-measurable-add*) *auto*

have $(\int \omega. (\text{if } \omega \in S \text{ then } f \ \omega \text{ else } g \ \omega) \ \partial M) = (\int \omega. ?h \ \omega \ \partial M)$

by (*rule integral-cong-AE[OF meas-if meas-h eq-ae]*)

also have $\dots = (\int \omega. \text{indicat-real } S \ \omega * f \ \omega \ \partial M) + (\int \omega. \text{indicat-real } (\text{space } M - S) \ \omega * g \ \omega \ \partial M)$

by (*rule Bochner-Integration.integral-add[OF int-Sf int-Sg]*)

also have $\dots = \text{set-lebesgue-integral } M \ S \ f + \text{set-lebesgue-integral } M \ (\text{space } M - S) \ g$

unfolding *set-lebesgue-integral-def* **by** (*simp add: scaleR-conv-of-real*)

finally show *?thesis* .

qed

end

end

3 Fair Games Theorem

```
theory Optional-Stopping
  imports
    Piecewise-Stopping-Time
    Martingales.Martingale
begin
```

The optional stopping theorem (fair games theorem): an adapted integrable process is a submartingale if and only if for all bounded stopping times τ and π with $\tau \leq \pi$, the expected stopped value at τ is at most that at π . We also prove that the stopped process of a submartingale is a submartingale.

```
context nat-sigma-finite-filtered-measure
begin
```

3.1 Helper lemmas

```
lemma indicator-set-in-F:
  assumes  $\tau$ -st: stopping-time  $\tau$  and  $\pi$ -st: stopping-time  $\pi$ 
  shows  $\{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \in \text{sets } (F i)$ 
proof -
  have *: Measurable.pred (F i) ( $\lambda\omega. \tau \omega \leq i \wedge \neg \pi \omega \leq i$ )
    by (simp add:  $\pi$ -st  $\tau$ -st pred-intros-logic stopping-timeD)
  have eq:  $\{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} = \{\omega \in \text{space } (F i). \tau \omega \leq i \wedge \neg \pi \omega \leq i\}$ 
    by auto
  show ?thesis using Measurable.predE[OF *] sets-F-subset[of i]
    by (auto simp: eq)
qed
```

3.2 Forward direction

If X is a submartingale and $\tau \leq \pi$ are bounded stopping times, then $\text{integral}^L M (\text{stopped-value } X \tau) \leq \text{integral}^L M (\text{stopped-value } X \pi)$. This corresponds to `Submartingale.expected_stoppedValue_mono` in Mathlib.

```
theorem expected-stopped-value-mono:
  fixes  $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$ 
  assumes sub: submartingale-linorder  $M F 0 X$ 
    and  $\tau$ -st: stopping-time  $\tau$  and  $\pi$ -st: stopping-time  $\pi$ 
    and le:  $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq \pi \omega$ 
    and bnd:  $\bigwedge \omega. \omega \in \text{space } M \implies \pi \omega \leq N$ 
  shows  $(\int \omega. \text{stopped-value } X \tau \omega \partial M) \leq (\int \omega. \text{stopped-value } X \pi \omega \partial M)$ 
proof -
```

from sub interpret S : *submartingale-linorder* $M F 0 X$.
have τ -*bnd*: $\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq N$
using *le bnd by (meson order-trans)*
— Integrability of stopped values
obtain *int- τ* : *integrable* M (*stopped-value* $X \tau$) **and** *int- π* : *integrable* M (*stopped-value* $X \pi$)
by (*meson S.integrable τ -bnd τ -st π -st bnd integrable-stopped-value zero-le*)
— Suffices to show the difference is non-negative
have $0 \leq (\int \omega. \text{stopped-value } X \pi \omega \partial M) - (\int \omega. \text{stopped-value } X \tau \omega \partial M)$
proof —
have $(\int \omega. \text{stopped-value } X \pi \omega \partial M) - (\int \omega. \text{stopped-value } X \tau \omega \partial M) =$
 $(\int \omega. \text{stopped-value } X \pi \omega - \text{stopped-value } X \tau \omega \partial M)$
by (*rule Bochner-Integration.integral-diff[OF int- π int- τ , symmetric]*)
— Apply the telescoping identity AE
also have $\dots = (\int \omega. (\sum_{i \leq N}. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\}$
 $\omega * (X (Suc i) \omega - X i \omega)) \partial M)$
proof (*rule Bochner-Integration.integral-cong-AE*)
show $(\lambda \omega. \text{stopped-value } X \pi \omega - \text{stopped-value } X \tau \omega) \in \text{borel-measurable } M$
using *int- π int- τ by (intro borel-measurable-diff) auto*
show $(\lambda \omega. \sum_{i \leq N}. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \omega * (X (Suc i) \omega - X i \omega)) \in \text{borel-measurable } M$
proof (*intro borel-measurable-sum borel-measurable-times borel-measurable-indicator borel-measurable-diff borel-measurable-integrable*)
fix i **assume** $i \in \{..N\}$
show $\{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \in \text{sets } M$
using *indicator-set-in-F[OF τ -st π -st] sets-F-subset by blast*
qed (*auto simp: S.integrable*)
show *AE ω in M . stopped-value $X \pi \omega - \text{stopped-value } X \tau \omega =$*
 $(\sum_{i \leq N}. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \omega * (X (Suc i) \omega - X i \omega))$
by (*rule AE-I2 (simp add: stopped-value-sub-eq-sum[OF τ -st π -st le bnd]*)
qed
— Exchange integral and sum
also have $\dots = (\sum_{i \leq N}. \int \omega. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\}$
 $\omega * (X (Suc i) \omega - X i \omega) \partial M)$
proof (*rule Bochner-Integration.integral-sum*)
fix i **assume** $i \in \{..N\}$
have $\{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\} \in \text{sets } M$
using *indicator-set-in-F[OF τ -st π -st] sets-F-subset by blast*
then show *integrable* M $(\lambda \omega. \text{indicator } \{\omega \in \text{space } M. \tau \omega \leq i \wedge i < \pi \omega\}$
 $\omega * (X (Suc i) \omega - X i \omega))$
by (*simp add: S.integrable integrable-real-mult-indicator mult.commute*)
qed
— Each summand is non-negative via *set-integral-le*
also have $\dots \geq 0$
proof (*rule sum-nonneg*)
fix i **assume** $i \in \{..N\}$

```

let ?A = {ω ∈ space M. τ ω ≤ i ∧ i < π ω}
have A-Fi: ?A ∈ sets (F i)
  by (rule indicator-set-in-F[OF τ-st π-st])
— The summand equals  $\int_A X(\text{Suc } i) - \int_A X i$ 
have eq: ( $\int \omega. \text{indicat-real } ?A \omega * (X (\text{Suc } i) \omega - X i \omega) \partial M$ ) =
  set-lebesgue-integral M ?A (X (Suc i)) - set-lebesgue-integral M ?A (X i)
proof -
  have ?A ∈ sets M
    using A-Fi sets-F-subset by blast
  then have int: integrable M ( $\lambda x. \text{indicat-real } ?A x * X j x$ ) for j
    by (simp add: S.integrable integrable-real-mult-indicator mult.commute)
  have ( $\int \omega. \text{indicat-real } ?A \omega * (X (\text{Suc } i) \omega - X i \omega) \partial M$ ) =
    ( $\int \omega. \text{indicat-real } ?A \omega * X (\text{Suc } i) \omega - \text{indicat-real } ?A \omega * X i \omega \partial M$ )
    by (simp add: right-diff-distrib)
  also have ... = ( $\int \omega. \text{indicat-real } ?A \omega * X (\text{Suc } i) \omega \partial M$ ) - ( $\int \omega. \text{indicat-real } ?A \omega * X i \omega \partial M$ )
    using Bochner-Integration.integral-diff int by blast
  also have ... = set-lebesgue-integral M ?A (X (Suc i)) - set-lebesgue-integral
M ?A (X i)
    unfolding set-lebesgue-integral-def by (simp add: scaleR-conv-of-real)
  finally show ?thesis .
qed
— Apply the submartingale set-integral-le property
have set-lebesgue-integral M ?A (X i) ≤ set-lebesgue-integral M ?A (X (Suc
i))
  by (rule S.set-integral-le[OF A-Fi]) simp-all
then show 0 ≤ ( $\int \omega. \text{indicat-real } ?A \omega * (X (\text{Suc } i) \omega - X i \omega) \partial M$ )
  unfolding eq by simp
qed
finally show ?thesis by simp
qed
then show ?thesis by simp
qed

```

3.3 Converse direction

If an adapted integrable process satisfies the monotonicity of expected stopped values for all bounded stopping times, then it is a submartingale. This corresponds to `submartingale_of_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-of-expected-stopped-value-mono*:

fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$

assumes *adapted*: *adapted-process* M F 0 X

and *integrable*: $\bigwedge i. \text{integrable } M (X i)$

and *mono*: $\bigwedge \tau \pi N. \text{stopping-time } \tau \implies \text{stopping-time } \pi \implies$

$(\bigwedge \omega. \omega \in \text{space } M \implies \tau \omega \leq \pi \omega) \implies (\bigwedge \omega. \omega \in \text{space } M \implies \pi \omega \leq N) \implies$

$(\int \omega. \text{stopped-value } X \tau \omega \partial M) \leq (\int \omega. \text{stopped-value } X \pi \omega \partial M)$

shows *submartingale* M F 0 X

proof (rule *submartingale-of-set-integral-le-Suc*[OF *adapted integrable*])

fix A :: 'a set **and** i :: nat

```

assume  $A-Fi$ :  $A \in sets (F i)$ 
— Construct piecewise stopping times
define  $\tau :: 'a \Rightarrow nat$  where  $\tau \omega = (if \omega \in A then i else Suc i)$  for  $\omega$ 
define  $\pi :: 'a \Rightarrow nat$  where  $\pi \omega = Suc i$  for  $\omega$ 
have  $\tau-st$ : stopping-time  $\tau$ 
  unfolding  $\tau-def$  by (rule stopping-time-piecewise-const) (simp-all add: A-Fi)
have  $\pi-st$ : stopping-time  $\pi$ 
  unfolding  $\pi-def$  by (rule stopping-time-const) simp
have  $\tau-le-\pi$ :  $\bigwedge \omega. \omega \in space M \implies \tau \omega \leq \pi \omega$ 
  unfolding  $\tau-def$   $\pi-def$  by simp
have  $\pi-bnd$ :  $\bigwedge \omega. \omega \in space M \implies \pi \omega \leq Suc i$ 
  unfolding  $\pi-def$  by simp
— Apply the monotonicity hypothesis
have ineq:  $(\int \omega. stopped-value X \tau \omega \partial M) \leq (\int \omega. stopped-value X \pi \omega \partial M)$ 
  by (rule mono[OF  $\tau-st$   $\pi-st$   $\tau-le-\pi$   $\pi-bnd$ ])
— Decompose stopped values
have  $A-sub$ :  $A \subseteq space M$ 
  using  $A-Fi$  sets-F-subset sets.sets-into-space by blast
have  $sv-\tau$ : stopped-value  $X \tau = (\lambda \omega. if \omega \in A then X i \omega else X (Suc i) \omega)$ 
  unfolding  $\tau-def$  by (rule stopped-value-piecewise-const[OF A-sub])
have  $sv-\pi$ : stopped-value  $X \pi = X (Suc i)$ 
  unfolding  $\pi-def$  stopped-value-def by simp
— Decompose the integral of the piecewise function
have  $A-meas$ :  $A \in sets M$ 
  using  $A-Fi$  sets-F-subset by blast
have lhs:  $(\int \omega. stopped-value X \tau \omega \partial M) =$ 
  set-lebesgue-integral  $M A (X i) + set-lebesgue-integral M (space M - A) (X$ 
   $(Suc i))$ 
  unfolding  $sv-\tau$  by (rule integral-piecewise[OF A-meas integrable integrable])
have rhs:  $(\int \omega. stopped-value X \pi \omega \partial M) =$ 
  set-lebesgue-integral  $M A (X (Suc i)) + set-lebesgue-integral M (space$ 
   $M - A) (X (Suc i))$ 
proof —
  have  $(\int \omega. stopped-value X \pi \omega \partial M) = (\int \omega. X (Suc i) \omega \partial M)$ 
    unfolding  $sv-\pi$  ..
  also have ... =  $(\int \omega. (if \omega \in A then X (Suc i) \omega else X (Suc i) \omega) \partial M)$ 
    by simp
  also have ... = set-lebesgue-integral  $M A (X (Suc i)) +$ 
    set-lebesgue-integral  $M (space M - A) (X (Suc i))$ 
    by (rule integral-piecewise[OF A-meas integrable integrable])
  finally show ?thesis .
qed
show set-lebesgue-integral  $M A (X i) \leq set-lebesgue-integral M A (X (Suc i))$ 
  using ineq lhs rhs by simp
qed

```

3.4 The optional stopping theorem (iff)

The full characterization: an adapted integrable process is a submartingale iff expected stopped values are monotone for all bounded stopping times. This corresponds to `submartingale_iff_expected_stoppedValue_mono` in Mathlib.

theorem *submartingale-iff-expected-stopped-value-mono*:

fixes $X :: nat \Rightarrow 'a \Rightarrow real$

assumes *adapted-process* $M F 0 X \wedge i$. *integrable* $M (X i)$

shows *submartingale* $M F 0 X \iff$

$(\forall \tau \pi. \textit{stopping-time } \tau \longrightarrow \textit{stopping-time } \pi \longrightarrow$

$(\forall \omega \in \textit{space } M. \tau \omega \leq \pi \omega) \longrightarrow (\exists N. \forall \omega \in \textit{space } M. \pi \omega \leq N) \longrightarrow$

$(\int \omega. \textit{stopped-value } X \tau \omega \partial M) \leq (\int \omega. \textit{stopped-value } X \pi \omega \partial M))$ (**is** $?L =$

$?R$)

proof

show $?L \implies ?R$

by (*metis expected-stopped-value-mono submartingale-linorder-def*)

show $?R \implies ?L$

by (*intro submartingale-of-expected-stopped-value-mono[OF assms]*) *blast*

qed

3.5 Stopped process of a submartingale

The stopped process of a submartingale with respect to a stopping time is a submartingale. This corresponds to `Submartingale.stoppedProcess` in Mathlib.

We first show the stopped process is adapted. The proof proceeds by induction: $X^\tau 0 = X 0$ is trivially $F 0$ -measurable, and $X^\tau (Suc n) = if \tau \leq n \textit{ then } X^\tau n \textit{ else } X (Suc n)$ is $F (Suc n)$ -measurable by the induction hypothesis, adaptedness of X , and the stopping time property.

lemma *adapted-stopped-process*:

fixes $X :: nat \Rightarrow 'a \Rightarrow real$

assumes *adapted*: *adapted-process* $M F 0 X$ **and** τ -*st*: *stopping-time* τ

shows *adapted-process* $M F 0$ (*stopped-process* $X \tau$)

proof (*rule adapted-process.intro[OF filtered-measure-axioms]*)

show *adapted-process-axioms* $F 0$ (*stopped-process* $X \tau$)

proof (*rule adapted-process-axioms.intro*)

fix $i :: nat$ **assume** $0 \leq i$

show *stopped-process* $X \tau i \in \textit{borel-measurable } (F i)$

proof (*induction i*)

case 0

have *stopped-process* $X \tau 0 = X 0$

unfolding *stopped-process-def* **by** *simp*

then show $?case$

using *adapted-process.adapted*[*OF adapted, of 0*] **by** *simp*

next

case (*Suc n*)

— The stopped process at $Suc\ n$ equals a piecewise function on *space* M
have $eq: \bigwedge \omega. \text{stopped-process } X\ \tau\ (Suc\ n)\ \omega =$
 (if $\tau\ \omega \leq n$ then $\text{stopped-process } X\ \tau\ n\ \omega$ else $X\ (Suc\ n)\ \omega$)
unfolding $\text{stopped-process-def}$ **by** (*auto simp: min-def*)
 — $F\ n$ is a sub-sigma-algebra of $F\ (Suc\ n)$
have $subalg: \text{subalgebra } (F\ (Suc\ n))\ (F\ n)$
unfolding subalgebra-def **using** $\text{space-}F\ \text{sets-}F\ \text{mono}[of\ n\ Suc\ n]$ **by** *auto*
 — The induction hypothesis gives $F\ n$ -measurability, lift to $F\ (Suc\ n)$
have $meas-n: \text{stopped-process } X\ \tau\ n \in \text{borel-measurable } (F\ (Suc\ n))$
by (*rule measurable-from-subalg[OF subalg Suc.IH]*)
 — $X\ (Suc\ n)$ is $F\ (Suc\ n)$ -measurable by adaptedness
have $meas-Sn: X\ (Suc\ n) \in \text{borel-measurable } (F\ (Suc\ n))$
using $\text{adapted-process.adapted}[OF\ adapted]$ **by** *simp*
 — The set $\{\omega. \tau\ \omega \leq n\}$ is in $\text{sets } (F\ (Suc\ n))$
have $set-Sn: \{\omega \in \text{space } (F\ (Suc\ n)). \tau\ \omega \leq n\} \in \text{sets } (F\ (Suc\ n))$
using $\tau\text{-st order-less-imp-le predE stopping-time-measurable-le}$ **by** *blast*
 — The piecewise function is $F\ (Suc\ n)$ -measurable
let $?A = \{\omega \in \text{space } (F\ (Suc\ n)). \tau\ \omega \leq n\}$
have $A\text{-sets}: ?A \in \text{sets } (F\ (Suc\ n))$ **by** (*rule set-Sn*)
have $A\text{-sub}: ?A \subseteq \text{space } (F\ (Suc\ n))$ **using** $\text{sets.sets-into-space}[OF\ A\text{-sets}]$.
have $meas-pw: (\lambda \omega. \text{if } \omega \in ?A \text{ then } \text{stopped-process } X\ \tau\ n\ \omega \text{ else } X\ (Suc\ n)\ \omega)$
 $\in \text{borel-measurable } (F\ (Suc\ n))$
using $\text{measurable-If-set meas-n meas-Sn set-Sn}$ **by** *blast*
 — Transfer: the stopped process agrees with the piecewise function on *space*
 $(F\ (Suc\ n))$
show $?case$
using $\text{measurable-cong}[of\ F\ (Suc\ n)\ \text{stopped-process } X\ \tau\ (Suc\ n)]$
 $\lambda \omega. \text{if } \omega \in ?A \text{ then } \text{stopped-process } X\ \tau\ n\ \omega \text{ else } -\ \omega]$
 $eq\ \text{meas-pw}$ **by** *fastforce*
qed
qed
qed

theorem *stopped-process-submartingale*:
fixes $X :: \text{nat} \Rightarrow 'a \Rightarrow \text{real}$
assumes $sub: \text{submartingale-linorder } M\ F\ 0\ X$
and $\tau\text{-st}: \text{stopping-time } \tau$
shows $\text{submartingale } M\ F\ 0\ (\text{stopped-process } X\ \tau)$
proof —
let $?sv = \text{stopped-value } (\text{stopped-process } X\ \tau)$
from sub **interpret** $S: \text{submartingale-linorder } M\ F\ 0\ X$.
 — Use the converse direction: suffices to show monotonicity of expected stopped values
show $?thesis$
proof (*rule submartingale-of-expected-stopped-value-mono*)
show $\text{adapted-process } M\ F\ 0\ (\text{stopped-process } X\ \tau)$
by (*simp add: S.adapted-process-axioms τ -st adapted-stopped-process*)
show $\bigwedge i. \text{integrable } M\ (\text{stopped-process } X\ \tau\ i)$

```

    by (simp add: S.integrable  $\tau$ -st integrable-stopped-process)
next
fix  $\sigma \varrho :: 'a \Rightarrow \text{nat}$  and  $N :: \text{nat}$ 
assume  $\sigma$ -st: stopping-time  $\sigma$  and  $\varrho$ -st: stopping-time  $\varrho$ 
    and  $\bigwedge \omega. \omega \in \text{space } M \implies \sigma \omega \leq \varrho \omega$ 
    and bnd:  $\bigwedge \omega. \omega \in \text{space } M \implies \varrho \omega \leq N$ 
then have le':  $\bigwedge \omega. \omega \in \text{space } M \implies \min (\sigma \omega) (\tau \omega) \leq \min (\varrho \omega) (\tau \omega)$ 
    by (simp add: min-le-iff-disj)
— stopped value of the stopped process equals that of  $X$  with  $\min$ 
have sv:  $?sv \sigma = \text{stopped-value } X (\lambda \omega. \min (\sigma \omega) (\tau \omega)) \ ?sv \varrho = \text{stopped-value}$ 
 $X (\lambda \omega. \min (\varrho \omega) (\tau \omega))$ 
    unfolding stopped-value-def stopped-process-def by auto
—  $\lambda \omega. \min (\sigma \omega) (\tau \omega)$  and  $\lambda \omega. \min (\varrho \omega) (\tau \omega)$  are stopping times
have st- $\sigma'$ : stopping-time  $(\lambda \omega. \min (\sigma \omega) (\tau \omega))$ 
    by (intro stopping-time-min  $\sigma$ -st  $\tau$ -st)
have st- $\varrho'$ : stopping-time  $(\lambda \omega. \min (\varrho \omega) (\tau \omega))$ 
    by (intro stopping-time-min  $\varrho$ -st  $\tau$ -st)
— Apply the forward direction
show  $(\int \omega. ?sv \sigma \omega \partial M) \leq (\int \omega. ?sv \varrho \omega \partial M)$ 
    using expected-stopped-value-mono[OF sub st- $\sigma'$  st- $\varrho'$  le'] sv bnd min-le-iff-disj
    by metis
qed
qed
end
end
end

```