

The Exponential and Logarithmic Integral

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Abstract

This entry provides three closely related special functions: the exponential integral Ei , the logarithmic integral li , and the complementary exponential integral Ein . All three functions are defined both on the reals and in the complex plane.

Basic properties are shown, e.g.:

- derivatives, continuity, limits, etc.
- $Ei(x) = \int_{-\infty}^x e^t/t dt$ and $li(x) = \int_0^x 1/\ln t dt$ for real x , where for $x > 0$ and $x > 1$, respectively, the integrals must be interpreted as Cauchy principal values
- asymptotic expansions for Ei and li
- the relationship to the incomplete gamma function via $\Gamma(0, x) = -Ei(-x)$ and $\Gamma(0, z) = Ein(z) - \ln z - \gamma$

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1 Exponential and logarithmic integrals

theory Exp_Log_Integral

imports

"HOL-Complex_Analysis.Complex_Analysis"

"Zeta_Function.Hadjicostas_Chapman"

"Incomplete_Gamma.More_Dominated_Convergence"

"Generalized_Hypergeometric_Series.Generalized_Hypergeometric_Series"

"Incomplete_Gamma.Incomplete_Gamma"

begin

1.1 Auxiliary material

lemmas [simp del] = div_add div_diff div_mult_self1 div_mult_self2 div_mult_self3
div_mult_self4

lemma integral_bigo:

fixes f g g' :: "real \Rightarrow real"

assumes "f $\in O(g')$ " and "filterlim g at_top at_top"

assumes " $\bigwedge a' x. a \leq a' \implies a' \leq x \implies f$ absolutely_integrable_on
{a'..x}"

assumes deriv: " $\bigwedge x. x \geq a \implies (g$ has_field_derivative g' x) (at x
within {a..})"

assumes cont: "continuous_on {a..} g'"

assumes nonneg: "eventually ($\lambda x. g' x \geq 0$) at_top"

shows " $(\lambda x. \text{integral } \{a..x\} f) \in O(g)$ "

<proof>

lemma real_nonpos_Reals_eq [simp]: " $\mathbb{R}_{\leq 0} = \{x::\text{real}. x \leq 0\}$ "

<proof>

lemma interval_lebesgue_integrable_cong:

assumes "M = M'" "a = a'" "b = b'" " $\bigwedge x. \min a b < \text{ereal } x \implies \text{ereal}$
x < max a b $\implies f x = f' x$ "

shows "interval_lebesgue_integrable M a b f \longleftrightarrow interval_lebesgue_integrable
M' a' b' f'"

<proof>

lemma set_integrable_complex_of_real_iff:

"set_integrable lborel A ($\lambda x. \text{complex_of_real } (f x)$) \longleftrightarrow set_integrable
lborel A f"

<proof>

lemma interval_lebesgue_integrable_complex_of_real_iff:

"interval_lebesgue_integrable lborel a b ($\lambda x. \text{complex_of_real } (f x)$)
 \longleftrightarrow

interval_lebesgue_integrable lborel a b f"

```

⟨proof⟩

lemma filterlim_abs_real_at_bot:
  "filterlim (abs :: real ⇒ real) at_top at_bot"
  ⟨proof⟩

lemmas filterlim_abs_real' [tendsto_intros] =
  filterlim_abs_real [THEN filterlim_compose]
lemmas filterlim_abs_real_at_bot' [tendsto_intros] =
  filterlim_abs_real_at_bot [THEN filterlim_compose]

lemma filterlim_abs_real_at_infinity:
  assumes "filterlim f at_infinity F"
  shows "filterlim (λx. abs (f x :: real)) at_top F"
  ⟨proof⟩

lemma has_field_derivative_abs:
  assumes "(x::real) ≠ 0"
  shows "(abs has_field_derivative sgn x) (at x within A)"
  ⟨proof⟩

lemmas has_field_derivative_abs' [derivative_intros] =
  has_field_derivative_abs [THEN DERIV_chain2]

lemma Ln_not_in_nonpos_Reals:
  assumes "Im x ≠ 0 ∨ Re x > 1"
  shows "Ln x ∉ ℝ≤0"
  ⟨proof⟩

lemma analytic_on_Ln [analytic_intros]:
  assumes "S ∩ ℝ≤0 = {}"
  shows "Ln analytic_on S"
  ⟨proof⟩

lemma analytic_on_Ln' [analytic_intros]:
  "(∧z. z ∈ A ⇒ f z ∉ ℝ≤0) ⇒ f analytic_on A ⇒ (λz. Ln (f z))
  analytic_on A"
  ⟨proof⟩

lemma continuous_Ln [continuous_intros]:
  assumes "continuous (at x within A) f" "f x ∉ ℝ≤0"
  shows "continuous (at x within A) (λx. Ln (f x))"
  ⟨proof⟩

lemma contour_integral_primitive':

```

```

    assumes "\x. x ∈ s ⇒ (f has_field_derivative f' x) (at x within
s)"
    and "valid_path g" "path_image g ⊆ s" "pathfinish g = b" "pathstart
g = a"
    shows "(f' has_contour_integral (f b - f a)) g"
⟨proof⟩

```

1.2 Cauchy principal value for reals

```

lemma Cauchy_principal_value_Icc_realI:
  fixes a l r :: real and S :: "real set"
  assumes "finite S"
  assumes lim: "(λε. F (a + ε) - F (a - ε)) -> 0"
  assumes deriv: "\x. x ∈ {l..r} - {a} - S ⇒ (F has_field_derivative
f x) (at x)"
  assumes cont: "continuous_on ({l..r}-{a}) F"
  assumes lr: "l ≠ a" "r ≠ a" "l ≤ r"
  shows "((λε. integral ({l..r} - {a-ε..a+ε}) f) → F r - F l) (at_right
0)"
⟨proof⟩

```

```

lemma Cauchy_principal_value_Iic_realI:
  fixes a l r :: real and S :: "real set"
  assumes "finite S"
  assumes lim: "(λε. F (a + ε) - F (a - ε)) -> 0"
  assumes deriv: "\x. x ∈ {..r} - {a} - S ⇒ (F has_field_derivative
f x) (at x)"
  assumes cont: "continuous_on ({..r}-{a}) F"
  assumes int: "\x. x < a ⇒ (f has_integral F x) {..x}"
  assumes r: "r ≠ a"
  shows "((λε. integral ({..r} - {a-ε..a+ε}) f) → F r) (at_right
0)"
⟨proof⟩

```

1.3 Integral representations for the Euler–Mascheroni constant γ

```

context
  fixes f g :: "real ⇒ real"
  defines "f ≡ (λu. 1 / (1 - u) + 1 / ln u :: real)"
  defines "g ≡ (λx. 1 / (exp x - 1) - exp (-x) / x)"
begin

```

```

lemma euler_mascheroni_integral_01:
  "f absolutely_integrable_on {0<..<1}"
  "integral {0<..<1} f = euler_mascheroni"
⟨proof⟩

```

```

lemma euler_mascheroni_integral_0_inf:

```

"g absolutely_integrable_on {0<..}" "integral {0<..} g = euler_mascheroni"
 <proof>

end

1.4 The complementary exponential integral Ein

We start by defining the complementary exponential integral Ein, which is an entire function that can be defined as follows:

$$\text{Ein}(z) = \int_0^z \frac{1 - e^{-u}}{u} du$$

We instead define it via the generalised hypergeometric series as $\text{Ein}(z) = z \cdot {}_2F_2(1, 1; 2, 2; -z)$. This has the advantage that we get properties like holomorphicity and continuity for free.

The Ein function will later allow us to define the exponential integral Ei.

As a literature reference for Ein and the other functions that we will define, see §6.2 of the NIST Digital Library of Mathematical Functions [1].

definition Ein :: "'a :: {banach, real_normed_field} ⇒ 'a" where
 "Ein z = z * hypergeo_F [1,1] [2,2] (-z)"

definition Ein_coefs :: "nat ⇒ real" where
 "Ein_coefs n = (if n = 0 then 0 else (-1)^(n+1) / (n * fact n))"

lemma converges_Ein: "(λn. Ein_coefs n *_R z ^ n) sums Ein z"
 <proof>

lemma Ein_0 [simp]: "Ein 0 = (0 :: 'a :: {banach, real_normed_field})"
 <proof>

lemma Ein_of_real [simp]: "Ein (of_real x) = of_real (Ein x)"
 <proof>

lemma has_field_derivative_Ein:
 "(Ein has_field_derivative (if z = 0 then 1 else (1 - exp (- z)) / z))
 (at z within A)"
 (is "?has_deriv A")
 <proof>

lemmas has_field_derivative_Ein' [derivative_intros] =
 has_field_derivative_Ein [THEN DERIV_chain2]

lemma holomorphic_Ein [holomorphic_intros]:
 "f holomorphic_on A ⇒ (λx. Ein (f x)) holomorphic_on A"
 <proof>

lemma analytic_Ein [analytic_intros]: "f analytic_on A ⇒ (λx. Ein (f x)) analytic_on A"

```

    <proof>

lemma continuous_on_Ein [continuous_intros]:
  "continuous_on A f  $\implies$  continuous_on A ( $\lambda x. \text{Ein } (f x) :: 'a :: \{\text{banach}, \text{real\_normed\_field}\}$ )"
  <proof>

lemma tendsto_Ein [tendsto_intros]:
  "(f  $\longrightarrow$  x) F  $\implies$  (( $\lambda x. \text{Ein } (f x)$ )  $\longrightarrow$  Ein x) F"
  <proof>

lemma continuous_Ein [continuous_intros]:
  "continuous (at x within A) f  $\implies$  continuous (at x within A) ( $\lambda x. \text{Ein } (f x)$ )"
  <proof>

lemma Ein_neg_upper_bound:
  assumes "z::real > 0"
  shows "Ein (-z)  $\leq$  1 + (1 - exp z) / z"
  <proof>

lemma Ein_at_bot:
  "filterlim (Ein :: real  $\Rightarrow$  real) at_bot at_bot"
  <proof>

lemmas Ein_at_bot' [tendsto_intros] = Ein_at_bot [THEN filterlim_compose]

theorem has_integral_Ein_nonneg_real:
  assumes "x  $\geq$  0"
  shows "(( $\lambda t::\text{real}. (1 - \exp (-t)) / t$ ) has_integral Ein x) {0..x}"
  <proof>

lemma has_integral_Ein_nonpos_real:
  assumes "x  $\leq$  0"
  shows "(( $\lambda t::\text{real}. (1 - \exp (-t)) / t$ ) has_integral (-Ein x)) {x..0}"
  <proof>

lemma Ein_diff_ge:
  fixes x y :: real
  assumes xy: "x > 0" "x  $\leq$  y"
  shows "(1 - exp (-x)) * (ln y - ln x)  $\leq$  Ein y - Ein x"
  <proof>

lemma Ein_at_top: "filterlim (Ein :: real  $\Rightarrow$  real) at_top at_top"
  <proof>

```

lemmas *Ein_at_top'* [*tendsto_intros*] = *Ein_at_top* [*THEN filterlim_compose*]

theorem *has_contour_integral_Ein*:

"(($\lambda z :: \text{complex. } (1 - \exp (-z)) / z$) *has_contour_integral Ein z*) (*linepath*
0 z)"

<proof>

1.5 The exponential integral Ei

The Ei function is more tricky to define. The basic idea is that we want to have $\text{Ei}(z) = \int_{-\infty}^z e^t/t dt$. However, this definition is problematic: for real $z > 0$, we integrate straight over the pole of the integrand and the integral must be interpreted as a Cauchy principal value. For complex z , the integration path matters and one has to deal with a branch cut (which, in the standard version, lies on the negative real axis..

We therefore define Ei in a slightly more awkward way and derive the integral and Cauchy principal value representations as derived facts later. : In the complex plane, we define

$$\text{Ei}(z) = \gamma + \frac{1}{2}(\ln z - \ln \frac{1}{z}) - \text{Ein}(-z)$$

where γ is the Euler–Mascheroni constant. For real x , this simplifies to:

$$\text{Ei}(x) = \gamma + \ln |x| - \text{Ein}(-x)$$

As we will show, this gives us a function with the desired properties:

- It is analytic away from the non-positive reals with derivative e^z/z
- It vanishes as $z \rightarrow -\infty$ on the real line.
- In particular, for real $x < 0$, we get $\text{Ei}(x) = \int_{-\infty}^x e^t/t dt$.

Note that unlike the NIST DLMF [1], we do not define the E_1 function. If so desired, E_1 can be defined as $E_1(x) = -\text{Ei}(-x)$ for real x and as $E_1(z) = \text{Ein}(z) - \ln z - \gamma$, or one can simply use the incomplete gamma function and define $E_1(z) = \Gamma(0, z)$ (see also the lemmas proving this relationship below).

1.5.1 Definition and basic properties

definition *Ei* :: "'a \Rightarrow 'a :: {*banach, real_normed_field, ln*}" **where**
"*Ei z* = (*if z* = 0 *then* 0 *else* *euler_mascheroni* +
(*if z* $\in \mathbb{R}_{\leq 0}$ *then* $\ln (-z)$ *else* ($\ln z - \ln (1/z)$) / 2) - *Ein*
(-z))"

lemma *Ei_real_def*:

```

  "Ei (x :: real) = (if x = 0 then 0 else euler_mascheroni + ln (abs x)
- Ein (-x))"
  ⟨proof⟩

```

```

lemma Ei_complex_def:
  "Ei (z :: complex) = (if z = 0 then 0 else euler_mascheroni + (ln z
- ln (1/z)) / 2 - Ein (-z))"
  ⟨proof⟩

```

```

lemma Ei_complex_of_real: "Ei (complex_of_real x) = complex_of_real (Ei
x)"
  ⟨proof⟩

```

```

lemma Ei_real_conv_complex: "Ei x = Re (Ei (of_real x))"
  ⟨proof⟩

```

We now calculate the derivative of Ei and show basic facts such as continuity and holomorphicity.

```

lemma has_field_derivative_Ei_real:
  assumes "(x::real) ≠ 0"
  shows "(Ei has_field_derivative (exp x / x)) (at x within A)"
  ⟨proof⟩

```

```

lemma has_field_derivative_Ei_complex:
  assumes "(z :: complex) ∉ ℝ≤0"
  shows "(Ei has_field_derivative (exp z / z)) (at z within A)"
  ⟨proof⟩

```

```

lemmas has_field_derivative_Ei_real' [derivative_intros] =
  has_field_derivative_Ei_real [THEN DERIV_chain2]

```

```

lemmas has_field_derivative_Ei_complex' [derivative_intros] =
  has_field_derivative_Ei_complex [THEN DERIV_chain2]

```

```

lemma holomorphic_Ei_complex [holomorphic_intros]:
  assumes "f holomorphic_on A" "∧z. z ∈ A ⇒ f z ∉ ℝ≤0"
  shows "(λx. Ei (f x)) holomorphic_on A"
  ⟨proof⟩

```

```

lemma analytic_Ei_complex [analytic_intros]:
  assumes "f analytic_on A" "∧z. z ∈ A ⇒ f z ∉ ℝ≤0"
  shows "(λx. Ei (f x)) analytic_on A"
  ⟨proof⟩

```

```

lemma continuous_on_Ei_complex [continuous_intros]:
  assumes "continuous_on A f" "∧z. z ∈ A ⇒ f z ∉ ℝ≤0"
  shows "continuous_on A (λx. Ei (f x :: complex))"
  ⟨proof⟩

```

```

lemma continuous_Ei_complex [continuous_intros]:
  fixes f :: "'a :: t2_space  $\Rightarrow$  complex"
  assumes "continuous (at x within A) f" "f x  $\notin \mathbb{R}_{\leq 0}$ "
  shows "continuous (at x within A) ( $\lambda x. Ei (f x)$ )"
  <proof>

```

```

lemma continuous_on_Ei_real [continuous_intros]:
  assumes "continuous_on A f" " $\bigwedge z. z \in A \implies f z \neq 0$ "
  shows "continuous_on A ( $\lambda x. Ei (f x :: real)$ )"
  <proof>

```

```

lemma continuous_Ei_real [continuous_intros]:
  fixes f :: "'a :: t2_space  $\Rightarrow$  real"
  assumes "continuous (at x within A) f" "f x  $\neq 0$ "
  shows "continuous (at x within A) ( $\lambda x. Ei (f x)$ )"
  <proof>

```

```

lemma tendsto_Ei_complex [tendsto_intros]:
  assumes "filterlim f (nhds c) F" "c  $\notin \mathbb{R}_{\leq 0}$ "
  shows "filterlim ( $\lambda x. Ei (f x) :: complex$ ) (nhds (Ei c)) F"
  <proof>

```

```

lemma tendsto_Ei_real [tendsto_intros]:
  assumes "filterlim f (nhds c) F" "c  $\neq 0$ "
  shows "filterlim ( $\lambda x. Ei (f x) :: real$ ) (nhds (Ei c)) F"
  <proof>

```

1.5.2 Limits and relationship to $\int e^t/t dt$

Next, we examine the limits of Ei at the origin and at $\pm\infty$.

On the real line, the logarithm causes $Ei(x)$ to go to $-\infty$ as $x \rightarrow 0$.

```

lemma Ei_real_at_0: "filterlim Ei at_bot (at (0 :: real) within A)"
  <proof>

```

Since the logarithm and $-Ein(-x)$ both go to ∞ as $x \rightarrow \infty$, so does Ei .

```

lemma Ei_at_top:
  "filterlim (Ei :: real  $\Rightarrow$  real) at_top at_top"
  <proof>

```

```

lemmas Ei_at_top' = Ei_at_top [THEN filterlim_compose]

```

Using some manipulation of integrals, we can see that $Ein(x) - \ln x \rightarrow \gamma$ as $x \rightarrow \infty$.

```

lemma tendsto_Ein_minus_ln_at_top: " $((\lambda t :: real. Ein t - \ln t) \longrightarrow euler\_mascheroni)$ 
  at_top"
  <proof>

```

It follows that $\text{Ei}(x) \rightarrow 0$ as $x \rightarrow -\infty$, which then means that indeed we have $\text{Ei}(x) = \int_{-\infty}^x e^t/t dt$ for all real x .

lemma *Ei_at_bot*: "(Ei \longrightarrow (0::real)) at_bot"
 ⟨proof⟩

lemmas *Ei_at_bot'* = *Ei_at_bot* [THEN filterlim_compose]

lemma *absolutely_integrable_exp_over_x*:
 assumes "x < (0::real)"
 shows "(λt. exp t / t) absolutely_integrable_on {...x}"
 ⟨proof⟩

The following now shows that we indeed have $\text{Ei}(x) = \int_{-\infty}^x e^t/t dt$, at least for negative real x . For positive real x , this relationship only holds in terms of a Cauchy principal value, which we will show afterwards.

theorem *has_integral_Ei_real*:
 assumes "x < (0::real)"
 shows "(λt. exp t / t) has_integral Ei x) {...x}"
 ⟨proof⟩

lemma *has_integral_Ei_real'*:
 assumes x: "x > (0::real)"
 shows "(λt. exp (-t) / t) has_integral (-Ei (-x)) {x..}"
 ⟨proof⟩

lemma *Ei_neg_real*:
 assumes "x < (0::real)"
 shows "Ei x < 0"
 ⟨proof⟩

lemma *Ei_at_bot_strong*: "filterlim Ei (at_left (0 :: real)) at_bot"
 ⟨proof⟩

We now show that Ei is indeed the Cauchy principal value of the integral $\int_{-\infty}^x e^t/t dt$ for any real $x \neq 0$. The first step here is to show that the divergent parts really cancel out, i.e. that $\text{Ei}(x) - \text{Ei}(-x)$ vanishes as $x \rightarrow 0$, which follows from continuity.

lemma *Ei_minus_neg_Ei_tendsto_0_real*: "(λx. Ei x - Ei (-x) :: real) $\rightarrow 0$ "
 ⟨proof⟩

theorem *Ei_principal_value_real*:
 assumes "x \neq (0::real)"
 shows "(λε. integral ({...x} - {-ε..ε}) (λt. exp t / t)) \longrightarrow Ei x) (at_right 0)"
 ⟨proof⟩

1.6 Relationship to the incomplete gamma function

lemma *Gamma_incu_0_conv_Ei_real*:
 assumes $x: "x > (0::real)"$
 shows $"Gamma_incu\ 0\ x = -Ei\ (-x)"$
<proof>

lemma *Gamma_incu_0_conv_Ei_complex*:
 assumes $z: "(z::complex) \notin \mathbb{R}_{\leq 0}"$
 shows $"Gamma_incu\ 0\ z = Ein\ z - \ln\ z - euler_mascheroni"$
<proof>

1.6.1 Asymptotic series

By using iterated integration by parts, we can derive a (divergent) asymptotic series expansion for $Ei(x)$ as $x \rightarrow \infty$.

The remainder term is of the form $\int_c^x e^t/t^n dt$. We arbitrarily choose $c = 1$. Note that this integral is also referred to as the *generalised exponential integral* sometimes, and one could in principle also build more theory around it.

definition *Ei_remainder* :: $"nat \Rightarrow real \Rightarrow real"$
 where $"Ei_remainder\ n\ x = integral\ \{1..x\}\ (\lambda t. exp\ t / t \wedge n)"$

We apply integration by parts to related $Ei_remainder\ n\ x$ to $Ei_remainder\ (n + 1)\ x$:

lemma *Ei_remainder_rec*:
 assumes $x: "x \geq 1"$
 shows $"Ei_remainder\ n\ x = exp\ x / x \wedge n - exp\ 1 + real\ n * Ei_remainder\ (Suc\ n)\ x"$
<proof>

It is obvious that $Ei_remainder\ n\ x$ is $O(e^x/x^{n+1})$. However, by unfolding the above recurrence once and *then* doing the estimate, we obtain the sharper estimate $O(e^x/x^n)$.

lemma *Ei_remainder_bigo*: $"Ei_remainder\ n \in O(\lambda x. exp\ x / x \wedge n)"$
<proof>

Unfolding the recurrence n times yields the following expression. It is a bit unwieldy due to the presence of constants (which are of course asymptotically irrelevant).

lemma *Ei_asymptotics_at_top_explicit*:
 assumes $x: "x \geq 1"$
 shows $"Ei\ x = exp\ x / x * ((\sum k < n. fact\ k / x \wedge k)) + Ei\ 1 - exp\ 1 * (\sum k < n. fact\ k) + fact\ n * Ei_remainder\ (Suc\ n)\ x"$
<proof>

We finally obtain the full asymptotic expansion:

theorem *Ei_asymptotics_at_top*:
 " $(\lambda x::\text{real}. \text{Ei } x - \exp x / x * (\sum k < n. \text{fact } k / x ^ k)) \in O(\lambda x. \exp x / x ^ \text{Suc } n)$ "
<proof>

corollary *Ei_asymp_equiv_real*: " $\text{Ei} \sim[\text{at_top}] (\lambda x::\text{real}. \exp x / x)$ "
<proof>

We now do the same for the function $\text{Ei}(-x)$ as $x \rightarrow \infty$. Here we use the integral $\int_x^\infty e^{-t}/t^n dt$, which avoids the additive constants that plagued us before.

definition *Ei_neg_remainder* :: " $\text{nat} \Rightarrow \text{real} \Rightarrow \text{real}$ "
 where " $\text{Ei_neg_remainder } n \ x = \text{integral } \{x..\} (\lambda t. \exp (-t) / t ^ n)$ "

lemma *absolutely_integrable_on_Ei_neg_remainder*:
 assumes " $a > 0$ "
 shows " $(\lambda t::\text{real}. \exp (-t) * t \text{ powr } c)$ absolutely_integrable_on $\{a..\}$ "
<proof>

lemma *absolutely_integrable_on_Ei_neg_remainder'*:
 assumes " $a > 0$ "
 shows " $(\lambda t::\text{real}. \exp (-t) / t ^ n)$ absolutely_integrable_on $\{a..\}$ "
<proof>

The integration by parts becomes a bit more tedious since there is no lemma for integration by parts where one of the integration bounds is ∞ . We can, however, simply use a version of the fundamental theorem of calculus that allows ∞ as an integration bound.

lemma *Ei_neg_remainder_rec*:
 assumes $x: "x > 0"$
 shows " $\text{Ei_neg_remainder } n \ x = \exp (-x) / x ^ n - \text{real } n * \text{Ei_neg_remainder } (\text{Suc } n) \ x$ "
<proof>

lemma *Ei_neg_remainder_nonneg*: " $\text{Ei_neg_remainder } n \ x \geq 0$ " if $x: "x > 0"$ for x
<proof>

lemma *Ei_neg_remainder_le*: " $\text{Ei_neg_remainder } n \ x \leq \exp (-x) / x ^ n$ " if $x: "x > 0"$ for x
<proof>

lemma *Ei_neg_remainder_bigo*: " $\text{Ei_neg_remainder } n \in O(\lambda x. \exp (-x) / x ^ n)$ "
<proof>

lemma *Ei_neg_asymptotics_at_top_explicit*:
 assumes $x: "x > 0"$

shows $Ei(-x) = -\exp(-x) / x * (\sum_{k < n} (-1)^k * fact k / x^k) - (-1)^n * fact n * Ei_neg_remainder(Suc n) x$
 <proof>

lemma *Ei_neg_asymptotics_at_top*:
 $(\lambda x :: real. Ei(-x) + \exp(-x) / x * (\sum_{k < n} (-1)^k * fact k / x^k)) \in O(\lambda x. \exp(-x) / x^{Suc n})$
 <proof>

The asymptotic expansion for $Ei(x)$ as $x \rightarrow -\infty$ is in fact the same one as before for $x \rightarrow \infty$; the behaviour, however, is of course very different, since $\exp(x)$ for $x \rightarrow \infty$ grows very quickly whereas it quickly vanishes for $x \rightarrow -\infty$.

theorem *Ei_asymptotics_at_bot*:
 $(\lambda x :: real. Ei x - \exp x / x * (\sum_{k < n} fact k / x^k)) \in O[at_bot](\lambda x. \exp x / x^{Suc n})$
 <proof>

1.7 The logarithmic integral li

Similarly to the exponential integral, the logarithmic integral $li(z)$ is defined as $li(z) = \int_0^z \frac{1}{\ln u} du$, but this definition is plagued by the same issues as the exponential integral.

The easiest way to define it is simply as $Ei(\ln z)$, which yields a function that is holomorphic away from the strip $z \leq 1$ with derivative $1/\ln z$, and for real $x \in (0, 1)$ the above integral definition does indeed hold even without considering the principal value.

definition *li* :: "'a :: {banach, real_normed_field, ln} \Rightarrow 'a"
 where "li x = (if x = 0 then 0 else Ei (ln x))"

lemma *li_0 [simp]*: "li 0 = 0"
 <proof>

lemma *li_neg_real*: "x \in {0 <.. <1 } $\implies li(x :: real) < 0$ "
 <proof>

lemma *has_field_derivative_li_real*:
 assumes "(x :: real) > 0" "x \neq 1"
 shows "(li has_field_derivative (1 / ln x)) (at x within A)"
 <proof>

lemma *has_field_derivative_li_complex*:
 assumes "Im x \neq 0 \vee Re x > 1"
 shows "(li has_field_derivative (1 / ln x)) (at x within A)"
 <proof>

```

lemmas has_field_derivative_li_real' [derivative_intros] =
  has_field_derivative_li_real [THEN DERIV_chain2]

lemmas has_field_derivative_li_complex' [derivative_intros] =
  has_field_derivative_li_complex [THEN DERIV_chain2]

lemma holomorphic_li_complex [holomorphic_intros]:
  assumes "f holomorphic_on A" " $\bigwedge z. z \in A \implies \text{Im } (f z) \neq 0 \vee \text{Re } (f z) > 1$ "
  shows " $(\lambda x. \text{li } (f x))$  holomorphic_on A"
  <proof>

lemma analytic_li_complex [analytic_intros]:
  assumes "f analytic_on A" " $\bigwedge z. z \in A \implies \text{Im } (f z) \neq 0 \vee \text{Re } (f z) > 1$ "
  shows " $(\lambda x. \text{li } (f x))$  analytic_on A"
  <proof>

lemma continuous_on_li_complex [continuous_intros]:
  assumes "continuous_on A f" " $\bigwedge z. z \in A \implies \text{Im } (f z) \neq 0 \vee \text{Re } (f z) > 1$ "
  shows "continuous_on A  $(\lambda x. \text{li } (f x :: \text{complex}))$ "
  <proof>

lemma continuous_li_complex [continuous_intros]:
  fixes f :: "'a :: t2_space  $\Rightarrow$  complex"
  assumes "continuous (at x within A) f" " $\text{Im } (f x) \neq 0 \vee \text{Re } (f x) > 1$ "
  shows "continuous (at x within A)  $(\lambda x. \text{li } (f x))$ "
  <proof>

lemma tendsto_li_complex [tendsto_intros]:
  assumes "filterlim f (nhds c) F" " $\text{Im } c \neq 0 \vee \text{Re } c > 1$ "
  shows "filterlim  $(\lambda x. \text{li } (f x) :: \text{complex})$  (nhds (li c)) F"
  <proof>

lemma li_real_at_1_real: "filterlim li at_bot (at (1 :: real))"
  <proof>

lemma li_at_top_real: "filterlim (li :: real  $\Rightarrow$  real) at_top at_top"
  <proof>

lemma li_at_right_0_real: "filterlim li (at_left (0 :: real)) (at_right 0)"
  <proof>

lemma continuous_on_li_real: "continuous_on  $(\{0..\} - \{1\})$  (li :: real  $\Rightarrow$  real)"
  <proof>

```

```

lemma continuous_on_li_real' [continuous_intros]:
  assumes "continuous_on A f" "\z. z \in A \implies f z \ge 0 \wedge f z \neq 1"
  shows "continuous_on A (\lambda x. li (f x :: real))"
<proof>

```

```

lemma tendsto_li_real [tendsto_intros]:
  assumes "filterlim f (nhds c) F" "c \ge 0" "c \neq 1"
  assumes "c = 0 \implies eventually (\lambda x. f x \ge 0) F"
  shows "filterlim (\lambda x. li (f x) :: real) (nhds (li c)) F"
<proof>

```

```

lemma continuous_li_real [continuous_intros]:
  fixes f :: "'a :: t2_space \Rightarrow real"
  assumes "continuous (at x within A) f" "f x \ge 0" "f x \neq 1"
  assumes "f x = 0 \implies eventually (\lambda y. f y \ge 0) (at x within A)"
  shows "continuous (at x within A) (\lambda x. li (f x))"
<proof>

```

Similarly to Ei, we find that $\text{li}(x) = \int_0^x 1/\ln t dt$ for any positive real $x \neq 1$, with the caveat that we have to take the Cauchy principal value if $x > 1$

```

theorem has_integral_li_real_pos:
  fixes x :: real
  assumes x: "x \in {0<..<1}"
  shows "( \lambda t. 1 / ln t ) absolutely_integrable_on {0<..

```

```

lemma li_minus_neg_li_tendsto_0_real: "( \lambda x. li (1 + x) - li (1 - x) ::
real) -0 \to 0"
<proof>

```

```

theorem li_principal_value_real:
  fixes x :: real
  assumes "x > 0" "x \neq 1"
  shows "( (\lambda \epsilon. integral ({0..

```

The asymptotic expansions for $\text{li}(x)$ as $x \rightarrow \infty$ and $x \rightarrow 0^+$ are easily obtained from the corresponding ones for Ei:

```

theorem li_asymptotics_at_top:
  "( \lambda x :: real. li x - x / ln x * ( \sum k < n. fact k / ln x ^ k ) ) \in O( \lambda x. x
/ ln x ^ Suc n )"
<proof>

```

```

corollary li_asymp_equiv_real: "li \sim[at_top] (\lambda x :: real. x / ln x)"
<proof>

```

theorem *li_asymptotics_at_right_0*:
 "(\(\lambda x :: \text{real}. \text{li } x - x / \ln x * (\sum k < n. \text{fact } k / \ln x ^ k)\) \(\in O[\text{at_right } 0](\lambda x. x / \ln x ^ \text{Suc } n)\)"
 <proof>

1.8 The shifted logarithmic integral Li

The shifted logarithmic integral

$$\text{Li}(z) = \text{li}(z) - \text{li}(2) = \int_2^z \frac{dx}{\log x}$$

is sometimes of interest, since the integral is well-defined for any real $z > 1$. This function also shows up in the Prime Number Theorem.

definition *Li* :: "'a :: {banach, real_normed_field, ln} \(\Rightarrow 'a"
 where "Li z = li z - li 2"

lemma *Li_2 [simp]*: "Li 2 = 0"
 <proof>

lemma *has_field_derivative_Li_real*:
 assumes "(x :: real) > 0" "x \(\neq 1"
 shows "(Li has_field_derivative (1 / ln x)) (at x within A)"
 <proof>

lemma *has_field_derivative_Li_complex*:
 assumes "Im x \(\neq 0 \vee \text{Re } x > 1"
 shows "(Li has_field_derivative (1 / ln x)) (at x within A)"
 <proof>

lemmas *has_field_derivative_Li_real'* [derivative_intros] =
 has_field_derivative_Li_real [THEN DERIV_chain2]

lemmas *has_field_derivative_Li_complex'* [derivative_intros] =
 has_field_derivative_Li_complex [THEN DERIV_chain2]

lemma *holomorphic_Li_complex [holomorphic_intros]*:
 assumes "f holomorphic_on A" "\(\wedge z. z \(\in A \(\Rightarrow \text{Im } (f z) \(\neq 0 \vee \text{Re } (f z) > 1"
 shows "(\(\lambda x. \text{Li } (f x)) \text{holomorphic_on } A"
 <proof>

lemma *analytic_Li_complex [analytic_intros]*:
 assumes "f analytic_on A" "\(\wedge z. z \(\in A \(\Rightarrow \text{Im } (f z) \(\neq 0 \vee \text{Re } (f z) > 1"
 shows "(\(\lambda x. \text{Li } (f x)) \text{analytic_on } A"
 <proof>

lemma *continuous_on_Li_complex [continuous_intros]*:

```

    assumes "continuous_on A f" " $\bigwedge z. z \in A \implies \text{Im } (f z) \neq 0 \vee \text{Re } (f z) > 1$ "
  shows "continuous_on A ( $\lambda x. \text{Li } (f x :: \text{complex})$ )"
    <proof>

lemma continuous_Li_complex [continuous_intros]:
  fixes f :: "'a :: t2_space  $\Rightarrow$  complex"
  assumes "continuous (at x within A) f" " $\text{Im } (f x) \neq 0 \vee \text{Re } (f x) > 1$ "
  shows "continuous (at x within A) ( $\lambda x. \text{Li } (f x)$ )"
    <proof>

lemma tendsto_Li_complex [tendsto_intros]:
  assumes "filterlim f (nhds c) F" " $\text{Im } c \neq 0 \vee \text{Re } c > 1$ "
  shows "filterlim ( $\lambda x. \text{Li } (f x) :: \text{complex}$ ) (nhds (Li c)) F"
    <proof>

lemma continuous_on_Li_real [continuous_intros]:
  " $\text{continuous\_on } A f \implies (\bigwedge z. z \in A \implies f z \geq 0 \wedge f z \neq 1) \implies$ 
  continuous_on A ( $\lambda x. \text{Li } (f x :: \text{real})$ )"
    <proof>

lemma continuous_Li_real [continuous_intros]:
  fixes f :: "'a :: t2_space  $\Rightarrow$  real"
  assumes "continuous (at x within A) f" " $f x \geq 0$ " " $f x \neq 1$ "
  assumes " $f x = 0 \implies \text{eventually } (\lambda x. f x \geq 0)$  (at x within A)"
  shows "continuous (at x within A) ( $\lambda x. \text{Li } (f x)$ )"
    <proof>

lemma tendsto_Li_real [tendsto_intros]:
  assumes "filterlim f (nhds c) F" " $c \geq 0$ " " $c \neq 1$ "
  assumes " $c = 0 \implies \text{eventually } (\lambda x. f x \geq 0)$  F"
  shows "filterlim ( $\lambda x. \text{Li } (f x) :: \text{real}$ ) (nhds (Li c)) F"
    <proof>

lemma has_integral_Li_real:
  assumes " $x \geq 2$ "
  shows " $(\lambda t :: \text{real}. 1 / \ln t)$  has_integral Li x {2..x}"
    <proof>

lemma Li_real_conv_integral: " $x \geq 2 \implies \text{Li } (x :: \text{real}) = \text{integral } \{2..x\}$ 
( $\lambda t. 1 / \ln t$ )"
    <proof>

end

```

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