Completeness of Decreasing Diagrams for the Least Uncountable Cardinality

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Abstract

In [8] it was formally proved that the decreasing diagrams method [7] is sound for proving confluence: if a binary relation r has LD property defined in [8], then it has CR property defined in [6].

In this formal theory it is proved that if the cardinality of r does not exceed the first uncountable cardinal, then r has CR property if and only if r has LD property. As a consequence, the decreasing diagrams method is complete for proving confluence of relations of the least uncountable cardinality.

A paper that describes details of this proof has been submitted to the FSCD 2025 conference. This formalization extends formalizations [1, 5, 4, 2] and the paper [3].

Contents

1 Preliminaries

1.1 Formal definition of finite levels of the DCR hierarchy

theory *Finite-DCR-Hierarchy*
  imports *Main*
begin

1.1.1 Auxiliary definitions

definition *confl-rel*
  where *confl-rel* $r \equiv (\forall a\ b\ c.\ (a,b) \in r\hat{}* \wedge (a,c) \in r\hat{}* \longrightarrow (\exists\ d.\ (b,d) \in r\hat{}* \wedge (c,d) \in r\hat{}*)\ )$

definition *jn00* $::$ $'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn00\ r0\ b\ c \equiv (\exists\ d.\ (b,d) \in r0\hat{}= \wedge (c,d) \in r0\hat{}=)$

definition *jn01* $::$ $'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn01\ r0\ r1\ b\ c \equiv (\exists b'\ d.\ (b,b') \in r1\hat{}= \wedge (b',d) \in r0\hat{}* \wedge (c,d) \in r0\hat{}*)$

definition *jn10* $::$ $'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn10\ r0\ r1\ b\ c \equiv (\exists c'\ d.\ (b,d) \in r0\hat{}* \wedge (c,c') \in r1\hat{}= \wedge (c',d) \in r0\hat{}*)$

definition *jn11* $::$ $'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn11\ r0\ r1\ b\ c \equiv (\exists b'\ b''\ c'\ c''\ d.\ \ (b,b') \in r0\hat{}* \wedge (b',b'') \in r1\hat{}= \wedge (b'',d) \in r0\hat{}*$
  $\wedge (c,c') \in r0\hat{}* \wedge (c',c'') \in r1\hat{}= \wedge (c'',d) \in r0\hat{}*)$

definition *jn02* $::$ $'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn02\ r0\ r1\ r2\ b\ c \equiv (\exists b'\ d.\ \ (b,b') \in r2\hat{}= \wedge (b',d) \in (r0 \cup r1)\hat{}* \wedge (c,d) \in (r0 \cup r1)\hat{}*\ )$

definition *jn12* $::$ $'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn12\ r0\ r1\ r2\ b\ c \equiv (\exists b'\ b''\ d.\ \ (b,b') \in (r0)\hat{}* \wedge (b',b'') \in r2\hat{}= \wedge (b'',d) \in (r0 \cup r1)\hat{}*$
  $\wedge (c,d) \in (r0 \cup r1)\hat{}*)$

definition *jn22* $::$ $'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a\ rel \Rightarrow\ 'a \Rightarrow\ 'a \Rightarrow bool$
where
  $jn22\ r0\ r1\ r2\ b\ c \equiv (\exists b'\ b''\ c'\ c''\ d.\ \ (b,b') \in (r0 \cup r1)\hat{}* \wedge (b',b'') \in r2\hat{}= \wedge (b'',d) \in (r0 \cup r1)\hat{}*$

$\wedge\ (c,c') \in (r0 \cup r1)\widehat{\ }* \wedge (c',c'') \in r2\widehat{\ }= \wedge (c'',d) \in (r0 \cup r1)\widehat{\ }*)$

definition *LD2* :: *'a rel* ⇒ *'a rel* ⇒ *'a rel* ⇒ *bool*
where
LD2 r r0 r1 ≡ (*r* = *r0* ∪ *r1*
∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r0* ⟶ *jn00 r0 b c*)
∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r1* ⟶ *jn01 r0 r1 b c*)
∧ (∀ *a b c*. (*a*,*b*) ∈ *r1* ∧ (*a*,*c*) ∈ *r1* ⟶ *jn11 r0 r1 b c*))

definition *LD3* :: *'a rel* ⇒ *'a rel* ⇒ *'a rel* ⇒ *'a rel* ⇒ *bool*
where
LD3 r r0 r1 r2 ≡ (*r* = *r0* ∪ *r1* ∪ *r2*
∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r0* ⟶ *jn00 r0 b c*)
∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r1* ⟶ *jn01 r0 r1 b c*)
∧ (∀ *a b c*. (*a*,*b*) ∈ *r1* ∧ (*a*,*c*) ∈ *r1* ⟶ *jn11 r0 r1 b c*)
∧ (∀ *a b c*. (*a*,*b*) ∈ *r0* ∧ (*a*,*c*) ∈ *r2* ⟶ *jn02 r0 r1 r2 b c*)
∧ (∀ *a b c*. (*a*,*b*) ∈ *r1* ∧ (*a*,*c*) ∈ *r2* ⟶ *jn12 r0 r1 r2 b c*)
∧ (∀ *a b c*. (*a*,*b*) ∈ *r2* ∧ (*a*,*c*) ∈ *r2* ⟶ *jn22 r0 r1 r2 b c*))

definition *DCR2* :: *'a rel* ⇒ *bool*
where
DCR2 r ≡ (∃*r0 r1*. *LD2 r r0 r1*)

definition *DCR3* :: *'a rel* ⇒ *bool*
where
DCR3 r ≡ (∃*r0 r1 r2*. *LD3 r r0 r1 r2*)

definition $\mathfrak{L}$*1* :: (*nat* ⇒ *'U rel*) ⇒ *nat* ⇒ *'U rel*
where
$\mathfrak{L}1\ g\ \alpha \equiv \bigcup\ \{A.\ \exists\ \alpha'.\ (\alpha' < \alpha) \wedge A = g\ \alpha'\}$

definition $\mathfrak{L}$*v* :: (*nat* ⇒ *'U rel*) ⇒ *nat* ⇒ *nat* ⇒ *'U rel*
where
$\mathfrak{L}v\ g\ \alpha\ \beta \equiv \bigcup\ \{A.\ \exists\ \alpha'.\ (\alpha' < \alpha \vee \alpha' < \beta) \wedge A = g\ \alpha'\}$

definition $\mathfrak{D}$ :: (*nat* ⇒ *'U rel*) ⇒ *nat* ⇒ *nat* ⇒ (*'U* × *'U* × *'U* × *'U*) *set*
where
$\mathfrak{D}\ g\ \alpha\ \beta = \{(b,b',b'',d).\ (b,b') \in (\mathfrak{L}1\ g\ \alpha)\widehat{\ }* \wedge (b',b'') \in (g\ \beta)\widehat{\ }= \wedge (b'',d) \in (\mathfrak{L}v\ g\ \alpha\ \beta)\widehat{\ }*\}$

definition *DCR-generating* :: (*nat* ⇒ *'U rel*) ⇒ *bool*
where
$DCR\text{-}generating\ g \equiv (\forall\ \alpha\ \beta\ a\ b\ c.\ (a,b) \in (g\ \alpha) \wedge (a,c) \in (g\ \beta)$
$\longrightarrow\ (\exists\ b'\ b''\ c'\ c''\ d.\ (b,b',b'',d) \in (\mathfrak{D}\ g\ \alpha\ \beta) \wedge (c,c',c'',d) \in (\mathfrak{D}\ g\ \beta\ \alpha)\ ))$

1.1.2 Result

The next definition formalizes the condition "an ARS with a reduction relation r belongs to the class DCR_n", where n is a natural number.

definition *DCR* :: *nat* ⇒ *'U rel* ⇒ *bool*
where
DCR n r ≡ (∃ *g*::(*nat* ⇒ *'U rel*). *DCR-generating g* ∧ *r* = ⋃ { *r'*. ∃ *α'*. *α'* < *n* ∧ *r'* = *g α'* })

end

1.2 Completeness of the DCR3 method for proving confluence of relations of the least uncountable cardinality

theory *DCR3-Method*
imports
HOL−Cardinals.Cardinals
Abstract−Rewriting.Abstract-Rewriting
Finite-DCR-Hierarchy
begin

1.2.1 Auxiliary definitions

abbreviation *ω-ord* **where** *ω-ord* ≡ *natLeq*

definition *sc-ord*::*'U rel* ⇒ *'U rel* ⇒ *bool*
where *sc-ord α α'* ≡ (*α* <*o* *α'* ∧ (∀ *β*::*'U rel*. *α* <*o* *β* ⟶ *α'* ≤*o* *β*))

definition *lm-ord*::*'U rel* ⇒ *bool*
where *lm-ord α* ≡ *Well-order α* ∧ ¬ (*α* = {} ∨ *isSuccOrd α*)

definition *nord* :: *'U rel* ⇒ *'U rel* **where** *nord α* = (*SOME α'*::*'U rel*. *α'* =*o* *α*)

definition 𝒪::*'U rel set* **where** 𝒪 ≡ *nord* ‘ {*α*. *Well-order α*}

definition *oord*::*'U rel rel* **where** *oord* ≡ (*Restr ordLeq* 𝒪)

definition *CCR* :: *'U rel* ⇒ *bool*
where
CCR r = (∀ *a* ∈ *Field r*. ∀ *b* ∈ *Field r*. ∃ *c* ∈ *Field r*. (*a*,*c*) ∈ *r*^* ∧ (*b*,*c*) ∈ *r*^*)

definition *Conelike* :: *'U rel* ⇒ *bool*
where
Conelike r = (*r* = {} ∨ (∃ *m* ∈ *Field r*. ∀ *a* ∈ *Field r*. (*a*,*m*) ∈ *r*^*))

definition *dncl* :: *'U rel* ⇒ *'U set* ⇒ *'U set*
where
dncl r A = ((*r*^*)^−1)‘‘*A*

definition $Inv :: 'U\ rel \Rightarrow 'U\ set\ set$
where
$Inv\ r = \{\ A :: 'U\ set\ .\ r\ ``\ A \subseteq A\ \}$

definition $SF :: 'U\ rel \Rightarrow 'U\ set\ set$
where
$SF\ r = \{\ A :: 'U\ set.\ Field\ (Restr\ r\ A) = A\ \}$

definition $SCF::'U\ rel \Rightarrow ('U\ set)\ set$ **where**
$SCF\ r \equiv \{\ B::('U\ set)\ .\ B \subseteq Field\ r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in B.\ (a,b) \in r\hat{}*)\ \}$

definition $cfseq :: 'U\ rel \Rightarrow (nat \Rightarrow 'U) \Rightarrow bool$
where
$cfseq\ r\ xi \equiv ((\forall\ a \in Field\ r.\ \exists\ i.\ (a,\ xi\ i) \in r\hat{}*) \wedge (\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r))$

definition $rpth :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow nat \Rightarrow (nat \Rightarrow 'U)\ set$
where
$rpth\ r\ a\ b\ n \equiv \{\ f::(nat \Rightarrow 'U).\ f\ 0 = a \wedge f\ n = b \wedge (\forall\ i<n.\ (f\ i,\ f(Suc\ i)) \in r)$
$\}$

definition $\mathcal{F} :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow 'U\ set\ set$
where
$\mathcal{F}\ r\ a\ b \equiv \{\ F::'U\ set.\ \exists\ n::nat.\ \exists\ f \in rpth\ r\ a\ b\ n.\ F = f`\{i.\ i \le n\}\ \}$

definition $\mathfrak{f} :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow 'U\ set$
where
$\mathfrak{f}\ r\ a\ b \equiv (if\ (\mathcal{F}\ r\ a\ b \neq \{\})\ then\ (SOME\ F.\ F \in \mathcal{F}\ r\ a\ b)\ else\ \{\})$

definition $dnEsc :: 'U\ rel \Rightarrow 'U\ set \Rightarrow 'U \Rightarrow 'U\ set\ set$
where
$dnEsc\ r\ A\ a \equiv \{\ F.\ \exists\ b.\ ((b \notin dncl\ r\ A) \wedge (F \in \mathcal{F}\ r\ a\ b) \wedge (F \cap A = \{\}))\ \}$

definition $dnesc :: 'U\ rel \Rightarrow 'U\ set \Rightarrow 'U \Rightarrow 'U\ set$
where
$dnesc\ r\ A\ a = (if\ (dnEsc\ r\ A\ a \neq \{\})\ then\ (SOME\ F.\ F \in dnEsc\ r\ A\ a)\ else\ \{$
$a\ \})$

definition $escl :: 'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow 'U\ set$
where
$escl\ r\ A\ B = \bigcup\ ((dnesc\ r\ A)\ `\ B)$

definition $clterm$ **where** $clterm\ s'\ r \equiv (Conelike\ s' \longrightarrow Conelike\ r)$

definition $spthlen::'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow nat$
where
$spthlen\ r\ a\ b \equiv (LEAST\ n::nat.\ (a,b) \in r\hat{}\hat{}n)$

definition $spth :: 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow (nat \Rightarrow 'U)\ set$
where

$spth\ r\ a\ b = rpth\ r\ a\ b\ (spthlen\ r\ a\ b)$

definition $\mathfrak{U}$::$'U\ rel \Rightarrow ('U\ rel)\ set$ **where**
$\mathfrak{U}\ r \equiv \{\ s::('U\ rel)\ .\ CCR\ s \wedge s \subseteq r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in Field\ s.\ (a,b) \in r\hat{}*)\ \}$

definition $RCC\text{-}rel$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow bool$ **where**
$RCC\text{-}rel\ r\ \alpha \equiv (\mathfrak{U}\ r = \{\} \wedge \alpha = \{\}) \vee (\exists\ s \in \mathfrak{U}\ r.\ |s| =o\ \alpha \wedge (\ \forall\ s' \in \mathfrak{U}\ r.\ |s| \leq o\ |s'|\ ))$

definition RCC :: $'U\ rel \Rightarrow 'U\ rel\ (\|\text{-}\|)$
where $\|r\| \equiv (SOME\ \alpha.\ RCC\text{-}rel\ r\ \alpha)$

definition Den::$'U\ rel \Rightarrow ('U\ set)\ set$ **where**
$Den\ r \equiv \{\ B::('U\ set)\ .\ B \subseteq Field\ r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in B.\ (a,b) \in r\hat{}=)\ \}$

definition $Span$::$'U\ rel \Rightarrow ('U\ rel)\ set$ **where**
$Span\ r \equiv \{\ s.\ s \subseteq r \wedge Field\ s = Field\ r\ \}$

definition $scf\text{-}rel$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow bool$ **where**
$scf\text{-}rel\ r\ \alpha \equiv (\exists\ B \in SCF\ r.\ |B| =o\ \alpha \wedge (\ \forall\ B' \in SCF\ r.\ |B| \leq o\ |B'|\ ))$

definition scf :: $'U\ rel \Rightarrow 'U\ rel$
where $scf\ r \equiv (SOME\ \alpha.\ scf\text{-}rel\ r\ \alpha)$

definition $w\text{-}dncl$ :: $'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$
where
$w\text{-}dncl\ r\ A = \{\ a \in dncl\ r\ A.\ \forall\ b.\ \forall\ F \in \mathcal{F}\ r\ a\ b.\ (\ b \notin dncl\ r\ A \longrightarrow F \cap A \neq \{\}\ )\ \}$

definition $\mathfrak{L}$ :: $('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set$
where
$\mathfrak{L}\ f\ \alpha \equiv \bigcup\ \{A.\ \exists\ \alpha'.\ \alpha' <o\ \alpha \wedge A = f\ \alpha'\}$

definition Dbk :: $('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set\ (\ \nabla\ \text{-}\ \text{-}\ )$
where
$\nabla\ f\ \alpha \equiv f\ \alpha - (\mathfrak{L}\ f\ \alpha)$

definition $\mathcal{Q}$ :: $'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set$
where
$\mathcal{Q}\ r\ f\ \alpha \equiv (f\ \alpha - (dncl\ r\ (\mathfrak{L}\ f\ \alpha)))$

definition $\mathcal{W}$ :: $'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U\ set$
where
$\mathcal{W}\ r\ f\ \alpha \equiv (f\ \alpha - (w\text{-}dncl\ r\ (\mathfrak{L}\ f\ \alpha)))$

definition $\mathcal{N}1$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}1\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha\ \alpha'.\ (\ \alpha \leq o\ \alpha 0 \wedge \alpha' \leq o\ \alpha\ ) \longrightarrow (f\ \alpha') \subseteq (f\ \alpha)\ \}$

definition $\mathcal{N}2$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}2\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge \neg\ (\alpha = \{\} \vee isSuccOrd\ \alpha)\ ) \longrightarrow (\nabla\ f\ \alpha) = \{\}\ \}$

definition $\mathcal{N}3$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}3\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge (\alpha = \{\} \vee isSuccOrd\ \alpha)\ ) \longrightarrow$
$(\ \omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha| \longrightarrow ((escl\ r\ (\mathfrak{L}\ f\ \alpha)\ (f\ \alpha) \subseteq (f\ \alpha)) \wedge (clterm\ (Restr\ r\ (f\ \alpha))\ r))\ )\ \}$

definition $\mathcal{N}4$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}4\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge (\alpha = \{\} \vee isSuccOrd\ \alpha)\ ) \longrightarrow$
$(\ \forall a \in (\mathfrak{L}\ f\ \alpha).\ (\ r``\{a\} \subseteq w\text{-}dncl\ r\ (\mathfrak{L}\ f\ \alpha)\ ) \vee (\ r``\{a\} \cap (\mathcal{W}\ r\ f\ \alpha) \neq \{\}\ )\ )\ \}$

definition $\mathcal{N}5$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}5\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow (f\ \alpha) \in SF\ r\ \}$

definition $\mathcal{N}6$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}6\ r\ \alpha 0 \equiv \{\ f.\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow CCR\ (Restr\ r\ (f\ \alpha))\ \}$

definition $\mathcal{N}7$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}7\ r\ \alpha 0 \equiv \{\ f.\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow (\ \alpha < o\ \omega\text{-}ord \longrightarrow |f\ \alpha| < o\ \omega\text{-}ord\ ) \wedge (\omega\text{-}ord \leq o\ \alpha \longrightarrow |f\ \alpha| \leq o\ \alpha)\ \}$

definition $\mathcal{N}8$ :: $'U\ rel \Rightarrow 'U\ set\ set \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}8\ r\ Ps\ \alpha 0 \equiv \{\ f.\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \wedge (\alpha = \{\} \vee isSuccOrd\ \alpha) \wedge (\ (\exists\ P.\ Ps = \{P\}) \vee (\neg\ finite\ Ps \wedge |Ps| \leq o\ |f\ \alpha|\ )) \longrightarrow$
$(\forall\ P \in Ps.\ ((f\ \alpha) \cap P) \in SCF\ (Restr\ r\ (f\ \alpha)))\ \}$

definition $\mathcal{N}9$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}9\ r\ \alpha 0 \equiv \{\ f\ .\ \omega\text{-}ord \leq o\ \alpha 0 \longrightarrow Field\ r \subseteq (f\ \alpha 0)\ \}$

definition $\mathcal{N}10$ :: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}10\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ \alpha \leq o\ \alpha 0 \longrightarrow ((\exists\ y{::}'U.\ \mathcal{Q}\ r\ f\ \alpha = \{y\}) \longrightarrow (Field\ r \subseteq dncl\ r\ (f\ \alpha)))\ \}$

definition $\mathcal{N}11$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}11\ r\ \alpha 0 \equiv \{\ f\ .\ \forall \alpha.\ (\ \alpha \leq o\ \alpha 0 \wedge isSuccOrd\ \alpha) \longrightarrow \mathcal{Q}\ r\ f\ \alpha = \{\} \longrightarrow (Field$

$r \subseteq dncl\ r\ (f\ \alpha))$ }

definition $\mathcal{N}12$:: $'U\ rel \Rightarrow 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}12\ r\ \alpha0 \equiv \{\ f\ .\ \forall\alpha.\ \alpha \leq o\ \alpha0 \longrightarrow \omega\text{-}ord \leq o\ \alpha \longrightarrow \omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|\ \}$

definition $\mathcal{N}$ :: $'U\ rel \Rightarrow 'U\ set\ set \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{N}\ r\ Ps \equiv \{\ f \in (\mathcal{N}1\ r\ |Field\ r|\ ) \cap (\mathcal{N}2\ r\ |Field\ r|\ ) \cap (\mathcal{N}3\ r\ |Field\ r|\ ) \cap (\mathcal{N}4$
$r\ |Field\ r|\ )$
$\cap\ (\mathcal{N}5\ r\ |Field\ r|\ ) \cap (\mathcal{N}6\ r\ |Field\ r|\ ) \cap (\mathcal{N}7\ r\ |Field\ r|\ ) \cap (\mathcal{N}8\ r\ Ps$
$|Field\ r|\ )$
$\cap\ (\mathcal{N}9\ r\ |Field\ r| \cap \mathcal{N}10\ r\ |Field\ r| \cap \mathcal{N}11\ r\ |Field\ r| \cap \mathcal{N}12\ r\ |Field\ r|\ ).$
$(\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow f\ \alpha = f\ \beta)\ \}$

definition $\mathcal{T}$ :: $('U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set) \Rightarrow ('U\ rel \Rightarrow 'U\ set)\ set$
where
$\mathcal{T}\ F \equiv \{\ f::'U\ rel \Rightarrow 'U\ set\ .$
$f\ \{\} = \{\}$
$\wedge\ (\forall\ \alpha0\ \alpha::'U\ rel.\ (sc\text{-}ord\ \alpha0\ \alpha \longrightarrow f\ \alpha = F\ \alpha0\ (f\ \alpha0)))$
$\wedge\ (\forall\ \alpha.\ (lm\text{-}ord\ \alpha \longrightarrow f\ \alpha = \bigcup\ \{\ D.\ \exists\ \beta.\ \beta <o\ \alpha \wedge D = f\ \beta\ \}))$
$\wedge\ (\forall\alpha\ \beta.\ \alpha =o\ \beta \longrightarrow f\ \alpha = f\ \beta)\ \}$

definition $\mathcal{E}p$ **where** $\mathcal{E}p\ r\ Ps\ A\ A' \equiv$
$(((\exists\ P.\ Ps = \{P\}) \vee ((\neg\ finite\ Ps) \wedge |Ps| \leq o\ |A|\ ))$
$\longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A')\ ))$

definition $\mathcal{E}$ :: $'U\ rel \Rightarrow 'U \Rightarrow 'U\ set \Rightarrow 'U\ set\ set \Rightarrow 'U\ set\ set$
where
$\mathcal{E}\ r\ a\ A\ Ps \equiv \{\ A'.$
$(a \in Field\ r \longrightarrow a \in A') \wedge A \subseteq A'$
$\wedge\ (\ |A| <o\ \omega\text{-}ord \longrightarrow |A'| <o\ \omega\text{-}ord\ ) \wedge (\ \omega\text{-}ord \leq o\ |A| \longrightarrow |A'| \leq o\ |A|\ )$
$\wedge\ (A \in SF\ r \longrightarrow ($
$A' \in SF\ r$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ (\ \forall\ a{\in}A.\ (r``\{a\} \subseteq w\text{-}dncl\ r\ A) \vee (r``\{a\} \cap (A' - w\text{-}dncl\ r\ A) \neq \{\})$
$)$
$\wedge\ ((\exists\ y.\ A' -\ dncl\ r\ A \subseteq \{y\}) \longrightarrow (Field\ r \subseteq (dncl\ r\ A')))$
$\wedge\ \mathcal{E}p\ r\ Ps\ A\ A'$
$\wedge\ (\ \omega\text{-}ord \leq o\ |A| \longrightarrow escl\ r\ A\ A' \subseteq A' \wedge clterm\ (Restr\ r\ A')\ r))\ )\ \}$

definition $wbase$::$'U\ rel \Rightarrow 'U\ set \Rightarrow ('U\ set)\ set$ **where**
$wbase\ r\ A \equiv \{\ B::'U\ set.\ A \subseteq w\text{-}dncl\ r\ B\ \}$

definition $wrank\text{-}rel$ :: $'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ rel \Rightarrow bool$ **where**
$wrank\text{-}rel\ r\ A\ \alpha \equiv (\exists\ B \in wbase\ r\ A.\ |B| =o\ \alpha \wedge (\ \forall\ B' \in wbase\ r\ A.\ |B| \leq o$
$|B'|\ ))$

definition $wrank$ :: $'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ rel$

where $wrank\ r\ A \equiv (SOME\ \alpha.\ wrank\text{-}rel\ r\ A\ \alpha)$

definition $Mwn :: 'U\ rel \Rightarrow 'U\ rel \Rightarrow 'U\ set$
where
$Mwn\ r\ \alpha = \{\ a \in Field\ r.\ \alpha <o\ wrank\ r\ (r\ ``\{a\})\ \}$

definition $Mwnm :: 'U\ rel \Rightarrow 'U\ set$
where
$Mwnm\ r = \{\ a \in Field\ r.\ \|r\| \leq o\ wrank\ r\ (r\ ``\{a\})\ \}$

definition $wesc\text{-}rel :: 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U \Rightarrow 'U \Rightarrow bool$
where
$wesc\text{-}rel\ r\ f\ \alpha\ a\ b \equiv (\ b \in \mathcal{W}\ r\ f\ \alpha \wedge (a,b) \in (Restr\ r\ (\mathcal{W}\ r\ f\ \alpha))\hat{}*$
$\wedge (\forall \beta.\ \alpha <o\ \beta \wedge \beta <o\ |Field\ r| \wedge (\beta = \{\} \vee isSuccOrd\ \beta) \longrightarrow (r``\{b\} \cap (\mathcal{W}\ r\ f\ \beta) \neq \{\}))\ )$

definition $wesc :: 'U\ rel \Rightarrow ('U\ rel \Rightarrow 'U\ set) \Rightarrow 'U\ rel \Rightarrow 'U \Rightarrow 'U$
where
$wesc\ r\ f\ \alpha\ a \equiv (SOME\ b.\ wesc\text{-}rel\ r\ f\ \alpha\ a\ b)$

definition $cardLeN1::'a\ set \Rightarrow bool$
where
$cardLeN1\ A \equiv (\forall\ B \subseteq A.$
$(\ \forall\ C \subseteq B\ .\ ((\exists\ D\ f.\ D \subset C \wedge C \subseteq f`D\ ) \longrightarrow (\ \exists\ f.\ B \subseteq f`C\ ))\ )$
$\vee (\ \exists\ g\ .\ A \subseteq g`B\ )\ )$

1.2.2 Auxiliary lemmas

lemma *lem-Ldo-ldogen-ord*:
assumes $\forall \alpha\ \beta\ a\ b\ c.\ \alpha \leq \beta \longrightarrow (a,\ b) \in g\ \alpha \wedge (a,\ c) \in g\ \beta \longrightarrow$
$(\exists b'\ b''\ c'\ c''\ d.\ (b,\ b',\ b'',\ d) \in \mathfrak{D}\ g\ \alpha\ \beta \wedge (c,\ c',\ c'',\ d) \in \mathfrak{D}\ g\ \beta\ \alpha)$
shows *DCR-generating g*
⟨*proof*⟩

lemma *lem-rtr-field*: $(x,y) \in r\hat{}* \Longrightarrow (x = y) \vee (x \in Field\ r \wedge y \in Field\ r)$
⟨*proof*⟩

lemma *lem-fin-fl-rel*: $finite\ (Field\ r) = finite\ r$
⟨*proof*⟩

lemma *lem-Relprop-fld-sat*:
fixes $r\ s::'U\ rel$
assumes $a1$: $s \subseteq r$ **and** $a2$: $s' = Restr\ r\ (Field\ s)$
shows $s \subseteq s' \wedge Field\ s' = Field\ s$
⟨*proof*⟩

lemma *lem-Relprop-sat-un*:
fixes $r::'U\ rel$ **and** $S::'U\ set\ set$ **and** $A'::'U\ set$
assumes $a1$: $\forall A{\in}S.\ Field\ (Restr\ r\ A) = A$ **and** $a2$: $A' = \bigcup\ S$

shows *Field* (*Restr* *r* *A'*) = *A'*
⟨*proof*⟩

lemma *lem-nord-r*: $Well\text{-}order\ \alpha \Longrightarrow nord\ \alpha =o\ \alpha$ ⟨*proof*⟩

lemma *lem-nord-l*: $Well\text{-}order\ \alpha \Longrightarrow \alpha =o\ nord\ \alpha$ ⟨*proof*⟩

lemma *lem-nord-eq*: $\alpha =o\ \beta \Longrightarrow nord\ \alpha = nord\ \beta$ ⟨*proof*⟩

lemma *lem-nord-req*: $Well\text{-}order\ \alpha \Longrightarrow Well\text{-}order\ \beta \Longrightarrow nord\ \alpha = nord\ \beta \Longrightarrow \alpha =o\ \beta$
⟨*proof*⟩

lemma *lem-Onord*: $\alpha \in \mathcal{O} \Longrightarrow \alpha = nord\ \alpha$ ⟨*proof*⟩

lemma *lem-Oeq*: $\alpha \in \mathcal{O} \Longrightarrow \beta \in \mathcal{O} \Longrightarrow \alpha =o\ \beta \Longrightarrow \alpha = \beta$ ⟨*proof*⟩

lemma *lem-Owo*: $\alpha \in \mathcal{O} \Longrightarrow Well\text{-}order\ \alpha$ ⟨*proof*⟩

lemma *lem-fld-oord*: $Field\ oord = \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nord-less*: $\alpha <o\ \beta \Longrightarrow nord\ \beta \neq nord\ \alpha \wedge (nord\ \alpha,\ nord\ \beta) \in oord$
⟨*proof*⟩

lemma *lem-nord-ls*: $\alpha <o\ \beta \Longrightarrow nord\ \alpha <o\ nord\ \beta$
⟨*proof*⟩

lemma *lem-nord-le*: $\alpha \leq o\ \beta \Longrightarrow nord\ \alpha \leq o\ nord\ \beta$
⟨*proof*⟩

lemma *lem-nordO-ls-l*: $\alpha <o\ \beta \Longrightarrow nord\ \alpha \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nordO-ls-r*: $\alpha <o\ \beta \Longrightarrow nord\ \beta \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nordO-le-l*: $\alpha \leq o\ \beta \Longrightarrow nord\ \alpha \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nordO-le-r*: $\alpha \leq o\ \beta \Longrightarrow nord\ \beta \in \mathcal{O}$ ⟨*proof*⟩

lemma *lem-nord-ls-r*: $\alpha <o\ \beta \Longrightarrow \alpha <o\ nord\ \beta$
⟨*proof*⟩

lemma *lem-nord-ls-l*: $\alpha <o\ \beta \Longrightarrow nord\ \alpha <o\ \beta$
⟨*proof*⟩

lemma *lem-nord-le-r*: $\alpha \leq o\ \beta \Longrightarrow \alpha \leq o\ nord\ \beta$
⟨*proof*⟩

lemma *lem-nord-le-l*: $\alpha \leq o\ \beta \Longrightarrow nord\ \alpha \leq o\ \beta$
⟨*proof*⟩

lemma *lem-oord-wo*: *Well-order oord*
⟨*proof*⟩

lemma *lem-lmord-inf*:
fixes α::'*U rel*
assumes *lm-ord* α
shows $\neg$ *finite* (*Field* α)
⟨*proof*⟩

lemma *lem-sucord-ex*:
fixes α β::'*U rel*
assumes $\alpha <o\ \beta$
shows $\exists\ \alpha'$::'*U rel*. *sc-ord* $\alpha\ \alpha'$
⟨*proof*⟩

lemma *lem-osucc-eq*: *isSuccOrd* $\alpha \Longrightarrow \alpha =o\ \beta \Longrightarrow$ *isSuccOrd* β
⟨*proof*⟩

lemma *lem-ord-subemp*: (α::'*a rel*) $\leq o$ ({}::'*b rel*) $\Longrightarrow \alpha$ = {}
⟨*proof*⟩

lemma *lem-ordint-sucord*:
fixes $\alpha 0$::'*a rel* **and** α::'*b rel*
assumes $\alpha 0 <o\ \alpha \wedge (\forall\ \gamma$::'*b rel*. $\alpha 0 <o\ \gamma \longrightarrow \alpha \leq o\ \gamma)$
shows *isSuccOrd* α
⟨*proof*⟩

lemma *lem-sucord-ordint*:
fixes α::'*U rel*
assumes *Well-order* $\alpha \wedge$ *isSuccOrd* α
shows $\exists\ \alpha 0$::'*U rel*. $\alpha 0 <o\ \alpha \wedge (\forall\ \gamma$::'*U rel*. $\alpha 0 <o\ \gamma \longrightarrow \alpha \leq o\ \gamma)$
⟨*proof*⟩

lemma *lem-sclm-ordind*:
fixes P::'*U rel* $\Rightarrow$ *bool*
assumes *a1*: P {}
  and *a2*: $\forall\ \alpha 0\ \alpha$::'*U rel*. (*sc-ord* $\alpha 0\ \alpha \wedge P\ \alpha 0 \longrightarrow P\ \alpha$)
  and *a3*: $\forall\ \alpha$. ((*lm-ord* $\alpha \wedge (\forall\ \beta.\ \beta <o\ \alpha \longrightarrow P\ \beta)) \longrightarrow P\ \alpha$)
shows $\forall\ \alpha$. *Well-order* $\alpha \longrightarrow P\ \alpha$
⟨*proof*⟩

lemma *lem-ordseq-rec-sets*:
fixes E::'*U set* **and** F::'*U rel* $\Rightarrow$ '*U set* $\Rightarrow$ '*U set*
assumes $\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow F\ \alpha = F\ \beta$
shows $\exists\ f$::('*U rel* $\Rightarrow$ '*U set*).
    f {} = E
  $\wedge$ ($\forall\ \alpha 0\ \alpha$::'*U rel*. (*sc-ord* $\alpha 0\ \alpha \longrightarrow f\ \alpha = F\ \alpha 0\ (f\ \alpha 0)$))
  $\wedge$ ($\forall\ \alpha$. *lm-ord* $\alpha \longrightarrow f\ \alpha = \bigcup$ { D. $\exists\ \beta.\ \beta <o\ \alpha \wedge D = f\ \beta$ })

$\wedge$ ($\forall$ α β. α =o β $\longrightarrow$ f α = f β)
⟨proof⟩

lemma *lem-lmord-prec*:
fixes α::'a rel **and** α'::'b rel
assumes *a1*: α' <o α **and** *a2*: *isLimOrd* α
shows $\exists$ β::('a rel). α' <o β $\wedge$ β <o α
⟨proof⟩

lemma *lem-inford-ge-w*:
fixes α::'U rel
assumes *Well-order* α **and** $\neg$ *finite* (*Field* α)
shows ω-ord $\leq$o α
⟨proof⟩

lemma *lem-ge-w-inford*:
fixes α::'U rel
assumes ω-ord $\leq$o α
shows $\neg$ *finite* (*Field* α)
⟨proof⟩

lemma *lem-fin-card*: *finite* |A| = *finite* A
⟨proof⟩

lemma *lem-cardord-emp*: *Card-order* ({}::'U rel)
⟨proof⟩

lemma *lem-card-emprel*: |{}::'U rel| =o ({}::'U rel)
⟨proof⟩

lemma *lem-cord-lin*: *Card-order* α $\Longrightarrow$ *Card-order* β $\Longrightarrow$ (α $\leq$o β) = ($\neg$ (β <o α)) ⟨proof⟩

lemma *lem-co-one-ne-min*:
fixes α::'U rel **and** a::'a
assumes *Well-order* α **and** $\alpha \neq$ {}
shows |{a}| $\leq$o α
⟨proof⟩

lemma *lem-rel-inf-fld-card*:
fixes r::'U rel
assumes $\neg$ *finite* r
shows |*Field* r| =o |r|
⟨proof⟩

lemma *lem-cardreleq-cardfldeq-inf*:
fixes r1 r2:: 'U rel
assumes *a1*: |r1| =o |r2| **and** *a2*: $\neg$ *finite* r1 $\vee$ $\neg$ *finite* r2
shows |*Field* r1| =o |*Field* r2|

$\langle proof \rangle$

lemma *lem-card-un-bnd*:
fixes $S::'a\ set\ set$ **and** $\alpha::'U\ rel$
assumes $a3$: $\forall A \in S.\ |A| \leq o\ \alpha$ **and** $a4$: $|S| \leq o\ \alpha$ **and** $a5$: $\omega\text{-}ord \leq o\ \alpha$
shows $|\ \bigcup\ S\ | \leq o\ \alpha$
$\langle proof \rangle$

lemma *lem-ord-suc-ge-w*:
fixes $\alpha 0\ \ \alpha::'U\ rel$
assumes $a1$: $\omega\text{-}ord \leq o\ \alpha$ **and** $a2$: $sc\text{-}ord\ \alpha 0\ \alpha$
shows $\omega\text{-}ord \leq o\ \alpha 0$
$\langle proof \rangle$

lemma *lem-restr-ordbnd*:
fixes $r::'U\ rel$ **and** $A::'U\ set$ **and** $\alpha::'U\ rel$
assumes $a1$: $\omega\text{-}ord \leq o\ \alpha$ **and** $a2$: $|A| \leq o\ \alpha$
shows $|Restr\ r\ A| \leq o\ \alpha$
$\langle proof \rangle$

lemma *lem-card-inf-lim*:
fixes $r::'U\ rel$
assumes $a1$: $Card\text{-}order\ \alpha$ **and** $a2$: $\omega\text{-}ord \leq o\ \alpha$
shows $\neg(\ \alpha = \{\} \vee isSuccOrd\ \alpha\ )$
$\langle proof \rangle$

lemma *lem-card-nreg-inf-osetlm*:
fixes $\alpha::'U\ rel$
assumes $a1$: $Card\text{-}order\ \alpha$ **and** $a2$: $\neg\ regularCard\ \alpha$ **and** $a3$: $\neg\ finite\ (Field\ \alpha)$
shows $\exists\ \ S::'U\ rel\ set.\ |S| <o\ \alpha \wedge (\forall\ \ \alpha' \in S.\ \alpha' <o\ \alpha) \wedge (\forall\ \ \alpha'::'U\ rel.\ \alpha' <o\ \alpha$
$\longrightarrow (\exists\ \ \beta \in S.\ \alpha' \leq o\ \beta))$
$\langle proof \rangle$

lemma *lem-card-un-bnd-stab*:
fixes $S::'a\ set\ set$ **and** $\alpha::'U\ rel$
assumes $stable\ \alpha$ **and** $\forall A \in S.\ |A| <o\ \alpha$ **and** $|S| <o\ \alpha$
shows $|\ \bigcup\ S\ | <o\ \alpha$
$\langle proof \rangle$

lemma *lem-finwo-cardord*: $finite\ \alpha \Longrightarrow Well\text{-}order\ \alpha \Longrightarrow Card\text{-}order\ \alpha$
$\langle proof \rangle$

lemma *lem-finwo-le-w*: $finite\ \alpha \Longrightarrow Well\text{-}order\ \alpha \Longrightarrow \alpha <o\ natLeq$
$\langle proof \rangle$

lemma *lem-wolew-fin*: $\alpha <o\ natLeq \Longrightarrow finite\ \alpha$
$\langle proof \rangle$

lemma *lem-wolew-nat*:

assumes *a1*: $\alpha <o\ natLeq$ **and** *a2*: $n = card\ (Field\ \alpha)$
shows $\alpha =o\ (natLeq\text{-}on\ n)$
$\langle proof \rangle$

lemma *lem-cntset-enum*: $|A| =o\ natLeq \Longrightarrow (\exists\ f.\ A = f\ `\ (UNIV::nat\ set))$
$\langle proof \rangle$

lemma *lem-oord-int-card-le-inf*:
fixes $\alpha::'U\ rel$
assumes $\omega\text{-}ord \leq o\ \alpha$
shows $|\{\ \gamma \in \mathcal{O}::'U\ rel\ set.\ \gamma <o\ \alpha\ \}| \leq o\ \alpha$
$\langle proof \rangle$

lemma *lem-oord-card-le-int-inf*:
fixes $\alpha::'U\ rel$
assumes *a1*: $Card\text{-}order\ \alpha$ **and** *a2*: $\omega\text{-}ord \leq o\ \alpha$
shows $\alpha \leq o\ |\{\ \gamma \in \mathcal{O}::'U\ rel\ set.\ \gamma <o\ \alpha\ \}|$
$\langle proof \rangle$

lemma *lem-ord-int-card-le-inf*:
fixes $\alpha::'U\ rel$ **and** $f :: 'U\ rel \Rightarrow 'a$
assumes $\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow f\ \alpha = f\ \beta$ **and** $\omega\text{-}ord \leq o\ \alpha$
shows $|f\ `\ \{\ \gamma::'U\ rel.\ \gamma <o\ \alpha\ \}| \leq o\ \alpha$
$\langle proof \rangle$

lemma *lem-card-setcv-inf-stab*:
fixes $\alpha::'U\ rel$ **and** $A::'U\ set$
assumes *a1*: $Card\text{-}order\ \alpha$ **and** *a2*: $\omega\text{-}ord \leq o\ \alpha$ **and** *a3*: $|A| \leq o\ \alpha$
shows $\exists\ f::('U\ rel \Rightarrow 'U).\ A \subseteq f\ `\{\ \gamma::'U\ rel.\ \gamma <o\ \alpha\ \} \wedge (\forall\ \gamma 1\ \gamma 2.\ \gamma 1 =o\ \gamma 2$
$\longrightarrow f\ \gamma 1 = f\ \gamma 2)$
$\langle proof \rangle$

lemma *lem-jnfix-gen*:
fixes $I::'i\ set$ **and** $leI::'i\ rel$ **and** $L::'l\ set$
and $t::'i \times 'l \Rightarrow 'i \Rightarrow 'n$ **and** $jnN::'n \Rightarrow 'n \Rightarrow 'n$
assumes *a1*: $\neg\ finite\ L$
and *a2*: $|L| <o\ |I|$
and *a3*: $\forall \alpha \in I.\ (\alpha,\alpha) \in leI$
and *a4*: $\forall \alpha \in I.\ \forall \beta \in I.\ \forall \gamma \in I.\ (\alpha,\beta) \in leI \wedge (\beta,\gamma) \in leI \longrightarrow (\alpha,\gamma) \in leI$
and *a5*: $\forall \alpha \in I.\ \forall \beta \in I.\ (\alpha,\beta) \in leI \vee (\beta,\alpha) \in leI$
and *a6*: $\forall \beta \in I.\ |\{\alpha \in I.\ (\alpha,\beta) \in leI\}| \leq o\ |L|$
and *a7*: $\forall \alpha \in I.\ \exists \alpha' \in I.\ (\alpha,\alpha') \in leI \wedge (\alpha',\alpha) \notin leI$
shows $\exists\ h.\ \forall \alpha \in I.\ \forall \beta \in I.\ \forall i \in L.\ \forall j \in L.\ \exists\ \gamma \in I.\ (\alpha,\gamma) \in leI \wedge (\beta,\gamma) \in leI \wedge (\gamma,\alpha) \notin leI$
$\wedge\ (\gamma,\beta) \notin leI$
$\wedge\ h\ \gamma = jnN\ (t\ (\alpha,i)\ \gamma)\ (t\ (\beta,j)\ \gamma)$
$\langle proof \rangle$

lemma *lem-jnfix-card*:
fixes $\kappa::'U\ rel$ **and** $L::'l\ set$ **and** $t::('U\ rel) \times 'l \Rightarrow 'U\ rel \Rightarrow 'n$ **and** $jnN::'n \Rightarrow 'n$

$\Rightarrow$ $'n$
  and S::$'U$ *rel set*
assumes *a1*: *Card-order* κ **and** *a2*: $\neg$ *finite* L **and** *a3*: $|L| <o\ \kappa$
    and *a4*: $\forall\ \alpha \in S.\ |Field\ \alpha| \leq o\ |L|$
    and *a5*: $S \subseteq \mathcal{O}$ **and** *a6*: $|\{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}| \leq o\ |S|$
    and *a7*: $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$
shows $\exists\ h.\ \forall\ \alpha \in S.\ \forall\ \beta \in S.\ \forall i{\in}L.\ \forall j{\in}L.$
    $(\exists\ \gamma \in S.\ \alpha <o\ \gamma \wedge \beta <o\ \gamma \wedge h\ \gamma = jnN\ (t\ (\alpha,i)\ \gamma)\ (t\ (\beta,j)\ \gamma)\ )$
⟨*proof*⟩

lemma *lem-cardsuc-ls-fldcard*:
fixes κ::$'a$ *rel* **and** α::$'b$ *rel*
assumes *a1*: *Card-order* κ **and** *a2*: $\alpha <o\ cardSuc\ \kappa$
shows $|Field\ \alpha| \leq o\ \kappa$
⟨*proof*⟩

lemma *lem-jnfix-cardsuc*:
fixes L::$'l$ *set* **and** κ::$'U$ *rel* **and** t::$('U\ rel){\times}'l \Rightarrow\ 'U\ rel \Rightarrow\ 'n$ **and** jnN::$'n \Rightarrow\ 'n$
$\Rightarrow$ $'n$
  and S::$'U$ *rel set*
assumes *a1*: $\neg$ *finite* L **and** *a2*: $\kappa =o\ cardSuc\ |L|$
    and *a3*: $S \subseteq \{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}$ **and** *a4*: $|\{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o$
$\kappa\}| \leq o\ |S|$
    and *a5*: $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$
shows $\exists\ h.\ \forall\ \alpha \in S.\ \forall\ \beta \in S.\ \forall i{\in}L.\ \forall j{\in}L.$
    $(\exists\ \gamma \in S.\ \alpha <o\ \gamma \wedge \beta <o\ \gamma \wedge h\ \gamma = jnN\ (t\ (\alpha,i)\ \gamma)\ (t\ (\beta,j)\ \gamma)\ )$
⟨*proof*⟩

lemma *lem-Relprop-cl-ccr*:
fixes r::$'U$ *rel*
shows *Conelike* $r \Longrightarrow$ *CCR* r
  ⟨*proof*⟩

lemma *lem-Relprop-ccr-confl*:
fixes r::$'U$ *rel*
shows *CCR* $r \Longrightarrow$ *confl-rel* r
  ⟨*proof*⟩

lemma *lem-Relprop-fin-ccr*:
fixes r::$'U$ *rel*
shows *finite* $r \Longrightarrow$ *CCR* r = *Conelike* r
⟨*proof*⟩

lemma *lem-Relprop-ccr-ch-un*:
fixes S::$'U$ *rel set*
assumes *a1*: $\forall s{\in}S.$ *CCR* s **and** *a2*: $\forall s1{\in}S.\ \forall s2{\in}S.\ s1 \subseteq s2 \vee s2 \subseteq s1$
shows *CCR* $(\bigcup\ S)$
⟨*proof*⟩

lemma *lem-Relprop-restr-ch-un*:
fixes $C::'U\ set\ set$ **and** $r::'U\ rel$
assumes $\forall A1\in C.\ \forall A2\in C.\ A1 \subseteq A2 \vee A2 \subseteq A1$
shows $Restr\ r\ (\bigcup\ C) = \bigcup\ \{\ s.\ \exists\ A \in C.\ s = Restr\ r\ A\ \}$
$\langle proof \rangle$

lemma *lem-Inv-restr-rtr*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $A \in Inv\ r$
shows $r\hat{}* \cap (A\times(UNIV::'U\ set)) \subseteq (Restr\ r\ A)\hat{}*$
$\langle proof \rangle$

lemma *lem-Inv-restr-rtr2*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $A \in Inv\ r$
shows $r\hat{}* \cap (A\times(UNIV::'U\ set)) \subseteq (Restr\ r\ A)\hat{}* \cap ((UNIV::'U\ set)\times A)$
$\langle proof \rangle$

lemma *lem-inv-rtr-mem*:
fixes $r::'U\ rel$ **and** $A::'U\ set$ **and** $a\ b::'U$
assumes $A \in Inv\ r$ **and** $a \in A$ **and** $(a,b) \in r\hat{}*$
shows $b \in A$
$\langle proof \rangle$

lemma *lem-Inv-ccr-restr*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $CCR\ r$ **and** $A \in Inv\ r$
shows $CCR\ (Restr\ r\ A)$
$\langle proof \rangle$

lemma *lem-Inv-cl-restr*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $Conelike\ r$ **and** $A \in Inv\ r$
shows $Conelike\ (Restr\ r\ A)$
$\langle proof \rangle$

lemma *lem-Inv-ccr-restr-invdiff*:
fixes $r::'U\ rel$ **and** $A\ B::'U\ set$
assumes $a1$: $CCR\ (Restr\ r\ A)$ **and** $a2$: $B \in Inv\ (r\hat{}-1)$
shows $CCR\ (Restr\ r\ (A - B))$
$\langle proof \rangle$

lemma *lem-Inv-dncl-invbk*: $dncl\ r\ A \in Inv\ (r\hat{}-1)$
$\langle proof \rangle$

lemma *lem-inv-sf-ext*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
assumes $A \subseteq Field\ r$
shows $\exists\ A' \in SF\ r.\ A \subseteq A' \wedge (finite\ A \longrightarrow finite\ A') \wedge ((\neg\ finite\ A) \longrightarrow |A'| =o$

$|A|$)
⟨*proof*⟩

lemma *lem-inv-sf-un*:
assumes $S \subseteq SF\ r$
shows $(\bigcup\ S) \in SF\ r$
⟨*proof*⟩

lemma *lem-Inv-ccr-sf-inv-diff*:
fixes r::$'U\ rel$ **and** $A\ B$::$'U\ set$
assumes *a1*: $A \in SF\ r$ **and** *a2*: $CCR\ (Restr\ r\ A)$ **and** *a3*: $B \in Inv\ (r\hat{\ }{-1})$
shows $(A{-}B) \in SF\ r \vee (\exists\ y{::}'U.\ (A{-}B) = \{y\})$
⟨*proof*⟩

lemma *lem-Inv-ccr-sf-dn-diff*:
fixes r::$'U\ rel$ **and** $A\ D\ A'$::$'U\ set$
assumes *a1*: $A \in SF\ r$ **and** *a2*: $CCR\ (Restr\ r\ A)$ **and** *a3*: $A' = (A - (dncl\ r\ D))$
shows $((A' \in SF\ r) \wedge CCR\ (Restr\ r\ A')) \vee (\exists\ y{::}'U.\ A' = \{y\})$
⟨*proof*⟩

lemma *lem-rseq-tr*:
fixes r::$'U\ rel$ **and** xi::$nat \Rightarrow\ 'U$
assumes $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\forall\ i\ j.\ i < j \longrightarrow (xi\ i \in Field\ r \wedge (xi\ i,\ xi\ j) \in r\hat{\ }{+})$
⟨*proof*⟩

lemma *lem-rseq-rtr*:
fixes r::$'U\ rel$ **and** xi::$nat \Rightarrow\ 'U$
assumes $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\forall\ i\ j.\ i \le j \longrightarrow (xi\ i \in Field\ r \wedge (xi\ i,\ xi\ j) \in r\hat{\ }{*})$
⟨*proof*⟩

lemma *lem-rseq-svacyc-inv-tr*:
fixes r::$'U\ rel$ **and** xi::$nat \Rightarrow\ 'U$ **and** a::$'U$
assumes *a1*: *single-valued* r **and** *a2*: $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\bigwedge\ i.\ (xi\ i,\ a) \in r\hat{\ }{+} \Longrightarrow (\exists\ j.\ i{<}j \wedge a = xi\ j)$
⟨*proof*⟩

lemma *lem-rseq-svacyc-inv-rtr*:
fixes r::$'U\ rel$ **and** xi::$nat \Rightarrow\ 'U$ **and** a::$'U$
assumes *a1*: *single-valued* r **and** *a2*: $\forall\ i.\ (xi\ i,\ xi\ (Suc\ i)) \in r$
shows $\bigwedge\ i.\ (xi\ i,\ a) \in r\hat{\ }{*} \Longrightarrow (\exists\ j.\ i{\le}j \wedge a = xi\ j)$
⟨*proof*⟩

lemma *lem-ccrsv-cfseq*:
fixes r::$'U\ rel$
assumes *a1*: $r \neq \{\}$ **and** *a2*: $CCR\ r$ **and** *a3*: *single-valued* r **and** *a4*: $\forall\, x{\in}Field\ r.\ r``\{x\} \neq \{\}$
shows $\exists\ xi.\ cfseq\ r\ xi$

⟨*proof*⟩

lemma *lem-cfseq-fld*: *cfseq r xi* $\Longrightarrow$ *xi* ' *UNIV* $\subseteq$ *Field r*
⟨*proof*⟩

lemma *lem-cfseq-inv*: *cfseq r xi* $\Longrightarrow$ *single-valued r* $\Longrightarrow$ *xi* ' *UNIV* $\in$ *Inv r*
⟨*proof*⟩

lemma *lem-scfinv-scf-int*: $A \in$ *SCF r* $\cap$ *Inv r* $\Longrightarrow$ $B \in$ *SCF r* $\Longrightarrow$ $(A \cap B) \in$ *SCF r*
⟨*proof*⟩

lemma *lem-scf-minr*: $a \in$ *Field r* $\Longrightarrow$ $B \in$ *SCF r* $\Longrightarrow$ $\exists\ b \in B.\ (a,b) \in (r \cap ((UNIV-B) \times UNIV))\hat{}*$
⟨*proof*⟩

lemma *lem-cfseq-ncl*:
fixes *r*::*'U rel* **and** *xi*::*nat* ⇒ *'U*
assumes *a1*: *cfseq r xi* **and** *a2*: ¬ *Conelike r*
shows $\forall\ n.\ \exists\ k.\ n \le k \wedge (xi\ (Suc\ k),\ xi\ k) \notin r\hat{}*$
⟨*proof*⟩

lemma *lem-cfseq-inj*:
fixes *r*::*'U rel* **and** *xi*::*nat* ⇒ *'U*
assumes *a1*: *cfseq r xi* **and** *a2*: *acyclic r*
shows *inj xi*
⟨*proof*⟩

lemma *lem-cfseq-rmon*:
fixes *r*::*'U rel* **and** *xi*::*nat* ⇒ *'U*
assumes *a1*: *cfseq r xi* **and** *a2*: *single-valued r* **and** *a3*: *acyclic r*
shows $\forall\ i\ j.\ (xi\ i,\ xi\ j) \in r\hat{}+ \longrightarrow i < j$
⟨*proof*⟩

lemma *lem-rseq-hd*:
assumes $\forall i<n.\ (f\ i,\ f\ (Suc\ i)) \in r$
shows $\forall i \le n.\ (f\ 0,\ f\ i) \in r\hat{}*$
⟨*proof*⟩

lemma *lem-rseq-tl*:
assumes $\forall i<n.\ (f\ i,\ f\ (Suc\ i)) \in r$
shows $\forall i \le n.\ (f\ i,\ f\ n) \in r\hat{}*$
⟨*proof*⟩

lemma *lem-ccext-ntr-rpth*: $(a,b) \in r\hat{}\hat{}n = (rpth\ r\ a\ b\ n \neq \{\})$
⟨*proof*⟩

lemma *lem-ccext-rtr-rpth*: $(a,b) \in r\hat{}* \Longrightarrow \exists\ n.\ rpth\ r\ a\ b\ n \neq \{\}$
⟨*proof*⟩

lemma *lem-ccext-rpth-rtr*: *rpth r a b n* $\neq$ {} $\Longrightarrow$ (*a*,*b*) $\in$ *r*^*
⟨*proof*⟩

lemma *lem-ccext-rtr-Fne*:
fixes *r*::′*U rel* **and** *a b*::′*U*
shows (*a*,*b*) $\in$ *r*^* = ($\mathcal{F}$ *r a b* $\neq$ {})
⟨*proof*⟩

lemma *lem-ccext-fprop*: $\mathcal{F}$ *r a b* $\neq$ {} $\Longrightarrow$ $\mathfrak{f}$ *r a b* $\in$ $\mathcal{F}$ *r a b* ⟨*proof*⟩

lemma *lem-ccext-ffin*: *finite* ($\mathfrak{f}$ *r a b*)
⟨*proof*⟩

lemma *lem-ccr-fin-subr-ext*:
fixes *r s*::′*U rel*
assumes *a1*: *CCR r* **and** *a2*: *s* $\subseteq$ *r* **and** *a3*: *finite s*
shows $\exists$ *s*′::(′*U rel*). *finite s*′ $\wedge$ *CCR s*′ $\wedge$ *s* $\subseteq$ *s*′ $\wedge$ *s*′ $\subseteq$ *r*
⟨*proof*⟩

lemma *lem-Ccext-fint*:
fixes *r s*::′*U rel* **and** *a b*::′*U*
assumes *a1*: *Restr r* ($\mathfrak{f}$ *r a b*) $\subseteq$ *s* **and** *a2*: (*a*,*b*) $\in$ *r*^*
shows {*a*, *b*} $\subseteq$ $\mathfrak{f}$ *r a b* $\wedge$ ($\forall$ *c* $\in$ $\mathfrak{f}$ *r a b*. (*a*,*c*) $\in$ *s*^* $\wedge$ (*c*,*b*) $\in$ *s*^*)
⟨*proof*⟩

lemma *lem-Ccext-subccr-eqfld*:
fixes *r r*′::′*U rel*
assumes *CCR r* **and** *r* $\subseteq$ *r*′ **and** *Field r*′ = *Field r*
shows *CCR r*′
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-pext*:
fixes *r s*::′*U rel* **and** *x*::′*U*
assumes *a1*: *CCR r* **and** *a2*: *s* $\subseteq$ *r* **and** *a3*: *finite s* **and** *a5*: *x* $\in$ *Field r*
shows $\exists$ *s*′::(′*U rel*). *finite s*′ $\wedge$ *CCR s*′ $\wedge$ *s* $\subseteq$ *s*′ $\wedge$ *s*′ $\subseteq$ *r* $\wedge$ *x* $\in$ *Field s*′
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-dext*:
fixes *r*::′*U rel* **and** *A*::′*U set*
assumes *a1*: *CCR r* **and** *a2*: *A* $\subseteq$ *Field r* **and** *a3*: *finite A*
shows $\exists$ *s*::(′*U rel*). *finite s* $\wedge$ *CCR s* $\wedge$ *s* $\subseteq$ *r* $\wedge$ *A* $\subseteq$ *Field s*
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-pext*:
fixes *r s*::′*U rel* **and** *x*::′*U*
assumes *a1*: *CCR r* **and** *a2*: *s* $\subseteq$ *r* **and** *a3*: $\neg$ *finite s* **and** *a5*: *x* $\in$ *Field r*
shows $\exists$ *s*′::(′*U rel*). *CCR s*′ $\wedge$ *s* $\subseteq$ *s*′ $\wedge$ *s*′ $\subseteq$ *r* $\wedge$ |*s*′| =*o* |*s*| $\wedge$ *x* $\in$ *Field s*′
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-set-ext*:
fixes $r\ s::'U\ rel$ **and** $A::'U\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $s \subseteq r$ **and** *a3*: $finite\ s$ **and** *a4*: $A \subseteq Field\ r$ **and** *a5*: $finite\ A$
shows $\exists\ s'::('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge finite\ s' \wedge A \subseteq Field\ s'$
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-set-ext*:
fixes $r\ s::'U\ rel$ **and** $A::'U\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $s \subseteq r$ **and** *a3*: $\neg\ finite\ s$ **and** *a4*: $A \subseteq Field\ r$ **and** *a5*: $|A| \leq o\ |Field\ s|$
shows $\exists\ s'::('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge |s'| =o\ |s| \wedge A \subseteq Field\ s'$
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-pext5*:
fixes $r::'U\ rel$ **and** $A\ B::'U\ set$ **and** $x::'U$
assumes *a1*: $CCR\ r$ **and** *a2*: $finite\ A$ **and** *a3*: $A \in SF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge finite\ A'$
$\wedge\ (\forall\, a{\in}A.\ r``\{a\}{\subseteq}B \vee r``\{a\}{\cap}(A'{-}B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y::'U.\ A'{-}B = \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B))$
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-pext5*:
fixes $r::'U\ rel$ **and** $A\ B::'U\ set$ **and** $x::'U$
assumes *a1*: $CCR\ r$ **and** *a2*: $\neg\ finite\ A$ **and** *a3*: $A \in SF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge |A'| =o\ |A|$
$\wedge\ (\forall\, a{\in}A.\ r``\{a\}{\subseteq}B \vee r``\{a\}{\cap}(A'{-}B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y::'U.\ A'{-}B = \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B))$
⟨*proof*⟩

lemma *lem-Ccext-subccr-pext5*:
fixes $r::'U\ rel$ **and** $A\ B::'U\ set$ **and** $x::'U$
assumes $CCR\ r$ **and** $A \in SF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A')$
$\wedge\ A \subseteq A'$
$\wedge\ A' \in SF\ r$
$\wedge\ (\forall\, a{\in}A.\ ((r``\{a\}{\subseteq}B) \vee (r``\{a\}{\cap}(A'{-}B) \neq \{\})))$
$\wedge\ ((\exists\ y::'U.\ A'{-}B = \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B))$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ ((finite\ A \longrightarrow finite\ A') \wedge (\ (\neg\ finite\ A) \longrightarrow |A'| =o\ |A|\ ))$
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-set-ext-scf*:
fixes $r\ s::'U\ rel$ **and** $A\ P::'U\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $s \subseteq r$ **and** *a3*: $finite\ s$ **and** *a4*: $A \subseteq Field\ r$ **and** *a5*: $finite\ A$

and *a6*: $P \in SCF\ r$
shows $\exists\ s'::('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge finite\ s' \wedge A \subseteq Field\ s'$
$\wedge\ ((Field\ s' \cap P) \in SCF\ s')$
$\langle proof \rangle$

lemma *lem-ccext-scf-sat*:
assumes $s \subseteq r$ **and** $Field\ s = Field\ r$
shows $SCF\ s \subseteq SCF\ r$
$\langle proof \rangle$

lemma *lem-Ccext-infsubccr-set-ext-scf2*:
fixes $r\ s::'U\ rel$ **and** $A::'U\ set$ **and** $Ps::'U\ set\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $s \subseteq r$ **and** *a3*: $\neg\ finite\ s$ **and** *a4*: $A \subseteq Field\ r$
and *a5*: $|A| \leq o\ |Field\ s|$ **and** *a6*: $Ps \subseteq SCF\ r \wedge |Ps| \leq o\ |Field\ s|$
shows $\exists\ s'::('U\ rel).\ CCR\ s' \wedge s \subseteq s' \wedge s' \subseteq r \wedge |s'| =o\ |s| \wedge A \subseteq Field\ s'$
$\wedge\ (\forall\ P\in Ps.\ (Field\ s' \cap P) \in SCF\ s')$
$\langle proof \rangle$

lemma *lem-Ccext-finsubccr-pext5-scf2*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $finite\ A$ **and** *a3*: $A \in SF\ r$ **and** *a4*: $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge$
$finite\ A'$
$\wedge\ (\forall a\in A.\ r``\{a\}\subseteq B \vee r``\{a\}\cap(A'-B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y::'U.\ A'-B' = \{y\}) \longrightarrow Field\ r \subseteq (A'\cup B'))$
$\wedge\ ((\exists\ P.\ Ps = \{P\}) \longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r$
$A')))$
$\langle proof \rangle$

lemma *lem-Ccext-infsubccr-pext5-scf2*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes *a1*: $CCR\ r$ **and** *a2*: $\neg\ finite\ A$ **and** *a3*: $A \in SF\ r$ **and** *a4*: $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A') \wedge A \subseteq A' \wedge CCR\ (Restr\ r\ A') \wedge$
$|A'| =o\ |A|$
$\wedge\ (\forall a\in A.\ r``\{a\}\subseteq B \vee r``\{a\}\cap(A'-B) \neq \{\}) \wedge A' \in SF\ r$
$\wedge\ ((\exists\ y::'U.\ A'-B' = \{y\}) \longrightarrow Field\ r \subseteq (A'\cup B'))$
$\wedge\ (\ |Ps| \leq o\ |A| \longrightarrow (\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A'))\ )$
$\langle proof \rangle$

lemma *lem-Ccext-subccr-pext5-scf2*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$
assumes $CCR\ r$ **and** $A \in SF\ r$ **and** $Ps \subseteq SCF\ r$
shows $\exists\ A'::('U\ set).\ (x \in Field\ r \longrightarrow x \in A')$
$\wedge\ A \subseteq A'$
$\wedge\ A' \in SF\ r$
$\wedge\ (\forall a\in A.\ ((r``\{a\}\subseteq B) \vee (r``\{a\}\cap(A'-B) \neq \{\})))$
$\wedge\ ((\exists\ y::'U.\ A'-B' = \{y\}) \longrightarrow Field\ r \subseteq (A'\cup B'))$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ ((finite\ A \longrightarrow finite\ A') \wedge (\ (\neg\ finite\ A) \longrightarrow |A'| =o\ |A|\ ))$

$\wedge$ ((($\exists$ *P*. *Ps* = {*P*}) $\vee$ (($\neg$ *finite Ps*) $\wedge$ |*Ps*| $\leq o$ |*A*|)) $\longrightarrow$
($\forall$ *P* $\in$ *Ps*. (*A'* $\cap$ *P*) $\in$ *SCF* (*Restr r A'*)))
⟨*proof*⟩

lemma *lem-dnEsc-el*: *F* $\in$ *dnEsc r A a* $\Longrightarrow$ *a* $\in$ *F* $\wedge$ *finite F* ⟨*proof*⟩

lemma *lem-dnEsc-emp*: *dnEsc r A a* = {} $\Longrightarrow$ *dnesc r A a* = { *a* } ⟨*proof*⟩

lemma *lem-dnEsc-ne*: *dnEsc r A a* $\neq$ {} $\Longrightarrow$ *dnesc r A a* $\in$ *dnEsc r A a*
⟨*proof*⟩

lemma *lem-dnesc-in*: *a* $\in$ *dnesc r A a* $\wedge$ *finite* (*dnesc r A a*)
⟨*proof*⟩

lemma *lem-escl-incr*: *B* $\subseteq$ *escl r A B* ⟨*proof*⟩

lemma *lem-escl-card*: (*finite B* $\longrightarrow$ *finite* (*escl r A B*)) $\wedge$ ($\neg$ *finite B* $\longrightarrow$ |*escl r A B*| $\leq o$ |*B*|)
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-set-ext-scf3*:
fixes *r s*::*'U rel* **and** *A A0*::*'U set* **and** *Ps*::*'U set set*
assumes *a1*: *CCR r* **and** *a2*: *s* $\subseteq$ *r* **and** *a3*: $\neg$ *finite s* **and** *a4*: *A* $\subseteq$ *Field r*
and *a5*: |*A*| $\leq o$ |*Field s*| **and** *a6*: *Ps* $\subseteq$ *SCF r* $\wedge$ |*Ps*| $\leq o$ |*Field s*|
shows $\exists$ *s'*::(*'U rel*). *CCR s'* $\wedge$ *s* $\subseteq$ *s'* $\wedge$ *s'* $\subseteq$ *r* $\wedge$ |*s'*| $=o$ |*s*| $\wedge$ *A* $\subseteq$ *Field s'*
$\wedge$ ($\forall$ *P*$\in$*Ps*. (*Field s'* $\cap$ *P*) $\in$ *SCF s'*) $\wedge$ (*escl r A0* (*Field s'*) $\subseteq$ *Field s'*)
$\wedge$ ($\exists$ *D*. *s'* = *Restr r D*) $\wedge$ (*Conelike s'* $\longrightarrow$ *Conelike r*)
⟨*proof*⟩

lemma *lem-Ccext-infsubccr-pext5-scf3*:
fixes *r*::*'U rel* **and** *A B B'*::*'U set* **and** *x*::*'U* **and** *Ps*::*'U set set*
assumes *a1*: *CCR r* **and** *a2*: $\neg$ *finite A* **and** *a3*: *A* $\in$ *SF r* **and** *a4*: *Ps* $\subseteq$ *SCF r*
shows $\exists$ *A'*::(*'U set*). (*x* $\in$ *Field r* $\longrightarrow$ *x* $\in$ *A'*) $\wedge$ *A* $\subseteq$ *A'* $\wedge$ *CCR* (*Restr r A'*) $\wedge$ |*A'*| $=o$ |*A*|
$\wedge$ ($\forall$ *a*$\in$*A*. *r*‘‘{*a*}$\subseteq$*B* $\vee$ *r*‘‘{*a*}$\cap$(*A'*−*B*) $\neq$ {}) $\wedge$ *A'* $\in$ *SF r*
$\wedge$ (($\exists$ *y*::*'U*. *A'*−*B'* $\subseteq$ {*y*}) $\longrightarrow$ *Field r* $\subseteq$ (*A'*$\cup$*B'*))
$\wedge$ (|*Ps*| $\leq o$ |*A*| $\longrightarrow$ ($\forall$ *P* $\in$ *Ps*. (*A'* $\cap$ *P*) $\in$ *SCF* (*Restr r A'*)))
$\wedge$ (*escl r A A'* $\subseteq$ *A'*) $\wedge$ *clterm* (*Restr r A'*) *r*
⟨*proof*⟩

lemma *lem-Ccext-finsubccr-pext5-scf3*:
fixes *r*::*'U rel* **and** *A B B'*::*'U set* **and** *x*::*'U* **and** *Ps*::*'U set set*
assumes *a1*: *CCR r* **and** *a2*: *finite A* **and** *a3*: *A* $\in$ *SF r* **and** *a4*: *Ps* $\subseteq$ *SCF r*
shows $\exists$ *A'*::(*'U set*). (*x* $\in$ *Field r* $\longrightarrow$ *x* $\in$ *A'*) $\wedge$ *A* $\subseteq$ *A'* $\wedge$ *CCR* (*Restr r A'*) $\wedge$ *finite A'*
$\wedge$ ($\forall$ *a*$\in$*A*. *r*‘‘{*a*}$\subseteq$*B* $\vee$ *r*‘‘{*a*}$\cap$(*A'*−*B*) $\neq$ {}) $\wedge$ *A'* $\in$ *SF r*
$\wedge$ (($\exists$ *y*::*'U*. *A'*−*B'* $\subseteq$ {*y*}) $\longrightarrow$ *Field r* $\subseteq$ (*A'*$\cup$*B'*))
$\wedge$ (($\exists$ *P*. *Ps* = {*P*}) $\longrightarrow$ ($\forall$ *P* $\in$ *Ps*. (*A'* $\cap$ *P*) $\in$ *SCF* (*Restr r A'*)))

$\langle proof \rangle$

lemma *lem-Ccext-subccr-pext5-scf3*:
fixes $r::'U\ rel$ **and** $A\ B\ B'::'U\ set$ **and** $x::'U$ **and** $Ps::'U\ set\ set$ **and** $C::'U\ set \Rightarrow bool$
assumes *a1*: $CCR\ r$ **and** *a2*: $A \in SF\ r$ **and** *a3*: $Ps \subseteq SCF\ r$
and *a4*: $C = (\lambda\ A'::'U\ set.\ (x \in Field\ r \longrightarrow x \in A')$
$\wedge\ A \subseteq A'$
$\wedge\ A' \in SF\ r$
$\wedge\ (\forall\ a{\in}A.\ ((r``\{a\}{\subseteq}B) \vee (r``\{a\}{\cap}(A'{-}B) \neq \{\})))$
$\wedge\ ((\exists\ y::'U.\ A'{-}B' \subseteq \{y\}) \longrightarrow Field\ r \subseteq (A'{\cup}B'))$
$\wedge\ CCR\ (Restr\ r\ A')$
$\wedge\ ((finite\ A \longrightarrow finite\ A') \wedge (\ (\neg\ finite\ A) \longrightarrow |A'| =o\ |A|\ ))$
$\wedge\ (\ ((\exists\ P.\ Ps = \{P\}) \vee ((\neg\ finite\ Ps) \wedge |Ps| \leq o\ |A|\ )) \longrightarrow$
$(\forall\ P \in Ps.\ (A' \cap P) \in SCF\ (Restr\ r\ A')))$
$\wedge\ (\ (\neg\ finite\ A) \longrightarrow ((escl\ r\ A\ A' \subseteq A') \wedge (clterm\ (Restr\ r\ A')\ r)))\ )$
shows $\exists\ A'::('U\ set).\ C\ A'$
$\langle proof \rangle$

lemma *lem-acyc-un-emprd*:
fixes $r\ s::\ 'U\ rel$
assumes *a1*: $acyclic\ r \wedge acyclic\ s$ **and** *a2*: $(Range\ r) \cap (Domain\ s) = \{\}$
shows $acyclic\ (r \cup s)$
$\langle proof \rangle$

lemma *lem-spthlen-rtr*: $(a,b) \in r\hat{}* \Longrightarrow (a,b) \in r\hat{}\hat{}(spthlen\ r\ a\ b)$
$\langle proof \rangle$

lemma *lem-spthlen-tr*: $(a,b) \in r\hat{}* \wedge a \neq b \Longrightarrow (a,b) \in r\hat{}\hat{}(spthlen\ r\ a\ b) \wedge spthlen\ r\ a\ b > 0$
$\langle proof \rangle$

lemma *lem-spthlen-min*: $(a,b) \in r\hat{}\hat{}n \Longrightarrow spthlen\ r\ a\ b \leq n$
$\langle proof \rangle$

lemma *lem-spth-inj*:
fixes $r::'U\ rel$ **and** $a\ b::'U$ **and** $f::nat \Rightarrow 'U$ **and** $n::nat$
assumes *a1*: $f \in spth\ r\ a\ b$ **and** *a2*: $n = spthlen\ r\ a\ b$
shows $inj\text{-}on\ f\ \{i.\ i{\leq}n\}$
$\langle proof \rangle$

lemma *lem-rtn-rpth-inj*: $(a,b) \in r\hat{}\hat{}n \Longrightarrow n = spthlen\ r\ a\ b \Longrightarrow \exists\ f\ .\ f \in rpth\ r\ a\ b\ n \wedge inj\text{-}on\ f\ \{i.\ i \leq n\}$
$\langle proof \rangle$

lemma *lem-rtr-rpth-inj*: $(a,b) \in r\hat{}* \Longrightarrow \exists\ f\ n\ .\ f \in rpth\ r\ a\ b\ n \wedge inj\text{-}on\ f\ \{i.\ i \leq n\}$

⟨*proof*⟩

lemma *lem-sum-ind-ex*:
assumes *a1*: $g = (\lambda n::nat.\ \sum i{<}n.\ f\ i)$
and *a2*:$\forall\, i::nat.\ f\ i > 0$
shows $\exists\ n\ k.\ (m::nat) = g\ n + k \wedge k < f\ n$
⟨*proof*⟩

lemma *lem-sum-ind-un*:
assumes *a1*: $g = (\lambda n::nat.\ \sum i{<}n.\ f\ i)$
and *a2*: $\forall\, i::nat.\ f\ i > 0$
and *a3*: $(m::nat) = g\ n + k \wedge k < f\ n$
and *a4*: $m = g\ n' + k' \wedge k' < f\ n'$
shows $n = n' \wedge k = k'$
⟨*proof*⟩

lemma *lem-flatseq*:
fixes $r::'U\ rel$ **and** $xi::nat \Rightarrow\ 'U$
assumes $\forall\, n.\ (xi\ n,\ xi\ (Suc\ n)) \in r\hat{\ }* \wedge (xi\ n \neq xi\ (Suc\ n))$
shows $\exists\ g\ yi.\ (\ \forall\, n.\ (yi\ n,\ yi\ (Suc\ n)) \in r\ )$
$\wedge\ (\forall\ i::nat.\ \forall\ j::nat.\ i < j \longleftrightarrow g\ i < g\ j\ )$
$\wedge\ (\forall\ i::nat.\ yi\ (g\ i) = xi\ i)$
$\wedge\ (\forall\ i::nat.\ inj\text{-}on\ yi\ \{\ k.\ g\ i \leq k \wedge k \leq g\ (Suc\ i)\ \}\ )$
$\wedge\ (\forall\ k::nat.\ \exists\ i::nat.\ g\ i \leq k \wedge Suc\ k \leq g\ (Suc\ i))$
$\wedge\ (\forall\ k\ i\ i'.\ g\ i \leq k \wedge Suc\ k \leq g\ (Suc\ i) \wedge g\ i' \leq k \wedge Suc\ k \leq g\ (Suc\ i') \longrightarrow i = i'\ )$
⟨*proof*⟩

lemma *lem-sv-un3*:
fixes $r1\ r2\ r3::'U\ rel$
assumes $single\text{-}valued\ (r1 \cup r3)$ **and** $single\text{-}valued\ (r2 \cup r3)$ **and** $Field\ r1 \cap Field\ r2 = \{\}$
shows $single\text{-}valued\ (r1 \cup r2 \cup r3)$
⟨*proof*⟩

lemma *lem-cfcomp-d2uset*:
fixes $\kappa::'U\ rel$ **and** $r::'U\ rel$ **and** $W::'U\ rel \Rightarrow\ 'U\ set$ **and** $R::'U\ rel \Rightarrow\ 'U\ rel$
and $S::'U\ rel\ set$
assumes *a1*: $\kappa =o\ cardSuc\ |UNIV::nat\ set|$
and *a3*: $T = \{\ t::'U\ rel.\ t \neq \{\} \wedge CCR\ t \wedge single\text{-}valued\ t \wedge acyclic\ t \wedge (\forall\, x{\in}Field\ t.\ t``\{x\} \neq \{\})\ \}$
and *a4*: $Refl\ r$

and *a5*: $S \subseteq \{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}$
and *a6*: $|\{\alpha \in \mathcal{O}::'U\ rel\ set.\ \alpha <o\ \kappa\}| \leq o\ |S|$
and *a7*: $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$

and *a8*: $Field\ r = (\bigcup \alpha{\in}S.\ W\ \alpha)$ **and** *a9*: $\forall\, \alpha{\in}S.\ \forall\ \beta{\in}S.\ \alpha \neq \beta \longrightarrow W\ \alpha \cap W\ \beta = \{\}$

and *a10*: $\bigwedge \alpha.\ \alpha \in S \Longrightarrow R\ \alpha \in T \wedge R\ \alpha \subseteq r \wedge |W\ \alpha| \leq o\ |UNIV::nat\ set|$
$\wedge$ *Field* $(R\ \alpha) = W\ \alpha \wedge \neg$ *Conelike* (*Restr* r $(W\ \alpha)$)
and *a11*: $\bigwedge \alpha\ x.\ \alpha \in S \Longrightarrow x \in W\ \alpha \Longrightarrow \exists\ a.$
$((x,a) \in (Restr\ r\ (W\ \alpha))\hat{}* \wedge (\forall\ \beta \in S.\ \alpha <o\ \beta \longrightarrow (r``\{a\} \cap W\ \beta)$
$\neq \{\}))$
shows $\exists\ r'.\ CCR\ r' \wedge DCR\ 2\ r' \wedge r' \subseteq r \wedge (\forall\ a \in Field\ r.\ \exists\ b \in Field\ r'.\ (a,b)$
$\in r\hat{}*)$
⟨*proof*⟩

lemma *lem-uset-cl-ext*:
fixes $r::'U\ rel$ **and** $s::'U\ rel$
assumes $s \in \mathfrak{U}\ r$ **and** *Conelike* s
shows *Conelike* r
⟨*proof*⟩

lemma *lem-uset-cl-singleton*:
fixes $r::'U\ rel$
assumes *Conelike* r **and** $r \neq \{\}$
shows $\exists\ m::'U.\ \exists\ m'::'U.\ \{(m',m)\} \in \mathfrak{U}\ r$
⟨*proof*⟩

lemma *lem-rcc-emp*: $\|\{\}\| = \{\}$
⟨*proof*⟩

lemma *lem-rcc-rccrel*:
fixes $r::'U\ rel$
shows *RCC-rel* $r\ \|r\|$
⟨*proof*⟩

lemma *lem-rcc-uset-ne*:
assumes $\mathfrak{U}\ r \neq \{\}$
shows $\exists\ s \in \mathfrak{U}\ r.\ |s| =o\ \|r\| \wedge (\ \forall\ s' \in \mathfrak{U}\ r.\ |s| \leq o\ |s'|\ )$
⟨*proof*⟩

lemma *lem-rcc-uset-emp*:
assumes $\mathfrak{U}\ r = \{\}$
shows $\|r\| = \{\}$
⟨*proof*⟩

lemma *lem-rcc-uset-mem-bnd*:
assumes $s \in \mathfrak{U}\ r$
shows $\|r\| \leq o\ |s|$
⟨*proof*⟩

lemma *lem-rcc-cardord*: *Card-order* $\|r\|$
⟨*proof*⟩

lemma *lem-uset-ne-rcc-inf*:
fixes $r::'U\ rel$

assumes $\neg$ ($\|r\|$ $<o$ ω-*ord*)
shows $\mathfrak{U}$ $r \neq \{\}$
⟨*proof*⟩

lemma *lem-rcc-inf*: (ω-*ord* $\leq o$ $\|r\|$) = ($\neg$ ($\|r\|$ $<o$ ω-*ord*))
⟨*proof*⟩

lemma *lem-Rcc-eq1-12*:
fixes r::$'U$ *rel*
shows *CCR* $r \Longrightarrow r \in \mathfrak{U}$ r
⟨*proof*⟩

lemma *lem-Rcc-eq1-23*:
fixes r::$'U$ *rel*
assumes $r \in \mathfrak{U}$ r
shows (r = ({}::$'U$ *rel*)) $\vee$ (({}::$'U$ *rel*) $<o$ $\|r\|$)
⟨*proof*⟩

lemma *lem-Rcc-eq1-31*:
fixes r::$'U$ *rel*
assumes (r = ({}::$'U$ *rel*)) $\vee$ (({}::$'U$ *rel*) $<o$ $\|r\|$)
shows *CCR* r
⟨*proof*⟩

lemma *lem-Rcc-eq2-12*:
fixes r::$'U$ *rel* **and** a::$'a$
assumes *Conelike* r
shows $\|r\|$ $\leq o$ $|\{a\}|$
⟨*proof*⟩

lemma *lem-Rcc-eq2-23*:
fixes r::$'U$ *rel* **and** a::$'a$
assumes $\|r\|$ $\leq o$ $|\{a\}|$
shows $\|r\|$ $<o$ ω-*ord*
⟨*proof*⟩

lemma *lem-Rcc-eq2-31*:
fixes r::$'U$ *rel*
assumes *CCR* r **and** $\|r\|$ $<o$ ω-*ord*
shows *Conelike* r
⟨*proof*⟩

lemma *lem-Rcc-range*:
fixes r::$'U$ *rel*
shows $\|r\|$ $\leq o$ $|$*UNIV*::($'U$ *set*)$|$
⟨*proof*⟩

lemma *lem-rcc-nccr*:
fixes r::$'U$ *rel*

assumes ¬ (*CCR r*)
shows $\|r\| = \{\}$
⟨*proof*⟩

lemma *lem-Rcc-relcard-bnd*:
fixes *r*::*'U rel*
shows $\|r\| \leq o\ |r|$
⟨*proof*⟩

lemma *lem-Rcc-inf-lim*:
fixes *r*::*'U rel*
assumes ω-*ord* $\leq o\ \|r\|$
shows ¬($\|r\| = \{\}$ ∨ *isSuccOrd* $\|r\|$)
⟨*proof*⟩

lemma *lem-rcc-uset-ne-ccr*:
fixes *r*::*'U rel*
assumes $\mathfrak{U}\ r \neq \{\}$
shows *CCR r*
⟨*proof*⟩

lemma *lem-rcc-uset-tr*:
fixes *r s t*::*'U rel*
assumes *a1*: $s \in \mathfrak{U}\ r$ **and** *a2*: $t \in \mathfrak{U}\ s$
shows $t \in \mathfrak{U}\ r$
⟨*proof*⟩

lemma *lem-scf-emp*: *scf* {} = {}
⟨*proof*⟩

lemma *lem-scf-scfrel*:
fixes *r*::*'U rel*
shows *scf-rel r* (*scf r*)
⟨*proof*⟩

lemma *lem-scf-uset*:
shows ∃ $A \in$ *SCF r*. $|A| =o$ *scf r* ∧ (∀ $B \in$ *SCF r*. $|A| \leq o\ |B|$)
⟨*proof*⟩

lemma *lem-scf-uset-mem-bnd*:
assumes $B \in$ *SCF r*
shows *scf r* $\leq o\ |B|$
⟨*proof*⟩

lemma *lem-scf-cardord*: *Card-order* (*scf r*)
⟨*proof*⟩

lemma *lem-scf-inf*: (ω-*ord* $\leq o$ (*scf r*)) = (¬ ((*scf r*) $<o$ ω-*ord*))
⟨*proof*⟩

lemma *lem-scf-eq1-12*:
fixes *r*::*'U rel*
shows *Field r* $\in$ *SCF r*
⟨*proof*⟩

lemma *lem-scf-range*:
fixes *r*::*'U rel*
shows (*scf r*) $\leq$*o* |*UNIV*::(*'U set*)|
⟨*proof*⟩

lemma *lem-scf-relfldcard-bnd*:
fixes *r*::*'U rel*
shows (*scf r*) $\leq$*o* |*Field r*|
⟨*proof*⟩

lemma *lem-scf-ccr-scf-rcc-eq*:
fixes *r*::*'U rel*
assumes *CCR r*
shows ‖*r*‖ =*o* (*scf r*)
⟨*proof*⟩

lemma *lem-scf-ccr-scf-uset*:
fixes *r*::*'U rel*
assumes *CCR r* **and** ¬ *Conelike r*
shows ∃ *s* ∈ $\mathfrak{U}$ *r*. (¬ *finite s*) ∧ |*Field s*| =*o* (*scf r*)
⟨*proof*⟩

lemma *lem-Scf-scfprops*:
fixes *r*::*'U rel*
shows ((*scf r*) $\leq$*o* |*UNIV*::(*'U set*)|) ∧ ((*scf r*) $\leq$*o* |*Field r*|)
⟨*proof*⟩

lemma *lem-scf-ccr-finscf-cl*:
assumes *CCR r*
shows *finite* (*Field* (*scf r*)) = *Conelike r*
⟨*proof*⟩

lemma *lem-sv-uset-sv-span*:
fixes *r s*::*'U rel*
assumes *a1*: *s* ∈ $\mathfrak{U}$ *r* **and** *a2*: *single-valued s*
shows ∃ *r1*. *r1* ∈ *Span r* ∧ *CCR r1* ∧ *single-valued r1* ∧ *s* ⊆ *r1* ∧ (*acyclic s* ⟶ *acyclic r1*)
⟨*proof*⟩

lemma *lem-incrfun-nat*: ∀ *i*::*nat*. *f i* < *f* (*Suc i*) ⟹ ∀ *i j*. *i* ≤ *j* ⟶ *f i* + (*j*−*i*) ≤ *f j*
⟨*proof*⟩

lemma *lem-sv-uset-rcceqw*:
fixes $r::'U\ rel$
assumes $a1$: $\|r\| =o\ \omega\text{-}ord$
shows $\exists\ r1 \in \mathfrak{U}\ r.\ single\text{-}valued\ r1 \wedge acyclic\ r1 \wedge (\forall\ x \in Field\ r1.\ r1``\{x\} \neq \{\})$
⟨*proof*⟩

lemma *lem-sv-span-scflew*:
fixes $r::'U\ rel$
assumes $CCR\ r$ **and** $scf\ r \leq o\ \omega\text{-}ord$
shows $\exists\ r1.\ r1 \in Span\ r \wedge CCR\ r1 \wedge single\text{-}valued\ r1$
⟨*proof*⟩

lemma *lem-sv-span-scfeqw*:
fixes $r::'U\ rel$
assumes $CCR\ r$ **and** $scf\ r =o\ \omega\text{-}ord$
shows $\exists\ r1.\ r1 \in Span\ r \wedge r1 \neq \{\} \wedge CCR\ r1 \wedge single\text{-}valued\ r1 \wedge acyclic\ r1 \wedge (\forall x{\in}Field\ r1.\ r1``\{x\} \neq \{\})$
⟨*proof*⟩

lemma *lem-Ldo-den-ccr-uset*:
fixes $r\ s::'U\ rel$
assumes $CCR\ s$ **and** $s \subseteq r \wedge Field\ s \in Den\ r$
shows $s \in \mathfrak{U}\ r$
⟨*proof*⟩

lemma *lem-Ldo-ds-reduc*:
fixes $r\ s::'U\ rel$ **and** $n0::nat$
assumes $a1$: $CCR\ s \wedge DCR\ n0\ s$ **and** $a2$: $s \subseteq r$ **and** $a3$: $Field\ s \in Den\ r$ **and** $a4$: $Field\ s \in Inv\ (r - s)$
shows $CCR\ r \wedge DCR\ (Suc\ n0)\ r$
⟨*proof*⟩

lemma *lem-Ldo-sat-reduc*:
fixes $r\ s::'U\ rel$ **and** $n::nat$
assumes $a1$: $s \in Span\ r$ **and** $a2$: $CCR\ s \wedge DCR\ n\ s$
shows $CCR\ r \wedge DCR\ (Suc\ n)\ r$
⟨*proof*⟩

lemma *lem-Ldo-uset-reduc*:
fixes $r\ s::'U\ rel$ **and** $n0::nat$
assumes $a1$: $s \in \mathfrak{U}\ r$ **and** $a2$: $DCR\ n0\ s$ **and** $a3$: $n0 \neq 0$
shows $DCR\ (Suc\ n0)\ r$
⟨*proof*⟩

lemma *lem-Ldo-addid*:
fixes $r::'U\ rel$ **and** $r'::'U\ rel$ **and** $n0::nat$ **and** $A::'U\ set$
assumes $a1$: $DCR\ n0\ r$ **and** $a2$: $r' = r \cup \{(a,b).\ a = b \wedge a \in A\}$ **and** $a3$: $n0 \neq 0$
shows $DCR\ n0\ r'$

⟨*proof*⟩

lemma *lem-Ldo-removeid*:
fixes $r::'U\ rel$ **and** $r'::'U\ rel$ **and** $n0::nat$
assumes *a1*: $DCR\ n0\ r$ **and** *a2*: $r' = r - \{(a,b).\ a = b\}$
shows $DCR\ n0\ r'$
⟨*proof*⟩

lemma *lem-Ldo-eqid*:
fixes $r::'U\ rel$ **and** $r'::'U\ rel$ **and** $n::nat$
assumes *a1*: $DCR\ n\ r$ **and** *a2*: $r' - \{(a,b).\ a = b\} = r - \{(a,b).\ a = b\}$ **and** *a3*: $n \neq 0$
shows $DCR\ n\ r'$
⟨*proof*⟩

lemma *lem-wdn-range-lb*: $A \subseteq w\text{-}dncl\ r\ A$
⟨*proof*⟩

lemma *lem-wdn-range-ub*: $w\text{-}dncl\ r\ A \subseteq dncl\ r\ A$ ⟨*proof*⟩

lemma *lem-wdn-mon*: $A \subseteq A' \Longrightarrow w\text{-}dncl\ r\ A \subseteq w\text{-}dncl\ r\ A'$ ⟨*proof*⟩

lemma *lem-wdn-compl*:
fixes $r::'U\ rel$ **and** $A::'U\ set$
shows $UNIV - w\text{-}dncl\ r\ A = \{a.\ \exists\ b.\ b \notin dncl\ r\ A \wedge (a,b) \in (Restr\ r\ (UNIV{-}A))\hat{}*\}$
⟨*proof*⟩

lemma *lem-cowdn-uset*:
fixes $r::'U\ rel$ **and** $A\ A'\ W::'U\ set$
assumes *a1*: $CCR\ (Restr\ r\ A')$ **and** *a2*: $escl\ r\ A\ A' \subseteq A'$
and *a3*: $Q = A' - dncl\ r\ A$ **and** *a4*: $W = A' - w\text{-}dncl\ r\ A$ **and** *a5*: $Q \in SF\ r$
shows $Restr\ r\ Q \in \mathfrak{U}\ (Restr\ r\ W)$
⟨*proof*⟩

lemma *lem-shrel-L-eq*:
fixes $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$ **and** $\beta::'U\ rel$
assumes $\alpha =o\ \beta$
shows $\mathfrak{L}\ f\ \alpha = \mathfrak{L}\ f\ \beta$
⟨*proof*⟩

lemma *lem-shrel-dbk-eq*:
fixes $f::'U\ rel \Rightarrow 'U\ set$ **and** $Ps::'U\ set\ set$ **and** $\alpha::'U\ rel$ **and** $\beta::'U\ rel$
assumes $f \in \mathcal{N}\ r\ Ps$ **and** $\alpha =o\ \beta$ **and** $\alpha \leq o\ |Field\ r|$ **and** $\beta \leq o\ |Field\ r|$
shows $(\nabla\ f\ \alpha) = (\nabla\ f\ \beta)$
⟨*proof*⟩

lemma *lem-L-emp*: $\alpha =o\ (\{\}::'U\ rel) \Longrightarrow \mathfrak{L}\ f\ \alpha = \{\}$
⟨*proof*⟩

lemma *lem-der-qinv1*:
fixes $r::'U\ rel$ **and** $\alpha::'U\ rel$ **and** $x\ y::'U$
assumes *a1*: $x \in \mathcal{Q}\ r\ f\ \alpha$ **and** *a2*: $(x,y) \in r\hat{}*$ **and** *a3*: $y \in (f\ \alpha)$
shows $y \in \mathcal{Q}\ r\ f\ \alpha$
⟨*proof*⟩

lemma *lem-der-qinv2*:
fixes $r::'U\ rel$ **and** $\alpha::'U\ rel$ **and** $x\ y::'U$
assumes *a1*: $x \in \mathcal{Q}\ r\ f\ \alpha$ **and** *a2*: $(x,y) \in (Restr\ r\ (f\ \alpha))\hat{}*$ **and** *a3*: $y \in (f\ \alpha)$
shows $(x,y) \in (Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha))\hat{}*$
⟨*proof*⟩

lemma *lem-der-qinv3*:
fixes $r::'U\ rel$ **and** $\alpha::'U\ rel$
assumes *a1*: $A \subseteq (f\ \alpha)$ **and** *a2*: $\forall\ x \in (f\ \alpha).\ \exists\ y \in A.\ (x,y) \in (Restr\ r\ (f\ \alpha))\hat{}*$
shows $\forall\ x \in (\mathcal{Q}\ r\ f\ \alpha).\ \exists\ y \in (A \cap (\mathcal{Q}\ r\ f\ \alpha)).\ (x,y) \in (Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha))\hat{}*$
⟨*proof*⟩

lemma *lem-der-inf-qrestr-ccr1*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $\alpha::'U\ rel$
assumes $f \in \mathcal{N}\ r\ Ps$ **and** $\alpha \leq o\ |Field\ r|$
shows $CCR\ (Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha))$
⟨*proof*⟩

lemma *lem-Nfdn-aemp*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $CCR\ r$ **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ scf\ r$ **and** *a4*: $Field\ r \subseteq dncl\ r\ (f\ \alpha)$
shows $\alpha = \{\}$
⟨*proof*⟩

lemma *lem-der-qccr-lscf-sf*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $CCR\ r$ **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ scf\ r$
shows $(\mathcal{Q}\ r\ f\ \alpha) \in SF\ r$
⟨*proof*⟩

lemma *lem-der-q-uset*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $\alpha::'U\ rel$
assumes *a1*: $CCR\ r$ **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ scf\ r$ **and** *a4*: $isSuccOrd\ \alpha$
shows $Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha) \in \mathfrak{U}\ (Restr\ r\ (f\ \alpha))$
⟨*proof*⟩

lemma *lem-qw-range*: $f \in \mathcal{N}\ r\ Ps \Longrightarrow \alpha \leq o\ |Field\ r| \Longrightarrow \mathcal{W}\ r\ f\ \alpha \subseteq Field\ r$
⟨*proof*⟩

lemma *lem-der-qw-eq*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$ **and** $\alpha\ \beta::'U\ rel$

assumes $f \in \mathcal{N}\ r\ Ps$ **and** $\alpha =o\ \beta$
shows $\mathcal{W}\ r\ f\ \alpha = \mathcal{W}\ r\ f\ \beta$
⟨*proof*⟩

lemma *lem-Der-inf-qw-disj*:
fixes r::'*U rel* **and** $\alpha\ \beta$::'*U rel*
assumes *Well-order* α **and** *Well-order* β
shows $(\neg\ (\alpha =o\ \beta)) \longrightarrow (\mathcal{W}\ r\ f\ \alpha) \cap (\mathcal{W}\ r\ f\ \beta) = \{\}$
⟨*proof*⟩

lemma *lem-der-inf-qw-restr-card*:
fixes r::'*U rel* **and** *Ps*::'*U set set* **and** α::'*U rel*
assumes *a1*: ¬ *finite r* **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ |Field\ r|$
shows $|Restr\ r\ (\mathcal{W}\ r\ f\ \alpha)| <o\ |Field\ r|$
⟨*proof*⟩

lemma *lem-QS-subs-WS*: $\mathcal{Q}\ r\ f\ \alpha \subseteq \mathcal{W}\ r\ f\ \alpha$
⟨*proof*⟩

lemma *lem-WS-limord*:
fixes r::'*U rel* **and** *Ps*::'*U set set* **and** f::'*U rel* ⇒ '*U set* **and** α::'*U rel*
assumes *a1*: ¬ *finite r* **and** *a2*: $f \in \mathcal{N}\ r\ Ps$ **and** *a3*: $\alpha <o\ |Field\ r|$
and *a4*: $\neg\ (\alpha = \{\} \vee isSuccOrd\ \alpha)$
shows $\mathcal{W}\ r\ f\ \alpha = \{\}$
⟨*proof*⟩

lemma *lem-der-inf-qw-restr-uset*:
fixes r::'*U rel* **and** *Ps*::'*U set set* **and** f::'*U rel* ⇒ '*U set* **and** α::'*U rel*
assumes *a1*: *Refl r* ∧ ¬ *finite r* **and** *a2*: $f \in \mathcal{N}\ r\ Ps$
and *a3*: $\alpha <o\ |Field\ r|$ **and** *a4*: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$
shows $Restr\ r\ (\mathcal{Q}\ r\ f\ \alpha) \in \mathfrak{U}\ (Restr\ r\ (\mathcal{W}\ r\ f\ \alpha))$
⟨*proof*⟩

lemma *lem-der-inf-qw-restr-ccr*:
fixes r::'*U rel* **and** *Ps*::'*U set set* **and** f::'*U rel* ⇒ '*U set* **and** α::'*U rel*
assumes *a1*: *Refl r* ∧ ¬ *finite r* **and** *a2*: $f \in \mathcal{N}\ r\ Ps$
and *a3*: $\alpha <o\ |Field\ r|$ **and** *a4*: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$
shows $CCR\ (Restr\ r\ (\mathcal{W}\ r\ f\ \alpha))$
⟨*proof*⟩

lemma *lem-der-qw-uset*:
fixes r::'*U rel* **and** *Ps*::'*U set set* **and** f::'*U rel* ⇒ '*U set* **and** α::'*U rel*
assumes *a1*: *CCR r* ∧ *Refl r* ∧ ¬ *finite r* **and** *a2*: $f \in \mathcal{N}\ r\ Ps$
and *a3*: $\alpha <o\ scf\ r$ **and** *a4*: $\omega\text{-}ord \leq o\ |\mathfrak{L}\ f\ \alpha|$ **and** *a5*: *isSuccOrd* α
shows $Restr\ r\ (\mathcal{W}\ r\ f\ \alpha) \in \mathfrak{U}\ (Restr\ r\ (f\ \alpha))$
⟨*proof*⟩

lemma *lem-Shinf-N1*:
fixes r::'*U rel* **and** F::'*U rel* ⇒ '*U set* ⇒ '*U set* **and** f::'*U rel* ⇒ '*U set*

assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
shows $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}1\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N2*:
fixes r::$'U\ rel$ **and** F::$'U\ rel \Rightarrow\ 'U\ set \Rightarrow\ 'U\ set$ **and** f::$'U\ rel \Rightarrow\ 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
shows $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}2\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N3*:
fixes r::$'U\ rel$ **and** F::$'U\ rel \Rightarrow\ 'U\ set \Rightarrow\ 'U\ set$ **and** f::$'U\ rel \Rightarrow\ 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a3*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \in \textit{SF}\ r \longrightarrow$
    $(\omega\textit{-ord} \leq o\ |A| \longrightarrow \textit{escl}\ r\ A\ (F\ \alpha\ A) \subseteq (F\ \alpha\ A) \wedge \textit{clterm}\ (\textit{Restr}\ r\ (F$
$\alpha\ A))\ r)$
shows $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}3\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N4*:
fixes r::$'U\ rel$ **and** F::$'U\ rel \Rightarrow\ 'U\ set \Rightarrow\ 'U\ set$ **and** f::$'U\ rel \Rightarrow\ 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a4*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \in \textit{SF}\ r \longrightarrow (\forall\ a{\in}A.\ r``\{a\} \subseteq \textit{w-dncl}\ r\ A$
$\vee\ r``\{a\} \cap (F\ \alpha\ A\ -\ \textit{w-dncl}\ r\ A) \neq \{\})$
shows $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}4\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N5*:
fixes r::$'U\ rel$ **and** F::$'U\ rel \Rightarrow\ 'U\ set \Rightarrow\ 'U\ set$ **and** f::$'U\ rel \Rightarrow\ 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
assumes *a5*: $\forall\ \alpha\ A.\ (\textit{Well-order}\ \alpha \wedge A \in \textit{SF}\ r) \longrightarrow (F\ \alpha\ A) \in \textit{SF}\ r$
shows $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N6*:
fixes r::$'U\ rel$ **and** F::$'U\ rel \Rightarrow\ 'U\ set \Rightarrow\ 'U\ set$ **and** f::$'U\ rel \Rightarrow\ 'U\ set$
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a6*: $\forall\ \alpha\ A.\ \textit{Well-order}\ \alpha \longrightarrow A \in \textit{SF}\ r \longrightarrow \textit{CCR}\ (\textit{Restr}\ r\ (F\ \alpha\ A))$
shows $\forall \alpha.\ \textit{Well-order}\ \alpha \longrightarrow f \in \mathcal{N}6\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N7*:

fixes r::$'U$ *rel* **and** F::$'U$ *rel* ⇒ $'U$ *set* ⇒ $'U$ *set* **and** f::$'U$ *rel* ⇒ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a7*: $\forall\ \alpha\ A.$ $(\ |A| <o\ \omega\text{-}ord \longrightarrow |F\ \alpha\ A| <o\ \omega\text{-}ord\ )$
    $\wedge\ (\ \omega\text{-}ord \leq o\ |A| \longrightarrow |F\ \alpha\ A| \leq o\ |A|\ )$
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}7\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N8*:
fixes r::$'U$ *rel* **and** F::$'U$ *rel* ⇒ $'U$ *set* ⇒ $'U$ *set* **and** f::$'U$ *rel* ⇒ $'U$ *set* **and** Ps::$'U$ *set set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a7*: $\forall\ \alpha\ A.$ $(\ |A| <o\ \omega\text{-}ord \longrightarrow |F\ \alpha\ A| <o\ \omega\text{-}ord\ )$
    $\wedge\ (\ \omega\text{-}ord \leq o\ |A| \longrightarrow |F\ \alpha\ A| \leq o\ |A|\ )$
  and *a8*: $\forall\alpha\ A.\ A \in SF\ r \longrightarrow \mathcal{E}p\ r\ Ps\ A\ (F\ \alpha\ A)$
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}8\ r\ Ps\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N9*:
fixes r::$'U$ *rel* **and** g::$'U$ *rel* ⇒ $'U$
 and F::$'U$ *rel* ⇒ $'U$ *set* ⇒ $'U$ *set* **and** f::$'U$ *rel* ⇒ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a2*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow g\ \alpha \in$ *Field* $r \longrightarrow g\ \alpha \in F\ \alpha\ A$
  and *a11*: $\omega\text{-}ord \leq o\ |Field\ r| \longrightarrow Field\ r \subseteq g\ `\ \{\ \gamma::'U\ rel.\ \gamma <o\ |Field\ r|\ \}$
shows $f \in \mathcal{N}9\ r\ |Field\ r|$
⟨*proof*⟩

lemma *lem-Shinf-N10*:
fixes r::$'U$ *rel* **and** F::$'U$ *rel* ⇒ $'U$ *set* ⇒ $'U$ *set* **and** f::$'U$ *rel* ⇒ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a10*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \in SF\ r \longrightarrow$
    $((\exists\,y.\ \ (F\ \alpha\ A) - dncl\ r\ A \subseteq \{y\}) \longrightarrow (Field\ r \subseteq dncl\ r\ (F\ \alpha\ A)))$
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}10\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N11*:
fixes r::$'U$ *rel* **and** F::$'U$ *rel* ⇒ $'U$ *set* ⇒ $'U$ *set* **and** f::$'U$ *rel* ⇒ $'U$ *set*
assumes *a0*: $f \in \mathcal{T}\ F$
  and *a1*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \subseteq F\ \alpha\ A$
  and *a5*: $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}5\ r\ \alpha$
  and *a10*: $\forall\ \alpha\ A.$ *Well-order* $\alpha \longrightarrow A \in SF\ r \longrightarrow$
    $((\exists\,y.\ \ (F\ \alpha\ A) - dncl\ r\ A \subseteq \{y\}) \longrightarrow (Field\ r \subseteq dncl\ r\ (F\ \alpha\ A)))$
shows $\forall\alpha.$ *Well-order* $\alpha \longrightarrow f \in \mathcal{N}11\ r\ \alpha$
⟨*proof*⟩

lemma *lem-Shinf-N12*:
fixes $r::'U\ rel$ **and** $g::'U\ rel \Rightarrow 'U$
and $F::'U\ rel \Rightarrow 'U\ set \Rightarrow 'U\ set$ **and** $f::'U\ rel \Rightarrow 'U\ set$
assumes $a0$: $f \in \mathcal{T}\ F$
and $a1$: $\forall\alpha.\ Well\text{-}order\ \alpha \longrightarrow f \in \mathcal{N}1\ r\ \alpha$
and $a2$: $\forall\ \alpha\ A.\ Well\text{-}order\ \alpha \longrightarrow g\ \alpha \in Field\ r \longrightarrow g\ \alpha \in F\ \alpha\ A$
and $a11$: $\omega\text{-}ord \leq o\ |Field\ r| \longrightarrow Field\ r = g\ `\ \{\ \gamma::'U\ rel.\ \gamma <o\ |Field\ r|\ \}$
and $a2'$: $\forall\alpha::'U\ rel.\ \omega\text{-}ord \leq o\ \alpha \wedge \alpha \leq o\ |Field\ r| \longrightarrow \omega\text{-}ord \leq o\ |g\ `\ \{\gamma.\ \gamma <o$
$\alpha\}|$
shows $f \in \mathcal{N}12\ r\ |Field\ r|$
⟨*proof*⟩

lemma *lem-Shinf-E-ne*:
fixes $r::'U\ rel$ **and** $a0::'U$ **and** $A::'U\ set$ **and** $Ps::'U\ set\ set$
assumes $a2$: $CCR\ r$ **and** $a3$: $Ps \subseteq SCF\ r$
shows $\mathcal{E}\ r\ a0\ A\ Ps \neq \{\}$
⟨*proof*⟩

lemma *lem-oseq-fin-inj*:
fixes $g::'U\ rel \Rightarrow 'a$ **and** $I::'U\ rel \Rightarrow 'U\ rel\ set$ **and** $A::'a\ set$
assumes $a1$: $I = (\lambda\ \alpha'.\ \{\ \alpha::'U\ rel.\ \alpha <o\ \alpha'\ \})$
and $a2$: $\omega\text{-}ord \leq o\ |A|$
and $a3$: $\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow g\ \alpha = g\ \beta$
shows $\exists\ h.\ (\forall\ \alpha'.\ g`(I\ \alpha') \subseteq h`(I\ \alpha') \wedge h`(I\ \alpha') \subseteq g`(I\ \alpha') \cup A)$
$\wedge\ (\forall\ \alpha'.\ \omega\text{-}ord \leq o\ \alpha' \longrightarrow \omega\text{-}ord \leq o\ |h`(I\ \alpha')|\ )$
$\wedge\ (\forall\ \alpha\ \beta.\ \alpha =o\ \beta \longrightarrow h\ \alpha = h\ \beta)$
⟨*proof*⟩

lemma *lem-Shinf-N-ne*:
fixes $r::'U\ rel$ **and** $Ps::'U\ set\ set$
assumes $CCR\ r$ **and** $Ps \subseteq SCF\ r$
shows $\mathcal{N}\ r\ Ps \neq \{\}$
⟨*proof*⟩

lemma *lem-wrankrel-eq*: $wrank\text{-}rel\ r\ A0\ \alpha \Longrightarrow \alpha =o\ \beta \Longrightarrow wrank\text{-}rel\ r\ A0\ \beta$
⟨*proof*⟩

lemma *lem-wrank-wrankrel*:
fixes $r::'U\ rel$ **and** $A0::'U\ set$
shows $wrank\text{-}rel\ r\ A0\ (wrank\ r\ A0)$
⟨*proof*⟩

lemma *lem-wrank-uset*:
fixes $r::'U\ rel$ **and** $A0::'U\ set$
shows $\exists\ A \in wbase\ r\ A0.\ |A| =o\ wrank\ r\ A0 \wedge (\ \forall\ B \in wbase\ r\ A0.\ |A| \leq o\ |B|$
$)$
⟨*proof*⟩

lemma *lem-wrank-uset-mem-bnd*:
fixes r::$'U$ *rel* **and** *A0 B*::$'U$ *set*
assumes $B \in$ *wbase r A0*
shows *wrank r A0* $\leq o$ $|B|$
⟨*proof*⟩

lemma *lem-wrank-cardord*: *Card-order* (*wrank r A0*)
⟨*proof*⟩

lemma *lem-wrank-ub*: *wrank r A0* $\leq o$ $|A0|$
⟨*proof*⟩

lemma *lem-card-un2-bnd*: ω-*ord* $\leq o$ $\alpha \Longrightarrow |A| \leq o$ $\alpha \Longrightarrow |B| \leq o$ $\alpha \Longrightarrow |A \cup B|$ $\leq o$ α
⟨*proof*⟩

lemma *lem-card-un2-lsbnd*: ω-*ord* $\leq o$ $\alpha \Longrightarrow |A| < o$ $\alpha \Longrightarrow |B| < o$ $\alpha \Longrightarrow |A \cup B|$ $< o$ α
⟨*proof*⟩

lemma *lem-wrank-un-bnd*:
fixes r::$'U$ *rel* **and** S::$'U$ *set set* **and** α::$'U$ *rel*
assumes *a1*: $\forall$ $A \in S$. *wrank r A* $\leq o$ α **and** *a2*: $|S| \leq o$ α **and** *a3*: ω-*ord* $\leq o$ α
shows *wrank r* $(\bigcup S) \leq o$ α
⟨*proof*⟩

lemma *lem-wrank-un-bnd-stab*:
fixes r::$'U$ *rel* **and** S::$'U$ *set set* **and** α::$'U$ *rel*
assumes *a1*: $\forall$ $A \in S$. *wrank r A* $< o$ α **and** *a2*: $|S| < o$ α **and** *a3*: *stable* α
shows *wrank r* $(\bigcup S) < o$ α
⟨*proof*⟩

lemma *lem-wrank-fw*:
fixes r::$'U$ *rel* **and** K::$'U$ *set* **and** α::$'U$ *rel*
assumes *a1*: ω-*ord* $\leq o$ α **and** *a2*: *wrank r K* $\leq o$ α **and** *a3*: $\forall$ $b \in K$. *wrank r* $(r``\{b\}) \leq o$ α
shows *wrank r* $(\bigcup b \in K. (r``\{b\})) \leq o$ α
⟨*proof*⟩

lemma *lem-wrank-fw-stab*:
fixes r::$'U$ *rel* **and** K::$'U$ *set* **and** α::$'U$ *rel*
assumes *a1*: ω-*ord* $\leq o$ α $\wedge$ *stable* α **and** *a2*: *wrank r K* $< o$ α **and** *a3*: $\forall$ $b \in K$. *wrank r* $(r``\{b\}) < o$ α
shows *wrank r* $(\bigcup b \in K. (r``\{b\})) < o$ α
⟨*proof*⟩

lemma *lem-wnb-neib*:
fixes r::$'U$ *rel* **and** α::$'U$ *rel*
assumes *a1*: ω-*ord* $\leq o$ α **and** *a2*: $\alpha < o$ $\|r\|$

shows ∀ $a \in$ *Field r*. ∃ $b \in$ *Mwn r* α. $(a,b) \in r\hat{}*$
⟨*proof*⟩

lemma *lem-wnb-neib3*:
fixes *r*::′*U rel*
assumes *a1*: ω-*ord* <*o* ‖*r*‖ **and** *a2*: *stable* ‖*r*‖
shows ∀ $a \in$ *Field r*. ∃ $b \in$ *Mwnm r*. $(a,b) \in r\hat{}*$
⟨*proof*⟩

lemma *lem-scfgew-ncl*: ω-*ord* ≤*o scf r* ⟹ ¬ *Conelike r*
⟨*proof*⟩

lemma *lem-wnb-P-ncl-reg-grw*:
fixes *r*::′*U rel*
assumes *a1*: *CCR r* **and** *a2*: ω-*ord* <*o scf r* **and** *a3*: *regularCard* (*scf r*)
shows ∃ $P \in$ *SCF r*. (∀ α::′*U rel*. α <*o scf r* ⟶ (∀ $a \in P$. α <*o wrank r* (*r*‘‘{*a*})
))
⟨*proof*⟩

lemma *lem-wnb-P-ncl-nreg*:
fixes *r*::′*U rel*
assumes *a1*: *CCR r* **and** *a2*: ω-*ord* ≤*o scf r* **and** *a3*: ¬ *regularCard* (*scf r*)
shows ∃ *Ps*::′*U set set*. *Ps* ⊆ *SCF r* ∧ |*Ps*| <*o scf r*
∧ (∀ α::′*U rel*. α <*o scf r* ⟶ (∃ $P \in Ps$. ∀ $a \in P$. α <*o wrank*
r (*r*‘‘{*a*})))
⟨*proof*⟩

lemma *lem-Wf-ext-arc*:
fixes *r*::′*U rel* **and** *Ps*::′*U set set* **and** *f*::′*U rel* ⇒ ′*U set* **and** α::′*U rel* **and** *a*::′*U*
assumes *a1*: *scf r* =*o* |*Field r*| **and** *a2*: $f \in \mathcal{N}$ *r Ps*
and *a3*: ∀ γ::′*U rel*. γ <*o scf r* ⟶ (∀ $a \in P$. γ <*o wrank r* (*r*‘‘{*a*}))
and *a4*: ω-*ord* ≤*o* α **and** *a5*: $a \in f\ \alpha \cap P$
shows ⋀ β. α <*o* β ∧ β <*o* |*Field r*| ∧ (β = {} ∨ *isSuccOrd* β) ⟹ (*r*‘‘{*a*} ∩
($\mathcal{W}$ *r f* β) ≠ {})
⟨*proof*⟩

lemma *lem-Wf-esc-pth*:
fixes *r*::′*U rel* **and** *Ps*::′*U set set* **and** *f*::′*U rel* ⇒ ′*U set* **and** α::′*U rel*
assumes *a1*: *Refl r* ∧ ¬ *finite r* **and** *a2*: $f \in \mathcal{N}$ *r Ps*
and *a3*: ω-*ord* ≤*o* |$\mathfrak{L}$ *f* α| **and** *a4*: α <*o* |*Field r*|
shows ⋀ *F*. *F* ∈ *SCF* (*Restr r* (*f* α)) ⟹
∀ $a \in \mathcal{W}$ *r f* α. ∃ $b \in$ (*F* ∩ ($\mathcal{W}$ *r f* α)). $(a,b) \in$ (*Restr r* ($\mathcal{W}$ *r f* α))$\hat{}*$
⟨*proof*⟩

lemma *lem-Nf-lewfbnd*:
assumes *a1*: $f \in \mathcal{N}$ *r Ps* **and** *a2*: α ≤*o* |*Field r*| **and** *a3*: ω-*ord* ≤*o* |$\mathfrak{L}$ *f* α|
shows ω-*ord* ≤*o* α
⟨*proof*⟩

lemma *lem-regcard-iso*: $\kappa =o\ \kappa' \Longrightarrow regularCard\ \kappa' \Longrightarrow regularCard\ \kappa$
⟨*proof*⟩

lemma *lem-cardsuc-inf-gwreg*: $\neg$ *finite* $A \Longrightarrow \kappa =o\ cardSuc\ |A| \Longrightarrow \omega\text{-}ord <o\ \kappa$
$\wedge$ *regularCard* κ
⟨*proof*⟩

lemma *lem-ccr-rcscf-struct*:
fixes r::$'U$ *rel*
assumes *a1*: *Refl r* **and** *a2*: *CCR r* **and** *a3*: ω-*ord* $<o$ *scf r* **and** *a4*: *regularCard* (*scf r*)
and *a5*: *scf r* $=o$ |*Field r*|
shows $\exists$ *Ps*. $\exists$ $f \in \mathcal{N}$ *r Ps*.
$\forall\alpha$. ω-*ord* $\leq o$ $|\mathfrak{L}\ f\ \alpha| \wedge \alpha <o$ |*Field r*| $\wedge$ *isSuccOrd* $\alpha \longrightarrow$
CCR (*Restr r* ($\mathcal{W}$ *r f* α)) $\wedge$ |*Restr r* ($\mathcal{W}$ *r f* α)| $<o$ |*Field r*|
$\wedge$ ($\forall a \in \mathcal{W}$ *r f* α. *wesc-rel r f* α *a* (*wesc r f* α *a*))
⟨*proof*⟩

lemma *lem-oint-infcard-sc-cf*:
fixes $\alpha 0$::$'a$ *rel* **and** κ::$'U$ *rel* **and** S::$'U$ *rel set*
assumes *a1*: *Card-order* κ **and** *a2*: ω-*ord* $\leq o\ \kappa$
and *a3*: $S = \{\alpha \in \mathcal{O}$::$'U$ *rel set*. $\alpha 0 \leq o\ \alpha \wedge$ *isSuccOrd* $\alpha \wedge \alpha <o\ \kappa\}$
shows $\forall\ \alpha \in S.\ \exists\ \beta \in S.\ \alpha <o\ \beta$
⟨*proof*⟩

lemma *lem-oint-infcard-gew-sc-cfbnd*:
fixes $\alpha 0$::$'a$ *rel* **and** κ::$'U$ *rel* **and** S::$'U$ *rel set*
assumes *a1*: *Card-order* κ **and** *a2*: ω-*ord* $\leq o\ \kappa$ **and** *a3*: $\alpha 0 <o\ \kappa$ **and** *a4*: $\alpha 0$ $=o\ \omega$-*ord*
and *a5*: $S = \{\alpha \in \mathcal{O}$::$'U$ *rel set*. $\alpha 0 \leq o\ \alpha \wedge$ *isSuccOrd* $\alpha \wedge \alpha <o\ \kappa\}$
shows $|\{\alpha \in \mathcal{O}$::$'U$ *rel set*. $\alpha <o\ \kappa\}| \leq o\ |S|$
$\wedge$ ($\exists$ *f*. ($\forall\ \alpha \in \mathcal{O}$::$'U$ *rel set*. $\alpha 0 \leq o\ \alpha \wedge \alpha <o\ \kappa \longrightarrow \alpha \leq o\ f\ \alpha \wedge f\ \alpha \in S$))
⟨*proof*⟩

lemma *lem-rcc-uset-rcc-bnd*:
assumes $s \in \mathfrak{U}\ r$
shows $\|r\| \leq o\ \|s\|$
⟨*proof*⟩

lemma *lem-dc2-ccr-scf-lew*:
fixes r::$'U$ *rel*
assumes *a1*: *CCR r* **and** *a2*: *scf r* $\leq o$ ω-*ord*
shows *DCR 2 r*
⟨*proof*⟩

lemma *lem-dc3-ccr-refl-scf-wsuc*:
fixes r::$'U$ *rel*
assumes *a1*: *Refl r* **and** *a2*: *CCR r*
and *a3*: |*Field r*| $=o$ *cardSuc* |*UNIV*::*nat set*| **and** *a4*: *scf r* $=o$ |*Field r*|

shows *DCR 3 r*
⟨*proof*⟩

lemma *lem-dc3-ccr-scf-lewsuc*:
fixes *r*::′*U rel*
assumes *a1*: *CCR r* **and** *a2*: |*Field r*| ≤*o cardSuc* |*UNIV*::*nat set*|
shows *DCR 3 r*
⟨*proof*⟩

lemma *lem-Cprf-conf-ccr-decomp*:
fixes *r*::′*U rel*
assumes *confl-rel r*
shows ∃ *S*::(′*U rel set*). (∀ *s*∈*S*. *CCR s*) ∧ (*r* = ⋃ *S*) ∧ (∀ *s1*∈*S*. ∀ *s2*∈*S*. *s1* ≠ *s2* ⟶ *Field s1* ∩ *Field s2* = {})
⟨*proof*⟩

lemma *lem-Cprf-dc-disj-fld-un*:
fixes *S*::′*U rel set* **and** *n*::*nat*
assumes *a1*: ∀ *s1*∈*S*. ∀ *s2*∈*S*. *s1*≠*s2* ⟶ *Field s1* ∩ *Field s2* = {}
and *a2*: ∀ *s*∈*S*. *DCR n s*
shows *DCR n* (⋃ *S*)
⟨*proof*⟩

lemma *lem-dc3-to-d3*:
fixes *r*::′*U rel*
assumes *DCR 3 r*
shows *DCR3 r*
⟨*proof*⟩

lemma *lem-dc3-confl-lewsuc*:
fixes *r*::′*U rel*
assumes *a1*: *confl-rel r* **and** *a2*: |*Field r*| ≤*o cardSuc* |*UNIV*::*nat set*|
shows *DCR 3 r*
⟨*proof*⟩

lemma *lem-cle-eqdef*: |*A*| ≤*o* |*B*| = (∃ *g* . *A* ⊆ *g*‘*B*)
⟨*proof*⟩

lemma *lem-cardLeN1-eqdef*:
fixes *A*::′*a set*
shows *cardLeN1 A* = (|*A*| ≤*o cardSuc* |{*n*::*nat* . *True*}|)
⟨*proof*⟩

lemma *lem-cleN1-eqdef*:
fixes *r*::(′*U*×′*U*) *set*
shows (|*r*| ≤*o cardSuc* |{*n*::*nat* . *True*}|)
⟷ (∀ *s* ⊆ *r*. ((∀ *t* ⊆ *s* . ((∃ *t*′ *f*. *t*′ ⊂ *t* ∧ *t* ⊆ *f*‘*t*′) ⟶ (∃ *f*. *s* ⊆ *f*‘*t*)))
∨ (∃ *g* . *r* ⊆ *g*‘*s*)
))

⟨*proof*⟩

1.2.3 Result

The next theorem has the following meaning: if the cardinality of a confluent binary relation r does not exceed the first uncountable cardinal, then confluence of r can be proved with the help of the decreasing diagrams method using no more than 3 labels (e.g. 0, 1, 2 ordered in the usual way).

theorem *thm-main*:
fixes r::($'U \times 'U$) *set*
assumes $\forall\ a\ b\ c\ .\ (a,b) \in r\hat{\ }* \wedge (a,c) \in r\hat{\ }* \longrightarrow (\exists\ d.\ (b,d) \in r\hat{\ }* \wedge (c,d) \in r\hat{\ }*)$
  and $|r| \leq o\ cardSuc\ |\{n::nat\ .\ True\}|$
shows $\exists\ r0\ r1\ r2\ .\ ($
  $(\ r = (r0 \cup r1 \cup r2)\ )$
  $\wedge\ (\ \forall\ a\ b\ c.\ (a,b) \in r0 \wedge (a,c) \in r0$
    $\longrightarrow (\exists\ d.$
      $(b,d) \in r0\hat{\ }=$
      $\wedge\ (c,d) \in r0\hat{\ }=\ )\ )$
  $\wedge\ (\ \forall\ a\ b\ c.\ (a,b) \in r0 \wedge (a,c) \in r1$
    $\longrightarrow (\exists\ b'\ d.$
      $(b,b') \in r1\hat{\ }= \wedge (b',d) \in r0\hat{\ }*$
      $\wedge\ (c,d) \in r0\hat{\ }*\ )\ )$
  $\wedge\ (\ \forall\ a\ b\ c.\ (a,b) \in r1 \wedge (a,c) \in r1$
    $\longrightarrow (\exists\ b'\ b''\ c'\ c''\ d.$
      $(b,b') \in r0\hat{\ }* \wedge (b',b'') \in r1\hat{\ }= \wedge (b'',d) \in r0\hat{\ }*$
      $\wedge\ (c,c') \in r0\hat{\ }* \wedge (c',c'') \in r1\hat{\ }= \wedge (c'',d) \in r0\hat{\ }*\ )\ )$
  $\wedge\ (\ \forall\ a\ b\ c.\ (a,b) \in r0 \wedge (a,c) \in r2$
    $\longrightarrow (\exists\ b'\ d.$
      $(b,b') \in r2\hat{\ }= \wedge (b',d) \in (r0 \cup r1)\hat{\ }*$
      $\wedge\ (c,d) \in (r0 \cup r1)\hat{\ }*\ )\ )$
  $\wedge\ (\ \forall\ a\ b\ c.\ (a,b) \in r1 \wedge (a,c) \in r2$
    $\longrightarrow (\ \exists\ b'\ b''\ d.$
      $(b,b') \in r0\hat{\ }* \wedge (b',b'') \in r2\hat{\ }= \wedge (b'',d) \in (r0 \cup r1)\hat{\ }*$
      $\wedge\ (c,d) \in (r0 \cup r1)\hat{\ }*\ )\ )$
  $\wedge\ (\ \forall\ a\ b\ c.\ (a,b) \in r2 \wedge (a,c) \in r2$
    $\longrightarrow (\exists\ b'\ b''\ c'\ c''\ d.$
      $(b,b') \in (r0 \cup r1)\hat{\ }* \wedge (b',b'') \in r2\hat{\ }= \wedge (b'',d) \in (r0 \cup r1)\hat{\ }*$
      $\wedge\ (c,c') \in (r0 \cup r1)\hat{\ }* \wedge (c',c'') \in r2\hat{\ }= \wedge (c'',d) \in (r0 \cup r1)\hat{\ }*$
$)\ )$
  $)$
⟨*proof*⟩

end

1.3 Optimality of the DCR3 method for proving confluence of relations of the least uncountable cardinality

theory *DCR3-Optimality*
  imports

HOL−Cardinals.Cardinals
Finite-DCR-Hierarchy
begin

1.3.1 Auxiliary definitions

datatype $Lev = l0 \mid l1 \mid l2 \mid l3 \mid l4 \mid l5 \mid l6 \mid l7 \mid l8$

type-synonym $'U\ rD = Lev \times 'U\ set \times 'U\ set \times 'U\ set$

fun $rP :: Lev \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow Lev \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow bool$
where
$rP\ l0\ A\ B\ C\ n'\ A'\ B'\ C' = (A = \{\} \wedge B = \{\} \wedge C = \{\} \wedge n' = l1 \wedge finite\ A' \wedge B' = \{\} \wedge C' = \{\})$
$\mid rP\ l1\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge B = \{\} \wedge C = \{\} \wedge n' = l2 \wedge A' = A \wedge B' = \{\} \wedge C' = \{\})$
$\mid rP\ l2\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge B = \{\} \wedge C = \{\} \wedge n' = l3 \wedge A' = A \wedge finite\ B' \wedge C' = \{\})$
$\mid rP\ l3\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge finite\ B \wedge C = \{\} \wedge n' = l4 \wedge A' = A \wedge B' = B \wedge C' = \{\})$
$\mid rP\ l4\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge finite\ B \wedge C = \{\} \wedge n' = l5 \wedge A' = A \wedge B' = B \wedge finite\ C')$
$\mid rP\ l5\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge finite\ B \wedge finite\ C \wedge n' = l6 \wedge A' = A \wedge B' = B \wedge C' = C)$
$\mid rP\ l6\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge finite\ B \wedge finite\ C \wedge n' = l7 \wedge A' = A \cup B \cup C \wedge B' = A' \wedge C' = A')$
$\mid rP\ l7\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge B = A \wedge C = A \wedge n' = l8 \wedge A' = A \wedge B' = A' \wedge C' = A')$
$\mid rP\ l8\ A\ B\ C\ n'\ A'\ B'\ C' = (finite\ A \wedge B = A \wedge C = A \wedge n' = l7 \wedge A \subset A' \wedge finite\ A' \wedge B' = A' \wedge C' = A')$

definition $rC :: 'U\ set \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow 'U\ set \Rightarrow bool$
where
$rC\ S\ A\ B\ C = (A \subseteq S \wedge B \subseteq S \wedge C \subseteq S)$

definition $rE :: 'U\ set \Rightarrow ('U\ rD)\ rel$
where
$rE\ S = \{\ ((n1,\ A1,\ B1,\ C1),\ (n2,\ A2,\ B2,\ C2)).\ rP\ n1\ A1\ B1\ C1\ n2\ A2\ B2\ C2 \wedge rC\ S\ A1\ B1\ C1 \wedge rC\ S\ A2\ B2\ C2\ \}$

fun $lev\text{-}next :: Lev \Rightarrow Lev$
where
$lev\text{-}next\ l0 = l1$
$\mid lev\text{-}next\ l1 = l2$
$\mid lev\text{-}next\ l2 = l3$
$\mid lev\text{-}next\ l3 = l4$
$\mid lev\text{-}next\ l4 = l5$
$\mid lev\text{-}next\ l5 = l6$

| *lev-next* l*6* = l*7*
| *lev-next* l*7* = l*8*
| *lev-next* l*8* = l*7*

fun *levrd* :: *'U rD* ⇒ *Lev*
where
levrd (*n*, *A*, *B*, *C*) = *n*

fun *wrd* :: *'U rD* ⇒ *'U set*
where
wrd (*n*, *A*, *B*, *C*) = *A* ∪ *B* ∪ *C*

definition *Wrd* :: *'U rD set* ⇒ *'U set*
where
Wrd S = (⋃ (*wrd* ' *S*))

definition *bkset* :: *'a rel* ⇒ *'a set* ⇒ *'a set*
where
bkset r A = ((*r*^*)^−*1*)''*A*

1.3.2 Auxiliary lemmas

lemma *lem-rtr-field*: (*x*,*y*) ∈ *r*^* ⟹ (*x* = *y*) ∨ (*x* ∈ *Field r* ∧ *y* ∈ *Field r*)
⟨*proof*⟩

lemma *lem-fin-fl-rel*: *finite* (*Field r*) = *finite r*
⟨*proof*⟩

lemma *lem-rel-inf-fld-card*:
fixes *r*::*'U rel*
assumes ¬ *finite r*
shows |*Field r*| =*o* |*r*|
⟨*proof*⟩

lemma *lem-confl-field*: *confl-rel r* = (∀ *a* ∈ *Field r*. ∀ *b* ∈ *Field r*. ∀ *c* ∈ *Field r*.
(*a*,*b*) ∈ *r*^* ∧ (*a*,*c*) ∈ *r*^* ⟶
(∃ *d* ∈ *Field r*. (*b*,*d*) ∈ *r*^* ∧ (*c*,*d*) ∈ *r*^*))
⟨*proof*⟩

lemma *lem-d2-to-dc2*:
fixes *r*::*'U rel*
assumes *DCR2 r*
shows *DCR 2 r*
⟨*proof*⟩

lemma *lem-dc2-to-d2*:
fixes *r*::*'U rel*
assumes *DCR 2 r*
shows *DCR2 r*

⟨*proof*⟩

lemma *lem-rP-inv*: $rP\ n\ A\ B\ C\ n'\ A'\ B'\ C' \Longrightarrow (\ A \subseteq A' \wedge B \subseteq B' \wedge C \subseteq C'$
$\wedge$ *finite* $A \wedge$ *finite* $B \wedge$ *finite* $C \wedge$ *finite* $A' \wedge$ *finite* $B' \wedge$ *finite* C')
⟨*proof*⟩

lemma *lem-infset-finext*:
fixes S::$'U$ *set* **and** A::$'U$ *set*
assumes $\neg$ *finite* S **and** *finite* A **and** $A \subseteq S$
shows $\exists\ B.\ B \subseteq S \wedge A \subset B \wedge$ *finite* B
⟨*proof*⟩

lemma *lem-rE-df*:
fixes S::$'U$ *set*
shows $(u,v) \in rE\ S \Longrightarrow (u,w) \in rE\ S \Longrightarrow (v,t) \in (rE\ S)\hat{}{=} \Longrightarrow (w,t) \in (rE\ S)\hat{}{=} \Longrightarrow v = w$
⟨*proof*⟩

lemma *lem-rE-succ-lev*:
fixes S::$'U$ *set*
assumes $(u,v) \in rE\ S$
shows *levrd* v = (*lev-next* (*levrd* u))
⟨*proof*⟩

lemma *lem-rE-levset-inv*:
fixes S::$'U$ *set* **and** $L\ u\ v$
assumes *a1*: $(u,v) \in (rE\ S)\hat{}*$ **and** *a2*: *levrd* $u \in L$ **and** *a3*: *lev-next* ‘ $L \subseteq L$
shows *levrd* $v \in L$
⟨*proof*⟩

lemma *lem-rE-levun*:
fixes S::$'U$ *set*
shows $u \in$ *Domain* $(rE\ S) \Longrightarrow$ *levrd* $u \in \{l1, l3, l5\} \Longrightarrow \exists\ v.\ (rE\ S)``\{u\} \subseteq \{v\}$
⟨*proof*⟩

lemma *lem-rE-domfield*:
fixes S::$'U$ *set*
assumes $\neg$ *finite* S
shows *Domain* $(rE\ S)$ = *Field* $(rE\ S)$
⟨*proof*⟩

lemma *lem-wrd-fin-field-rE*:
fixes S::$'U$ *set*
assumes $\neg$ *finite* S
shows $u \in$ *Field* $(rE\ S) \Longrightarrow$ *finite* (*wrd* u)
⟨*proof*⟩

lemma *lem-rE-rtr-wrd-mon*:
fixes S::$'U$ *set* **and** $u\ v$::$'U\ rD$

shows $(u,v) \in (rE\ S)\hat{}* \Longrightarrow wrd\ u \subseteq wrd\ v$
⟨*proof*⟩

lemma *lem-Wrd-bkset-rE*: $Wrd\ (bkset\ (rE\ S)\ U) = Wrd\ U$
⟨*proof*⟩

lemma *lem-Wrd-rE-field-subs-cnt*:
fixes $S::'U\ set$ **and** $U::('U\ rD)\ set$
assumes $\neg$ *finite S*
shows $U \subseteq Field\ (rE\ S) \Longrightarrow |U| \leq o\ |UNIV::nat\ set| \Longrightarrow |Wrd\ U| \leq o\ |UNIV::nat\ set|$
⟨*proof*⟩

lemma *lem-rE-dn-cnt*:
fixes $S::'U\ set$ **and** $U::('U\ rD)\ set$
assumes $\neg$ *finite S*
shows $U \subseteq Field\ (rE\ S) \Longrightarrow |U| \leq o\ |UNIV::nat\ set| \Longrightarrow V \subseteq bkset\ (rE\ S)\ U \Longrightarrow |Wrd\ V| \leq o\ |UNIV::nat\ set|$
⟨*proof*⟩

lemma *lem-rE-succ-Wrd-univ*: $(u,w) \in (rE\ S) \Longrightarrow levrd\ u \in \{l0, l2, l4\} \Longrightarrow S - wrd\ w \subseteq Wrd\ (((rE\ S)``\{u\}) - \{w\})$
⟨*proof*⟩

lemma *lem-rE-succ-nocntbnd*:
fixes $S::'U\ set$ **and** $u0::'U\ rD$ **and** $v0::'U\ rD$ **and** $U::('U\ rD)\ set$
assumes *a0*: $\neg\ |S| \leq o\ |UNIV::nat\ set|$ **and** *a1*: $(u0,\ v0) \in (rE\ S)$ **and** *a2*: $levrd\ u0 \in \{l0, l2, l4\}$
  and *a3*: $U \subseteq Field\ (rE\ S)$ **and** *a4*: $((rE\ S)\ ``\ \{u0\}) - \{v0\} \subseteq bkset\ (rE\ S)\ U$
shows $\neg\ |U| \leq o\ |UNIV::nat\ set|$
⟨*proof*⟩

lemma *lem-rE-succ-nocntbnd2*:
fixes $S::'U\ set$ **and** $u0::'U\ rD$ **and** $v0::'U\ rD$
assumes *a0*: $\neg\ |S| \leq o\ |UNIV::nat\ set|$
  and *a1*: $(u0,\ v0) \in (rE\ S)$ **and** *a2*: $levrd\ u0 \in \{l0, l2, l4\}$
  and *a3*: $r \subseteq (rE\ S)$ **and** *a4*: $\forall\ u.\ |r``\{u\}| \leq o\ |UNIV::nat\ set|$
  and *a5*: $((rE\ S)\ ``\ \{u0\}) - \{v0\} \subseteq bkset\ (rE\ S)\ ((r\hat{}*)``\{u0\})$
shows *False*
⟨*proof*⟩

lemma *lem-rE-diamsubr-un*:
fixes $S::'U\ set$
assumes *a1*: $r0 \subseteq (rE\ S)$ **and** *a2*: $\forall\ a\ b\ c.\ (a,b) \in r0 \wedge (a,c) \in r0 \longrightarrow (\exists\ d.\ (b,d) \in r0\hat{}= \wedge (c,d) \in r0\hat{}=)$
shows $\forall\ u.\ \exists\ v.\ r0``\{u\} \subseteq \{v\}$
⟨*proof*⟩

lemma *lem-rE-succ-nocntbnd3*:

fixes $S::'U\ set$ **and** $u0::'U\ rD$ **and** $v0::'U\ rD$
assumes $a0$: $\neg\ |S| \leq o\ |UNIV::nat\ set|$
  and $a1$: $LD2\ (rE\ S)\ r0\ r1$
  and $a2$: $(u0,\ v0) \in (rE\ S)$ **and** $a3$: $levrd\ u0 \in \{l0,\ l2,\ l4\}$
  and $a4$: $r = \{(u,v) \in rE\ S.\ u = v0\} \cup r0$
  and $a5$: $((rE\ S)\ ``\ \{u0\}) - \{v0\} \subseteq bkset\ (rE\ S)\ ((r\hat{}*)``\{u0\})$
shows $False$
$\langle proof \rangle$

lemma *lem-rE-one*:
fixes $S::'U\ set$ **and** $u0::'U\ rD$ **and** $v0::'U\ rD$
assumes $a0$: $\neg\ |S| \leq o\ |UNIV::nat\ set|$ **and** $a1$: $LD2\ (rE\ S)\ r0\ r1$
  and $a2$: $(u0,\ v0) \in r0$ **and** $a3$: $levrd\ u0 \in \{l0,\ l2,\ l4\}$
shows $False$
$\langle proof \rangle$

lemma *lem-rE-jn0*:
fixes $S::'U\ set$ **and** $u1::'U\ rD$ **and** $u2::'U\ rD$ **and** $v::'U\ rD$
assumes $a1$: $(u1,v) \in (rE\ S)$ **and** $a2$: $(u2,v) \in (rE\ S)$ **and** $a3$: $u1 \neq u2$
shows $levrd\ v \in \{l7,\ l8\}$
$\langle proof \rangle$

lemma *lem-rE-jn1*:
fixes $S::'U\ set$ **and** $u1::'U\ rD$ **and** $u2::'U\ rD$ **and** $v::'U\ rD$
assumes $a1$: $(u1,v) \in (rE\ S)$ **and** $a2$: $(u2,v) \in (rE\ S)\hat{}*$ **and** $a3$: $(u1,u2) \notin (rE\ S) \wedge (u2,u1) \notin (rE\ S)\hat{}*$
shows $levrd\ v \in \{l7,\ l8\}$
$\langle proof \rangle$

lemma *lem-rE-jn2*:
fixes $S::'U\ set$ **and** $u1::'U\ rD$ **and** $u2::'U\ rD$ **and** $v::'U\ rD$
assumes $a1$: $(u1,v) \in (rE\ S)\hat{}*$ **and** $a2$: $(u2,v) \in (rE\ S)\hat{}*$ **and** $a3$: $(u1,u2) \notin (rE\ S)\hat{}* \wedge (u2,u1) \notin (rE\ S)\hat{}*$
shows $levrd\ v \in \{l7,\ l8\}$
$\langle proof \rangle$

lemma *lem-rel-pow2fw*: $(u,u1) \in r \wedge (u1,v) \in r \longrightarrow (u,v) \in r\hat{}\hat{}2$
$\langle proof \rangle$

lemma *lem-rel-pow3fw*: $(u,u1) \in r \wedge (u1,u2) \in r \wedge (u2,v) \in r \longrightarrow (u,v) \in r\hat{}\hat{}3$
$\langle proof \rangle$

lemma *lem-rel-pow3*: $(u,v) \in r\hat{}\hat{}3 \Longrightarrow \exists\ u1\ u2.\ (u,u1) \in r \wedge (u1,u2) \in r \wedge (u2,v) \in r$
$\langle proof \rangle$

lemma *lem-rel-pow4*: $(u,v) \in r\hat{}\hat{}4 \Longrightarrow \exists\ u1\ u2\ u3.\ (u,u1) \in r \wedge (u1,u2) \in r \wedge (u2,u3) \in r \wedge (u3,v) \in r$
$\langle proof \rangle$

lemma *lem-rel-pow5*: $(u,v) \in r^{\frown\frown}5 \Longrightarrow \exists\ u1\ u2\ u3\ u4.\ (u,u1) \in r \wedge (u1,u2) \in r \wedge (u2,u3) \in r \wedge (u3,u4) \in r \wedge (u4,v) \in r$
⟨*proof*⟩

lemma *lem-rE-l1-l78-dist*:
fixes $S::'U\ set$
assumes *a1*: $levrd\ u = \mathrm{l}1$ **and** *a2*: $levrd\ v \in \{\mathrm{l}7, \mathrm{l}8\}$ **and** *a3*: $n \leq 5$
shows $(u,v) \notin (rE\ S)^{\frown\frown}n$
⟨*proof*⟩

lemma *lem-rE-notLD2*:
fixes $S::'U\ set$ **and** $r0\ r1::('U\ rD)\ rel$
assumes *a0*: $\neg\ |S| \leq o\ |UNIV::nat\ set|$ **and** *a1*: $LD2\ (rE\ S)\ r0\ r1$
shows *False*
⟨*proof*⟩

lemma *lem-rE-dominv*:
fixes $S::'U\ set$
assumes $\neg$ *finite S*
shows $u \in Domain\ (rE\ S) \Longrightarrow (u,v) \in (rE\ S)^{\frown}{*} \Longrightarrow v \in Domain\ (rE\ S)$
⟨*proof*⟩

lemma *lem-rE-next*:
fixes $S::'U\ set$
assumes $\neg$ *finite S* **and** $u \in Domain\ (rE\ S)$
shows $\exists\ v.\ (u,v) \in (rE\ S) \wedge v \in Domain\ (rE\ S) \wedge levrd\ v = (lev\text{-}next\ (levrd\ u))$
⟨*proof*⟩

lemma *lem-rE-reachl8*:
fixes $S::'U\ set$
assumes $\neg$ *finite S* **and** $u \in Domain\ (rE\ S)$
shows $\exists\ v.\ (u,v) \in (rE\ S)^{\frown}{*} \wedge v \in Domain\ (rE\ S) \wedge levrd\ v = \mathrm{l}8$
⟨*proof*⟩

lemma *lem-rE-jn*:
fixes $S::'U\ set$
assumes *a0*: $\neg$ *finite S* **and** *a1*: $u1 \in Domain\ (rE\ S)$ **and** *a2*: $u2 \in Domain\ (rE\ S)$
shows $\exists\ t.\ (u1,t) \in (rE\ S)^{\frown}{*} \wedge (u2,t) \in (rE\ S)^{\frown}{*}$
⟨*proof*⟩

lemma *lem-rE-confl*:
fixes $S::'U\ set$
assumes $\neg$ *finite S*
shows *confl-rel* $(rE\ S)$
⟨*proof*⟩

lemma *lem-rE-dc3dc2*:

fixes *S*::′*U set*
assumes ¬ |*S*| ≤*o* |*UNIV*::*nat set*|
shows *confl-rel* (*rE S*) ∧ (¬ *DCR2* (*rE S*))
⟨*proof*⟩

lemma *lem-rE-cardbnd*:
fixes *S*::′*U set*
assumes ¬ *finite S*
shows |*rE S*| ≤*o* |*S*|
⟨*proof*⟩

lemma *lem-fmap-rel*:
fixes *f r s a0 b0*
assumes *a1*: (*a0*, *b0*) ∈ *r*^∗ **and** *a2*: ∀ *a b*. (*a*,*b*) ∈ *r* ⟶ (*f a*, *f b*) ∈ *s*
shows (*f a0*, *f b0*) ∈ *s*^∗
⟨*proof*⟩

lemma *lem-fmap-confl*:
fixes *r*::′*a rel* **and** *f*::′*a* ⇒ ′*b*
assumes *a1*: *inj-on f* (*Field r*) **and** *a2*: *confl-rel r*
shows *confl-rel* {(*u*,*v*). ∃ *a b*. *u* = *f a* ∧ *v* = *f b* ∧ (*a*,*b*) ∈ *r*}
⟨*proof*⟩

lemma *lem-fmap-dcn*:
fixes *r*::′*a rel* **and** *f*::′*a* ⇒ ′*b*
assumes *a1*: *inj-on f* (*Field r*) **and** *a2*: *DCR n r*
shows *DCR n* {(*u*,*v*). ∃ *a b*. *u* = *f a* ∧ *v* = *f b* ∧ (*a*,*b*) ∈ *r*}
⟨*proof*⟩

lemma *lem-not-dcr2*:
assumes *cardSuc* |*UNIV*::*nat set*| ≤*o* |*UNIV*::′*U set*|
shows ∃ *r*::′*U rel*. *confl-rel r* ∧ |*r*| ≤*o cardSuc* |*UNIV*::*nat set*| ∧ (¬ *DCR2 r*)
⟨*proof*⟩

1.3.3 Result

The next theorem has the following meaning: if the set of elements of type ′*U* is uncountable, then there exists a confluent binary relation *r* on ′*U* such that the cardinality of *r* does not exceed the first uncountable cardinal and confluence of *r* cannot be proved using the decreasing diagrams method with 2 labels.

theorem *thm-example-not-dcr2*:
assumes *cardSuc* |{*n*::*nat*. *True*}| ≤*o* |{*x*::′*U*. *True*}|
shows ∃ *r*::′*U rel*. (
    (∀ *a b c*. (*a*,*b*) ∈ *r*^∗ ∧ (*a*,*c*) ∈ *r*^∗ ⟶ (∃ *d*. (*b*,*d*) ∈ *r*^∗ ∧ (*c*,*d*) ∈ *r*^∗)
)
  ∧ |*r*| ≤*o cardSuc* |{*n*::*nat*. *True*}|
  ∧ (¬ (∃ *r0 r1*. (

```
        ( r = (r0 ∪ r1) )
      ∧ (∀ a b c. (a,b) ∈ r0 ∧ (a,c) ∈ r0
          ⟶ (∃ d.
                (b,d) ∈ r0^=
              ∧ (c,d) ∈ r0^=) )
      ∧ (∀ a b c. (a,b) ∈ r0 ∧ (a,c) ∈ r1
          ⟶ (∃ b' d.
                (b,b') ∈ r1^= ∧ (b',d) ∈ r0^*
              ∧ (c,d) ∈ r0^*) )
      ∧ (∀ a b c. (a,b) ∈ r1 ∧ (a,c) ∈ r1
          ⟶ (∃ b' b'' c' c'' d.
                (b,b') ∈ r0^* ∧ (b',b'') ∈ r1^= ∧ (b'',d) ∈ r0^*
              ∧ (c,c') ∈ r0^* ∧ (c',c'') ∈ r1^= ∧ (c'',d) ∈ r0^*) ) ) )
     ) )
⟨proof⟩
```

corollary *cor-example-not-dcr2*:

```
shows ∃ r::(nat set) rel. (
     ( ∀ a b c. (a,b) ∈ r^* ∧ (a,c) ∈ r^* ⟶ (∃ d. (b,d) ∈ r^* ∧ (c,d) ∈ r^*)
)
    ∧ |r| ≤o cardSuc |{n::nat. True}|
    ∧ (¬ ( ∃r0 r1. (
        ( r = (r0 ∪ r1) )
      ∧ (∀ a b c. (a,b) ∈ r0 ∧ (a,c) ∈ r0
          ⟶ (∃ d.
                (b,d) ∈ r0^=
              ∧ (c,d) ∈ r0^=) )
      ∧ (∀ a b c. (a,b) ∈ r0 ∧ (a,c) ∈ r1
          ⟶ (∃ b' d.
                (b,b') ∈ r1^= ∧ (b',d) ∈ r0^*
              ∧ (c,d) ∈ r0^*) )
      ∧ (∀ a b c. (a,b) ∈ r1 ∧ (a,c) ∈ r1
          ⟶ (∃ b' b'' c' c'' d.
                (b,b') ∈ r0^* ∧ (b',b'') ∈ r1^= ∧ (b'',d) ∈ r0^*
              ∧ (c,c') ∈ r0^* ∧ (c',c'') ∈ r1^= ∧ (c'',d) ∈ r0^*) ) ) )
     ) )
⟨proof⟩
```

end

1.4 DCR implies LD Property

theory *Main-Result-DCR-N1*
  imports
    DCR3-Method
    Decreasing−Diagrams.Decreasing-Diagrams
begin

1.4.1 Auxiliary definitions

definition *map-seq-labels* :: $('b \Rightarrow 'c) \Rightarrow ('a,'b)\ seq \Rightarrow ('a,'c)\ seq$
where
map-seq-labels $f\ \sigma = (fst\ \sigma,\ map\ (\lambda(\alpha,a).\ (f\ \alpha,\ a))\ (snd\ \sigma))$

fun *map-diag-labels* :: $('b \Rightarrow 'c) \Rightarrow$
$('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq \Rightarrow$
$('a,'c)\ seq \times ('a,'c)\ seq \times ('a,'c)\ seq \times ('a,'c)\ seq$
where
map-diag-labels $f\ (\tau,\sigma,\sigma',\tau') = ((map\text{-}seq\text{-}labels\ f\ \tau), (map\text{-}seq\text{-}labels\ f\ \sigma), (map\text{-}seq\text{-}labels\ f\ \sigma'), (map\text{-}seq\text{-}labels\ f\ \tau'))$

fun *f-to-ls* :: $(nat \Rightarrow 'a) \Rightarrow nat \Rightarrow 'a\ list$
where
f-to-ls $f\ 0 = []$
| *f-to-ls* $f\ (Suc\ n) = (f\text{-}to\text{-}ls\ f\ n)\ @\ [(f\ n)]$

1.4.2 Auxiliary lemmas

lemma *lem-ftofs-len*: $length\ (f\text{-}to\text{-}ls\ f\ n) = n\ \langle proof\rangle$

lemma *lem-irr-inj-im-irr*:
fixes $r::'a\ rel$ **and** $r'::'b\ rel$ **and** $f::'a \Rightarrow 'b$
assumes *irrefl* r **and** *inj-on* $f\ (Field\ r)$
and $r' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \land b' = f\ b \land (a,b) \in r\}$
shows *irrefl* r'
$\langle proof\rangle$

lemma *lem-tr-inj-im-tr*:
fixes $r::'a\ rel$ **and** $r'::'b\ rel$ **and** $f::'a \Rightarrow 'b$
assumes *trans* r **and** *inj-on* $f\ (Field\ r)$
and $r' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \land b' = f\ b \land (a,b) \in r\}$
shows *trans* r'
$\langle proof\rangle$

lemma *lem-lpeak-expr*: *local-peak* $lrs\ (\tau,\ \sigma) = (\exists\ a\ b\ c\ \alpha\ \beta.\ (a,\alpha,b) \in lrs \land (a,\beta,c) \in lrs \land \tau = (a,[(\alpha,b)]) \land \sigma = (a,[(\beta,c)]))$
$\langle proof\rangle$

lemma *lem-map-seq*:
fixes $lrs::('a,'b)\ lars$ **and** $f::'b \Rightarrow 'c$ **and** $lrs'::('a,'c)\ lars$ **and** $\sigma::('a,'b)\ seq$
assumes *a1*: $lrs' = \{(a,l',b).\ \exists\, l.\ l' = f\ l \land (a,l,b) \in lrs\ \}$
and *a2*: $\sigma \in$ *Decreasing-Diagrams.seq* lrs
shows $(map\text{-}seq\text{-}labels\ f\ \sigma) \in$ *Decreasing-Diagrams.seq* lrs'
$\langle proof\rangle$

lemma *lem-map-diag*:
fixes $lrs::('a,'b)\ lars$ **and** $f::'b \Rightarrow 'c$ **and** $lrs'::('a,'c)\ lars$
and $d::('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq \times ('a,'b)\ seq$

assumes *a1*: $lrs' = \{(a,l',b).\ \exists l.\ l' = f\ l \wedge (a,l,b) \in lrs\ \}$
and *a2*: *diagram lrs d*
shows *diagram lrs'* (*map-diag-labels f d*)
⟨*proof*⟩

lemma *lem-map-D-loc*:
fixes *cmp cmp' s1 s2 s3 s4 f*
assumes *a1*: *Decreasing-Diagrams.D cmp s1 s2 s3 s4*
and *a2*: *trans cmp* **and** *a3*: *irrefl cmp* **and** *a4*: *inj-on f* (*Field cmp*)
and *a5*: $cmp' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \wedge b' = f\ b \wedge (a,b) \in cmp\}$
and *a6*: *length s1* = *1* **and** *a7*: *length s2* = *1*
shows *Decreasing-Diagrams.D cmp'* (*map f s1*) (*map f s2*) (*map f s3*) (*map f s4*)
⟨*proof*⟩

lemma *lem-map-DD-loc*:
fixes *lrs*::('*a*,'*b*) *lars* **and** *cmp*::'*b rel* **and** *lrs'*::('*a*,'*c*) *lars* **and** *cmp'*::'*c rel* **and** *f*::'*b* ⇒ '*c*
assumes *a1*: *trans cmp* **and** *a2*: *irrefl cmp* **and** *a3*: *inj-on f* (*Field cmp*)
and *a4*: $cmp' = \{(a',b').\ \exists\ a\ b.\ a' = f\ a \wedge b' = f\ b \wedge (a,b) \in cmp\}$
and *a5*: $lrs' = \{(a,l',b).\ \exists l.\ l' = f\ l \wedge (a,l,b) \in lrs\ \}$
and *a6*: *length* (*snd* (*fst d*)) = *1* **and** *a7*: *length* (*snd* (*fst* (*snd d*))) = *1*
and *a8*: *DD lrs cmp d*
shows *DD lrs' cmp'* (*map-diag-labels f d*)
⟨*proof*⟩

lemma *lem-ddseq-mon*: $lrs1 \subseteq lrs2 \Longrightarrow$ *Decreasing-Diagrams.seq lrs1* ⊆ *Decreasing-Diagrams.seq lrs2*
⟨*proof*⟩

lemma *lem-dd-D-mon*:
fixes *cmp1 cmp2* α β *s1 s2*
assumes *a1*: *trans cmp1* ∧ *irrefl cmp1* **and** *a2*: *trans cmp2* ∧ *irrefl cmp2* **and** *a3*: *cmp1* ⊆ *cmp2*
and *a4*: *Decreasing-Diagrams.D cmp1* [α] [β] *s1 s2*
shows *Decreasing-Diagrams.D cmp2* [α] [β] *s1 s2*
⟨*proof*⟩

1.4.3 Result

The next lemma has the following meaning: every ARS in the finite DCR hierarchy has the LD property.

lemma *lem-dcr-to-ld*:
fixes *n*::*nat* **and** *r*::'*U rel*
assumes *DCR n r*
shows *LD* (*UNIV*::*nat set*) *r*
⟨*proof*⟩

2 Main theorem

The next theorem has the following meaning: if the cardinality of a binary relation r does not exceed the first uncountable cardinal ($cardSuc\ |UNIV::nat\ set|$), then the following two conditions are equivalent:

1. r is confluent ($Abstract\text{-}Rewriting.CR\ r$)

2. r can be proven confluent using the decreasing diagrams method with natural numbers as labels ($Decreasing\text{-}Diagrams.LD\ (UNIV::nat\ set)\ r$).

theorem *N1-completeness*:
fixes $r::'a\ rel$
assumes $|r| \leq o\ cardSuc\ |UNIV::nat\ set|$
shows $Abstract\text{-}Rewriting.CR\ r = Decreasing\text{-}Diagrams.LD\ (UNIV::nat\ set)\ r$
$\langle proof \rangle$

end

References

[1] I. Ivanov. Formal proof of completeness of the decreasing diagrams method for proving confluence of relations of the least uncountable cardinality, 2024. https://doi.org/10.5281/zenodo.14254256, Formal proof development.

[2] I. Ivanov. Formalization of an abstract rewriting system in the class $DCR_3 \backslash DCR_2$, 2024. https://doi.org/10.5281/zenodo.11571490, Formal proof development.

[3] I. Ivanov. On non-triviality of the hierarchy of decreasing Church-Rosser abstract rewriting systems. In *Proceedings of the 13th International Workshop on Confluence*, pages 30–35, 2024.

[4] I. Ivanov. Formalization of a confluent abstract rewriting system of the least uncountable cardinality outside of the class DCR_2, 2025. https://doi.org/10.5281/zenodo.14740062, Formal proof development.

[5] I. Ivanov. Modified version of a formal proof of completeness of the decreasing diagrams method for proving confluence of relations of the least uncountable cardinality, 2025. https://doi.org/10.5281/zenodo.15190469, Formal proof development.

[6] C. Sternagel and R. Thiemann. Abstract rewriting. *Archive of Formal Proofs*, June 2010. https://isa-afp.org/entries/Abstract-Rewriting.html, Formal proof development.

[7] V. Van Oostrom. Confluence by decreasing diagrams. *Theoretical computer science*, 126(2):259–280, 1994.

[8] H. Zankl. Decreasing diagrams. *Archive of Formal Proofs*, November 2013. https://isa-afp.org/entries/Decreasing-Diagrams.html, Formal proof development.