The Dottie Number

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Abstract

The Dottie number is the unique fixed point d of the cosine function: $\cos d = d$. It is approximately 0.739085133215 and has no known closed form.

This theory establishes the Dottie number's key properties: the fixed point exists (by the intermediate value theorem) and is unique (because $\cos x - x$ has a strictly negative derivative). Next, the value of d to 12 decimal places is shown using the `approximation` proof method. Two more properties of d are also shown: first, that it is *transcendental* (via the Hermite–Lindemann–Weierstrass theorem); second, that it is a *universal attractor*, in the sense that iterating the cosine function from any real starting point converges to it.

Contents

1 The Dottie Number

theory *Dottie*
  imports *HOL−Analysis.Analysis*
    HOL−Decision-Procs.Approximation
    Hermite-Lindemann.Hermite-Lindemann

begin

The Dottie number, approximately 0.739085133215, is the unique fixed point of the cosine function.

This theory establishes the Dottie number's basic theory. We first show that the fixed point *exists* (by the intermediate value theorem) and is *unique* (because $\cos x - x$ has a strictly negative derivative), justifying the definition of *dottie*, and we pin it down to twelve decimal places. We then prove three further results: that the Dottie number is a *universal attractor*, in the sense that iterating cosine from any real starting point converges to it; the trigonometric identity $\sin(\mathit{dottie}) = \sqrt{1 - \mathit{dottie}^2}$; and, using the Hermite–Lindemann–Weierstraß theorem, that the Dottie number is *transcendental.*

definition *dottie* :: *real* **where**
  dottie ≡ *THE x. cos x = x*

lemma *cos-1-lt-1*: *cos* (*1*::*real*) < *1*
  using *cos-monotone-0-pi pi-gt3* **by** *force*

We shall reason about the function $g(x) = \cos x - x$. The locale provides a scope for g and its properties, which are used by several of the lemmas below.

locale *Dottie* =
  fixes *g* :: *real* ⇒ *real*
  defines *g* ≡ λ*x*::*real*. *cos x* − *x*

begin

lemma *g-has-negative-deriv*:
  assumes |*t*| ≤ *1*
  shows ∃ *d*. (*g has-real-derivative d*) (*at t*) ∧ *d* < *0*
proof (*intro exI conjI*)
  show (*g has-real-derivative* (− *sin t* − *1*)) (*at t*)
    unfolding *g-def* **by** (*auto intro*!: *derivative-eq-intros*)
  show − *sin t* − *1* < *0*
    using *assms pi-gt3 le-arcsin-iff* [*of - t*] **by** *fastforce*
qed

1.1 Existence

We have $g(0) = 1 > 0$ and $g(1) = \cos 1 - 1 < 0$. Since g is continuous, the intermediate value theorem gives a point $x \in (0, 1)$ where $g(x) = 0$, i.e.

$\cos x = x$.

lemma *dottie-exists*: $\exists x$::*real*. $0 < x \wedge x < 1 \wedge \cos x = x$
proof –
  — Apply the IVT to g on the unit interval at 0.
  have *g-cont*: *continuous-on* $\{0..1\}$ *g*
    unfolding *g-def* **by** (*intro continuous-intros*)
  obtain $g\ 0 = 1$ $g\ 1 < 0$ **using** *cos-1-lt-1* **by** (*simp add*: *g-def*)
  with *IVT2'*[*of g 1 0 0*] *g-cont*
  obtain *x* **where** *hx*: $0 \leq x$ $x \leq 1$ $g\ x = 0$
    by (*metis less-eq-real-def zero-le-one*)
  hence *cos-eq*: $\cos x = x$ **by** (*simp add*: *g-def*)
  with *hx* **show** *?thesis*
    by (*metis cos-1-lt-1 cos-zero order-less-le*)
qed

1.2 Uniqueness

The function $g(x) = \cos x - x$ has derivative $g'(x) = -\sin x - 1$, which is strictly negative for $x \in [-1, 1]$ (since $\sin x \geq 0$ there). A function with strictly negative derivative is strictly decreasing, so g can have at most one zero. We can extend uniqueness to the entire real line.

lemma *dottie-unique*:
  fixes *x y* :: *real*
  assumes $\cos x = x$ $\cos y = y$
  shows $x = y$
proof (*rule ccontr*)
  assume $x \neq y$
  have *gx*: $g\ x = 0$ **and** *gy*: $g\ y = 0$ **using** *assms* **by** (*auto simp*: *g-def*)
  — The derivative of g is $\lambda x.\ -\ \sin x\ -\ 1$, which is negative on $\{-\ 1..1\}$.
  show *False*
  proof (*cases* $|x| > 1 \vee |y| > 1$)
    case *True*
    then show *?thesis*
      by (*metis assms abs-cos-le-one not-less*)
  next
    case *False*
    then have $|x| \leq 1 \wedge |y| \leq 1$
      by *simp*
    moreover have $x < y \vee y < x$ **using** ‹$x \neq y$› **by** *linarith*
    ultimately show *?thesis*
      using *DERIV-neg-imp-decreasing* [*OF - g-has-negative-deriv*] *gx gy*
      by *force*
  qed
qed

lemma *facts*: $0 <$ *dottie* *dottie* < 1 $\cos$ *dottie* $=$ *dottie*
proof –
  obtain *x* :: *real* **where** *hx*: $0 < x$ $x < 1$ $\cos x = x$

using *dottie-exists* **by** *blast*
have *unique*: $y = x$ **if** *cos* $y = y$ **for** y :: *real*
by (*simp add*: *dottie-unique* ‹*cos* $x = x$› *that*)
have *the-eq*: *dottie* $= x$
unfolding *dottie-def* **using** ‹*cos* $x = x$› *unique* **by** *blast*
then show *0* < *dottie dottie* < *1 cos dottie* = *dottie*
using *hx* **by** (*auto simp*: *g-def*)
qed

1.3 Approximation

We pin down the Dottie number to 12 decimal places. Note that g is decreasing. We check that $\cos(lb) > lb$ (so the fixed point is above lb) and $\cos(ub) < u$ (so it is below ub).

definition *lb*::*real* **where** *lb* ≡ *0.739085133215*

definition *ub*::*real* **where** *ub* ≡ *0.739085133216*

lemma *lb-gt*: *cos lb* > *lb*
unfolding *lb-def*
by (*approximation 50*)

lemma *ub-lt*: *cos ub* < *ub*
unfolding *ub-def*
by (*approximation 50*)

lemma *lb*: *lb* < *dottie*
proof (*rule ccontr*)
assume *neg*: ¬ *lb* < *dottie*
have *gd*: *g lb* > *0*
using *facts lb-gt* **by** (*auto simp*: *g-def*)
show *False*
using *DERIV-neg-imp-decreasing* [*OF* - *g-has-negative-deriv*] *facts neg*
by (*smt* (*verit*, *ccfv-SIG*) *cos-le-one cos-monotone-0-pi lb-gt pi-gt3*)
qed

lemma *ub*: *ub* > *dottie*
proof (*rule ccontr*)
assume *neg*: ¬ *ub* > *dottie*
have *gd*: *g ub* < *0*
using *facts ub-lt* **by** (*auto simp*: *g-def*)
show *False*
using *DERIV-neg-imp-decreasing* [*OF* - *g-has-negative-deriv*] *facts neg*
by (*smt* (*verit*) *cos-ge-minus-one ub-lt gd g-def*)
qed

1.4 The Dottie number is a universal attractor

Iterating cosine from *any* real starting point converges to the Dottie number. The key fact is that cos is a contraction on $[-1, 1]$ with Lipschitz constant $\sin 1 < 1$ (since $|\cos' x| = |\sin x| \leq \sin 1$ there), and that cos maps all of $\mathbb{R}$ into $[-1, 1]$.

lemma *sin1-bounds*: *0* < *sin* (*1*::*real*) *sin* (*1*::*real*) < *1*
proof −
  have *lt*: (*1*::*real*) < *pi* **using** *pi-gt3* **by** *simp*
  have *0* < (*1*::*real*) **by** *simp*
  from *sin-gt-zero*[*OF this lt*] **show** *0* < *sin* (*1*::*real*) **.**
next
  have *sin 1* < *sin* (*pi*/*2*)
    using *sin-monotone-2pi*[*of 1 pi*/*2*] *pi-gt3* **by** *simp*
  then show *sin* (*1*::*real*) < *1* **by** *simp*
qed

lemma *abs-sin-le-sin1*:
  assumes |*t*| ≤ *1* **shows** |*sin t*| ≤ *sin* (*1*::*real*)
proof −
  have *1* < *pi*/*2* **using** *pi-gt3* **by** *simp*
  then show *?thesis*
    by (*smt* (*verit*, *best*) *assms sin-minus sin-monotone-2pi-le*)
qed

The mean value theorem turns the derivative bound into a Lipschitz bound.

lemma *cos-contraction-lt*:
  fixes *x y* :: *real*
  assumes *x* < *y* |*x*| ≤ *1* |*y*| ≤ *1*
  shows |*cos x* − *cos y*| ≤ *sin 1* ∗ |*x* − *y*|
proof −
  have *cont*: *continuous-on* {*x*..*y*} *cos* **by** (*intro continuous-intros*)
  have *deriv*: ((*cos*::*real*⇒*real*) *has-derivative* (∗) (− *sin u*)) (*at u*) **for** *u* :: *real*
    using *DERIV-cos*[*of u*] **unfolding** *has-field-derivative-def* **by** *simp*
  have ∃*ξ*∈{*x*<..<*y*}. *norm* (*cos y* − *cos x*) ≤ *norm* ((∗) (− *sin ξ*) (*y* − *x*))
    by (*rule mvt-general*[*OF* ‹*x* < *y*› *cont*]) (*use deriv* **in** *blast*)
  then obtain *ξ* **where** *ξ*: *ξ* ∈ {*x*<..<*y*} *norm* (*cos y* − *cos x*) ≤ *norm* (− *sin ξ* ∗ (*y* − *x*))
    by *auto*
  have |*ξ*| ≤ *1* **using** *ξ assms* **by** *auto*
  then have *absxi*: |*sin ξ*| ≤ *sin 1* **by** (*rule abs-sin-le-sin1*)
  have |*cos y* − *cos x*| ≤ |*sin ξ*| ∗ |*y* − *x*|
    using *ξ*(*2*) **by** (*simp add*: *abs-mult*)
  also have ... ≤ *sin 1* ∗ |*y* − *x*|
    using *absxi* **by** (*simp add*: *mult-right-mono*)
  finally show *?thesis*
    by (*simp add*: *abs-minus-commute*)
qed

lemma *cos-contraction*:
fixes *x y* :: *real*
assumes $|x| \le 1$ $|y| \le 1$
shows $|cos\ x - cos\ y| \le sin\ 1 * |x - y|$
using *cos-contraction-lt*[*of x y*] *cos-contraction-lt*[*of y x*] *assms*
by (*cases x y rule*: *linorder-cases*) (*auto simp*: *abs-minus-commute*)

lemma *dottie-in-pm1*: $|dottie| \le 1$
using *facts* **by** *simp*

lemma *cos-step-to-dottie*:
assumes $|w| \le 1$
shows $|cos\ w - dottie| \le sin\ 1 * |w - dottie|$
using *facts* **by** (*metis assms cos-contraction dottie-in-pm1*)

After one step the iteration lands in $[-1, 1]$ and stays there.

lemma *cos-funpow-in-pm1*:
fixes *x0* :: *real*
assumes $n \ge 1$
shows $|(cos\ \hat{}\hat{}\ n)\ x0| \le 1$
proof −
obtain *m* **where** $n = Suc\ m$ **using** *assms not0-implies-Suc* **by** *force*
then have $(cos\ \hat{}\hat{}\ n)\ x0 = cos\ ((cos\ \hat{}\hat{}\ m)\ x0)$
by (*simp add*: *funpow-swap1*)
then show *?thesis* **by** (*simp add*: *abs-cos-le-one*)
qed

From a start in $[-1, 1]$, the distance to the fixed point decays geometrically.

lemma *cos-funpow-bound*:
fixes *y0* :: *real*
assumes $|y0| \le 1$
shows $|(cos\ \hat{}\hat{}\ n)\ y0 - dottie| \le (sin\ 1)\hat{}n * |y0 - dottie|$
proof (*induction n*)
case *0*
show *?case* **by** *simp*
next
case (*Suc n*)
have *inpm1*: $|(cos\ \hat{}\hat{}\ n)\ y0| \le 1$
using *cos-funpow-in-pm1*[*of n y0*] *assms*
by (*metis funpow-0 less-one not-le*)
have $|(cos\ \hat{}\hat{}\ Suc\ n)\ y0 - dottie| = |cos\ ((cos\ \hat{}\hat{}\ n)\ y0) - dottie|$
by (*simp add*: *funpow-swap1*)
also have $\ldots \le sin\ 1 * |(cos\ \hat{}\hat{}\ n)\ y0 - dottie|$
using *cos-step-to-dottie*[*OF inpm1*] **.**
also have $\ldots \le sin\ 1 * ((sin\ 1)\hat{}n * |y0 - dottie|)$
using *Suc.IH sin1-bounds*(*1*) **by** (*simp add*: *mult-left-mono*)
also have $\ldots = (sin\ 1)\hat{}(Suc\ n) * |y0 - dottie|$

```
    by (simp add: mult.assoc)
  finally show ?case .
qed

lemma cos-iter-tendsto-unit:
  fixes y0 :: real
  assumes |y0| ≤ 1
  shows (λn. (cos ^^ n) y0) ⟶ dottie
proof −
  have pow0: (λn. (sin 1)^n) ⟶ (0::real)
    using sin1-bounds by (intro LIMSEQ-realpow-zero) auto
  have null: (λn. (sin 1)^n ∗ |y0 − dottie|) ⟶ 0
    using tendsto-mult[OF pow0 tendsto-const, of |y0 − dottie|] by simp
  have (λn. |(cos ^^ n) y0 − dottie|) ⟶ 0
   using tendsto-sandwich[OF - - tendsto-const null] cos-funpow-bound[OF assms]
by auto
  then have (λn. (cos ^^ n) y0 − dottie) ⟶ 0
    by (rule tendsto-rabs-zero-cancel)
  then show ?thesis
    using Lim-null by blast
qed

theorem cos-iter-tendsto-dottie:
  fixes x0 :: real
  shows (λn. (cos ^^ n) x0) ⟶ dottie
proof −
  have |cos x0| ≤ 1 by (simp add: abs-cos-le-one)
  from cos-iter-tendsto-unit[OF this]
  have (λn. (cos ^^ n) (cos x0)) ⟶ dottie .
  moreover have ⋀n. (cos ^^ n) (cos x0) = (cos ^^ Suc n) x0
    by (simp add: funpow-swap1)
  ultimately have (λn. (cos ^^ Suc n) x0) ⟶ dottie by simp
  then show ?thesis
    using filterlim-sequentially-Suc by blast
qed
```

1.5 A trigonometric identity

Since $\cos(dottie) = dottie$ and $dottie \in (0, 1)$, the Pythagorean identity gives $\sin(dottie) = \sqrt{1 - dottie^2}$.

```
lemma sin-dottie: sin dottie = sqrt (1 − dottie^2)
proof −
  have 0 < dottie dottie < pi using facts pi-gt3 by auto
  then have pos: 0 < sin dottie by (rule sin-gt-zero)
  have (sin dottie)^2 = 1 − (cos dottie)^2
    using sin-cos-squared-add[of dottie] by (simp add: algebra-simps)
  also have ... = 1 − dottie^2
    using facts(3) by simp
  finally have (sin dottie)^2 = 1 − dottie^2 .
```

```
  then show ?thesis
    using pos by (simp add: real-sqrt-unique)
qed
```

1.6 Transcendence

By the Hermite–Lindemann–Weierstraß theorem, $\cos z$ is transcendental for every nonzero algebraic z. If the Dottie number were algebraic, then cos(*dottie*) = *dottie* would be both algebraic and transcendental.

```
theorem dottie-transcendental: ¬ algebraic dottie
proof
  assume alg: algebraic dottie
  then have ¬ algebraic (cos (complex-of-real dottie))
    using facts transcendental-cos by auto
  moreover have cos (complex-of-real dottie) = complex-of-real dottie
    using facts by (simp add: cos-of-real)
  ultimately show False using alg by simp
qed

end
```

We make key facts available outside the locale

```
lemmas dottie-fp = Dottie.facts(3)
lemmas dottie-bounds = Dottie.lb Dottie.ub
lemmas dottie-attractor = Dottie.cos-iter-tendsto-dottie
lemmas dottie-sin = Dottie.sin-dottie
lemmas dottie-transcendental = Dottie.dottie-transcendental

end
```