

Countable Infinite Version of Dilworth’s Theorem

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Abstract

Dilworth’s Theorem states that in any finite partially ordered set, the size of a largest antichain equals the minimum number of chains needed to cover the set. We report a complete, machine-checked formalization of Dilworth’s Theorem and of its extension to countable partial orders of finite width in the Isabelle/HOL proof assistant. The finite case is obtained from a formalization of the König–Egerváry Theorem on the construction of a bipartite digraph associated with the partial order. For finite width, the countable case reduces to the finite one through a compactness argument: the incomparability graph of the partial order is shown to be k -colorable whenever every finite induced subgraph is, by appealing to a formalization of the De Bruijn–Erdős coloring theorem, itself a consequence of the compactness of propositional logic. We describe the formal statements, the central definitions, and the two top-level theorems together with their Isar proofs. This development continues the previous work on the mechanization of combinatorial theorems over countable infinite structures derived from propositional compactness.

Keywords: Dilworth’s Theorem · Isabelle/HOL · Partial Order Theory · Chains and Antichains · De Bruijn–Erdős Theorem · König–Egerváry Theorem · Propositional Compactness.

1 Introduction

In the PhD thesis, an Isabelle/HOL formalization of the theory of propositional logic that includes landmark theorems such as the compactness theorem was developed [11], further adapted and included in the Isabelle theory on compactness and combinatorial applications [15]. This theory includes formalizations of combinatorial theorems extended to the countable (infinite) case though the constructive application of the compactness theorem,

which include the first formalization of the countable version of Hall’s Theorem (cf. [4]) for countable families of sets, the countable graph-version of Hall’s Theorem, as well as the de Bruijn–Erdős k -coloring theorem (cf. [2]) for infinite graphs [12, 13]. The formalizations use standard compactness arguments, building models for finite substructures from which the combinatorial theorem for infinite structures follows. For instance, the formalization of the infinite version of Hall’s Theorem uses the completeness theorem and the finite formalization of Hall’s Theorem developed by Jiang and Nipkow [5, 6].

Following this line of research, this paper addresses the countable version of Dilworth’s Theorem for partial orders of *finite width*, a case that — in contrast to the situation for infinite width discussed below — does follow from the finite theorem by a standard compactness argument; our contribution is a complete machine-checked formalization of it. A *chain* in a partially ordered set $P = (A, \leq)$ is a subset that is totally ordered by $\leq$, and an *antichain* is a subset of pairwise incomparable elements. Dilworth’s Theorem [3] asserts that, for a finite P , the maximum cardinality of an antichain (the *width* of P) equals the least number of chains into which A can be partitioned. The result is a cornerstone of order theory and combinatorics and admits several classical proofs, including one through the König-Egerváry Theorem on bipartite graphs. Indeed, also in his seminal paper [3], Dilworth proved the general theorem that if an infinite poset has a finite maximum antichain size w , it can be partitioned into exactly w chains. Dilworth did not use a standard compactness approach. Instead, he found an ad hoc proof that constructed the infinite decomposition directly by applying Zorn’s Lemma.

In [8], Milner observes that,

Nowadays, we should probably say that the infinite case of Dilworth’s decomposition theorem [3] follows from the finite case by “a standard compactness argument.” Depending upon one’s upbringing, what we would have in mind is an application of Gödel’s compactness theorem in logic (cf Church [1]), Tychonoff’s theorem on compact topological spaces, or (if one is a combinatorialist) Rado’s selection lemma [10]. For example, we might argue as follows. Let $\mathcal{P} = (P, \leq)$ be a (partially) ordered set with no antichain of size $k + 1$. Then, by the finite case of Dilworth’s Theorem, any finite subcollection of the set of sentences

$$S = \{ p_{x,1} \vee \cdots \vee p_{x,k} : x \in P \} \cup \{ \neg(p_{x,i} \wedge p_{y,i}) : x \perp y, 1 \leq i \leq k \},$$

has a model (where $x \perp y$ means x and y incomparable in $\mathcal{P}$, and $p_{x,i}$ is a propositional variable with the intended interpretation “ x

belongs to the i^{th} chain"). Hence S has a model and P is the union of the k chains $C_i = \{x \in P : p_{x,i} \text{ is true}\}$ ($1 \leq i \leq k$).

Micha A. Perles' famous counterexample, published in [9], shows that Dilworth's Theorem fails for posets with infinite width. The construction utilizes the Cartesian product of the first uncountable ordinal with itself, denoted as $P = \omega_1 \times \omega_1$, equipped with the standard component-wise product order where $(\alpha, \beta) \leq (\gamma, \delta)$ if and only if $\alpha \leq \gamma$ and $\beta \leq \delta$. An antichain in this poset requires that elements be strictly increasing in one coordinate and strictly decreasing in the other. Because ω_1 is a well-ordered set, it cannot contain an infinite strictly decreasing sequence, which implies that every antichain in P must be finite.

On the other side, Perles's poset cannot be decomposed into a countable collection of chains, thereby shattering the classical finite intuition. By definition, any two elements in a chain $C \subseteq P$ must be comparable, meaning both coordinates must move in the same direction simultaneously. This geometric constraint implies that a chain can intersect any vertical slice $\{\alpha\} \times \omega_1$ at most once, effectively behaving as a partial function from ω_1 to ω_1 . Suppose P could be partitioned into countably many chains $\{C_n\}_{n \in \omega}$, the set of all vertical coordinates covered at a fixed horizontal position α would be a countable union of single points, which is at most countable. However, since the vertical axis ω_1 is inherently uncountable, a vast majority of the grid's surface area remains entirely untouched, proving that a countable chain cover is mathematically impossible.

Concerning previous formal developments on this topic, the only other Mechanization of Dilworth's Theorem in Isabelle/HOL that we are aware of is the recent Archive of Formal Proofs entry *Formal Proof of Dilworth's Theorem* by Maadoori, Meesum, Pillai, Prathamesh and Swami [7], which formalises the *finite* case following Perles's proof and Singh's earlier Coq development [14]. Our work instead addresses the countable-infinite case via propositional compactness.

As previously mentioned, for infinite posets, Dilworth's Theorem requires care: the largest antichain and the smallest (finite) chain decomposition have equal cardinality, whenever there is a finite bound on the size of antichains. In this case, the finite version of Dilworth's Theorem can be combined with a compactness argument based on a graph-coloring reformulation.

Main Contributions. The main contributions of this paper are the following.

- A formal proof of the **countable infinite version of Dilworth's Theorem** obtained from the finite case through a formalization of the k -coloring De Bruijn-Erdős theorem [2] and the propositional compactness theorem.

- As lemmas necessary to obtain the formalization of the infinite version of Dilworth’s Theorem, we have two relevant and nontrivial results:
 - the **formalization of König-Egerváry’s Theorem**, also known as the bipartite matching duality theorem, which states that the size of a maximal matching equals the size of a minimum vertex cover in finite bipartite digraphs. This theorem was formalized using the countable version of Hall’s Theorem for graphs, described in [13], developed as an application of the compactness theorem and Hall’s Theorem for sets [12], and part of the theory on propositional compactness in the Archive of Formal Proofs [15]; and
 - the formalization of the **finite version of Dilworth’s Theorem**, obtained from König-Egerváry’s theorem by associating a bipartite digraph with the partial order and constructing, from a maximum matching and a minimum vertex cover, a smallest chain decomposition and a largest antichain of the same cardinality.

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2 König-Egervary theorem for finite graphs

theory *Koenig-Egervary-finite*

This theory formalizes the fact that the Cardinality of a maximal matching equals the Cardinality of a minimum Vertex Cover in finite bipartite digraphs. The formalization uses the countable graph version of Hall Theorem proved in the **Prop_Compactness** as a consequence of the compactness theorem.

Goal: apply the finite version of König-Egervary theorem in the proof of a countable (infinite) version of Dilworth’s theorem.

The countable version of Dilworth’s theorem follows the standard approach

which applies the finite version of Dilworth's theorem, De Bruijn-Erdős k-colouring theorem.

From the finite graph version of Hall theorem follows the finite version of König-Egervary theorem, and then, the finite version of Dilworth's theorem.

Formalization following Jonathan L. Gross' textbook, "Graph Theory and its Applications", 3rd edition, pages 477-478 Hall's theorem implies König-Egervary's theorem

Definition 1 *A vertex cover of a graph G is a set $C \subseteq V(G)$ that contains at least one endpoint of every edge. The vertices in C cover $E(G)$. The next definition is relative to a subset of edges. Then, C such that $\text{vertex_cover } G \ C \ E(G)$ is a vertex cover covering all edges of G .*

definition $\text{vertex-cover} :: ('a, 'b) \text{ pre-digraph} \Rightarrow 'b \text{ set} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$
where
 $\text{vertex-cover } G \ E \ C \equiv C \subseteq (\text{verts } G) \wedge E \subseteq (\text{arcs } G) \wedge$
 $(\forall e \in E. (\text{head } G \ e) \in C \vee (\text{tail } G \ e) \in C)$

Since no vertex can cover two edges of a matching, the size of every vertex cover is at least the size of every matching.

Lemma 1 *Let M be a matching in a graph G , and let C be a vertex cover of G . Then $\text{card } M \leq \text{card } C$.*

lemma $\text{vertex-in-matching}$:
assumes $\text{dirBD-matching } G \ X \ Y \ E$
shows $\forall e1 \in E. \forall e2 \in E. e1 \neq e2 \longrightarrow$
 $((\text{tail } G \ e1) = v \vee (\text{head } G \ e1) = v) \longrightarrow ((\text{tail } G \ e2) \neq v \wedge (\text{head } G \ e2) \neq v))$
 $\langle \text{proof} \rangle$

fun $\text{f-matching-cover} :: ('a, 'b) \text{ pre-digraph} \Rightarrow 'a \text{ set} \Rightarrow 'b \text{ set} \Rightarrow ('a \Rightarrow 'b)$
where
 $\text{f-matching-cover } G \ V \ E = (\lambda v. (\text{THE } e. e \in E \wedge (\text{tail } G \ e = v \vee \text{head } G \ e = v)))$

lemma $\text{function-f-matching-cover}$:
assumes $\text{dirBD-matching } G \ X \ Y \ E$
and $a \in E$ **and** $(\text{head } G \ a) = v \vee (\text{tail } G \ a) = v$
shows $\text{f-matching-cover } G \ V \ E \ v = a$
 $\langle \text{proof} \rangle$

lemma $\text{surj-f-matching-cover1}$:
assumes $\text{dirBD-matching } G \ X \ Y \ E$ **and** $\text{vertex-cover } G \ E \ C$

shows $\forall e \in E. \exists v \in C. f\text{-matching-cover } G \ C \ E \ v = e$
 $\langle \text{proof} \rangle$

lemma *surj-f-matching-cover*:
assumes *dirBD-matching* $G \ X \ Y \ E$ **and** *vertex-cover* $G \ E \ C$
shows $E \subseteq (f\text{-matching-cover } G \ C \ E) \text{ ‘ } C$
 $\langle \text{proof} \rangle$

lemma *card-matching-cover*:
assumes *dirBD-matching* $G \ X \ Y \ E$ **and** *vertex-cover* $G \ E \ C$ **and** *finite* C
shows $\text{card } E \leq \text{card } ((f\text{-matching-cover } G \ C \ E) \text{ ‘ } C)$
 $\langle \text{proof} \rangle$

lemma *card-matching-cover0*:
assumes *finite* C
shows $\text{card } ((f\text{-matching-cover } G \ C \ E) \text{ ‘ } C) \leq \text{card } C \langle \text{proof} \rangle$

lemma *card-matching-cover1*:
assumes *dirBD-matching* $G \ X \ Y \ E$ **and** *vertex-cover* $G \ E \ C$ **and** *finite* C
shows $\text{card } E \leq \text{card } C \langle \text{proof} \rangle$

Definition 2 *A minimum vertex cover is a vertex cover of minimum size among all vertex cover of the graph.*

definition *minimum-vertex-cover*:: $('a, 'b) \text{ pre-digraph} \Rightarrow 'b \text{ set} \Rightarrow 'a \text{ set} \Rightarrow \text{bool}$
where *minimum-vertex-cover* $G \ E \ C$
 $\equiv (\text{vertex-cover } G \ E \ C) \wedge (\forall \ C1. \text{vertex-cover } G \ E \ C1 \longrightarrow \text{card } C \leq \text{card } C1)$

Definition 3 *A maximum matching is a matching of maximum size among all matchings of the graph.*

definition *maximum-dirBD-matching*:: $('a, 'b) \text{ pre-digraph} \Rightarrow 'a \text{ set} \Rightarrow 'a \text{ set} \Rightarrow 'b \text{ set} \Rightarrow \text{bool}$
where *maximum-dirBD-matching* $G \ X \ Y \ E \equiv$
 $\text{dirBD-matching } G \ X \ Y \ E \wedge (\forall \ E1. \text{dirBD-matching } G \ X \ Y \ E1 \longrightarrow \text{card } E1 \leq \text{card } E)$

Let M be a matching in a graph G , and let C be a vertex cover of G such that $|M| = |C|$. Then M is a maximum matching and C is a minimum vertex cover.

lemma *matching-cover-sub1*:
assumes *vertex-cover* $G \ E \ C$ **and** $E1 \subseteq E$
shows *vertex-cover* $G \ E1 \ C \langle \text{proof} \rangle$

corollary *matching-cover-sub2*:

assumes *vertex-cover* G (*arcs* G) C **and** $E1 \subseteq (\text{arcs } G)$

shows *vertex-cover* G $E1$ C $\langle \text{proof} \rangle$

lemma *card-matching-cover2*:

assumes *dirBD-matching* G X Y E **and**

vertex-cover G (*arcs* G) C **and**

finite (*verts* G) **and**

$\text{card } E = \text{card } C$

shows *maximum-dirBD-matching* G X Y $E \wedge \text{minimum-vertex-cover } G$ (*arcs* G) C
 $\langle \text{proof} \rangle$

Remark: The converse of the above property does not hold in general; however, it does hold for bipartite graphs.

Theorem 1 (König-Egervary) *If G is a bipartite graph, then the maximum size of a matching in G equals the minimum size of a vertex cover G .*

lemma *vertex-subset-BD-decomposition*:

assumes *bipartite-digraph* G X Y **and** $C \subseteq (\text{verts } G)$

shows $C = (C \cap X) \cup (C \cap Y)$

$\langle \text{proof} \rangle$

lemma *cover-descomposition*:

assumes *bipartite-digraph* G X Y **and** *vertex-cover* G (*arcs* G) C

shows $C = (C \cap X) \cup (C \cap Y)$

$\langle \text{proof} \rangle$

lemma *cover-descomposition1*:

assumes *bipartite-digraph* G X Y **and** *minimum-vertex-cover* G (*arcs* G) C

shows $C = (C \cap X) \cup (C \cap Y)$

$\langle \text{proof} \rangle$

lemma *card-neighbourhood*:

assumes *finite* $(\bigcup (\text{neighbourhood } G'W))$ **and**

$\text{card } (\bigcup (\text{neighbourhood } G'(W - \{w\}))) = \text{card } (\bigcup (\text{neighbourhood } G'W))$

shows $(\bigcup (\text{neighbourhood } G'(W - \{w\}))) = (\bigcup (\text{neighbourhood } G'W))$ $\langle \text{proof} \rangle$

lemma *neighbourhood-set*:

assumes *finite* W **and** *finite* $(\bigcup (\text{neighbourhood } G'W))$ **and**

$(\text{card } W) > \text{card } (\bigcup (\text{neighbourhood } G'W))$

shows $\exists w \in W. (\bigcup (\text{neighbourhood } G'(W - \{w\}))) = (\bigcup (\text{neighbourhood } G'W))$

$\langle \text{proof} \rangle$

definition *induced-subgraph*:: (a, b) pre-digraph $\Rightarrow a$ set $\Rightarrow (a, b)$ pre-digraph

where

induced-subgraph $G \ V \equiv$
 $(| \text{verts} = V,$
 $\text{arcs} = \{e \mid e. e \in (\text{arcs } G) \wedge (\text{tail } G \ e) \in V \wedge (\text{head } G \ e) \in V\},$
 $\text{tail} = (\lambda e. \text{tail } G \ e),$
 $\text{head} = (\lambda e. \text{head } G \ e)$
 $|)$

lemma *induced-Bipartite-Digraph:*

assumes *bipartite-digraph* $G \ X \ Y$ **and** $X1 \subseteq X$ **and** $Y1 \subseteq Y$
shows *bipartite-digraph* (*induced-subgraph* $G \ (X1 \cup Y1)$) $X1 \ Y1$
 $\langle \text{proof} \rangle$

lemma *perfect-matching-induced-subgraph:*

fixes $G :: ('a::\text{countable}, 'b::\text{countable}) \text{pre-digraph}$
assumes *dir-bipartite-digraph* (*induced-subgraph* $G \ V$) $X \ Y$
and $\forall i \in X. \text{finite } (\text{neighbourhood } (\text{induced-subgraph } G \ V) \ i)$
and $\forall W \subseteq X. \text{card } W \leq \text{card } (\bigcup (\text{neighbourhood } (\text{induced-subgraph } G \ V) \ `W))$
shows $(\exists E. \text{dirBD-perfect-matching } (\text{induced-subgraph } G \ V) \ X \ Y \ E)$
 $\langle \text{proof} \rangle$

lemma *vertex-cover-refining:*

assumes *dir-bipartite-digraph* $G \ X \ Y$
and *finite* (*verts* G)
and *vertex-cover* $G \ (\text{arcs } G) \ C$
and $W \subseteq C \cap X$
and $\text{card}(W) > \text{card}(\bigcup (\text{neighbourhood } (\text{induced-subgraph } G \ ((C \cap X) \cup (Y - (C \cap Y)))) \ `W))$
shows *vertex-cover* $G \ (\text{arcs } G)$
 $((C \cap X - W) \cup ((C \cap Y) \cup \bigcup (\text{neighbourhood } (\text{induced-subgraph } G \ ((C \cap X) \cup (Y - (C \cap Y)))) \ `W)))$
 $\langle \text{proof} \rangle$

lemma *vertex-cover-refining1:*

assumes *dir-bipartite-digraph* $G \ X \ Y$ **and** *finite* (*verts* G) **and** *vertex-cover* $G \ (\text{arcs } G) \ C$
and $W \subseteq C \cap X$
and $\text{card}(W) > \text{card}(\bigcup (\text{neighbourhood } (\text{induced-subgraph } G \ ((C \cap X) \cup (Y - (C \cap Y)))) \ `W))$
shows $\text{card } ((C \cap X) - W \cup ((C \cap Y) \cup \bigcup (\text{neighbourhood } (\text{induced-subgraph } G \ ((C \cap X) \cup (Y - (C \cap Y)))) \ `W))) < (\text{card } C)$
 $\langle \text{proof} \rangle$

corollary *vc-refinable-not-minimal:*

assumes *dir-bipartite-digraph* $G \ X \ Y$
and *finite* (*verts* G)

and *vertex-cover* G (*arcs* G) C
and $W \subseteq C \cap X$
and $\text{card}(W) > \text{card}(\bigcup (\text{neighbourhood } (\text{induced-subgraph } G ((C \cap X) \cup (Y - (C \cap Y)))) 'W))$
shows $\neg \text{minimum-vertex-cover } G$ (*arcs* G) C
 $\langle \text{proof} \rangle$

theorem *Konig-Egervary1:*

fixes $G :: ('a::\text{countable}, 'b::\text{countable}) \text{pre-digraph}$
assumes *maximum-dirBD-matching* G X Y E **and** *minimum-vertex-cover* G (*arcs* G) C
and *finite* (*verts* G) **and** *finite* C **and** *dir-bipartite-digraph* G X Y
and $\forall i \in X. \text{finite } (\text{neighbourhood } G i)$
and *dir-bipartite-digraph* (*induced-subgraph* $G ((C \cap X) \cup (Y - (C \cap Y)))) (C \cap X) (Y - (C \cap Y))$
shows $(\exists E. \text{dirBD-perfect-matching}(\text{induced-subgraph } G (C \cap X \cup (Y - C \cap Y))) (C \cap X) (Y - C \cap Y) E)$
 $\langle \text{proof} \rangle$

lemma *bd-induced-subgraph:*

assumes *bipartite-digraph* G X Y **and** $X1 \subseteq X$ **and** $Y1 \subseteq Y$
shows *bipartite-digraph* (*induced-subgraph* (*inverse-digraph* G) ($X1 \cup Y1$)) $Y1$ $X1$
 $\langle \text{proof} \rangle$

lemma *minimum-vertex-cover-inverse:*

assumes *minimum-vertex-cover* G (*arcs* G) C
shows *minimum-vertex-cover* (*inverse-digraph* G) (*arcs* G) C
 $\langle \text{proof} \rangle$

lemma *vc-inv-digraph-is-vc:*

assumes *vertex-cover* G (*arcs* G) C
shows *vertex-cover* (*inverse-digraph* G) (*arcs* G) C
 $\langle \text{proof} \rangle$

corollary *vertex-cover-refining2a:*

assumes *dir-bipartite-digraph* (*inverse-digraph* G) Y X
and *finite* (*verts* G)
and *vertex-cover* (*inverse-digraph* G) (*arcs* G) C
and $W \subseteq C \cap Y$
and $\text{card } W > \text{card}(\bigcup (\text{neighbourhood } (\text{induced-subgraph } (\text{inverse-digraph } G) (C \cap Y \cup (X - C \cap X))) 'W))$
shows $\neg \text{minimum-vertex-cover } (\text{inverse-digraph } G)$ (*arcs* G) C
 $\langle \text{proof} \rangle$

theorem *Konig-Egervary2:*

fixes $G :: ('a::\text{countable}, 'b::\text{countable}) \text{pre-digraph}$
assumes *maximum-dirBD-matching* G X Y E
and *minimum-vertex-cover* (*inverse-digraph* G) (*arcs* G) C

and *finite* (*verts*(*inverse-digraph* *G*))
and *finite* *C*
and *dir-bipartite-digraph* *G* *X* *Y*
and $\forall i \in Y. \text{finite}(\text{neighbourhood}(\text{inverse-digraph } G) i)$
and *dir-bipartite-digraph* (*induced-subgraph* (*inverse-digraph* *G*) ((*C* $\cap$ *Y*) $\cup$ (*X* $-$ (*C* $\cap$ *X*)))) (*C* $\cap$ *Y*) (*X* $-$ (*C* $\cap$ *X*))
and *verts* *G* $\subseteq$ *X* $\cup$ *Y*
shows ($\exists E. \text{dirBD-perfect-matching}(\text{induced-subgraph}(\text{inverse-digraph } G)(C \cap Y \cup (X - C \cap X)))(C \cap Y)(X - C \cap X) E$)
 $\langle \text{proof} \rangle$

lemma *union-dirBd-matching*:

assumes *dir-bipartite-digraph* *G* *X* *Y* **and** *C* = (*C* $\cap$ *X*) $\cup$ (*C* $\cap$ *Y*)
and *dirBD-perfect-matching* (*induced-subgraph* *G* (*C* $\cap$ *X* $\cup$ (*Y* $-$ *C* $\cap$ *Y*)))
(*C* $\cap$ *X*) (*Y* $-$ *C* $\cap$ *Y*) *E1*
and *dirBD-perfect-matching* (*induced-subgraph* (*inverse-digraph* *G*) (*C* $\cap$ *Y* $\cup$ (*X* $-$ *C* $\cap$ *X*))) (*C* $\cap$ *Y*)
(*X* $-$ *C* $\cap$ *X*) *E2*
shows *dirBD-matching* *G* *X* *Y* (*E1* $\cup$ *E2*)
 $\langle \text{proof} \rangle$

lemma *card-prop-induced-dirBD-perfect-matching*:

assumes *dirBD-perfect-matching* (*induced-subgraph* *G* (*C* $\cap$ *X* $\cup$ (*Y* $-$ *C* $\cap$ *Y*))) (*C* $\cap$ *X*) (*Y* $-$ *C* $\cap$ *Y*) *E*
and *finite* (*arcs* *G*)
shows *card* (*C* $\cap$ *X*) = *card* *E*
 $\langle \text{proof} \rangle$

lemma *card-prop-induced-inv-dirBD-perfect-matching*:

assumes *dirBD-perfect-matching* (*induced-subgraph* (*inverse-digraph* *G*) (*C* $\cap$ *Y* $\cup$ (*X* $-$ *C* $\cap$ *X*))) (*C* $\cap$ *Y*) (*X* $-$ *C* $\cap$ *X*) *E*
shows *card* (*C* $\cap$ *Y*) = *card* *E* $\langle \text{proof} \rangle$

lemma *finite-neighbourhood-aux*:

assumes *dir-bipartite-digraph* *G* *X* *Y* **and** *finite* (*verts* *G*)
shows $\forall i \in (\text{verts } G). \text{finite}(\text{neighbourhood } G i)$
 $\langle \text{proof} \rangle$

lemma *finite-neighbourhood*:

assumes *dir-bipartite-digraph* *G* *X* *Y* **and** *finite* (*verts* *G*)
shows $\forall i \in X. \text{finite}(\text{neighbourhood } G i)$
 $\langle \text{proof} \rangle$

lemma *finite-inverse-neighbourhood*:

assumes *dir-bipartite-digraph* *G* *X* *Y* **and** *finite* (*verts* *G*)
shows $\forall i \in Y. \text{finite}(\text{neighbourhood}(\text{inverse-digraph } G) i)$
 $\langle \text{proof} \rangle$

lemma *finite-vertex-cover*:

```

    assumes dir-bipartite-digraph  $G$   $X$   $Y$  and finite (verts  $G$ ) and vertex-cover  $G$ 
    (arcs  $G$ )  $C$ 
    shows finite  $C$ 
    <proof>

```

```

theorem Konig-Egervary:
  fixes  $G :: ('a::countable, 'b::countable)$  pre-digraph
  assumes dir-bipartite-digraph  $G$   $X$   $Y$  and finite (verts  $G$ )
    and maximum-dirBD-matching  $G$   $X$   $Y$   $E$  and minimum-vertex-cover  $G$  (arcs
     $G$ )  $C$ 
  shows  $\text{card } E = \text{card } C$ 
  <proof>

```

3 Dilworth Finite Theorem

In a finite graph, the Cardinality of the smallest chain decomposition and the largest anti-chain are equal.

theory *Dilworth-Finite*

imports

Main

HOL-Library.Disjoint-Sets

Koenig-Egervary-finite

begin

```

locale part-order =
  fixes  $A :: 'a$  set
  fixes  $r :: 'a$  rel
  assumes p-o-translation : partial-order-on  $A$   $r$ 

```

```

locale part-order-countable = part-order  $A$   $r$ 
  for  $A :: 'a::countable$  set and  $r :: 'a$  rel

```

```

definition chain::  $'a$  set  $\Rightarrow$   $'a$  set  $\Rightarrow$   $'a$  rel  $\Rightarrow$  bool
  where chain  $B$   $A$   $r \equiv B \subseteq A \wedge$  partial-order-on  $A$   $r \wedge$  total-on  $B$   $r$ 

```

```

definition chain-decomposition::  $'a$  set  $\Rightarrow$   $'a$  rel  $\Rightarrow$   $'a$  set set  $\Rightarrow$  bool
  where chain-decomposition  $A$   $r$   $C \equiv$  partition-on  $A$   $C \wedge (\forall B \in C. (\text{chain } B \ A \ r))$ 

```

```

lemma chain-decomposition  $\{\}$   $r$   $\{\}$ 
  <proof>

```

```

definition smallest-chain-decomposition::  $'a$  set  $\Rightarrow$   $'a$  rel  $\Rightarrow$   $'a$  set set  $\Rightarrow$  bool
  where smallest-chain-decomposition  $A$   $r$   $C \equiv$ 
    (chain-decomposition  $A$   $r$   $C$ )  $\wedge (\forall P. \text{chain-decomposition } A \ r \ P \longrightarrow \text{card } C \leq \text{card } P)$ 

```

definition *anti-total* :: 'a set $\Rightarrow$ 'a rel $\Rightarrow$ bool

where *anti-total* $A\ r \equiv (\forall x \in A. \forall y \in A. x \neq y \longrightarrow (x,y) \notin r \wedge (y,x) \notin r)$

definition *anti-chain* :: 'a set $\Rightarrow$ 'a rel $\Rightarrow$ 'a set $\Rightarrow$ bool

where *anti-chain* $A\ r\ B \equiv B \subseteq A \wedge \text{partial-order-on } A\ r \wedge \text{anti-total } B\ r$

definition *largest-antichain* :: 'a set $\Rightarrow$ 'a rel $\Rightarrow$ 'a set $\Rightarrow$ bool

where *largest-antichain* $A\ r\ B \equiv \text{anti-chain } A\ r\ B \wedge (\forall C. \text{anti-chain } A\ r\ C \longrightarrow \text{card } C \leq \text{card } B)$

definition *relation-to-digraph* :: 'a set $\Rightarrow$ 'a rel $\Rightarrow ((\text{'a} + \text{'a}), (\text{'a} + \text{'a}) \times (\text{'a} + \text{'a}))\text{pre-digraph}$

where

relation-to-digraph $A\ r \equiv$

$(\mid \text{verts} = A <+> A,$

$\text{arcs} = \{(Inl\ x, Inr\ y) \mid x\ y. x \in A \wedge y \in A \wedge (x,y) \in r \wedge x \neq y\},$

$\text{tail} = \lambda\ (x,y). x,$

$\text{head} = \lambda\ (x,y). y$

$\mid)$

lemma *rel-to-digraph-is-dir-bipartite*:

assumes $r \subseteq A \times A$

shows *dir-bipartite-digraph* (*relation-to-digraph* $A\ r$) (*Inl* ' A) (*Inr* ' A)

<proof>

lemma *exists-min-vertex-cover*:

assumes *dir-bipartite-digraph* $G\ X\ Y$ **and** *finite* $(X \cup Y)$

shows $\exists C. \text{minimum-vertex-cover } G\ (\text{arcs } G)\ C$

<proof>

definition *roots* :: 'a set $\Rightarrow ((\text{'a} + \text{'a}) \times (\text{'a} + \text{'a}))\ \text{set} \Rightarrow$ 'a set **where**

roots $A\ M \equiv \{x \in A. \forall y. (Inl\ y, Inr\ x) \notin M\}$

lemma *not-root-has-pred*:

assumes $x \in A$ **and** $x \notin \text{roots } A\ M$

shows $\exists y. (Inl\ y, Inr\ x) \in M$

<proof>

definition *match-rel* ::

$((\text{'a} + \text{'a}) \times (\text{'a} + \text{'a}))\ \text{set} \Rightarrow (\text{'a} \times \text{'a})\ \text{set}$

where

match-rel $M =$

$\{(x,y). (Inl\ x, Inr\ y) \in M\}$

lemma *matching-edges-strict*:

assumes *dirBD-matching* (*relation-to-digraph* $A\ r$) (*Inl* ' A) (*Inr* ' A) M **and**

$(x, y) \in \text{match-rel } M$

shows $x \in A \wedge y \in A \wedge (x,y) \in r \wedge x \neq y$

$\langle proof \rangle$

definition *chain-from-root* **where**

chain-from-root $a\ A\ M =$
 $\{x \in A. (a, x) \in (match\text{-}rel\ M)^{\wedge*}\}$

lemma *successor-unique*:

assumes $(x, y1) \in match\text{-}rel\ M$ **and** $(x, y2) \in match\text{-}rel\ M$
and *dirBD-matching* (*relation-to-digraph* $A\ r$) (*Inl* ' A) (*Inr* ' A) M
shows $y1 = y2$

$\langle proof \rangle$

lemma *predecessor-unique*:

assumes $(x1, y) \in match\text{-}rel\ M$ **and** $(x2, y) \in match\text{-}rel\ M$
and *dirBD-matching* (*relation-to-digraph* $A\ r$) (*Inl* ' A) (*Inr* ' A) M
shows $x1 = x2$

$\langle proof \rangle$

lemma *root-unique*:

assumes
 $ax: (a, x) \in (match\text{-}rel\ M)^{\wedge*}$ **and**
 $bx: (b, x) \in (match\text{-}rel\ M)^{\wedge*}$ **and**
 $ar: a \in roots\ A\ M$ **and**
 $br: b \in roots\ A\ M$ **and**
 $mt: dirBD\text{-}matching\ (relation\text{-}to\text{-}digraph\ A\ r)\ (Inl\ 'A)\ (Inr\ 'A)\ M$
shows $a = b$

$\langle proof \rangle$

definition *pred-rel*:: $((a + 'a) \times (a + 'a))\ set \Rightarrow 'a\ set \Rightarrow ('a \times 'a)\ set$ **where**

pred-rel $M\ A = \{(x, y). (Inl\ x, Inr\ y) \in M \wedge x \in A \wedge y \in A\}$

lemma *pred-rel-subset-strict*:

assumes *dirBD-matching* (*relation-to-digraph* $A\ r$) (*Inl* ' A) (*Inr* ' A) M
shows *pred-rel* $M\ A \subseteq r - Id$

$\langle proof \rangle$

lemma (*in part-order*) *strict-order-wf*:

assumes *finite* A
shows *wf* $(r - Id)$

$\langle proof \rangle$

lemma (*in part-order*) *finite-pred-wf*:

assumes *finite* A **and**
dirBD-matching (*relation-to-digraph* $A\ r$) (*Inl* ' A) (*Inr* ' A) M
shows *wf* (*pred-rel* $M\ A$)

$\langle proof \rangle$

lemma (*in part-order*) *exists-root-reaching*:

assumes *finite* A

and *dirBD-matching* (*relation-to-digraph* A r) (*Inl* ' A) (*Inr* ' A) M
and $x \in A$
shows $\exists a \in \text{roots } A \ M. (a, x) \in (\text{match-rel } M)^{\wedge*}$
 $\langle \text{proof} \rangle$

lemma *single-valued-rfl-trnstv-closure* :
assumes *single-valued* r
and $(a, x) \in r^{\wedge*}$
and $(a, y) \in r^{\wedge*}$
shows $x = y \vee (x, y) \in r^{\wedge*} \vee (y, x) \in r^{\wedge*}$
 $\langle \text{proof} \rangle$

lemma *functional-rtrancl-linear*:
assumes
func: $\bigwedge x \ y1 \ y2. (x, y1) \in r \implies (x, y2) \in r \implies y1 = y2$
and *ax*: $(a, x) \in r^{\wedge*}$
and *ay*: $(a, y) \in r^{\wedge*}$
shows $x = y \vee (x, y) \in r^{\wedge*} \vee (y, x) \in r^{\wedge*}$
 $\langle \text{proof} \rangle$

lemma *reachable-linear-from-root*:
assumes
 $(a, x) \in (\text{match-rel } M)^{\wedge*}$ **and**
 $(a, y) \in (\text{match-rel } M)^{\wedge*}$ **and**
dirBD-matching (*relation-to-digraph* A r) (*Inl* ' A) (*Inr* ' A) M
shows
 $x = y \vee (x, y) \in (\text{match-rel } M)^{\wedge*} \vee (y, x) \in (\text{match-rel } M)^{\wedge*}$
 $\langle \text{proof} \rangle$

lemma *match-rel-subset-r*:
assumes *dirBD-matching* (*relation-to-digraph* A r) (*Inl* ' A) (*Inr* ' A) M
shows *match-rel* $M \subseteq r$
 $\langle \text{proof} \rangle$

definition *chain-decomposition-order* :: ' a set $\Rightarrow$ ' a rel $\Rightarrow ((a + 'a) \times (a + 'a))$
 $\text{set} \Rightarrow 'a \text{ set set}$
where
chain-decomposition-order A r $M =$
 $\{\text{chain-from-root } a \ A \ M \mid a. a \in (\text{roots } A \ M)\}$

lemma *match-rel-not-empty*: $a \in A \longrightarrow (a, a) \in (\text{match-rel } M)^{\wedge*}$
 $\langle \text{proof} \rangle$

context *part-order*
begin
lemma *match-rel-rtrancl-in-r*:
assumes

$\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$
and $a \in A$ **and** $(a, x) \in (\text{match-rel } M)^{\wedge*}$
shows $(a, x) \in r$
 $\langle \text{proof} \rangle$

lemma *reachable-comparable:*

assumes
 $(a, x) \in (\text{match-rel } M)^{\wedge*}$ **and**
 $(a, y) \in (\text{match-rel } M)^{\wedge*}$ **and**
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$ **and**
 $a \in A$
shows
 $(x, y) \in r \vee (y, x) \in r$
 $\langle \text{proof} \rangle$

lemma *chain-from-root-is-chain:*

assumes
 $\text{finite } A$ **and**
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$ **and** $a \in A$
shows $\text{chain } (\text{chain-from-root } a \ A \ M) \ A \ r$
 $\langle \text{proof} \rangle$

lemma *chain-decomposition-cover:*

assumes
 $\text{finite } A$
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$
shows
 $\bigcup (\text{chain-decomposition-order } A \ r \ M) = A$
 $\langle \text{proof} \rangle$

lemma *chain-decomposition-disjoint:*

assumes $\text{finite } A$ **and** $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$
shows $\text{disjoint } (\text{chain-decomposition-order } A \ r \ M)$
 $\langle \text{proof} \rangle$

lemma *chain-decomposition-non-empty:*

assumes $\text{roots } A \ M \neq \{\}$ **and**
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$
shows $\{\} \notin \text{chain-decomposition-order } A \ r \ M$
 $\langle \text{proof} \rangle$

lemma *chain-decomposition-partition:*

assumes $r \subseteq A \times A$ **and**
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$
and $\text{finite } A$
shows $\text{partition-on } A \ (\text{chain-decomposition-order } A \ r \ M)$
 $\langle \text{proof} \rangle$

lemma *exist-chain-decomposition*:

assumes $r \subseteq A \times A$ **and**
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ M$
and *finite* A
shows $\text{chain-decomposition } A \ r \ (\text{chain-decomposition-order } A \ r \ M)$
 $\langle \text{proof} \rangle$
end

lemma *max-card-in-set*:

fixes $M :: 'a \text{ set set}$
assumes *finite* M **and** $M \neq \{\}$
shows $\exists m \in M. \forall m' \in M. \text{card } m' \leq \text{card } m$
 $\langle \text{proof} \rangle$

lemma *maximum-matching-digraph*:

assumes $\text{dir-bipartite-digraph } G \ X \ Y$ **and** *finite* $(\text{arcs } G)$
shows $\exists M. \text{maximum-dirBD-matching } G \ X \ Y \ M$
 $\langle \text{proof} \rangle$

lemma *chain-from-root-inj*:

assumes
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ M$
shows
 $\text{inj-on } (\lambda a. \text{chain-from-root } a \ A \ M) \ (\text{roots } A \ M)$
 $\langle \text{proof} \rangle$

lemma *chain-decomposition-as-image*:

$\text{chain-decomposition-order } A \ r \ M = (\lambda a. \text{chain-from-root } a \ A \ M) ' \text{roots } A \ M$
 $\langle \text{proof} \rangle$

lemma *card-chain-decomposition*:

assumes
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ M$ **and** *finite*
 $(\text{roots } A \ M)$
shows
 $\text{card } (\text{chain-from-root } ' \text{roots } A \ M) = \text{card } (\text{roots } A \ M)$
 $\langle \text{proof} \rangle$

fun $f\text{-match-rel} :: ('a \times 'a) \Rightarrow (('a + 'a) \times ('a + 'a))$

where

$f\text{-match-rel } (x,y) = (\text{Inl } x, \text{Inr } y)$

lemma *inj-match-rel*:

assumes $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ M$
shows $\text{inj } f\text{-match-rel}$
 $\langle \text{proof} \rangle$

lemma *surj-match-rel*:

assumes $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ M$

shows $(\forall z. \forall w. (z, w) \in M \longrightarrow (\exists x. \exists y. (x, y) \in \text{match-rel } M \wedge (\text{Inl } x, \text{Inr } y) = (z, w)))$
 $\langle \text{proof} \rangle$

lemma *surj-match-rel1*:
assumes *dirBD-matching* (*relation-to-digraph* A r) ($\text{Inl} \text{ ' } A$) ($\text{Inr} \text{ ' } A$) M
shows $f\text{-match-rel}'\text{match-rel } M = M$
 $\langle \text{proof} \rangle$

lemma *card-match-rel-eq*:
assumes *dirBD-matching* (*relation-to-digraph* A r) ($\text{Inl} \text{ ' } A$) ($\text{Inr} \text{ ' } A$) M **and** *finite* M
and *finite* ($\text{match-rel } M$)
shows $\text{card } (\text{match-rel } M) = \text{card } M$
 $\langle \text{proof} \rangle$

definition *has-pred*:: $\text{'a set} \Rightarrow ((\text{'a} + \text{'a}) \times (\text{'a} + \text{'a})) \text{ set} \Rightarrow \text{'a set}$ **where**
 $\text{has-pred } A \ M = \{x \in A. (\exists y. (y, x) \in \text{match-rel } M)\}$

lemma *roots-partition*:
 $\text{roots } A \ M = A - \text{has-pred } A \ M$
 $\langle \text{proof} \rangle$

lemma *disjoint-roots-pred*:
 $\text{roots } A \ M \cap \text{has-pred } A \ M = \{\}$
 $\langle \text{proof} \rangle$

lemma *union-roots-pred*:
 $\text{roots } A \ M \cup \text{has-pred } A \ M = A$
 $\langle \text{proof} \rangle$

lemma *card-partition*:
assumes *finite* A
shows $\text{card } A = \text{card } (\text{roots } A \ M) + \text{card } (\text{has-pred } A \ M)$
 $\langle \text{proof} \rangle$

lemma *has-pred-image*:
assumes *dirBD-matching* (*relation-to-digraph* A r) ($\text{Inl} \text{ ' } A$) ($\text{Inr} \text{ ' } A$) M
shows $\text{has-pred } A \ M = \text{snd ' } (\text{match-rel } M)$
 $\langle \text{proof} \rangle$

lemma *inj-snd-match-rel*:
assumes
dirBD-matching (*relation-to-digraph* A r) ($\text{Inl} \text{ ' } A$) ($\text{Inr} \text{ ' } A$) M
shows *inj-on* snd ($\text{match-rel } M$)
 $\langle \text{proof} \rangle$

lemma *card-has-pred*:
assumes

$\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M \text{ and } \text{finite } (\text{match-rel } M)$
shows $\text{card } (\text{has-pred } A \ M) = \text{card } (\text{match-rel } M)$
 $\langle \text{proof} \rangle$

lemma *card-roots*:

assumes
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M \text{ and } \text{finite } A$
and $\text{finite } M$ **and** $\text{finite } (\text{match-rel } M)$
shows
 $\text{card } (\text{roots } A \ M) = \text{card } A - \text{card } M$
 $\langle \text{proof} \rangle$

lemma *finite-arcs*:

assumes $\text{finite } A$ **and** $r \subseteq A \times A$
shows $\text{finite } (\text{arcs } (\text{relation-to-digraph } A \ r))$
 $\langle \text{proof} \rangle$

lemma *finite-verts*:

assumes $\text{finite } A$
shows $\text{finite } (\text{verts } (\text{relation-to-digraph } A \ r))$
 $\langle \text{proof} \rangle$

lemma *finite-matching*:

assumes
 $r \subseteq A \times A$ **and**
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$ **and** $\text{finite } A$
shows $\text{finite } M$ $\langle \text{proof} \rangle$

lemma *(in part-order)chain-decomposition-cardinality*:

assumes
 $\text{dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } ' A) \ (\text{Inr } ' A) \ M$ **and** $\text{finite } A$
shows
 $\text{card } (\text{chain-decomposition-order } A \ r \ M) = \text{card } A - \text{card } M$
 $\langle \text{proof} \rangle$

lemma *cover-minimum*:

assumes $\text{minimum-vertex-cover } G \ E \ C$ **and** $\text{finite } C$
shows $\forall c \in C. \exists e \in E. \ c = (\text{head } G \ e) \vee c = (\text{tail } G \ e)$
 $\langle \text{proof} \rangle$

context *part-order*

begin

lemma *vertex-cover-reduction*:

assumes $(\forall e \in \text{arcs } (\text{relation-to-digraph } A \ r)).$
 $\text{head } (\text{relation-to-digraph } A \ r) \ e \in C \vee$
 $\text{tail } (\text{relation-to-digraph } A \ r) \ e \in C)$
and $\text{Inl } x \in C$ **and** $\text{Inr } x \in C$
shows $(\forall e \in \text{arcs } (\text{relation-to-digraph } A \ r)).$

$head (relation\text{-}to\text{-}digraph\ A\ r)\ e \in C - \{Inr\ x\} \vee$
 $tail (relation\text{-}to\text{-}digraph\ A\ r)\ e \in C - \{Inr\ x\} \vee$
 $(\forall e \in arcs\ (relation\text{-}to\text{-}digraph\ A\ r)).$
 $head (relation\text{-}to\text{-}digraph\ A\ r)\ e \in C - \{Inl\ x\} \vee$
 $tail (relation\text{-}to\text{-}digraph\ A\ r)\ e \in C - \{Inl\ x\}$
 $\langle proof \rangle$

lemma *vertex-cover-subset*:

assumes *vertex-cover* $(relation\text{-}to\text{-}digraph\ A\ r)\ (arcs\ (relation\text{-}to\text{-}digraph\ A\ r))$
 C
and $Inl\ x \in C \wedge Inr\ x \in C$
shows *vertex-cover* $(relation\text{-}to\text{-}digraph\ A\ r)\ (arcs\ (relation\text{-}to\text{-}digraph\ A\ r))$
 $(C - \{Inl\ x\})$
 $\vee$ *vertex-cover* $(relation\text{-}to\text{-}digraph\ A\ r)\ (arcs\ (relation\text{-}to\text{-}digraph\ A\ r))$
 $(C - \{Inr\ x\})$
 $\langle proof \rangle$

lemma *matching-vertex-cover*:

assumes *minimum-vertex-cover* $(relation\text{-}to\text{-}digraph\ A\ r)\ (arcs\ (relation\text{-}to\text{-}digraph\ A\ r))$
 C
and *finite* A
shows $Inl\ x \notin C \vee Inr\ x \notin C$
 $\langle proof \rangle$

lemma *L-R-disjoint*:

assumes *minimum-vertex-cover* $(relation\text{-}to\text{-}digraph\ A\ r)\ (arcs\ (relation\text{-}to\text{-}digraph\ A\ r))$
 C
and *finite* A
shows $\{x \in A. Inl\ x \in C\} \cap \{x \in A. Inr\ x \in C\} = \{\}$ $\langle proof \rangle$

lemma *card-projection*:

assumes *minimum-vertex-cover* $(relation\text{-}to\text{-}digraph\ A\ r)\ (arcs\ (relation\text{-}to\text{-}digraph\ A\ r))$
 C **and**
finite A
shows
 $card\ \{x \in A. Inl\ x \in C \vee Inr\ x \in C\} = card\ C$
 $\langle proof \rangle$

lemma *cover-antichain*:

assumes *minimum-vertex-cover* $(relation\text{-}to\text{-}digraph\ A\ r)\ (arcs\ (relation\text{-}to\text{-}digraph\ A\ r))$
 C **and**
finite A
shows *anti-chain* $A\ r\ \{x \in A. Inl\ x \notin C \wedge Inr\ x \notin C\} \wedge$
 $card\ \{x \in A. Inl\ x \notin C \wedge Inr\ x \notin C\} = card\ A - card\ C$
 $\langle proof \rangle$

lemma *antichain-le-chain-decomposition*:

assumes *chain-decomposition* $A\ r\ C$ **and** *anti-chain* $A\ r\ X$
and *finite* X **and** *finite* C

shows $\text{card } X \leq \text{card } C$
 $\langle \text{proof} \rangle$
end

lemma *Konig-Egervary-relation:*

fixes $A:: 'a::\text{countable set}$
assumes
 $\text{maximum-dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ E$ **and**
 $\text{minimum-vertex-cover } (\text{relation-to-digraph } A \ r) \ (\text{arcs } (\text{relation-to-digraph } A \ r)) \ C$ **and**
 $\text{finite } (\text{verts } (\text{relation-to-digraph } A \ r))$
shows $\text{card } E = \text{card } C$ $\langle \text{proof} \rangle$

lemma *(in part-order-countable)exists-max-antichain-from-matching:*

assumes $\text{finite } A$ **and** $\text{maximum-dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ M$
shows $\exists X. \text{largest-antichain } A \ r \ X \wedge (\text{card } X = \text{card } A - \text{card } M)$
 $\langle \text{proof} \rangle$

lemma *(in part-order-countable)smallest-chain-decomposition-order:*

assumes $\text{finite } A$ **and** $\text{maximum-dirBD-matching } (\text{relation-to-digraph } A \ r) \ (\text{Inl } 'A) \ (\text{Inr } 'A) \ M$
shows $\exists DC. (\text{smallest-chain-decomposition } A \ r \ DC) \wedge (\text{card } DC = \text{card } A - \text{card } M)$
 $\langle \text{proof} \rangle$

lemma *chain-unit:*

assumes $r \subseteq A \times A$ **and** $r \neq \{\}$ **and** $\forall a \ b. (a, b) \in r \longrightarrow a = b$
shows $(\text{chain } B \ A \ r) \wedge B \neq \{\} \longrightarrow (\exists x \in A. B = \{x\})$
 $\langle \text{proof} \rangle$

lemma *chain-descomposition-unit:*

assumes $r \subseteq A \times A$ **and** $\forall a \ b. (a, b) \in r \longrightarrow a = b$ **and** $\text{partial-order-on } A$
 r **and** $r \neq \{\}$
shows $(\text{chain-decomposition } A \ r \ \{\{x\} | x. x \in A\}) \wedge ((\text{chain-decomposition } A \ r \ D) \longrightarrow D = \{\{x\} | x. x \in A\})$
 $\langle \text{proof} \rangle$

lemma *antichain-on:*

assumes $r \subseteq A \times A$ **and** $\forall a \ b. (a, b) \in r \longrightarrow a = b$ **and** $\text{partial-order-on } A$
 r **and**
 $r \neq \{\}$
shows $\text{anti-chain } A \ r \ A$
 $\langle \text{proof} \rangle$

lemma *largest-antichain-on:*

assumes $r \subseteq A \times A$ **and** $\forall a \ b. (a, b) \in r \longrightarrow a = b$ **and** $\text{partial-order-on } A$
 r **and**
 $r \neq \{\}$ **and** $\text{finite } A$

shows *largest-antichain* A r A $\langle \text{proof} \rangle$

lemma *card-smCD-lgAC1*:
assumes *finite* A **shows** $\text{card } A = \text{card } \{\{x\} \mid x. x \in A\}$
 $\langle \text{proof} \rangle$

lemma *arcs-empty*:
assumes $r \subseteq A \times A$ **and** *arcs* (*relation-to-digraph* A r) = $\{\}$
shows $\forall a\ b. (a, b) \in r \longrightarrow a = b$
 $\langle \text{proof} \rangle$

lemma *card-smCD-lgAC*:
assumes $r \subseteq A \times A$ **and** $\forall a\ b. (a, b) \in r \longrightarrow a = b$ **and** *partial-order-on* A
 r **and**
 $r \neq \{\}$ **and** *finite* A **and**
smallest-chain-decomposition A r *smCD* **and** *largest-antichain* A r *lgAC*
shows $\text{card } \text{smCD} = \text{card } \text{lgAC}$
 $\langle \text{proof} \rangle$

lemma (*in part-order*) *Dilworth-empty*:
assumes *arcs* (*relation-to-digraph* A r) = $\{\}$ **and** *finite* A **and**
smallest-chain-decomposition A r *smCD* **and** *largest-antichain* A r *lgAC*
shows $\text{card } \text{smCD} = \text{card } \text{lgAC}$
 $\langle \text{proof} \rangle$

theorem (*in part-order-countable*) *Dilworth-Finite*:
assumes *finite* A
shows $\exists \text{smCD}. \exists \text{lgAC}. \text{smallest-chain-decomposition } A \text{ } r \text{ smCD} \wedge \text{largest-antichain } A \text{ } r \text{ lgAC}$
 $\wedge \text{card } \text{smCD} = \text{card } \text{lgAC}$
 $\langle \text{proof} \rangle$

end

4 Dilworth Countable Theorem

theory *DilworthCountable*
imports
Dilworth-Finite
Prop-Compactness.k-coloring
begin

The countable infinite version of Dilworth's theorem for partially ordered sets states that, whenever the "width" of the partial order is finite, i.e., whenever there exists a finite upper bound on the size of anti-chains, there exists a finite minimal chain decomposition with size equal to the size of the largest anti-chain.

definition *incomparability-graph* :: $'a \text{ rel} \Rightarrow 'a \text{ set} \Rightarrow 'a \text{ digraph}$

where

incomparability-graph $r A \equiv (A, \{(x, y) \mid x y. x \in A \wedge y \in A \wedge (x, y) \notin r \wedge (y, x) \notin r \wedge x \neq y\})$

lemma *chain-decomposition-colorable*:

assumes *partition-on* $A P$ **and** *finite* P **and** $(\forall B \in P. (\text{total-on } B r))$

shows (*colorable* (*incomparability-graph* $r A$) (*card* P))

$\langle \text{proof} \rangle$

lemma *non-empty-set-color*: **assumes** *coloring* $c k$ (*incomparability-graph* $r A$) **and** $j \in c'A$

shows $\{x \in A. c x = j\} \neq \{\}$

$\langle \text{proof} \rangle$

lemma *coloring-particion*:

assumes *coloring* $c k$ (*incomparability-graph* $r A$)

shows *partition-on* $A \{\{x \in A. c(x) = j\} \mid j. j \in c'A\}$

$\langle \text{proof} \rangle$

lemma (*in part-order*) *coloring-total*:

assumes *coloring* $c k$ (*incomparability-graph* $r A$) **and** $j \in c'A$

shows *total-on* $\{x \in A. c x = j\} r$

$\langle \text{proof} \rangle$

lemma *partition-card-le-colors*:

assumes *col*: *coloring* $c k G$

shows *card* $(\{\{x \in A. c x = j\} \mid j. j \in c'(V[G])\}) \leq k$

$\langle \text{proof} \rangle$

lemma (*in part-order*) *coloring-chain-decomposition1*:

assumes *coloring* $c k$ (*incomparability-graph* $r A$)

shows *chain-decomposition* $A r \{\{x \in A. c x = j\} \mid j. j \in c'A\} \wedge$

card $(\{\{x \in A. c x = j\} \mid j. j \in c'A\}) \leq k$

$\langle \text{proof} \rangle$

lemma *coloring-chain-decomposition-finite*:

assumes *coloring* $c k$ (*incomparability-graph* $r A$)

shows *finite* $(\{\{x \in A. c x = j\} \mid j. j \in c'A\})$

$\langle \text{proof} \rangle$

lemma (*in part-order*) *coloring-chain-decomposition*:

assumes *coloring* $c k$ (*incomparability-graph* $r A$)

shows $(\exists CD. \text{chain-decomposition } A r CD \wedge \text{card } CD \leq k \wedge \text{finite } CD)$

$\langle \text{proof} \rangle$

lemma (*in part-order-countable*) *width-coloring-finite*:

assumes $(\forall AC. \text{anti-chain } A r AC \longrightarrow \text{card } AC \leq m)$ **and** *finite* A

shows $(\exists c. \exists k. \text{coloring } c k (\text{incomparability-graph } r A) \wedge k \leq m)$

$\langle \text{proof} \rangle$

lemma (in *part-order-countable*) *partial-order-restrc*:

assumes $B \subseteq A$

shows *partial-order-on* B (*Restr* r B)

<proof>

lemma (in *part-order-countable*) *antichain-restrc*:

assumes *anti-chain* A r AC **and** $B \subseteq A$

shows *anti-chain* B (*Restr* r B) ($AC \cap B$)

<proof>

lemma (in *part-order-countable*) *width-induced-subgraph*:

assumes $\forall AC. \text{anti-chain } A \text{ } r \text{ } AC \longrightarrow \text{card } AC \leq w$ **and** *finite* B

shows $\forall ACB. \text{anti-chain } B \text{ } (Restr \text{ } r \text{ } B) \text{ } ACB \longrightarrow \text{card } ACB \leq w$ *<proof>*

lemma *graph-incomparability*:

is-graph (*incomparability-graph* r A)

<proof>

lemma *induced-subgraph-incomparability*:

assumes *is-induced-subgraph* H (*incomparability-graph* r A) $\wedge$ *finite-graph* H

shows $\exists B. (\text{finite } B) \wedge B = V[H] \wedge (E[H] = E[(\text{incomparability-graph } r \text{ } A)] \cap (B \times B)) \wedge B \subseteq A$

<proof>

lemma *incomparability-subgraph*:

assumes *is-graph* (*incomparability-graph* r A)

and *is-subgraph* H (*incomparability-graph* r A) $\wedge$ *finite-graph* H

shows $\exists B. (\text{finite } B) \wedge B = V[H] \wedge (E[H] \subseteq E[(\text{incomparability-graph } r \text{ } A)] \cap (B \times B)) \wedge B \subseteq A$

<proof>

lemma (in *part-order-countable*) *colorable-induced-subgraph*:

assumes $\forall AC. \text{anti-chain } A \text{ } r \text{ } AC \longrightarrow \text{card } AC \leq m$

shows $(\forall H. \text{is-induced-subgraph } H \text{ } (incomparability-graph \text{ } r \text{ } A) \wedge \text{finite-graph } H \longrightarrow \text{colorable } H \text{ } m)$

<proof>

lemma *exists-smallest-chain-decomposition*:

assumes $\exists C. \text{chain-decomposition } A \text{ } r \text{ } C$

shows $\exists C. \text{smallest-chain-decomposition } A \text{ } r \text{ } C$

<proof>

lemma (in *part-order-countable*) *exists-largest-antichain*:

assumes $\forall AC. \text{anti-chain } A \text{ } r \text{ } AC \longrightarrow \text{card } AC \leq m$

shows $\exists \text{lg}AC. \text{largest-antichain } A \text{ } r \text{ } \text{lg}AC$

<proof>

theorem (in *part-order-countable*) *Dilworth-countable-aux*:

assumes *largest-antichain* $A \ r \ lgAC$
shows $(\exists CD. (chain-decomposition \ A \ r \ CD \wedge finite \ CD) \wedge card \ CD \leq card \ lgAC)$
 $\langle proof \rangle$

definition *largest-finite-antichain* $:: 'a \ set \Rightarrow 'a \ rel \Rightarrow 'a \ set \Rightarrow bool$
where *largest-finite-antichain* $A \ r \ B \equiv$
 $anti-chain \ A \ r \ B \wedge finite \ B \wedge (\forall C. anti-chain \ A \ r \ C \wedge finite \ C \longrightarrow card \ C \leq card \ B)$

lemma (*in part-order-countable*) *exists-largest-finite-antichain*:
assumes $H: \forall AC. anti-chain \ A \ r \ AC \wedge finite \ AC \longrightarrow card \ AC \leq m$
shows $\exists lgAC. largest-finite-antichain \ A \ r \ lgAC$
 $\langle proof \rangle$

definition *finite-smallest-chain-decomposition* $:: 'a \ set \Rightarrow 'a \ rel \Rightarrow 'a \ set \ set \Rightarrow bool$
where
finite-smallest-chain-decomposition $A \ r \ CD \equiv$
 $chain-decomposition \ A \ r \ CD \wedge$
 $finite \ CD \wedge$
 $(\forall P. chain-decomposition \ A \ r \ P \wedge finite \ P \longrightarrow card \ CD \leq card \ P)$

lemma *exists-smallest-finite-chain-decomposition*:
assumes $\exists CD. chain-decomposition \ A \ r \ CD \wedge finite \ CD$
shows $\exists sCD. finite-smallest-chain-decomposition \ A \ r \ sCD$
 $\langle proof \rangle$

theorem (*in part-order-countable*) *Dilworth-countable*:
assumes $\exists m. \forall AC. anti-chain \ A \ r \ AC \wedge finite \ AC \longrightarrow card \ AC \leq m$
shows $\exists smD. \exists lgAC. finite-smallest-chain-decomposition \ A \ r \ smD \wedge largest-finite-antichain \ A \ r \ lgAC \wedge$
 $card \ smD = card \ lgAC$
 $\langle proof \rangle$

end

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