

Cubical Categories

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February 6, 2026

Abstract

This AFP entry formalises cubical ω -categories and cubical ω -categories with connections in the style of single-set categories. Cubical categories, and the cubical sets on which they are based, have their origins and main applications in algebraic topology. Applications in computer science include homotopy type theory, higher-dimensional automata in concurrency theory and higher-dimensional rewriting. The single-set axiomatisation, introduced in these components and a companion paper, allows a formalisation based on Isabelle’s type classes.

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1 Introductory Remarks

Based on a formalisation of catoids and single-set categories in the AFP [2] we develop single-set axiomatisations for cubical ω -categories with and without connections. A detailed explanation of the single-set approach, the classical approach to cubical ω -categories and the proof of equivalence of the single-set and the classical approach can be found in a companion article [1]. Isabelle, with its high degree of proof automation, has been instrumental for developing the single-set axioms introduced in this article.

2 Indexed Catoids

theory *ICatoids*
 imports *Catoids.Catoid*

begin

All categories considered in this component are single-set categories.

no-notation *src* ($\langle\sigma\rangle$)

notation *True* ($\langle tt\rangle$)

notation *False* ($\langle ff\rangle$)

abbreviation *Fix* :: ($'a \Rightarrow 'a$) \Rightarrow $'a$ set **where**
 $Fix\ f \equiv \{x. f\ x = x\}$

First we lift locality to powersets.

lemma (**in** *local-catoid*) *locality-lifting*: $(X \star Y \neq \{\}) = (Tgt\ X \cap Src\ Y \neq \{\})$
 $\langle proof \rangle$

The following lemma about functional catoids is useful in proofs.

lemma (**in** *functional-catoid*) *pcomp-def-var4*: $\Delta\ x\ y \Longrightarrow x \odot y = \{x \otimes y\}$
 $\langle proof \rangle$

2.1 Indexed catoids and categories

class *face-map-op* =
 fixes *fmap* :: $nat \Rightarrow bool \Rightarrow 'a \Rightarrow 'a$ ($\langle\partial\rangle$)

begin

abbreviation *Face* :: $nat \Rightarrow bool \Rightarrow 'a$ set $\Rightarrow 'a$ set ($\langle\partial\partial\rangle$) **where**
 $\partial\partial\ i\ \alpha \equiv image\ (\partial\ i\ \alpha)$

abbreviation *face-fix* :: $nat \Rightarrow 'a$ set **where**
 $face-fix\ i \equiv Fix\ (\partial\ i\ ff)$

abbreviation *fFx* $i\ x \equiv (\partial\ i\ ff\ x = x)$

abbreviation *FFx* $i\ X \equiv (\forall x \in X. fFx\ i\ x)$

end

class *icomp-op* =
 fixes *icomp* :: $'a \Rightarrow nat \Rightarrow 'a \Rightarrow 'a$ set ($\langle\cdot\odot\cdot\rangle[70,70,70]70$)

class *imultisemigroup* = *icomp-op* +
 assumes *iassoc*: $(\bigcup v \in y \odot_i z. x \odot_i v) = (\bigcup v \in x \odot_i y. v \odot_i z)$

begin

sublocale *ims*: *multisemigroup* $\lambda x y. x \odot_i y$
<proof>

abbreviation $DD \equiv ims.\Delta$

abbreviation *iconv* :: 'a set \Rightarrow nat \Rightarrow 'a set \Rightarrow 'a set ($\langle \cdot \star \cdot \rangle [70, 70, 70] 70$) **where**
 $X \star_i Y \equiv ims.conv\ i\ X\ Y$

end

class *icatoid* = *imultisemigroup* + *face-map-op* +
assumes *iDst*: $DD\ i\ x\ y \Longrightarrow \partial\ i\ tt\ x = \partial\ i\ ff\ y$
and *is-absorb* [*simp*]: $(\partial\ i\ ff\ x) \odot_i x = \{x\}$
and *it-absorb* [*simp*]: $x \odot_i (\partial\ i\ tt\ x) = \{x\}$

begin

Every indexed catoid is a catoid.

sublocale *icid*: *catoid* $\lambda x y. x \odot_i y\ \partial\ i\ ff\ \partial\ i\ tt$
<proof>

lemma *lFace-Src*: $\partial\ \partial\ i\ ff = icid.Src\ i$
<proof>

lemma *uFace-Tgt*: $\partial\ \partial\ i\ tt = icid.Tgt\ i$
<proof>

lemma *face-fix-sfix*: *face-fix* = *icid.sfix*
<proof>

lemma *face-fix-tfix*: *face-fix* = *icid.tfix*
<proof>

lemma *face-fix-prop* [*simp*]: $x \in face-fix\ i = (\partial\ i\ \alpha\ x = x)$
<proof>

lemma *fFx-prop*: $fFx\ i\ x = (\partial\ i\ \alpha\ x = x)$
<proof>

end

class *icategory* = *icatoid* +
assumes *locality*: $\partial\ i\ tt\ x = \partial\ i\ ff\ y \Longrightarrow DD\ i\ x\ y$
and *functionality*: $z \in x \odot_i y \Longrightarrow z' \in x \odot_i y \Longrightarrow z = z'$

begin

Every indexed category is a (single-set) category.

sublocale *icat*: *single-set-category* $\lambda x y. x \odot_i y \partial i \text{ff} \partial i \text{tt}$
 ⟨*proof*⟩

abbreviation *ipcomp* :: 'a \Rightarrow nat \Rightarrow 'a \Rightarrow 'a ($\langle - \otimes - \rangle [70, 70, 70] 70$) **where**
 $x \otimes_i y \equiv \text{icat.pcomp } i x y$

lemma *iconv-prop*: $X \star_i Y = \{x \otimes_i y \mid x y. x \in X \wedge y \in Y \wedge DD i x y\}$
 ⟨*proof*⟩

abbreviation *dim-bound* $k x \equiv (\forall i. k \leq i \longrightarrow fFx i x)$

abbreviation *fin-dim* $x \equiv (\exists k. \text{dim-bound } k x)$

end

end

3 Cubical Categories

theory *CubicalCategories*
imports *ICatoids*

begin

All categories considered in this component are single-set categories.

3.1 Semi-cubical ω -categories

We first define a class for cubical ω -categories without symmetries.

class *semi-cubical-omega-category* = *icategory* +
assumes *face-comm*: $i \neq j \Longrightarrow \partial i \alpha \circ \partial j \beta = \partial j \beta \circ \partial i \alpha$
and *face-func*: $i \neq j \Longrightarrow DD j x y \Longrightarrow \partial i \alpha (x \otimes_j y) = \partial i \alpha x \otimes_j \partial i \alpha y$
and *interchange*: $i \neq j \Longrightarrow DD i w x \Longrightarrow DD i y z \Longrightarrow DD j w y \Longrightarrow DD j x z$
 $\Longrightarrow (w \otimes_i x) \otimes_j (y \otimes_i z) = (w \otimes_j y) \otimes_i (x \otimes_j z)$
and *fin-fix*: $\exists k. \forall i. k \leq i \longrightarrow fFx i x$

begin

lemma *pcomp-face-func-DD*: $i \neq j \Longrightarrow DD j x y \Longrightarrow DD j (\partial i \alpha x) (\partial i \alpha y)$
 ⟨*proof*⟩

lemma *comp-face-func*: $i \neq j \Longrightarrow (\partial \partial i \alpha) (x \odot_j y) \subseteq \partial i \alpha x \odot_j \partial i \alpha y$
 ⟨*proof*⟩

lemma *interchange-var*:
assumes $i \neq j$
and $(w \odot_i x) \star_j (y \odot_i z) \neq \{\}$
and $(w \odot_j y) \star_i (x \odot_j z) \neq \{\}$

shows $(w \odot_i x) \star_j (y \odot_i z) = (w \odot_j y) \star_i (x \odot_j z)$
 ⟨proof⟩

lemma *interchange-var2*:

assumes $i \neq j$
and $(\bigcup a \in w \odot_i x. \bigcup b \in y \odot_i z. a \odot_j b) \neq \{\}$
and $(\bigcup c \in w \odot_j y. \bigcup d \in x \odot_j z. c \odot_i d) \neq \{\}$
shows $(\bigcup a \in w \odot_i x. \bigcup b \in y \odot_i z. a \odot_j b) = (\bigcup c \in w \odot_j y. \bigcup d \in x \odot_j z. c \odot_i d)$
 ⟨proof⟩

lemma *face-compat*: $\partial i \alpha \circ \partial i \beta = \partial i \beta$
 ⟨proof⟩

lemma *face-compat-var* [simp]: $\partial i \alpha (\partial i \beta x) = \partial i \beta x$
 ⟨proof⟩

lemma *face-comm-var*: $i \neq j \implies \partial i \alpha (\partial j \beta x) = \partial j \beta (\partial i \alpha x)$
 ⟨proof⟩

lemma *face-comm-lift*: $i \neq j \implies \partial \partial i \alpha (\partial \partial j \beta X) = \partial \partial j \beta (\partial \partial i \alpha X)$
 ⟨proof⟩

lemma *face-func-lift*: $i \neq j \implies (\partial \partial i \alpha) (X \star_j Y) \subseteq \partial \partial i \alpha X \star_j \partial \partial i \alpha Y$
 ⟨proof⟩

lemma *pcomp-lface*: $DD i x y \implies \partial i \text{ff} (x \otimes_i y) = \partial i \text{ff} x$
 ⟨proof⟩

lemma *pcomp-uface*: $DD i x y \implies \partial i \text{tt} (x \otimes_i y) = \partial i \text{tt} y$
 ⟨proof⟩

lemma *interchange-DD1*:

assumes $i \neq j$
and $DD i w x$
and $DD i y z$
and $DD j w y$
and $DD j x z$
shows $DD j (w \otimes_i x) (y \otimes_i z)$
 ⟨proof⟩

lemma *interchange-DD2*:

assumes $i \neq j$
and $DD i w x$
and $DD i y z$
and $DD j w y$
and $DD j x z$
shows $DD i (w \otimes_j y) (x \otimes_j z)$
 ⟨proof⟩

lemma *face-idem1*: $\partial i \alpha x = \partial i \beta y \implies \partial i \alpha x \odot_i \partial i \beta y = \{\partial i \alpha x\}$
 ⟨proof⟩

lemma *face-pidem1*: $\partial i \alpha x = \partial i \beta y \implies \partial i \alpha x \otimes_i \partial i \beta y = \partial i \alpha x$
 ⟨proof⟩

lemma *face-pidem2*: $\partial i \alpha x \neq \partial i \beta y \implies \partial i \alpha x \odot_i \partial i \beta y = \{\}$
 ⟨proof⟩

lemma *face-fix-comp-var*: $i \neq j \implies \partial \partial i \alpha (\partial i \alpha x \odot_j \partial i \alpha y) = \partial i \alpha x \odot_j \partial i \alpha y$
 ⟨proof⟩

lemma *interchange-lift-aux*: $x \in X \implies y \in Y \implies DD i x y \implies x \otimes_i y \in X \star_i Y$
 ⟨proof⟩

lemma *interchange-lift1*:

assumes $i \neq j$

and $\exists w \in W. \exists x \in X. \exists y \in Y. \exists z \in Z. DD i w x \wedge DD i y z \wedge DD j w y \wedge DD j x z$

shows $((W \star_i X) \star_j (Y \star_i Z)) \cap ((W \star_j Y) \star_i (X \star_j Z)) \neq \{\}$
 ⟨proof⟩

lemma *interchange-lift2*:

assumes $i \neq j$

and $\forall w \in W. \forall x \in X. \forall y \in Y. \forall z \in Z. DD i w x \wedge DD i y z \wedge DD j w y \wedge DD j x z$

shows $((W \star_i X) \star_j (Y \star_i Z)) = ((W \star_j Y) \star_i (X \star_j Z))$
 ⟨proof⟩

lemma *double-fix-prop*: $(\partial i \alpha (\partial j \beta x) = x) = (fFx i x \wedge fFx j x)$
 ⟨proof⟩

end

3.2 Type classes for cubical ω -categories

abbreviation *diffSup* :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat} \Rightarrow \text{bool}$ **where**

diffSup $i j k \equiv (i - j \geq k \vee j - i \geq k)$

class *symmetry-ops* =

fixes *symmetry* :: $\text{nat} \Rightarrow 'a \Rightarrow 'a$ ($\langle \sigma \rangle$)

and *inv-symmetry* :: $\text{nat} \Rightarrow 'a \Rightarrow 'a$ ($\langle \vartheta \rangle$)

begin

abbreviation $\sigma \sigma i \equiv \text{image } (\sigma i)$

abbreviation $\vartheta\vartheta i \equiv \text{image } (\vartheta i)$

$\text{symcomp } i j$ composes the symmetry maps from index i to index $i+j-1$.

primrec $\text{symcomp} :: \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a \ (\langle \Sigma \rangle)$ **where**
 $\Sigma i \ 0 \ x = x$
 $|\ \Sigma i \ (\text{Suc } j) \ x = \sigma (i + j) (\Sigma i j x)$

$\text{inv-symcomp } i j$ composes the inverse symmetries from $i+j-1$ to i .

primrec $\text{inv-symcomp} :: \text{nat} \Rightarrow \text{nat} \Rightarrow 'a \Rightarrow 'a \ (\langle \Theta \rangle)$ **where**
 $\Theta i \ 0 \ x = x$
 $|\ \Theta i \ (\text{Suc } j) \ x = \Theta i j (\vartheta (i + j) x)$

end

Next we define a class for cubical ω -categories.

class $\text{cubical-omega-category} = \text{semi-cubical-omega-category} + \text{symmetry-ops} +$
assumes $\text{sym-type}: \sigma \sigma i \ (\text{face-fix } i) \subseteq \text{face-fix } (i + 1)$
and $\text{inv-sym-type}: \vartheta\vartheta i \ (\text{face-fix } (i + 1)) \subseteq \text{face-fix } i$
and $\text{sym-inv-sym}: \text{fF}x \ (i + 1) \ x \Longrightarrow \sigma i \ (\vartheta i x) = x$
and $\text{inv-sym-sym}: \text{fF}x \ i \ x \Longrightarrow \vartheta i \ (\sigma i x) = x$
and $\text{sym-face1}: \text{fF}x \ i \ x \Longrightarrow \partial i \ \alpha \ (\sigma i x) = \sigma i \ (\partial (i + 1) \ \alpha \ x)$
and $\text{sym-face2}: i \neq j \Longrightarrow i \neq j + 1 \Longrightarrow \text{fF}x \ j \ x \Longrightarrow \partial i \ \alpha \ (\sigma j x) = \sigma j \ (\partial i \ \alpha \ x)$
and $\text{sym-func}: i \neq j \Longrightarrow \text{fF}x \ i \ x \Longrightarrow \text{fF}x \ i \ y \Longrightarrow \text{DD } j \ x \ y \Longrightarrow$
 $\sigma i \ (x \otimes_j y) = (\text{if } j = i + 1 \text{ then } \sigma i \ x \otimes_i \sigma i \ y \text{ else } \sigma i \ x \otimes_j \sigma i \ y)$
and $\text{sym-fix}: \text{fF}x \ i \ x \Longrightarrow \text{fF}x \ (i + 1) \ x \Longrightarrow \sigma i \ x = x$
and $\text{sym-sym-braid}: \text{diffSup } i \ j \ 2 \Longrightarrow \text{fF}x \ i \ x \Longrightarrow \text{fF}x \ j \ x \Longrightarrow \sigma i \ (\sigma j x) = \sigma j \ (\sigma i x)$

begin

First we prove variants of the axioms.

lemma $\text{sym-type-var}: \text{fF}x \ i \ x \Longrightarrow \text{fF}x \ (i + 1) \ (\sigma i x)$
 $\langle \text{proof} \rangle$

lemma $\text{sym-type-var1} \ [\text{simp}]: \partial (i + 1) \ \alpha \ (\sigma i \ (\partial i \ \alpha \ x)) = \sigma i \ (\partial i \ \alpha \ x)$
 $\langle \text{proof} \rangle$

lemma $\text{sym-type-var2} \ [\text{simp}]: \partial (i + 1) \ \alpha \circ \sigma i \circ \partial i \ \alpha = \sigma i \circ \partial i \ \alpha$
 $\langle \text{proof} \rangle$

lemma $\text{sym-type-var-lift-var} \ [\text{simp}]: \partial\partial (i + 1) \ \alpha \ (\sigma\sigma i \ (\partial\partial i \ \alpha \ X)) = \sigma\sigma i \ (\partial\partial i \ \alpha \ X)$
 $\langle \text{proof} \rangle$

lemma $\text{sym-type-var-lift} \ [\text{simp}]:$
assumes $\text{fF}x \ i \ X$
shows $\partial\partial (i + 1) \ \alpha \ (\sigma\sigma i \ X) = \sigma\sigma i \ X$

<proof>

lemma *inv-sym-type-var*: $fFx (i + 1) x \implies fFx i (\vartheta i x)$
<proof>

lemma *inv-sym-type-var1* [*simp*]: $\partial i \alpha (\vartheta i (\partial (i + 1) \alpha x)) = \vartheta i (\partial (i + 1) \alpha x)$
<proof>

lemma *inv-sym-type-var2* [*simp*]: $\partial i \alpha \circ \vartheta i \circ \partial (i + 1) \alpha = \vartheta i \circ \partial (i + 1) \alpha$
<proof>

lemma *inv-sym-type-lift-var* [*simp*]: $\partial \partial i \alpha (\vartheta \vartheta i (\partial \partial (i + 1) \alpha X)) = \vartheta \vartheta i (\partial \partial (i + 1) \alpha X)$
<proof>

lemma *inv-sym-type-lift*:
assumes $FFx (i + 1) X$
shows $\partial \partial i \alpha (\vartheta \vartheta i X) = \vartheta \vartheta i X$
<proof>

lemma *sym-inv-sym-var1* [*simp*]: $\sigma i (\vartheta i (\partial (i + 1) \alpha x)) = \partial (i + 1) \alpha x$
<proof>

lemma *sym-inv-sym-var2* [*simp*]: $\sigma i \circ \vartheta i \circ \partial (i + 1) \alpha = \partial (i + 1) \alpha$
<proof>

lemma *sym-inv-sym-lift-var*: $\sigma \sigma i (\vartheta \vartheta i (\partial \partial (i + 1) \alpha X)) = \partial \partial (i + 1) \alpha X$
<proof>

lemma *sym-inv-sym-lift*:
assumes $FFx (i + 1) X$
shows $\sigma \sigma i (\vartheta \vartheta i X) = X$
<proof>

lemma *inv-sym-sym-var1* [*simp*]: $\vartheta i (\sigma i (\partial i \alpha x)) = \partial i \alpha x$
<proof>

lemma *inv-sym-sym-var2* [*simp*]: $\vartheta i \circ \sigma i \circ \partial i \alpha = \partial i \alpha$
<proof>

lemma *inv-sym-sym-lift-var* [*simp*]: $\vartheta \vartheta i (\sigma \sigma i (\partial \partial i \alpha X)) = \partial \partial i \alpha X$
<proof>

lemma *inv-sym-sym-lift*:
assumes $FFx i X$
shows $\vartheta \vartheta i (\sigma \sigma i X) = X$
<proof>

lemma *sym-fix-var1* [*simp*]: $\sigma i (\partial i \alpha (\partial (i + 1) \beta x)) = \partial i \alpha (\partial (i + 1) \beta x)$
 ⟨*proof*⟩

lemma *sym-fix-var2* [*simp*]: $\sigma i \circ \partial i \alpha \circ \partial (i + 1) \beta = \partial i \alpha \circ \partial (i + 1) \beta$
 ⟨*proof*⟩

lemma *sym-fix-lift-var*: $\sigma \sigma i (\partial \partial i \alpha (\partial \partial (i + 1) \beta X)) = \partial \partial i \alpha (\partial \partial (i + 1) \beta X)$
 ⟨*proof*⟩

lemma *sym-fix-lift*:
 assumes *FFx i X*
 and *FFx (i + 1) X*
 shows $\sigma \sigma i X = X$
 ⟨*proof*⟩

lemma *sym-face1-var1*: $\partial i \alpha (\sigma i (\partial i \beta x)) = \sigma i (\partial (i + 1) \alpha (\partial i \beta x))$
 ⟨*proof*⟩

lemma *sym-face1-var2*: $\partial i \alpha \circ \sigma i \circ \partial i \beta = \sigma i \circ \partial (i + 1) \alpha \circ \partial i \beta$
 ⟨*proof*⟩

lemma *sym-face1-lift-var*: $\partial \partial i \alpha (\sigma \sigma i (\partial \partial i \beta X)) = \sigma \sigma i (\partial \partial (i + 1) \alpha (\partial \partial i \beta X))$
 ⟨*proof*⟩

lemma *sym-face1-lift*:
 assumes *FFx i X*
 shows $\partial \partial i \alpha (\sigma \sigma i X) = \sigma \sigma i (\partial \partial (i + 1) \alpha X)$
 ⟨*proof*⟩

lemma *sym-face2-var1*:
 assumes $i \neq j$
 and $i \neq j + 1$
 shows $\partial i \alpha (\sigma j (\partial j \beta x)) = \sigma j (\partial i \alpha (\partial j \beta x))$
 ⟨*proof*⟩

lemma *sym-face2-var2*:
 assumes $i \neq j$
 and $i \neq j + 1$
 shows $\partial i \alpha \circ \sigma j \circ \partial j \beta = \sigma j \circ \partial i \alpha \circ \partial j \beta$
 ⟨*proof*⟩

lemma *sym-face2-lift-var*:
 assumes $i \neq j$
 and $i \neq j + 1$
 shows $\partial \partial i \alpha (\sigma \sigma j (\partial \partial j \beta X)) = \sigma \sigma j (\partial \partial i \alpha (\partial \partial j \beta X))$
 ⟨*proof*⟩

lemma *sym-face2-lift*:

assumes $i \neq j$
and $i \neq j + 1$
and $FFx\ j\ X$
shows $\partial\partial\ i\ \alpha\ (\sigma\sigma\ j\ X) = \sigma\sigma\ j\ (\partial\partial\ i\ \alpha\ X)$
<proof>

lemma *sym-sym-braid-var1*:

assumes $diffSup\ i\ j\ 2$
shows $\sigma\ i\ (\sigma\ j\ (\partial\ i\ \alpha\ (\partial\ j\ \beta\ x))) = \sigma\ j\ (\sigma\ i\ (\partial\ i\ \alpha\ (\partial\ j\ \beta\ x)))$
<proof>

lemma *sym-sym-braid-var2*:

assumes $diffSup\ i\ j\ 2$
shows $\sigma\ i\ \circ\ \sigma\ j\ \circ\ \partial\ i\ \alpha\ \circ\ \partial\ j\ \beta = \sigma\ j\ \circ\ \sigma\ i\ \circ\ \partial\ i\ \alpha\ \circ\ \partial\ j\ \beta$
<proof>

lemma *sym-sym-braid-lift-var*:

assumes $diffSup\ i\ j\ 2$
shows $\sigma\sigma\ i\ (\sigma\sigma\ j\ (\partial\partial\ i\ \alpha\ (\partial\partial\ j\ \beta\ X))) = \sigma\sigma\ j\ (\sigma\sigma\ i\ (\partial\partial\ i\ \alpha\ (\partial\partial\ j\ \beta\ X)))$
<proof>

lemma *sym-sym-braid-lift*:

assumes $diffSup\ i\ j\ 2$
and $FFx\ i\ X$
and $FFx\ j\ X$
shows $\sigma\sigma\ i\ (\sigma\sigma\ j\ X) = \sigma\sigma\ j\ (\sigma\sigma\ i\ X)$
<proof>

lemma *sym-func2*:

assumes $fFx\ i\ x$
and $fFx\ i\ y$
and $DD\ (i + 1)\ x\ y$
shows $\sigma\ i\ (x \otimes_{(i+1)} y) = \sigma\ i\ x \otimes_i \sigma\ i\ y$
<proof>

lemma *sym-func3*:

assumes $i \neq j$
and $j \neq i + 1$
and $fFx\ i\ x$
and $fFx\ i\ y$
and $DD\ j\ x\ y$
shows $\sigma\ i\ (x \otimes_j y) = \sigma\ i\ x \otimes_j \sigma\ i\ y$
<proof>

lemma *sym-func2-var1*:

assumes $DD\ (i + 1)\ (\partial\ i\ \alpha\ x)\ (\partial\ i\ \beta\ y)$
shows $\sigma\ i\ (\partial\ i\ \alpha\ x \otimes_{(i+1)} \partial\ i\ \beta\ y) = \sigma\ i\ (\partial\ i\ \alpha\ x) \otimes_i \sigma\ i\ (\partial\ i\ \beta\ y)$
<proof>

lemma *sym-func3-var1*:

assumes $i \neq j$

and $j \neq i + 1$

and $DD\ j\ (\partial\ i\ \alpha\ x)\ (\partial\ i\ \beta\ y)$

shows $\sigma\ i\ (\partial\ i\ \alpha\ x\ \otimes_j\ \partial\ i\ \beta\ y) = \sigma\ i\ (\partial\ i\ \alpha\ x)\ \otimes_j\ \sigma\ i\ (\partial\ i\ \beta\ y)$

<proof>

lemma *sym-func2-DD*:

assumes $fFx\ i\ x$

and $fFx\ i\ y$

shows $DD\ (i + 1)\ x\ y = DD\ i\ (\sigma\ i\ x)\ (\sigma\ i\ y)$

<proof>

lemma *func2-var2*: $\sigma\ i\ (\partial\ i\ \alpha\ x\ \odot_{(i+1)}\ \partial\ i\ \beta\ y) = \sigma\ i\ (\partial\ i\ \alpha\ x)\ \odot_i\ \sigma\ i\ (\partial\ i\ \beta\ y)$

<proof>

lemma *sym-func2-lift-var1*: $\sigma\ i\ (\partial\partial\ i\ \alpha\ X\ \star_{(i+1)}\ \partial\partial\ i\ \beta\ Y) = \sigma\ i\ (\partial\partial\ i\ \alpha\ X)\ \star_i\ \sigma\ i\ (\partial\partial\ i\ \beta\ Y)$

<proof>

lemma *sym-func2-lift*:

assumes $FFx\ i\ X$

and $FFx\ i\ Y$

shows $\sigma\ i\ (X\ \star_{(i+1)}\ Y) = \sigma\ i\ X\ \star_i\ \sigma\ i\ Y$

<proof>

lemma *func3-var1*:

assumes $i \neq j$

and $j \neq i + 1$

shows $\sigma\ i\ (\partial\ i\ \alpha\ x\ \odot_j\ \partial\ i\ \beta\ y) = \sigma\ i\ (\partial\ i\ \alpha\ x)\ \odot_j\ \sigma\ i\ (\partial\ i\ \beta\ y)$

<proof>

lemma *sym-func3-lift-var1*:

assumes $i \neq j$

and $j \neq i + 1$

shows $\sigma\ i\ (\partial\partial\ i\ \alpha\ X\ \star_j\ \partial\partial\ i\ \beta\ Y) = \sigma\ i\ (\partial\partial\ i\ \alpha\ X)\ \star_j\ \sigma\ i\ (\partial\partial\ i\ \beta\ Y)$

<proof>

lemma *sym-func3-lift*:

assumes $i \neq j$

and $j \neq i + 1$

and $FFx\ i\ X$

and $FFx\ i\ Y$

shows $\sigma\ i\ (X\ \star_j\ Y) = \sigma\ i\ X\ \star_j\ \sigma\ i\ Y$

<proof>

lemma *sym-func3-var2*: $i \neq j \implies \sigma\ i\ (\partial\ i\ \alpha\ x\ \odot_j\ \partial\ i\ \beta\ y) = (\text{if } j = i + 1 \text{ then}$

$\sigma i (\partial i \alpha x) \odot_i \sigma i (\partial i \beta y)$ else $\sigma i (\partial i \alpha x) \odot_j \sigma i (\partial i \beta y)$
 ⟨proof⟩

Symmetries and inverse symmetries form a bijective pair on suitable fix-points of the face maps.

lemma *sym-inj*: *inj-on* (σi) (*face-fix* i)
 ⟨proof⟩

lemma *sym-inj-var*:
assumes $fFx i x$
and $fFx i y$
and $\sigma i x = \sigma i y$
shows $x = y$
 ⟨proof⟩

lemma *inv-sym-inj*: *inj-on* (ϑi) (*face-fix* ($i + 1$))
 ⟨proof⟩

lemma *inv-sym-inj-var*:
assumes $fFx (i + 1) x$
and $fFx (i + 1) y$
and $\vartheta i x = \vartheta i y$
shows $x = y$
 ⟨proof⟩

lemma *surj-sym*: *image* (σi) (*face-fix* i) = *face-fix* ($i + 1$)
 ⟨proof⟩

lemma *surj-inv-sym*: *image* (ϑi) (*face-fix* ($i + 1$)) = *face-fix* i
 ⟨proof⟩

lemma *sym-adj*:
assumes $fFx i x$
and $fFx (i + 1) y$
shows $(\sigma i x = y) = (x = \vartheta i y)$
 ⟨proof⟩

Next we list properties for inverse symmetries corresponding to the axioms.

lemma *inv-sym*:
assumes $fFx i x$
and $fFx (i + 1) x$
shows $\vartheta i x = x$
 ⟨proof⟩

lemma *inv-sym-face2*:
assumes $i \neq j$
and $i \neq j + 1$
and $fFx (j + 1) x$
shows $\partial i \alpha (\vartheta j x) = \vartheta j (\partial i \alpha x)$

$\langle proof \rangle$

lemma *sym-braid*:

assumes $fFx\ i\ x$

and $fFx\ (i + 1)\ x$

shows $\sigma\ i\ (\sigma\ (i + 1)\ (\sigma\ i\ x)) = \sigma\ (i + 1)\ (\sigma\ i\ (\sigma\ (i + 1)\ x))$

$\langle proof \rangle$

lemma *inv-sym-braid*:

assumes $fFx\ (i + 1)\ x$

and $fFx\ (i + 2)\ x$

shows $\vartheta\ i\ (\vartheta\ (i + 1)\ (\vartheta\ i\ x)) = \vartheta\ (i + 1)\ (\vartheta\ i\ (\vartheta\ (i + 1)\ x))$

$\langle proof \rangle$

lemma *sym-inv-sym-braid*:

assumes $diffSup\ i\ j\ 2$

and $fFx\ (j + 1)\ x$

and $fFx\ i\ x$

shows $\sigma\ i\ (\vartheta\ j\ x) = \vartheta\ j\ (\sigma\ i\ x)$

$\langle proof \rangle$

lemma *sym-func1*:

assumes $fFx\ i\ x$

and $fFx\ i\ y$

and $DD\ i\ x\ y$

shows $\sigma\ i\ (x \otimes_i y) = \sigma\ i\ x \otimes_{(i + 1)} \sigma\ i\ y$

$\langle proof \rangle$

lemma *sym-func1-var1*: $\sigma\ \sigma\ i\ (\partial\ i\ \alpha\ x \odot_i \partial\ i\ \beta\ y) = \sigma\ i\ (\partial\ i\ \alpha\ x) \odot_{(i + 1)} \sigma\ i\ (\partial\ i\ \beta\ y)$

$\langle proof \rangle$

lemma *inv-sym-func2-var1*: $\vartheta\ \vartheta\ i\ (\partial\ (i + 1)\ \alpha\ x \odot_i \partial\ (i + 1)\ \beta\ y) = \vartheta\ i\ (\partial\ (i + 1)\ \alpha\ x) \odot_{(i + 1)} \vartheta\ i\ (\partial\ (i + 1)\ \beta\ y)$

$\langle proof \rangle$

lemma *inv-sym-func3-var1*: $\vartheta\ \vartheta\ i\ ((\partial\ (i + 1)\ \alpha\ x) \odot_{(i + 1)} (\partial\ (i + 1)\ \beta\ y)) = \vartheta\ i\ (\partial\ (i + 1)\ \alpha\ x) \odot_i \vartheta\ i\ (\partial\ (i + 1)\ \beta\ y)$

$\langle proof \rangle$

lemma *inv-sym-func-var1*:

assumes $i \neq j$

and $j \neq i + 1$

shows $\vartheta\ \vartheta\ i\ ((\partial\ (i + 1)\ \alpha\ x) \odot_j (\partial\ (i + 1)\ \beta\ y)) = \vartheta\ i\ (\partial\ (i + 1)\ \alpha\ x) \odot_j \vartheta\ i\ (\partial\ (i + 1)\ \beta\ y)$

$\langle proof \rangle$

lemma *inv-sym-func2*:

assumes $fFx\ (i + 1)\ x$

and $fFx (i + 1) y$
and $DD i x y$
shows $\vartheta i (x \otimes_i y) = \vartheta i x \otimes_{(i+1)} \vartheta i y$
 $\langle proof \rangle$

lemma *inv-sym-func3*:
assumes $fFx (i + 1) x$
and $fFx (i + 1) y$
and $DD (i + 1) x y$
shows $\vartheta i (x \otimes_{(i+1)} y) = \vartheta i x \otimes_i \vartheta i y$
 $\langle proof \rangle$

lemma *inv-sym-func*:
assumes $i \neq j$
and $j \neq i + 1$
and $fFx (i + 1) x$
and $fFx (i + 1) y$
and $DD j x y$
shows $\vartheta i (x \otimes_j y) = \vartheta i x \otimes_j \vartheta i y$
 $\langle proof \rangle$

The following properties are related to faces and braids.

lemma *sym-face3*:
assumes $fFx i x$
shows $\partial (i + 1) \alpha (\sigma i x) = \sigma i (\partial i \alpha x)$
 $\langle proof \rangle$

lemma *sym-face3-var1*: $\partial (i + 1) \alpha (\sigma i (\partial i \beta x)) = \sigma i (\partial i \alpha (\partial i \beta x))$
 $\langle proof \rangle$

lemma *sym-face3-simp* [*simp*]:
assumes $fFx i x$
shows $\partial (i + 1) \alpha (\sigma i x) = \sigma i x$
 $\langle proof \rangle$

lemma *sym-face3-simp-var1* [*simp*]: $\partial (i + 1) \alpha (\sigma i (\partial i \beta x)) = \sigma i (\partial i \beta x)$
 $\langle proof \rangle$

lemma *inv-sym-face3*:
assumes $fFx (i + 1) x$
shows $\partial i \alpha (\vartheta i x) = \vartheta i (\partial (i + 1) \alpha x)$
 $\langle proof \rangle$

lemma *inv-sym-face3-var1*: $\partial i \alpha (\vartheta i (\partial (i + 1) \beta x)) = \vartheta i (\partial (i + 1) \alpha (\partial (i + 1) \beta x))$
 $\langle proof \rangle$

lemma *inv-sym-face3-simp*:
assumes $fFx (i + 1) x$

shows $\partial i \alpha (\vartheta i x) = \vartheta i x$
 $\langle \text{proof} \rangle$

lemma *inv-sym-face3-simp-var1* [simp]: $\partial i \alpha (\vartheta i (\partial (i + 1) \beta x)) = \vartheta i (\partial (i + 1) \beta x)$
 $\langle \text{proof} \rangle$

lemma *inv-sym-face1*:
assumes $fFx (i + 1) x$
shows $\partial (i + 1) \alpha (\vartheta i x) = \vartheta i (\partial i \alpha x)$
 $\langle \text{proof} \rangle$

lemma *inv-sym-face1-var1*: $\partial (i + 1) \alpha (\vartheta i (\partial (i + 1) \beta x)) = \vartheta i (\partial i \alpha (\partial (i + 1) \beta x))$
 $\langle \text{proof} \rangle$

lemma *inv-sym-sym-braid*:
assumes $\text{diffSup } i j \ 2$
and $fFx j x$
and $fFx (i + 1) x$
shows $\vartheta i (\sigma j x) = \sigma j (\vartheta i x)$
 $\langle \text{proof} \rangle$

lemma *inv-sym-sym-braid-var1*: $\text{diffSup } i j \ 2 \implies \vartheta i (\sigma j (\partial (i + 1) \alpha (\partial j \beta x))) = \sigma j (\vartheta i (\partial (i + 1) \alpha (\partial j \beta x)))$
 $\langle \text{proof} \rangle$

lemma *inv-sym-inv-sym-braid*:
assumes $\text{diffSup } i j \ 2$
and $fFx (i + 1) x$
and $fFx (j + 1) x$
shows $\vartheta i (\vartheta j x) = \vartheta j (\vartheta i x)$
 $\langle \text{proof} \rangle$

lemma *inv-sym-inv-sym-braid-var1*: $\text{diffSup } i j \ 2 \implies \vartheta i (\vartheta j (\partial (i + 1) \alpha (\partial (j + 1) \beta x))) = \vartheta j (\vartheta i (\partial (i + 1) \alpha (\partial (j + 1) \beta x)))$
 $\langle \text{proof} \rangle$

The following properties are related to symcomp and inv-symcomp.

lemma *symcomp-type-var*:
assumes $fFx i x$
shows $fFx (i + j) (\Sigma i j x) \langle \text{proof} \rangle$

lemma *symcomp-type*: $\text{image } (\Sigma i j) (\text{face-fix } i) \subseteq \text{face-fix } (i + j)$
 $\langle \text{proof} \rangle$

lemma *symcomp-type-var1* [simp]: $\partial (i + j) \alpha (\Sigma i j (\partial i \beta x)) = \Sigma i j (\partial i \beta x)$
 $\langle \text{proof} \rangle$

lemma *inv-symcomp-type-var*:
assumes $fFx (i + j) x$
shows $fFx i (\Theta i j x)$ *<proof>*

lemma *inv-symcomp-type: image* $(\Theta i j) (\text{face-fix } (i + j)) \subseteq \text{face-fix } i$
<proof>

lemma *inv-symcomp-type-var1* [simp]: $\partial i \alpha (\Theta i j (\partial (i + j) \beta x)) = \Theta i j (\partial (i + j) \beta x)$
<proof>

lemma *symcomp-inv-symcomp*:
assumes $fFx (i + j) x$
shows $\Sigma i j (\Theta i j x) = x$ *<proof>*

lemma *inv-symcomp-symcomp*:
assumes $fFx i x$
shows $\Theta i j (\Sigma i j x) = x$ *<proof>*

lemma *symcomp-adj*:
assumes $fFx i x$
and $fFx (i + j) y$
shows $(\Sigma i j x = y) = (x = \Theta i j y)$
<proof>

lemma *decomp-symcomp1*:
assumes $k \leq j$
and $fFx i x$
shows $\Sigma i j x = \Sigma (i + k) (j - k) (\Sigma i k x)$ *<proof>*

lemma *decomp-symcomp2*:
assumes $1 \leq k$
and $k \leq j$
and $fFx i x$
shows $\Sigma i j x = \Sigma (i + k) (j - k) (\sigma (i + k - 1) (\Sigma i (k - 1) x))$
<proof>

lemma *decomp-symcomp3*:
assumes $i \leq l$
and $l + 1 \leq i + j$
and $fFx i x$
shows $\Sigma i j x = \Sigma (l + 1) (i + j - l - 1) (\sigma l (\Sigma i (l - i) x))$
<proof>

lemma *symcomp-face2*:
assumes $l < i \vee i + j < l$
and $fFx i x$
shows $\partial l \alpha (\Sigma i j x) = \Sigma i j (\partial l \alpha x)$ *<proof>*

lemma *symcomp-face3*: $fFx\ i\ x \implies \partial\ (i + j)\ \alpha\ (\Sigma\ i\ j\ x) = \Sigma\ i\ j\ (\partial\ i\ \alpha\ x)$
 ⟨*proof*⟩

lemma *symcomp-face1*:
 assumes $i \leq l$
 and $l < i + j$
 and $fFx\ i\ x$
 shows $\partial\ l\ \alpha\ (\Sigma\ i\ j\ x) = \Sigma\ i\ j\ (\partial\ (l + 1)\ \alpha\ x)$
 ⟨*proof*⟩

lemma *inv-symcomp-face2*:
 assumes $l < i \vee i + j < l$
 and $fFx\ (i + j)\ x$
 shows $\partial\ l\ \alpha\ (\Theta\ i\ j\ x) = \Theta\ i\ j\ (\partial\ l\ \alpha\ x)$ ⟨*proof*⟩

lemma *inv-symcomp-face3*: $fFx\ (i + j)\ x \implies \partial\ i\ \alpha\ (\Theta\ i\ j\ x) = \Theta\ i\ j\ (\partial\ (i + j)\ \alpha\ x)$
 ⟨*proof*⟩

lemma *inv-symcomp-face1*:
 assumes $i < l$
 and $l \leq i + j$
 and $fFx\ (i + j)\ x$
 shows $\partial\ l\ \alpha\ (\Theta\ i\ j\ x) = \Theta\ i\ j\ (\partial\ (l - 1)\ \alpha\ x)$
 ⟨*proof*⟩

lemma *symcomp-comp1*:
 assumes $fFx\ i\ x$
 and $fFx\ i\ y$
 and $DD\ i\ x\ y$
 shows $\Sigma\ i\ j\ (x \otimes_i y) = \Sigma\ i\ j\ x \otimes_{(i + j)} \Sigma\ i\ j\ y$
 ⟨*proof*⟩

lemma *symcomp-comp2*:
 assumes $k < i$
 and $fFx\ i\ x$
 and $fFx\ i\ y$
 and $DD\ k\ x\ y$
 shows $\Sigma\ i\ j\ (x \otimes_k y) = \Sigma\ i\ j\ x \otimes_k \Sigma\ i\ j\ y$
 ⟨*proof*⟩

lemma *symcomp-comp3*:
 assumes $i + j < k$
 and $fFx\ i\ x$
 and $fFx\ i\ y$
 and $DD\ k\ x\ y$
 shows $\Sigma\ i\ j\ (x \otimes_k y) = \Sigma\ i\ j\ x \otimes_k \Sigma\ i\ j\ y$ ⟨*proof*⟩

lemma *fix-comp*:

assumes $i \neq j$
and $fFx\ i\ x$
and $fFx\ i\ y$
and $DD\ j\ x\ y$
shows $fFx\ i\ (x \otimes_j y)$
 $\langle proof \rangle$

lemma *symcomp-comp4*:

assumes $i < k$
and $k \leq i + j$
and $fFx\ i\ x$
and $fFx\ i\ y$
and $DD\ k\ x\ y$
shows $\Sigma\ i\ j\ (x \otimes_k y) = \Sigma\ i\ j\ x \otimes_{(k-1)} \Sigma\ i\ j\ y$
 $\langle proof \rangle$

lemma *symcomp-comp*:

assumes $fFx\ i\ x$
and $fFx\ i\ y$
and $DD\ k\ x\ y$
shows $\Sigma\ i\ j\ (x \otimes_k y) =$ (if $k = i$ then $\Sigma\ i\ j\ x \otimes_{(i+j)} \Sigma\ i\ j\ y$
else (if $(i < k \wedge k \leq i + j)$ then $\Sigma\ i\ j\ x \otimes_{(k-1)} \Sigma\ i\ j\ y$
else $\Sigma\ i\ j\ x \otimes_k \Sigma\ i\ j\ y$)
 $\langle proof \rangle$

lemma *inv-symcomp-comp1*:

assumes $fFx\ (i + j)\ x$
and $fFx\ (i + j)\ y$
and $DD\ (i + j)\ x\ y$
shows $\Theta\ i\ j\ (x \otimes_{(i+j)} y) = \Theta\ i\ j\ x \otimes_i \Theta\ i\ j\ y$
 $\langle proof \rangle$

lemma *inv-symcomp-comp2*:

assumes $k < i$
and $fFx\ (i + j)\ x$
and $fFx\ (i + j)\ y$
and $DD\ k\ x\ y$
shows $\Theta\ i\ j\ (x \otimes_k y) = \Theta\ i\ j\ x \otimes_k \Theta\ i\ j\ y$
 $\langle proof \rangle$

lemma *inv-symcomp-comp3*:

assumes $i + j < k$
and $fFx\ (i + j)\ x$
and $fFx\ (i + j)\ y$
and $DD\ k\ x\ y$
shows $\Theta\ i\ j\ (x \otimes_k y) = \Theta\ i\ j\ x \otimes_k \Theta\ i\ j\ y$
 $\langle proof \rangle$

lemma *inv-symcomp-comp4*:

```

assumes  $i \leq k$ 
and  $k < i + j$ 
and  $fFx (i + j) x$ 
and  $fFx (i + j) y$ 
and  $DD k x y$ 
shows  $\Theta i j (x \otimes_k y) = \Theta i j x \otimes_{(k+1)} \Theta i j y$ 
⟨proof⟩

end

end

```

4 Cubical Categories with Connections

```

theory CubicalCategoriesConnections
imports CubicalCategories

```

```

begin

```

All categories considered in this component are single-set categories.

```

class connection-ops =
fixes connection ::  $nat \Rightarrow bool \Rightarrow 'a \Rightarrow 'a$  ( $\langle \Gamma \rangle$ )

```

```

abbreviation (in connection-ops)  $\Gamma \Gamma i \alpha \equiv image (\Gamma i \alpha)$ 

```

We define a class for cubical ω -categories with connections.

```

class cubical-omega-category-connections = cubical-omega-category + connection-ops
+
assumes conn-face1:  $fFx j x \Longrightarrow \partial j \alpha (\Gamma j \alpha x) = x$ 
and conn-face2:  $fFx j x \Longrightarrow \partial (j + 1) \alpha (\Gamma j \alpha x) = \sigma j x$ 
and conn-face3:  $i \neq j \Longrightarrow i \neq j + 1 \Longrightarrow fFx j x \Longrightarrow \partial i \alpha (\Gamma j \beta x) = \Gamma j \beta$ 
( $\partial i \alpha x$ )
and conn-corner1:  $fFx i x \Longrightarrow fFx i y \Longrightarrow DD (i + 1) x y \Longrightarrow \Gamma i tt (x \otimes_{(i+1)} y) = (\Gamma i tt x \otimes_{(i+1)} \sigma i x) \otimes_i (x \otimes_{(i+1)} \Gamma i tt y)$ 
and conn-corner2:  $fFx i x \Longrightarrow fFx i y \Longrightarrow DD (i + 1) x y \Longrightarrow \Gamma i ff (x \otimes_{(i+1)} y) = (\Gamma i ff x \otimes_{(i+1)} y) \otimes_i (\sigma i y \otimes_{(i+1)} \Gamma i ff y)$ 
and conn-corner3:  $j \neq i \wedge j \neq i + 1 \Longrightarrow fFx i x \Longrightarrow fFx i y \Longrightarrow DD j x y \Longrightarrow \Gamma i \alpha (x \otimes_j y) = \Gamma i \alpha x \otimes_j \Gamma i \alpha y$ 
and conn-fix:  $fFx i x \Longrightarrow fFx (i + 1) x \Longrightarrow \Gamma i \alpha x = x$ 
and conn-zigzag1:  $fFx i x \Longrightarrow \Gamma i tt x \otimes_{(i+1)} \Gamma i ff x = x$ 
and conn-zigzag2:  $fFx i x \Longrightarrow \Gamma i tt x \otimes_i \Gamma i ff x = \sigma i x$ 
and conn-conn-braid:  $diffSup i j 2 \Longrightarrow fFx j x \Longrightarrow fFx i x \Longrightarrow \Gamma i \alpha (\Gamma j \beta x) = \Gamma j \beta (\Gamma i \alpha x)$ 
and conn-shift:  $fFx i x \Longrightarrow fFx (i + 1) x \Longrightarrow \sigma (i + 1) (\sigma i (\Gamma (i + 1) \alpha x)) = \Gamma i \alpha (\sigma (i + 1) x)$ 

```

```

begin

```

lemma *conn-face4*: $fFx\ j\ x \implies \partial\ j\ \alpha\ (\Gamma\ j\ (\neg\alpha)\ x) = \partial\ (j + 1)\ \alpha\ x$
 ⟨*proof*⟩

lemma *conn-face1-lift*: $FFx\ j\ X \implies \partial\partial\ j\ \alpha\ (\Gamma\Gamma\ j\ \alpha\ X) = X$
 ⟨*proof*⟩

lemma *conn-face4-lift*: $FFx\ j\ X \implies \partial\partial\ j\ \alpha\ (\Gamma\Gamma\ j\ (\neg\alpha)\ X) = \partial\partial\ (j + 1)\ \alpha\ X$
 ⟨*proof*⟩

lemma *conn-face2-lift*: $FFx\ j\ X \implies \partial\partial\ (j + 1)\ \alpha\ (\Gamma\Gamma\ j\ \alpha\ X) = \sigma\sigma\ j\ X$
 ⟨*proof*⟩

lemma *conn-face3-lift*: $i \neq j \implies i \neq j + 1 \implies FFx\ j\ X \implies \partial\partial\ i\ \alpha\ (\Gamma\Gamma\ j\ \beta\ X)$
 $= \Gamma\Gamma\ j\ \beta\ (\partial\partial\ i\ \alpha\ X)$
 ⟨*proof*⟩

lemma *conn-fix-lift*: $FFx\ i\ X \implies FFx\ (i + 1)\ X \implies \Gamma\Gamma\ i\ \alpha\ X = X$
 ⟨*proof*⟩

lemma *conn-conn-braid-lift*:
 assumes *diffSup* $i\ j\ 2$
 and $FFx\ j\ X$
 and $FFx\ i\ X$
 shows $\Gamma\Gamma\ i\ \alpha\ (\Gamma\Gamma\ j\ \beta\ X) = \Gamma\Gamma\ j\ \beta\ (\Gamma\Gamma\ i\ \alpha\ X)$
 ⟨*proof*⟩

lemma *conn-sym-braid*:
 assumes *diffSup* $i\ j\ 2$
 and $fFx\ i\ x$
 and $fFx\ j\ x$
 shows $\Gamma\ i\ \alpha\ (\sigma\ j\ x) = \sigma\ j\ (\Gamma\ i\ \alpha\ x)$
 ⟨*proof*⟩

lemma *conn-zigzag1-var* [*simp*]: $\Gamma\ i\ tt\ (\partial\ i\ \alpha\ x) \odot_{(i + 1)} \Gamma\ i\ ff\ (\partial\ i\ \alpha\ x) = \{\partial\ i\ \alpha\ x\}$
 ⟨*proof*⟩

lemma *conn-zigzag1-lift*:
 assumes $FFx\ i\ X$
 shows $\Gamma\Gamma\ i\ tt\ X \star_{(i + 1)} \Gamma\Gamma\ i\ ff\ X = X$
 ⟨*proof*⟩

lemma *conn-zigzag2-var*: $\Gamma\ i\ tt\ (\partial\ i\ \alpha\ x) \odot_i \Gamma\ i\ ff\ (\partial\ i\ \alpha\ x) = \{\sigma\ i\ (\partial\ i\ \alpha\ x)\}$
 ⟨*proof*⟩

lemma *conn-zigzag2-lift*:
 assumes $FFx\ i\ X$
 shows $\Gamma\Gamma\ i\ tt\ X \star_i \Gamma\Gamma\ i\ ff\ X = \sigma\sigma\ i\ X$

<proof>

lemma *conn-sym-braid-lift*: $\text{diffSup } i \ j \ 2 \implies \text{FFx } i \ X \implies \text{FFx } j \ X \implies \Gamma \Gamma \ i \ \alpha$
 $(\sigma \sigma \ j \ X) = \sigma \sigma \ j \ (\Gamma \Gamma \ i \ \alpha \ X)$

<proof>

lemma *conn-corner1-DD*:

assumes $\text{fFx } i \ x$

and $\text{fFx } i \ y$

and $\text{DD } (i + 1) \ x \ y$

shows $\text{DD } i \ (\Gamma \ i \ \text{tt } x \otimes_{(i+1)} \sigma \ i \ x) \ (x \otimes_{(i+1)} \Gamma \ i \ \text{tt } y)$

<proof>

lemma *conn-corner1-var*: $\Gamma \Gamma \ i \ \text{tt} \ (\partial \ i \ \alpha \ x \odot_{(i+1)} \partial \ i \ \beta \ y) = (\Gamma \ i \ \text{tt} \ (\partial \ i \ \alpha \ x)$
 $\odot_{(i+1)} \sigma \ i \ (\partial \ i \ \alpha \ x)) \ \star_i \ (\partial \ i \ \alpha \ x \odot_{(i+1)} \Gamma \ i \ \text{tt} \ (\partial \ i \ \beta \ y))$

<proof>

lemma *conn-corner1-lift-aux*: $\text{fFx } i \ x \implies \partial \ (i + 1) \ \text{ff} \ (\Gamma \ i \ \text{tt } x) = \partial \ (i + 1) \ \text{ff } x$

<proof>

lemma *conn-corner1-lift*:

assumes $\text{FFx } i \ X$

and $\text{FFx } i \ Y$

shows $\Gamma \Gamma \ i \ \text{tt} \ (X \ \star_{(i+1)} \ Y) = (\Gamma \Gamma \ i \ \text{tt} \ X \ \star_{(i+1)} \ \sigma \sigma \ i \ X) \ \star_i \ (X \ \star_{(i+1)} \Gamma \Gamma$
 $i \ \text{tt} \ Y)$

<proof>

lemma *conn-corner2-DD*:

assumes $\text{fFx } i \ x$

and $\text{fFx } i \ y$

and $\text{DD } (i + 1) \ x \ y$

shows $\text{DD } i \ (\Gamma \ i \ \text{ff } x \otimes_{(i+1)} y) \ (\sigma \ i \ y \otimes_{(i+1)} \Gamma \ i \ \text{ff } y)$

<proof>

lemma *conn-corner2-var*: $\Gamma \Gamma \ i \ \text{ff} \ (\partial \ i \ \alpha \ x \odot_{(i+1)} \partial \ i \ \beta \ y) = (\Gamma \ i \ \text{ff} \ (\partial \ i \ \alpha \ x)$
 $\odot_{(i+1)} \partial \ i \ \beta \ y) \ \star_i \ (\sigma \ i \ (\partial \ i \ \beta \ y) \odot_{(i+1)} \Gamma \ i \ \text{ff} \ (\partial \ i \ \beta \ y))$

<proof>

lemma *conn-corner2-lift*:

assumes $\text{FFx } i \ X$

and $\text{FFx } i \ Y$

shows $\Gamma \Gamma \ i \ \text{ff} \ (X \ \star_{(i+1)} \ Y) = (\Gamma \Gamma \ i \ \text{ff} \ X \ \star_{(i+1)} \ Y) \ \star_i \ (\sigma \sigma \ i \ Y \ \star_{(i+1)} \Gamma \Gamma$
 $i \ \text{ff} \ Y)$

<proof>

lemma *conn-corner3-var*:

assumes $j \neq i \wedge j \neq i + 1$

shows $\Gamma \Gamma \ i \ \alpha \ (\partial \ i \ \beta \ x \odot_j \partial \ i \ \gamma \ y) = \Gamma \ i \ \alpha \ (\partial \ i \ \beta \ x) \odot_j \Gamma \ i \ \alpha \ (\partial \ i \ \gamma \ y)$

<proof>

lemma *conn-corner3-lift*:

assumes $j \neq i$

and $j \neq i + 1$

and $fFx\ i\ X$

and $fFx\ i\ Y$

shows $\Gamma\Gamma\ i\ \alpha\ (X \star_j Y) = \Gamma\Gamma\ i\ \alpha\ X \star_j \Gamma\Gamma\ i\ \alpha\ Y$

<proof>

lemma *conn-face5 [simp]*: $\partial\ (j + 1)\ \alpha\ (\Gamma\ j\ (-\alpha)\ (\partial\ j\ \gamma\ x)) = \partial\ (j + 1)\ \alpha\ (\partial\ j\ \gamma\ x)$

<proof>

lemma *conn-inv-sym-braid*:

assumes $diffSup\ i\ j\ 2$

shows $\Gamma\ i\ \alpha\ (\vartheta\ j\ (\partial\ i\ \beta\ (\partial\ (j + 1)\ \gamma\ x))) = \vartheta\ j\ (\Gamma\ i\ \alpha\ (\partial\ i\ \beta\ (\partial\ (j + 1)\ \gamma\ x)))$

<proof>

lemma *conn-corner4*: $\Gamma\Gamma\ i\ tt\ (\partial\ i\ \alpha\ x \odot_{(i+1)} \partial\ i\ \beta\ y) = (\Gamma\ i\ tt\ (\partial\ i\ \alpha\ x) \odot_i \partial\ i\ \beta\ y) \star_{(i+1)} (\sigma\ i\ (\partial\ i\ \alpha\ x) \odot_i \Gamma\ i\ tt\ (\partial\ i\ \beta\ y))$

<proof>

lemma *conn-corner5*: $\Gamma\Gamma\ i\ ff\ (\partial\ i\ \alpha\ x \odot_{(i+1)} \partial\ i\ \beta\ y) = (\Gamma\ i\ ff\ (\partial\ i\ \alpha\ x) \odot_i \sigma\ i\ (\partial\ i\ \beta\ y)) \star_{(i+1)} (\partial\ i\ \beta\ y \odot_i \Gamma\ i\ ff\ (\partial\ i\ \beta\ y))$

<proof>

lemma *conn-corner3-alt*: $j \neq i \implies j \neq i + 1 \implies \Gamma\Gamma\ i\ \alpha\ (\partial\ i\ \beta\ x \odot_j \partial\ i\ \gamma\ y) = \Gamma\ i\ \alpha\ (\partial\ i\ \beta\ x) \odot_j \Gamma\ i\ \alpha\ (\partial\ i\ \gamma\ y)$

<proof>

lemma *conn-shift2*:

assumes $fFx\ i\ x$

and $fFx\ (i + 2)\ x$

shows $\vartheta\ i\ (\vartheta\ (i + 1)\ (\Gamma\ i\ \alpha\ x)) = \Gamma\ (i + 1)\ \alpha\ (\vartheta\ (i + 1)\ x)$

<proof>

end

end

5 Cubical $(\omega, 0)$ -Categories with Connections

theory *CubicalOmegaZeroCategoriesConnections*

imports *CubicalCategoriesConnections*

begin

All categories considered in this component are single-set categories.

First we define shell-invertibility.

abbreviation (in *cubical-omega-category-connections*) $ri\text{-}inv\ i\ x\ y \equiv (DD\ i\ x\ y \wedge DD\ i\ y\ x \wedge x \otimes_i y = \partial\ i\ ff\ x \wedge y \otimes_i x = \partial\ i\ tt\ x)$

abbreviation (in *cubical-omega-category-connections*) $ri\text{-}inv\text{-}shell\ k\ i\ x \equiv (\forall j\ \alpha.\ j + 1 \leq k \wedge j \neq i \longrightarrow (\exists y.\ ri\text{-}inv\ i\ (\partial\ j\ \alpha\ x)\ y))$

Next we define the class of cubical $(\omega, 0)$ -categories with connections.

class *cubical-omega-zero-category-connections* = *cubical-omega-category-connections* +
assumes $ri\text{-}inv: k \geq 1 \implies i \leq k - 1 \implies dim\text{-}bound\ k\ x \implies ri\text{-}inv\text{-}shell\ k\ i\ x \implies \exists y.\ ri\text{-}inv\ i\ x\ y$

begin

Finally, to show our axiomatisation at work we prove Proposition 2.4.7 from our companion paper, namely that every cell in an $(\omega, 0)$ -category is ri-invertible for each natural number i . This requires some background theory engineering.

lemma *ri-inv-fix*:
assumes $fFx\ i\ x$
shows $\exists y.\ ri\text{-}inv\ i\ x\ y$
<proof>

lemma *ri-inv2*:
assumes $k \geq 1$
assumes $dim\text{-}bound\ k\ x$
and $ri\text{-}inv\text{-}shell\ k\ i\ x$
shows $\exists y.\ ri\text{-}inv\ i\ x\ y$
<proof>

lemma *ri-inv3*:
assumes $dim\text{-}bound\ k\ x$
and $ri\text{-}inv\text{-}shell\ k\ i\ x$
shows $\exists y.\ ri\text{-}inv\ i\ x\ y$
<proof>

lemma *ri-unique*: $(\exists y.\ ri\text{-}inv\ i\ x\ y) = (\exists!y.\ ri\text{-}inv\ i\ x\ y)$
<proof>

lemma *ri-unique-var*: $ri\text{-}inv\ i\ x\ y \implies ri\text{-}inv\ i\ x\ z \implies y = z$
<proof>

definition $ri\ i\ x = (THE\ y.\ ri\text{-}inv\ i\ x\ y)$

lemma *ri-inv-ri*: $ri\text{-}inv\ i\ x\ y \implies (y = ri\ i\ x)$

<proof>

lemma *ri-def-prop*:

assumes *dim-bound k x*

and *ri-inv-shell k i x*

shows $DD\ i\ x\ (ri\ i\ x) \wedge DD\ i\ (ri\ i\ x)\ x \wedge x \otimes_i (ri\ i\ x) = \partial\ i\ ff\ x \wedge (ri\ i\ x) \otimes_i x = \partial\ i\ tt\ x$

<proof>

lemma *ri-right*:

assumes *dim-bound k x*

and *ri-inv-shell k i x*

shows $x \otimes_i ri\ i\ x = \partial\ i\ ff\ x$

<proof>

lemma *ri-right-set*:

assumes *dim-bound k x*

and *ri-inv-shell k i x*

shows $x \odot_i ri\ i\ x = \{\partial\ i\ ff\ x\}$

<proof>

lemma *ri-left*:

assumes *dim-bound k x*

and *ri-inv-shell k i x*

shows $ri\ i\ x \otimes_i x = \partial\ i\ tt\ x$

<proof>

lemma *ri-left-set*:

assumes *dim-bound k x*

and *ri-inv-shell k i x*

shows $ri\ i\ x \odot_i x = \{\partial\ i\ tt\ x\}$

<proof>

lemma *dim-face*: $dim-bound\ k\ x \implies dim-bound\ k\ (\partial\ i\ \alpha\ x)$

<proof>

lemma *dim-ri-inv*:

assumes *dim-bound k x*

and *ri-inv i x y*

shows *dim-bound k y*

<proof>

lemma *every-dim-k-ri-inv*:

assumes *dim-bound k x*

shows $\forall i. \exists y. ri-inv\ i\ x\ y$ *<proof>*

We can now show that every cell is ri-invertible in every direction i.

lemma *every-ri-inv*: $\exists y. ri-inv\ i\ x\ y$

<proof>

end

end

References

- [1] P. Malbos, T. Massacrier, and G. Struth. Single-set cubical categories and their formalisation with a proof assistant. 2024. <http://arxiv.org/abs/2401.10553v1>.
- [2] G. Struth. Catoids, categories, groupoids. *Arch. Formal Proofs*, 2023, 2023.