The Dottie Number

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Abstract

The Dottie number is the unique fixed point d of the cosine function: $\cos d = d$. It is approximately 0.739085133215 and has no known closed form.

This theory establishes the Dottie number's key properties: the fixed point exists (by the intermediate value theorem) and is unique (because $\cos x - x$ has a strictly negative derivative). Next, the value of d to 12 decimal places is shown using the `approximation` proof method. Two more properties of d are also shown: first, that it is *transcendental* (via the Hermite–Lindemann–Weierstrass theorem); second, that it is a *universal attractor*, in the sense that iterating the cosine function from any real starting point converges to it.

Contents

1 The Dottie Number

theory *Dottie*
  imports *HOL−Analysis.Analysis*
    HOL−Decision-Procs.Approximation
    Hermite-Lindemann.Hermite-Lindemann

begin

The Dottie number, approximately 0.739085133215, is the unique fixed point of the cosine function.

This theory establishes the Dottie number's basic theory. We first show that the fixed point *exists* (by the intermediate value theorem) and is *unique* (because $\cos x - x$ has a strictly negative derivative), justifying the definition of *dottie*, and we pin it down to twelve decimal places. We then prove three further results: that the Dottie number is a *universal attractor*, in the sense that iterating cosine from any real starting point converges to it; the trigonometric identity $\sin(\mathit{dottie}) = \sqrt{1 - \mathit{dottie}^2}$; and, using the Hermite–Lindemann–Weierstraß theorem, that the Dottie number is *transcendental*.

definition *dottie* :: *real* **where**
  dottie ≡ *THE x. cos x = x*

lemma *cos-1-lt-1*: *cos (1::real) < 1*
  ⟨*proof*⟩

We shall reason about the function $g(x) = \cos x - x$. The locale provides a scope for g and its properties, which are used by several of the lemmas below.

locale *Dottie* =
  fixes *g* :: *real* ⇒ *real*
  defines *g* ≡ *λx::real. cos x − x*

begin

lemma *g-has-negative-deriv*:
  assumes |*t*| ≤ *1*
  shows ∃ *d*. (*g has-real-derivative d*) (*at t*) ∧ *d* < *0*
⟨*proof*⟩

1.1 Existence

We have $g(0) = 1 > 0$ and $g(1) = \cos 1 - 1 < 0$. Since g is continuous, the intermediate value theorem gives a point $x \in (0, 1)$ where $g(x) = 0$, i.e. $\cos x = x$.

lemma *dottie-exists*: ∃ *x::real. 0 < x ∧ x < 1 ∧ cos x = x*
⟨*proof*⟩

1.2 Uniqueness

The function $g(x) = \cos x - x$ has derivative $g'(x) = -\sin x - 1$, which is strictly negative for $x \in [-1, 1]$ (since $\sin x \geq 0$ there). A function with strictly negative derivative is strictly decreasing, so g can have at most one zero. We can extend uniqueness to the entire real line.

lemma *dottie-unique*:
  fixes *x y* :: *real*
  assumes *cos x = x cos y = y*
  shows *x = y*
⟨*proof*⟩

lemma *facts*: *0 < dottie dottie < 1 cos dottie = dottie*
⟨*proof*⟩

1.3 Approximation

We pin down the Dottie number to 12 decimal places. Note that g is decreasing. We check that $\cos(lb) > lb$ (so the fixed point is above lb) and $\cos(ub) < u$ (so it is below ub).

definition *lb*::*real* **where** *lb* ≡ *0.739085133215*

definition *ub*::*real* **where** *ub* ≡ *0.739085133216*

lemma *lb-gt*: *cos lb > lb*
  ⟨*proof*⟩

lemma *ub-lt*: *cos ub < ub*
  ⟨*proof*⟩

lemma *lb*: *lb < dottie*
⟨*proof*⟩

lemma *ub*: *ub > dottie*
⟨*proof*⟩

1.4 The Dottie number is a universal attractor

Iterating cosine from *any* real starting point converges to the Dottie number. The key fact is that cos is a contraction on $[-1, 1]$ with Lipschitz constant $\sin 1 < 1$ (since $|\cos' x| = |\sin x| \leq \sin 1$ there), and that cos maps all of $\mathbb{R}$ into $[-1, 1]$.

lemma *sin1-bounds*: *0 < sin (1::real) sin (1::real) < 1*
⟨*proof*⟩

lemma *abs-sin-le-sin1*:
  assumes *|t| ≤ 1* **shows** *|sin t| ≤ sin (1::real)*

⟨*proof*⟩

The mean value theorem turns the derivative bound into a Lipschitz bound.

lemma *cos-contraction-lt*:
fixes $x\ y$ :: *real*
assumes $x < y$ $|x| \leq 1$ $|y| \leq 1$
shows $|cos\ x - cos\ y| \leq sin\ 1 * |x - y|$
⟨*proof*⟩

lemma *cos-contraction*:
fixes $x\ y$ :: *real*
assumes $|x| \leq 1$ $|y| \leq 1$
shows $|cos\ x - cos\ y| \leq sin\ 1 * |x - y|$
⟨*proof*⟩

lemma *dottie-in-pm1*: $|dottie| \leq 1$
⟨*proof*⟩

lemma *cos-step-to-dottie*:
assumes $|w| \leq 1$
shows $|cos\ w - dottie| \leq sin\ 1 * |w - dottie|$
⟨*proof*⟩

After one step the iteration lands in $[-1, 1]$ and stays there.

lemma *cos-funpow-in-pm1*:
fixes *x0* :: *real*
assumes $n \geq 1$
shows $|(cos\ \hat{}\hat{}\ n)\ x0| \leq 1$
⟨*proof*⟩

From a start in $[-1, 1]$, the distance to the fixed point decays geometrically.

lemma *cos-funpow-bound*:
fixes *y0* :: *real*
assumes $|y0| \leq 1$
shows $|(cos\ \hat{}\hat{}\ n)\ y0 - dottie| \leq (sin\ 1)\hat{}n * |y0 - dottie|$
⟨*proof*⟩

lemma *cos-iter-tendsto-unit*:
fixes *y0* :: *real*
assumes $|y0| \leq 1$
shows $(\lambda n.\ (cos\ \hat{}\hat{}\ n)\ y0) \longrightarrow dottie$
⟨*proof*⟩

theorem *cos-iter-tendsto-dottie*:
fixes *x0* :: *real*
shows $(\lambda n.\ (cos\ \hat{}\hat{}\ n)\ x0) \longrightarrow dottie$
⟨*proof*⟩

1.5 A trigonometric identity

Since $\cos(dottie) = dottie$ and $dottie \in (0, 1)$, the Pythagorean identity gives $\sin(dottie) = \sqrt{1 - dottie^2}$.

lemma *sin-dottie*: *sin dottie* = *sqrt* (*1* − *dottie*^{2})
⟨*proof*⟩

1.6 Transcendence

By the Hermite–Lindemann–Weierstraß theorem, $\cos z$ is transcendental for every nonzero algebraic z. If the Dottie number were algebraic, then $\cos(dottie) = dottie$ would be both algebraic and transcendental.

theorem *dottie-transcendental*: ¬ *algebraic dottie*
⟨*proof*⟩

end

We make key facts available outside the locale

lemmas *dottie-fp* = *Dottie.facts(3)*
lemmas *dottie-bounds* = *Dottie.lb Dottie.ub*
lemmas *dottie-attractor* = *Dottie.cos-iter-tendsto-dottie*
lemmas *dottie-sin* = *Dottie.sin-dottie*
lemmas *dottie-transcendental* = *Dottie.dottie-transcendental*

end