The Error Function

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Abstract

This entry provides the definitions and basic properties of the complex and real error function erf and the complementary error function erfc. Additionally, it gives their full asymptotic expansions.

Contents

1 The complex and real error function

theory *Error-Function*
imports *HOL−Complex-Analysis.Complex-Analysis HOL−Library.Landau-Symbols*
begin

1.1 Auxiliary Facts

lemma *tendsto-sandwich-mono*:
assumes $(\lambda n.\ f\ (real\ n)) \longlongrightarrow (c::real)$
assumes *eventually* $(\lambda x.\ \forall y\ z.\ x \leq y \land y \leq z \longrightarrow f\ y \leq f\ z)$ *at-top*
shows $(f \longrightarrow c)$ *at-top*
⟨*proof*⟩

lemma *tendsto-sandwich-antimono*:
assumes $(\lambda n.\ f\ (real\ n)) \longlongrightarrow (c::real)$
assumes *eventually* $(\lambda x.\ \forall y\ z.\ x \leq y \land y \leq z \longrightarrow f\ y \geq f\ z)$ *at-top*
shows $(f \longrightarrow c)$ *at-top*
⟨*proof*⟩

lemma *has-bochner-integral-completion* [*intro*]:
fixes $f :: 'a \Rightarrow 'b::\{banach,\ second\text{-}countable\text{-}topology\}$
shows *has-bochner-integral M f I* $\Longrightarrow$ *has-bochner-integral* (*completion M*) *f I*
⟨*proof*⟩

lemma *has-bochner-integral-imp-has-integral*:
has-bochner-integral lebesgue $(\lambda x.\ indicator\ S\ x *_R f\ x)\ I \Longrightarrow$
$(f$ *has-integral* $(I :: 'b :: euclidean\text{-}space))\ S$
⟨*proof*⟩

lemma *has-bochner-integral-imp-has-integral'*:
has-bochner-integral lborel $(\lambda x.\ indicator\ S\ x *_R f\ x)\ I \Longrightarrow$
$(f$ *has-integral* $(I :: 'b :: euclidean\text{-}space))\ S$
⟨*proof*⟩

lemma *has-bochner-integral-erf-aux*:
has-bochner-integral lborel $(\lambda x.\ indicator\ \{0..\}\ x *_R exp\ (-\ x^2))\ (sqrt\ pi\ /\ 2)$
⟨*proof*⟩

lemma *has-integral-erf-aux*: $((\lambda t::real.\ exp\ (-t^2))$ *has-integral* $(sqrt\ pi\ /\ 2))\ \{0..\}$
⟨*proof*⟩

lemma *contour-integrable-on-linepath-neg-exp-squared* [*simp*, *intro*]:
$(\lambda t.\ exp\ (-(t\hat{}2)))$ *contour-integrable-on linepath 0 z*
⟨*proof*⟩

lemma *holomorphic-on-chain*:
g holomorphic-on t $\Longrightarrow$ *f holomorphic-on s* $\Longrightarrow$ $f\ `\ s \subseteq t \Longrightarrow$
$(\lambda x.\ g\ (f\ x))$ *holomorphic-on s*
⟨*proof*⟩

lemma *holomorphic-on-chain-UNIV*:
g holomorphic-on UNIV $\Longrightarrow$ *f holomorphic-on s* $\Longrightarrow$
($\lambda x.$ *g* (*f x*)) *holomorphic-on s*
⟨*proof*⟩

lemmas *holomorphic-on-exp*′ [*holomorphic-intros*] =
holomorphic-on-exp [*THEN holomorphic-on-chain-UNIV*]

lemma *leibniz-rule-field-derivative-real*:
fixes *f*::′*a*::{*real-normed-field*, *banach*} ⇒ *real* ⇒ ′*a*
assumes *fx*: $\bigwedge x\ t.\ x \in U \Longrightarrow t \in \{a..b\} \Longrightarrow$ (($\lambda x.$ *f x t*) *has-field-derivative fx x t*) (*at x within U*)
assumes *integrable-f2*: $\bigwedge x.\ x \in U \Longrightarrow$ (*f x*) *integrable-on* {*a*..*b*}
assumes *cont-fx*: *continuous-on* (*U* × {*a*..*b*}) (λ(*x*, *t*). *fx x t*)
assumes *U*: *x0* ∈ *U convex U*
shows (($\lambda x.$ *integral* {*a*..*b*} (*f x*)) *has-field-derivative integral* {*a*..*b*} (*fx x0*)) (*at x0 within U*)
⟨*proof*⟩

lemma *has-vector-derivative-linepath-within* [*derivative-intros*]:
assumes [*derivative-intros*]:
(*f has-vector-derivative f*′) (*at x within S*) (*g has-vector-derivative g*′) (*at x within S*)
(*h has-real-derivative h*′) (*at x within S*)
shows (($\lambda x.$ *linepath* (*f x*) (*g x*) (*h x*)) *has-vector-derivative*
(*1* − *h x*) $*_R$ *f*′ + *h x* $*_R$ *g*′ − *h*′ $*_R$ (*f x* − *g x*)) (*at x within S*)
⟨*proof*⟩

lemma *has-field-derivative-linepath-within* [*derivative-intros*]:
assumes [*derivative-intros*]:
(*f has-field-derivative f*′) (*at x within S*) (*g has-field-derivative g*′) (*at x within S*)
(*h has-real-derivative h*′) (*at x within S*)
shows (($\lambda x.$ *linepath* (*f x*) (*g x*) (*h x*)) *has-field-derivative*
(*1* − *h x*) $*_R$ *f*′ + *h x* $*_R$ *g*′ − *h*′ $*_R$ (*f x* − *g x*)) (*at x within S*)
⟨*proof*⟩

lemma *continuous-on-linepath*′ [*continuous-intros*]:
assumes [*continuous-intros*]: *continuous-on A f continuous-on A g continuous-on A h*
shows *continuous-on A* ($\lambda x.$ *linepath* (*f x*) (*g x*) (*h x*))
⟨*proof*⟩

lemma *contour-integral-has-field-derivative*:
assumes *A*: *open A convex A a* ∈ *A z* ∈ *A*
assumes *integrable*: $\bigwedge z.\ z \in A \Longrightarrow$ *f contour-integrable-on linepath a z*
assumes *holo*: *f holomorphic-on A*
shows (($\lambda z.$ *contour-integral* (*linepath a z*) *f*) *has-field-derivative f z*) (*at z*

within B)
⟨*proof*⟩

1.2 Definition of the error function

definition *erf-coeffs* :: *nat* ⇒ *real* **where**
erf-coeffs n =
(*if odd n then 2 / sqrt pi* ∗ (−*1*) ^ (*n div 2*) / (*of-nat n* ∗ *fact* (*n div 2*))
else 0)

lemma *summable-erf*:
fixes *z* :: ′*a* :: {*real-normed-div-algebra*, *banach*}
shows *summable* (λ*n*. *of-real* (*erf-coeffs n*) ∗ *z* ^ *n*)
⟨*proof*⟩

definition *erf* :: (′*a* :: {*real-normed-field*, *banach*}) ⇒ ′*a* **where**
erf z = (∑*n*. *of-real* (*erf-coeffs n*) ∗ *z* ^ *n*)

lemma *erf-converges*: (λ*n*. *of-real* (*erf-coeffs n*) ∗ *z* ^ *n*) *sums erf z*
⟨*proof*⟩

lemma *erf-0* [*simp*]: *erf 0* = *0*
⟨*proof*⟩

lemma *erf-minus* [*simp*]: *erf* (−*z*) = − *erf z*
⟨*proof*⟩

lemma *erf-of-real* [*simp*]: *erf* (*of-real x*) = *of-real* (*erf x*)
⟨*proof*⟩

lemma *of-real-erf-numeral* [*simp*]: *of-real* (*erf* (*numeral n*)) = *erf* (*numeral n*)
⟨*proof*⟩

lemma *of-real-erf-1* [*simp*]: *of-real* (*erf 1*) = *erf 1*
⟨*proof*⟩

lemma *erf-has-field-derivative*:
(*erf has-field-derivative of-real* (*2 / sqrt pi*) ∗ *exp* (−(*z*^*2*))) (*at z within A*)
⟨*proof*⟩

lemmas *erf-has-field-derivative*′ [*derivative-intros*] =
erf-has-field-derivative [*THEN DERIV-chain2*]

lemma *erf-continuous-on*: *continuous-on A erf*
⟨*proof*⟩

lemma *continuous-on-compose2-UNIV*:
continuous-on UNIV g ⟹ *continuous-on s f* ⟹ *continuous-on s* (λ*x*. *g* (*f x*))

⟨*proof*⟩

lemmas *erf-continuous-on′* [*continuous-intros*] =
erf-continuous-on [*THEN continuous-on-compose2-UNIV*]

lemma *erf-continuous* [*continuous-intros*]: *continuous* (*at x within A*) *erf*
⟨*proof*⟩

lemmas *erf-continuous′* [*continuous-intros*] =
continuous-within-compose2[*OF - erf-continuous*]

lemmas *tendsto-erf* [*tendsto-intros*] = *isCont-tendsto-compose*[*OF erf-continuous*]

lemma *erf-cnj* [*simp*]: *erf* (*cnj z*) = *cnj* (*erf z*)
⟨*proof*⟩

lemma *integral-exp-minus-squared-real*:
assumes $a \leq b$
shows $((\lambda t.\ exp\ (-(t\hat{\ }2)))\ has\text{-}integral\ (sqrt\ pi\ /\ 2 * (erf\ b - erf\ a)))\ \{a..b\}$
⟨*proof*⟩

lemma *erf-real-altdef-nonneg*:
$x \geq 0 \Longrightarrow erf\ (x::real) = 2\ /\ sqrt\ pi * integral\ \{0..x\}\ (\lambda t.\ exp\ (-(t\hat{\ }2)))$
⟨*proof*⟩

lemma *erf-real-altdef-nonpos*:
$x \leq 0 \Longrightarrow erf\ (x::real) = -2\ /\ sqrt\ pi * integral\ \{0..-x\}\ (\lambda t.\ exp\ (-(t\hat{\ }2)))$
⟨*proof*⟩

lemma *less-imp-erf-real-less*:
assumes $a < (b::real)$
shows $erf\ a < erf\ b$
⟨*proof*⟩

lemma *le-imp-erf-real-le*: $a \leq (b::real) \Longrightarrow erf\ a \leq erf\ b$
⟨*proof*⟩

lemma *erf-real-less-cancel* [*simp*]: $(erf\ (a :: real) < erf\ b) \longleftrightarrow a < b$
⟨*proof*⟩

lemma *erf-real-eq-iff* [*simp*]: $erf\ (a::real) = erf\ b \longleftrightarrow a = b$
⟨*proof*⟩

lemma *erf-real-le-cancel* [*simp*]: $(erf\ (a :: real) \leq erf\ b) \longleftrightarrow a \leq b$
⟨*proof*⟩

lemma *inj-on-erf-real* [*intro*]: $inj\text{-}on\ (erf :: real \Rightarrow real)\ A$
⟨*proof*⟩

lemma *strict-mono-erf-real* [*intro*]: *strict-mono* (*erf* :: *real* ⇒ *real*)
⟨*proof*⟩

lemma *mono-erf-real* [*intro*]: *mono* (*erf* :: *real* ⇒ *real*)
⟨*proof*⟩

lemma *erf-real-ge-0-iff* [*simp*]: *erf* (*x*::*real*) ≥ *0* ⟷ *x* ≥ *0*
⟨*proof*⟩

lemma *erf-real-le-0-iff* [*simp*]: *erf* (*x*::*real*) ≤ *0* ⟷ *x* ≤ *0*
⟨*proof*⟩

lemma *erf-real-gt-0-iff* [*simp*]: *erf* (*x*::*real*) > *0* ⟷ *x* > *0*
⟨*proof*⟩

lemma *erf-real-less-0-iff* [*simp*]: *erf* (*x*::*real*) < *0* ⟷ *x* < *0*
⟨*proof*⟩

lemma *erf-at-top* [*tendsto-intros*]: ((*erf* :: *real* ⇒ *real*) ⟶ *1*) *at-top*
⟨*proof*⟩

lemma *erf-at-bot* [*tendsto-intros*]: ((*erf* :: *real* ⇒ *real*) ⟶ −*1*) *at-bot*
⟨*proof*⟩

lemmas *tendsto-erf-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-top*]
lemmas *tendsto-erf-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erf-at-bot*]

1.3 The complimentary error function

definition *erfc* **where** *erfc z* = *1* − *erf z*

lemma *erf-conv-erfc*: *erf z* = *1* − *erfc z* ⟨*proof*⟩

lemma *erfc-0* [*simp*]: *erfc 0* = *1*
⟨*proof*⟩

lemma *erfc-minus*: *erfc* (−*z*) = *2* − *erfc z*
⟨*proof*⟩

lemma *erfc-of-real* [*simp*]: *erfc* (*of-real x*) = *of-real* (*erfc x*)
⟨*proof*⟩

lemma *of-real-erfc-numeral* [*simp*]: *of-real* (*erfc* (*numeral n*)) = *erfc* (*numeral n*)
⟨*proof*⟩

lemma *of-real-erfc-1* [*simp*]: *of-real* (*erfc 1*) = *erfc 1*

⟨*proof*⟩

lemma *less-imp-erfc-real-less*: $a < (b{::}real) \Longrightarrow erfc\ a > erfc\ b$
⟨*proof*⟩

lemma *le-imp-erfc-real-le*: $a \le (b{::}real) \Longrightarrow erfc\ a \ge erfc\ b$
⟨*proof*⟩

lemma *erfc-real-less-cancel* [*simp*]: $(erfc\ (a :: real) < erfc\ b) \longleftrightarrow a > b$
⟨*proof*⟩

lemma *erfc-real-eq-iff* [*simp*]: $erfc\ (a{::}real) = erfc\ b \longleftrightarrow a = b$
⟨*proof*⟩

lemma *erfc-real-le-cancel* [*simp*]: $(erfc\ (a :: real) \le erfc\ b) \longleftrightarrow a \ge b$
⟨*proof*⟩

lemma *inj-on-erfc-real* [*intro*]: $inj\text{-}on\ (erfc :: real \Rightarrow real)\ A$
⟨*proof*⟩

lemma *antimono-erfc-real* [*intro*]: $antimono\ (erfc :: real \Rightarrow real)$
⟨*proof*⟩

lemma *erfc-real-ge-0-iff* [*simp*]: $erfc\ (x{::}real) \ge 1 \longleftrightarrow x \le 0$
⟨*proof*⟩

lemma *erfc-real-le-0-iff* [*simp*]: $erfc\ (x{::}real) \le 1 \longleftrightarrow x \ge 0$
⟨*proof*⟩

lemma *erfc-real-gt-0-iff* [*simp*]: $erfc\ (x{::}real) > 1 \longleftrightarrow x < 0$
⟨*proof*⟩

lemma *erfc-real-less-0-iff* [*simp*]: $erfc\ (x{::}real) < 1 \longleftrightarrow x > 0$
⟨*proof*⟩

lemma *erfc-has-field-derivative*:
$(erfc\ has\text{-}field\text{-}derivative\ -of\text{-}real\ (2\ /\ sqrt\ pi) * exp\ (-(z\hat{}2)))\ (at\ z\ within\ A)$
⟨*proof*⟩

lemmas *erfc-has-field-derivative′* [*derivative-intros*] =
erfc-has-field-derivative [*THEN DERIV-chain2*]

lemma *erfc-continuous-on*: *continuous-on A erfc*
⟨*proof*⟩

lemmas *erfc-continuous-on′* [*continuous-intros*] =
erfc-continuous-on [*THEN continuous-on-compose2-UNIV*]

lemma *erfc-continuous* [*continuous-intros*]: *continuous* (*at x within A*) *erfc*
⟨*proof*⟩

lemmas *erfc-continuous'* [*continuous-intros*] =
continuous-within-compose2[*OF - erfc-continuous*]

lemmas *tendsto-erfc* [*tendsto-intros*] = *isCont-tendsto-compose*[*OF erfc-continuous*]

lemma *erfc-at-top* [*tendsto-intros*]: ((*erfc* :: *real* ⇒ *real*) ⟶ *0*) *at-top*
⟨*proof*⟩

lemma *erfc-at-bot* [*tendsto-intros*]: ((*erfc* :: *real* ⇒ *real*) ⟶ *2*) *at-bot*
⟨*proof*⟩

lemmas *tendsto-erfc-at-top* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-top*]
lemmas *tendsto-erfc-at-bot* [*tendsto-intros*] = *filterlim-compose*[*OF erfc-at-bot*]

lemma *integrable-exp-minus-squared*:
assumes $A \subseteq \{0..\}$ $A \in$ *sets lborel*
shows *set-integrable lborel A* (λt::*real*. *exp* $(-t^2)$) (**is** *?thesis1*)
and (λt::*real*. *exp* $(-t^2)$) *integrable-on A* (**is** *?thesis2*)
⟨*proof*⟩

lemma
assumes $x \geq 0$
shows *erfc-real-altdef-nonneg*: *erfc x* = *2* / *sqrt pi* ∗ *integral* {*x*..} (λt. *exp* $(-t^2)$)
and *has-integral-erfc*: ((λt. *exp* $(-t^2)$) *has-integral* (*sqrt pi* / *2* ∗ *erfc x*)) {*x*..}
⟨*proof*⟩

lemma *erfc-real-gt-0* [*simp*, *intro*]: *erfc* (*x*::*real*) > *0*
⟨*proof*⟩

lemma *erfc-real-less-2* [*intro*]: *erfc* (*x*::*real*) < *2*
⟨*proof*⟩

lemma *erf-real-gt-neg1* [*intro*]: *erf* (*x*::*real*) > −*1*
⟨*proof*⟩

lemma *erf-real-less-1* [*intro*]: *erf* (*x*::*real*) < *1*
⟨*proof*⟩

lemma *erfc-cnj* [*simp*]: *erfc* (*cnj z*) = *cnj* (*erfc z*)
⟨*proof*⟩

1.4 Specific facts about the complex case

lemma *erf-complex-altdef*:
erf z = of-real (2 / sqrt pi) ∗ contour-integral (linepath 0 z) (λt. exp (−(t^2)))
⟨*proof*⟩

lemma *erf-holomorphic-on*: *erf holomorphic-on A*
⟨*proof*⟩

lemmas *erf-holomorphic-on′* [*holomorphic-intros*] =
erf-holomorphic-on [*THEN holomorphic-on-chain-UNIV*]

lemma *erf-analytic-on*: *erf analytic-on A*
⟨*proof*⟩

lemma *erf-analytic-on′* [*analytic-intros*]:
assumes *f analytic-on A*
shows (*λx. erf* (*f x*)) *analytic-on A*
⟨*proof*⟩

lemma *erfc-holomorphic-on*: *erfc holomorphic-on A*
⟨*proof*⟩

lemmas *erfc-holomorphic-on′* [*holomorphic-intros*] =
erfc-holomorphic-on [*THEN holomorphic-on-chain-UNIV*]

lemma *erfc-analytic-on*: *erfc analytic-on A*
⟨*proof*⟩

lemma *erfc-analytic-on′* [*analytic-intros*]:
assumes *f analytic-on A*
shows (*λx. erfc* (*f x*)) *analytic-on A*
⟨*proof*⟩

end

1.5 Asymptotics

theory *Error-Function-Asymptotics*
imports *Error-Function Landau-Symbols.Landau-More*
begin

lemma *real-powr-eq-powerI*:
x > 0 ⟹ y = real y′ ⟹ x powr y = x ^ y′
⟨*proof*⟩

definition *erf-remainder-integral* **where**
erf-remainder-integral n x =
lim (λm. integral {x..x + real m} (λt. exp (−(t^2)) / t ^ (2∗n)))

The following is the remainder term in the asymptotic expansion of *erfc*.

definition *erf-remainder* **where**
erf-remainder n x =
((−1)^n ∗ 2 ∗ fact (2∗n)) / (sqrt pi ∗ 4 ^ n ∗ fact n) ∗
erf-remainder-integral n x

lemma *erf-remainder-integral-aux-nonneg*:
x > 0 ⟹ integral {x..x + real m} (λt. exp (−(t^2)) / t ^(2∗n)) ≥ 0
⟨*proof*⟩

lemma *erf-remainder-integral-aux-bound*:
assumes *x > 0*
shows *norm (integral {x..x + real m} (λt. exp (−t^2) / t ^(2∗n))) ≤ exp (−x^2) / x ^(2∗n+1)*
and *integral {x..x + real m} (λt. exp (−t^2) / t ^(2∗n)) ≤ exp (−x^2) / x ^(2∗n+1)*
⟨*proof*⟩

lemma *convergent-erf-remainder-integral*:
assumes *x > 0*
shows *convergent (λm. integral {x..x + real m} (λt. exp (−(t^2)) / t ^(2∗n)))*
⟨*proof*⟩

lemma *LIMSEQ-erf-remainder-integral*:
x > 0 ⟹ (λm. integral {x..x + real m} (λt. exp (−(t^2)) / t ^(2∗n))) ⟶
erf-remainder-integral n x
⟨*proof*⟩

We show some bounds on the remainder term.

lemma
assumes *x > 0*
shows *erf-remainder-integral-nonneg*: *erf-remainder-integral n x ≥ 0*
and *erf-remainder-integral-bound*: *erf-remainder-integral n x ≤ exp (−x^2) / x ^(2∗n+1)*
⟨*proof*⟩

lemma *erf-remainder-integral-bigo*:
erf-remainder-integral n ∈ O(λx. exp (−x^2) / x ^(2∗n+1))
⟨*proof*⟩

theorem *erf-remainder-bigo*: *erf-remainder n ∈ O(λx. exp (−x^2) / x ^(2∗n+1))*
⟨*proof*⟩

Next, we unroll the remainder term to develop the asymptotic expansion.

lemma *erf-remainder-integral-0-conv-erfc*:
assumes *(x::real) > 0*
shows *erf-remainder-integral 0 x = sqrt pi / 2 ∗ erfc x*
⟨*proof*⟩

The first remainder is the *erfc* function itself.

lemma *erf-remainder-0-conv-erfc*: $x > 0 \Longrightarrow$ *erf-remainder* 0 x = *erfc* x
⟨*proof*⟩

Also, the following recurrence allows us to get the next term of the asymptotic expansion.

lemma *erf-remainder-integral-conv-Suc*:
assumes $x > 0$
shows *erf-remainder-integral* n x = *exp* $(-x^2)$ / $(2*x$^$(2*n+1))$ −
real $(2*n+1)$ / 2 ∗ *erf-remainder-integral* (*Suc* n) x
⟨*proof*⟩

lemma *erf-remainder-conv-Suc*:
assumes $x > 0$
shows *erf-remainder* n x = $(-\ 1)$ ^ n ∗ *fact* $(2 * n)$ / (*sqrt pi* ∗ 4 ^ n ∗ *fact* n) ∗
exp $(-\ x^2)$ / $(x$ ^ $(2 * n + 1))$ + *erf-remainder* (*Suc* n) x
⟨*proof*⟩

Finally, this gives us the full asymptotic expansion for *erfc*:

theorem *erfc-unroll*:
assumes $x > 0$
shows *erfc* x = *exp* $(-x^2)$ / *sqrt pi* ∗
$(\sum i<n.\ (-1)$^i ∗ *fact* $(2*i)$ / $(4$^$i*$*fact* $i)$ / x^$(2*i+1))$ + *erf-remainder* n x
⟨*proof*⟩

For convenience, we define another auxiliary function that is more suitable for use in an automated expansion framework, since it has a simple asymptotic expansion in powers of x.

definition *erfc-aux* **where** *erfc-aux* x = *exp* (x^2) ∗ *sqrt pi* ∗ *erfc* x
definition *erf-remainder′* **where** *erf-remainder′* n x = *exp* (x^2) ∗ *sqrt pi* ∗ *erf-remainder* n x

lemma *erfc-aux-unroll*:
$x > 0 \Longrightarrow$
erfc-aux x = $(\sum i<n.\ (-1)$^i ∗ *fact* $(2*i)$ / $(4$^$i*$*fact* $i)$ / x^$(2*i+1))$ + *erf-remainder′* n x
⟨*proof*⟩

lemma *erf-remainder′-bigo*: *erf-remainder′* $n \in O(\lambda x.\ 1\ /\ x$ ^ $(2*n+1))$
⟨*proof*⟩

lemma *has-field-derivative-erfc-aux*:
(*erfc-aux has-field-derivative* $(2 * x *$ *erfc-aux* $x - 2)$) (*at* x)
⟨*proof*⟩

end