Efficient Mergesort

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Abstract

We provide a formalization of the mergesort algorithm as used in GHC's Data.List module, proving correctness and stability. Furthermore, experimental data suggests that generated (Haskell-)code for this algorithm is much faster than for previous algorithms available in the Isabelle distribution.

theory *Efficient-Sort*
  imports *HOL−Library.Multiset*
begin

1 GHC Version of Mergesort

In the following we show that the mergesort implementation used in GHC (see http://hackage.haskell.org/package/base-4.11.1.0/docs/src/Data.OldList.html#sort) is a correct and stable sorting algorithm. Furthermore, experimental data suggests that generated code for this implementation is much more efficient than for the implementation provided by *HOL−Library.Multiset*.

A high-level overview of an older version of this formalization as well as some experimental data is to be found in [1].

1.1 Definition of Natural Mergesort

context
  fixes *key* :: *'a* ⇒ *'k*::*linorder*
begin

Split a list into chunks of ascending and descending parts, where descending parts are reversed on the fly. Thus, the result is a list of sorted lists.

fun *sequences* :: *'a list* ⇒ *'a list list*
  and *asc* :: *'a* ⇒ (*'a list* ⇒ *'a list*) ⇒ *'a list* ⇒ *'a list list*
  and *desc* :: *'a* ⇒ *'a list* ⇒ *'a list* ⇒ *'a list list*
  where
    sequences (*a* # *b* # *xs*) =

```
    (if key a > key b then desc b [a] xs else asc b ((#) a) xs)
| sequences [x] = [[x]]
| sequences [] = []
| asc a as (b # bs) =
    (if key a ≤ key b then asc b (λys. as (a # ys)) bs
    else as [a] # sequences (b # bs))
| asc a as [] = [as [a]]
| desc a as (b # bs) =
    (if key a > key b then desc b (a # as) bs
    else (a # as) # sequences (b # bs))
| desc a as [] = [a # as]

fun merge :: 'a list ⇒ 'a list ⇒ 'a list
  where
    merge (a # as) (b # bs) =
      (if key a > key b then b # merge (a # as) bs else a # merge as (b # bs))
  | merge [] bs = bs
  | merge as [] = as

fun merge-pairs :: 'a list list ⇒ 'a list list
  where
    merge-pairs (a # b # xs) = merge a b # merge-pairs xs
  | merge-pairs xs = xs

lemma length-merge [simp]:
  length (merge xs ys) = length xs + length ys
  by (induct xs ys rule: merge.induct) simp-all

lemma length-merge-pairs [simp]:
  length (merge-pairs xs) = (1 + length xs) div 2
  by (induct xs rule: merge-pairs.induct) simp-all

fun merge-all :: 'a list list ⇒ 'a list
  where
    merge-all [] = []
  | merge-all [x] = x
  | merge-all xs = merge-all (merge-pairs xs)

fun msort-key :: 'a list ⇒ 'a list
  where
    msort-key xs = merge-all (sequences xs)
```

1.2 The Functional Argument of *local.asc*

f is a function that only adds some prefix to a given list.

```
definition ascP f = (∀ xs. f xs = f [] @ xs)

lemma ascP-Cons [simp]: ascP ((#) x) by (simp add: ascP-def)
```

lemma *ascP-comp-append-Cons* [*simp*]:
ascP (λ*xs*. *f* [] @ *x* # *xs*)
by (*auto simp*: *ascP-def*)

lemma *ascP-f-Cons*:
assumes *ascP f*
shows *f* (*x* # *xs*) = *f* [] @ *x* # *xs*
using ‹*ascP f*› [*unfolded ascP-def*, *THEN spec*, *of x* # *xs*] **.**

lemma *ascP-comp-Cons* [*simp*]:
assumes *ascP f*
shows *ascP* (λ*ys*. *f* (*x* # *ys*))
proof (*unfold ascP-def*, *intro allI*)
fix *xs* **show** *f* (*x* # *xs*) = *f* [*x*] @ *xs*
using *assms* **by** (*simp add*: *ascP-f-Cons*)
qed

lemma *ascP-f-singleton*:
assumes *ascP f*
shows *f* [*x*] = *f* [] @ [*x*]
by (*rule ascP-f-Cons* [*OF assms*])

1.3 Facts about Lengths

lemma
shows *length-sequences*: *length* (*sequences xs*) ≤ *length xs*
and *length-asc*: *ascP f* ⟹ *length* (*asc a f ys*) ≤ *1* + *length ys*
and *length-desc*: *length* (*desc a xs ys*) ≤ *1* + *length ys*
by (*induct xs* **and** *a f ys* **and** *a xs ys rule*: *sequences-asc-desc.induct*) *auto*

lemma *length-concat-merge-pairs* [*simp*]:
length (*concat* (*merge-pairs xss*)) = *length* (*concat xss*)
by (*induct xss rule*: *merge-pairs.induct*) *simp-all*

1.4 Functional Correctness

lemma *mset-merge* [*simp*]:
mset (*merge xs ys*) = *mset xs* + *mset ys*
by (*induct xs ys rule*: *merge.induct*) *simp-all*

lemma *set-merge* [*simp*]:
set (*merge xs ys*) = *set xs* ∪ *set ys*
by (*simp flip*: *set-mset-mset*)

lemma *mset-concat-merge-pairs* [*simp*]:
mset (*concat* (*merge-pairs xs*)) = *mset* (*concat xs*)
by (*induct xs rule*: *merge-pairs.induct*) *auto*

lemma *set-concat-merge-pairs* [*simp*]:
set (*concat* (*merge-pairs xs*)) = *set* (*concat xs*)

by (*simp flip*: *set-mset-mset*)

lemma *mset-merge-all* [*simp*]:
mset (*merge-all xs*) = *mset* (*concat xs*)
by (*induct xs rule*: *merge-all.induct*) *simp-all*

lemma *set-merge-all* [*simp*]:
set (*merge-all xs*) = *set* (*concat xs*)
by (*simp flip*: *set-mset-mset*)

lemma
shows *mset-seqeuences* [*simp*]: *mset* (*concat* (*sequences xs*)) = *mset xs*
and *mset-asc*: *ascP f* ⟹ *mset* (*concat* (*asc x f ys*)) = {#*x*#} + *mset* (*f* []) + *mset ys*
and *mset-desc*: *mset* (*concat* (*desc x xs ys*)) = {#*x*#} + *mset xs* + *mset ys*
by (*induct xs* **and** *x f ys* **and** *x xs ys rule*: *sequences-asc-desc.induct*)
(*auto simp*: *ascP-f-singleton*)

lemma *mset-msort-key*:
mset (*msort-key xs*) = *mset xs*
by (*auto*)

lemma *sorted-merge* [*simp*]:
assumes *sorted* (*map key xs*) **and** *sorted* (*map key ys*)
shows *sorted* (*map key* (*merge xs ys*))
using *assms* **by** (*induct xs ys rule*: *merge.induct*) (*auto*)

lemma *sorted-merge-pairs* [*simp*]:
assumes ∀ *x*∈*set xs*. *sorted* (*map key x*)
shows ∀ *x*∈*set* (*merge-pairs xs*). *sorted* (*map key x*)
using *assms* **by** (*induct xs rule*: *merge-pairs.induct*) *simp-all*

lemma *sorted-merge-all*:
assumes ∀ *x*∈*set xs*. *sorted* (*map key x*)
shows *sorted* (*map key* (*merge-all xs*))
using *assms* **by** (*induct xs rule*: *merge-all.induct*) *simp-all*

lemma
shows *sorted-sequences*: ∀ *x* ∈ *set* (*sequences xs*). *sorted* (*map key x*)
and *sorted-asc*: *ascP f* ⟹ *sorted* (*map key* (*f* [])) ⟹ ∀ *x*∈*set* (*f* []). *key x* ≤ *key a* ⟹ ∀ *x*∈*set* (*asc a f ys*). *sorted* (*map key x*)
and *sorted-desc*: *sorted* (*map key xs*) ⟹ ∀ *x*∈*set xs*. *key a* ≤ *key x* ⟹ ∀ *x*∈*set* (*desc a xs ys*). *sorted* (*map key x*)
by (*induct xs* **and** *a f ys* **and** *a xs ys rule*: *sequences-asc-desc.induct*)
(*auto simp*: *ascP-f-singleton sorted-append not-less dest*: *order-trans*, *fastforce*)

lemma *sorted-msort-key*:
sorted (*map key* (*msort-key xs*))
by (*unfold msort-key.simps*) (*intro sorted-merge-all sorted-sequences*)

1.5 Stability

lemma
shows *filter-by-key-sequences* [*simp*]: [*y*←*concat* (*sequences xs*). *key y* = *k*] = [*y*←*xs*. *key y* = *k*]
and *filter-by-key-asc*: *ascP f* ⟹ [*y*←*concat* (*asc a f ys*). *key y* = *k*] = [*y*←*f* [*a*] @ *ys*. *key y* = *k*]
and *filter-by-key-desc*: *sorted* (*map key xs*) ⟹ ∀ *x*∈*set xs*. *key a* ≤ *key x* ⟹ [*y*←*concat* (*desc a xs ys*). *key y* = *k*] = [*y*←*a* # *xs* @ *ys*. *key y* = *k*]
proof (*induct xs* **and** *a f ys* **and** *a xs ys rule*: *sequences-asc-desc.induct*)
case (*4 a f b bs*)
then show *?case*
by (*auto simp*: *o-def ascP-f-Cons* [**where** *f* = *f*])
next
case (*6 a as b bs*)
then show *?case*
proof (*cases key b* < *key a*)
case *True*
with *6* **have** [*y*←*concat* (*desc b* (*a* # *as*) *bs*). *key y* = *k*] = [*y*←*b* # (*a* # *as*) @ *bs*. *key y* = *k*]
by (*auto simp*: *less-le order-trans*)
then show *?thesis*
using *True* **and** *6*
by (*cases key a* = *k*, *cases key b* = *k*)
(*auto simp*: *Cons-eq-append-conv intro*!: *filter-False*)
qed *auto*
qed *auto*

lemma *filter-by-key-merge-is-append* [*simp*]:
assumes *sorted* (*map key xs*)
shows [*y*←*merge xs ys*. *key y* = *k*] = [*y*←*xs*. *key y* = *k*] @ [*y*←*ys*. *key y* = *k*]
using *assms*
by (*induct xs ys rule*: *merge.induct*) (*auto simp*: *Cons-eq-append-conv leD intro*!: *filter-False*)

lemma *filter-by-key-merge-pairs* [*simp*]:
assumes ∀ *xs*∈*set xss*. *sorted* (*map key xs*)
shows [*y*←*concat* (*merge-pairs xss*). *key y* = *k*] = [*y*←*concat xss*. *key y* = *k*]
using *assms* **by** (*induct xss rule*: *merge-pairs.induct*) *simp-all*

lemma *filter-by-key-merge-all* [*simp*]:
assumes ∀ *xs*∈*set xss*. *sorted* (*map key xs*)
shows [*y*←*merge-all xss*. *key y* = *k*] = [*y*←*concat xss*. *key y* = *k*]
using *assms* **by** (*induct xss rule*: *merge-all.induct*) *simp-all*

lemma *filter-by-key-merge-all-sequences* [*simp*]:
[*x*←*merge-all* (*sequences xs*) . *key x* = *k*] = [*x*←*xs* . *key x* = *k*]
using *sorted-sequences* [*of xs*] **by** *simp*

lemma *msort-key-stable*:

$[x \leftarrow msort\text{-}key\ xs.\ key\ x = k] = [x \leftarrow xs.\ key\ x = k]$
by *auto*

lemma *sort-key-msort-key-conv*:
sort-key key = *msort-key*
using *msort-key-stable* [*of key x* **for** *x*]
by (*intro ext properties-for-sort-key mset-msort-key sorted-msort-key*)
(*metis* (*mono-tags*, *lifting*) *filter-cong*)

end

Replace existing code equations for *sort-key* by *msort-key*.

declare *sort-key-by-quicksort-code* [*code del*]
declare *sort-key-msort-key-conv* [*code*]

end

theory *Mergesort-Complexity*
imports
Efficient-Sort
Complex-Main
begin

lemma *log2-mono*:
$x > 0 \Longrightarrow x \leq y \Longrightarrow log\ 2\ x \leq log\ 2\ y$
by *auto*

2 Counting the Number of Comparisons

context
fixes *key* :: *'a* ⇒ *'k*::*linorder*
begin

fun *c-merge* :: *'a list* ⇒ *'a list* ⇒ *nat*
where
c-merge (*x* # *xs*) (*y* # *ys*) =
1 + (*if key y* < *key x then c-merge* (*x* # *xs*) *ys else c-merge xs* (*y* # *ys*))
| *c-merge* [] *ys* = *0*
| *c-merge xs* [] = *0*

fun *c-merge-pairs* :: *'a list list* ⇒ *nat*
where
c-merge-pairs (*xs* # *ys* # *zss*) = *c-merge xs ys* + *c-merge-pairs zss*
| *c-merge-pairs* [] = *0*
| *c-merge-pairs* [*x*] = *0*

fun *c-merge-all* :: *'a list list* ⇒ *nat*
where

```
    c-merge-all [] = 0
  | c-merge-all [x] = 0
  | c-merge-all xss = c-merge-pairs xss + c-merge-all (merge-pairs key xss)

fun c-sequences :: 'a list ⇒ nat
  and c-asc :: 'a ⇒ 'a list ⇒ nat
  and c-desc :: 'a ⇒ 'a list ⇒ nat
  where
    c-sequences (x # y # zs) = 1 + (if key y < key x then c-desc y zs else c-asc y
zs)
  | c-sequences [] = 0
  | c-sequences [x] = 0
  | c-asc x (y # ys) = 1 + (if ¬ key y < key x then c-asc y ys else c-sequences (y
# ys))
  | c-asc x [] = 0
  | c-desc x (y # ys) = 1 + (if key y < key x then c-desc y ys else c-sequences (y
# ys))
  | c-desc x [] = 0

fun c-msort :: 'a list ⇒ nat
  where
    c-msort xs = c-sequences xs + c-merge-all (sequences key xs)

lemma c-merge:
  c-merge xs ys ≤ length xs + length ys
  by (induct xs ys rule: c-merge.induct) simp-all

lemma c-merge-pairs:
  c-merge-pairs xss ≤ length (concat xss)
proof (induct xss rule: c-merge-pairs.induct)
  case (1 xs ys zss)
  then show ?case using c-merge [of xs ys] by simp
qed simp-all

lemma c-merge-all:
  c-merge-all xss ≤ length (concat xss) * ⌈log 2 (length xss)⌉
proof (induction xss rule: c-merge-all.induct)
  case (3 xs ys zss)
  let ?clen = λxs. length (concat xs)
  let ?xss = xs # ys # zss
  let ?xss2 = merge-pairs key ?xss

  have *: ⌈log 2 (real n + 2)⌉ = ⌈log 2 (Suc n div 2 + 1)⌉ + 1 for n :: nat
    using ceiling-log2-div2 [of n + 2] by (simp add: algebra-simps)

  have c-merge-all ?xss = c-merge-pairs ?xss + c-merge-all ?xss2 by simp
  also have ... ≤ ?clen ?xss + c-merge-all ?xss2
    using c-merge [of xs ys] and c-merge-pairs [of ?xss] by auto
  also have ... ≤ ?clen ?xss + ?clen ?xss2 * ⌈log 2 (length ?xss2)⌉
```

using *3.IH* **by** *simp*
also have $\ldots \leq$ *?clen ?xss* $*$ $\lceil$*log 2* (*length ?xss*)$\rceil$
by (*auto simp*: $*$ *algebra-simps*)
finally show *?case* **by** *simp*
qed *simp-all*

lemma
shows *c-sequences*: *c-sequences xs* $\leq$ *length xs* $-$ *1*
and *c-asc*: *c-asc x ys* $\leq$ *length ys*
and *c-desc*: *c-desc x ys* $\leq$ *length ys*
by (*induct xs* **and** *x ys* **and** *x ys rule*: *c-sequences-c-asc-c-desc.induct*) *simp-all*

lemma
shows *length-concat-sequences* [*simp*]: *length* (*concat* (*sequences key xs*)) = *length xs*
and *length-concat-asc*: *ascP f* $\Longrightarrow$ *length* (*concat* (*asc key a f ys*)) = *1* + *length* (*f* []) + *length ys*
and *length-concat-desc*: *length* (*concat* (*desc key a xs ys*)) = *1* + *length xs* + *length ys*
by (*induct xs* **and** *a f ys* **and** *a xs ys rule*: *sequences-asc-desc.induct*)
(*auto simp*: *ascP-f-singleton*)

lemma
shows *sequences-ne*: *xs* $\neq$ [] $\Longrightarrow$ *sequences key xs* $\neq$ []
and *asc-ne*: *ascP f* $\Longrightarrow$ *asc key a f ys* $\neq$ []
and *desc-ne*: *desc key a xs ys* $\neq$ []
by (*induct xs* **and** *a f ys* **and** *a xs ys taking*: *key rule*: *sequences-asc-desc.induct*) *simp-all*

lemma *c-msort*:
assumes [*simp*]: *length xs* = *n*
shows *c-msort xs* $\leq$ *n* + *n* $*$ $\lceil$*log 2 n*$\rceil$
proof $-$
have [*simp*]: *xs* = [] $\longleftrightarrow$ *length xs* = *0* **by** *blast*
have *int* (*c-merge-all* (*sequences key xs*)) $\leq$ *int n* $*$ $\lceil$*log 2* (*length* (*sequences key xs*))$\rceil$
using *c-merge-all* [*of sequences key xs*] **by** *simp*
also have $\ldots \leq$ *int n* $*$ $\lceil$*log 2 n*$\rceil$
using *length-sequences* [*of key xs*]
by (*cases n*) (*auto intro*!: *sequences-ne mult-mono ceiling-mono log2-mono*)
finally have *int* (*c-merge-all* (*sequences key xs*)) $\leq$ *int n* $*$ $\lceil$*log 2 n*$\rceil$ **.**
moreover have *c-sequences xs* $\leq$ *n* **using** *c-sequences* [*of xs*] **by** *auto*
ultimately show *?thesis* **by** (*auto intro*: *add-mono*)
qed

end

end

theory *Natural-Mergesort*
imports *HOL−Data-Structures.Sorting*
begin

2.1 Auxiliary Results

lemma *C-merge-adj'*:
C-merge-adj xss ≤ *length* (*concat xss*)
proof (*induct xss rule*: *C-merge-adj.induct*)
case (*3 xs ys zss*)
then show *?case* **using** *C-merge-ub* [*of xs ys*] **by** *simp*
qed *simp-all*

lemma *length-concat-merge-adj*:
length (*concat* (*merge-adj xss*)) = *length* (*concat xss*)
by (*induct xss rule*: *merge-adj.induct*) (*simp-all add*: *length-merge*)

lemma *C-merge-all'*:
C-merge-all xss ≤ *length* (*concat xss*) ∗ ⌈*log 2* (*length xss*)⌉
proof (*induction xss rule*: *C-merge-all.induct*)
case (*3 xs ys zss*)
let *?xss* = *xs* # *ys* # *zss*
let *?m* = *length* (*concat ?xss*)

have ∗: ⌈*log 2* (*real n* + *2*)⌉ = ⌈*log 2* (*Suc n div 2* + *1*)⌉ + *1* **for** *n* :: *nat*
using *ceiling-log2-div2* [*of n* + *2*] **by** (*simp add*: *algebra-simps*)

have *C-merge-all ?xss* = *C-merge-adj ?xss* + *C-merge-all* (*merge-adj ?xss*) **by** *simp*
also have ... ≤ *?m* + *C-merge-all* (*merge-adj ?xss*)
using *C-merge-adj'* [*of ?xss*] **by** *auto*
also have ... ≤ *?m* + *length* (*concat* (*merge-adj ?xss*)) ∗ ⌈*log 2* (*length* (*merge-adj ?xss*))⌉
using *3.IH* **by** *simp*
also have ... = *?m* + *?m* ∗ ⌈*log 2* (*length* (*merge-adj ?xss*))⌉
by (*simp only*: *length-concat-merge-adj*)
also have ... ≤ *?m* ∗ ⌈*log 2* (*length ?xss*)⌉
by (*auto simp*: ∗ *algebra-simps*)
finally show *?case* **by** *simp*
qed *simp-all*

2.2 Definition of Natural Mergesort

Partition input into ascending and descending subsequences. (The latter are reverted on the fly.)

fun *runs* :: ('*a*::*linorder*) *list* ⇒ '*a list list* **and**
asc :: '*a* ⇒ ('*a list* ⇒ '*a list*) ⇒ '*a list* ⇒ '*a list list* **and**
desc :: '*a* ⇒ '*a list* ⇒ '*a list* ⇒ '*a list list*

where
runs (*a* # *b* # *xs*) = (*if a* > *b then desc b* [*a*] *xs else asc b* ((#) *a*) *xs*)
| *runs* [*x*] = [[*x*]]
| *runs* [] = []
| *asc a as* (*b* # *bs*) = (*if* ¬ *a* > *b then asc b* (*as* ∘ (#) *a*) *bs else as* [*a*] # *runs* (*b* # *bs*))
| *asc a as* [] = [*as* [*a*]]
| *desc a as* (*b* # *bs*) = (*if a* > *b then desc b* (*a* # *as*) *bs else* (*a* # *as*) # *runs* (*b* # *bs*))
| *desc a as* [] = [*a* # *as*]

definition *nmsort* :: ('*a*::*linorder*) *list* ⇒ '*a list*
where
nmsort xs = *merge-all* (*runs xs*)

2.3 Functional Correctness

definition *ascP f* = (∀ *xs ys*. *f* (*xs* @ *ys*) = *f xs* @ *ys*)

lemma *ascP-Cons* [*simp*]: *ascP* ((#) *x*) **by** (*simp add*: *ascP-def*)

lemma *ascP-comp-Cons* [*simp*]: *ascP f* ⟹ *ascP* (*f* ∘ (#) *x*)
by (*auto simp*: *ascP-def simp flip*: *append-Cons*)

lemma *ascP-simp* [*simp*]:
assumes *ascP f*
shows *f* [*x*] = *f* [] @ [*x*]
using *assms* [*unfolded ascP-def*, *THEN spec*, *THEN spec*, *of* [] [*x*]] **by** *simp*

lemma
shows *mset-runs*: $\sum_{\#}$ (*image-mset mset* (*mset* (*runs xs*))) = *mset xs*
and *mset-asc*: *ascP f* ⟹ $\sum_{\#}$ (*image-mset mset* (*mset* (*asc x f ys*))) = {#*x*#} + *mset* (*f* []) + *mset ys*
and *mset-desc*: $\sum_{\#}$ (*image-mset mset* (*mset* (*desc x xs ys*))) = {#*x*#} + *mset xs* + *mset ys*
by (*induct xs* **and** *x f ys* **and** *x xs ys rule*: *runs-asc-desc.induct*) *auto*

lemma *mset-nmsort*:
mset (*nmsort xs*) = *mset xs*
by (*auto simp*: *mset-merge-all nmsort-def mset-runs*)

lemma
shows *sorted-runs*: ∀ *x*∈*set* (*runs xs*). *sorted x*
and *sorted-asc*: *ascP f* ⟹ *sorted* (*f* []) ⟹ ∀ *x*∈*set* (*f* []). *x* ≤ *a* ⟹ ∀ *x*∈*set* (*asc a f ys*). *sorted x*
and *sorted-desc*: *sorted xs* ⟹ ∀ *x*∈*set xs*. *a* ≤ *x* ⟹ ∀ *x*∈*set* (*desc a xs ys*). *sorted x*
by (*induct xs* **and** *a f ys* **and** *a xs ys rule*: *runs-asc-desc.induct*)
(*auto simp*: *sorted-append not-less dest*: *order-trans*, *fastforce*)

lemma *sorted-nmsort*:
sorted (*nmsort xs*)
by (*auto intro*: *sorted-merge-all simp*: *nmsort-def sorted-runs*)

2.4 Running Time Analysis

fun *C-runs* :: ('*a*::*linorder*) *list* ⇒ *nat* **and**
C-asc :: ('*a*::*linorder*) ⇒ '*a list* ⇒ *nat* **and**
C-desc :: ('*a*::*linorder*) ⇒ '*a list* ⇒ *nat*
where
C-runs (*a* # *b* # *xs*) = *1* + (*if a* > *b then C-desc b xs else C-asc b xs*)
| *C-runs xs* = *0*
| *C-asc a* (*b* # *bs*) = *1* + (*if* ¬ *a* > *b then C-asc b bs else C-runs* (*b* # *bs*))
| *C-asc a* [] = *0*
| *C-desc a* (*b* # *bs*) = *1* + (*if a* > *b then C-desc b bs else C-runs* (*b* # *bs*))
| *C-desc a* [] = *0*

fun *C-nmsort* :: ('*a*::*linorder*) *list* ⇒ *nat*
where
C-nmsort xs = *C-runs xs* + *C-merge-all* (*runs xs*)

lemma
fixes *a* :: '*a*::*linorder* **and** *xs ys* :: '*a list*
shows *C-runs*: *C-runs xs* ≤ *length xs* − *1*
and *C-asc*: *C-asc a ys* ≤ *length ys*
and *C-desc*: *C-desc a ys* ≤ *length ys*
by (*induct xs* **and** *a ys* **and** *a ys rule*: *C-runs-C-asc-C-desc.induct*) *auto*

lemma
shows *length-runs*: *length* (*runs xs*) ≤ *length xs*
and *length-asc*: *ascP f* ⟹ *length* (*asc a f ys*) ≤ *1* + *length ys*
and *length-desc*: *length* (*desc a xs ys*) ≤ *1* + *length ys*
by (*induct xs* **and** *a f ys* **and** *a xs ys rule*: *runs-asc-desc.induct*) *auto*

lemma
shows *length-concat-runs* [*simp*]: *length* (*concat* (*runs xs*)) = *length xs*
and *length-concat-asc*: *ascP f* ⟹ *length* (*concat* (*asc a f ys*)) = *1* + *length* (*f* []) + *length ys*
and *length-concat-desc*: *length* (*concat* (*desc a xs ys*)) = *1* + *length xs* + *length ys*
by (*induct xs* **and** *a f ys* **and** *a xs ys rule*: *runs-asc-desc.induct*) *auto*

lemma *log2-mono*:
x > *0* ⟹ *x* ≤ *y* ⟹ *log 2 x* ≤ *log 2 y*
by *auto*

lemma
shows *runs-ne*: *xs* ≠ [] ⟹ *runs xs* ≠ []

```
    and ascP f ⟹ asc a f ys ≠ []
    and desc a xs ys ≠ []
  by (induct xs and a f ys and a xs ys rule: runs-asc-desc.induct) simp-all

lemma C-nmsort:
  assumes [simp]: length xs = n
  shows C-nmsort xs ≤ n + n * ⌈log 2 n⌉
proof −
  have [simp]: xs = [] ⟷ length xs = 0 by blast
  have int (C-merge-all (runs xs)) ≤ int n * ⌈log 2 (length (runs xs))⌉
    using C-merge-all' [of runs xs] by simp
  also have ... ≤ int n * ⌈log 2 n⌉
    using length-runs [of xs]
    by (cases n) (auto intro!: runs-ne mult-mono ceiling-mono log2-mono)
  finally have int (C-merge-all (runs xs)) ≤ int n * ⌈log 2 n⌉ .
  moreover have C-runs xs ≤ n using C-runs [of xs] by auto
  ultimately show ?thesis by (auto intro: add-mono)
qed

end
```

References

[1] Christian Sternagel. Proof pearl - a mechanized proof of GHC's mergesort. *Journal of Automated Reasoning*, 2012. doi:10.1007/s10817-012-9260-7.