

# The Euler–MacLaurin summation formula

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## Abstract

The Euler–MacLaurin formula relates the value of a discrete sum  $\sum_{i=a}^b f(i)$  to that of the integral  $\int_a^b f(x) dx$  in terms of the derivatives of  $f$  at  $a$  and  $b$  and a remainder term. Since the remainder term is often very small as  $b$  grows, this can be used to compute asymptotic expansions for sums.

This entry contains a proof of this formula for functions from the reals to an arbitrary Banach space. Two variants of the formula are given: the standard textbook version and a variant outlined in *Concrete Mathematics* [3] that is more useful for deriving asymptotic estimates.

As example applications, we use that formula to derive the full asymptotic expansion of the harmonic numbers and the sum of inverse squares.

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# 1 The Euler–MacLaurin summation formula

**theory** *Euler-MacLaurin*

**imports**

*HOL-Complex-Analysis.Complex-Analysis*

*Bernoulli.Periodic-Bernpoly*

*Bernoulli.Bernoulli-FPS*

**begin**

## 1.1 Auxiliary facts

**lemma** *pbernpoly-of-int [simp]: pbernpoly n (of-int a) = bernoulli n*  
*<proof>*

**lemma** *continuous-on-bernpoly' [continuous-intros]:*

**assumes** *continuous-on A f*

**shows** *continuous-on A ( $\lambda x. \text{bernpoly } n (f x) :: 'a :: \text{real-normed-algebra-1}$ )*

*<proof>*

**lemma** *sum-atLeastAtMost-int-last:*

**assumes** *a < (b :: int)*

**shows** *sum f {a..b} = sum f {a..**b**} + f b*

*<proof>*

**lemma** *sum-atLeastAtMost-int-head:*

**assumes** *a < (b :: int)*

**shows** *sum f {a..b} = f a + sum f {a<..**b**}*

*<proof>*

**lemma** *not-in-nonpos-Reals-imp-add-nonzero:*

**assumes** *z  $\notin$   $\mathbb{R}_{\leq 0}$  x  $\geq$  0*

**shows** *z + of-real x  $\neq$  0*

*<proof>*

**lemma** *negligible-atLeastAtMostI: b  $\leq$  a  $\implies$  negligible {a..(b::real)}*

*<proof>*

**lemma** *integrable-on-negligible:*

*negligible A  $\implies$  (f :: 'n :: euclidean-space  $\Rightarrow$  'a :: banach) integrable-on A*

*<proof>*

**lemma** *Union-atLeastAtMost-real-of-int:*

**assumes** *a < b*

**shows** *( $\bigcup n \in \{a..<b\}. \{ \text{real-of-int } n.. \text{real-of-int } (n + 1) \}$ ) = {real-of-int a..real-of-int*

*b}*

*<proof>*

## 1.2 The remainder terms

The following describes the remainder term in the classical version of the Euler–MacLaurin formula.

**definition** *EM-remainder'* ::  $\text{nat} \Rightarrow (\text{real} \Rightarrow 'a :: \text{banach}) \Rightarrow \text{real} \Rightarrow \text{real} \Rightarrow 'a$   
**where**

$EM\text{-remainder}'\ n\ f\ a\ b = ((-1) \wedge \text{Suc}\ n / \text{fact}\ n) *_{\mathbb{R}} \text{integral}\ \{a..b\}\ (\lambda t. \text{pbernpoly}\ n\ t *_{\mathbb{R}} f\ t)$

Next, we define the remainder term that occurs when one lets the right bound of summation in the Euler–MacLaurin formula tend to infinity.

**definition** *EM-remainder-converges* ::  $\text{nat} \Rightarrow (\text{real} \Rightarrow 'a :: \text{banach}) \Rightarrow \text{int} \Rightarrow \text{bool}$   
**where**

$EM\text{-remainder-converges}\ n\ f\ a \iff (\exists L. ((\lambda x. EM\text{-remainder}'\ n\ f\ a\ (\text{of-int}\ x)) \longrightarrow L)\ \text{at-top})$

**definition** *EM-remainder* ::  $\text{nat} \Rightarrow (\text{real} \Rightarrow 'a :: \text{banach}) \Rightarrow \text{int} \Rightarrow 'a$  **where**

$EM\text{-remainder}\ n\ f\ a =$

(if *EM-remainder-converges*  $n\ f\ a$  then

$\text{Lim}\ \text{at-top}\ (\lambda x. EM\text{-remainder}'\ n\ f\ a\ (\text{of-int}\ x))$  else 0)

The following lemmas are fairly obvious – but tedious to prove – properties of the remainder terms.

**lemma** *EM-remainder-eqI*:

**fixes**  $L$

**assumes**  $((\lambda x. EM\text{-remainder}'\ n\ f\ b\ (\text{of-int}\ x)) \longrightarrow L)\ \text{at-top}$

**shows**  $EM\text{-remainder}\ n\ f\ b = L$

*<proof>*

**lemma** *integrable-EM-remainder'-int*:

**fixes**  $a\ b :: \text{int}$  **and**  $f :: \text{real} \Rightarrow 'a :: \text{banach}$

**assumes** *continuous-on*  $\{\text{of-int}\ a.. \text{of-int}\ b\}\ f$

**shows**  $(\lambda t. \text{pbernpoly}\ n\ t *_{\mathbb{R}} f\ t)\ \text{integrable-on}\ \{a..b\}$

*<proof>*

**lemma** *integrable-EM-remainder'*:

**fixes**  $a\ b :: \text{real}$  **and**  $f :: \text{real} \Rightarrow 'a :: \text{banach}$

**assumes** *continuous-on*  $\{a..b\}\ f$

**shows**  $(\lambda t. \text{pbernpoly}\ n\ t *_{\mathbb{R}} f\ t)\ \text{integrable-on}\ \{a..b\}$

*<proof>*

**lemma** *EM-remainder'-bounded-linear*:

**assumes** *bounded-linear*  $h$

**assumes** *continuous-on*  $\{a..b\}\ f$

**shows**  $EM\text{-remainder}'\ n\ (\lambda x. h\ (f\ x))\ a\ b = h\ (EM\text{-remainder}'\ n\ f\ a\ b)$

*<proof>*

**lemma** *EM-remainder-converges-of-real*:

**assumes** *EM-remainder-converges*  $n$   $f$   $a$  *continuous-on*  $\{of-int\ a..\}$   $f$   
**shows** *EM-remainder-converges*  $n$   $(\lambda x. of-real (f\ x))$   $a$   
 $\langle proof \rangle$

**lemma** *EM-remainder-converges-of-real-iff*:

**fixes**  $f :: real \Rightarrow real$   
**assumes** *continuous-on*  $\{of-int\ a..\}$   $f$   
**shows** *EM-remainder-converges*  $n$   $(\lambda x. of-real (f\ x)) ::$   
 $'a :: \{banach, real-normed-algebra-1, real-inner\}$   $a \longleftrightarrow$  *EM-remainder-converges*  
 $n$   $f$   $a$   
 $\langle proof \rangle$

**lemma** *EM-remainder-of-real*:

**assumes** *continuous-on*  $\{a..\}$   $f$   
**shows** *EM-remainder*  $n$   $(\lambda x. of-real (f\ x)) ::$   
 $'a :: \{banach, real-normed-algebra-1, real-inner\}$   $a =$   
 $of-real (EM-remainder\ n\ f\ a)$   
 $\langle proof \rangle$

**lemma** *EM-remainder'-cong*:

**assumes**  $\bigwedge x. x \in \{a..b\} \implies f\ x = g\ x$   $n = n'$   $a = a'$   $b = b'$   
**shows** *EM-remainder'*  $n$   $f$   $a$   $b = EM-remainder'$   $n'$   $g$   $a'$   $b'$   
 $\langle proof \rangle$

**lemma** *EM-remainder-converges-cong*:

**assumes**  $\bigwedge x. x \geq of-int\ a \implies f\ x = g\ x$   $n = n'$   $a = a'$   
**shows** *EM-remainder-converges*  $n$   $f$   $a = EM-remainder-converges$   $n'$   $g$   $a'$   
 $\langle proof \rangle$

**lemma** *EM-remainder-cong*:

**assumes**  $\bigwedge x. x \geq of-int\ a \implies f\ x = g\ x$   $n = n'$   $a = a'$   
**shows** *EM-remainder*  $n$   $f$   $a = EM-remainder$   $n'$   $g$   $a'$   
 $\langle proof \rangle$

**lemma** *EM-remainder-converges-cnj*:

**assumes** *continuous-on*  $\{a..\}$   $f$  **and** *EM-remainder-converges*  $n$   $f$   $a$   
**shows** *EM-remainder-converges*  $n$   $(\lambda x. cnj (f\ x))$   $a$   
 $\langle proof \rangle$

**lemma** *EM-remainder-converges-cnj-iff*:

**assumes** *continuous-on*  $\{of-int\ a..\}$   $f$   
**shows** *EM-remainder-converges*  $n$   $(\lambda x. cnj (f\ x))$   $a \longleftrightarrow$  *EM-remainder-converges*  
 $n$   $f$   $a$   
 $\langle proof \rangle$

**lemma** *EM-remainder-cnj*:

**assumes** *continuous-on*  $\{a..\}$   $f$   
**shows** *EM-remainder*  $n$   $(\lambda x. cnj (f\ x))$   $a = cnj (EM-remainder\ n\ f\ a)$   
 $\langle proof \rangle$

**lemma** *EM-remainder'-combine*:

**fixes**  $f :: \text{real} \Rightarrow 'a :: \text{banach}$   
**assumes** [*continuous-intros*]: *continuous-on*  $\{a..c\} f$   
**assumes**  $a \leq b \leq c$   
**shows**  $EM\text{-remainder}' n f a b + EM\text{-remainder}' n f b c = EM\text{-remainder}' n f a c$   
*<proof>*

**lemma** *uniformly-convergent-EM-remainder'*:

**fixes**  $f :: 'a \Rightarrow \text{real} \Rightarrow 'b :: \{\text{banach}, \text{real-normed-algebra}\}$   
**assumes** *deriv*:  $\bigwedge y. a \leq y \implies (G \text{ has-real-derivative } g y) \text{ (at } y \text{ within } \{a..\})$   
**assumes** *integrable*:  $\bigwedge a' b y. y \in A \implies a \leq a' \implies a' \leq b \implies$   
 $(\lambda t. \text{pbernpoly } n t *_R f y t) \text{ integrable-on } \{a'..b\}$   
**assumes** *conv*: *convergent*  $(\lambda y. G (\text{real } y))$   
**assumes** *bound*: *eventually*  $(\lambda x. \forall y \in A. \text{norm } (f y x) \leq g x) \text{ at-top}$   
**shows** *uniformly-convergent-on*  $A (\lambda b s. EM\text{-remainder}' n (f s) a b)$   
*<proof>*

**lemma** *uniform-limit-EM-remainder*:

**fixes**  $f :: 'a \Rightarrow \text{real} \Rightarrow 'b :: \{\text{banach}, \text{real-normed-algebra}\}$   
**assumes** *deriv*:  $\bigwedge y. a \leq y \implies (G \text{ has-real-derivative } g y) \text{ (at } y \text{ within } \{a..\})$   
**assumes** *integrable*:  $\bigwedge a' b y. y \in A \implies a \leq a' \implies a' \leq b \implies$   
 $(\lambda t. \text{pbernpoly } n t *_R f y t) \text{ integrable-on } \{a'..b\}$   
**assumes** *conv*: *convergent*  $(\lambda y. G (\text{real } y))$   
**assumes** *bound*: *eventually*  $(\lambda x. \forall y \in A. \text{norm } (f y x) \leq g x) \text{ at-top}$   
**shows** *uniform-limit*  $A (\lambda b s. EM\text{-remainder}' n (f s) a b)$   
 $(\lambda s. EM\text{-remainder } n (f s) a) \text{ sequentially}$   
*<proof>*

**lemma** *tendsto-EM-remainder*:

**fixes**  $f :: \text{real} \Rightarrow 'b :: \{\text{banach}, \text{real-normed-algebra}\}$   
**assumes** *deriv*:  $\bigwedge y. a \leq y \implies (G \text{ has-real-derivative } g y) \text{ (at } y \text{ within } \{a..\})$   
**assumes** *integrable*:  $\bigwedge a' b . a \leq a' \implies a' \leq b \implies$   
 $(\lambda t. \text{pbernpoly } n t *_R f t) \text{ integrable-on } \{a'..b\}$   
**assumes** *conv*: *convergent*  $(\lambda y. G (\text{real } y))$   
**assumes** *bound*: *eventually*  $(\lambda x. \text{norm } (f x) \leq g x) \text{ at-top}$   
**shows** *filterlim*  $(\lambda b. EM\text{-remainder}' n f a b)$   
 $(\text{nhds } (EM\text{-remainder } n f a)) \text{ sequentially}$   
*<proof>*

**lemma** *EM-remainder-0* [*simp*]:  $EM\text{-remainder } n (\lambda x. 0) a = 0$

*<proof>*

**lemma** *holomorphic-EM-remainder'*:

**assumes** *deriv*:  
 $\bigwedge z t. z \in U \implies t \in \{a..x\} \implies$   
 $((\lambda z. f z t) \text{ has-field-derivative } f' z t) \text{ (at } z \text{ within } U)$   
**assumes** *int*:  $\bigwedge b c z e. a \leq b \implies c \leq x \implies z \in U \implies$

$(\lambda t. \text{of-real } (\text{bernpoly } n (t - e)) * f z t) \text{ integrable-on } \{b..c\}$   
**assumes** *cont*: *continuous-on*  $(U \times \{a..x\}) (\lambda(z, t). f' z t)$   
**assumes** *convex*  $U$   
**shows**  $(\lambda s. \text{EM-remainder}' n (f s) a x) \text{ holomorphic-on } U$   
 $\langle \text{proof} \rangle$

**lemma**

**assumes** *deriv*:  $\bigwedge y. a \leq y \implies (G \text{ has-real-derivative } g y) \text{ (at } y \text{ within } \{a..\})$   
**assumes** *deriv'*:  
 $\bigwedge z t x. z \in U \implies x \geq a \implies t \in \{a..x\} \implies$   
 $((\lambda z. f z t) \text{ has-field-derivative } f' z t) \text{ (at } z \text{ within } U)$   
**assumes** *cont*: *continuous-on*  $(U \times \{\text{of-int } a..\}) (\lambda(z, t). f' z t)$   
**assumes** *int*:  $\bigwedge b c z e. a \leq b \implies z \in U \implies$   
 $(\lambda t. \text{of-real } (\text{bernpoly } n (t - e)) * f z t) \text{ integrable-on } \{b..c\}$   
**assumes** *int'*:  $\bigwedge a' b y. y \in U \implies a \leq a' \implies a' \leq b \implies$   
 $(\lambda t. \text{pbernpoly } n t *_R f y t) \text{ integrable-on } \{a'..b\}$   
**assumes** *conv*: *convergent*  $(\lambda y. G (\text{real } y))$   
**assumes** *bound*: *eventually*  $(\lambda x. \forall y \in U. \text{norm } (f y x) \leq g x) \text{ at-top}$   
**assumes** *open*  $U$   
**shows** *analytic-EM-remainder*:  $(\lambda s :: \text{complex}. \text{EM-remainder } n (f s) a) \text{ analytic-on } U$   
**and** *holomorphic-EM-remainder*:  $(\lambda s :: \text{complex}. \text{EM-remainder } n (f s) a) \text{ holomorphic-on } U$   
 $\langle \text{proof} \rangle$

The following lemma is the first step in the proof of the Euler–MacLaurin formula: We show the relationship between the first-order remainder term and the difference of the integral and the sum.

**context**

**fixes**  $f f' :: \text{real} \Rightarrow 'a :: \text{banach}$   
**fixes**  $a b :: \text{int}$  **and**  $I S :: 'a$   
**fixes**  $Y :: \text{real set}$   
**assumes**  $a \leq b$   
**assumes** *fin*: *finite*  $Y$   
**assumes** *cont*: *continuous-on*  $\{\text{real-of-int } a..\text{real-of-int } b\} f$   
**assumes** *deriv* [*derivative-intros*]:  
 $\bigwedge x :: \text{real}. x \in \{a..b\} - Y \implies (f \text{ has-vector-derivative } f' x) \text{ (at } x)$   
**defines** *S-def*:  $S \equiv (\sum i \in \{a <.. b\}. f i)$  **and** *I-def*:  $I \equiv \text{integral } \{a..b\} f$   
**begin**

**lemma**

*diff-sum-integral-has-integral-int*:  
 $((\lambda t. (\text{frac } t - 1/2) *_R f' t) \text{ has-integral } (S - I - (f b - f a) /_R 2)) \{a..b\}$   
 $\langle \text{proof} \rangle$

**lemma** *diff-sum-integral-has-integral-int'*:

$((\lambda t. \text{pbernpoly } 1 t *_R f' t) \text{ has-integral } (S - I - (f b - f a) /_R 2)) \{a..b\}$   
 $\langle \text{proof} \rangle$

**lemma** *EM-remainder'-Suc-0*:  $EM\text{-remainder}' (Suc\ 0) f' a\ b = S - I - (f\ b - f\ a) /_R\ 2$   
 ⟨proof⟩

**end**

Next, we show that the  $n$ -th-order remainder can be expressed in terms of the  $n + 1$ -th-order remainder term. Iterating this essentially yields the Euler–MacLaurin formula.

**context**

**fixes**  $f\ f' :: real \Rightarrow 'a :: banach$  **and**  $a\ b :: int$  **and**  $n :: nat$  **and**  $A :: real\ set$   
**assumes**  $ab: a \leq b$  **and**  $n: n > 0$   
**assumes**  $fin: finite\ A$   
**assumes**  $cont: continuous\ on\ \{of\ int\ a..of\ int\ b\}\ f$   
**assumes**  $cont': continuous\ on\ \{of\ int\ a..of\ int\ b\}\ f'$   
**assumes**  $deriv: \bigwedge x. x \in \{of\ int\ a < .. < of\ int\ b\} - A \implies (f\ has\ vector\ derivative\ f'\ x)\ (at\ x)$

**begin**

**lemma** *EM-remainder'-integral-conv-Suc*:

**shows**  $integral\ \{a..b\}\ (\lambda t. pbernpoly\ n\ t\ *_R\ f\ t) =$   
 $(bernoulli\ (Suc\ n) / real\ (Suc\ n)) *_R\ (f\ b - f\ a) -$   
 $integral\ \{a..b\}\ (\lambda t. pbernpoly\ (Suc\ n)\ t\ *_R\ f'\ t) /_R\ real\ (Suc\ n)$   
 ⟨proof⟩

**lemma** *EM-remainder'-conv-Suc*:

$EM\text{-remainder}'\ n\ f\ a\ b =$   
 $((-1) \wedge Suc\ n * bernoulli\ (Suc\ n) / fact\ (Suc\ n)) *_R\ (f\ b - f\ a) +$   
 $EM\text{-remainder}'\ (Suc\ n)\ f'\ a\ b$   
 ⟨proof⟩

**end**

**context**

**fixes**  $f\ f' :: real \Rightarrow 'a :: banach$  **and**  $a :: int$  **and**  $n :: nat$  **and**  $A :: real\ set$  **and**  $C$   
**assumes**  $n: n > 0$   
**assumes**  $fin: finite\ A$   
**assumes**  $cont: continuous\ on\ \{of\ int\ a..\}\ f$   
**assumes**  $cont': continuous\ on\ \{of\ int\ a..\}\ f'$   
**assumes**  $lim: (f \longrightarrow C)\ at\ top$   
**assumes**  $deriv: \bigwedge x. x \in \{of\ int\ a < ..\} - A \implies (f\ has\ vector\ derivative\ f'\ x)\ (at\ x)$

**begin**

**lemma**

**shows** *EM-remainder-converges-iff-Suc-converges*:  
 $EM\text{-remainder-converges}\ n\ f\ a \longleftrightarrow EM\text{-remainder-converges}\ (Suc\ n)\ f'\ a$   
**and** *EM-remainder-conv-Suc*:

$EM\text{-remainder-converges } n \ f \ a \implies$   
 $EM\text{-remainder } n \ f \ a =$   
 $((-1) \wedge \text{Suc } n * \text{bernoulli } (\text{Suc } n) / \text{fact } (\text{Suc } n)) *_R (C - f \ a) +$   
 $EM\text{-remainder } (\text{Suc } n) \ f' \ a$   
 <proof>  
 end

### 1.3 The conventional version of the Euler–MacLaurin formula

The following theorems are the classic Euler–MacLaurin formula that can be found, with slight variations, in many sources (e. g. [1, 2, 3]).

**context**

**fixes**  $f :: \text{real} \Rightarrow 'a :: \text{banach}$   
**fixes**  $fs :: \text{nat} \Rightarrow \text{real} \Rightarrow 'a$   
**fixes**  $a \ b :: \text{int}$  **assumes**  $ab: a \leq b$   
**fixes**  $N :: \text{nat}$  **assumes**  $N: N > 0$   
**fixes**  $Y :: \text{real set}$  **assumes**  $fin: \text{finite } Y$   
**assumes**  $fs\text{-}0$  [*simp*]:  $fs \ 0 = f$   
**assumes**  $fs\text{-}cont$  [*continuous-intros*]:  
 $\bigwedge k. k \leq N \implies \text{continuous-on } \{\text{real-of-int } a..\text{real-of-int } b\} (fs \ k)$   
**assumes**  $fs\text{-}deriv$  [*derivative-intros*]:  
 $\bigwedge k \ x. k < N \implies x \in \{a..b\} - Y \implies (fs \ k \ \text{has-vector-derivative } fs \ (\text{Suc } k) \ x)$   
 (at  $x$ )  
**begin**

**theorem** *euler-maclaurin-raw-strong-int*:

**defines**  $S \equiv (\sum i \in \{a <..b\}. f \ (\text{of-int } i))$   
**defines**  $I \equiv \text{integral } \{\text{of-int } a..\text{of-int } b\} f$   
**defines**  $c' \equiv \lambda k. (\text{bernoulli}' (\text{Suc } k) / \text{fact } (\text{Suc } k)) *_R (fs \ k \ b - fs \ k \ a)$   
**shows**  $S - I = (\sum k < N. c' \ k) + EM\text{-remainder}' \ N \ (fs \ N) \ a \ b$   
 <proof>

end

**theorem** *euler-maclaurin-strong-raw-nat*:

**assumes**  $a \leq b \ 0 < N \ \text{finite } Y \ fs \ 0 = f$   
 $(\bigwedge k. k \leq N \implies \text{continuous-on } \{\text{real } a..\text{real } b\} (fs \ k))$   
 $(\bigwedge k \ x. k < N \implies x \in \{\text{real } a..\text{real } b\} - Y \implies$   
 $(fs \ k \ \text{has-vector-derivative } fs \ (\text{Suc } k) \ x) \ (\text{at } x))$   
**shows**  $(\sum i \in \{a <..b\}. f \ (\text{real } i)) - \text{integral } \{\text{real } a..\text{real } b\} f =$   
 $(\sum k < N. (\text{bernoulli}' (\text{Suc } k) / \text{fact } (\text{Suc } k)) *_R (fs \ k \ (\text{real } b) - fs \ k \ (\text{real } a))) +$   
 $EM\text{-remainder}' \ N \ (fs \ N) \ (\text{real } a) \ (\text{real } b)$   
 <proof>

## 1.4 The “Concrete Mathematics” version of the Euler–MacLaurin formula

As explained in *Concrete Mathematics* [3], the above form of the formula has some drawbacks: When applying it to determine the asymptotics of some concrete function, one is usually left with several different unwieldy constant terms that are difficult to get rid of.

There is no general way to determine what these constant terms are, but in concrete applications, they can often be determined or estimated by other means. We can therefore simply group all the constant terms into a single constant and have the user provide a proof of what it is.

```

locale euler-maclaurin-int =
  fixes F f :: real ⇒ 'a :: banach
  fixes fs :: nat ⇒ real ⇒ 'a
  fixes a :: int
  fixes N :: nat assumes N: N > 0
  fixes C :: 'a
  fixes Y :: real set assumes fin: finite Y
  assumes fs-0 [simp]: fs 0 = f
  assumes fs-cont [continuous-intros]:
    ∧k. k ≤ N ⇒ continuous-on {real-of-int a..} (fs k)
  assumes fs-deriv [derivative-intros]:
    ∧k x. k < N ⇒ x ∈ {of-int a..} − Y ⇒ (fs k has-vector-derivative fs (Suc
k) x) (at x)
  assumes F-cont [continuous-intros]: continuous-on {of-int a..} F
  assumes F-deriv [derivative-intros]:
    ∧x. x ∈ {of-int a..} − Y ⇒ (F has-vector-derivative f x) (at x)
  assumes limit:
    ((λb. (∑ k=a..b. f k) − F (of-int b) −
      (∑ i<N. (bernoulli' (Suc i) / fact (Suc i)) *R fs i (of-int b))) → C)
  at-top
begin

context
  fixes C' T
  defines C' ≡ −f a + F a + C + (∑ k<N. (bernoulli' (Suc k) / fact (Suc k))
*_R (fs k (of-int a)))
  and T ≡ (λx. ∑ i<N. (bernoulli' (Suc i) / fact (Suc i)) *_R fs i x)
begin

lemma euler-maclaurin-strong-int-aux:
  assumes ab: a ≤ b
  defines S ≡ (∑ k=a..b. f (of-int k))
  shows S − F (of-int b) − T (of-int b) = EM-remainder' N (fs N) (of-int a)
(of-int b) + (C − C')
  ⟨proof⟩

lemma EM-remainder-limit:

```

**assumes**  $ab: a \leq b$   
**defines**  $D \equiv EM\text{-remainder}' N (fs N) (of\text{-int } a) (of\text{-int } b)$   
**shows**  $EM\text{-remainder } N (fs N) b = C' - D$   
**and**  $EM\text{-remainder-converges}: EM\text{-remainder-converges } N (fs N) b$   
 ⟨proof⟩

**theorem** *euler-maclaurin-strong-int*:  
**assumes**  $ab: a \leq b$   
**defines**  $S \equiv (\sum_{k=a..b} f (of\text{-int } k))$   
**shows**  $S = F (of\text{-int } b) + C + T (of\text{-int } b) - EM\text{-remainder } N (fs N) b$   
 ⟨proof⟩

**end**  
**end**

The following version of the formula removes all the terms where the associated Bernoulli numbers vanish.

**locale** *euler-maclaurin-int'* =  
**fixes**  $F f :: real \Rightarrow 'a :: banach$   
**fixes**  $fs :: nat \Rightarrow real \Rightarrow 'a$   
**fixes**  $a :: int$   
**fixes**  $N :: nat$   
**fixes**  $C :: 'a$   
**fixes**  $Y :: real\ set$  **assumes**  $fin: finite\ Y$   
**assumes**  $fs\text{-}0$  [*simp*]:  $fs\ 0 = f$   
**assumes**  $fs\text{-}cont$  [*continuous-intros*]:  
 $\bigwedge k. k \leq 2*N+1 \implies continuous\text{-}on\ \{real\text{-}of\text{-}int\ a..\} (fs\ k)$   
**assumes**  $fs\text{-}deriv$  [*derivative-intros*]:  
 $\bigwedge k\ x. k \leq 2*N \implies x \in \{of\text{-}int\ a..\} - Y \implies (fs\ k\ has\text{-}vector\text{-}derivative\ fs\ (Suc\ k)\ x)\ (at\ x)$   
**assumes**  $F\text{-}cont$  [*continuous-intros*]:  $continuous\text{-}on\ \{of\text{-}int\ a..\} F$   
**assumes**  $F\text{-}deriv$  [*derivative-intros*]:  
 $\bigwedge x. x \in \{of\text{-}int\ a..\} - Y \implies (F\ has\text{-}vector\text{-}derivative\ f\ x)\ (at\ x)$   
**assumes** *limit*:  
 $((\lambda b. (\sum_{k=a..b} f\ k) - F (of\text{-}int\ b) -$   
 $(\sum_{i < 2*N+1} (bernoulli' (Suc\ i) / fact (Suc\ i)) *_R fs\ i (of\text{-}int\ b))) \longrightarrow$   
 $C) \text{ at-top}$   
**begin**

**sublocale** *euler-maclaurin-int*  $F f fs a 2*N+1 C Y$   
 ⟨proof⟩

**theorem** *euler-maclaurin-strong-int'*:  
**assumes**  $a \leq b$   
**shows**  $(\sum_{k=a..b} f (of\text{-int } k)) =$   
 $F (of\text{-int } b) + C + (1 / 2) *_R f (of\text{-int } b) +$   
 $(\sum_{i=1..N} (bernoulli (2*i) / fact (2*i)) *_R fs (2*i-1) (of\text{-int } b)) -$   
 $EM\text{-remainder } (2*N+1) (fs (2*N+1)) b$   
 ⟨proof⟩

**end**

For convenience, we also offer a version where the sum ranges over natural numbers instead of integers.

**lemma** *sum-atLeastAtMost-of-int-nat-transfer*:

$$(\sum k=int\ a..int\ b.\ f\ (of-int\ k)) = (\sum k=a..b.\ f\ (of-nat\ k))$$

*<proof>*

**lemma** *euler-maclaurin-nat-int-transfer*:

**fixes**  $F$  **and**  $f :: real \Rightarrow 'a :: real-normed-vector$   
**assumes**  $((\lambda b. (\sum k=a..b.\ f\ (real\ k)) - F\ (real\ b) - T\ (real\ b)) \longrightarrow C)$  *at-top*  
**shows**  $((\lambda b. (\sum k=int\ a..b.\ f\ (of-int\ k)) - F\ (of-int\ b) - T\ (of-int\ b)) \longrightarrow C)$  *at-top*  
*<proof>*

**locale** *euler-maclaurin-nat* =

**fixes**  $F\ f :: real \Rightarrow 'a :: banach$   
**fixes**  $fs :: nat \Rightarrow real \Rightarrow 'a$   
**fixes**  $a :: nat$   
**fixes**  $N :: nat$  **assumes**  $N: N > 0$   
**fixes**  $C :: 'a$   
**fixes**  $Y :: real\ set$  **assumes**  $fin: finite\ Y$   
**assumes**  $fs-0$  [*simp*]:  $fs\ 0 = f$   
**assumes**  $fs-cont$  [*continuous-intros*]:  
 $\bigwedge k. k \leq N \implies continuous-on\ \{real\ a..\}\ (fs\ k)$   
**assumes**  $fs-deriv$  [*derivative-intros*]:  
 $\bigwedge k\ x. k < N \implies x \in \{real\ a..\} - Y \implies (fs\ k\ has-vector-derivative\ fs\ (Suc\ k))$   
*x) (at\ x)*  
**assumes**  $F-cont$  [*continuous-intros*]:  $continuous-on\ \{real\ a..\}\ F$   
**assumes**  $F-deriv$  [*derivative-intros*]:  
 $\bigwedge x. x \in \{real\ a..\} - Y \implies (F\ has-vector-derivative\ f\ x)\ (at\ x)$   
**assumes** *limit*:  
 $((\lambda b. (\sum k=a..b.\ f\ k) - F\ (real\ b) -$   
 $(\sum i < N. (bernoulli'\ (Suc\ i) / fact\ (Suc\ i)) *_R\ fs\ i\ (real\ b))) \longrightarrow C)$  *at-top*  
**begin**

**sublocale** *euler-maclaurin-int*  $F\ f\ fs\ int\ a\ N\ C\ Y$   
*<proof>*

**theorem** *euler-maclaurin-strong-nat*:

**assumes**  $ab: a \leq b$   
**defines**  $S \equiv (\sum k=a..b.\ f\ (real\ k))$   
**shows**  $S = F\ (real\ b) + C + (\sum i < N. (bernoulli'\ (Suc\ i) / fact\ (Suc\ i)) *_R\ fs\ i\ (real\ b)) -$   
 $EM-remainder\ N\ (fs\ N)\ (int\ b)$   
*<proof>*

**end**

```

locale euler-maclaurin-nat' =
  fixes F f :: real ⇒ 'a :: banach
  fixes fs :: nat ⇒ real ⇒ 'a
  fixes a :: nat
  fixes N :: nat
  fixes C :: 'a
  fixes Y :: real set assumes fin: finite Y
  assumes fs-0 [simp]: fs 0 = f
  assumes fs-cont [continuous-intros]:
    ∧k. k ≤ 2*N+1 ⇒ continuous-on {real a..} (fs k)
  assumes fs-deriv [derivative-intros]:
    ∧k x. k ≤ 2*N ⇒ x ∈ {real a..} - Y ⇒ (fs k has-vector-derivative fs (Suc
k) x) (at x)
  assumes F-cont [continuous-intros]: continuous-on {real a..} F
  assumes F-deriv [derivative-intros]:
    ∧x. x ∈ {real a..} - Y ⇒ (F has-vector-derivative f x) (at x)
  assumes limit:
    ((λb. (∑ k=a..b. f k) - F (real b) -
      (∑ i<2*N+1. (bernoulli' (Suc i) / fact (Suc i)) *R fs i (real b))) → C)
at-top
begin

```

```

sublocale euler-maclaurin-int' F f fs int a N C Y
  ⟨proof⟩

```

```

theorem euler-maclaurin-strong-nat':
  assumes a ≤ b
  shows (∑ k=a..b. f (real k)) =
    F (real b) + C + (1 / 2) *R f (real b) +
    (∑ i=1..N. (bernoulli (2*i) / fact (2*i)) *R fs (2*i-1) (real b)) -
    EM-remainder (2*N+1) (fs (2*N+1)) b
  ⟨proof⟩

```

**end**

## 1.5 Bounds on the remainder term

The following theorems provide some simple means to bound the remainder terms. In practice, better bounds can often be obtained e.g. for the  $n$ -th remainder term by expanding it to the sum of the first discarded term in the expansion and the  $n + 1$ -th remainder term.

```

lemma
  fixes f :: real ⇒ 'a :: {real-normed-field, banach}
  and g g' :: real ⇒ real
  assumes fin: finite Y
  assumes pbernpoly-bound: ∀ x. |pbernpoly n x| ≤ D
  assumes cont-f: continuous-on {a..} f

```

**assumes** *cont-g*: *continuous-on* {*a..*} *g*  
**assumes** *cont-g'*: *continuous-on* {*a..*} *g'*  
**assumes** *limit-g*: (*g*  $\longrightarrow$  *C*) *at-top*  
**assumes** *f-bound*:  $\bigwedge x. x \geq a \implies \text{norm } (f x) \leq g' x$   
**assumes** *deriv*:  $\bigwedge x. x \in \{a..\} - Y \implies (g \text{ has-field-derivative } g' x) (at x)$   
**shows** *norm-EM-remainder-le-strong-int*:  
 $\forall x. \text{of-int } x \geq a \longrightarrow \text{norm } (EM\text{-remainder } n f x) \leq D / \text{fact } n * (C -$   
*g x)*  
**and** *norm-EM-remainder-le-strong-nat*:  
 $\forall x. \text{real } x \geq a \longrightarrow \text{norm } (EM\text{-remainder } n f (\text{int } x)) \leq D / \text{fact } n * (C$   
 $- g x)$   
*<proof>*

**lemma**

**fixes** *f* :: *real*  $\Rightarrow$  '*a* :: {*real-normed-field, banach*}  
**and** *g g'* :: *real*  $\Rightarrow$  *real*  
**assumes** *fin*: *finite* *Y*  
**assumes** *pbernpoly-bound*:  $\forall x. |pbernpoly n x| \leq D$   
**assumes** *cont-f*: *continuous-on* {*a..*} *f*  
**assumes** *cont-g*: *continuous-on* {*a..*} *g*  
**assumes** *cont-g'*: *continuous-on* {*a..*} *g'*  
**assumes** *limit-g*: (*g*  $\longrightarrow$  0) *at-top*  
**assumes** *f-bound*:  $\bigwedge x. x \geq a \implies \text{norm } (f x) \leq g' x$   
**assumes** *deriv*:  $\bigwedge x. x \in \{a..\} - Y \implies (g \text{ has-field-derivative } -g' x) (at x)$   
**shows** *norm-EM-remainder-le-strong-int'*:  
 $\forall x. \text{of-int } x \geq a \longrightarrow \text{norm } (EM\text{-remainder } n f x) \leq D / \text{fact } n * g x$   
**and** *norm-EM-remainder-le-strong-nat'*:  
 $\forall x. \text{real } x \geq a \longrightarrow \text{norm } (EM\text{-remainder } n f (\text{int } x)) \leq D / \text{fact } n * g x$   
*<proof>*

**lemma** *norm-EM-remainder'-le*:

**fixes** *f* :: *real*  $\Rightarrow$  '*a* :: {*real-normed-field, banach*}  
**and** *g g'* :: *real*  $\Rightarrow$  *real*  
**assumes** *cont-f*: *continuous-on* {*a..*} *f*  
**assumes** *cont-g'*: *continuous-on* {*a..*} *g'*  
**assumes** *f-bigo*: *eventually* ( $\lambda x. \text{norm } (f x) \leq g' x$ ) *at-top*  
**assumes** *deriv*: *eventually* ( $\lambda x. (g \text{ has-field-derivative } g' x) (at x)$ ) *at-top*  
**obtains** *C D* **where**  
 $\text{eventually } (\lambda x. \text{norm } (EM\text{-remainder}' n f a x) \leq C + D * g x)$  *at-top*  
*<proof>*

## 1.6 Application to harmonic numbers

As a first application, we can apply the machinery we have developed to the harmonic numbers.

**definition** *harm-remainder* :: *nat*  $\Rightarrow$  *nat*  $\Rightarrow$  *real* **where**

$\text{harm-remainder } N n = EM\text{-remainder } (2*N+1) (\lambda x. -\text{fact } (2*N+1) / x \wedge$   
 $(2*N+2)) (\text{int } n)$

**lemma** *harm-expansion*:  
**assumes**  $n: n > 0$  **and**  $N: N > 0$   
**shows**  $\text{harm } n = \ln n + \text{euler-mascheroni} + 1 / (2*n) -$   
 $(\sum_{i=1..N}. \text{bernoulli } (2*i) / ((2*i) * n^{(2*i)})) - \text{harm-remainder}$   
 $N n$   
 $\langle \text{proof} \rangle$

**lemma** *of-nat-ge-1-iff*: *of-nat*  $x \geq (1 :: 'a :: \text{linordered-semidom}) \iff x \geq 1$   
 $\langle \text{proof} \rangle$

**lemma** *harm-remainder-bound*:  
**fixes**  $N :: \text{nat}$   
**assumes**  $N: N > 0$   
**shows**  $\exists C. \forall n \geq 1. \text{norm } (\text{harm-remainder } N n) \leq C / \text{real } n^{(2*N+1)}$   
 $\langle \text{proof} \rangle$

## 1.7 Application to sums of inverse squares

In the same vein, we can derive the asymptotics of the partial sum of inverse squares.

**lemma** *sum-inverse-squares-expansion*:  
**assumes**  $n: n > 0$  **and**  $N: N > 0$   
**shows**  $(\sum_{k=1..n}. 1 / \text{real } k^2) =$   
 $\pi^2 / 6 - 1 / \text{real } n + 1 / (2 * \text{real } n^2) -$   
 $(\sum_{i=1..N}. \text{bernoulli } (2*i) / n^{(2*i+1)}) -$   
 $\text{EM-remainder } (2*N+1) (\lambda x. -\text{fact } (2*N+2) / x^{(2*N+3)})$   
 $(\text{int } n)$   
 $\langle \text{proof} \rangle$

**lemma** *sum-inverse-squares-remainder-bound*:  
**fixes**  $N :: \text{nat}$   
**assumes**  $N: N > 0$   
**defines**  $R \equiv (\lambda n. \text{EM-remainder } (2*N+1) (\lambda x. -\text{fact } (2*N+2) / x^{(2*N+3)}))$   
 $(\text{int } n)$   
**shows**  $\exists C. \forall n \geq 1. \text{norm } (R n) \leq C / \text{real } n^{(2*N+2)}$   
 $\langle \text{proof} \rangle$

**end**

## 2 Connection of Euler–MacLaurin summation to Landau symbols

**theory** *Euler-MacLaurin-Landau*  
**imports**  
 $\text{Euler-MacLaurin}$   
 $\text{Landau-Symbols.Landau-More}$   
**begin**

## 2.1 O-bound for the remainder term

Landau symbols allow us to state the bounds on the remainder terms from the Euler–MacLaurin formula a bit more nicely.

**lemma**

**fixes**  $f :: \text{real} \Rightarrow 'a :: \{\text{real-normed-field}, \text{banach}\}$   
**and**  $g\ g' :: \text{real} \Rightarrow \text{real}$   
**assumes**  $\text{fin}: \text{finite } Y$   
**assumes**  $\text{cont-f}: \text{continuous-on } \{a..\} f$   
**assumes**  $\text{cont-g}: \text{continuous-on } \{a..\} g$   
**assumes**  $\text{cont-g}': \text{continuous-on } \{a..\} g'$   
**assumes**  $\text{limit-g}: (g \longrightarrow 0) \text{ at-top}$   
**assumes**  $\text{f-bound}: \bigwedge x. x \geq a \implies \text{norm } (f\ x) \leq g'\ x$   
**assumes**  $\text{deriv}: \bigwedge x. x \in \{a..\} - Y \implies (g \text{ has-field-derivative } -g'\ x) \text{ (at } x)$   
**shows**  $\text{EM-remainder-strong-bigo-int}: (\lambda x::\text{int}. \text{norm } (\text{EM-remainder } n\ f\ x)) \in O(g)$   
**and**  $\text{EM-remainder-strong-bigo-nat}: (\lambda x::\text{nat}. \text{norm } (\text{EM-remainder } n\ f\ x)) \in O(g)$   
 $\langle \text{proof} \rangle$

## 2.2 Asymptotic expansion of the harmonic numbers

We can now show the asymptotic expansion

$$H_n = \ln n + \gamma + \frac{1}{2n} - \sum_{i=1}^m \frac{B_{2i}}{2i} n^{-2i} + O(n^{-2m-2})$$

**lemma** *harm-remainder-bigo*:

**assumes**  $N > 0$   
**shows**  $\text{harm-remainder } N \in O(\lambda n. 1 / \text{real } n \wedge (2 * N + 1))$   
 $\langle \text{proof} \rangle$

**lemma** *harm-expansion-bigo*:

**fixes**  $N :: \text{nat}$   
**defines**  $T \equiv \lambda n. \ln n + \text{euler-mascheroni} + 1 / (2 * n) -$   
 $(\sum_{i=1..N}. \text{bernoulli } (2 * i) / ((2 * i) * n \wedge (2 * i)))$   
**defines**  $S \equiv (\lambda n. \text{bernoulli } (2 * (\text{Suc } N)) / ((2 * \text{Suc } N) * \text{real } n \wedge (2 * \text{Suc } N)))$   
**shows**  $(\lambda n. \text{harm } n - T\ n) \in O(\lambda n. 1 / \text{real } n \wedge (2 * N + 2))$   
 $\langle \text{proof} \rangle$

**lemma** *harm-expansion-bigo-simple1*:

$(\lambda n. \text{harm } n - (\ln n + \text{euler-mascheroni} + 1 / (2 * n))) \in O(\lambda n. 1 / n \wedge 2)$   
 $\langle \text{proof} \rangle$

**lemma** *harm-expansion-bigo-simple2*:

$(\lambda n. \text{harm } n - (\ln n + \text{euler-mascheroni})) \in O(\lambda n. 1 / n)$   
 $\langle \text{proof} \rangle$

**lemma** *harm-expansion-bigo-simple'*:

*harm* = *o* ( $\lambda n. \ln n + \text{euler-mascheroni} + 1 / (2 * n)$ ) + *o* ( $O(\lambda n. 1 / n \wedge 2)$ )

*<proof>*

## 2.3 Asymptotic expansion of the sum of inverse squares

Similarly to before, we show

$$\sum_{i=1}^n \frac{1}{i^2} = \frac{\pi^2}{6} - \frac{1}{n} + \frac{1}{2n^2} - \sum_{i=1}^m B_{2i} n^{-2i-1} + O(n^{-2m-3})$$

**context**

**fixes**  $R :: \text{nat} \Rightarrow \text{nat} \Rightarrow \text{real}$

**defines**  $R \equiv (\lambda N n. \text{EM-remainder } (2*N+1) (\lambda x. -\text{fact } (2*N+2) / x \wedge (2*N+3)))$   
(*int n*)

**begin**

**lemma** *sum-inverse-squares-remainder-bigo*:

**assumes**  $N > 0$

**shows**  $R N \in O(\lambda n. 1 / \text{real } n \wedge (2 * N + 2))$

*<proof>*

**lemma** *sum-inverse-squares-expansion-bigo*:

**fixes**  $N :: \text{nat}$

**defines**  $T \equiv \lambda n. \pi \wedge 2 / 6 - 1 / n + 1 / (2 * n \wedge 2) -$   
 $(\sum_{i=1..N}. \text{bernoulli } (2*i) / (n \wedge (2*i+1)))$

**defines**  $S \equiv (\lambda n. \text{bernoulli } (2*(\text{Suc } N)) / (\text{real } n \wedge (2*N+3)))$

**shows** ( $\lambda n. (\sum_{i=1..n}. 1 / \text{real } i \wedge 2) - T n$ )  $\in O(\lambda n. 1 / \text{real } n \wedge (2 * N + 3))$

*<proof>*

**lemma** *sum-inverse-squares-expansion-bigo-simple*:

( $\lambda n. (\sum_{i=1..n}. 1 / \text{real } i \wedge 2) - (\pi \wedge 2 / 6 - 1 / n + 1 / (2 * n \wedge 2))$ )  $\in O(\lambda n. 1 / n \wedge 3)$

*<proof>*

**lemma** *sum-inverse-squares-expansion-bigo-simple'*:

( $\lambda n. (\sum_{i=1..n}. 1 / \text{real } i \wedge 2)$ ) = *o* ( $\lambda n. \pi \wedge 2 / 6 - 1 / n + 1 / (2 * n \wedge 2)$ ) + *o* ( $O(\lambda n. 1 / n \wedge 3)$ )

*<proof>*

**end**

**end**

## References

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