Countable Sums and Discrete (Sub)Distributions

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Abstract

We provide elementary formalizations of countable sums over positive real numbers, and of discrete probabilistic subdistributions and distributions. This is intended as a lightweight alternative to the corresponding concepts from the Isabelle distribution, which are defined using their continuous counterparts (namely Lebesgue integral and general probability distributions) and therefore have significant dependencies.

Contents

1 Infinite Sums of Positive Reals

This is a theory of infinite sums of positive reals defined as limits of finite sums. The goal is to make reasoning about these infinite sums almost as easy as that about finite sums.

theory *Infinite-Sums-of-Positive-Reals*
imports *Complex-Main HOL−Library.Countable-Set*
begin

1.1 Preliminaries

lemma *real-pm-iff*:
$\bigwedge a\ b\ c.\ (a{::}real) + b \le c \longleftrightarrow a \le c - b$
$\bigwedge a\ b\ c.\ (a{::}real) + b \le c \longleftrightarrow b \le c - a$

$\bigwedge a\ b\ c.\ (a{::}real) \leq b - c \longleftrightarrow c \leq b - a$
⟨*proof*⟩

lemma *real-md-iff*:
$\bigwedge a\ b\ c.\ a \geq 0 \Longrightarrow b > 0 \Longrightarrow c \geq 0 \Longrightarrow (a{::}real) * b \leq c \longleftrightarrow a \leq c\ /\ b$
$\bigwedge a\ b\ c.\ a > 0 \Longrightarrow b \geq 0 \Longrightarrow c \geq 0 \Longrightarrow (a{::}real) * b \leq c \longleftrightarrow b \leq c\ /\ a$
$\bigwedge a\ b\ c.\ a > 0 \Longrightarrow b \geq 0 \Longrightarrow c > 0 \Longrightarrow (a{::}real) \leq b\ /\ c \longleftrightarrow c \leq b\ /\ a$
⟨*proof*⟩

lemma *disjoint-finite-aux*:
$\forall i{\in}I.\ \forall j{\in}I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\} \Longrightarrow B \subseteq \bigcup\ (A\ `\ I) \Longrightarrow finite\ B \Longrightarrow finite\ \{i \in I.\ B \cap A\ i \neq \{\}\}$
⟨*proof*⟩

lemma *incl-UNION-aux*: $B \subseteq \bigcup\ (A\ `\ I) \Longrightarrow B = \bigcup\ ((\lambda i.\ (B \cap A\ i))\ `\ \{i \in I.\ B \cap A\ i \neq \{\}\})$
⟨*proof*⟩

lemma *incl-UNION-aux2*: $B \subseteq \bigcup\ (A\ `\ I) \longleftrightarrow B = \bigcup\ ((\lambda i.\ (B \cap A\ i))\ `\ I)$
⟨*proof*⟩

lemma *sum-singl*[*simp*]: $sum\ f\ \{a\} = f\ a$
⟨*proof*⟩

lemma *sum-two*[*simp*]: $a1 \neq a2 \Longrightarrow sum\ f\ \{a1,a2\} = f\ a1 + f\ a2$
⟨*proof*⟩

lemma *sum-three*[*simp*]: $a1 \neq a2 \Longrightarrow a1 \neq a3 \Longrightarrow a2 \neq a3 \Longrightarrow$
$sum\ f\ \{a1,a2,a3\} = f\ a1 + f\ a2 + f\ a3$
⟨*proof*⟩

lemma *Sup-leq*:
$A \neq \{\} \Longrightarrow \forall a{\in}A.\ \exists b{\in}B.\ (a{::}real) \leq b \Longrightarrow bdd\text{-}above\ B \Longrightarrow Sup\ A \leq Sup\ B$
⟨*proof*⟩

lemma *Sup-image-leq*:
$A \neq \{\} \Longrightarrow \forall a{\in}A.\ \exists b{\in}B.\ (f\ a{::}real) \leq g\ b \Longrightarrow bdd\text{-}above\ (g\ `\ B) \Longrightarrow$
$Sup\ (f\ `\ A) \leq Sup\ (g\ `\ B)$
⟨*proof*⟩

lemma *Sup-cong*:
assumes $A \neq \{\} \lor B \neq \{\}$ $\forall a{\in}A.\ \exists b{\in}B.\ (a{::}real) \leq b$ $\forall b{\in}B.\ \exists a{\in}A.\ (b{::}real) \leq a$
$bdd\text{-}above\ A \lor bdd\text{-}above\ B$
shows $Sup\ A = Sup\ B$
⟨*proof*⟩

lemma *Sup-image-cong*:
$A \neq \{\} \lor B \neq \{\} \Longrightarrow \forall a{\in}A.\ \exists b{\in}B.\ (f\ a{::}real) \leq g\ b \Longrightarrow \forall b{\in}B.\ \exists a{\in}A.\ (g\ b{::}real) \leq f\ a \Longrightarrow$
$bdd\text{-}above\ (f\ `\ A) \lor bdd\text{-}above\ (g\ `\ B) \Longrightarrow$
$Sup\ (f\ `\ A) = Sup\ (g\ `\ B)$
⟨*proof*⟩

lemma *Sup-congL*:
$A \neq \{\} \Longrightarrow \forall a{\in}A.\ \exists b{\in}B.\ (a{::}real) \leq b \Longrightarrow \forall b{\in}B.\ b \leq Sup\ A \Longrightarrow Sup\ A = Sup\ B$
⟨*proof*⟩

lemma *Sup-image-congL*:
$A \neq \{\} \Longrightarrow \forall a{\in}A.\ \exists b{\in}B.\ (f\ a{::}real) \leq g\ b \Longrightarrow \forall b{\in}B.\ g\ b \leq Sup\ (f\ `\ A) \Longrightarrow Sup\ (f\ `\ A) = Sup\ (g\ `\ B)$
⟨*proof*⟩

lemma *Sup-congR*:
$B \neq \{\} \Longrightarrow \forall a{\in}A.\ a \leq Sup\ B \Longrightarrow \forall b{\in}B.\ \exists a{\in}A.\ (b{::}real) \leq a \Longrightarrow Sup\ A = Sup\ B$
⟨*proof*⟩

lemma *Sup-image-congR*:
$B \neq \{\} \Longrightarrow \forall a{\in}A.\ f\ a \leq Sup\ (g\ `\ B) \Longrightarrow \forall b{\in}B.\ \exists a{\in}A.\ (g\ b{::}real) \leq f\ a \Longrightarrow Sup\ (f\ `\ A) = Sup\ (g\ `\ B)$
⟨*proof*⟩

lemma *Sup-eq-0-iff*:
assumes $A \neq \{\}$ *bdd-above* A $(\forall a{\in}A.\ (a{::}real) \geq 0)$
shows $Sup\ A = 0 \longleftrightarrow (\forall a{\in}A.\ a = 0)$
⟨*proof*⟩

lemma *plus-Sup-commute*:
assumes *f1*: $\{f1\ b1 \mid b1.\ \varphi 1\ b1\} \neq \{\}$ *bdd-above* $\{f1\ b1 \mid b1.\ \varphi 1\ b1\}$ **and**
f2: $\{f2\ b2 \mid b2.\ \varphi 2\ b2\} \neq \{\}$ *bdd-above* $\{f2\ b2 \mid b2.\ \varphi 2\ b2\}$
shows
$Sup\ \{(f1\ b1{::}real) \mid b1\ .\ \varphi 1\ b1\} + Sup\ \{f2\ b2 \mid b2\ .\ \varphi 2\ b2\} =$
$Sup\ \{f1\ b1 + f2\ b2 \mid b1\ b2.\ \varphi 1\ b1 \land \varphi 2\ b2\}$ (**is** $?L1 + ?L2 = ?R$)
⟨*proof*⟩

lemma *plus-Sup-commute′*:
assumes *f1*: $A1 \neq \{\}$ *bdd-above* $A1$ **and**
f2: $A2 \neq \{\}$ *bdd-above* $A2$
shows $Sup\ A1 + Sup\ A2 = Sup\ \{(a1{::}real) + a2 \mid a1\ a2.\ a1 \in A1 \land a2 \in A2\}$

⟨*proof*⟩

lemma *plus-SupR*: $A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow$ *Sup* $A + (b::real) =$ *Sup* $\{a + b \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *plus-SupL*: $A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow (b::real) +$ *Sup* $A =$ *Sup* $\{b + a \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *mult-Sup-commute*:
assumes *f1*: $\{f1\ b1 \mid b1.\ \varphi 1\ b1\} \neq \{\}$ *bdd-above* $\{f1\ b1 \mid b1.\ \varphi 1\ b1\}$ $\forall b1.\ \varphi 1\ b1 \longrightarrow f1\ b1 \geq 0$ **and**
f2: $\{f2\ b2 \mid b2.\ \varphi 2\ b2\} \neq \{\}$ *bdd-above* $\{f2\ b2 \mid b2.\ \varphi 2\ b2\}$ $\forall b2.\ \varphi 2\ b2 \longrightarrow f2\ b2 \geq 0$
shows
Sup $\{(f1\ b1::real) \mid b1\ .\ \varphi 1\ b1\} *$ *Sup* $\{f2\ b2 \mid b2\ .\ \varphi 2\ b2\} =$
Sup $\{f1\ b1 * f2\ b2 \mid b1\ b2.\ \varphi 1\ b1 \wedge \varphi 2\ b2\}$ (**is** $?L1 * ?L2 = ?R$)
⟨*proof*⟩

lemma *mult-Sup-commute′*:
assumes $A1 \neq \{\}$ *bdd-above* $A1$ $\forall a1 \in A1.\ a1 \geq 0$ **and**
$A2 \neq \{\}$ *bdd-above* $A2$ $\forall a2 \in A2.\ a2 \geq 0$
shows *Sup* $A1 *$ *Sup* $A2 =$ *Sup* $\{(a1::real) * a2 \mid a1\ a2.\ a1 \in A1 \wedge a2 \in A2\}$

⟨*proof*⟩

lemma *mult-SupR*: $A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow \forall a \in A.\ a \geq 0 \Longrightarrow b \geq 0 \Longrightarrow$
Sup $A * (b::real) =$ *Sup* $\{a * b \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *mult-SupL*: $A \neq \{\} \Longrightarrow$ *bdd-above* $A \Longrightarrow \forall a \in A.\ a \geq 0 \Longrightarrow b \geq 0 \Longrightarrow$
$(b::real) *$ *Sup* $A =$ *Sup* $\{b * a \mid a.\ a \in A\}$
⟨*proof*⟩

lemma *sum-mono3*:
finite $B \Longrightarrow A \subseteq B \Longrightarrow (\bigwedge b.\ b \in B - A \Longrightarrow 0 \leq g\ b) \Longrightarrow (\bigwedge a.\ a \in A \Longrightarrow (f\ a::real) \leq g\ a) \Longrightarrow$
sum $f\ A \leq$ *sum* $g\ B$
⟨*proof*⟩

lemma *sum-Sup-commute*:
fixes $h :: 'a \Rightarrow real$
assumes *finite* J **and** $\forall i \in J.\ \{h\ b \mid b.\ \varphi\ i\ b\} \neq \{\} \wedge$ *bdd-above* $\{h\ b \mid b.\ \varphi\ i\ b\}$
shows *sum* $(\lambda i.$ *Sup* $\{h\ b \mid b.\ \varphi\ i\ b\})\ J =$
Sup $\{$*sum* $(\lambda i.\ h\ (b\ i))\ J \mid b\ .\ \forall i \in J.\ \varphi\ i\ (b\ i)\}$

⟨*proof*⟩

1.2 Positivity, boundedness and infinite summation

definition *positive* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow bool$ **where**
$positive\ f\ A \equiv \forall\, a{\in}A.\ f\ a \geq 0$

definition *sbounded* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow bool$ **where**
$sbounded\ f\ A \equiv \exists\, r.\ \forall\, B.\ B \subseteq A \wedge finite\ B \longrightarrow sum\ f\ B \leq r$

definition *isum* :: $('a \Rightarrow real) \Rightarrow 'a\ set \Rightarrow real$ **where**
$isum\ f\ A \equiv Sup\ (sum\ f\ `\ \{B \mid B\ .\ B \subseteq A \wedge finite\ B\})$

lemma *positive-mono*: $positive\ p\ A \Longrightarrow B \subseteq A \Longrightarrow positive\ p\ B$
⟨*proof*⟩

lemma *positive-eq*:
assumes $positive\ f\ A$ **and** $\forall\, a{\in}A.\ f1\ a = f\ a$
shows $positive\ f1\ A$
⟨*proof*⟩

lemma *sbounded-eq*:
assumes $sbounded\ f\ A$ **and** $\forall\, a{\in}A.\ f1\ a = f\ a$
shows $sbounded\ f1\ A$
⟨*proof*⟩

lemma *finite-imp-sbounded*: $positive\ f\ A \Longrightarrow finite\ A \Longrightarrow sbounded\ f\ A$
⟨*proof*⟩

lemma *sbounded-empty*[*simp*,*intro*!]: $sbounded\ f\ \{\}$
⟨*proof*⟩

lemma *sbounded-insert*[*simp*]: $sbounded\ f\ (insert\ a\ A) \longleftrightarrow sbounded\ f\ A$
⟨*proof*⟩

lemma *sbounded-Un*[*simp*]: $sbounded\ f\ (A1 \cup A2) \longleftrightarrow sbounded\ f\ A1 \wedge sbounded\ f\ A2$
⟨*proof*⟩

lemma *sbounded-UNION*:
assumes $finite\ I$ **shows** $sbounded\ f\ (\bigcup i{\in}I.\ A\ i) \longleftrightarrow (\forall\, i{\in}I.\ sbounded\ f\ (A\ i))$
⟨*proof*⟩

lemma *sbounded-mono*: $A \subseteq B \Longrightarrow sbounded\ f\ B \Longrightarrow sbounded\ f\ A$
⟨*proof*⟩

lemma *sbounded-reindex*: *sbounded* (*f o u*) *A* $\Longrightarrow$ *sbounded f* (*u* ‘ *A*)
⟨*proof*⟩

lemma *sbounded-reindex-inj-on*: *inj-on u A* $\Longrightarrow$ *sbounded f* (*u* ‘ *A*) $\longleftrightarrow$ *sbounded* (*f o u*) *A*
⟨*proof*⟩

lemma *sbounded-swap*:
sbounded (λ(*a*,*b*). *f a b*) (*A* × *B*) $\longleftrightarrow$ *sbounded* (λ(*b*,*a*). *f a b*) (*B* × *A*)
⟨*proof*⟩

lemma *sbounded-constant-0*:
assumes ∀ *a*∈*A*. *f a* = (*0*::*real*)
shows *sbounded f A*
⟨*proof*⟩

lemma *sbounded-setminus*:
assumes *sbounded f A* **and** ∀ *b*∈*B*−*A*. *f b* = *0*
shows *sbounded f B*
⟨*proof*⟩

lemma *isum-eq-sum*:
positive f A $\Longrightarrow$ *finite A* $\Longrightarrow$ *isum f A* = *sum f A*
⟨*proof*⟩

lemma *isum-cong*:
assumes *A* = *B* **and** ⋀*x*. *x* ∈ *B* $\Longrightarrow$ *g x* = *h x*
shows *isum g A* = *isum h B*
⟨*proof*⟩

lemma *isum-mono*:
assumes *sbounded h A* **and** ⋀*x*. *x* ∈ *A* $\Longrightarrow$ *g x* ≤ *h x*
shows *isum g A* ≤ *isum h A*
⟨*proof*⟩

lemma *isum-mono′*:
assumes *sbounded g B* **and** *A* ⊆ *B*
shows *isum g A* ≤ *isum g B*
⟨*proof*⟩

lemma *isum-empty*[*simp*]: *isum g* {} = *0*

⟨*proof*⟩

lemma *isum-const-zero*[*simp*]: $isum\ (\lambda x.\ 0)\ A = 0$
⟨*proof*⟩

lemma *isum-const-zero'*: $\forall x{\in}A.\ g\ x = 0 \Longrightarrow isum\ g\ A = 0$
⟨*proof*⟩

lemma *isum-eq-0-iff*: $positive\ f\ A \Longrightarrow sbounded\ f\ A \Longrightarrow isum\ f\ A = 0 \longleftrightarrow (\forall a{\in}A.\ f\ a = 0)$
⟨*proof*⟩

lemma *isum-reindex*: $inj\text{-}on\ h\ A \Longrightarrow isum\ g\ (h\ `\ A) = isum\ (g \circ h)\ A$
⟨*proof*⟩

lemma *isum-reindex-cong*: $inj\text{-}on\ l\ B \Longrightarrow A = l\ `\ B \Longrightarrow (\bigwedge x.\ x \in B \Longrightarrow g\ (l\ x) = h\ x) \Longrightarrow isum\ g\ A = isum\ h\ B$
⟨*proof*⟩

lemma *isum-reindex-cong'*:
$(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \neq y \Longrightarrow h\ x = h\ y \Longrightarrow g\ (h\ x) = 0) \Longrightarrow isum\ g\ (h\ `\ A) = isum\ (g \circ h)\ A$
⟨*proof*⟩

lemma *isum-zeros-cong*:
assumes $sbounded\ g\ (S \cap T) \vee sbounded\ h\ (S \cap T)$
and $(\bigwedge i.\ i \in T - S \Longrightarrow h\ i = 0)$ **and** $(\bigwedge i.\ i \in S - T \Longrightarrow g\ i = 0)$
and $(\bigwedge x.\ x \in S \cap T \Longrightarrow g\ x = h\ x)$
shows $isum\ g\ S = isum\ h\ T$
⟨*proof*⟩

lemma *isum-zeros-congL*:
$sbounded\ g\ S \Longrightarrow S \subseteq T \Longrightarrow \forall i{\in}T - S.\ g\ i = 0 \Longrightarrow isum\ g\ S = isum\ g\ T$
⟨*proof*⟩

lemma *isum-zeros-congR*:
$sbounded\ g\ S \Longrightarrow S \subseteq T \Longrightarrow \forall i{\in}T - S.\ g\ i = 0 \Longrightarrow isum\ g\ T = isum\ g\ S$
⟨*proof*⟩

lemma *isum-singl*[*simp*]: $f\ a \geq (0::real) \Longrightarrow isum\ f\ \{a\} = f\ a$
⟨*proof*⟩

lemma *isum-two*[*simp*]: $a1 \neq a2 \Longrightarrow f\ a1 \geq (0::real) \Longrightarrow f\ a2 \geq 0 \Longrightarrow isum\ f\ \{a1,a2\} = f\ a1 + f\ a2$
⟨*proof*⟩

lemma *isum-three*[*simp*]: $a1 \neq a2 \Longrightarrow a1 \neq a3 \Longrightarrow a2 \neq a3 \Longrightarrow f\ a1 \geq 0 \Longrightarrow f\ a2 \geq (0::real) \Longrightarrow f\ a3 \geq 0 \Longrightarrow$
$isum\ f\ \{a1,a2,a3\} = f\ a1 + f\ a2 + f\ a3$
⟨*proof*⟩

lemma *isum-ge-0*: $positive\ f\ A \Longrightarrow sbounded\ f\ A \Longrightarrow isum\ f\ A \geq 0$
⟨*proof*⟩

lemma *in-le-isum*: $positive\ f\ A \Longrightarrow sbounded\ f\ A \Longrightarrow a \in A \Longrightarrow f\ a \leq isum\ f\ A$
⟨*proof*⟩

lemma *isum-eq-singl*:
assumes *fx*: $f\ a = x$ **and** *f*: $\forall a'.\ a' \neq a \longrightarrow f\ a' = 0$ **and** *x*: $x \geq 0$ **and** *a*: $a \in A$
shows $isum\ f\ A = x$
⟨*proof*⟩

lemma *isum-le-singl*:
assumes *fx*: $f\ a \leq x$ **and** *f*: $\forall a'.\ a' \neq a \longrightarrow f\ a' = 0$ **and** *x*: $f\ a \geq 0$ **and** *a*: $a \in A$
shows $isum\ f\ A \leq x$
⟨*proof*⟩

lemma *isum-insert*[*simp*]: $a \notin A \Longrightarrow sbounded\ f\ A \Longrightarrow f\ a \geq 0 \Longrightarrow isum\ f\ (insert\ a\ A) = isum\ f\ A + f\ a$
⟨*proof*⟩

lemma *isum-UNION*:
assumes *dsj*: $\forall i{\in}I.\ \forall j{\in}I.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$ **and** *sb*: $sbounded\ g\ (\bigcup\ (A\ `\ I))$
shows $isum\ g\ (\bigcup\ (A\ `\ I)) = isum\ (\lambda i.\ isum\ g\ (A\ i))\ I$
⟨*proof*⟩

lemma *isum-Un*[*simp*]:
assumes *positive f A1 sbounded f A1 positive f A2 sbounded f A2* $A1 \cap A2 = \{\}$
shows $isum\ f\ (A1 \cup A2) = isum\ f\ A1 + isum\ f\ A2$
⟨*proof*⟩

lemma *isum-Sigma*:
assumes *sbd*: $sbounded\ (\lambda(a,b).\ f\ a\ b)\ (Sigma\ A\ Bs)$

shows *isum* (λ(*a*,*b*). *f a b*) (*Sigma A Bs*) = *isum* (λ*a*. *isum* (*f a*) (*Bs a*)) *A*
⟨*proof*⟩

lemma *isum-Times*:
assumes *sbounded* (λ(*a*,*b*). *f a b*) (*A* × *B*)
shows *isum* (λ(*a*,*b*). *f a b*) (*A* × *B*) = *isum* (λ*a*. *isum* (*f a*) *B*) *A*
⟨*proof*⟩

lemma *isum-swap*:
assumes *sbounded* (λ(*a*,*b*). *f a b*) (*A* × *B*)
shows *isum* (λ*a*. *isum* (*f a*) *B*) *A* = *isum* (λ*b*. *isum* (λ*a*. *f a b*) *A*) *B* (**is** *?L* = *?R*)
⟨*proof*⟩

lemma *isum-plus*:
assumes *f1*: *positive f1 A sbounded f1 A*
and *f2*: *positive f2 A sbounded f2 A*
shows *isum* (λ*a*. *f1 a* + *f2 a*) *A* = *isum f1 A* + *isum f2 A*
⟨*proof*⟩

lemma *sbounded-product*:
assumes *f*: *positive f A sbounded f A* **and** *g*: *positive g B sbounded g B*
shows *sbounded* (λ(*a*,*b*). *f a* ∗ *g b*) (*A* × *B*)
⟨*proof*⟩

lemma *sbounded-multL*: *x* ≥ *0* ⟹ *sbounded f A* ⟹ *sbounded* (λ*a*. *x* ∗ *f a*) *A*
⟨*proof*⟩

lemma *sbounded-multL-strict*[*simp*]:
assumes *x*: *x* > *0*
shows *sbounded* (λ*a*. *x* ∗ *f a*) *A* ⟷ *sbounded f A*
⟨*proof*⟩

lemma *sbounded-multR*: *x* ≥ *0* ⟹ *sbounded f A* ⟹ *sbounded* (λ*a*. *f a* ∗ *x*) *A*
⟨*proof*⟩

lemma *sbounded-multR-strict*[*simp*]:
assumes *x*: *x* > *0*
shows *sbounded* (λ*a*. *f a* ∗ *x*) *A* ⟷ *sbounded f A*
⟨*proof*⟩

lemma *positive-sbounded-multL*:
assumes *f*: *positive f A sbounded f A* **and** *g*: ∀ *a*∈*A*. *g a* ≤ *x*

shows *sbounded* $(\lambda a.\ f\ a * g\ a)\ A$
⟨*proof*⟩

lemma *positive-sbounded-multR*:
assumes *f*: *positive f A sbounded f A* **and** *g*: $\forall\, a \in A.\ g\ a \leq x$
shows *sbounded* $(\lambda a.\ g\ a * f\ a)\ A$
⟨*proof*⟩

lemma *isum-product-Times*:
assumes *f*: *positive f A sbounded f A* **and** *g*: *positive g B sbounded g B*
shows *isum f A* $*$ *isum g B* = *isum* $(\lambda(a,b).\ f\ a * g\ b)\ (A \times B)$
⟨*proof*⟩

lemma *isum-product*:
assumes *f*: *positive f A sbounded f A* **and** *g*: *positive g B sbounded g B*
shows *isum f A* $*$ *isum g B* = *isum* $(\lambda a.$ *isum* $(\lambda b.\ f\ a * g\ b)\ B)\ A$
⟨*proof*⟩

lemma *isum-distribR*:
assumes *f*: *positive f* $(A{::}'a\ set)$ *sbounded f A* **and** *r*: $r \geq 0$
shows *isum f A* $* r$ = *isum* $(\lambda a.\ f\ a * r)\ A$
⟨*proof*⟩

lemma *isum-distribL*:
assumes *f*: *positive f* $(A{::}'a\ set)$ *sbounded f A* **and** *r*: $r \geq 0$
shows $r *$ *isum f A* = *isum* $(\lambda a.\ r * f\ a)\ A$
⟨*proof*⟩

end

2 Discrete Subdistributions and Distributions

This theory defines countably discrete probability (sub)distributions and their monadic operators, namely:

- Kleisli extension, "ext"
- functorial action, the lifting operator "lift"
- monad unit, the indicator function "ind"
- monad counit, the flattening operators "flat" for subdistributions and "dflat" for subdistributions

Basic facts about them are proved, including the monadic laws.

In all operators except the monad counit (flattening/averaging), the operators for distributions are restrictions of those for subdistributions. For flattening, as explained later we must use two distinct operators "flat" and "dflat".

We also define the expectation operator, "expd", which is the Lebesgue integral for the discrete case.

theory *Discrete-Subdistributions-and-Distributions*
  imports *Infinite-Sums-of-Positive-Reals*
begin

2.1 Definitions and Basic Properties

definition *Subdis* :: $'a\ set \Rightarrow ('a \Rightarrow real)\ set$ **where**
  $Subdis\ A \equiv \{p.\ positive\ p\ A \wedge sbounded\ p\ A \wedge isum\ p\ A \leq 1\}$

definition *Dis* :: $'a\ set \Rightarrow ('a \Rightarrow real)\ set$ **where**
  $Dis\ A \equiv \{p.\ p \in Subdis\ A \wedge isum\ p\ A \geq 1\}$

lemma *Dis-incl-Subdis*: $Dis\ A \subseteq Subdis\ A$ ⟨*proof*⟩

lemma *Subdis-mono*: $p \in Subdis\ A \Longrightarrow B \subseteq A \Longrightarrow p \in Subdis\ B$
  ⟨*proof*⟩

lemma *Subdis-Dis2*: $Subdis\ (Subdis\ A) \subseteq Subdis\ (Dis\ A)$
  ⟨*proof*⟩

lemma *Subdis-ge-0*: $p \in Subdis\ A \Longrightarrow a \in A \Longrightarrow p\ a \geq 0$
  ⟨*proof*⟩

lemma *Subdis-le-1*: $p \in Subdis\ A \Longrightarrow a \in A \Longrightarrow p\ a \leq 1$
  ⟨*proof*⟩

lemma *Subdis-eq*:
  assumes $p \in Subdis\ A$ **and** $\forall\, a{\in}A.\ p1\ a = p\ a$
  shows $p1 \in Subdis\ A$
  ⟨*proof*⟩

lemma *Dis-Subdis-mono*: $p \in Dis\ A \Longrightarrow B \subseteq A \Longrightarrow p \in Subdis\ B$
  ⟨*proof*⟩

lemma *Dis-zeros-mono*: $p \in Dis\ A \Longrightarrow B \subseteq A \Longrightarrow \forall\, a{\in}A{-}B.\ p\ a = 0 \Longrightarrow p \in Dis\ B$
  ⟨*proof*⟩

lemma *Dis-ge-0*: $p \in Dis\ A \Longrightarrow a \in A \Longrightarrow p\ a \geq 0$

⟨*proof*⟩

lemma *Dis-le-1*: $p \in Dis\ A \Longrightarrow a \in A \Longrightarrow p\ a \le 1$
⟨*proof*⟩

lemma *Dis-isum-1*: $p \in Dis\ A \Longrightarrow isum\ p\ A = 1$
⟨*proof*⟩

lemma *Dis-sum-1*: $p \in Dis\ A \Longrightarrow finite\ A \Longrightarrow sum\ p\ A = 1$
⟨*proof*⟩

lemma *Dis-eq*:
assumes $p \in Dis\ A$ **and** $\forall a{\in}A.\ p1\ a = p\ a$
shows $p1 \in Dis\ A$
⟨*proof*⟩

lemma *Subdis-le-1-eq-1*: $p \in Subdis\ A \Longrightarrow 1 \le isum\ p\ A \Longrightarrow isum\ p\ A = 1$
⟨*proof*⟩

lemma *Subdis-sum-le-1*: $p \in Subdis\ A \Longrightarrow finite\ A \Longrightarrow sum\ p\ A \le 1$
⟨*proof*⟩

lemma *Subdis-sum-ge-0*: $p \in Subdis\ A \Longrightarrow finite\ A \Longrightarrow sum\ p\ A \ge 0$
⟨*proof*⟩

lemma *Subdis-sum-ge-0-sub*: $p \in Subdis\ A \Longrightarrow B \subseteq A \Longrightarrow finite\ B \Longrightarrow sum\ p\ B \ge 0$
⟨*proof*⟩

lemma *Subdis-sum-le-1-sub*: $p \in Subdis\ A \Longrightarrow B \subseteq A \Longrightarrow finite\ B \Longrightarrow sum\ p\ B \le 1$
⟨*proof*⟩

lemma *Subdis-sboundedL*:
assumes $p \in Subdis\ A\ \forall a{\in}A.\ g\ a \le x$
shows $sbounded\ (\lambda a.\ p\ a * g\ a)\ A$
⟨*proof*⟩

lemma *Subdis-sboundedR*:
assumes $p \in Subdis\ A\ \forall a{\in}A.\ g\ a \le x$
shows $sbounded\ (\lambda a.\ g\ a * p\ a)\ A$
⟨*proof*⟩

lemma *Subdis-isum-leL*:
assumes *p*: $p \in Subdis\ A$ **and** *g*: $positive\ g\ A\ \forall a{\in}A.\ g\ a \le x$ **and** *x*: $x \ge 0$
shows $isum\ (\lambda a.\ p\ a * g\ a)\ A \le x$
⟨*proof*⟩

lemma *Subdis-isum-leR*:
assumes *p*: $p \in$ *Subdis A* **and** *g*: *positive g A* $\forall a{\in}A.\ g\ a \leq x$ **and** *x*: $x \geq 0$
shows *isum* $(\lambda a.\ g\ a * p\ a)\ A \leq x$
⟨*proof*⟩

lemma *Subdis-sum-le-Max*:
assumes *finite A* $p \in$ *Subdis A positive g A* $A \neq \{\}$
shows $(\sum a{\in}A.\ p\ a * g\ a) \leq Max\ (g\ `\ A)$
⟨*proof*⟩

lemma *Subdis-sum-le*:
assumes *finite A* $p \in$ *Subdis A positive g A* $A \neq \{\}\ \forall a{\in}A.\ g\ a \leq x$
shows $(\sum a{\in}A.\ p\ a * g\ a) \leq x$
⟨*proof*⟩

2.2 Monadic structure

definition *ind* :: $'a \Rightarrow ('a \Rightarrow real)$ **where**
ind a $\equiv \lambda a'.$ *if* $a' = a$ *then 1 else 0*

lemma *ind-simps*[*simp*]: $\bigwedge a.\ ind\ a\ a = 1$
$\bigwedge a\ a'.\ a' \neq a \Longrightarrow ind\ a'\ a = 0$
⟨*proof*⟩

lemma *ind-eq-0-iff*[*simp*]: $ind\ a\ a' = 0 \longleftrightarrow a \neq a'$
⟨*proof*⟩

lemma *ind-eq-1-iff*[*simp*]: $ind\ a\ a' = 1 \longleftrightarrow a = a'$
⟨*proof*⟩

lemma *ind-ge-0*: $ind\ a\ a' \geq 0$
⟨*proof*⟩

lemma *ind-le-1*: $ind\ a\ a' \leq 1$
⟨*proof*⟩

lemma *positive-ind*[*simp*]: *positive* (*ind a*) *A*
⟨*proof*⟩

lemma *sbounded-ind*[*simp*]: *sbounded* (*ind a*) *A*
⟨*proof*⟩

lemma *sum-ind*[*simp*]: $\bigwedge a\ B.\ finite\ B \Longrightarrow a \in B \Longrightarrow sum\ (ind\ a)\ B = 1$
$\bigwedge a\ B.\ finite\ B \Longrightarrow a \notin B \Longrightarrow sum\ (ind\ a)\ B = 0$
⟨*proof*⟩

lemma *isum-ind*[*simp*]: $\bigwedge a\ A.\ a \in A \Longrightarrow isum\ (ind\ a)\ A = 1$
$\bigwedge a\ A.\ a \notin A \Longrightarrow isum\ (ind\ a)\ A = 0$
⟨*proof*⟩

lemma *ind-Subdis*[*simp*, *intro*!]: *ind a* ∈ *Subdis A*
⟨*proof*⟩

lemma *Dis-ind*[*simp*, *intro*!]: *a* ∈ *A* ⟹ *ind a* ∈ *Dis A*
⟨*proof*⟩

lemma *ind-mult-SubdisL*:
assumes *p*: *p* ∈ *Subdis A*
shows (λ*a*. *p a* ∗ *ind* (*f a*) *a*′) ∈ *Subdis A*
⟨*proof*⟩

lemma *ind-mult-SubdisR*:
assumes *p*: *p* ∈ *Subdis A*
shows (λ*a*. *ind* (*f a*) *a*′ ∗ *p a*) ∈ *Subdis A*
⟨*proof*⟩

lemma *isum-ind-multL*: *a*′ ∈ *A* ⟹ *f a*′ ≥ *0* ⟹ *isum* (λ*a*. *f a* ∗ *ind a*′ *a*) *A* = *f a*′
⟨*proof*⟩

lemma *isum-ind-multR*: *a*′ ∈ *A* ⟹ *f a*′ ≥ *0* ⟹ *isum* (λ*a*. *ind a*′ *a* ∗ *f a*) *A* = *f a*′
⟨*proof*⟩

definition *ext* :: ′*a set* ⇒ (′*a* ⇒ (′*b* ⇒ *real*)) ⇒ ((′*a* ⇒ *real*) ⇒ (′*b* ⇒ *real*))
where
ext A f ≡ λ*p b*. *isum* (λ*a*. *p a* ∗ *f a b*) *A*

lemma *ext-ge-0*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *Subdis B* **and** *p*: *p* ∈ *Subdis A* **and** *b*: *b* ∈ *B*
shows *ext A f p b* ≥ *0*
⟨*proof*⟩

lemma *Subdis-sum-isum-le-1*:
assumes *B*: *finite B* **and** *f*: ∀ *a*∈*A*. *f a* ∈ *Subdis B* **and** *p*: *p* ∈ *Subdis A*
shows (∑ *b*∈*B*. *isum* (λ*a*. *p a* ∗ *f a b*) *A*) ≤ *1*
⟨*proof*⟩

lemma *sbounded-prod-Subdis*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *Subdis B* **and** *p*: *p* ∈ *Subdis A*
shows *sbounded* (λ(*a*, *b*). *p b* ∗ *f b a*) (*B* × *A*)
⟨*proof*⟩

lemma *ext-eq*: $\forall a \in A.\ p1\ a = p2\ a \Longrightarrow \forall a \in A.\ \forall b \in B.\ f1\ a\ b = f2\ a\ b \Longrightarrow b \in B \Longrightarrow ext\ A\ f1\ p1\ b = ext\ A\ f2\ p2\ b$
⟨*proof*⟩

lemma *ext-Subdis*:
assumes *f*: $\forall a \in A.\ f\ a \in Subdis\ B$ **and** *p*: $p \in Subdis\ A$
shows $ext\ A\ f\ p \in Subdis\ B$
⟨*proof*⟩

lemma *ext-Dis*:
assumes *f*: $\forall a \in A.\ f\ a \in Dis\ B$ **and** *p*: $p \in Dis\ A$
shows $ext\ A\ f\ p \in Dis\ B$
⟨*proof*⟩

lemma *ext-ind*: $p \in Subdis\ A \Longrightarrow a \in A \Longrightarrow ext\ A\ ind\ p\ a = p\ a$
⟨*proof*⟩

lemma *ext-o*:
assumes *f*: $\forall a \in A.\ f\ a \in B$ **and** *gg*: $\forall b \in B.\ gg\ b \in Subdis\ C$ **and** *p*: $p \in Subdis\ A$ **and** *c*: $c \in C$
shows $ext\ A\ (gg\ o\ f)\ p\ c = ext\ B\ gg\ (ext\ A\ (ind\ o\ f)\ p)\ c$
⟨*proof*⟩

definition *lift* :: $'a\ set \Rightarrow ('a \Rightarrow 'b) \Rightarrow ('a \Rightarrow real) \Rightarrow ('b \Rightarrow real)$ **where**
$lift\ A\ f\ p \equiv \lambda b.\ isum\ (\lambda a.\ p\ a)\ \{a.\ a \in A \wedge f\ a = b\}$

lemma *lift-ext*:
assumes *p*: $p \in Subdis\ A$
shows $lift\ A\ f\ p = ext\ A\ (ind\ o\ f)\ p$
⟨*proof*⟩

lemma *lift-eq*:
assumes *f*: $\forall a \in A.\ f1\ a = f2\ a$ **and** *p*: $\forall a \in A.\ p1\ a = p2\ a$ **and** *b*: $b \in B$
shows $lift\ A\ f1\ p1\ b = lift\ A\ f2\ p2\ b$
⟨*proof*⟩

lemma *lift-Subdis*:
assumes *p*: $p \in Subdis\ A$

shows *lift A f p* ∈ *Subdis B*
⟨*proof*⟩

lemma *lift-Dis*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *B* **and** *p*: *p* ∈ *Dis A*
shows *lift A f p* ∈ *Dis B*
⟨*proof*⟩

lemma *lift-id*[*simp*]:
assumes *p*: *p* ∈ *Subdis A* **and** *a*∈*A*
shows *lift A id p a* = *p a*
⟨*proof*⟩

lemma *lift-o*[*simp*]:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *B* **and** *g*: ∀ *b*∈*B*. *g b* ∈ *C* **and** *p*: *p* ∈ *Subdis A* **and** *c*: *c*∈*C*
shows *lift A* (*g o f*) *p c* = *lift B g* (*lift A f p*) *c*
⟨*proof*⟩

lemma *lift-ind*:
assumes *a*: *a* ∈ *A*
shows *lift A f* (*ind a*) = *ind* (*f a*)
⟨*proof*⟩

lemma *isum-lift*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *B* **and** *p*: *p* ∈ *Subdis A*
shows *isum* (*lift A f p*) *B* = *isum p A*
⟨*proof*⟩

lemma *lift-reflects-Dis*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *B* **and** *p*: *p* ∈ *Subdis A*
shows *lift A f p* ∈ *Dis B* ⟷ *p* ∈ *Dis A*
⟨*proof*⟩

definition *flatP* :: (*'a* ⇒ *real*) *set* ⇒
((*'a* ⇒ *real*) ⇒ *real*) ⇒ (*'a* ⇒ *real*) **where**

flatP Da pp ≡ *λa. isum* (*λp. pp p* ∗ *p a*) *Da*

lemma *flatP-ext*: *flatP Da* = *ext Da id*
⟨*proof*⟩

lemma *flatP-eq*: ∀ *p*∈*Da. pp1 p* = *pp2 p* ⟹ *a* ∈ *A* ⟹ *flatP Da pp1 a* = *flatP Da pp2 a*
⟨*proof*⟩

lemma *flatP-Subdis*: *Da* ⊆ *Subdis A* ⟹
pp ∈ *Subdis Da* ⟹ *flatP Da pp* ∈ *Subdis A*
⟨*proof*⟩

lemma *flatP-Da*: ∀ *pp*∈*Dis Da. ext Da id pp* ∈ *Da* ⟹
pp ∈ *Dis Da* ⟹ *flatP Da pp* ∈ *Da*
⟨*proof*⟩

lemma *flatP-lift-ind*:
assumes *Da*: *Da* ⊆ *Subdis A ind* ' *A* ⊆ *Da*
and *p*: *p* ∈ *Subdis A* **and** *a*: *a* ∈ *A*
shows *flatP Da* (*lift A ind p*) *a* = *p a*
⟨*proof*⟩

lemma *flatP-ind*:
assumes *Da*: *Da* ⊆ *Subdis A*
and *p* ∈ *Da* **and** *a* ∈ *A*
shows *flatP Da* (*ind p*) *a* = *p a*
⟨*proof*⟩

lemma *flatP-lift*:
assumes *Da*: *Da* ⊆ *Subdis A*
and *Db*: *Db* ⊆ *Subdis B*
and *Dab*: ∀ *p*∈*Da. lift A f p* ∈ *Db*
assumes *f*: ∀ *a*∈*A. f a* ∈ *B* **and** *pp*: *pp* ∈ *Subdis Da* **and** *b*: *b* ∈ *B*
shows *flatP Db* (*lift Da* (*lift A f*) *pp*) *b* = *lift A f* (*flatP Da pp*) *b*
⟨*proof*⟩

lemma *flatP-flatP-lift*:

assumes *Da*: $Da \subseteq Subdis\ A$
and *fDa*: $\forall\, pp \in Daa.\ flatP\ Da\ pp \in Da$
and *Daa*: $Daa \subseteq Subdis\ Da$
assumes *ppp*: $ppp \in Subdis\ Daa$ **and** *a*: $a \in A$
shows $flatP\ Da\ (flatP\ Daa\ ppp)\ a = flatP\ Da\ (lift\ Daa\ (flatP\ Da)\ ppp)\ a$
⟨*proof*⟩

definition *flat* :: $'a\ set \Rightarrow (('a \Rightarrow real) \Rightarrow real) \Rightarrow ('a \Rightarrow real)$ **where**
$flat\ A\ pp \equiv \lambda a.\ isum\ (\lambda p.\ pp\ p * p\ a)\ (Subdis\ A)$

lemma *flat-flatP*: $flat\ A = flatP\ (Subdis\ A)$
⟨*proof*⟩

lemma *flat-ext*: $flat\ A = ext\ (Subdis\ A)\ id$
⟨*proof*⟩

lemma *flat-eq*: $\forall\, p{\in}Subdis\ A.\ pp1\ p = pp2\ p \Longrightarrow a \in A \Longrightarrow flat\ A\ pp1\ a = flat\ A\ pp2\ a$
⟨*proof*⟩

lemma *flat-Subdis*: $pp \in Subdis\ (Subdis\ A) \Longrightarrow flat\ A\ pp \in Subdis\ A$
⟨*proof*⟩

lemma *flat-lift-ind*:
assumes *p*: $p \in Subdis\ A$ **and** *a*: $a \in A$
shows $flat\ A\ (lift\ A\ ind\ p)\ a = p\ a$
⟨*proof*⟩

lemma *flat-ind*:
assumes $p \in Subdis\ A$ **and** $a \in A$
shows $flat\ A\ (ind\ p)\ a = p\ a$
⟨*proof*⟩

lemma *flat-lift*:
assumes *f*: $\forall\, a{\in}A.\ f\ a \in B$ **and** *pp*: $pp \in Subdis\ (Subdis\ A)$ **and** *b*: $b \in B$
shows $flat\ B\ (lift\ (Subdis\ A)\ (lift\ A\ f)\ pp)\ b = lift\ A\ f\ (flat\ A\ pp)\ b$
⟨*proof*⟩

lemma *flat-flat-lift*:
assumes *ppp*: *ppp* ∈ *Subdis* (*Subdis* (*Subdis A*)) **and** *a*: *a* ∈ *A*
shows *flat A* (*flat* (*Subdis A*) *ppp*) *a* = *flat A* (*lift* (*Subdis* (*Subdis A*)) (*flat A*) *ppp*) *a*
⟨*proof*⟩

definition *dflat* :: *'a set* ⇒ ((*'a* ⇒ *real*) ⇒ *real*) ⇒ (*'a* ⇒ *real*) **where**
dflat A pp ≡ λ*a*. *isum* (λ*p*. *pp p* ∗ *p a*) (*Dis A*)

lemma *dflat-flatP*: *dflat A* = *flatP* (*Dis A*)
⟨*proof*⟩

lemma *dflat-ext*: *dflat A* = *ext* (*Dis A*) *id*
⟨*proof*⟩

lemma *dflat-eq*: ∀ *p*∈*Dis A*. *pp1 p* = *pp2 p* ⟹ *a* ∈ *A* ⟹ *dflat A pp1 a* = *dflat A pp2 a*
⟨*proof*⟩

lemma *dflat-Subdis*: *pp* ∈ *Subdis* (*Dis A*) ⟹ *dflat A pp* ∈ *Subdis A*
⟨*proof*⟩

lemma *dflat-Dis*: *pp* ∈ *Dis* (*Dis A*) ⟹ *dflat A pp* ∈ *Dis A*
⟨*proof*⟩

lemma *dflat-lift-ind*:
assumes *p*: *p* ∈ *Dis A* **and** *a*: *a* ∈ *A*
shows *dflat A* (*lift A ind p*) *a* = *p a*
⟨*proof*⟩

lemma *dflat-ind*:
assumes *p* ∈ *Dis A* **and** *a* ∈ *A*
shows *dflat A* (*ind p*) *a* = *p a*
⟨*proof*⟩

lemma *dflat-lift-Subdis*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *B* **and** *pp*: *pp* ∈ *Subdis* (*Dis A*) **and** *b*: *b* ∈ *B*
shows *dflat B* (*lift* (*Dis A*) (*lift A f*) *pp*) *b* = *lift A f* (*dflat A pp*) *b*
⟨*proof*⟩

corollary *dflat-lift*:
assumes *f*: ∀ *a*∈*A*. *f a* ∈ *B* **and** *pp*: *pp* ∈ *Dis* (*Dis A*) **and** *b*: *b* ∈ *B*
shows *dflat B* (*lift* (*Dis A*) (*lift A f*) *pp*) *b* = *lift A f* (*dflat A pp*) *b*
⟨*proof*⟩

lemma *dflat-dflat-lift-Subdis*:
assumes *ppp*: *ppp* ∈ *Subdis* (*Dis* (*Dis A*)) **and** *a*: *a* ∈ *A*
shows *dflat A* (*dflat* (*Dis A*) *ppp*) *a* = *dflat A* (*lift* (*Dis* (*Dis A*)) (*dflat A*) *ppp*) *a*
⟨*proof*⟩

corollary *dflat-dflat-lift*:
assumes *ppp*: *ppp* ∈ *Dis* (*Dis* (*Dis A*)) **and** *a*: *a* ∈ *A*
shows *dflat A* (*dflat* (*Dis A*) *ppp*) *a* = *dflat A* (*lift* (*Dis* (*Dis A*)) (*dflat A*) *ppp*) *a*
⟨*proof*⟩

lemma *dflat-from-flat*:
assumes *pp*: *pp* ∈ *Subdis* (*Dis A*) **and** *a*: *a* ∈ *A*
shows *dflat A pp a* = *flat A* (λ*p*. *if p* ∈ *Dis A then pp p else 0*) *a*
⟨*proof*⟩

lemma *dflat-flat*:
assumes *pp*: *pp* ∈ *Subdis* (*Dis A*) **and** *a*: *a* ∈ *A* **and** ∀ *p*∈*Subdis A* − *Dis A*. *pp p* = *0*
shows *dflat A pp a* = *flat A pp a*
⟨*proof*⟩

lemma *dflat-flat′*:
assumes *pp*: *pp* ∈ *Dis* (*Dis A*) **and** *a*: *a* ∈ *A* **and** ∀ *p*∈*Subdis A* − *Dis A*. *pp p* = *0*
shows *dflat A pp a* = *flat A pp a*
⟨*proof*⟩

2.3 Expectation

definition *expd* :: ′*a set* ⇒ (′*a* ⇒ *real*) ⇒ (′*a* ⇒ *real*) ⇒ *real* **where**
expd A p X ≡ *isum* (λ*a*. *p a* ∗ *X a*) *A*

lemma *ext-expd*: *ext A f p b* = *expd A p* (λ*a*. *f a b*)
⟨*proof*⟩

lemma *expd-ge-0'*:
assumes $p \in$ *Subdis A* **and** *positive f A* **and** *sbounded* $(\lambda a.\ p\ a * f\ a)$ *A*
shows *expd A p f* ≥ 0
⟨*proof*⟩

lemma *expd-ge-0*:
assumes *p*: $p \in$ *Subdis A* **and** *f*: *positive f A* $\forall a \in A.\ f\ a \leq x$
shows *expd A p f* ≥ 0
⟨*proof*⟩

lemma *expd-le-upper*:
assumes *p*: $p \in$ *Subdis A* **and** *f*: *positive f A* $\forall a \in A.\ f\ a \leq x$ **and** *x*: $x \geq 0$
shows *expd A p f* $\leq x$
⟨*proof*⟩

lemma *expd-ge-lower-Subdis*:
assumes *p*: $p \in$ *Subdis A* **and** *f*: $\forall a \in A.\ f\ a \geq x$ **and** *x*: $x \geq 0$
and *pf*: *sbounded* $(\lambda a.\ p\ a * f\ a)$ *A*
shows *expd A p f* $\geq x *$ *isum p A*
⟨*proof*⟩

lemma *expd-ge-lower-Dis'*:
assumes *p*: $p \in$ *Dis A* **and** *f*: $\forall a \in A.\ f\ a \geq x$ **and** *x*: $x \geq 0$
and *pf*: *sbounded* $(\lambda a.\ p\ a * f\ a)$ *A*
shows *expd A p f* $\geq x$
⟨*proof*⟩

lemma *expd-ge-lower-Dis*:
assumes *p*: $p \in$ *Dis A* **and** *f*: $\forall a \in A.\ f\ a \geq x$ $\forall a \in A.\ f\ a \leq y$
and *xy*: $x \geq 0$ $y \geq 0$
shows *expd A p f* $\geq x$
⟨*proof*⟩

lemma *expd-ge01*:
assumes *p*: $p \in$ *Subdis A* **and** *f*: $\forall a \in A.\ f\ a \geq 0$ $\forall a \in A.\ f\ a \leq 1$
shows *expd A p f* ≥ 0
⟨*proof*⟩

lemma *expd-le01*:
assumes *p*: $p \in$ *Subdis A* **and** *f*: $\forall a \in A.\ f\ a \geq 0$ $\forall a \in A.\ f\ a \leq 1$
shows *expd A p f* ≤ 1
⟨*proof*⟩

lemma *expd-const-Subdis*[*simp*]:
assumes *p*: $p \in$ *Subdis A* **and** $c \geq 0$
shows *expd A p* $(\lambda\text{-}.\ c) = c *$ *isum p A*

⟨*proof*⟩

lemma *expd-const-le*:
assumes *p*: $p \in Subdis\ A$ **and** $c \geq 0$
shows $expd\ A\ p\ (\lambda\text{-}.\ c) \leq c$
⟨*proof*⟩

lemma *expd-const-Dis*[*simp*]:
assumes *p*: $p \in Dis\ A$ **and** $c \geq 0$
shows $expd\ A\ p\ (\lambda\text{-}.\ c) = c$
⟨*proof*⟩

lemma *expd-eq-ct-iff* [*simp*]:
assumes $p \in Subdis\ A\ \ c > 0$
shows $expd\ A\ p\ (\lambda\text{-}.\ c) = c \longleftrightarrow p \in Dis\ A$
⟨*proof*⟩

lemma *expd-0* [*simp*]: $expd\ A\ p\ (\lambda\text{-}.\ 0) = 0$
⟨*proof*⟩

lemma *expd-1-le-1*: $p \in Subdis\ A \Longrightarrow expd\ A\ p\ (\lambda\text{-}.\ 1) \leq 1$
⟨*proof*⟩

lemma *expd-1-eq-1* [*simp*]: $p \in Dis\ A \Longrightarrow expd\ A\ p\ (\lambda\text{-}.\ 1) = 1$
⟨*proof*⟩

lemma *expd-plus′*:
assumes *p*: $p \in Subdis\ A$
and *f1*: *positive f1 A sbounded* $(\lambda a.\ p\ a * f1\ a)\ A$
and *f2*: *positive f2 A sbounded* $(\lambda a.\ p\ a * f2\ a)\ A$
shows $expd\ A\ p\ (\lambda a.\ f1\ a + f2\ a) = expd\ A\ p\ f1 + expd\ A\ p\ f2$
⟨*proof*⟩

lemma *expd-plus*:
assumes *p*: $p \in Subdis\ A$
and *f1*: *positive f1 A bdd-above* (*f1‘A*)
and *f2*: *positive f2 A bdd-above* (*f2‘A*)
shows $expd\ A\ p\ (\lambda a.\ f1\ a + f2\ a) = expd\ A\ p\ f1 + expd\ A\ p\ f2$
⟨*proof*⟩

lemma *expd-mult′*:
assumes *p*: $p \in Subdis\ A$
and *f*: *positive f A sbounded* $(\lambda a.\ p\ a * f\ a)\ A$ **and** *c*: $c \geq 0$
shows $expd\ A\ p\ (\lambda a.\ c * f\ a) = c * expd\ A\ p\ f$
⟨*proof*⟩

lemma *expd-mult*:
assumes *p*: $p \in Subdis\ A$
and *f*: *positive f A bdd-above* (*f‘A*) **and** *c*: $c \geq 0$

shows *expd A p* $(\lambda a.\ c * f\ a) = c * expd\ A\ p\ f$
⟨*proof*⟩

end