Extreme Value Theory

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Abstract

This entry is a formalization of extreme value theory (EVT). EVT is a branch of probability theory and statistics that studies *rare events*. We formalize two main results of EVT, mainly following the books by de Haan and Ferreira [2] and by Embrechts et al. [3].

- The Fisher–Tippett–Gnedenko theorem, stating that if the properly normalized maximum of i.i.d. random variables converges in distribution to a non-degenerate limit, then the limit is an extreme value distribution.
- The Pickands–Balkema–de Haan theorem, stating that the conditional distribution of the excesses over a threshold u converges uniformly in the excess, with a suitable scale $\beta(u)$, to a generalized Pareto distribution as u tends to the right endpoint of the support, if and only if the underlying distribution belongs to the maximum domain of attraction of an extreme value distribution.

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1 Lemmas

theory *Lemmas-EVT*
imports
HOL−Probability.Probability
Regular-Variation.Regular-Variation
begin

lemma *power-of-nat-log-less-Suc*:
fixes $Z\ t :: real$
assumes Z: $1 < Z$
shows $t < Z \hat{} (nat\ \lfloor log\ Z\ t \rfloor + 1)$
⟨*proof*⟩

1.1 Lemmas for Measure Theory

lemma *measure-density-on*:
assumes *integ*:*set-integrable M A f*
and [*measurable*]: $f \in$ *borel-measurable M*
and [*measurable*]:$A \in$ *sets M*
and *f-nonneg*:$AE\ x \in A$ *in M*. $0 \leq f\ x$
shows *measure* (*density M f*) $A = (\int x{\in}A.\ f\ x\ \partial M)$
⟨*proof*⟩

corollary *measure-density*:

assumes *integ*[*measurable*]:*integrable M f*
and [*measurable*]:$A \in$ *sets M*
and *f-nonneg*:*AE x in M*. $0 \leq f\ x$
shows *measure* (*density M f*) $A = (\int x{\in}A.\ f\ x\ \partial M)$
⟨*proof*⟩

lemma *nn-integral-FTC-greaterThan*:
fixes f :: *real* ⇒ *real* **and** F :: *real* ⇒ *real*
assumes *f-borel*: $f \in$ *borel-measurable borel*
and *cont*: *continuous* (*at-right a*) F
and f: $\bigwedge x.\ a < x \Longrightarrow$ *DERIV F x* :> $f\ x$
and *nonneg*: $\bigwedge x.\ a < x \Longrightarrow 0 \leq f\ x$
and *lim*: $(F \longrightarrow T)$ *at-top*
shows $(\int^+ x.\ ennreal\ (f\ x) * indicator\ \{a <..\}\ x\ \partial lborel) = T - F\ a$
⟨*proof*⟩

lemma *nn-integral-FTC-lessThan*:
fixes f :: *real* ⇒ *real* **and** F :: *real* ⇒ *real*
assumes *f-borel*: $f \in$ *borel-measurable borel*
and *cont*: *continuous* (*at-left b*) F
and f: $\bigwedge x.\ x < b \Longrightarrow$ *DERIV F x* :> $f\ x$
and *nonneg*: $\bigwedge x.\ x < b \Longrightarrow 0 \leq f\ x$
and *lim*: $(F \longrightarrow S)$ *at-bot*
shows $(\int^+ x.\ ennreal\ (f\ x) * indicator\ \{..< b\}\ x\ \partial lborel) = F\ b - S$
⟨*proof*⟩

lemma
fixes f :: *real* ⇒ *real* **and** F :: *real* ⇒ *real*
assumes *f-borel*[*measurable*]: $f \in$ *borel-measurable borel*
and *cont*: *continuous* (*at-right a*) F
and f: $\bigwedge x.\ a < x \Longrightarrow$ *DERIV F x* :> $f\ x$
and *nonneg*: $\bigwedge x.\ a < x \Longrightarrow 0 \leq f\ x$
and *lim*: $(F \longrightarrow T)$ *at-top*
shows *set-integral-FTC-atLeast*:$(\int x{\in}\{a..\}.\ f\ x\ \partial lborel) = T - F\ a$
and *set-integrable-FTC-atLeast*:*set-integrable lborel* $\{a..\}\ f$
⟨*proof*⟩

lemma
fixes f :: *real* ⇒ *real* **and** F :: *real* ⇒ *real*
assumes *f-borel*[*measurable*]: $f \in$ *borel-measurable borel*
assumes *cont*: *continuous* (*at-left b*) F
assumes f: $\bigwedge x.\ x < b \Longrightarrow$ *DERIV F x* :> $f\ x$
assumes *nonneg*: $\bigwedge x.\ x < b \Longrightarrow 0 \leq f\ x$
assumes *lim*: $(F \longrightarrow S)$ *at-bot*
shows *set-integral-FTC-atMost*:$(\int x{\in}\{..b\}.\ f\ x\ \partial lborel) = F\ b - S$
and *set-integrable-FTC-atMost*:*set-integrable lborel* $\{..b\}\ f$
⟨*proof*⟩

1.2 Lemmas for Limits

lemma *filterlim-log-at-top-gt-1*:
fixes $a :: real$
assumes $a{:}1 < a$
shows $filterlim\ (log\ a)\ at\text{-}top\ at\text{-}top$
⟨*proof*⟩

lemma *LIMSEQ-power-at-top*:
fixes $c :: real$
assumes $1 < c$
shows $filterlim\ (\lambda n.\ c\ \hat{}\ n)\ at\text{-}top\ sequentially$
⟨*proof*⟩

lemma *floor-log-at-top*:
fixes $Z :: real$
assumes $Z{:}\ 1 < Z$
shows $filterlim\ (\lambda t.\ nat\ \lfloor log\ Z\ t \rfloor)\ at\text{-}top\ at\text{-}top$
⟨*proof*⟩

lemma *squeeze-tendsto-cont-points*:
fixes $f :: real \Rightarrow real$ **and** $h :: 'a \Rightarrow real$ **and** $gr :: real \Rightarrow 'a \Rightarrow real$
and $F :: 'a\ filter$ **and** $y :: real$
and $A :: real\ set$
assumes *mono*: $mono\text{-}on\ A\ f$
and $y{:}y \in interior\ A$
and *cont*: $isCont\ f\ y$
and *conv*: $\bigwedge y'.\ y' \in A \Longrightarrow isCont\ f\ y' \Longrightarrow ((gr\ y') \longrightarrow f\ y')\ F$
and *low*: $\bigwedge y'.\ y' < y \Longrightarrow y' \in A \Longrightarrow \forall_F\ t\ in\ F.\ gr\ y'\ t \leq h\ t$
and *up*: $\bigwedge y'.\ y < y' \Longrightarrow y' \in A \Longrightarrow \forall_F\ t\ in\ F.\ h\ t \leq gr\ y'\ t$
shows $(h \longrightarrow f\ y)\ F$
⟨*proof*⟩

lemma *squeeze-tendsto-local*:
fixes $f :: real \Rightarrow real$ **and** $h :: 'a \Rightarrow real$ **and** $gr :: real \Rightarrow 'a \Rightarrow real$
and $F :: 'a\ filter$ **and** $y\ \eta :: real$
assumes *eta*: $0 < \eta$
and *cont*: $isCont\ f\ y$
and *conv*: $\bigwedge y'.\ |y' - y| < \eta \Longrightarrow ((gr\ y') \longrightarrow f\ y')\ F$
and *low*: $\bigwedge y'.\ y - \eta < y' \Longrightarrow y' < y \Longrightarrow \forall_F\ t\ in\ F.\ gr\ y'\ t \leq h\ t$
and *up*: $\bigwedge y'.\ y < y' \Longrightarrow y' < y + \eta \Longrightarrow \forall_F\ t\ in\ F.\ h\ t \leq gr\ y'\ t$
shows $(h \longrightarrow f\ y)\ F$
⟨*proof*⟩

corollary *squeeze-tendsto-cont-points-pos*:
fixes $f :: real \Rightarrow real$ **and** $h :: 'a \Rightarrow real$ **and** $gr :: real \Rightarrow 'a \Rightarrow real$
and $F :: 'a\ filter$ **and** $y :: real$
assumes *mono*: $mono\text{-}on\ \{0{<}..\}\ f$
and $y{:}\ 0 < y$ **and** *cont*: $isCont\ f\ y$
and *conv*: $\bigwedge r.\ 0 < r \Longrightarrow isCont\ f\ r \Longrightarrow ((gr\ r) \longrightarrow f\ r)\ F$

and *low*: $\bigwedge r.\ 0 < r \Longrightarrow r < y \Longrightarrow \forall_F\ t\ in\ F.\ gr\ r\ t \leq h\ t$
and *up*: $\bigwedge r.\ y < r \Longrightarrow \forall_F\ t\ in\ F.\ h\ t \leq gr\ r\ t$
shows $(h \longrightarrow f\ y)\ F$
⟨*proof*⟩

corollary *squeeze-tendsto-cont-points-UNIV*:
fixes $f :: real \Rightarrow real$ **and** $h :: 'a \Rightarrow real$ **and** $gr :: real \Rightarrow 'a \Rightarrow real$
and $F :: 'a\ filter$ **and** $y :: real$
assumes *mono*: *mono f*
and *cont*: *isCont f y*
and *conv*: $\bigwedge y'.\ isCont\ f\ y' \Longrightarrow ((gr\ y') \longrightarrow f\ y')\ F$
and *low*: $\bigwedge y'.\ y' < y \Longrightarrow \forall_F\ t\ in\ F.\ gr\ y'\ t \leq h\ t$
and *up*: $\bigwedge y'.\ y < y' \Longrightarrow \forall_F\ t\ in\ F.\ h\ t \leq gr\ y'\ t$
shows $(h \longrightarrow f\ y)\ F$
⟨*proof*⟩

lemma *left-cont-agree-cocountable-eq*:
fixes $f\ h :: real \Rightarrow real$ **and** $N :: real\ set$ **and** $y :: real$
assumes N: *countable N* **and** y: $0 < y$
and *left-cont*: $(f \longrightarrow f\ y)\ (at\text{-}left\ y)$
and *left-cont-h*: $(h \longrightarrow h\ y)\ (at\text{-}left\ y)$
and *eq*: $\bigwedge x.\ 0 < x \Longrightarrow x < y \Longrightarrow x \notin N \Longrightarrow f\ x = h\ x$
shows $f\ y = h\ y$
⟨*proof*⟩

lemma *power-tendsto-iff-tendsto-tail*:
fixes $p :: nat \Rightarrow real$ **and** $q :: real$
assumes *p-nonneg*: $\bigwedge n.\ 0 \leq p\ n$ **and** *p-le1*: $\bigwedge n.\ p\ n \leq 1$
and *q-pos*: $0 < q$ **and** *q-le1*: $q \leq 1$
shows $(\lambda n.\ p\ n\ \hat{}\ n) \longrightarrow q \longleftrightarrow (\lambda n.\ real\ n * (1 - p\ n)) \longrightarrow -\ ln\ q$
⟨*proof*⟩

lemma *filterlim-shift-scale-at-top*:
fixes $\mu\ \sigma\ \xi :: real$
assumes [*arith*]: $0 < \sigma\ \ 0 < \xi$
shows *LIM x at-top.* $1 + \xi * ((x - \mu)\ /\ \sigma) :> at\text{-}top$
⟨*proof*⟩

lemma *filterlim-shift-scale-at-bot*:
fixes $\mu\ \sigma\ \xi :: real$
assumes *sigma*: $0 < \sigma$ **and** *xi-neg*: $\xi < 0$
shows *LIM x at-bot.* $1 + \xi * ((x - \mu)\ /\ \sigma) :> at\text{-}top$
⟨*proof*⟩

1.3 Lemmas for Weak Convergence

lemma *weak-conv-ignore-initial-segment*:
assumes *weak-conv a A*
shows *weak-conv* $(\lambda n.\ a\ (n + k))\ A$

⟨*proof*⟩

lemma *weak-conv-offset*:
assumes *weak-conv* (λ*n*. *a* (*n* + *k*)) *A*
shows *weak-conv a A*
⟨*proof*⟩

lemma *weak-conv-cong*:
assumes *ev*:⋀*x*. *isCont A x* ⟹ ∀ *F n in sequentially*. *a n x* = *b n x*
shows *weak-conv a A* ⟷ *weak-conv b A*
⟨*proof*⟩

lemma *LIMSEQ-imp-weak-conv*:
(⋀*x*. (λ*n*. *a n x*) ⟶ *A x*) ⟹ *weak-conv a A*
⟨*proof*⟩

1.4 Generalized Inverses

definition *ginv* :: ('*a* :: *conditionally-complete-linorder* ⇒ '*a*) ⇒ '*a* ⇒ '*a* (-⁻¹ [*81*])
where
ginv F y = *Inf* {*x*. *y* ≤ *F x*}

The constant *ginv* is the same as *right-continuous-mono.I*. We define *ginv* because *right-continuous-mono.I* is defined inside the locale *right-continuous-mono*, which makes it inconvenient to use.

lemma *bdd-below-ginv-set*:
assumes *mono*:*mono F* **and** *Fw*:*F w* < *y*
shows *bdd-below* {*x*::- :: *linorder*. *y* ≤ *F x*}
⟨*proof*⟩

lemma *ginv-le-iff1*:
fixes *y* :: '*a*::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *mono*: *mono F* **and** *cont*: ⋀*x*. *continuous* (*at-right x*) *F*
and *yFz*: *y* ≤ *F z* **and** *Fwy*: *F w* < *y*
shows *F*⁻¹ *y* ≤ *x* ⟷ *y* ≤ *F x*
⟨*proof*⟩

lemma *ginv-mono*:
assumes *mono*: *mono F* **and** *y12*: *y1* ≤ *y2*
and *ex-z*: ∃ *z*. *y2* ≤ *F z* **and** *ex-w*: ∃ *w*. *F w* < *y1*
shows *ginv F y1* ≤ *ginv F y2*
⟨*proof*⟩

lemma *ginv-compose-mono*:
assumes *mono*: *mono F* **and** *monog*:*mono-on A g*
and *ex*:∀ *s*∈*A*. ∃ *z*. *g s* ≤ *F z* ∀ *r*∈*A*. ∃ *w*. *F w* < *g r*
shows *mono-on A* (λ*x*. *ginv F* (*g x*))
⟨*proof*⟩

lemma *ginv-attained1*:
fixes y :: *'a*::{*linorder-topology*,*conditionally-complete-linorder*}
assumes *mono*: *mono F* **and** *cont*: $\bigwedge z$. *continuous* (*at-right z*) *F*
and *ex-z*: $\exists\, z.\ y \leq F\ z$ **and** *ex-w*: $\exists\, w.\ F\ w < y$
shows $y \leq F$ (*ginv F y*)
⟨*proof*⟩

Convergence of generalized inverses (Lemma 1.1.1 of the book by de Haan and Ferreira [2]).

lemma *tendsto-ginv*:
fixes *Fn* :: *nat* ⇒ *real* ⇒ *real* **and** *g* :: *real* ⇒ *real* **and** *y* :: *real*
assumes *mono-Fn*: $\bigwedge n$. *mono* (*Fn n*) **and** *mono-g*: *mono g*
and *wc'*:*weak-conv Fn g*
and *cont-ginv*: *isCont* (*ginv g*) *y*
and *yz*: $y < g\ z$ **and** *wy*: $g\ w < y$
shows $(\lambda n.\ (Fn\ n)^{\leftarrow}\ y) \longrightarrow g^{\leftarrow}\ y$
⟨*proof*⟩

context *right-continuous-mono*
begin

lemma *ex-gt-b*:
assumes $p < b$
shows $\exists\, z.\ p < f\ z$
⟨*proof*⟩

lemma *ex-ge-b*:
assumes $p < b$
shows $\exists\, z.\ p \leq f\ z$
⟨*proof*⟩

lemma *ex-less-a*:
assumes $a < p$
shows $\exists\, w.\ f\ w < p$
⟨*proof*⟩

lemma *ex-le-a*:
assumes $a < p$
shows $\exists\, w.\ f\ w \leq p$
⟨*proof*⟩

lemma *a-le-b*:$a \leq b$
⟨*proof*⟩

lemma *ginv-le-iff*:
assumes $a < p\ p < b$
shows *ginv f p* $\leq x \longleftrightarrow p \leq f\ x$
⟨*proof*⟩

lemma *ginv-attained*:
assumes $a < p$ $p < b$
shows $p \leq f\ (ginv\ f\ p)$
⟨*proof*⟩

lemma
shows *a-le-f*: $a \leq f\ x$
and *f-le-b*: $f\ x \leq b$
⟨*proof*⟩

lemma *ginv-left-cont*:
assumes *ap*: $a < p$ **and** *pb*:$p < b$
shows $(ginv\ f \longrightarrow ginv\ f\ p)\ (at\text{-}left\ p)$
⟨*proof*⟩

lemma *f-eq-b*:
assumes $f\ y = b$
and $y \leq x$
shows $f\ x = b$
⟨*proof*⟩

lemma *f-eq-a*:
assumes $f\ y = a$
and $x \leq y$
shows $f\ x = a$
⟨*proof*⟩

corollary *regularly-varying-less-b*:
assumes *regularly-varying* $(\lambda x.\ b - f\ x)\ \varrho$
shows $f\ x < b$
⟨*proof*⟩

lemma *ex-gt-less*:
assumes *z*: $f\ z < c$
shows $\exists\, w{>}z.\ f\ w < c$
⟨*proof*⟩

lemma *upper-semicontinuous*: *upper-semicontinuous-map euclidean f*
⟨*proof*⟩

corollary *open-less*: *open* $\{x.\ f\ x < c\}$
⟨*proof*⟩

lemma *strict-mono-of-deriv*:
assumes *deriv*: $\bigwedge z.\ x0 < z \Longrightarrow f\ z < c \Longrightarrow (f\ has\text{-}real\text{-}derivative\ f'\ z)\ (at\ z)$
and *pos*: $\bigwedge z.\ x0 < z \Longrightarrow f\ z < c \Longrightarrow 0 < f'\ z$
and *x*: $x0 < x$ **and** *xy*: $x < y$ **and** *fy*: $f\ y < c$
shows $f\ x < f\ y$
⟨*proof*⟩

lemma *ginv-f-eq*:
assumes *fmono*: $\bigwedge x\ y.\ x0 < x \Longrightarrow x < y \Longrightarrow f\ y < b \Longrightarrow f\ x < f\ y$
and *z*: $x0 < z$ **and** *fz*: $f\ z < b$
shows $ginv\ f\ (f\ z) = z$
⟨*proof*⟩

end

sublocale *real-distribution* ⊆ *right-continuous-mono cdf M 0 1*
⟨*proof*⟩

corollary *right-continuous-mono-cdf*:
assumes *real-distribution M*
shows *right-continuous-mono* (*cdf M*) *0 1*
⟨*proof*⟩

lemma *one-less-of-inverse-tail-less*:
fixes $F :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1* **and** *x0*: $F\ x0 < 1$
and *t*: $1\ /\ (1 - F\ x0) < t$
shows $1 < t$
⟨*proof*⟩

1.5 Lemmas for Infinite Sums

The following are used to prove relation (1.2.21) of [2].

lemma *geometric-ratio-chain-ge*:
fixes $d :: nat \Rightarrow real$
assumes *pos*: $\bigwedge n.\ 0 < d\ n$
and *r*: $0 < r$
and *bound*: $\bigwedge j.\ M \leq j \Longrightarrow r \leq d\ (Suc\ j)\ /\ d\ j$
and *k*: $M \leq k$
shows $d\ k * r\ \hat{}\ m \leq d\ (k + m)$
⟨*proof*⟩

lemma *geometric-ratio-chain-le*:
fixes $d :: nat \Rightarrow real$
assumes *pos*: $\bigwedge n.\ 0 < d\ n$
and *r*: $0 < r$
and *bound*: $\bigwedge j.\ M \leq j \Longrightarrow d\ (Suc\ j)\ /\ d\ j \leq r$
and *k*: $M \leq k$
shows $d\ (k + m) \leq d\ k * r\ \hat{}\ m$
⟨*proof*⟩

lemma *geometric-ratio-tailsum-tendsto*:
fixes $d :: nat \Rightarrow real$ **and** $L :: real$
assumes *pos*: $\bigwedge n.\ 0 < d\ n$
and *L*: $0 < L\ L < 1$

and *ratio*: $(\lambda n.\ d\ (Suc\ n)\ /\ d\ n) \longrightarrow L$
shows *geometric-ratio-tailsum-summable*: *summable d*
and *geometric-ratio-tailsum-div*: $(\lambda n.\ (\sum k.\ d\ (k + n))\ /\ d\ n) \longrightarrow 1\ /\ (1 - L)$
and *geometric-ratio-tailsum-ratio*: $(\lambda n.\ (\sum k.\ d\ (k + Suc\ n))\ /\ (\sum k.\ d\ (k + n))) \longrightarrow L$
⟨*proof*⟩

lemma *geometric-ratio-sum-tendsto*:
fixes $d :: nat \Rightarrow real$ **and** $L :: real$
assumes *pos*: $\bigwedge n.\ 0 < d\ n$
and *L*: $1 < L$
and *ratio*: $(\lambda n.\ d\ (Suc\ n)\ /\ d\ n) \longrightarrow L$
shows *geometric-ratio-sum-div-last*: $(\lambda n.\ (\sum k{\le}n.\ d\ k)\ /\ d\ n) \longrightarrow L\ /\ (L - 1)$
and *geometric-ratio-sum-ratio*: $(\lambda n.\ (\sum k{\le}Suc\ n.\ d\ k)\ /\ (\sum k{\le}n.\ d\ k)) \longrightarrow L$
⟨*proof*⟩

lemma *geometric-ratio-forward-telescope*:
fixes $g :: real \Rightarrow real$ **and** $Z\ c :: real$
assumes *power-ratio*: $\bigwedge t0.\ 0 < t0 \Longrightarrow (\lambda n.\ g\ (t0 * Z \hat{}\ Suc\ n)\ /\ g\ (t0 * Z \hat{}\ n)) \longrightarrow c$
and *Z*: $1 < Z$ **and** *c*: $c \neq 0$
shows $(\lambda n.\ g\ (Z \hat{}\ (n + j))\ /\ g\ (Z \hat{}\ n)) \longrightarrow c \hat{}\ j$
⟨*proof*⟩

lemma *monotone-geometric-ratio-imp-ratio*:
fixes $g :: real \Rightarrow real$ **and** $\varrho\ y :: real$
assumes *gmono*: $\bigwedge s\ t.\ 1 < s \Longrightarrow s \le t \Longrightarrow g\ s \le g\ t$
and *gpos*: $\forall_F\ t\ in\ at\text{-}top.\ 0 < g\ t$
and *rho*: $0 < \varrho$
and *power-ratio*: $\bigwedge Z\ t0.\ 1 < Z \Longrightarrow 0 < t0 \Longrightarrow$
$(\lambda n.\ g\ (t0 * Z \hat{}\ Suc\ n)\ /\ g\ (t0 * Z \hat{}\ n)) \longrightarrow Z\ powr\ \varrho$
and *y*: $0 < y$
shows $((\lambda t.\ g\ (t * y)\ /\ g\ t) \longrightarrow y\ powr\ \varrho)\ at\text{-}top$
⟨*proof*⟩

lemma *galois-inverse-ratio-tendsto*:
fixes $W\ s :: real \Rightarrow real$ **and** $\varrho\ x :: real$
assumes *W*: *filterlim W at-top at-top*
and *s*: *filterlim s at-top at-top*
and *s-pos*: $\bigwedge w.\ 0 < w \Longrightarrow 0 < s\ w$
and *le-iff*: $\bigwedge t\ w.\ 1 < t \Longrightarrow (W\ t \le w) = (t \le s\ w)$
and *rho*[*arith*]: $0 < \varrho$
and *lim*: $\bigwedge y.\ 0 < y \Longrightarrow ((\lambda t.\ W\ (t * y)\ /\ W\ t) \longrightarrow y\ powr\ \varrho)\ at\text{-}top$
and *x*[*arith*]: $0 < x$
shows $((\lambda u.\ s\ (u * x)\ /\ s\ u) \longrightarrow x\ powr\ (1\ /\ \varrho))\ at\text{-}top$
⟨*proof*⟩

end

2 Distributions

2.1 Generalized Extreme Value Distribution

theory *Extreme-Value-Distribution*
imports
Lemmas-EVT
begin

We define the generalized extreme value (GEV) density and cdf. Our definition includes a location parameter μ and a scale parameter σ.

definition *EVD-density* :: [*real*, *real*, *real*, *real*] ⇒ *real* **where**
EVD-density ξ μ σ x ≡ *let* $y = (x - \mu) / \sigma$ *in*
if $\xi = 0$ *then*
exp $(-y) * exp\ (-\ exp\ (-\ y)) / \sigma$
else
if $\xi * y > -\ 1$ *then*
$(1 + \xi * y)$ *powr* $(-\ 1 / \xi - 1)$
$*\ exp\ (-\ ((1 + \xi * y)\ powr\ (-\ 1 / \xi))) / \sigma$
else 0

lemma *EVD-density-nonneg*[*simp*]:
$0 \le \sigma \Longrightarrow 0 \le$ *EVD-density* ξ μ σ x
⟨*proof*⟩

lemma *EVD-density-xi-0*:
EVD-density 0 μ σ x = *exp* $(-\ ((x - \mu) / \sigma)) * exp\ (-\ exp\ (-\ ((x - \mu) / \sigma)))$
$/ \sigma$
⟨*proof*⟩

lemma
assumes $\xi \neq 0$
shows *EVD-density-xi-non0-1*:
$\xi * (x - \mu) / \sigma \ge -\ 1$
$\Longrightarrow$ *EVD-density* ξ μ σ x = $(1 + \xi * (x - \mu) / \sigma)$ *powr* $(-\ 1 / \xi - 1)$
$* exp\ (-\ ((1 + \xi * (x - \mu) / \sigma)\ powr\ (-\ 1 / \xi))) / \sigma$
and *EVD-density-xi-non0-2*:
$\xi * (x - \mu) / \sigma \le -\ 1 \Longrightarrow$ *EVD-density* ξ μ σ $x = 0$
⟨*proof*⟩

lemma *EVD-density-measurable*[*measurable*]:
EVD-density ξ μ $\sigma \in$ *borel-measurable borel*
⟨*proof*⟩

lemma
assumes *sigma*[*arith*]:$0 < \sigma$

shows *EVD-density-set-integrable0*:*set-integrable lborel* {..*a*} (*EVD-density 0* μ σ)
and *EVD-density-set-integral0*:
($\int x{\in}\{..a\}$. *EVD-density 0* μ σ *x* ∂*lborel*) = *exp* (− *exp* (− (*a* − μ) / σ))
and *EVD-density-set-nn-integral0*:
($\int^+ x{\in}\{..a\}$. *EVD-density 0* μ σ *x* ∂*lborel*) = *ennreal* (*exp* (− *exp* (− (*a* − μ) / σ)))
⟨*proof*⟩

lemma
fixes *a* μ σ
defines *y* ≡ (*a* − μ) / σ
assumes *sigma*[*arith*]:*0* < σ
and *xi*:$\xi \neq 0$
and *xiy*:$\xi * y \geq -1$
and *xiy-pos*:*0* < $\xi \Longrightarrow -1 < \xi * y$
shows *EVD-density-set-integrable1*:*set-integrable lborel* {..*a*} (*EVD-density* ξ μ σ)
and *EVD-density-set-integral1*:
($\int x{\in}\{..a\}$. *EVD-density* ξ μ σ *x* ∂*lborel*) = *exp* (− ((*1* + ξ ∗ *y*) *powr* (− *1* / ξ)))
and *EVD-density-set-nn-integral1*:
($\int^+ x{\in}\{..a\}$. *EVD-density* ξ μ σ *x* ∂*lborel*) = *ennreal* (*exp* (− ((*1* + ξ ∗ *y*) *powr* (− *1* / ξ))))
⟨*proof*⟩

lemma
fixes *a* μ σ
defines *y* ≡ (*a* − μ) / σ
assumes *sigma*[*arith*]:*0* < σ
and *xiy*:$\xi * y \leq -1$
and *xi*:$\xi > 0$
shows *EVD-density-set-integrable2*: *set-integrable lborel* {..*a*} (*EVD-density* ξ μ σ)
and *EVD-density-set-integral2*:
($\int x{\in}\{..a\}$. *EVD-density* ξ μ σ *x* ∂*lborel*) = *0*
and *EVD-density-set-nn-integral2*:
($\int^+ x{\in}\{..a\}$. *EVD-density* ξ μ σ *x* ∂*lborel*) = *0*
⟨*proof*⟩

lemma
fixes *a* μ σ
defines *y* ≡ (*a* − μ) / σ
assumes *sigma*[*arith*]:*0* < σ
and *xiy*:$\xi * y \leq -1$
and *xi*:$\xi < 0$
shows *EVD-density-set-integrable3*: *set-integrable lborel* {..*a*} (*EVD-density* ξ μ σ)
and *EVD-density-set-integral3*:

$(\int x{\in}\{..a\}.\ EVD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = 1$
and *EVD-density-set-nn-integral3*:
$(\int^{+} x{\in}\{..a\}.\ EVD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = 1$
⟨*proof*⟩

lemma *EVD-density-set-integrable*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows *set-integrable lborel* $\{..a\}$ (*EVD-density* $\xi\ \mu\ \sigma$)
⟨*proof*⟩

abbreviation *EVD-cdf* **where** *EVD-cdf* $\equiv \lambda\xi\ \mu\ \sigma\ x.$ *cdf* (*density lborel* (*EVD-density* $\xi\ \mu\ \sigma$)) x

lemma *EVD-cdf-set-integral*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows *EVD-cdf* $\xi\ \mu\ \sigma\ a = (\int x{\in}\{..a\}.\ EVD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel)$
⟨*proof*⟩

corollary *EVD-cdf-def*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows *EVD-cdf* $\xi\ \mu\ \sigma\ x =$
$(let\ y = (x - \mu)\ /\ \sigma\ in$
$if\ \xi = 0\ then\ exp\ (-\ exp\ (-\ y))$
$else\ if\ \xi * y > -\ 1\ then\ exp\ (-\ ((1 + \xi * y)\ powr\ (-\ 1\ /\ \xi)))$
$else\ if\ \xi > 0\ then\ 0\ else\ 1)$
⟨*proof*⟩

lemma *EVD-cdf-supp-bounds*:
fixes $x\ \gamma$ :: *real*
assumes [*arith*]:$1 + \gamma * (x - \mu)\ /\ \sigma > 0\ 0 < \sigma$
shows $0 <$ *EVD-cdf* $\gamma\ \mu\ \sigma\ x$ *EVD-cdf* $\gamma\ \mu\ \sigma\ x \leq 1$
⟨*proof*⟩

lemma *EVD-cdf-supp-bounds01*:
fixes $x\ \gamma$ :: *real*
assumes [*arith*]:$1 + \gamma * x > 0$
shows $0 <$ *EVD-cdf* $\gamma\ 0\ 1\ x$ *EVD-cdf* $\gamma\ 0\ 1\ x \leq 1$
⟨*proof*⟩

lemma *ln-EVD-cdf-0*:
assumes $0 < \sigma$
shows $-\ ln\ (EVD\text{-}cdf\ 0\ \mu\ \sigma\ x) = exp\ (-\ ((x - \mu)\ /\ \sigma))$
⟨*proof*⟩

lemma *ln-EVD-cdf-1*:
assumes $0 < \sigma\ 1 + \gamma * (x - \mu)\ /\ \sigma > 0\ \gamma \neq 0$
shows $-\ ln\ (EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x) = (1 + \gamma * (x - \mu)\ /\ \sigma)\ powr\ (-\ 1\ /\ \gamma)$
⟨*proof*⟩

lemma *EVD-cdf-location-scale*:
assumes [*arith*]:$0 < \sigma$
shows *EVD-cdf* ξ *0 1* $((x - \mu) \ / \ \sigma) =$ *EVD-cdf* ξ μ σ x
⟨*proof*⟩

lemma *EVD-cdf-at-bot-0*:
assumes *sigma*:$0 < \sigma$
shows (*EVD-cdf* ξ μ σ $\longrightarrow$ *0*) *at-bot*
⟨*proof*⟩

lemma *EVD-cdf-at-top-1*:
assumes *sigma*:$0 < \sigma$
shows (*EVD-cdf* ξ μ σ $\longrightarrow$ *1*) *at-top*
⟨*proof*⟩

lemma
assumes *sigma*[*arith*]:$0 < \sigma$
shows *EVD-density-integrable*:*integrable lborel* (*EVD-density* ξ μ σ)
and *EVD-density-integral*:$(\int x.$ *EVD-density* ξ μ σ x ∂*lborel*$) =$ *1*
and *EVD-density-nn-integral*:$(\int^+ x.$ *EVD-density* ξ μ σ x ∂*lborel*$) =$ *1*
⟨*proof*⟩

lemma *prob-space-EVD*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows *prob-space* (*density lborel* (*EVD-density* ξ μ σ))
⟨*proof*⟩

corollary *real-distribution-EVD*:
$0 < \sigma \Longrightarrow$ *real-distribution* (*density lborel* (*EVD-density* ξ μ σ))
⟨*proof*⟩

corollary *right-continuous-mono-EVD-cdf*:
$0 < \sigma \Longrightarrow$ *right-continuous-mono* (*EVD-cdf* ξ μ σ) *0 1*
⟨*proof*⟩

corollary *EVD-cdf-nonneg*:$0 < \sigma \Longrightarrow 0 \leq$ *EVD-cdf* ξ μ σ x
and *EVD-cdf-le-1*:$0 < \sigma \Longrightarrow$ *EVD-cdf* ξ μ σ $x \leq 1$
⟨*proof*⟩

lemma *isCont-EVD-cdf*:
assumes *sigma*[*arith*]: $0 < \sigma$
shows *isCont* (*EVD-cdf* ξ μ σ) x
⟨*proof*⟩

corollary *weak-conv-EVD-cdf-iff*:
assumes *sigma*:$0 < \sigma$
shows *weak-conv F* (*EVD-cdf* ξ μ σ) $\longleftrightarrow$ $(\forall x.$ $(\lambda n.$ *F n x*$)$ $\longrightarrow$ *EVD-cdf* ξ μ σ x)
⟨*proof*⟩

Max-stability of the GEV distribution (Exercise 1.2 of the book [2]). The book's argument is $G_\xi^n(a_n x + b_n) = G_\xi(x)$ for $a_n = n^\xi$ and $b_n = (n^\xi - 1)/\xi$, whereas we state the equivalent solved form $G_\xi^n(x) = G_\xi((x - b_n)/a_n)$, in which $(x - b_n)/a_n = n^{-\xi}x + (n^{-\xi} - 1)/\xi$.

lemma *EVD-cdf-max-stable*:
assumes $0 < n$
shows *EVD-cdf* ξ *0 1 x* ^ *n*
= *EVD-cdf* ξ *0 1* (*if* $\xi = 0$ *then x* − *ln n*
else n powr (− ξ) ∗ *x* + (*real n powr* (− ξ) − *1*) / ξ)
⟨*proof*⟩

end

2.2 Generalized Pareto Distribution

theory *Pareto-Distribution*
imports
Lemmas-EVT
Extreme-Value-Distribution
begin

We define the generalized Pareto distribution (GPD) density and cdf.

definition *GPD-density* :: [*real*, *real*, *real*, *real*] ⇒ *real* **where**
GPD-density ξ μ σ *x* ≡ *if* $\xi = 0$ *then*
if $\mu \le x$ *then exp* (− (*x* − μ) / σ) / σ *else 0*
else if $\xi > 0$ *then*
if $\mu \le x$ *then* ((*1* + ξ ∗ (*x* − μ) / σ) *powr* (− *1* − *1* / ξ)) / σ
else 0
else
if $\mu \le x \wedge x \le \mu - \sigma$ / ξ *then*
((*1* + ξ ∗ (*x* − μ) / σ) *powr* (− *1* − *1* / ξ)) / σ
else 0

lemma *GPD-density-nonneg*[*simp*]:*0* ≤ σ ⟹ *0* ≤ *GPD-density* ξ μ σ *x*
⟨*proof*⟩

lemma *GPD-density-measurable*[*measurable*]:
GPD-density ξ μ σ ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *GPD-density-eq-exponential-density*:
GPD-density 0 μ σ = (λ*x*. *exponential-density* (*1* / σ) (*x* − μ))
⟨*proof*⟩

lemma *GPD-density-xi-pos*:
0 < ξ ⟹ $\mu \le x$
⟹ *GPD-density* ξ μ σ *x* = (*1* + ξ ∗ (*x* − μ) / σ) *powr* (− *1* − *1* / ξ) / σ
⟨*proof*⟩

lemma *GPD-density-xi-neg*:
$\xi < 0 \Longrightarrow \mu \leq x \Longrightarrow x \leq \mu - \sigma \mathbin{/} \xi$
$\Longrightarrow$ *GPD-density* $\xi\ \mu\ \sigma\ x = (1 + \xi * (x - \mu) \mathbin{/} \sigma)\ powr\ (-1 - 1 \mathbin{/} \xi) \mathbin{/} \sigma$
⟨*proof*⟩

lemma *GPD-density-eq-0*:
$x < \mu \Longrightarrow$ *GPD-density* $\xi\ \mu\ \sigma\ x = 0$
⟨*proof*⟩

lemma *GPD-density-xi-neg-eq-0*:
$\xi < 0 \Longrightarrow \mu - \sigma \mathbin{/} \xi < x \Longrightarrow$ *GPD-density* $\xi\ \mu\ \sigma\ x = 0$
⟨*proof*⟩

lemma
fixes $a\ \mu\ \sigma$ :: *real*
assumes *sigma*[*arith*]:$0 < \sigma$
shows *GPD-density-set-integrable0*:*set-integrable lborel* $\{..a\}$ (*GPD-density* $0\ \mu\ \sigma$)
and *GPD-density-set-integral0*:
$(\int x{\in}\{..a\}.$ *GPD-density* $0\ \mu\ \sigma\ x\ \partial lborel) =$ (*if* $a < \mu$ *then* 0 *else* $1 - exp\ (-(a - \mu) \mathbin{/} \sigma)$)
and *GPD-density-set-nn-integral0*:
$(\int^{+} x{\in}\{..a\}.$ *GPD-density* $0\ \mu\ \sigma\ x\ \partial lborel)$
$=$ *ennreal* (*if* $a < \mu$ *then* 0 *else* $1 - exp\ (-(a - \mu) \mathbin{/} \sigma)$)
⟨*proof*⟩

lemma
fixes $a\ \mu\ \sigma$ :: *real*
defines $y \equiv (a - \mu) \mathbin{/} \sigma$
assumes *sigma*[*arith*]:$0 < \sigma$
and *amu*:$a \leq \mu$
shows *GPD-density-set-integrable2*:*set-integrable lborel* $\{..a\}$ (*GPD-density* $\xi\ \mu\ \sigma$)
and *GPD-density-set-integral2*:$(\int x{\in}\{..a\}.$ *GPD-density* $\xi\ \mu\ \sigma\ x\ \partial lborel) = 0$
and *GPD-density-set-nn-integral2*:$(\int^{+} x{\in}\{..a\}.$ *GPD-density* $\xi\ \mu\ \sigma\ x\ \partial lborel) = 0$
⟨*proof*⟩

lemma
fixes $a\ \mu\ \sigma$ :: *real*
defines $y \equiv (a - \mu) \mathbin{/} \sigma$
assumes *sigma*[*arith*]:$0 < \sigma$
and *xi*:$\xi \neq 0$
and *mua*:$\mu \leq a$
and *xiy*:$-1 \leq \xi * y$
shows *GPD-density-set-integrable1*:*set-integrable lborel* $\{..a\}$ (*GPD-density* $\xi\ \mu\ \sigma$)
and *GPD-density-set-integral1*:

$(\int x{\in}\{..a\}.\ GPD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = 1 - (1 + \xi * y)\ powr\ (-\ 1\ /\ \xi)$
and *GPD-density-set-nn-integral1*:
$(\int^{+} x{\in}\{..a\}.\ GPD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = ennreal\ (1 - (1 + \xi * y)\ powr\ (-\ 1\ /\ \xi))$
⟨*proof*⟩

lemma
fixes $a\ \mu\ \sigma :: real$
defines $y \equiv (a - \mu)\ /\ \sigma$
assumes *sigma*[*arith*]:$0 < \sigma$
and *xi*:$\xi < 0$
and *xiy*:$\xi * y \leq -\ 1$
shows *GPD-density-set-integrable3*:*set-integrable lborel* $\{..a\}$ (*GPD-density* $\xi\ \mu\ \sigma$)
and *GPD-density-set-integral3*:$(\int x{\in}\{..a\}.\ GPD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = 1$
and *GPD-density-set-nn-integral3*:$(\int^{+} x{\in}\{..a\}.\ GPD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = 1$
⟨*proof*⟩

lemma *GPD-density-set-integrable*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows *set-integrable lborel* $\{..a\}$ (*GPD-density* $\xi\ \mu\ \sigma$)
⟨*proof*⟩

abbreviation *GPD-cdf* **where** $GPD\text{-}cdf \equiv \lambda\xi\ \mu\ \sigma\ x.\ cdf\ (density\ lborel\ (GPD\text{-}density\ \xi\ \mu\ \sigma))\ x$

lemma *GPD-cdf-set-integral*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows $GPD\text{-}cdf\ \xi\ \mu\ \sigma\ x = (\int t{\in}\{..x\}.\ GPD\text{-}density\ \xi\ \mu\ \sigma\ t\ \partial lborel)$
⟨*proof*⟩

corollary *GPD-cdf-def*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows $GPD\text{-}cdf\ \xi\ \mu\ \sigma\ x =$
$(let\ y = (x - \mu)\ /\ \sigma\ in$
$if\ \xi = 0\ then$
$if\ x < \mu\ then\ 0\ else\ 1 - exp\ (-\ y)$
$else$
$if\ x < \mu\ then\ 0$
$else\ if\ -\ 1 \leq \xi * y\ then\ 1 - (1 + \xi * y)\ powr\ (-\ 1\ /\ \xi)$
$else\ 1)$
⟨*proof*⟩

lemma *GPD-cdf-location-scale*:
assumes *sigma*: $0 < \sigma$
shows $GPD\text{-}cdf\ \gamma\ 0\ \sigma\ x = GPD\text{-}cdf\ \gamma\ 0\ 1\ (x\ /\ \sigma)$
⟨*proof*⟩

lemma *EVD-tail-eq-GPD-cdf*:
assumes *supp*: $1 + \gamma * x > 0$ **and** x: $0 \leq x$
shows $-\ ln\ (EVD\text{-}cdf\ \gamma\ 0\ 1\ x) = 1 - GPD\text{-}cdf\ \gamma\ 0\ 1\ x$
⟨*proof*⟩

lemma *GPD-cdf-eq0*:
assumes $x \leq \mu$
shows $GPD\text{-}cdf\ \xi\ \mu\ \sigma\ x = 0$
⟨*proof*⟩

lemma *GPD-cdf-at-bot-0*: $(GPD\text{-}cdf\ \xi\ \mu\ \sigma \longrightarrow 0)\ at\text{-}bot$
⟨*proof*⟩

lemma *GPD-cdf-at-top-1*:
assumes *sigma*:$0 < \sigma$
shows $(GPD\text{-}cdf\ \xi\ \mu\ \sigma \longrightarrow 1)\ at\text{-}top$
⟨*proof*⟩

lemma
assumes *sigma*[*arith*]:$0 < \sigma$
shows *GPD-density-integrable*:*integrable lborel* $(GPD\text{-}density\ \xi\ \mu\ \sigma)$
and *GPD-density-integral*:$(\int x.\ GPD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = 1$
and *GPD-density-nn-integral*:$(\int^+ x.\ GPD\text{-}density\ \xi\ \mu\ \sigma\ x\ \partial lborel) = 1$
⟨*proof*⟩

lemma *prob-space-GPD*:
assumes *sigma*[*arith*]:$0 < \sigma$
shows *prob-space* $(density\ lborel\ (GPD\text{-}density\ \xi\ \mu\ \sigma))$
⟨*proof*⟩

corollary *real-distribution-GPD*:
$0 < \sigma \Longrightarrow real\text{-}distribution\ (density\ lborel\ (GPD\text{-}density\ \xi\ \mu\ \sigma))$
⟨*proof*⟩

lemma *isCont-GPD-cdf*:
assumes *sigma*[*arith*]: $0 < \sigma$
shows *isCont* $(GPD\text{-}cdf\ \xi\ \mu\ \sigma)\ x$
⟨*proof*⟩

lemma *right-continuous-mono-GPD-cdf*:
assumes *sigma*[*arith*]: $0 < \sigma$
shows *right-continuous-mono* $(GPD\text{-}cdf\ \xi\ \mu\ \sigma)\ 0\ 1$
⟨*proof*⟩

corollary *GPD-cdf-nonneg*:$0 < \sigma \Longrightarrow 0 \leq GPD\text{-}cdf\ \xi\ \mu\ \sigma\ x$
and *GPD-cdf-le-1*:$0 < \sigma \Longrightarrow GPD\text{-}cdf\ \xi\ \mu\ \sigma\ x \leq 1$
⟨*proof*⟩

lemma *GPD-normalization*:

fixes γ μ σ x :: *real*
assumes *sigma*[*arith*]: $0 < \sigma$
defines $a \equiv (\lambda n.\ \sigma * (if\ \gamma = 0\ then\ 1\ else\ real\ n\ powr\ \gamma))$
defines $b \equiv (\lambda n.\ \mu + \sigma * (if\ \gamma = 0\ then\ ln\ (real\ n)\ else\ (real\ n\ powr\ \gamma - 1)\ /\ \gamma))$
shows $(\lambda n.\ GPD\text{-}cdf\ \gamma\ \mu\ \sigma\ (a\ n * x + b\ n)\ \hat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ 0\ 1\ x$
⟨*proof*⟩

end

3 Domain of Attraction

theory *Domain-of-Attraction*
imports
HOL−Probability.Weak-Convergence
Cauchy-Functional-Equation.Cauchy-Functional-Equation
Pareto-Distribution
begin

This theory file is based on Chapter 1 of the book by de Haan and Ferreira [2].

3.1 Definitions

We work with distribution functions as plain functions *real* ⇒ *real*, as does the book. The corresponding statements for i.i.d. random variables are given in Section 5.

3.1.1 Degenerate Distributions

definition *degenerate-df* :: (*real* ⇒ *real*) ⇒ *bool* **where**
degenerate-df F ⟷ (∀ *x*. *F x* = *0* ∨ *F x* = *1*)

lemma *degenerate-df-iff-indicator*:
assumes *right-continuous-mono F 0 1*
shows *degenerate-df F* ⟷ (∃ *c*. *F* = (λ*x*. *indicat-real* {*c*..} *x*))
⟨*proof*⟩

lemma *EVD-cdf-nondegenerate*:
assumes [*arith*]:*0* < *σ*
shows ¬ *degenerate-df* (*EVD-cdf ξ μ σ*)
⟨*proof*⟩

3.1.2 Domain of Attraction

definition *domain-of-attraction* :: (*real* ⇒ *real*) ⇒ (*real* ⇒ *real*) ⇒ *bool*
(**infix** ‹(*in′-domain′-attraction′-of*)› *50*) **where**
domain-of-attraction F G ⟷

right-continuous-mono F 0 1 ∧ *right-continuous-mono G 0 1* ∧
(∃ *a b*. (∀ *n*::*nat*. *0* < *a n*) ∧ *weak-conv* (λ*n x*. *F* (*a n* ∗ *x* + *b n*) ^ *n*) *G*)

lemma *domain-of-attractionD*:
assumes *domain-of-attraction F G*
shows *right-continuous-mono F 0 1 right-continuous-mono G 0 1*
and ∃ *a b*. (∀ *n*::*nat*. *0* < *a n*) ∧ *weak-conv* (λ*n x*. *F* (*a n* ∗ *x* + *b n*) ^ *n*) *G*
⟨*proof*⟩

lemma *domain-of-attractionI-eventually-pos*:
assumes *F*:*right-continuous-mono F 0 1*
and *G*:*right-continuous-mono G 0 1*
and *a-pos*:∀ *F n in at-top*. *a n* > *0*
and *wc*:*weak-conv* (λ*n x*. *F* (*a n* ∗ *x* + *b n*) ^ *n*) *G*
shows *domain-of-attraction F G*
⟨*proof*⟩

lemma *domain-of-attraction-EVD-cdf-rescale*:
fixes γ *μ1 μ2 σ1 σ2* :: *real*
assumes *sigma1*[*arith*]: *0* < *σ1* **and** *sigma2*[*arith*]: *0* < *σ2*
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *μ1 σ1*)
shows *domain-of-attraction F* (*EVD-cdf* γ *μ2 σ2*)
⟨*proof*⟩

theorem *domain-of-attraction-EVD-cdf-iff*:
fixes γ *μ σ* :: *real*
assumes *sigma*[*arith*]: *0* < *σ*
shows *domain-of-attraction F* (*EVD-cdf* γ *μ σ*) ⟷ *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
⟨*proof*⟩

lemma *EVD-cdf-MDA*:
assumes *sigma*[*arith*]:*0* < *σ*
shows *domain-of-attraction* (*EVD-cdf* ξ *μ σ*) (*EVD-cdf* ξ *μ σ*)
⟨*proof*⟩

lemma *GPD-cdf-MDA*:
fixes γ *μ σ* :: *real*
assumes *0* < *σ*
shows *domain-of-attraction* (*GPD-cdf* γ *μ σ*) (*EVD-cdf* γ *0 1*)
⟨*proof*⟩

lemma *domain-of-attraction-affine*:
fixes *c d* :: *real*
assumes *c*[*arith*]:*0* < *c* **and** *dm*:*domain-of-attraction F G*
shows *domain-of-attraction* (λ*x*. *F* ((*x* − *d*) / *c*)) *G*
⟨*proof*⟩

3.2 Fisher–Tippett–Gnedenko Theorem

The following two definitions correspond to the functions U and D in equation (1.1.6) in the book.

abbreviation *tail-quantile* :: (*real* ⇒ *real*) ⇒ *real* ⇒ *real* **where**
tail-quantile F t ≡ *ginv F* (*1* − *1* / *t*)

tail-quantile is exactly the *return level* associated with *return period t*.

abbreviation *limit-quantile* :: (*real* ⇒ *real*) ⇒ *real* ⇒ *real* **where**
limit-quantile G y ≡ *ginv G* (*exp* (− (*1* / *y*)))

lemma *lt-tail-quantile-iff*:
fixes *F* :: *real* ⇒ *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *s*: *1* < *s*
shows *w* < *tail-quantile F s* ⟷ *1* / *s* < *1* − *F w*
⟨*proof*⟩

lemma *tail-quantile-le-of-lt-inv*:
assumes *F*: *right-continuous-mono F 0 1*
and *Ft*[*arith*]: *F t* < *1*
and *u1*[*arith*]: *1* < *u* **and** *u2*[*arith*]: *u* < *1* / (*1* − *F t*)
shows *tail-quantile F u* ≤ *t*
⟨*proof*⟩

lemma *t-lt-tail-quantile-of-gt-inv*:
fixes *F* :: *real* ⇒ *real* **and** *t u* :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *Ft*: *F t* < *1*
and *u*: *1* / (*1* − *F t*) < *u*
shows *t* < *tail-quantile F u*
⟨*proof*⟩

lemma *tail-quantile-at-top*:
fixes *F* :: *real* ⇒ *real*
assumes *F*: *right-continuous-mono F 0 1*
and *unbdd*: ⋀*x*. *F x* < *1*
shows *filterlim* (*tail-quantile F*) *at-top at-top*
⟨*proof*⟩

lemma *limit-quantile-left-cont*:
assumes *G*: *right-continuous-mono G 0 1* **and** *p*: *0* < *p*
shows (*limit-quantile G* ⟶ *limit-quantile G p*) (*at-left p*)
⟨*proof*⟩

lemma *limit-quantile-left-cont-scale*:
assumes *G*: *right-continuous-mono G 0 1* **and** *y*: *0* < *y* **and** *p*: *0* < *p*
shows ((λ*x*. *limit-quantile G* (*x* ∗ *y*)) ⟶ *limit-quantile G* (*p* ∗ *y*)) (*at-left p*)
⟨*proof*⟩

lemma *tail-quantile-mono*:
assumes *F*: *right-continuous-mono F 0 1* **and** *s*: *1 < s* **and** *st*: *s ≤ t*
shows *tail-quantile F s ≤ tail-quantile F t*
⟨*proof*⟩

lemma *limit-quantile-mono*:
assumes *G*: *right-continuous-mono G 0 1* **and** *y1*: *0 < y1* **and** *y12*: *y1 ≤ y2*
shows *limit-quantile G y1 ≤ limit-quantile G y2*
⟨*proof*⟩

lemma *limit-quantile-EVD-cdf*:
assumes *sigma*[*arith*]:*0 < σ*
and *ypos*[*arith*]: *0 < y*
shows *limit-quantile* (*EVD-cdf ξ μ σ*) *y*
= (*if ξ = 0 then σ ∗ ln y + μ else σ ∗* (*y powr ξ − 1*) */ ξ + μ*)
(**is** *ginv - ?p = ?xs*)
⟨*proof*⟩

corollary *isCont-limit-quantile-EVD-cdf*:
assumes *sigma*[*arith*]:*0 < σ*
and *ypos*[*arith*]: *0 < y*
shows *isCont* (*limit-quantile* (*EVD-cdf ξ μ σ*)) *y*
⟨*proof*⟩

lemma *dist-function-unique-limit-quantile*:
assumes *G*:*right-continuous-mono G 0 1* **and** *G′*:*right-continuous-mono G′ 0 1*
and *eq*:⋀*y*. *0 < y ⟹ limit-quantile G y = limit-quantile G′ y*
shows *G = G′*
⟨*proof*⟩

lemma *isCont-ginv-iff-isCont-limit-quantile*:
assumes *G*: *right-continuous-mono G 0 1*
and *y*: *0 < y*
shows *isCont* (*ginv G*) (*exp* (*− (1 / y)*)) ⟷ *isCont* (*limit-quantile G*) *y*
⟨*proof*⟩

lemma *seq-exp-quantile-tendsto*:
fixes *F G* :: *real ⇒ real* **and** *a b* :: *nat ⇒ real* **and** *y* :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *G*: *right-continuous-mono G 0 1*
and *a*: ⋀*n*. *0 < a n*
and *wc*: *weak-conv* (*λn x*. *F* (*a n ∗ x + b n*) ^ *n*) *G*
and *y*: *0 < y* **and** *cont*: *isCont* (*limit-quantile G*) *y*
shows (*λn*. (*limit-quantile F* (*real n ∗ y*) *− b n*) */ a n*) ⟶ *limit-quantile G y*
⟨*proof*⟩

lemma *seq-tail-quantile-tendsto*:
fixes *F G* :: *real ⇒ real* **and** *a b* :: *nat ⇒ real* **and** *y* :: *real*
assumes *F*:*right-continuous-mono F 0 1* **and** *G*:*right-continuous-mono G 0 1*

and *dg*: ¬ *degenerate-df G*
and *a*:⋀*n*. *0* < *a n*
and *wc*:*weak-conv* (λ*n x*. *F* (*a n* ∗ *x* + *b n*) ^ *n*) *G*
and *y*:*0* < *y* **and** *cont*:*isCont* (*limit-quantile G*) *y*
shows (λ*n*. (*tail-quantile F* (*real n* ∗ *y*) − *b n*) / *a n*) ⟶ *limit-quantile G y*
⟨*proof*⟩

lemma *nondegenerate-growth-points*:
assumes *G*: *right-continuous-mono G 0 1* **and** *deg*: ¬ *degenerate-df G*
and *N*: *countable N*
obtains *y1 y2* **where** *0* < *y1 y1* < *y2 y1* ∉ *N y2* ∉ *N*
isCont (*limit-quantile G*) *y1 isCont* (*limit-quantile G*) *y2*
limit-quantile G y1 < *limit-quantile G y2*
⟨*proof*⟩

lemma
fixes *F G* :: *real* ⇒ *real* **and** *a b* :: *nat* ⇒ *real*
assumes *F*:*right-continuous-mono F 0 1* **and** *G*:*right-continuous-mono G 0 1*
and *deg*:¬ *degenerate-df G*
and *a*:⋀*n*. *0* < *a n*
and *wc*:*weak-conv* (λ*n x*. *F* (*a n* ∗ *x* + *b n*) ^ *n*) *G*
shows *normalizing-ratio1*:(λ*n*. *a* (*Suc n*) / *a n*) ⟶ *1*
and *normalizing-ratio2*:(λ*n*. (*b* (*Suc n*) − *b n*) / *a n*) ⟶ *0*
⟨*proof*⟩

lemma *real-tail-quantile-tendsto*:
fixes *F G* :: *real* ⇒ *real* **and** *a b* :: *nat* ⇒ *real* **and** *y* :: *real*
assumes *F*:*right-continuous-mono F 0 1* **and** *G*:*right-continuous-mono G 0 1*
and *deg*:¬ *degenerate-df G*
and *a*:⋀*n*. *0* < *a n*
and *wc*:*weak-conv* (λ*n x*. *F* (*a n* ∗ *x* + *b n*) ^ *n*) *G*
and *y*:*0* < *y* **and** *cont*:*isCont* (*limit-quantile G*) *y*
shows ((λ*t*. (*tail-quantile F* (*t* ∗ *y*) − *b* (*nat* ⌊*t*⌋)) / *a* (*nat* ⌊*t*⌋))
⟶ *limit-quantile G y*) *at-top*
⟨*proof*⟩

lemma *scaling-limits-ex*:
fixes *F G* :: *real* ⇒ *real* **and** *a b* :: *nat* ⇒ *real*
assumes *F*:*right-continuous-mono F 0 1* **and** *G*:*right-continuous-mono G 0 1*
and *deg*: ¬ *degenerate-df G*
and *a*:⋀*n*. *0* < *a n*
and *wc*:*weak-conv* (λ*n x*. *F* (*a n* ∗ *x* + *b n*) ^ *n*) *G*
obtains *A B* **where**
⋀*y*. *0* < *y* ⟹ *0* < *A y*
and ⋀*y*. *0* < *y* ⟹ ((λ*t*. *a* (*nat* ⌊*t* ∗ *y*⌋) / *a* (*nat* ⌊*t*⌋)) ⟶ *A y*) *at-top*
and ⋀*y*. *0* < *y* ⟹
((λ*t*. (*b* (*nat* ⌊*t* ∗ *y*⌋) − *b* (*nat* ⌊*t*⌋)) / *a* (*nat* ⌊*t*⌋)) ⟶ *B y*) *at-top*
and *A* ∈ *borel-measurable* (*restrict-space borel* {*0*<..}⟩)
and ⋀*x y*. *0* < *x* ⟹ *0* < *y* ⟹ *A* (*x* ∗ *y*) = *A x* ∗ *A y*

and $\bigwedge x\ y.\ 0 < x \Longrightarrow 0 < y \Longrightarrow isCont\ (limit\text{-}quantile\ G)\ x$
$\Longrightarrow isCont\ (limit\text{-}quantile\ G)\ (x * y)$
$\Longrightarrow limit\text{-}quantile\ G\ (x * y) = limit\text{-}quantile\ G\ x * A\ y + B\ y$
⟨*proof*⟩

theorem *limit-quantile-form*:
fixes $F\ G :: real \Rightarrow real$ **and** $a\ b :: nat \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1* **and** *G*: *right-continuous-mono G 0 1*
and *deg*: $\neg$ *degenerate-df G*
and *a*: $\bigwedge n.\ 0 < a\ n$
and *wc*: *weak-conv* $(\lambda n\ x.\ F\ (a\ n * x + b\ n)\ \hat{}\ n)\ G$
obtains $\gamma\ c\ d$ **where** $0 < c$
and $\bigwedge y.\ 0 < y \Longrightarrow limit\text{-}quantile\ G\ y$
$= (if\ \gamma = 0\ then\ c * ln\ y + d\ else\ c * (y\ powr\ \gamma - 1)\ /\ \gamma + d)$
⟨*proof*⟩

The Fisher–Tippett–Gnedenko theorem, which is Theorem 1.1.3 of the book by de Haan and Ferreira [2] (Fisher and Tippett (1928) [4], Gnedenko (1943) [5]).

theorem *Fisher-Tippett-Gnedenko*:
fixes $F\ G :: real \Rightarrow real$ **and** $a\ b :: nat \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1* **and** *G*: *right-continuous-mono G 0 1*
and *deg*: $\neg$ *degenerate-df G*
and *a*: $\bigwedge n.\ 0 < a\ n$
and *wc*: *weak-conv* $(\lambda n\ x.\ F\ (a\ n * x + b\ n)\ \hat{}\ n)\ G$
obtains $\gamma\ \mu\ \sigma$ **where** $0 < \sigma$ **and** $G = EVD\text{-}cdf\ \gamma\ \mu\ \sigma$
⟨*proof*⟩

theorem *Fisher-Tippett-Gnedenko′*:
fixes $F\ G :: real \Rightarrow real$ **and** $a\ b :: nat \Rightarrow real$
assumes *right-continuous-mono F 0 1 right-continuous-mono G 0 1* $\neg$ *degenerate-df G*
and $\bigwedge n.\ 0 < a\ n$
and *weak-conv* $(\lambda n\ x.\ F\ (a\ n * x + b\ n)\ \hat{}\ n)\ G$
obtains $\gamma\ \alpha\ \beta$ **where** $0 < \alpha\ \bigwedge x.\ G\ x = EVD\text{-}cdf\ \gamma\ 0\ 1\ (\alpha * x + \beta)$
⟨*proof*⟩

corollary *Fisher-Tippett-Gnedenko-MDA′*:
assumes *deg*: $\neg$ *degenerate-df G*
and *dom*: *domain-of-attraction F G*
obtains $\gamma\ \mu\ \sigma$ **where** $0 < \sigma$ **and** $G = EVD\text{-}cdf\ \gamma\ \mu\ \sigma$
⟨*proof*⟩

corollary *Fisher-Tippett-Gnedenko-MDA*:
assumes $\neg$ *degenerate-df G*
and *d*:*domain-of-attraction F G*
shows $\exists \gamma.$ *domain-of-attraction F* $(EVD\text{-}cdf\ \gamma\ 0\ 1)$
⟨*proof*⟩

3.3 Alternative Characterizations of the Domain of Attraction

We formalize Theorems 1.1.2 and 1.1.6 of the book [2].

3.3.1 Theorem 1.1.2 Part

lemma *MDA-tail-probability-witness*:
fixes $F\ a :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *a-pos*: $\bigwedge t.\ 0 < a\ t$ **and** *sigma*[*arith*]:$0 < \sigma$
and *tail*: $\bigwedge x.\ 1 + \gamma * (x - \mu) / \sigma > 0 \Longrightarrow$
$((\lambda t.\ t * (1 - F\ (a\ t * x + b\ t))) \longrightarrow -\ ln\ (EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x))\ at\text{-}top$
shows $(\lambda n.\ F\ (a\ (real\ n) * x + b\ (real\ n))\ \widehat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x$
⟨*proof*⟩

lemma *MDA-tail-quantile-to-tail-probability*:
fixes $F\ a :: real \Rightarrow real$ **and** $\gamma\ x :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *a-pos*: $\bigwedge t.\ 0 < a\ t$
and *sigma*[*arith*]:$0 < \sigma$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - b\ t) / a\ t) \longrightarrow (if\ \gamma = 0\ then\ \sigma * ln\ x + \mu\ else\ \sigma * (x\ powr\ \gamma - 1) / \gamma + \mu))\ at\text{-}top$
(**is** $\bigwedge x.\ 0 < x \Longrightarrow (- \longrightarrow ?G\ x)\ at\text{-}top$)
and *supp*: $1 + \gamma * (x - \mu) / \sigma > 0$
shows $((\lambda t.\ t * (1 - F\ (a\ t * x + b\ t))) \longrightarrow -\ ln\ (EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x))\ at\text{-}top$
⟨*proof*⟩

lemma *MDA-tail-quantile-witness*:
fixes $F\ a :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *a-pos*: $\bigwedge t.\ 0 < a\ t$ **and** *sigma*:$0 < \sigma$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - b\ t) / a\ t) \longrightarrow (if\ \gamma = 0\ then\ \sigma * ln\ x + \mu\ else\ \sigma * (x\ powr\ \gamma - 1) / \gamma + \mu))\ at\text{-}top$
shows $(\lambda n.\ F\ (a\ (real\ n) * x + b\ (real\ n))\ \widehat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x$
⟨*proof*⟩

lemma *MDA-imp-tail-probability*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *G*: *right-continuous-mono G 0 1*

and a:$\bigwedge n.\ 0 < a\ n$
and deg:$\neg$ $degenerate\text{-}df\ G$
and lim:$\bigwedge y.\ (\lambda n.\ F\ (a\ n * y + b\ n)\ \widehat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ 0\ 1\ y$
and $sigma[arith]$:$0 < \sigma$
and $supp$:$1 + \gamma * x > 0$
shows$((\lambda t.\ t * (1 - F\ (a\ (nat\ \lfloor t \rfloor) * x + tail\text{-}quantile\ F\ t)))$
$\longrightarrow - ln\ (EVD\text{-}cdf\ \gamma\ 0\ 1\ x))\ at\text{-}top$
⟨*proof*⟩

lemma *MDA-imp-tail-quantile*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes F: *right-continuous-mono F 0 1*
and G: *right-continuous-mono G 0 1*
and a:$\bigwedge n.\ 0 < a\ n$
and deg:$\neg$ $degenerate\text{-}df\ G$
and lim:$\bigwedge y.\ (\lambda n.\ F\ (a\ n * y + b\ n)\ \widehat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ 0\ 1\ y$
and $sigma[arith]$:$0 < \sigma$
and x:$x > 0$
shows$((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a\ (nat\ (floor\ t)))$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))\ at\text{-}top$
⟨*proof*⟩

lemma *MDA-iff-tail-probability*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes F: *right-continuous-mono F 0 1*
and a:$\bigwedge n.\ 0 < a\ n$
and $sigma[arith]$:$0 < \sigma$
shows $(\forall x.\ (\lambda n.\ F\ (a\ n * x + b\ n)\ \widehat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x) \longleftrightarrow$
$(\forall x.\ (1 + \gamma * (x - \mu)\ /\ \sigma > 0 \longrightarrow$
$((\lambda t.\ t * (1 - F\ (a\ (nat\ \lfloor t \rfloor) * x + b\ (nat\ \lfloor t \rfloor)))))$
$\longrightarrow - ln\ (EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x))\ at\text{-}top))$
⟨*proof*⟩

lemma *MDA-iff-tail-quantile*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes F: *right-continuous-mono F 0 1*
and a:$\bigwedge n.\ 0 < a\ n$
and $sigma[arith]$:$0 < \sigma$
shows $(\forall x.\ (\lambda n.\ F\ (a\ n * x + b\ n)\ \widehat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x) \longleftrightarrow$
$(\forall x{>}0.\ ((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) -\ b\ (nat\ \lfloor t \rfloor)))\ /\ a\ (nat\ \lfloor t \rfloor))$
$\longrightarrow (if\ \gamma = 0\ then\ \sigma * ln\ x + \mu\ else\ \sigma * (x\ powr\ \gamma - 1)\ /\ \gamma +$
$\mu))\ at\text{-}top)$
⟨*proof*⟩

lemma *MDA-iff-tail-quantile′*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes F: *right-continuous-mono F 0 1*
shows *domain-of-attraction F* $(EVD\text{-}cdf\ \gamma\ 0\ 1)$
$\longleftrightarrow (\exists a.\ (\forall t.\ 0 < a\ t) \wedge$

$(\forall\, x{>}0.\ ((\lambda t.\ (\mathit{tail\text{-}quantile}\ F\ (t * x) - \mathit{tail\text{-}quantile}\ F\ t)\ /\ a\ t)$
$\longrightarrow (\mathit{if}\ \gamma = 0\ \mathit{then}\ \mathit{ln}\ x\ \mathit{else}\ (x\ \mathit{powr}\ \gamma - 1)\ /\ \gamma))\ \mathit{at\text{-}top))$
⟨*proof*⟩

lemma *MDA-iff-tail-probability'*:
fixes $F :: \mathit{real} \Rightarrow \mathit{real}$ **and** $\gamma :: \mathit{real}$
assumes *F*: *right-continuous-mono F 0 1*
shows *domain-of-attraction F (EVD-cdf γ 0 1)*
$\longleftrightarrow (\exists\, a.\ (\forall\, t.\ 0 < a\ t) \wedge$
$(\forall\, x.\ 1 + \gamma * x > 0 \longrightarrow$
$((\lambda t.\ t * (1 - F\ (a\ t * x + \mathit{tail\text{-}quantile}\ F\ t)))$
$\longrightarrow -\ \mathit{ln}\ (\mathit{EVD\text{-}cdf}\ \gamma\ 0\ 1\ x))\ \mathit{at\text{-}top))$
⟨*proof*⟩

3.3.2 Theorem 1.1.6 Part

definition *x-star* :: $(\mathit{real} \Rightarrow \mathit{real}) \Rightarrow \mathit{real}$ **where**
x-star F = *Sup* $\{x.\ F\ x < 1\}$

abbreviation *bdd-support* :: $(\mathit{real} \Rightarrow \mathit{real}) \Rightarrow \mathit{bool}$ **where**
bdd-support F ≡ *bdd-above* $\{x.\ F\ x < 1\}$

lemma *not-bdd-supportI*:
assumes $\bigwedge x.\ \exists\, y{>}x.\ F\ y < 1$
shows ¬ *bdd-support F*
⟨*proof*⟩

lemma *not-bdd-supportI'*:
assumes $\bigwedge x.\ F\ x < 1$
shows ¬ *bdd-support F*
⟨*proof*⟩

definition *endpoint-filter* :: $(\mathit{real} \Rightarrow \mathit{real}) \Rightarrow \mathit{real\ filter}$ **where**
endpoint-filter F = (*if bdd-support F then at-left* (*x-star F*) *else at-top*)

lemma *endpoint-filter*:
bdd-support F ⟹ *endpoint-filter F* = *at-left* (*x-star F*)
¬ *bdd-support F* ⟹ *endpoint-filter F* = *at-top*
⟨*proof*⟩

lemma *endpoint-filter-neq-bot*: *endpoint-filter F* ≠ ⊥
⟨*proof*⟩

lemma *F-x-star-eq-1*:
assumes *right-continuous-mono F 0 1*
and *bdd*:*bdd-above* $\{x.\ F\ x < 1\}$
shows *F* (*x-star F*) = *1*
⟨*proof*⟩

lemma *F-tendsto-1-endpoint*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *a*:$\bigwedge n.\ 0 < a\ n$
and *sigma*[*arith*]:$0 < \sigma$
and *wc*:$\bigwedge x.\ (\lambda n.\ F\ (a\ n * x + b\ n)\ \widehat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x$
shows $(F \longrightarrow 1)$ (*endpoint-filter F*)
⟨*proof*⟩

lemma *lt-x-star-iff-F-lt-1*:
fixes $F :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1* **and** *bdd*: *bdd-support F*
shows $y < x\text{-}star\ F \longleftrightarrow F\ y < 1$
⟨*proof*⟩

lemma *F-lt-1-unbdd-support*:
assumes *F*: *right-continuous-mono F 0 1* **and** *bdd*: ¬ *bdd-support F*
shows $F\ x < 1$
⟨*proof*⟩

lemma *F-lt-1-endpoint*:
fixes $F :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1*
shows $\forall_{F}\ t\ in\ endpoint\text{-}filter\ F.\ F\ t < 1$
⟨*proof*⟩

lemma *tail-quantile-tendsto-x-star*:
fixes $F :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1* **and** *bdd*: *bdd-support F*
shows $(tail\text{-}quantile\ F \longrightarrow x\text{-}star\ F)\ at\text{-}top$
⟨*proof*⟩

lemma *tail-quantile-lt-x-star*:
fixes $F :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1* **and** *bdd*: *bdd-support F*
and *Flim*: $(F \longrightarrow 1)$ (*endpoint-filter F*) **and** *t1*: $1 < t$
shows $tail\text{-}quantile\ F\ t < x\text{-}star\ F$
⟨*proof*⟩

lemma *tail-quantile-tendsto-endpoint*:
fixes $F :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1*
and *Flim*: $(F \longrightarrow 1)$ (*endpoint-filter F*)
shows *filterlim* (*tail-quantile F*) (*endpoint-filter F*) *at-top*
⟨*proof*⟩

lemma *F-tendsto-1-endpoint-of-lim*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*

and *f1*:$\bigwedge t$. *bdd-support F* $\Longrightarrow$ *t* $<$ *x-star F* $\Longrightarrow$ *0* $<$ *f t*
and *f2*:$\bigwedge t$. $\neg$ *bdd-support F* $\Longrightarrow$ *0* $<$ *f t*
and *sigma*[*arith*]: *0* $<$ σ
and *lim*:$\bigwedge x$. *0* $<$ *1* $+$ γ $*$ $(x - \mu)$ / σ
$\Longrightarrow$ $((\lambda t.\ (1 - F\ (t + (x - \mu) * f\ t))\ /\ (1 - F\ t))$
$\longrightarrow$ $-$ *ln* (*EVD-cdf* γ μ σ *x*)) (*endpoint-filter F*)
shows (*F* $\longrightarrow$ *1*) (*endpoint-filter F*)
⟨*proof*⟩

lemma *s-tendsto-at-top-endpoint*:
fixes *F* :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *a*:$\bigwedge n$. *0* $<$ *a n*
and *sigma*[*arith*]:*0* $<$ σ
and *wc*:$\bigwedge x$. $(\lambda n.\ F\ (a\ n * x + b\ n)\ \hat{}\ n) \longrightarrow$ *EVD-cdf* γ μ σ *x*
shows *filterlim* $(\lambda t.\ 1\ /\ (1 - F\ t))$ *at-top* (*endpoint-filter F*)
⟨*proof*⟩

lemma *t-div-a-tendsto-mu-endpoint*:
fixes *F* :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *a*:$\bigwedge n$. *0* $<$ *a n*
and *sigma*[*arith*]:*0* $<$ σ
and *wc*:$\bigwedge x$. $(\lambda n.\ F\ (a\ n * x + b\ n)\ \hat{}\ n) \longrightarrow$ *EVD-cdf* γ μ σ *x*
shows $((\lambda t.\ (t - b\ (nat\ \lfloor 1\ /\ (1 - F\ t)\rfloor))$
$/\ a\ (nat\ \lfloor 1\ /\ (1 - F\ t)\rfloor)) \longrightarrow \mu)$ (*endpoint-filter F*)
⟨*proof*⟩

lemma *MDA-imp-divide-tail*:
assumes *F*: *right-continuous-mono F 0 1*
and *a*:$\bigwedge n$. *0* $<$ *a n*
and *sigma*[*arith*]:*0* $<$ σ
and *wc*:$\bigwedge x$. $(\lambda n.\ F\ (a\ n * x + b\ n)\ \hat{}\ n) \longrightarrow$ *EVD-cdf* γ μ σ *x*
and *x*:*0* $<$ *1* $+$ γ $*$ $(x - \mu)$ / σ
shows $((\lambda t.\ (1 - F\ (t + (x - \mu) * a\ (nat\ \lfloor 1\ /\ (1 - F\ (t))\rfloor)))\ /\ (1 - F\ t))$
$\longrightarrow$ $-$ *ln* (*EVD-cdf* γ μ σ *x*)) (*endpoint-filter F*)
⟨*proof*⟩

lemma *tail-quantile-asymp-inverse-of-lim*:
fixes *F* :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *f1*:$\bigwedge t$. *bdd-support F* $\Longrightarrow$ *t* $<$ *x-star F* $\Longrightarrow$ *0* $<$ *f t*
and *f2*:$\bigwedge t$. $\neg$ *bdd-support F* $\Longrightarrow$ *0* $<$ *f t*
and *sigma*[*arith*]: *0* $<$ σ
and *lim*:$\bigwedge x$. *0* $<$ *1* $+$ γ $*$ $(x - \mu)$ / σ
$\Longrightarrow$ $((\lambda t.\ (1 - F\ (t + (x - \mu) * f\ t))\ /\ (1 - F\ t))$
$\longrightarrow$ $-$ *ln* (*EVD-cdf* γ μ σ *x*)) (*endpoint-filter F*)
shows $((\lambda t.\ t * (1 - F\ (tail\text{-}quantile\ F\ t))) \longrightarrow 1)$ *at-top*
⟨*proof*⟩

lemma *divide-tail-imp-MDA*:
assumes *F*: *right-continuous-mono F 0 1*
and *a*:$\bigwedge n.\ 0 < a\ n$
and *sigma*[*arith*]:$0 < \sigma$
and *f1*:$\bigwedge t.$ *bdd-support F* $\Longrightarrow t <$ *x-star F* $\Longrightarrow 0 < f\ t$
and *f2*:$\bigwedge t.\ \neg$ *bdd-support F* $\Longrightarrow 0 < f\ t$
and *lim*:$\bigwedge x.\ 0 < 1 + \gamma * (x - \mu)\ /\ \sigma$
$\Longrightarrow ((\lambda t.\ (1 - F\ (t + (x - \mu) * f\ t))\ /\ (1 - F\ t))$
$\longrightarrow - ln\ (EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x))\ (endpoint\text{-}filter\ F)$
shows $\exists\, a\ b.\ (\forall\, n.\ 0 < a\ n) \wedge (\forall\, x.\ (\lambda n.\ F\ (a\ n * x + b\ n)\ \hat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ \mu\ \sigma\ x)$
⟨*proof*⟩

lemma *MDA-iff-divide-tail*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
shows *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
$\longleftrightarrow (\exists f.\ (\forall\, t.\ 0 < f\ t) \wedge$
$(\forall\, x.\ 1 + \gamma * x > 0 \longrightarrow$
$((\lambda t.\ (1 - F\ (t + x * f\ t))\ /\ (1 - F\ t))$
$\longrightarrow - ln\ (EVD\text{-}cdf\ \gamma\ 0\ 1\ x))\ (endpoint\text{-}filter\ F)))$
⟨*proof*⟩

3.4 von Mises' Conditions

lemma
fixes $F :: real \Rightarrow real$ **and** *x0 t* :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *x0*: *F x0* < 1
and *Flim*: $(F \longrightarrow 1)$ (*endpoint-filter F*)
and *Fcont*: $\bigwedge x.\ x0 < x \Longrightarrow F\ x < 1 \Longrightarrow isCont\ F\ x$
and *t*: $1\ /\ (1 - F\ x0) < t$
shows *tail-quantile-tail-eq*: $1 - F\ (tail\text{-}quantile\ F\ t) = 1\ /\ t$
and *F-tail-quantile-lt-1*: $F\ (tail\text{-}quantile\ F\ t) < 1$
⟨*proof*⟩

context
fixes *F f* :: $real \Rightarrow real$ **and** *x0* :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *x0*: *F x0* < 1
and *Flim*: $(F \longrightarrow 1)$ (*endpoint-filter F*)
and *Fderiv*: $\bigwedge y.\ x0 < y \Longrightarrow F\ y < 1 \Longrightarrow (F\ has\text{-}real\text{-}derivative\ f\ y)\ (at\ y)$
and *fpos*: $\bigwedge y.\ x0 < y \Longrightarrow F\ y < 1 \Longrightarrow 0 < f\ y$
begin

lemma *tail-quantile-has-deriv*:
fixes *t* :: *real*
assumes *t*: $1\ /\ (1 - F\ x0) < t$

shows (*tail-quantile F has-real-derivative*
*1 / (t² * f (tail-quantile F t))*) *(at t)*
⟨*proof*⟩

lemma *f-tail-quantile-pos*:
fixes *t* :: *real*
assumes *t*: *1 / (1 − F x0) < t*
shows *0 < f (tail-quantile F t)*
⟨*proof*⟩

lemma *tail-quantile-deriv-has-deriv*:
fixes *f′* :: *real ⇒ real* **and** *t* :: *real*
assumes *fderiv*: *⋀x. x0 < x ⟹ F x < 1 ⟹ (f has-real-derivative f′ x) (at x)*
and *t*: *1 / (1 − F x0) < t*
shows *((λs. 1 / (s² * f (tail-quantile F s))) has-real-derivative*
*− (2 * t * f (tail-quantile F t)*
+ f′ (tail-quantile F t) / f (tail-quantile F t))
*/ (t² * f (tail-quantile F t))²) (at t)*
⟨*proof*⟩

lemma *tail-quantile-second-deriv-eq*:
fixes *f′ u u′* :: *real ⇒ real* **and** *t* :: *real*
assumes *fderiv*: *⋀x. x0 < x ⟹ F x < 1 ⟹ (f has-real-derivative f′ x) (at x)*
and *uderiv*: *⋀s. 1 / (1 − F x0) < s*
⟹ (tail-quantile F has-real-derivative u s) (at s)
and *u′deriv*: *⋀s. 1 / (1 − F x0) < s ⟹ (u has-real-derivative u′ s) (at s)*
and *t*: *1 / (1 − F x0) < t*
shows *t * u′ t / u t*
*= − 2 − (1 − F (tail-quantile F t)) * f′ (tail-quantile F t)*
/ (f (tail-quantile F t))²
⟨*proof*⟩

context
fixes *f′* :: *real ⇒ real* **and** *γ* :: *real*
assumes *fderiv*: *⋀y. x0 < y ⟹ F y < 1 ⟹ (f has-real-derivative f′ y) (at y)*
and *f′cont*: *continuous-on {y. x0 < y ∧ F y < 1} f′*
and *lim*: *((λy. (1 − F y) * f′ y / (f y)²) ⟶ − γ − 1) (endpoint-filter F)*
begin

lemma *von-Mises-quantile-diff*:
fixes *x* :: *real*
assumes *x*: *0 < x*
shows *((λt. (tail-quantile F (t * x) − tail-quantile F t)*
** (t * f (tail-quantile F t)))*
⟶ (if γ = 0 then ln x else (x powr γ − 1) / γ)) at-top
⟨*proof*⟩

lemma *von-Mises-normalizing-ex*:
shows *∃ a. (∀ t. 0 < a t)*

$\wedge$ ($\forall_F$ t *in at-top*. $a\ t = 1\ /\ (t * f\ (\textit{tail-quantile}\ F\ t))$)
$\wedge$ ($\forall x{>}0$. ((λt. (*tail-quantile* F $(t * x)$ − *tail-quantile* F t) / a t)
⟶ (*if* $\gamma = 0$ *then ln* x *else* $(x\ powr\ \gamma - 1)\ /\ \gamma$)) *at-top*)
⟨*proof*⟩

von Mises' condition (Theorem 1.1.8 of the book by de Haan and Ferreira [2]).

theorem *von-Mises-condition*: *domain-of-attraction* F (*EVD-cdf* γ 0 1)
⟨*proof*⟩

corollary *von-Mises-normalization*:
fixes x :: *real*
shows (λn. F (x / (*real* n * f (*tail-quantile* F (*real* n)))
+ *tail-quantile* F (*real* n)) ^ n) ⟶ *EVD-cdf* γ 0 1 x
⟨*proof*⟩

end

end

Relation (1.1.35) of the book by de Haan and Ferreira [2].

lemma *tail-quantile-deriv-ratio-pos*:
fixes F f u :: *real* ⇒ *real* **and** γ *x0 t0* :: *real*
assumes F: *right-continuous-mono F 0 1*
and *unbdd*: ¬ *bdd-support F*
and *gamma*: $0 < \gamma$
and *Fderiv*: ⋀x. *x0* < x ⟹ (F *has-real-derivative* f x) (*at* x)
and *fpos*: ⋀x. *x0* < x ⟹ $0 < f\ x$
and *uderiv*: ⋀t. *t0* < t ⟹ (*tail-quantile F has-real-derivative* u t) (*at* t)
and *lim*: ((λt. $t * f\ t\ /\ (1 - F\ t)$) ⟶ 1 / γ) *at-top*
shows ((λt. $t * u\ t\ /$ *tail-quantile* F t) ⟶ γ) *at-top*
⟨*proof*⟩

von Mises' condition for positive γ (Theorem 1.1.11 of the book by de Haan and Ferreira [2]).

theorem *von-Mises-condition-pos*:
fixes F f :: *real* ⇒ *real* **and** γ *x0* :: *real*
assumes F: *right-continuous-mono F 0 1*
and *unbdd*: ¬ *bdd-support F*
and *gamma*: $0 < \gamma$
and *Fderiv*: ⋀x. *x0* < x ⟹ (F *has-real-derivative* f x) (*at* x)
and *fcont*: *continuous-on* {*x0*<..} f
and *fpos*: ⋀x. *x0* < x ⟹ $0 < f\ x$
and *lim*: ((λt. $t * f\ t\ /\ (1 - F\ t)$) ⟶ 1 / γ) *at-top*
shows *domain-of-attraction* F (*EVD-cdf* γ 0 1)
⟨*proof*⟩

lemma *F-tendsto-1-endpoint-of-deriv*:

fixes $F\ f :: real \Rightarrow real$ **and** $x0\ c :: real$
assumes F: *right-continuous-mono F 0 1*
and *bdd*: *bdd-support F*
and *x0*: $x0 < x\text{-}star\ F$
and *Fderiv*: $\bigwedge y.\ x0 < y \Longrightarrow y < x\text{-}star\ F \Longrightarrow (F\ has\text{-}real\text{-}derivative\ f\ y)\ (at\ y)$
and *cpos*: $0 < c$
and *lim*: $((\lambda y.\ (x\text{-}star\ F - y) * f\ y\ /\ (1 - F\ y)) \longrightarrow c)\ (at\text{-}left\ (x\text{-}star\ F))$
shows $(F \longrightarrow 1)\ (endpoint\text{-}filter\ F)$
⟨*proof*⟩

Relation (1.1.38) of the book by de Haan and Ferreira [2].

lemma *tail-quantile-deriv-ratio-neg*:
fixes $F\ f\ u :: real \Rightarrow real$ **and** $\gamma\ x0\ t0 :: real$
assumes F: *right-continuous-mono F 0 1*
and *bdd*: *bdd-support F*
and *gamma*: $\gamma < 0$
and *x0*: $x0 < x\text{-}star\ F$
and *Fderiv*: $\bigwedge x.\ x0 < x \Longrightarrow x < x\text{-}star\ F \Longrightarrow (F\ has\text{-}real\text{-}derivative\ f\ x)\ (at\ x)$
and *fpos*: $\bigwedge x.\ x0 < x \Longrightarrow x < x\text{-}star\ F \Longrightarrow 0 < f\ x$
and *uderiv*: $\bigwedge t.\ t0 < t \Longrightarrow (tail\text{-}quantile\ F\ has\text{-}real\text{-}derivative\ u\ t)\ (at\ t)$
and *lim*: $((\lambda t.\ (x\text{-}star\ F - t) * f\ t\ /\ (1 - F\ t)) \longrightarrow -\ 1\ /\ \gamma)\ (at\text{-}left\ (x\text{-}star\ F))$
shows $((\lambda t.\ t * u\ t\ /\ (x\text{-}star\ F - tail\text{-}quantile\ F\ t)) \longrightarrow -\ \gamma)\ at\text{-}top$
⟨*proof*⟩

von Mises' condition for negative γ (Theorem 1.1.13 of the book by de Haan and Ferreira [2]).

theorem *von-Mises-condition-neg*:
fixes $F\ f :: real \Rightarrow real$ **and** $\gamma\ x0 :: real$
assumes F: *right-continuous-mono F 0 1*
and *bdd*: *bdd-support F*
and *gamma*: $\gamma < 0$
and *x0*: $x0 < x\text{-}star\ F$
and *Fderiv*: $\bigwedge x.\ x0 < x \Longrightarrow x < x\text{-}star\ F \Longrightarrow (F\ has\text{-}real\text{-}derivative\ f\ x)\ (at\ x)$
and *fcont*: *continuous-on* $\{x0<..<x\text{-}star\ F\}\ f$
and *fpos*: $\bigwedge x.\ x0 < x \Longrightarrow x < x\text{-}star\ F \Longrightarrow 0 < f\ x$
and *lim*: $((\lambda t.\ (x\text{-}star\ F - t) * f\ t\ /\ (1 - F\ t)) \longrightarrow -\ 1\ /\ \gamma)\ (at\text{-}left\ (x\text{-}star\ F))$
shows *domain-of-attraction F* $(EVD\text{-}cdf\ \gamma\ 0\ 1)$
⟨*proof*⟩

3.5 Relation to Regularly Varying Functions

We formalize results for $\gamma \neq 0$.

lemma *tail-quantile-S-at-top*:
fixes $F :: real \Rightarrow real$ **and** $\varrho :: real$
assumes F: *right-continuous-mono F 0 1*
and *bdd*: *bdd-support F*

and *rv*: *regularly-varying* $(\lambda s.\ 1 - F\ (x\text{-}star\ F - 1\ /\ s))\ \varrho$
and *rho*: $\varrho < 0$
shows *filterlim* $(\lambda t.\ 1\ /\ (x\text{-}star\ F - tail\text{-}quantile\ F\ t))$ *at-top at-top*
⟨*proof*⟩

lemma *tail-quantile-diff-scale-ratio*:
fixes $U\ a :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *apos*: $\bigwedge t.\ 0 < a\ t$
and *gamma*: $\gamma \neq 0$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (U\ (t * x) - U\ t)\ /\ a\ t) \longrightarrow (x\ powr\ \gamma - 1)\ /\ \gamma)$ *at-top*
and *y*: $0 < y$
shows $((\lambda t.\ a\ (t * y)\ /\ a\ t) \longrightarrow y\ powr\ \gamma)$ *at-top*
⟨*proof*⟩

lemma *tail-quantile-cont-ratio*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$ **and** $a :: real \Rightarrow real$
assumes *apos*: $\bigwedge t.\ 0 < a\ t$
and *gamma*: $\gamma \neq 0$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a\ t)$
$\longrightarrow (x\ powr\ \gamma - 1)\ /\ \gamma)$ *at-top*
and *Z*: $1 < Z$
shows $((\lambda t.\ (tail\text{-}quantile\ F\ (t * Z * Z) - tail\text{-}quantile\ F\ (t * Z))$
$/\ (tail\text{-}quantile\ F\ (t * Z) - tail\text{-}quantile\ F\ t)) \longrightarrow Z\ powr\ \gamma)$ *at-top*
⟨*proof*⟩

lemma *tail-quantile-diff-geometric-seq*:
fixes $F :: real \Rightarrow real$ **and** $\gamma\ Z\ t0 :: real$ **and** $a :: real \Rightarrow real$
assumes *apos*: $\bigwedge t.\ 0 < a\ t$
and *gamma*: $\gamma \neq 0$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a\ t)$
$\longrightarrow (x\ powr\ \gamma - 1)\ /\ \gamma)$ *at-top*
and *Z*: $1 < Z$ **and** *t0*: $0 < t0$
obtains *N* **where**
$\bigwedge k.\ tail\text{-}quantile\ F\ (t0 * Z\ \hat{}\ (N + k)) < tail\text{-}quantile\ F\ (t0 * Z\ \hat{}\ Suc\ (N + k))$
and $(\lambda k.\ (tail\text{-}quantile\ F\ (t0 * Z\ \hat{}\ Suc\ (N + Suc\ k))$
$- tail\text{-}quantile\ F\ (t0 * Z\ \hat{}\ (N + Suc\ k)))$
$/\ (tail\text{-}quantile\ F\ (t0 * Z\ \hat{}\ Suc\ (N + k)) - tail\text{-}quantile\ F\ (t0 * Z\ \hat{}\ (N + k))))$
$\longrightarrow Z\ powr\ \gamma$
⟨*proof*⟩

lemma *tail-quantile-neg-gap-power-ratio*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$ **and** $a :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$

and *gamma*: $\gamma < 0$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (\textit{tail-quantile}\ F\ (t * x) - \textit{tail-quantile}\ F\ t)\ /\ a\ t)$
$\longrightarrow (x\ \textit{powr}\ \gamma - 1)\ /\ \gamma)\ \textit{at-top}$
and *Uconv*: $(\textit{tail-quantile}\ F \longrightarrow \textit{x-star}\ F)\ \textit{at-top}$
and *Z*: $1 < Z$ **and** *t0*: $0 < t0$
shows $(\lambda n.\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} \textit{Suc}\ n))$
$/\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} n))) \longrightarrow Z\ \textit{powr}\ \gamma$
⟨*proof*⟩

lemma *tail-quantile-neg-gap-ratio-of-power-ratio*:
fixes F :: *real* ⇒ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*: $\gamma < 0$
and *gap-pos*: $\bigwedge t.\ 1 < t \Longrightarrow 0 < \textit{x-star}\ F - \textit{tail-quantile}\ F\ t$
and *gap-power-ratio*: $\bigwedge Z\ t0.\ 1 < Z \Longrightarrow 0 < t0 \Longrightarrow$
$(\lambda n.\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} \textit{Suc}\ n))$
$/\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} n))) \longrightarrow Z\ \textit{powr}\ \gamma$
shows $\bigwedge y.\ 0 < y \Longrightarrow$
$((\lambda t.\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ (t * y))\ /\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ t))$
$\longrightarrow y\ \textit{powr}\ \gamma)\ \textit{at-top}$
⟨*proof*⟩

lemma *tail-quantile-neg-regularly-varying*:
fixes F :: *real* ⇒ *real* **and** γ :: *real* **and** a :: *real* ⇒ *real*
assumes *F*: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$
and *gamma*: $\gamma < 0$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (\textit{tail-quantile}\ F\ (t * x) - \textit{tail-quantile}\ F\ t)\ /\ a\ t) \longrightarrow (x\ \textit{powr}\ \gamma -$
$1)\ /\ \gamma)\ \textit{at-top}$
shows *tail-quantile-neg-bdd*: *bdd-support F*
and *tail-quantile-neg-tendsto*: $(\textit{tail-quantile}\ F \longrightarrow \textit{x-star}\ F)\ \textit{at-top}$
and *tail-quantile-neg-ratio*: $\bigwedge y.\ 0 < y \Longrightarrow$
$((\lambda t.\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ (t * y))\ /\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ t))$
$\longrightarrow y\ \textit{powr}\ \gamma)\ \textit{at-top}$
⟨*proof*⟩

lemma *tail-quantile-gap-ratio-neg-imp-tail-regularly-varying*:
fixes F :: *real* ⇒ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*: $\gamma < 0$
and *bdd*: *bdd-support F*
and *gap-rv*: $\bigwedge y.\ 0 < y \Longrightarrow$
$((\lambda t.\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ (t * y))\ /\ (\textit{x-star}\ F - \textit{tail-quantile}\ F\ t))$
$\longrightarrow y\ \textit{powr}\ \gamma)\ \textit{at-top}$
shows *regularly-varying* $(\lambda s.\ 1 - F\ (\textit{x-star}\ F - 1\ /\ s))\ (1\ /\ \gamma)$
⟨*proof*⟩

lemma *tail-quantile-pos-power-ratio*:
fixes F :: *real* ⇒ *real* **and** γ :: *real* **and** a :: *real* ⇒ *real*

assumes F: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$
and *gamma*: $0 < \gamma$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (\textit{tail-quantile}\ F\ (t * x) - \textit{tail-quantile}\ F\ t)\ /\ a\ t)$
$\longrightarrow (x\ \textit{powr}\ \gamma - 1)\ /\ \gamma)\ \textit{at-top}$
and *Uinf*: *filterlim* (*tail-quantile F*) *at-top at-top*
and Z: $1 < Z$ **and** *t0*: $0 < t0$
shows $(\lambda n.\ \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} \textit{Suc}\ n)\ /\ \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} n))$
$\longrightarrow Z\ \textit{powr}\ \gamma$
⟨*proof*⟩

lemma *tail-quantile-pos-ratio-of-power-ratio*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$ **and** $\gamma :: \textit{real}$
assumes F: *right-continuous-mono F 0 1* **and** *gamma*: $0 < \gamma$
and *Uinf*: *filterlim* (*tail-quantile F*) *at-top at-top*
and *power-ratio*: $\bigwedge Z\ t0.\ 1 < Z \Longrightarrow 0 < t0 \Longrightarrow$
$(\lambda n.\ \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} \textit{Suc}\ n)\ /\ \textit{tail-quantile}\ F\ (t0 * Z \mathbin{\hat{}} n)) \longrightarrow$
$Z\ \textit{powr}\ \gamma$
shows $\bigwedge y.\ 0 < y \Longrightarrow$
$((\lambda t.\ \textit{tail-quantile}\ F\ (t * y)\ /\ \textit{tail-quantile}\ F\ t) \longrightarrow y\ \textit{powr}\ \gamma)\ \textit{at-top}$
⟨*proof*⟩

lemma *tail-quantile-pos-regularly-varying*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$ **and** $\gamma :: \textit{real}$ **and** $a :: \textit{real} \Rightarrow \textit{real}$
assumes F: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$
and *gamma*: $0 < \gamma$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (\textit{tail-quantile}\ F\ (t * x) - \textit{tail-quantile}\ F\ t)\ /\ a\ t)$
$\longrightarrow (x\ \textit{powr}\ \gamma - 1)\ /\ \gamma)\ \textit{at-top}$
shows *tail-quantile-pos-at-top*: *filterlim* (*tail-quantile F*) *at-top at-top*
and *tail-quantile-pos-ratio*: $\bigwedge y.\ 0 < y \Longrightarrow$
$((\lambda t.\ \textit{tail-quantile}\ F\ (t * y)\ /\ \textit{tail-quantile}\ F\ t) \longrightarrow y\ \textit{powr}\ \gamma)\ \textit{at-top}$
⟨*proof*⟩

lemma *tail-quantile-ratio-pos-imp-tail-regularly-varying*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$ **and** $\gamma :: \textit{real}$
assumes F: *right-continuous-mono F 0 1* **and** *gamma*: $0 < \gamma$
and *Uinf*: *filterlim* (*tail-quantile F*) *at-top at-top*
and *Urv*: $\bigwedge y.\ 0 < y \Longrightarrow$
$((\lambda t.\ \textit{tail-quantile}\ F\ (t * y)\ /\ \textit{tail-quantile}\ F\ t) \longrightarrow y\ \textit{powr}\ \gamma)\ \textit{at-top}$
shows $(\forall x.\ F\ x < 1) \wedge \textit{regularly-varying}\ (\lambda x.\ 1 - F\ x)\ (-\ 1\ /\ \gamma)$
⟨*proof*⟩

theorem *MDA-regularly-varying-pos*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$ **and** $\gamma :: \textit{real}$
assumes F: *right-continuous-mono F 0 1* **and** *gamma*[*arith*]: $0 < \gamma$
shows *domain-of-attraction F* (*EVD-cdf* γ *0 1*)

$\longleftrightarrow$ $(\forall\, x.\ F\ x < 1) \wedge$ *regularly-varying* $(\lambda x.\ 1 - F\ x)\ (-\ 1\ /\ \gamma)$
⟨*proof*⟩

theorem *MDA-regularly-varying-neg*:
fixes F :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*[*arith*]: $\gamma < 0$
shows *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
$\longleftrightarrow$ *bdd-support F* $\wedge$
regularly-varying $(\lambda s.\ 1 - F\ (x\text{-}star\ F - 1\ /\ s))\ (1\ /\ \gamma)$
⟨*proof*⟩

Corollary 1.2.4

lemma *MDA-aux-fun-cong*:
fixes *F a a'* :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *a*: $\bigwedge t.\ 0 < a\ t$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a\ t)$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))\ at\text{-}top$
and *equiv*: $((\lambda t.\ a'\ t\ /\ a\ t) \longrightarrow 1)\ at\text{-}top$
and *x*: $0 < x$
shows $((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a'\ t)$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))\ at\text{-}top$
⟨*proof*⟩

lemma *MDA-canonical-normalization*:
fixes *F* :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
obtains *a* :: *real* $\Rightarrow$ *real*
where $\bigwedge t.\ 0 < a\ t$
and $\bigwedge x.\ 0 < x \Longrightarrow$
$((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a\ t)$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))\ at\text{-}top$
and $\bigwedge x.\ (\lambda n.\ F\ (a\ (real\ n) * x + tail\text{-}quantile\ F\ (real\ n))\ \hat{}\ n) \longrightarrow EVD\text{-}cdf$
γ *0 1 x*
⟨*proof*⟩

lemma *tail-quantile-pos-aux*:
fixes *F* :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*: $0 < \gamma$
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
and *x*: $0 < x$
shows $((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ (\gamma * tail\text{-}quantile\ F\ t))$
$\longrightarrow (x\ powr\ \gamma - 1)\ /\ \gamma)\ at\text{-}top$
⟨*proof*⟩

lemma *tail-quantile-neg-gap*:
fixes *F* :: *real* $\Rightarrow$ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*: $\gamma < 0$

and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
shows *tail-quantile-neg-bdd-support*: *bdd-support F*
and *tail-quantile-neg-gap-pos*: $\forall_F$ *t in at-top. 0 < x-star F − tail-quantile F t*
⟨*proof*⟩

lemma *tail-quantile-neg-aux*:
fixes *F* :: *real* ⇒ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*: $\gamma < 0$
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
and *x*: *0 < x*
shows ((λ*t*. (*tail-quantile F* (*t* ∗ *x*) − *tail-quantile F t*)
/ (− γ ∗ (*x-star F* − *tail-quantile F t*)))
⟶ (*x powr* γ − *1*) / γ) *at-top*
⟨*proof*⟩

theorem *MDA-normalization-pos*:
fixes *F* :: *real* ⇒ *real* **and** γ *x* :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*: *0 < γ*
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
shows (λ*n*. *F* (*tail-quantile F* (*real n*) ∗ *x*) ^ *n*)
⟶ (*if 0 < x then exp* (− (*x powr* − (*1* / γ))) *else 0*)
⟨*proof*⟩

theorem *MDA-normalization-neg*:
fixes *F* :: *real* ⇒ *real* **and** γ *x* :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *gamma*: $\gamma < 0$
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
shows (λ*n*. *F* ((*x-star F* − *tail-quantile F* (*real n*)) ∗ *x* + *x-star F*) ^ *n*)
⟶ (*if x < 0 then exp* (− ((− *x*) *powr* − (*1* / γ))) *else 1*)
⟨*proof*⟩

lemma *tail-quantile-spacing-aux*:
fixes *F* :: *real* ⇒ *real* **and** γ *c* :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
and *c*: *1 < c* **and** *x*: *0 < x*
shows ((λ*t*. (*tail-quantile F* (*t* ∗ *x*) − *tail-quantile F t*)
/ ((*tail-quantile F* (*t* ∗ *c*) − *tail-quantile F t*)
/ (*if* γ = *0 then ln c else* (*c powr* γ − *1*) / γ)))
⟶ (*if* γ = *0 then ln x else* (*x powr* γ − *1*) / γ)) *at-top*
⟨*proof*⟩

theorem *MDA-normalization-zero-spacing*:
fixes *F* :: *real* ⇒ *real* **and** *c x* :: *real*
assumes *F*: *right-continuous-mono F 0 1* **and** *c*: *1 < c*
and *dom*: *domain-of-attraction F* (*EVD-cdf 0 0 1*)
shows (λ*n*. *F* ((*tail-quantile F* (*real n* ∗ *c*) − *tail-quantile F* (*real n*)) / *ln c* ∗ *x*
+ *tail-quantile F* (*real n*)) ^ *n*)
⟶ *EVD-cdf 0 0 1 x*

⟨*proof*⟩

end

4 Peaks Over Threshold

theory *Peaks-Over-Threshold*
imports
Domain-of-Attraction
begin

In this section, we formalize the Pickands–Balkema–de Haan theorem. Our formalization is stated in the form of Theorem 3.4.13(b) in the book by Embrechts et al. [3]. The original sources of the theorem are Theorems 3 and 4 in the paper by Balkema and de Haan [1] and Theorem 7 in the paper by Pickands [6].

4.1 Excess (Peaks-Over-Threshold) Distribution Function

definition *excess-cdf* :: (*real* ⇒ *real*) ⇒ *real* ⇒ *real* ⇒ *real* **where**
excess-cdf F u x = *1* − (*1* − *F* (*u* + *x*)) / (*1* − *F u*)

lemma *excess-cdf-def2*:*F u* < *1* ⟹ *excess-cdf F u x* = (*F* (*u* + *x*) − *F u*) / (*1* − *F u*)
⟨*proof*⟩

lemma *excess-cdf-0*[*simp*]: *F u* < *1* ⟹ *excess-cdf F u 0* = *0*
⟨*proof*⟩

lemma *excess-cdf-eq-1-of-F-eq-1*:
assumes *F u* = *1*
shows *excess-cdf F u x* = *1*
⟨*proof*⟩

lemma *excess-cdf-mono*:
assumes *F*: *right-continuous-mono F 0 1* **and** *xy*: *x* ≤ *y*
shows *excess-cdf F u x* ≤ *excess-cdf F u y*
⟨*proof*⟩

lemma *excess-cdf-bounds*:
assumes *F*: *right-continuous-mono F 0 1* **and** *x*: *0* ≤ *x*
shows *0* ≤ *excess-cdf F u x excess-cdf F u x* ≤ *1*
⟨*proof*⟩

lemma *excess-cdf-eq-1-beyond-support*:
assumes *F*: *right-continuous-mono F 0 1* **and** *bdd*: *bdd-support F*
and *xu*: *x-star F* ≤ *u* + *x*
shows *excess-cdf F u x* = *1*

⟨*proof*⟩

lemma *excess-tendsto-of-divide-tail*:
fixes $F\ f$ :: *real* ⇒ *real* **and** $\gamma\ x$ :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *fpos*: $\bigwedge t.\ 0 < f\ t$
and *lim*: $\bigwedge x.\ 1 + \gamma * x > 0 \Longrightarrow$
$((\lambda t.\ (1 - F\ (t + x * f\ t))\ /\ (1 - F\ t))$
$\longrightarrow - ln\ (EVD\text{-}cdf\ \gamma\ 0\ 1\ x))\ (endpoint\text{-}filter\ F)$
and *x*: $0 \leq x$
shows $((\lambda t.\ excess\text{-}cdf\ F\ t\ (f\ t * x)) \longrightarrow GPD\text{-}cdf\ \gamma\ 0\ 1\ x)\ (endpoint\text{-}filter\ F)$
⟨*proof*⟩

lemma *tail-probability-u-form*:
fixes F :: *real* ⇒ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *dom*: *domain-of-attraction F* $(EVD\text{-}cdf\ \gamma\ 0\ 1)$
shows $\exists \beta.\ (\forall u.\ 0 < \beta\ u) \wedge$
$(\forall x{\geq}0.\ ((\lambda u.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x) - GPD\text{-}cdf\ \gamma\ 0\ 1\ x)$
$\longrightarrow 0)\ (endpoint\text{-}filter\ F))$
⟨*proof*⟩

lemma *excess-pointwise-imp-uniform*:
fixes F :: *real* ⇒ *real* **and** β :: *real* ⇒ *real* **and** *FF* :: *real filter*
assumes *F*: *right-continuous-mono F 0 1* **and** *βpos*: $\bigwedge u.\ 0 < \beta\ u$
and *pconv*: $\bigwedge x.\ 0 \leq x \Longrightarrow ((\lambda u.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x)) \longrightarrow GPD\text{-}cdf\ \gamma\ 0\ 1\ x)\ FF$
shows *uniform-limit* $\{0..\}\ (\lambda u\ x.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x))\ (GPD\text{-}cdf\ \gamma\ 0\ 1)\ FF$
⟨*proof*⟩

lemma *tail-probability-u-form-uniform*:
fixes F :: *real* ⇒ *real* **and** γ :: *real*
assumes *F*: *right-continuous-mono F 0 1*
and *dom*: *domain-of-attraction F* $(EVD\text{-}cdf\ \gamma\ 0\ 1)$
shows $\exists \beta.\ (\forall u.\ 0 < \beta\ u) \wedge$
uniform-limit $\{0..\}\ (\lambda u\ x.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x))\ (GPD\text{-}cdf\ \gamma\ 0\ 1)$
$(endpoint\text{-}filter\ F)$
⟨*proof*⟩

4.2 Pickands–Balkema–de Haan Theorem

The proof of the direct half follows the proof of Theorem 3.4.13(b) in the same book: the pointwise statement is Theorem 1.1.6(4) of the book by de Haan and Ferreira [2], and uniformity follows since the limit is continuous. The converse half follows the route sketched in the proof of Theorem 3.4.5 of Embrechts et al. [3].

4.2.1 Direct Half

theorem *Pickands-Balkema-de-Haan-forward*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
shows $\exists \beta.\ (\forall u.\ 0 < \beta\ u) \wedge$
uniform-limit $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u x* $-$ *GPD-cdf* γ *0* $(\beta\ u)$ *x*)
$(\lambda x.\ 0)$ (*endpoint-filter F*)
⟨*proof*⟩

4.2.2 Converse Half

lemma *GPD-level-nonneg*:
fixes $\gamma\ x :: real$
assumes $1 \leq x$
shows $0 \leq$ (*if* $\gamma = 0$ *then ln x else* $(x\ powr\ \gamma - 1)\ /\ \gamma$)
⟨*proof*⟩

lemma *GPD-level-pos*:
fixes $\gamma\ x :: real$
assumes $1 < x$
shows $0 <$ (*if* $\gamma = 0$ *then ln x else* $(x\ powr\ \gamma - 1)\ /\ \gamma$)
⟨*proof*⟩

lemma *GPD-level-supp*:
fixes $\gamma\ x :: real$
assumes $1 \leq x$
shows $1 + \gamma * ($*if* $\gamma = 0$ *then ln x else* $(x\ powr\ \gamma - 1)\ /\ \gamma) = ($*if* $\gamma = 0$ *then 1 else x powr* γ)
⟨*proof*⟩

lemma *GPD-tail-eq*:
assumes *x*: $0 \leq x$
shows $1 -$ *GPD-cdf* γ *0 1 x*
$=$ (*if* $\gamma = 0$ *then exp* $(-\ x)$
else if $0 \leq 1 + \gamma * x$ *then* $(1 + \gamma * x)$ *powr* $(-\ 1\ /\ \gamma)$ *else 0*)
⟨*proof*⟩

lemma *GPD-tail-level*:
assumes *x*: $1 \leq x$
shows $1 -$ *GPD-cdf* γ *0 1* ((*if* $\gamma = 0$ *then ln x else* $(x\ powr\ \gamma - 1)\ /\ \gamma$)) $= 1\ /\ x$
⟨*proof*⟩

lemma *GPD-tail-strict-antimono*:
assumes *y*: $0 \leq y$ **and** *yz*: $y < z$ **and** *supp*: $0 < 1 + \gamma * y$
shows $1 -$ *GPD-cdf* γ *0 1 z* $< 1 -$ *GPD-cdf* γ *0 1 y*
⟨*proof*⟩

lemma *GPD-tail-gt-level*:
assumes *x*: $1 < x$ **and** *y*: $0 \leq y$ **and** *lt*: $y < (\textit{if } \gamma = 0 \textit{ then } \ln x \textit{ else } (x \textit{ powr } \gamma - 1) / \gamma)$
shows $1 / x < 1 - \textit{GPD-cdf } \gamma\ 0\ 1\ y$
⟨*proof*⟩

lemma *GPD-tail-lt-level*:
assumes *x*: $1 < x$ **and** *gt*: $(\textit{if } \gamma = 0 \textit{ then } \ln x \textit{ else } (x \textit{ powr } \gamma - 1) / \gamma) < y$
shows $1 - \textit{GPD-cdf } \gamma\ 0\ 1\ y < 1 / x$
⟨*proof*⟩
lemma *endpoint-filter-eventually-at-left*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$
assumes *ev*: *eventually P* (*endpoint-filter F*)
shows $\forall_F\ t \textit{ in endpoint-filter } F.\ \textit{eventually } P\ (\textit{at-left } t)$
⟨*proof*⟩

lemma *excess-uniform-scaled-iff*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$ **and** $\beta :: \textit{real} \Rightarrow \textit{real}$ **and** *FF* :: *real filter*
assumes *βpos*: $\bigwedge u.\ 0 < \beta\ u$
shows *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ \textit{excess-cdf } F\ u\ x - \textit{GPD-cdf } \gamma\ 0\ (\beta\ u)\ x)$ $(\lambda x.\ 0)$ *FF*
$\longleftrightarrow$ *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ \textit{excess-cdf } F\ u\ (\beta\ u * x))$ $(\textit{GPD-cdf } \gamma\ 0\ 1)$ *FF*
⟨*proof*⟩

lemma *excess-uniform-eventually*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$ **and** $\beta :: \textit{real} \Rightarrow \textit{real}$
assumes *F*: *right-continuous-mono F 0 1*
and *ul*: *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ \textit{excess-cdf } F\ u\ (\beta\ u * x))$ $(\textit{GPD-cdf } \gamma\ 0\ 1)$ (*endpoint-filter F*)
and δ: $0 < \delta$
shows $\forall_F\ t \textit{ in endpoint-filter } F.\ F\ t < 1 \wedge (\forall\, x \geq 0.\ |\textit{excess-cdf } F\ t\ (\beta\ t * x) - \textit{GPD-cdf } \gamma\ 0\ 1\ x| < \delta)$
⟨*proof*⟩

lemma *excess-iterate-tail-small*:
fixes $F :: \textit{real} \Rightarrow \textit{real}$ **and** $\beta :: \textit{real} \Rightarrow \textit{real}$
assumes *βpos*: $\bigwedge u.\ 0 < \beta\ u$
and *Y0*: $0 < Y0$ **and** *KY0*: $1 / 2 \leq \textit{GPD-cdf } \gamma\ 0\ 1\ Y0$
and *cT*: $c < T$
and *lt1*: $\bigwedge z.\ c < z \Longrightarrow z < T \Longrightarrow F\ z < 1$
and *near*: $\bigwedge z.\ c < z \Longrightarrow z < T \Longrightarrow |\textit{excess-cdf } F\ z\ (\beta\ z * Y0) - \textit{GPD-cdf } \gamma\ 0\ 1\ Y0| < 1 / 4$
and *step*: $\bigwedge z.\ c < z \Longrightarrow z < T \Longrightarrow z + \beta\ z * Y0 < T$
and δ: $0 < \delta$
shows $\exists z.\ c < z \wedge z < T \wedge 1 - F\ z < \delta$
⟨*proof*⟩

lemma *excess-tail-small-below-x-star*:

fixes $F :: real \Rightarrow real$ **and** $\beta :: real \Rightarrow real$
assumes F: *right-continuous-mono F 0 1* **and** βpos: $\bigwedge u.\ 0 < \beta\ u$
and ul: *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x))$ $(GPD\text{-}cdf\ \gamma\ 0\ 1)$ $(endpoint\text{-}filter\ F)$
and *bdd*: *bdd-support F* **and** δ: $0 < \delta$
shows $\exists z.\ z < x\text{-}star\ F \wedge 1 - F\ z < \delta$
⟨*proof*⟩

lemma *F-tendsto-1-endpoint-of-excess*:
fixes $F :: real \Rightarrow real$ **and** $\beta :: real \Rightarrow real$
assumes F: *right-continuous-mono F 0 1* **and** βpos: $\bigwedge u.\ 0 < \beta\ u$
and ul: *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x))$ $(GPD\text{-}cdf\ \gamma\ 0\ 1)$ $(endpoint\text{-}filter\ F)$
shows $(F \longrightarrow 1)$ $(endpoint\text{-}filter\ F)$
⟨*proof*⟩

lemma *tail-quantile-asymp-inverse-of-excess*:
fixes $F :: real \Rightarrow real$ **and** $\beta :: real \Rightarrow real$
assumes F: *right-continuous-mono F 0 1* **and** βpos: $\bigwedge u.\ 0 < \beta\ u$
and ul: *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x))$ $(GPD\text{-}cdf\ \gamma\ 0\ 1)$ $(endpoint\text{-}filter\ F)$
shows $((\lambda s.\ s * (1 - F\ (tail\text{-}quantile\ F\ s))) \longrightarrow 1)$ *at-top*
⟨*proof*⟩

lemma *tail-quantile-gap-of-excess*:
fixes $F :: real \Rightarrow real$ **and** $\beta :: real \Rightarrow real$
assumes F: *right-continuous-mono F 0 1* **and** βpos: $\bigwedge u.\ 0 < \beta\ u$
and ul: *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ excess\text{-}cdf\ F\ u\ (\beta\ u * x))$ $(GPD\text{-}cdf\ \gamma\ 0\ 1)$ $(endpoint\text{-}filter\ F)$
and x: $1 < x$
shows $((\lambda s.\ (tail\text{-}quantile\ F\ (s * x) - tail\text{-}quantile\ F\ s)\ /\ \beta\ (tail\text{-}quantile\ F\ s))$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))$ *at-top*
⟨*proof*⟩

lemma *GPD-level-mult*:
fixes $\gamma\ y :: real$
assumes y: $0 < y$
shows $(if\ \gamma = 0\ then\ ln\ (2 * y)\ else\ ((2 * y)\ powr\ \gamma - 1)\ /\ \gamma)$
$- (if\ \gamma = 0\ then\ ln\ y\ else\ (y\ powr\ \gamma - 1)\ /\ \gamma)$
$= y\ powr\ \gamma * (if\ \gamma = 0\ then\ ln\ 2\ else\ (2\ powr\ \gamma - 1)\ /\ \gamma)$
⟨*proof*⟩

lemma *GPD-level-inverse*:
fixes $\gamma\ x :: real$
assumes x: $0 < x$
shows $-((if\ \gamma = 0\ then\ ln\ (1\ /\ x)\ else\ ((1\ /\ x)\ powr\ \gamma - 1)\ /\ \gamma) * x\ powr\ \gamma)$
$= (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma)$
⟨*proof*⟩

lemma *tail-quantile-gap-of-excess-all*:
fixes $F :: real \Rightarrow real$ **and** $\beta :: real \Rightarrow real$
assumes *F*: *right-continuous-mono F 0 1* **and** *βpos*: $\bigwedge u.\ 0 < \beta\ u$
and *ul*: *uniform-limit* $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u* $(\beta\ u * x))$ $(GPD\text{-}cdf\ \gamma\ 0\ 1)$
(endpoint-filter F)
and *x*: $0 < x$
shows $((\lambda s.\ (tail\text{-}quantile\ F\ (s * x) - tail\text{-}quantile\ F\ s) / \beta\ (tail\text{-}quantile\ F\ s))$
$\longrightarrow$ *(if* $\gamma = 0$ *then ln x else* $(x\ powr\ \gamma - 1) / \gamma))$ *at-top*
⟨*proof*⟩

theorem *Pickands-Balkema-de-Haan-converse*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *hyp*: $\exists \beta.\ (\forall u.\ 0 < \beta\ u) \wedge$
uniform-limit $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u x* $-$ $GPD\text{-}cdf\ \gamma\ 0\ (\beta\ u)\ x)$
$(\lambda x.\ 0)$ *(endpoint-filter F)*
shows *domain-of-attraction F (EVD-cdf* $\gamma\ 0\ 1)$
⟨*proof*⟩

theorem *Pickands-Balkema-de-Haan*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
shows *domain-of-attraction F (EVD-cdf* $\gamma\ 0\ 1)$
$\longleftrightarrow$ $(\exists \beta.\ (\forall u.\ 0 < \beta\ u) \wedge$
uniform-limit $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u x* $-$ $GPD\text{-}cdf\ \gamma\ 0\ (\beta\ u)\ x)$
$(\lambda x.\ 0)$ *(endpoint-filter F))*
⟨*proof*⟩

4.2.3 Explicit Choice of β

lemma *excess-uniform-beta-cong*:
fixes $F :: real \Rightarrow real$ **and** $\beta\ \beta' :: real \Rightarrow real$ **and** *FF* :: *real filter*
assumes *ul*: *uniform-limit* $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u x* $-$ $GPD\text{-}cdf\ \gamma\ 0\ (\beta\ u)\ x)$ $(\lambda x.\ 0)$ *FF*
and *ev*: $\forall_F\ u$ *in FF*. $\beta'\ u = \beta\ u$
shows *uniform-limit* $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u x* $-$ $GPD\text{-}cdf\ \gamma\ 0\ (\beta'\ u)\ x)$ $(\lambda x.\ 0)$ *FF*
⟨*proof*⟩

Case: $\gamma > 0$

theorem *Pickands-Balkema-de-Haan-pos*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1* **and** γ: $0 < \gamma$
and *dom*: *domain-of-attraction F (EVD-cdf* $\gamma\ 0\ 1)$
shows *uniform-limit* $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u x* $-$ $GPD\text{-}cdf\ \gamma\ 0\ (\gamma * u)\ x)$ $(\lambda x.\ 0)$
(endpoint-filter F)
⟨*proof*⟩

Case: $\gamma < 0$

theorem *Pickands-Balkema-de-Haan-neg*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes F: *right-continuous-mono F 0 1* **and** γ: $\gamma < 0$
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
shows *uniform-limit* $\{0..\}$ $(\lambda u\ x.$ *excess-cdf F u x* $-$ *GPD-cdf* γ *0* $(-\gamma * ($*x-star F* $- u)) x)$
$(\lambda x.\ 0)$ (*endpoint-filter F*)
⟨*proof*⟩

Case: arbitrary γ

lemma *isCont-GPD-level*:
fixes $\gamma\ y ::$ *real*
assumes y: $0 < y$
shows *isCont* $(\lambda z.$ *if* $\gamma = 0$ *then ln z else* $(z$ *powr* $\gamma - 1) / \gamma) y$
⟨*proof*⟩

lemma *tail-quantile-witness-of-gap*:
fixes $F\ a ::$ *real* $\Rightarrow$ *real* **and** $\gamma ::$ *real*
assumes F: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$
and *quant*: $\bigwedge y.\ 0 < y \Longrightarrow ((\lambda t.$ (*tail-quantile F* $(t * y)$ $-$ *tail-quantile F t*) / *a t*)
$\longrightarrow$ (*if* $\gamma = 0$ *then ln y else* $(y$ *powr* $\gamma - 1) / \gamma)$) *at-top*
shows $(\lambda n.$ *F* $(a$ (*real n*) $* x +$ *tail-quantile F* (*real n*)) $\hat{}\ n) \longrightarrow$ *EVD-cdf* γ *0 1 x*
⟨*proof*⟩

lemma *t-div-a-tendsto-zero-endpoint*:
fixes $F\ a ::$ *real* $\Rightarrow$ *real* **and** $\gamma ::$ *real*
assumes F: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$
and *quant*: $\bigwedge y.\ 0 < y \Longrightarrow ((\lambda t.$ (*tail-quantile F* $(t * y)$ $-$ *tail-quantile F t*) / *a t*)
$\longrightarrow$ (*if* $\gamma = 0$ *then ln y else* $(y$ *powr* $\gamma - 1) / \gamma)$) *at-top*
shows $((\lambda t.$ $(t -$ *tail-quantile F* $(1 / (1 -$ *F t*$)))) / a\ (1 / (1 -$ *F t*$))) \longrightarrow 0)$
(*endpoint-filter F*)
⟨*proof*⟩

lemma *divide-tail-of-gap*:
fixes $F\ a ::$ *real* $\Rightarrow$ *real* **and** $\gamma\ x ::$ *real*
assumes F: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$
and *quant*: $\bigwedge y.\ 0 < y \Longrightarrow ((\lambda t.$ (*tail-quantile F* $(t * y)$ $-$ *tail-quantile F t*) / *a t*)
$\longrightarrow$ (*if* $\gamma = 0$ *then ln y else* $(y$ *powr* $\gamma - 1) / \gamma)$) *at-top*
and x: $0 < 1 + \gamma * x$
shows $((\lambda t.$ $(1 -$ *F* $(t + x *$ *a* $(1 / (1 -$ *F t*$))))) / (1 -$ *F t*$))$
$\longrightarrow -$ *ln* (*EVD-cdf* γ *0 1 x*)) (*endpoint-filter F*)
⟨*proof*⟩

theorem *Pickands-Balkema-de-Haan-normalization*:
fixes $F\ a :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *apos*: $\bigwedge t.\ 0 < a\ t$
and *quant*: $\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a\ t)$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))\ at\text{-}top$
shows *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ excess\text{-}cdf\ F\ u\ x - GPD\text{-}cdf\ \gamma\ 0\ (a\ (1\ /\ (1 - F\ u)))\ x)$
$(\lambda x.\ 0)$ $(endpoint\text{-}filter\ F)$
⟨*proof*⟩

corollary *Pickands-Balkema-de-Haan-normalization′*:
fixes $F :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *F*: *right-continuous-mono F 0 1*
and *dom*: *domain-of-attraction F* $(EVD\text{-}cdf\ \gamma\ 0\ 1)$
obtains $a :: real \Rightarrow real$
where $\bigwedge t.\ 0 < a\ t$
and $\bigwedge x.\ 0 < x \Longrightarrow ((\lambda t.\ (tail\text{-}quantile\ F\ (t * x) - tail\text{-}quantile\ F\ t)\ /\ a\ t)$
$\longrightarrow (if\ \gamma = 0\ then\ ln\ x\ else\ (x\ powr\ \gamma - 1)\ /\ \gamma))\ at\text{-}top$
and $\bigwedge x.\ (\lambda n.\ F\ (a\ (real\ n) * x + tail\text{-}quantile\ F\ (real\ n))\ \hat{}\ n) \longrightarrow EVD\text{-}cdf\ \gamma\ 0\ 1\ x$
and *uniform-limit* $\{0..\}$ $(\lambda u\ x.\ excess\text{-}cdf\ F\ u\ x - GPD\text{-}cdf\ \gamma\ 0\ (a\ (1\ /\ (1 - F\ u)))\ x)$
$(\lambda x.\ 0)$ $(endpoint\text{-}filter\ F)$
⟨*proof*⟩

end

5 Concrete Examples

theory *Examples*
imports
Peaks-Over-Threshold
begin

5.1 Maximum of Random Variables

In this subsection, we state results using i.i.d. random variables.

context *prob-space*
begin

lemma *sample-max-measurable*[*measurable*]:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes *indep-vars* $(\lambda\text{-}::nat.\ borel)\ X\ UNIV$
shows $(\lambda w.\ MAX\ i{\in}\{..{<}n::nat\}.\ X\ i\ w) \in borel\text{-}measurable\ M$
⟨*proof*⟩

lemma *prob-sample-max-le*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes *indep*: *indep-vars* (λ-. *borel*) X *UNIV*
and *cdf*: $\bigwedge i.$ *cdf* (*distr M borel* (X i)) = F
and n: $0 < n$
shows *prob* $\{w \in$ *space* M. (*MAX* $i \in \{..<n::nat\}$. X i $w) \leq x\} = F$ x ^ n
⟨*proof*⟩

corollary *cdf-distr-sample-max*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$
assumes *indep*: *indep-vars* (λ-. *borel*) X *UNIV*
and *cdf*: $\bigwedge i.$ *cdf* (*distr M borel* (X i)) = F
and n: $0 < n$
shows *cdf* (*distr M borel* ($\lambda w.$ (*MAX* $i \in \{..<n::nat\}$. X i w))) x = F x ^ n
⟨*proof*⟩

corollary *cdf-distr-sample-max-normalized*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$ **and** a $b :: nat \Rightarrow real$
assumes *indep*: *indep-vars* (λ-. *borel*) X *UNIV*
and *cdf*: $\bigwedge i.$ *cdf* (*distr M borel* (X i)) = F
and a: $\bigwedge n.$ $0 < a$ n
and n: $0 < n$
shows *cdf* (*distr M borel* ($\lambda w.$ ((*MAX* $i \in \{..<n::nat\}$. X i w) − b n) / a n)) x
= F (a n * x + b n) ^ n
(**is** *?lhs* = -)
⟨*proof*⟩

theorem *Fisher-Tippett-Gnedenko-sample-max*:
fixes $X :: nat \Rightarrow 'a \Rightarrow real$ **and** a $b :: nat \Rightarrow real$
assumes *indep*: *indep-vars* (λ-. *borel*) X *UNIV*
and *cdf*: $\bigwedge i.$ *cdf* (*distr M borel* (X i)) = F
and N: *real-distribution* N **and** *deg*: ¬ *degenerate-df* (*cdf* N)
and a: $\bigwedge n.$ $0 < a$ n
and *wcm*: *weak-conv-m* ($\lambda n.$ *distr M borel* ($\lambda w.$ ((*MAX* $i \in \{..<n::nat\}$. X i w) − b n) / a n)) N
obtains γ μ σ **where** $0 < \sigma$ **and** N = *density lborel* (*EVD-density* γ μ σ)
⟨*proof*⟩

end

5.2 Excesses of a Random Variable over a High Threshold

context *prob-space*
begin

lemma *prob-gt-pos-of-lt-x-star*:
fixes $X :: 'a \Rightarrow real$
assumes X[*measurable*]: *random-variable borel* X

and *cdf*: *cdf* (*distr M borel X*) = *F*
and *u*: *u* < *x-star F*
shows *0* < *prob* {*w* ∈ *space M*. *u* < *X w*}
⟨*proof*⟩

lemma *AE-le-x-star*:
fixes *X* :: *'a* ⇒ *real*
assumes *X*[*measurable*]: *random-variable borel X*
and *cdf*: *cdf* (*distr M borel X*) = *F*
and *bdd*: *bdd-support F*
shows *AE w in M*. *X w* ≤ *x-star F*
⟨*proof*⟩

lemma *cond-prob-excess-cdf*:
fixes *X* :: *'a* ⇒ *real*
assumes *X*[*measurable*]: *random-variable borel X*
and *cdf*: *cdf* (*distr M borel X*) = *F*
and *u*: *F u* < *1* **and** *x*:*0* ≤ *x*
shows *P*(*w in M*. *X w* − *u* ≤ *x* | *u* < *X w*) = *excess-cdf F u x*
⟨*proof*⟩

lemma *cdf-distr-excess*:
fixes *X* :: *'a* ⇒ *real*
assumes *X*[*measurable*]: *random-variable borel X*
and *cdf*: *cdf* (*distr M borel X*) = *F*
and *u*: *F u* < *1* **and** *x*:*0* ≤ *x*
shows *cdf* (*distr* (*uniform-measure M* {*w* ∈ *space M*. *u* < *X w*}) *borel* (λ*w*. *X w* − *u*)) *x*
= *excess-cdf F u x* (**is** *?lhs* = -)
⟨*proof*⟩

theorem *Pickands-Balkema-de-Haan-cond-prob*:
fixes *X* :: *'a* ⇒ *real* **and** *F* :: *real* ⇒ *real* **and** γ :: *real*
assumes *X*[*measurable*]: *random-variable borel X*
and *cdf*: *cdf* (*distr M borel X*) = *F*
and *dom*: *domain-of-attraction F* (*EVD-cdf* γ *0 1*)
obtains β :: *real* ⇒ *real*
where ⋀*u*. *0* < β *u*
and *uniform-limit* {*0*..}
(λ*u x*. *P*(*w in M*. *X w* − *u* ≤ *x* | *u* < *X w*) − *GPD-cdf* γ *0* (β *u*) *x*)
(λ*x*. *0*) (*endpoint-filter F*)
⟨*proof*⟩

end

lemma *excess-cdf-GPD*:
fixes γ σ *u x* :: *real*
assumes *sigma*: *0* < σ **and** *u*: *0* ≤ *u* **and** *supp*: *0* < σ + γ ∗ *u* **and** *x*: *0* ≤ *x*
shows *excess-cdf* (*GPD-cdf* γ *0* σ) *u x* = *GPD-cdf* γ *0* (σ + γ ∗ *u*) *x*

⟨*proof*⟩

corollary *excess-cdf-exponential*:
fixes *u x* :: *real*
assumes *0* ≤ *u* **and** *0* ≤ *x*
shows *excess-cdf* (*GPD-cdf 0 0 1*) *u x* = *GPD-cdf 0 0 1 x*
⟨*proof*⟩

corollary *excess-cdf-Pareto1*:
fixes *u x* :: *real*
assumes *u*: *1* ≤ *u* **and** *x*: *0* ≤ *x*
shows *excess-cdf* (*GPD-cdf 1 1 1*) *u x* = *GPD-cdf 1 0 u x*
⟨*proof*⟩

corollary *excess-cdf-uniform*:
fixes *u x* :: *real*
assumes *0* ≤ *u* **and** *u* < *1* **and** *0* ≤ *x*
shows *excess-cdf* (*GPD-cdf* (− *1*) *0 1*) *u x* = *GPD-cdf* (− *1*) *0* (*1* − *u*) *x*
⟨*proof*⟩

lemma *excess-cdf-EVD-tendsto-GPD*:
fixes *γ* :: *real*
obtains *β* :: *real* ⇒ *real*
where ⋀*u*. *0* < *β u*
and *uniform-limit* {*0*..}
(λ*u x*. *excess-cdf* (*EVD-cdf γ 0 1*) *u x* − *GPD-cdf γ 0* (*β u*) *x*)
(λ*x*. *0*) (*endpoint-filter* (*EVD-cdf γ 0 1*))
⟨*proof*⟩

corollary(**in** *prob-space*) *cond-prob-excess-GPD*:
fixes *X* :: *'a* ⇒ *real* **and** *γ σ u x* :: *real*
assumes *X*: *random-variable borel X*
and *X-cdf*: *cdf* (*distr M borel X*) = *GPD-cdf γ 0 σ*
and *sigma*: *0* < *σ* **and** *u*: *0* ≤ *u* **and** *supp*: *0* < *σ* + *γ* ∗ *u* **and** *x*: *0* ≤ *x*
shows *P*(*w in M*. *X w* − *u* ≤ *x* | *u* < *X w*) = *GPD-cdf γ 0* (*σ* + *γ* ∗ *u*) *x*
⟨*proof*⟩

5.3 Domain of Attraction

In this subsection we formalize concrete distributions, and determine the extreme value index γ of the extreme value distribution they are attracted to.

The results cover the following distributions, where $\mathcal{D}(G_\gamma)$ denotes (*in-domain-attraction-of*) for *EVD-cdf γ 0 1*.

- Normal distribution: $F_{\mathrm{N}} \in \mathcal{D}(G_0)$.
- Erlang distribution: $F_{\mathrm{E}} \in \mathcal{D}(G_0)$.

- Uniform distribution on an interval: $F_{\mathrm{U}} \in \mathcal{D}(G_{-1})$.
- Cauchy distribution: $F_{\mathrm{C}} \in \mathcal{D}(G_1)$.

5.3.1 Normal Distribution

lemma *real-distribution-normal*:
$0 < \sigma \Longrightarrow$ *real-distribution* (*density lborel* (*normal-density* μ σ))
⟨*proof*⟩

lemma *normal-density-affine-eq*:
assumes *sigma*: $0 < \sigma$
shows *normal-density* μ σ $(\mu + \sigma * y)$ = *std-normal-density* y / σ
⟨*proof*⟩

lemma *std-normal-tail-measure*:
$1 -$ *cdf std-normal-distribution* t
= *enn2real* ($\int^+ y \in \{t..\}$. *ennreal* (*std-normal-density* y) ∂*lborel*)
⟨*proof*⟩

lemma *DERIV-std-normal-density*:
DERIV std-normal-density y :> $-$ y $*$ *std-normal-density* y
⟨*proof*⟩

lemma *DERIV-mills-upper*:
fixes y :: *real*
assumes y: $0 < y$
shows *DERIV* ($\lambda z.$ $-$ (*std-normal-density* z / z)) y :> *std-normal-density* y $*$ $(1 + 1 / y^2)$
⟨*proof*⟩

lemma *DERIV-mills-lower*:
fixes y :: *real*
assumes y: $0 < y$
shows *DERIV* ($\lambda z.$ *std-normal-density* z / z ^ 3 $-$ *std-normal-density* z / z) y
:> *std-normal-density* y $*$ $(1 - 3 / y$ ^ $4)$
⟨*proof*⟩

lemma *tendsto-mills-upper*: $((\lambda z$::*real*. $-$ (*std-normal-density* z / z)) $\longrightarrow$ $0)$ *at-top*
⟨*proof*⟩

lemma *tendsto-mills-lower*:
($(\lambda z.$ *std-normal-density* z / z ^ 3 $-$ *std-normal-density* z / z) $\longrightarrow$ 0) *at-top*
⟨*proof*⟩

lemma *std-normal-nn-integral-upper*:
fixes t :: *real*
assumes t: $0 < t$

shows $(\int^+ y \in \{t..\}.\ ennreal\ (std\text{-}normal\text{-}density\ y * (1 + 1 / y^2))\ \partial lborel)$
$= ennreal\ (std\text{-}normal\text{-}density\ t / t)$
⟨*proof*⟩

lemma *std-normal-nn-integral-lower*:
fixes $t :: real$
assumes t: $2 \leq t$
shows $(\int^+ y \in \{t..\}.\ ennreal\ (std\text{-}normal\text{-}density\ y * (1 - 3 / y \hat{} 4))\ \partial lborel)$
$= ennreal\ (std\text{-}normal\text{-}density\ t / t - std\text{-}normal\text{-}density\ t / t \hat{} 3)$
⟨*proof*⟩

lemma *std-normal-tail-bounds*:
fixes $t :: real$
assumes t: $2 \leq t$
shows *std-normal-tail-upper*: $1 - cdf\ std\text{-}normal\text{-}distribution\ t \leq std\text{-}normal\text{-}density\ t / t$
and *std-normal-tail-lower*:
$std\text{-}normal\text{-}density\ t / t - std\text{-}normal\text{-}density\ t / t \hat{} 3$
$\leq 1 - cdf\ std\text{-}normal\text{-}distribution\ t$
⟨*proof*⟩

lemma *std-normal-tail-asymp-equiv*:
$(\lambda t.\ 1 - cdf\ std\text{-}normal\text{-}distribution\ t) \sim[at\text{-}top]\ (\lambda t.\ std\text{-}normal\text{-}density\ t / t)$
⟨*proof*⟩

lemma *std-normal-density-at-normalization*:
fixes $x\ t :: real$
assumes t: $exp\ 1 < t$
defines $c \equiv sqrt\ (2 * ln\ t)$
and $w \equiv x - (ln\ (ln\ t) + ln\ (4 * pi)) / 2$
shows $std\text{-}normal\text{-}density\ (c + w / c)$
$= exp\ (-x) * c * exp\ (-(w^2 / (4 * ln\ t))) / t$
⟨*proof*⟩

lemma *normal-cdf-location-scale*:
assumes $sigma[arith]$: $0 < \sigma$
shows $cdf\ (density\ lborel\ (normal\text{-}density\ \mu\ \sigma))\ x = cdf\ std\text{-}normal\text{-}distribution\ ((x - \mu) / \sigma)$
⟨*proof*⟩

theorem *std-normal-MDA*: $domain\text{-}of\text{-}attraction\ (cdf\ std\text{-}normal\text{-}distribution)\ (EVD\text{-}cdf\ 0\ 0\ 1)$
⟨*proof*⟩

corollary *normal-MDA*:
assumes $0 < \sigma$
shows $domain\text{-}of\text{-}attraction\ (cdf\ (density\ lborel\ (normal\text{-}density\ \mu\ \sigma)))\ (EVD\text{-}cdf\ 0\ 0\ 1)$
⟨*proof*⟩

theorem *std-normal-normalization*:
fixes $x :: real$ **and** $a\ b :: nat \Rightarrow real$
defines $a \equiv \lambda n.\ 1\ /\ sqrt\ (2 * ln\ (real\ n))$
and $b \equiv \lambda n.\ sqrt\ (2 * ln\ (real\ n))$
$- (ln\ (ln\ (real\ n)) + ln\ (4 * pi))\ /\ (2 * sqrt\ (2 * ln\ (real\ n)))$
shows $(\lambda n.\ cdf\ std\text{-}normal\text{-}distribution\ (a\ n * x + b\ n)\ \hat{}\ n) \longrightarrow EVD\text{-}cdf\ 0\ 0\ 1\ x$
⟨*proof*⟩

5.3.2 Erlang Distribution

lemma *cdf-erlang-density*:
assumes $0 < l$
shows $cdf\ (density\ lborel\ (erlang\text{-}density\ k\ l)) = erlang\text{-}CDF\ k\ l$
⟨*proof*⟩

corollary *real-distribution-erlang*:
$0 < l \Longrightarrow real\text{-}distribution\ (density\ lborel\ (erlang\text{-}density\ k\ l))$
⟨*proof*⟩

lemma *right-continuous-mono-erlang-CDF*:
assumes *l*: $0 < l$
shows $right\text{-}continuous\text{-}mono\ (erlang\text{-}CDF\ k\ l)\ 0\ 1$
⟨*proof*⟩

lemma *erlang-CDF-lt-1*:
assumes $l[arith]$: $0 < l$
shows $erlang\text{-}CDF\ k\ l\ t < 1$
⟨*proof*⟩

lemma *erlang-density-pos*:
assumes $0 < l$ **and** $0 < x$
shows $0 < erlang\text{-}density\ k\ l\ x$
⟨*proof*⟩

lemma *DERIV-erlang-sum*:
fixes $x :: real$
shows $((\lambda y.\ -\ (\sum n{\le}k.\ y\ \hat{}\ n * exp\ (-\ y)\ /\ fact\ n))\ has\text{-}real\text{-}derivative$
$x\ \hat{}\ k * exp\ (-\ x)\ /\ fact\ k)\ (at\ x)$
⟨*proof*⟩

lemma *DERIV-erlang-CDF-1*:
fixes $t :: real$
assumes *t*: $0 < t$
shows $(erlang\text{-}CDF\ k\ 1\ has\text{-}real\text{-}derivative\ erlang\text{-}density\ k\ 1\ t)\ (at\ t)$
⟨*proof*⟩

lemma *DERIV-erlang-CDF*:

fixes t :: *real*
assumes l: $0 < l$ **and** t: $0 < t$
shows (*erlang-CDF k l has-real-derivative erlang-density k l t*) (*at t*)
⟨*proof*⟩

lemma *DERIV-erlang-density*:
fixes t :: *real*
assumes t: $0 < t$
shows (*erlang-density k l has-real-derivative*
erlang-density k l t ∗ (*real k* / *t* − *l*)) (*at t*)
⟨*proof*⟩

lemma *continuous-on-erlang-density*: *continuous-on* {*0*<..} (*erlang-density k l*)
⟨*proof*⟩

lemma *erlang-tail-ratio*:
fixes t :: *real*
assumes l: $0 < l$ **and** t: $0 < t$
shows (*1* − *erlang-CDF k l t*) / ((*l* ∗ *t*) ^ *k* ∗ *exp* (− *l* ∗ *t*) / *fact k*)
= (∑ *n*≤*k*. *fact k* / *fact n* / (*l* ∗ *t*) ^ (*k* − *n*))
⟨*proof*⟩

lemma *erlang-tail-asymp-equiv*:
assumes *l*[*arith*]: *0* < *l*
shows (λ*t*. *1* − *erlang-CDF k l t*) ∼[*at-top*] (λ*t*. (*l* ∗ *t*) ^ *k* ∗ *exp* (− *l* ∗ *t*) / *fact k*)
⟨*proof*⟩

lemma *erlang-von-Mises-ratio*:
fixes t :: *real*
assumes l: $0 < l$ **and** t: $0 < t$
shows ((*l* ∗ *t*) ^ *k* ∗ *exp* (− *l* ∗ *t*) / *fact k*)
∗ (*erlang-density k l t* ∗ (*real k* / *t* − *l*)) / (*erlang-density k l t*)^2
= (*real k* / *t* − *l*) / *l*
⟨*proof*⟩

The Erlang distribution is our example of von Mises' condition. Note that the characterization by regular variation is available only for $\gamma > 0$, so for $\gamma = 0$ the condition on the density is the natural tool.

theorem *erlang-MDA*:
assumes l: $0 < l$
shows *domain-of-attraction* (*erlang-CDF k l*) (*EVD-cdf 0 0 1*)
⟨*proof*⟩

theorem *erlang-normalization*:
fixes x :: *real*
assumes *l*[*arith*]: *0* < *l*
shows (λ*n*. *erlang-CDF k l*
((*x* + *ln* (*real n*) + *real k* ∗ *ln* (*ln* (*real n*)) − *ln* (*fact k*)) / *l*) ^ *n*)

$\longrightarrow$ *EVD-cdf 0 0 1 x*
⟨*proof*⟩

5.3.3 Uniform Distribution

lemma *uniform-cdf-eq-GPD*:
fixes *a b* :: *real*
assumes [*arith*]:*a* < *b*
shows *cdf* (*uniform-measure lborel* {*a*..*b*}) = *GPD-cdf* (− *1*) *a* (*b* − *a*)
⟨*proof*⟩

theorem *uniform-MDA*:
fixes *a b* :: *real*
assumes *a* < *b*
shows *domain-of-attraction* (*cdf* (*uniform-measure lborel* {*a*..*b*})) (*EVD-cdf* (− *1*) *0 1*)
⟨*proof*⟩

5.3.4 Cauchy Distribution

definition *Cauchy-density* :: *real* ⇒ *real* ⇒ *real* ⇒ *real* **where**
Cauchy-density μ σ *x* ≡ σ / (*pi* ∗ (σ^2 + (*x* − μ)^2))

abbreviation *std-Cauchy-density* :: *real* ⇒ *real* **where**
std-Cauchy-density ≡ *Cauchy-density 0 1*

abbreviation *std-Cauchy-distribution* :: *real measure* **where**
std-Cauchy-distribution ≡ *density lborel std-Cauchy-density*

lemma *std-Cauchy-density-def*: *std-Cauchy-density x* = *1* / (*pi* ∗ (*1* + x^2))
⟨*proof*⟩

lemma *Cauchy-density-nonneg*[*simp*]: *0* < σ ⟹ *0* ≤ *Cauchy-density* μ σ *x*
⟨*proof*⟩

lemma *Cauchy-density-measurable*[*measurable*]:
Cauchy-density μ σ ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *Cauchy-density-uminus*: *Cauchy-density* μ σ (− *x*) = *Cauchy-density* (− μ) σ *x*
⟨*proof*⟩

lemma *arctan-div-pi-bounds*:
shows *arctan-div-pi-pos*: *0* < *1* / *2* + *arctan z* / *pi*
and *arctan-div-pi-lt-1*: *1* / *2* + *arctan z* / *pi* < *1*
⟨*proof*⟩

lemma *DERIV-Cauchy-cdf*:
assumes *sigma*[*arith*]: *0* < σ

shows *DERIV* (λ*y*. *arctan* ((*y* − *μ*) / *σ*) / *pi*) *x* :> *Cauchy-density μ σ x*
⟨*proof*⟩

lemma *Cauchy-nn-integral-atLeast*:
assumes *sigma*[*arith*]: *0* < *σ*
shows (∫⁺*y*∈{*x*..}. *Cauchy-density μ σ y* ∂*lborel*)
= *ennreal* (*1* / *2* − *arctan* ((*x* − *μ*) / *σ*) / *pi*)
⟨*proof*⟩

lemma *Cauchy-nn-integral-atMost*:
assumes *sigma*[*arith*]: *0* < *σ*
shows (∫⁺*y*∈{..*x*}. *Cauchy-density μ σ y* ∂*lborel*)
= *ennreal* (*1* / *2* + *arctan* ((*x* − *μ*) / *σ*) / *pi*)
⟨*proof*⟩

lemma *prob-space-Cauchy-density*:
assumes *sigma*[*arith*]: *0* < *σ*
shows *prob-space* (*density lborel* (*Cauchy-density μ σ*))
⟨*proof*⟩

corollary *real-distribution-Cauchy*:
0 < *σ* ⟹ *real-distribution* (*density lborel* (*Cauchy-density μ σ*))
⟨*proof*⟩

lemma *Cauchy-cdf-def*:
assumes *sigma*[*arith*]: *0* < *σ*
shows *cdf* (*density lborel* (*Cauchy-density μ σ*)) *x* = *1* / *2* + *arctan* ((*x* − *μ*) / *σ*) / *pi*
⟨*proof*⟩

lemma *std-Cauchy-cdf-def*: *cdf std-Cauchy-distribution x* = *1* / *2* + *arctan x* / *pi*
⟨*proof*⟩

lemma *Cauchy-cdf-location-scale*:
assumes *sigma*: *0* < *σ*
shows *cdf* (*density lborel* (*Cauchy-density μ σ*)) *x*
= *cdf std-Cauchy-distribution* ((*x* − *μ*) / *σ*)
⟨*proof*⟩

lemma *std-Cauchy-tail-asymp-equiv*:
(λ*t*. *1* − *cdf std-Cauchy-distribution t*) ∼[*at-top*] (λ*t*. *1* / (*pi* ∗ *t*))
⟨*proof*⟩

theorem *std-Cauchy-MDA*: *domain-of-attraction* (*cdf std-Cauchy-distribution*) (*EVD-cdf 1 0 1*)
⟨*proof*⟩

corollary *Cauchy-MDA*:

assumes $0 < \sigma$
shows *domain-of-attraction* (*cdf* (*density lborel* (*Cauchy-density* μ σ))) (*EVD-cdf 1 0 1*)
⟨*proof*⟩

theorem *std-Cauchy-normalization*:
fixes x :: *real*
shows (λn. *cdf std-Cauchy-distribution* (*real n / pi* $*$ x) ^ n) ⟶ *EVD-cdf 1 1 1 x*
⟨*proof*⟩

end

References

[1] A. A. Balkema and L. de Haan. Residual life time at great age. *The Annals of Probability*, 2(5):792–804, 1974.

[2] L. de Haan and A. Ferreira. *Extreme Value Theory: An Introduction*. Springer Series in Operations Research and Financial Engineering. Springer, 2006.

[3] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling Extremal Events for Insurance and Finance*, volume 33 of *Stochastic Modelling and Applied Probability*. Springer, 1997.

[4] R. A. Fisher and L. H. C. Tippett. Limiting forms of the frequency distribution of the largest or smallest member of a sample. *Mathematical Proceedings of the Cambridge Philosophical Society*, 24(2):180–190, 1928.

[5] B. Gnedenko. Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics*, 44(3):423–453, 1943.

[6] J. Pickands, III. Statistical inference using extreme order statistics. *The Annals of Statistics*, 3(1):119–131, 1975.