The Transcendence of e

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Abstract

This work contains a formalisation of the proof that Euler's number e is transcendental. The proof follows the standard approach of assuming that e is algebraic and then using a specific integer polynomial to derive two inconsistent bounds, leading to a contradiction.

This approach can be found in many different sources; this formalisation mostly follows a PlanetMath article [1] by Roger Lipsett.

Contents

1 Proof of the Transcendence of e

theory *E-Transcendental*
imports
HOL−Complex-Analysis.Complex-Analysis
HOL−Number-Theory.Number-Theory
HOL−Computational-Algebra.Polynomial
Polynomial-Interpolation.Ring-Hom-Poly
begin

hide-const (**open**) *UnivPoly.coeff* *UnivPoly.up-ring.monom*
hide-const (**open**) *Module.smult* *Coset.order*

1.1 Various auxiliary facts

lemma *fact-dvd-pochhammer*:
assumes $m \leq n + 1$
shows *fact m dvd pochhammer* (*int n* − *int m* + *1*) *m*
⟨*proof*⟩

lemma *prime-elem-int-not-dvd-neg1-power*:

prime-elem (*p* :: *int*) $\Longrightarrow$ $\neg$*p dvd* (*−1*) ^ *n*
⟨*proof*⟩

lemma *nat-fact* [*simp*]: *nat* (*fact n*) = *fact n*
⟨*proof*⟩

lemma *prime-dvd-fact-iff-int*:
p dvd fact n $\longleftrightarrow$ *p* $\leq$ *int n* **if** *prime p*
⟨*proof*⟩

lemma *power-over-fact-tendsto-0*:
(λ*n*. (*x* :: *real*) ^ *n* / *fact n*) $\longlongrightarrow$ *0*
⟨*proof*⟩

lemma *power-over-fact-tendsto-0'*:
(λ*n*. *c* ∗ (*x* :: *real*) ^ *n* / *fact n*) $\longlongrightarrow$ *0*
⟨*proof*⟩

1.2 General facts about polynomials

lemma *fact-dvd-higher-pderiv*:
[:*fact n* :: *int*:] *dvd* (*pderiv* ^^ *n*) *p*
⟨*proof*⟩

lemma *fact-dvd-poly-higher-pderiv-aux*:
(*fact n* :: *int*) *dvd poly* ((*pderiv* ^^ *n*) *p*) *x*
⟨*proof*⟩

lemma *fact-dvd-poly-higher-pderiv-aux'*:
m $\leq$ *n* $\Longrightarrow$ (*fact m* :: *int*) *dvd poly* ((*pderiv* ^^ *n*) *p*) *x*
⟨*proof*⟩

1.3 Main proof

lemma *lindemann-weierstrass-integral*:
fixes *u* :: *complex* **and** *f* :: *complex poly*
defines *df* $\equiv$ λ*n*. (*pderiv* ^^ *n*) *f*
defines *m* $\equiv$ *degree f*
defines *I* $\equiv$ λ*f u*. *exp u* ∗ ($\sum$*j*$\leq$*degree f*. *poly* ((*pderiv* ^^ *j*) *f*) *0*) −
($\sum$*j*$\leq$*degree f*. *poly* ((*pderiv* ^^ *j*) *f*) *u*)
shows ((λ*t*. *exp* (*u* − *t*) ∗ *poly f t*) *has-contour-integral I f u*) (*linepath 0 u*)
⟨*proof*⟩

locale *lindemann-weierstrass-aux* =
fixes *f* :: *complex poly*
begin

definition *I* :: *complex* $\Rightarrow$ *complex* **where**
I u = *exp u* ∗ ($\sum$*j*$\leq$*degree f*. *poly* ((*pderiv* ^^ *j*) *f*) *0*) −
($\sum$*j*$\leq$*degree f*. *poly* ((*pderiv* ^^ *j*) *f*) *u*)

lemma *lindemann-weierstrass-integral-bound*:
  fixes u :: *complex*
  assumes $C \geq 0$ $\bigwedge t.\ t \in$ *closed-segment* $0\ u \Longrightarrow$ *norm* (*poly f t*) $\leq C$
  shows *norm* ($I\ u$) $\leq$ *norm* u ∗ *exp* (*norm* u) ∗ C
⟨*proof*⟩

end

lemma *poly-higher-pderiv-aux1*:
  fixes c :: $'a$ :: *idom*
  assumes $k < n$
  shows *poly* ((*pderiv* ^^ k) ([:$-c$, 1:] ^ n ∗ p)) $c = 0$
  ⟨*proof*⟩

lemma *poly-higher-pderiv-aux1'*:
  fixes c :: $'a$ :: *idom*
  assumes $k < n$ [:$-c$, 1:] ^ n *dvd* p
  shows *poly* ((*pderiv* ^^ k) p) $c = 0$
⟨*proof*⟩

lemma *poly-higher-pderiv-aux2*:
  fixes c :: $'a$ :: {*idom*, *semiring-char-0*}
  shows *poly* ((*pderiv* ^^ n) ([:$-c$, 1:] ^ n ∗ p)) c = *fact* n ∗ *poly* p c
⟨*proof*⟩

lemma *poly-higher-pderiv-aux3*:
  fixes c :: $'a$ :: {*idom*,*semiring-char-0*}
  assumes $k \geq n$
  shows ∃ q. *poly* ((*pderiv* ^^ k) ([:$-c$, 1:] ^ n ∗ p)) c = *fact* n ∗ *poly* q c
  ⟨*proof*⟩

lemma *poly-higher-pderiv-aux3'*:
  fixes c :: $'a$ :: {*idom*, *semiring-char-0*}
  assumes $k \geq n$ [:$-c$, 1:] ^ n *dvd* p
  shows *fact* n *dvd* *poly* ((*pderiv* ^^ k) p) c
⟨*proof*⟩

lemma *e-transcendental-aux-bound*:
  obtains C **where** $C \geq 0$
    $\bigwedge x.\ x \in$ *closed-segment* 0 (*of-nat* n) $\Longrightarrow$
      norm ($\prod k \in \{1..n\}$. (x − *of-nat* k :: *complex*)) $\leq C$
⟨*proof*⟩

theorem *e-transcendental-complex*: ¬ *algebraic* (*exp* 1 :: *complex*)
⟨*proof*⟩

corollary *e-transcendental-real*: ¬ *algebraic* (*exp* 1 :: *real*)

$\langle proof \rangle$

end

References

[1] R. Lipsett. Planetmath. http://planetmath.org/proofoflindemannweierstrasstheoremandthateandpiaretranscendental, 2007.