Conway's Circle Theorem in Isabelle/HOL

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Abstract

We formalize Conway's circle theorem in Isabelle/HOL. For a non-degenerate Euclidean triangle ABC with side lengths $a = |BC|$, $b = |CA|$, and $c = |AB|$, extend the two side-lines meeting at each vertex past that vertex by the length of the side opposite that vertex: at A both adjacent sides are extended by a, at B by b, and at C by c. The six resulting endpoints lie on one circle. Its center is the incenter I of the triangle, and its radius is

$$\sqrt{r^2 + s^2},$$

where r is the inradius and $s = (a + b + c)/2$ is the semiperimeter.

We give the result in three equivalent forms: as six explicit distance equations, as a subset relation $\subseteq$ into the corresponding `sphere`, and as a single bounded-universal statement over the six-point set. A separate construction-correctness theorem records that each Conway point lies on the intended side line and at the declared extension length from the adjacent vertex.

AI assistance was used for proof engineering.

Contents

1 Introduction

Conway's construction is a compact Euclidean configuration: a triangle determines three side lengths, those lengths determine six points on the three side lines, and the resulting endpoints unexpectedly form a circle. The theorem is often presented as a visual fact about equal segments; the formal proof has to make explicit which directed lines are used, which extensions are external to the triangle, and how the metric computation is organized.

The development follows the elementary proof. Each constructed point lies on one of the three side lines. The perpendicular from the incenter I to that side has length r, and the signed distance from the foot of the perpendicular to either constructed endpoint is s. The Pythagorean theorem therefore gives the common distance

$$\sqrt{r^2 + s^2}$$

from I to all six endpoints.

2 Definitions

The formal statement works in the Euclidean plane over real coordinates. The vertices A, B, and C are assumed not to be collinear. The side lengths are named in the usual way:

$$\begin{aligned} a &= |BC|, \qquad b = |CA|, \quad c = |AB|, \\ s &= \frac{a+b+c}{2}. \end{aligned}$$

The incenter I is the point whose perpendicular distance to each side line is the common inradius r.

For the construction, each side line contributes two points. On the line AB, the point A_c is beyond A with $|AA_c| = a$, and the point B_c is beyond B with $|BB_c| = b$. Cyclically, the line AC gives A_b and C_b, and the line BC gives B_a and C_a. These prose names are only a guide to the geometric objects; the precise predicates and uniqueness facts are those rendered from the Isabelle theory below.

3 Proof Sketch

Let T_{AB} be the touch point of the incircle with the side AB. The standard tangent-length identities give

$$|AT_{AB}| = s - a, \qquad |BT_{AB}| = s - b.$$

Since A_c lies past A by length a, the directed distance from T_{AB} to A_c has absolute value s; similarly the directed distance from T_{AB} to B_c has absolute value s. The segment IT_{AB} is perpendicular to AB and has length r, so the two right triangles with hypotenuses IA_c and IB_c both have legs r and s. Thus

$$|IA_c| = |IB_c| = \sqrt{r^2 + s^2}.$$

The same argument on the side lines AC and BC proves the identical distance formula for A_b, C_b, B_a, and C_a. Hence all six constructed points lie on the circle with center I and radius $\sqrt{r^2 + s^2}$.

4 Historical Remark

The result is part of the informal triangle-geometry tradition associated with John H. Conway, and public expositions record the construction under his name [3]. We do not attempt a precise priority claim here. Later conic work by Evans fits the theorem into the broader triangle-center literature [2], and García Capitán subsequently showed that the same six-point pattern belongs to a family of conics, with the Conway circle recovered when the chosen point is the incenter [1]. Honsberger's collection gives related Euclidean-geometry background [4].

theory *Conway-Circle*
  imports *HOL−Analysis.Analysis*
begin

definition *triangle-semiperimeter* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real*
  where *triangle-semiperimeter A B C* =
    (*dist B C* + *dist C A* + *dist A B*) / *2*

definition *triangle-incenter* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *triangle-incenter A B C* =
    (*let a* = *dist B C*; *b* = *dist C A*; *c* = *dist A B in*
      inverse (*a* + *b* + *c*) $*_R$ (*a* $*_R$ *A* + *b* $*_R$ *B* + *c* $*_R$ *C*))

definition *conway-point-past-first* :: *'a::real-normed-vector* ⇒ *'a* ⇒ *real* ⇒ *'a*
  where *conway-point-past-first X Y d* = *X* + (*d* / *dist X Y*) $*_R$ (*X* − *Y*)

definition *conway-point-past-second* :: *'a::real-normed-vector* ⇒ *'a* ⇒ *real* ⇒ *'a*
  where *conway-point-past-second X Y d* = *Y* + (*d* / *dist X Y*) $*_R$ (*Y* − *X*)

definition *side-contact-point* :: *'a::real-normed-vector* ⇒ *'a* ⇒ *real* ⇒ *real* ⇒ *'a*
  where *side-contact-point X Y p q* =
    (*let L* = *dist X Y*; *s* = (*p* + *q* + *L*) / *2 in X* + ((*s* − *p*) / *L*) $*_R$ (*Y* − *X*))

definition *conway-A-c* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *conway-A-c A B C* = *conway-point-past-first A B* (*dist B C*)

definition *conway-B-c* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *conway-B-c A B C* = *conway-point-past-second A B* (*dist C A*)

definition *conway-A-b* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *conway-A-b A B C* = *conway-point-past-first A C* (*dist B C*)

definition *conway-C-b* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *conway-C-b A B C* = *conway-point-past-second A C* (*dist A B*)

definition *conway-B-a* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *conway-B-a A B C* = *conway-point-past-first B C* (*dist C A*)

definition *conway-C-a* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *conway-C-a A B C* = *conway-point-past-second B C* (*dist A B*)

definition *contact-AB* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *contact-AB A B C* = *side-contact-point A B* (*dist B C*) (*dist C A*)

definition *contact-AC* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *contact-AC A B C* = *side-contact-point A C* (*dist B C*) (*dist A B*)

definition *contact-BC* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2*
  where *contact-BC A B C* = *side-contact-point B C* (*dist C A*) (*dist A B*)

definition *triangle-inradius* :: *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real* ^ *2* ⇒ *real*
  where *triangle-inradius A B C* =
    dist (*triangle-incenter A B C*) (*contact-AB A B C*)

lemma *noncollinear-triangle-distinct*:
  assumes ¬ *collinear* {*A*, *B*, *C*}
  shows $A \neq B$ $A \neq C$ $B \neq C$
⟨*proof*⟩

lemma *inner-side-vectors*:
  fixes *X Y Z* :: *'a*::*real-inner*
  shows $(Z - X) \cdot (Y - X) =$
    $((dist\ Z\ X)^2 + (dist\ X\ Y)^2 - (dist\ Y\ Z)^2)\ /\ 2$
⟨*proof*⟩

lemma *weighted-incenter-side-contact-orthogonal*:
  fixes *X Y Z* :: *'a*::*real-inner*
  assumes $X \neq Y$
  defines $a \equiv$ *dist Y Z*
  defines $b \equiv$ *dist Z X*
  defines $c \equiv$ *dist X Y*
  defines $S \equiv a + b + c$
  defines $I \equiv$ *inverse* $S *_R (a *_R X + b *_R Y + c *_R Z)$
  defines $T \equiv$ *side-contact-point X Y a b*
  shows *orthogonal* $(I - T)$ $(Y - X)$

$\langle proof \rangle$

lemma *incenter-contact-dist-sq-scalar*:
  fixes $a\ b\ c\ S$ :: *real*
  assumes $c \neq 0$
    and $S \neq 0$
    and $S = a + b + c$
  shows
    $(inverse\ S * b - (b + c - a) / (2 * c))^2 * c^2 +$
      $(inverse\ S * c)^2 * b^2 +$
      $2 * (inverse\ S * b - (b + c - a) / (2 * c)) * (inverse\ S * c) *$
        $((b^2 + c^2 - a^2) / 2) =$
    $((b + c - a) * (a + c - b) * (a + b - c)) / (4 * S)$
$\langle proof \rangle$

lemma *weighted-incenter-side-contact-dist-sq*:
  fixes $X\ Y\ Z$ :: $'a$::*real-inner*
  assumes $X \neq Y$
  defines $a \equiv dist\ Y\ Z$
  defines $b \equiv dist\ Z\ X$
  defines $c \equiv dist\ X\ Y$
  defines $S \equiv a + b + c$
  defines $I \equiv inverse\ S *_R (a *_R X + b *_R Y + c *_R Z)$
  defines $T \equiv side\text{-}contact\text{-}point\ X\ Y\ a\ b$
  shows $(dist\ I\ T)^2 =$
    $((b + c - a) * (a + c - b) * (a + b - c)) / (4 * S)$
$\langle proof \rangle$

lemma *triangle-incenter-contact-AB-orthogonal*:
  fixes $A\ B\ C$ :: *real* ^ *2*
  assumes $A \neq B$
  shows *orthogonal* (*triangle-incenter* $A\ B\ C$ − *contact-AB* $A\ B\ C$) $(B - A)$
  $\langle proof \rangle$

lemma *triangle-incenter-contact-AC-orthogonal*:
  fixes $A\ B\ C$ :: *real* ^ *2*
  assumes $A \neq C$
  shows *orthogonal* (*triangle-incenter* $A\ B\ C$ − *contact-AC* $A\ B\ C$) $(C - A)$
  $\langle proof \rangle$

lemma *triangle-incenter-contact-BC-orthogonal*:
  fixes $A\ B\ C$ :: *real* ^ *2*
  assumes $B \neq C$
  shows *orthogonal* (*triangle-incenter* $A\ B\ C$ − *contact-BC* $A\ B\ C$) $(C - B)$
  $\langle proof \rangle$

lemma *triangle-inradius-nonneg* [*simp*]: $0 \leq$ *triangle-inradius* $A\ B\ C$
  $\langle proof \rangle$

lemma *triangle-incenter-contact-AB-dist*:
  dist (*triangle-incenter* A B C) (*contact-AB* A B C) = *triangle-inradius* A B C
  $\langle proof \rangle$

lemma *triangle-incenter-contact-AC-dist*:
  fixes A B C :: *real* ^ *2*
  assumes *triangle*: ¬ *collinear* $\{A, B, C\}$
  shows *dist* (*triangle-incenter* A B C) (*contact-AC* A B C) = *triangle-inradius* A B C
$\langle proof \rangle$

lemma *triangle-incenter-contact-BC-dist*:
  fixes A B C :: *real* ^ *2*
  assumes *triangle*: ¬ *collinear* $\{A, B, C\}$
  shows *dist* (*triangle-incenter* A B C) (*contact-BC* A B C) = *triangle-inradius* A B C
$\langle proof \rangle$

lemma *triangle-incenter-contacts-dist*:
  fixes A B C :: *real* ^ *2*
  assumes ¬ *collinear* $\{A, B, C\}$
  shows
    dist (*triangle-incenter* A B C) (*contact-AB* A B C) = *triangle-inradius* A B C
    dist (*triangle-incenter* A B C) (*contact-AC* A B C) = *triangle-inradius* A B C
    dist (*triangle-incenter* A B C) (*contact-BC* A B C) = *triangle-inradius* A B C
$\langle proof \rangle$

lemma *triangle-incenter-contacts-orthogonal*:
  fixes A B C :: *real* ^ *2*
  assumes ¬ *collinear* $\{A, B, C\}$
  shows
    orthogonal (*triangle-incenter* A B C − *contact-AB* A B C) (B − A)
    orthogonal (*triangle-incenter* A B C − *contact-AC* A B C) (C − A)
    orthogonal (*triangle-incenter* A B C − *contact-BC* A B C) (C − B)
$\langle proof \rangle$

lemma *dist-conway-point-past-first*:
  fixes X Y :: *'a::real-normed-vector*
  assumes $X \neq Y$ $0 \leq d$
  shows *dist* X (*conway-point-past-first* X Y d) = d
$\langle proof \rangle$

lemma *dist-conway-point-past-second*:
  fixes X Y :: *'a::real-normed-vector*
  assumes $X \neq Y$ $0 \leq d$
  shows *dist* Y (*conway-point-past-second* X Y d) = d
$\langle proof \rangle$

lemma *conway-point-past-first-contact*:

fixes $X\ Y$ :: $'a$::*real-normed-vector*
assumes $X \neq Y$
shows *conway-point-past-first* $X\ Y\ p$ =
side-contact-point $X\ Y\ p\ q - (((p + q + \mathit{dist}\ X\ Y) / 2) / \mathit{dist}\ X\ Y) *_R (Y - X)$
$\langle proof \rangle$

lemma *conway-point-past-second-contact*:
fixes $X\ Y$ :: $'a$::*real-normed-vector*
assumes $X \neq Y$
shows *conway-point-past-second* $X\ Y\ q$ =
side-contact-point $X\ Y\ p\ q + (((p + q + \mathit{dist}\ X\ Y) / 2) / \mathit{dist}\ X\ Y) *_R (Y - X)$
$\langle proof \rangle$

lemma *dist-side-contact-first*:
fixes $X\ Y$ :: $'a$::*real-normed-vector*
assumes $X \neq Y\ 0 \leq p\ 0 \leq q$
shows *dist* (*side-contact-point* $X\ Y\ p\ q$) (*conway-point-past-first* $X\ Y\ p$) =
$(p + q + \mathit{dist}\ X\ Y) / 2$
$\langle proof \rangle$

lemma *dist-side-contact-second*:
fixes $X\ Y$ :: $'a$::*real-normed-vector*
assumes $X \neq Y\ 0 \leq p\ 0 \leq q$
shows *dist* (*side-contact-point* $X\ Y\ p\ q$) (*conway-point-past-second* $X\ Y\ q$) =
$(p + q + \mathit{dist}\ X\ Y) / 2$
$\langle proof \rangle$

lemma *dist-side-contact-first'*:
fixes $X\ Y$ :: $'a$::*real-normed-vector*
assumes $X \neq Y\ 0 \leq p\ 0 \leq q$
shows *dist* (*conway-point-past-first* $X\ Y\ p$) (*side-contact-point* $X\ Y\ p\ q$) =
$(p + q + \mathit{dist}\ X\ Y) / 2$
$\langle proof \rangle$

lemma *dist-side-contact-second'*:
fixes $X\ Y$ :: $'a$::*real-normed-vector*
assumes $X \neq Y\ 0 \leq p\ 0 \leq q$
shows *dist* (*conway-point-past-second* $X\ Y\ q$) (*side-contact-point* $X\ Y\ p\ q$) =
$(p + q + \mathit{dist}\ X\ Y) / 2$
$\langle proof \rangle$

lemma *orthogonal-conway-point-past-first-contact*:
fixes $X\ Y\ v$ :: $'a$::*real-inner*
assumes $X \neq Y$ *orthogonal* $v\ (Y - X)$
shows *orthogonal* v (*conway-point-past-first* $X\ Y\ p$ − *side-contact-point* $X\ Y\ p\ q$)
$\langle proof \rangle$

lemma *orthogonal-conway-point-past-second-contact*:
  fixes X Y v :: $'a$::*real-inner*
  assumes $X \neq Y$ *orthogonal* v $(Y - X)$
  shows *orthogonal* v (*conway-point-past-second* X Y q $-$ *side-contact-point* X Y p q)
$\langle proof \rangle$

lemma *dist-from-orthogonal-foot*:
  fixes I T P :: $'a$::*real-inner*
  assumes *orth*: *orthogonal* $(I - T)$ $(P - T)$
    and *IT*: *dist* I $T = r$
    and *TP*: *dist* T $P = s$
    and *r-nonneg*: $0 \leq r$
    and *s-nonneg*: $0 \leq s$
  shows *dist* I P = *sqrt* $(r^2 + s^2)$
$\langle proof \rangle$

lemma *conway-points-have-declared-lengths*:
  assumes $A \neq B$ $A \neq C$ $B \neq C$
  shows
    dist A (*conway-A-c* A B C) = *dist* B C
    dist B (*conway-B-c* A B C) = *dist* C A
    dist A (*conway-A-b* A B C) = *dist* B C
    dist C (*conway-C-b* A B C) = *dist* A B
    dist B (*conway-B-a* A B C) = *dist* C A
    dist C (*conway-C-a* A B C) = *dist* A B
  $\langle proof \rangle$

lemma *conway-points-about-contacts*:
  assumes $A \neq B$ $A \neq C$ $B \neq C$
  defines $s \equiv$ *triangle-semiperimeter* A B C
  shows
    dist (*contact-AB* A B C) (*conway-A-c* A B C) = s
    dist (*contact-AB* A B C) (*conway-B-c* A B C) = s
    dist (*contact-AC* A B C) (*conway-A-b* A B C) = s
    dist (*contact-AC* A B C) (*conway-C-b* A B C) = s
    dist (*contact-BC* A B C) (*conway-B-a* A B C) = s
    dist (*contact-BC* A B C) (*conway-C-a* A B C) = s
  $\langle proof \rangle$

lemma *conway-points-on-circle-from-contact-feet*:
  fixes A B C I :: *real* ^ *2*
  assumes *sides*: $A \neq B$ $A \neq C$ $B \neq C$
    and *r-nonneg*: $0 \leq r$
    and *foot-AB*: *dist* I (*contact-AB* A B C) = r
      orthogonal $(I -$ *contact-AB* A B $C)$ $(B - A)$
    and *foot-AC*: *dist* I (*contact-AC* A B C) = r
      orthogonal $(I -$ *contact-AC* A B $C)$ $(C - A)$
    and *foot-BC*: *dist* I (*contact-BC* A B C) = r

orthogonal $(I - \textit{contact-BC } A\ B\ C)\ (C - B)$
defines $s \equiv$ *triangle-semiperimeter* $A\ B\ C$
shows
dist I (*conway-A-c* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-B-c* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-A-b* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-C-b* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-B-a* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-C-a* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
⟨*proof*⟩

lemma *conway-points-on-circle-from-noncollinear-contact-feet*:
fixes $A\ B\ C\ I$ :: *real* ^ *2*
assumes *triangle*: ¬ *collinear* $\{A, B, C\}$
and *r-nonneg*: $0 \le r$
and *foot-AB*: *dist* I (*contact-AB* $A\ B\ C$) = r
orthogonal $(I - \textit{contact-AB } A\ B\ C)\ (B - A)$
and *foot-AC*: *dist* I (*contact-AC* $A\ B\ C$) = r
orthogonal $(I - \textit{contact-AC } A\ B\ C)\ (C - A)$
and *foot-BC*: *dist* I (*contact-BC* $A\ B\ C$) = r
orthogonal $(I - \textit{contact-BC } A\ B\ C)\ (C - B)$
defines $s \equiv$ *triangle-semiperimeter* $A\ B\ C$
shows
dist I (*conway-A-c* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-B-c* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-A-b* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-C-b* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-B-a* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
dist I (*conway-C-a* $A\ B\ C$) = *sqrt* $(r^2 + s^2)$
⟨*proof*⟩

theorem *conway-circle-theorem*:
fixes $A\ B\ C$ :: *real* ^ *2*
assumes *triangle*: ¬ *collinear* $\{A, B, C\}$
defines $I \equiv$ *triangle-incenter* $A\ B\ C$
defines $r \equiv$ *triangle-inradius* $A\ B\ C$
defines $s \equiv$ *triangle-semiperimeter* $A\ B\ C$
defines *A-c* ≡ *conway-A-c* $A\ B\ C$
defines *B-c* ≡ *conway-B-c* $A\ B\ C$
defines *A-b* ≡ *conway-A-b* $A\ B\ C$
defines *C-b* ≡ *conway-C-b* $A\ B\ C$
defines *B-a* ≡ *conway-B-a* $A\ B\ C$
defines *C-a* ≡ *conway-C-a* $A\ B\ C$
shows
dist I *A-c* = *sqrt* $(r^2 + s^2)$
dist I *B-c* = *sqrt* $(r^2 + s^2)$
dist I *A-b* = *sqrt* $(r^2 + s^2)$
dist I *C-b* = *sqrt* $(r^2 + s^2)$
dist I *B-a* = *sqrt* $(r^2 + s^2)$

$\quad$ *dist I C-a* $=$ *sqrt* $(r^2 + s^2)$
$\langle proof \rangle$

4.1 Wrapper formulations

The headline theorem above proves six separate distance equations. For downstream use we package the result in two equivalent forms: as a subset of an explicit *sphere* centred at the incenter, and as a single bounded-universal statement over the explicit six-point set. We also expose the construction correctness as a separate theorem: each Conway point lies on the intended side line and is at the declared distance from the adjacent vertex.

lemma *conway-point-past-first-on-line*:
$\quad$ **fixes** $X\ Y$ :: *'a::real-normed-vector*
$\quad$ **shows** *conway-point-past-first* $X\ Y\ d = X + (- (d\ /\ \mathit{dist}\ X\ Y)) *_R (Y - X)$
$\quad$ $\langle proof \rangle$

lemma *conway-point-past-second-on-line*:
$\quad$ **fixes** $X\ Y$ :: *'a::real-normed-vector*
$\quad$ **shows** *conway-point-past-second* $X\ Y\ d$
$\qquad\qquad = X + (1 + d\ /\ \mathit{dist}\ X\ Y) *_R (Y - X)$
$\quad$ $\langle proof \rangle$

theorem *conway-construction-correct*:
$\quad$ **fixes** $A\ B\ C$ :: *real ^ 2*
$\quad$ **assumes** *triangle*: $\neg$ *collinear* $\{A, B, C\}$
$\quad$ **shows**
$\qquad$ — The six Conway points lie on the three side lines of the triangle.
$\qquad \exists\, t{::}real.$ *conway-A-c* $A\ B\ C = A + t *_R (B - A)$
$\qquad \exists\, t{::}real.$ *conway-B-c* $A\ B\ C = A + t *_R (B - A)$
$\qquad \exists\, t{::}real.$ *conway-A-b* $A\ B\ C = A + t *_R (C - A)$
$\qquad \exists\, t{::}real.$ *conway-C-b* $A\ B\ C = A + t *_R (C - A)$
$\qquad \exists\, t{::}real.$ *conway-B-a* $A\ B\ C = B + t *_R (C - B)$
$\qquad \exists\, t{::}real.$ *conway-C-a* $A\ B\ C = B + t *_R (C - B)$
$\qquad$ — Their distances from the adjacent vertex equal the opposite side length.
$\qquad$ *dist* A (*conway-A-c* $A\ B\ C$) $=$ *dist* $B\ C$
$\qquad$ *dist* B (*conway-B-c* $A\ B\ C$) $=$ *dist* $C\ A$
$\qquad$ *dist* A (*conway-A-b* $A\ B\ C$) $=$ *dist* $B\ C$
$\qquad$ *dist* C (*conway-C-b* $A\ B\ C$) $=$ *dist* $A\ B$
$\qquad$ *dist* B (*conway-B-a* $A\ B\ C$) $=$ *dist* $C\ A$
$\qquad$ *dist* C (*conway-C-a* $A\ B\ C$) $=$ *dist* $A\ B$
$\langle proof \rangle$

definition *conway-points* :: *real ^ 2* $\Rightarrow$ *real ^ 2* $\Rightarrow$ *real ^ 2* $\Rightarrow$ (*real ^ 2*) *set*
$\quad$ **where** *conway-points* $A\ B\ C =$
$\qquad$ {*conway-A-c* $A\ B\ C$, *conway-B-c* $A\ B\ C$, *conway-A-b* $A\ B\ C$,
$\qquad$ *conway-C-b* $A\ B\ C$, *conway-B-a* $A\ B\ C$, *conway-C-a* $A\ B\ C$}

corollary *conway-circle-theorem-sphere*:

fixes $A\ B\ C$ :: *real* ^ *2*
assumes *triangle*: $\neg$ *collinear* $\{A, B, C\}$
shows
conway-points $A\ B\ C$
$\subseteq$ *sphere* (*triangle-incenter* $A\ B\ C$)
(*sqrt* ((*triangle-inradius* $A\ B\ C)^2$
\+ (*triangle-semiperimeter* $A\ B\ C)^2$))
⟨*proof*⟩

corollary *conway-circle-theorem-set*:
fixes $A\ B\ C$ :: *real* ^ *2*
assumes *triangle*: $\neg$ *collinear* $\{A, B, C\}$
shows
$\forall P \in$ *conway-points* $A\ B\ C$.
dist (*triangle-incenter* $A\ B\ C$) P
= *sqrt* ((*triangle-inradius* $A\ B\ C)^2$
\+ (*triangle-semiperimeter* $A\ B\ C)^2$)
⟨*proof*⟩

end

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