Combinatorics on Words formalized
Basics

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theory *Arithmetical-Hints*
imports *Main*
begin

0.1 Arithmetical hints

In this section we give some specific auxiliary lemmas on natural numbers.

lemma *zero-diff-eq*: $i \leq j \Longrightarrow (0::nat) = j - i \Longrightarrow j = i$
⟨*proof*⟩

lemma *zero-less-diff'*: $i < j \Longrightarrow j - i \neq (0::nat)$
⟨*proof*⟩

lemma *nat-prod-le*: $m \neq (0 :: nat) \Longrightarrow m*n \leq k \Longrightarrow n \leq k$
⟨*proof*⟩

lemma *get-div*: $(p :: nat) < a \Longrightarrow m = (m * a + p)\ div\ a$
⟨*proof*⟩

lemma *get-mod*: $(p :: nat) < a \Longrightarrow p = (m * a + p)\ mod\ a$
⟨*proof*⟩

lemma *plus-one-between*: $(a :: nat) < b \Longrightarrow \neg\ b < a + 1$
⟨*proof*⟩

lemma *quotient-smaller*: $k \neq (0 :: nat) \Longrightarrow\ b \leq k * b$
⟨*proof*⟩

thm *mult-right-le-imp-le*

lemma *add-lessD2*: $k + m < (n::nat) \Longrightarrow m < n$
⟨*proof*⟩

lemma *mod-offset*: **assumes** $M \neq (0 :: nat)$
obtains k **where** $n\ mod\ M = (l + k)\ mod\ M$
⟨*proof*⟩

lemma assumes $q \neq (0::nat)$ **shows** $p \leq p + q - gcd\ p\ q$
⟨*proof*⟩

lemma *pos-div-gcd-pos* [*intro*]: **assumes** $(0 :: nat) < p$ **shows** $0 < p\ div\ gcd\ p\ q$
⟨*proof*⟩

lemma *less-mult-one*: **assumes** $(m-1)*k < k$ **obtains** $m = 0 \mid m = (1::nat)$
⟨*proof*⟩

lemmas *gcd-le2-pos* = *gcd-le2-nat*[*folded zero-order*(*4*)] **and**
gcd-le1-pos = *gcd-le1-nat*[*folded zero-order*(*4*)]

lemma *ge1-pos-conv*: $1 \leq k \longleftrightarrow 0 < (k::nat)$
⟨*proof*⟩

lemma *per-lemma-len-le*: **assumes** *le*: $p + q - gcd\ p\ q \leq (n :: nat)$ **and** $0 < q$
shows $p \leq n$
⟨*proof*⟩

lemma *Suc-less-iff-Suc-le*: $Suc\ n < k \longleftrightarrow Suc\ n \leq k - 1$
⟨*proof*⟩

lemma *nat-induct-pair*: $P\ 0\ 0 \Longrightarrow (\bigwedge m\ n.\ P\ m\ n \Longrightarrow P\ m\ (Suc\ n)) \Longrightarrow (\bigwedge m\ n.\ P\ m\ n \Longrightarrow P\ (Suc\ m)\ n) \Longrightarrow P\ m\ n$
⟨*proof*⟩

lemma *One-less-Two-le-iff*: $1 < k \longleftrightarrow 2 \leq (k :: nat)$
⟨*proof*⟩

lemma *at-least2-Suc*: **assumes** $2 \leq k$
obtains k' **where** $k = Suc(Suc\ k')$
⟨*proof*⟩

lemma *at-least3-Suc*: **assumes** $3 \leq k$
obtains k' **where** $k = Suc(Suc(Suc\ k'))$
⟨*proof*⟩

lemmas *not0-SucE*[*elim*] = *not0-implies-Suc*[*THEN exE*]

lemma *le1-SucE*: **assumes** $1 \leq n$
obtains k **where** $n = Suc\ k$ ⟨*proof*⟩

lemma *Suc-minus*: $k \neq 0 \Longrightarrow Suc\ (k - 1) = k$
⟨*proof*⟩

lemma *Suc-minus'*: $1 \leq k \Longrightarrow Suc(k - 1) = k$
⟨*proof*⟩

lemmas *Suc-minus-pos* = *Suc-diff-1*

lemma *Suc-minus2*: $2 \leq k \Longrightarrow Suc\ (Suc(k - 2)) = k$
⟨*proof*⟩

lemma *Suc-leE*: **assumes** $Suc\ k \leq n$ **obtains** m **where** $n = Suc\ m$ **and** $k \leq m$
⟨*proof*⟩

lemma *two-three-add-le-mult*: $2 \leq (l::nat) \Longrightarrow 3 \leq k \Longrightarrow k + l + 1 \leq k*l$
⟨*proof*⟩

lemma *almost-equal-equal*: **assumes** $(a::\ nat) \neq 0$ **and** $b \neq 0$ **and** *eq*: $k*(a+b) + a = m*(a+b) + b$
shows $k = m$ **and** $a = b$
⟨*proof*⟩

lemma *add-le-cancel-ge-right*: $a + b \leq c + d \Longrightarrow d \leq b \Longrightarrow a \leq (c :: nat)$ **for** $a\ b\ c\ d$
⟨*proof*⟩

lemma *add-less-cancel-ge-right*: $a + b \leq c + d \Longrightarrow d < b \Longrightarrow a < (c :: nat)$ **for** $a\ b\ c\ d$
⟨*proof*⟩

lemma *add-le-cancel-ge-left*: $a + b \leq c + d \Longrightarrow c \leq a \Longrightarrow b \leq (d :: nat)$ **for** $a\ b\ c\ d$
⟨*proof*⟩

lemma *add-less-cancel-ge-left*: $a + b \leq c + d \Longrightarrow c < a \Longrightarrow b < (d :: nat)$ **for** $a\ b\ c\ d$
⟨*proof*⟩

lemma *crossproduct-le*: **assumes** $(a::nat) \leq b$ **and** $c \leq d$
shows $a*d + b*c \leq a*c + b*d$
⟨*proof*⟩

lemma (**in** *linorder*) *le-less-cases*: $(a \leq b \Longrightarrow P) \Longrightarrow (b < a \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

end

theory *Reverse-Symmetry*

imports *Main*
begin

Chapter 1

Reverse symmetry

This theory deals with a mechanism which produces new facts on lists from already known facts by the reverse symmetry of lists, induced by the mapping *rev*. It constructs the rule attribute "reversed" which produces the symmetrical fact using so-called reversal rules, which are rewriting rules that may be applied to obtain the symmetrical fact. An example of such a reversal rule is the already existing *rev ys* @ *rev xs* = *rev* (*xs* @ *ys*). Some additional reversal rules are given in this theory.

The symmetrical fact 'A[reversed]' is constructed from the fact 'A' in the following manner: 1. each schematic variable *xs* of type *'a list* is instantiated by *rev xs*; 2. constant Nil is substituted by *rev* []; 3. each quantification of *'a list* type variable $\bigwedge xs.\ P\ xs$ is substituted by (logically equivalent) quantification $\bigwedge xs.\ P\ (rev\ xs)$, similarly for $\forall$, $\exists$ and $\exists!$ quantifiers; each bounded quantification of *'a list* type variable $\forall\, xs{\in}A.\ P\ xs$ is substituted by (logically equivalent) quantification $\forall\, xs{\in}rev\ `\ A.\ P\ (rev\ xs)$, similarly for bounded $\exists$ quantifier; 4. simultaneous rewrites according to a the current list of reversal rules are performed; 5. final correctional rewrites related to reversion of (#) are performed.

List of reversal rules is maintained by declaration attribute "reversal_rule" with standard "add" and "del" options.

See examples at the end of the file.

1.1 Quantifications and maps

lemma *all-surj-conv*: **assumes** *surj f* **shows** $(\bigwedge x.\ PROP\ P\ (f\ x)) \equiv (\bigwedge y.\ PROP\ P\ y)$
⟨*proof*⟩

lemma *All-surj-conv*: **assumes** *surj f* **shows** $(\forall\, x.\ P\ (f\ x)) \longleftrightarrow (\forall\, y.\ P\ y)$
⟨*proof*⟩

lemma *Ex-surj-conv*: **assumes** *surj f* **shows** $(\exists\, x.\ P\ (f\ x)) \longleftrightarrow (\exists\, y.\ P\ y)$
⟨*proof*⟩

lemma *Ex1-bij-conv*: **assumes** *bij f* **shows** $(\exists!x.\ P\ (f\ x)) \longleftrightarrow (\exists!y.\ P\ y)$
⟨*proof*⟩

lemma *Ball-inj-conv*: **assumes** *inj f* **shows** $(\forall\, y{\in}f\ `\ A.\ P\ (inv\ f\ y)) \longleftrightarrow (\forall\, x{\in}A.\ P\ x)$
⟨*proof*⟩

lemma *Bex-inj-conv*: **assumes** *inj f* **shows** $(\exists\, y{\in}f\ `\ A.\ P\ (inv\ f\ y)) \longleftrightarrow (\exists\, x{\in}A.\ P\ x)$
⟨*proof*⟩

1.1.1 Quantifications and reverse

lemma *rev-involution'*: *rev* ∘ *rev* = *id*
⟨*proof*⟩

lemma *rev-inv*: *inv rev* = *rev*
⟨*proof*⟩

1.2 Attributes

context
begin

1.2.1 Cons reversion

definition *snocs* :: *'a list* ⇒ *'a list* ⇒ *'a list*
where *snocs xs ys* = *xs* @ *ys*

1.2.2 Final corrections

lemma *snocs-snocs*: *snocs* (*snocs xs* (*y* # *ys*)) *zs* = *snocs xs* (*y* # *snocs ys zs*)
⟨*proof*⟩

lemma *snocs-Nil*: *snocs* [] *xs* = *xs*
⟨*proof*⟩

lemma *snocs-is-append*: *snocs xs ys* = *xs* @ *ys*
⟨*proof*⟩ **lemmas** *final-correct1* =
snocs-snocs

private lemmas *final-correct2* =
snocs-Nil

private lemmas *final-correct3* =
snocs-is-append

1.2.3 Declaration attribute *reversal-rule*

⟨*ML*⟩

1.2.4 Tracing attribute

⟨*ML*⟩

1.2.5 Rule attribute *reversed*

private lemma *rev-Nil*: *rev* [] ≡ []
⟨*proof*⟩ **lemma** *map-Nil*: *map f* [] ≡ []
⟨*proof*⟩ **lemma** *image-empty*: *f* ‘ *Set.empty* ≡ *Set.empty*
⟨*proof*⟩

definition *COMP* :: (′*b* ⇒ *prop*) ⇒ (′*a* ⇒ ′*b*) ⇒ ′*a* ⇒ *prop* (**infixl** *oo 55*)
where *F oo g* ≡ (λ*x*. *F* (*g x*))

lemma *COMP-assoc*: *F oo* (*f o g*) ≡ (*F oo f*) *oo g*
⟨*proof*⟩ **lemma** *image-comp-image*: (‘) *f* ∘ (‘) *g* ≡ (‘) (*f* ∘ *g*)
⟨*proof*⟩ **lemma** *rev-involution*: *rev* ∘ *rev* ≡ *id*
⟨*proof*⟩ **lemma** *map-involution*: **assumes** *f* ∘ *f* ≡ *id* **shows** (*map f*) ∘ (*map f*) ≡ *id*
⟨*proof*⟩ **lemma** *image-involution*: **assumes** *f* ∘ *f* ≡ *id* **shows** (*image f*) ∘ (*image f*) ≡ *id*
⟨*proof*⟩ **lemma** *rev-map-comm*: *rev* ∘ *map f* ≡ *map f* ∘ *rev*
⟨*proof*⟩ **lemma** *involut-comm-comp*: **assumes** *f o f* ≡ *id* **and** *g o g* ≡ *id* **and** *f o g* ≡ *g o f*
shows (*f* ∘ *g*) ∘ (*f* ∘ *g*) ≡ *id*
⟨*proof*⟩ **lemma** *rev-map-involution*: **assumes** *g o g* ≡ *id*
shows (*rev* ∘ *map g*) ∘ (*rev* ∘ *map g*) ≡ *id*
⟨*proof*⟩ **lemma** *prop-abs-subst*: **assumes** *f o f* ≡ *id* **shows** (λ*x*. *F* (*f x*)) *oo f* ≡ (λ*x*. *F x*)
⟨*proof*⟩ **lemma** *prop-abs-subst-comm*: **assumes** *f o f* ≡ *id* **and** *g o g* ≡ *id* **and** *f o g* ≡ *g o f*
shows (λ*x*. *F* (*f* (*g x*))) *oo* (*f o g*) ≡ (λ*x*. *F x*)
⟨*proof*⟩ **lemma** *prop-abs-subst-rev-map*: **assumes** *g o g* ≡ *id*
shows (λ*x*. *F* (*rev* (*map g x*))) *oo* (*rev o map g*) ≡ (λ*x*. *F x*)
⟨*proof*⟩ **lemma** *obj-abs-subst*: **assumes** *f o f* ≡ *id* **shows** (λ*x*. *F* (*f x*)) *o f* ≡ (λ*x*. *F x*)
⟨*proof*⟩ **lemma** *obj-abs-subst-comm*: **assumes** *f o f* ≡ *id* **and** *g o g* ≡ *id* **and** *f o g* ≡ *g o f*
shows (λ*x*. *F* (*f* (*g x*))) *o* (*f o g*) ≡ (λ*x*. *F x*)
⟨*proof*⟩ **lemma** *obj-abs-subst-rev-map*: **assumes** *g o g* ≡ *id*
shows (λ*x*. *F* (*rev* (*map g x*))) *o* (*rev o map g*) ≡ (λ*x*. *F x*)
⟨*proof*⟩

⟨*ML*⟩

end

1.3 Declaration of basic reversal rules

1.3.1 Pure

lemma *all-surj-conv'* [*reversal-rule*]: **assumes** *surj f* **shows** *Pure.all* (*P oo f*) ≡ *Pure.all P*
⟨*proof*⟩

1.3.2 *HOL.HOL*

lemmas [*reversal-rule*] = *rev-is-rev-conv inj-eq*

— *All*
lemma *All-surj-conv'* [*reversal-rule*]: **assumes** *surj f* **shows** *All* (*P o f*) = *All P*
⟨*proof*⟩
lemma *Ex-surj-conv'* [*reversal-rule*]: **assumes** *surj f* **shows** *Ex* (*P o f*) ⟷ *Ex P*
⟨*proof*⟩
lemma *Ex1-bij-conv'* [*reversal-rule*]: **assumes** *bij f* **shows** *Ex1* (*P o f*) ⟷ *Ex1 P*
⟨*proof*⟩
lemma *if-image* [*reversal-rule*]: (*if P then f u else f v*) = *f* (*if P then u else v*)
⟨*proof*⟩

1.3.3 *HOL.Set*

lemma *collect-image*: *Collect* (*P* ∘ *f*) = *f* −‘ (*Collect P*)
⟨*proof*⟩

lemma *collect-image'* [*reversal-rule*]: **assumes** *f* ∘ *f* = *id* **shows** *Collect* (*P* ∘ *f*) = *f* ‘ (*Collect P*)
⟨*proof*⟩
lemma *Ball-image* [*reversal-rule*]: **assumes** (*g* ∘ *f*) ‘ *A* = *A* **shows** *Ball* (*f* ‘ *A*) (*P* ∘ *g*) = *Ball A P*
⟨*proof*⟩
lemma *Bex-image-comp*: *Bex* (*f* ‘ *A*) *g* = *Bex A* (*g* ∘ *f*)
⟨*proof*⟩

lemma *Bex-image* [*reversal-rule*]: **assumes** (*g* ∘ *f*) ‘ *A* = *A* **shows** *Bex* (*f* ‘ *A*) (*P* ∘ *g*) = *Bex A P*
⟨*proof*⟩
lemma *insert-image* [*reversal-rule*]: *insert* (*f x*) (*f* ‘ *X*) = *f* ‘ (*insert x X*)
⟨*proof*⟩
lemmas [*reversal-rule*] = *inj-image-mem-iff*

— (⊆)
lemmas [*reversal-rule*] = *inj-image-subset-iff*

1.3.4 *HOL.List*

lemmas [*reversal-rule*] = *set-rev set-map*

— (#)
lemma *Cons-rev*: $a \# rev\ u = rev\ (snocs\ u\ [a])$
⟨*proof*⟩

lemma *Cons-map*: $(f\ x) \#\ (map\ f\ xs) = map\ f\ (x \# xs)$
⟨*proof*⟩

lemmas [*reversal-rule*] = *Cons-rev Cons-map*

— *hd*
lemmas [*reversal-rule*] = *hd-rev hd-map*

— *tl*
lemma *tl-rev*: $tl\ (rev\ xs) = rev\ (butlast\ xs)$
⟨*proof*⟩

lemmas [*reversal-rule*] = *tl-rev map-tl*[*symmetric*]

— *last*
lemmas [*reversal-rule*] = *last-rev last-map*

— *butlast*
lemmas [*reversal-rule*] = *butlast-rev map-butlast*[*symmetric*]

— *List.coset*
lemma *coset-rev*: $List.coset\ (rev\ xs) = List.coset\ xs$
⟨*proof*⟩

lemma *coset-map*: **assumes** *bij f* **shows** $List.coset\ (map\ f\ xs) = f\ `\ List.coset\ xs$
⟨*proof*⟩

lemmas [*reversal-rule*] = *coset-rev coset-map*

— (@)
lemmas [*reversal-rule*] = *rev-append*[*symmetric*] *map-append*[*symmetric*]

— *concat*
lemma *concat-rev-map-rev*: $concat\ (rev\ (map\ rev\ xs)) = rev\ (concat\ xs)$
⟨*proof*⟩

lemma *concat-rev-map-rev'*: $concat\ (rev\ (map\ (rev \circ f)\ xs)) = rev\ (concat\ (map\ f\ xs))$
⟨*proof*⟩

lemmas [*reversal-rule*] = *concat-rev-map-rev concat-rev-map-rev'*

— *drop*
lemmas [*reversal-rule*] = *drop-rev drop-map*

— *take*
lemmas [*reversal-rule*] = *take-rev take-map*

— (!)
lemmas [*reversal-rule*] = *rev-nth nth-map*

— *List.insert*
lemma *list-insert-map* [*reversal-rule*]:
assumes *inj f* **shows** *List.insert* (*f x*) (*map f xs*) = *map f* (*List.insert x xs*)
⟨*proof*⟩
lemma *list-union-map* [*reversal-rule*]:
assumes *inj f* **shows** *List.union* (*map f xs*) (*map f ys*) = *map f* (*List.union xs ys*)
⟨*proof*⟩
lemmas [*reversal-rule*] = *length-rev length-map*

— *rotate*
lemmas [*reversal-rule*] = *rotate-rev rotate-map*

— *lists*
lemma *rev-in-lists*: *rev u* ∈ *lists A* ⟷ *u* ∈ *lists A*
⟨*proof*⟩

lemma *map-in-lists*: *inj f* ⟹ *map f u* ∈ *lists* (*f* ‘ *A*) ⟷ *u* ∈ *lists A*
⟨*proof*⟩

lemmas [*reversal-rule*] = *rev-in-lists map-in-lists*

— *list-all*
lemmas [*reversal-rule*] = *list-all-rev*

— *list-ex*
lemmas [*reversal-rule*] = *list-ex-rev*

1.3.5 Reverse Symmetry

lemma *snocs-map* [*reversal-rule*]: *snocs* (*map f u*) [*f a*] = *map f* (*snocs u* [*a*])
⟨*proof*⟩

1.4

lemma *bij-rev*: *bij rev*
⟨*proof*⟩

lemma *bij-map*: *bij f* ⟹ *bij* (*map f*)
⟨*proof*⟩

lemma *surj-map*: *surj f* ⟹ *surj* (*map f*)

⟨*proof*⟩

lemma *bij-image*: *bij f* ⟹ *bij* (*image f*)
⟨*proof*⟩

lemma *inj-image*: *inj f* ⟹ *inj* (*image f*)
⟨*proof*⟩

lemma *surj-image*: *surj f* ⟹ *surj* (*image f*)
⟨*proof*⟩

lemmas [*simp*] =
bij-rev
bij-is-inj
bij-is-surj
bij-comp
inj-compose
comp-surj
bij-map
inj-mapI
surj-map
bij-image
inj-image
surj-image

1.5 Examples

context
begin

1.5.1 Cons and append

private lemma *example-Cons-append*:
assumes *xs* = [*a*, *b*] **and** *ys* = [*b*, *a*, *b*]
shows *xs* @ *xs* @ *xs* = *a* # *b* # *a* # *ys*
⟨*proof*⟩

thm
example-Cons-append
example-Cons-append[*reversed*]
example-Cons-append[*reversed*, *reversed*]

thm
not-Cons-self
not-Cons-self[*reversed*]

thm
neq-Nil-conv
neq-Nil-conv[*reversed*]

1.5.2 Induction rules

thm
list-nonempty-induct
list-nonempty-induct[*reversed*] *list-nonempty-induct*[*reversed*, **where** $P{=}\lambda x.\ P$ (*rev x*) **for** *P*, *unfolded rev-rev-ident*]

thm
list-induct2
list-induct2[*reversed*] *list-induct2*[*reversed*, **where** $P{=}\lambda x\ y.\ P$ (*rev x*) (*rev y*) **for** *P*, *unfolded rev-rev-ident*]

1.5.3 hd, tl, last, butlast

thm
hd-append
hd-append[*reversed*]
last-append

thm
length-tl
length-tl[*reversed*]
length-butlast

thm
hd-Cons-tl
hd-Cons-tl[*reversed*]
append-butlast-last-id
append-butlast-last-id[*reversed*]

1.5.4 set

thm
hd-in-set
hd-in-set[*reversed*]
last-in-set

thm
set-ConsD
set-ConsD[*reversed*]

thm
split-list-first
split-list-first[*reversed*]

thm
split-list-first-prop
split-list-first-prop[*reversed*]
split-list-first-prop[*reversed*, *unfolded append-assoc append-Cons append-Nil*]
split-list-last-prop

thm
split-list-first-propE
split-list-first-propE[*reversed*]
split-list-first-propE[*reversed*, *unfolded append-assoc append-Cons append-Nil*]
split-list-last-propE

1.5.5 rotate

private lemma *rotate1-hd-tl*: $xs \neq [] \Longrightarrow rotate\ 1\ xs = tl\ xs\ @\ [hd\ xs]$
⟨*proof*⟩

thm
rotate1-hd-tl
rotate1-hd-tl[*reversed*]

end

end

theory *CoWBasic*
imports *HOL−Library.Sublist Arithmetical-Hints Reverse-Symmetry HOL−Eisbach.Eisbach-Tools List-Power.List-Power*
begin

Chapter 2

Basics of Combinatorics on Words

Combinatorics on Words, as the name suggests, studies words, finite or infinite sequences of elements from a, usually finite, alphabet. The essential operation on finite words is the concatenation of two words, which is associative and noncommutative. This operation yields many simply formulated problems, often in terms of *equations on words*, that are mathematically challenging.

See for instance [1] for an introduction to Combinatorics on Words, and [**?**, 5, 6] as another reference for Combinatorics on Words. This theory deals exclusively with finite words and provides basic facts of the field which can be considered as folklore.

The most natural way to represent finite words is by the type *'a list*. From an algebraic viewpoint, lists are free monoids. On the other hand, any free monoid is isomoporphic to a monoid of lists of its generators. The algebraic point of view and the combinatorial point of view therefore overlap significantly in Combinatorics on Words.

2.1 Definitions and notations

First, we introduce elementary definitions and notations.

The concatenation (@) of two finite lists/words is the very basic operation in Combinatorics on Words, its notation is usually omitted. In this field, a common notation for this operation is ·, which we use and add here.

notation *append* (**infixr** · *65*)

lemmas *rassoc* = *append-assoc*
lemmas *lassoc* = *append-assoc*[*symmetric*]

We add a common notation for the length of a given word $|w|$.

notation
length (|-| *1000*) — note that it's bold |
notation (*latex* **output**)
length (|-|)
notation *longest-common-prefix* (**infixr** $\wedge_p$ *61*) — provided by Sublist.thy

2.1.1 Empty and nonempty word

As the word of length zero [] or [] will be used often, we adopt its frequent notation ε in this formalization.

notation *Nil* (ε)

named-theorems *emp-simps pow-list-Nil*
lemmas [*emp-simps*] = *append-Nil2 append-Nil list.map*(*1*) *list.size*(*3*)

2.1.2 Prefix

The property of being a prefix shall be frequently used, and we give it yet another frequent shorthand notation. Analogously, we introduce shorthand notations for non-empty prefix and strict prefix, and continue with suffixes and factors.

notation *prefix* (**infixl** $\leq p$ *50*)

lemmas *prefI'*[*intro*] = *prefixI*

lemma *prefI*[*intro*]: $p \cdot s = w \Longrightarrow p \leq p\ w$
⟨*proof*⟩

lemma *prefD*: $u \leq p\ v \Longrightarrow \exists\ z.\ v = u \cdot z$
⟨*proof*⟩

definition *prefix-comparable* :: *'a list* ⇒ *'a list* ⇒ *bool* (**infixl** $\bowtie$ *50*)
where (*prefix-comparable u v*) ≡ $u \leq p\ v \vee v \leq p\ u$

lemma *pref-compI1*: $u \leq p\ v \implies u \bowtie v$
⟨*proof*⟩

lemma *pref-compI2*: $v \leq p\ u \Longrightarrow u \bowtie v$
⟨*proof*⟩

lemma *pref-compE* [*elim*]: **assumes** $u \bowtie v$ **obtains** $u \leq p\ v \mid v \leq p\ u$
⟨*proof*⟩

lemma *pref-compI*[*intro*]: $u \leq p\ v \vee v \leq p\ u \Longrightarrow u \bowtie v$

⟨*proof*⟩

notation *strict-prefix* (**infixl** $<_p$ *50*)
notation (*latex* **output**) *strict-prefix* (**infixl** $<_p$ *50*)

lemmas [*simp*] = *strict-prefix-def*

interpretation *lcp*: *semilattice-order* ($\wedge_p$) *prefix strict-prefix*
⟨*proof*⟩

lemma *sprefI1*[*intro*]: $v = u \cdot z \Longrightarrow z \neq \varepsilon \Longrightarrow u <_p v$
⟨*proof*⟩

lemma *sprefI1'*[*intro*]: $u \cdot z = v \Longrightarrow z \neq \varepsilon \Longrightarrow u <_p v$
⟨*proof*⟩

lemma *sprefI2*[*intro*]: $u \leq_p v \Longrightarrow |u| < |v| \Longrightarrow u <_p v$
⟨*proof*⟩

lemmas *sprefI*[*intro*] = *strict-prefixI*

lemma *sprefD*: $u <_p v \Longrightarrow u \leq_p v \wedge u \neq v$
⟨*proof*⟩

lemmas *sprefD1*[*dest*] = *prefix-order.strict-implies-order* **and**
sprefD2 = *prefix-order.less-imp-neq*

lemmas *sprefE*[*elim?*] = *strict-prefixE*

lemma *spref-exE*[*elim*]: **assumes** $u <_p v$ **obtains** z **where** $u \cdot z = v$ **and** $z \neq \varepsilon$
⟨*proof*⟩

lemma *pref-comp-sprefI*: $u \bowtie v \Longrightarrow \neg\, u \leq_p v \Longrightarrow v <_p u$
⟨*proof*⟩

2.1.3 Suffix

notation *suffix* (**infixl** $\leq_s$ *50*)
notation (*latex* **output**) *suffix* ($\leq_s$)

lemma *sufI*[*intro*]: $p \cdot s = w \Longrightarrow s \leq_s w$
⟨*proof*⟩

lemma *sufD*[*elim*]: $u \leq_s v \Longrightarrow \exists\ z.\ z \cdot u = v$
⟨*proof*⟩

notation *strict-suffix* (**infixl** $<_s$ *50*)

notation (*latex* **output**) *strict-suffix* ($<_s$)
lemmas [*simp*] = *strict-suffix-def*

lemmas [*intro*] = *suffix-order.le-neq-trans*

lemmas *ssufI* = *suffix-order.le-neq-trans*

lemma *ssufI1*[*intro*]: $u \cdot v = w \Longrightarrow u \neq \varepsilon \Longrightarrow v <s w$
⟨*proof*⟩

lemma *ssufI2*[*intro*]: $u \leq s\ v \Longrightarrow$ *length* $u <$ *length* $v \Longrightarrow u <s\ v$
⟨*proof*⟩

lemma *ssufE*: $u <s\ v \Longrightarrow (u \leq s\ v \Longrightarrow u \neq v \Longrightarrow$ *thesis*$) \Longrightarrow$ *thesis*
⟨*proof*⟩

lemma *ssufD*[*elim*]: $u <s\ v \Longrightarrow u \leq s\ v \wedge u \neq v$
⟨*proof*⟩

lemmas *ssufD1*[*elim*] = *suffix-order.strict-implies-order* **and**
ssufD2[*elim*] = *suffix-order.less-imp-neq*

definition *suffix-comparable* :: *'a list* ⇒ *'a list* ⇒ *bool* (**infixl** $\bowtie_s$ *50*)
where (*suffix-comparable u v*) ⟷ (*rev u*) ⋈ (*rev v*)

lemma *suf-compI1*[*intro*]: $u \leq s\ v \Longrightarrow u \bowtie_s v$
⟨*proof*⟩

lemma *suf-compI2*[*intro*]: $v \leq s\ u \Longrightarrow u \bowtie_s v$
⟨*proof*⟩

2.1.4 Factor

A *sublist* of some word is in Combinatorics of Words called a factor. We adopt a common shorthand notation for the property of being a factor, strict factor and nonempty factor (the latter we also define).

notation *sublist* (**infixl** $\leq f$ *50*)
notation (*latex* **output**) *sublist* ($\leq_f$)
lemmas *fac-def* = *sublist-def*

notation *strict-sublist* (**infixl** $<f$ *50*)
notation (*latex* **output**) *strict-sublist* ($<_f$)
lemmas *strict-factor-def*[*simp*] = *strict-sublist-def*

definition *nonempty-factor* (**infixl** $\leq nf$ *60*) **where** *nonempty-factor-def*[*simp*]: $u \leq nf\ v \equiv u \neq \varepsilon \wedge (\exists\ p\ s.\ p \cdot u \cdot s = v)$

notation (*latex* **output**) *nonempty-factor* ($\leq_{nf}$)

lemmas *facI* = *sublist-appendI*

lemma *facI'*: $a \cdot u \cdot b = w \Longrightarrow u \leq f\ w$
⟨*proof*⟩

lemma *facE*[*elim*]: **assumes** $u \leq f\ v$
obtains $p\ s$ **where** $v = p \cdot u \cdot s$
⟨*proof*⟩

lemma *facE'*[*elim*]: **assumes** $u \leq f\ v$
obtains $p\ s$ **where** $p \cdot u \cdot s = v$
⟨*proof*⟩

2.2 Various elementary lemmas

thm *drop-eq-Nil*

lemma *exE2*: $\exists\ x\ y.\ P\ x\ y \Longrightarrow (\bigwedge x\ y.\ P\ x\ y \Longrightarrow thesis) \Longrightarrow thesis$
⟨*proof*⟩

lemmas *concat-morph* = *concat-append*

lemmas *cancel* = *same-append-eq* **and**
cancel-right = *append-same-eq*

lemmas *disjI* = *verit-and-neg*(*3*)

lemma *rev-in-conv*: $rev\ u \in A \longleftrightarrow u \in rev\ `\ A$
⟨*proof*⟩

lemmas *map-rev-involution*[*simp*] = *list.map-comp*[*of rev rev*, *unfolded rev-involution'* *list.map-id*]

lemma *map-rev-lists-rev*: $map\ rev\ `\ (lists\ (rev\ `\ A)) = lists\ A$
⟨*proof*⟩

lemma *inj-on-map-lists*: **assumes** *inj-on f A*
shows *inj-on* (*map f*) (*lists A*)
⟨*proof*⟩

lemma *bij-lists*: *bij-betw f X Y* $\Longrightarrow$ *bij-betw* (*map f*) (*lists X*) (*lists Y*)
⟨*proof*⟩

lemma *concat-sing'*: $concat\ [r] = r$
⟨*proof*⟩

lemma *concat-sing*: **assumes** $s = [a]$ **shows** $concat\ s = a$

⟨*proof*⟩

lemma *rev-sing*: $rev\ [x] = [x]$
⟨*proof*⟩

lemma *hd-word*: $a\#ws = [a] \cdot ws$
⟨*proof*⟩

lemma *hd-Cons-append*[*intro*,*simp*]: $hd\ ((a\#v) \cdot u) = a$
⟨*proof*⟩

lemma *pref-singE*: **assumes** $p \leq p\ [a]$ **obtains** $p = \varepsilon \mid p = [a]$
⟨*proof*⟩

lemma *map-hd*: $map\ f\ (a\#v) = [f\ a] \cdot (map\ f\ v)$
⟨*proof*⟩

lemma *hd-tl*: $w \neq \varepsilon \Longrightarrow [hd\ w] \cdot tl\ w = w$
⟨*proof*⟩

lemma *hd-tlE*: **assumes** $w \neq \varepsilon$
obtains $a\ w'$ **where** $w = a\#w'$
⟨*proof*⟩

lemma *hd-tl-lenE*: **assumes** $0 < |w|$
obtains $a\ w'$ **where** $w = a\#w'$
⟨*proof*⟩

lemma *long-list-tl*[*intro*]: **assumes** $1 < |w|$
shows $tl\ w \neq \varepsilon$
⟨*proof*⟩

lemma *long-last-tl*: **assumes** $1 < |w|$ **shows** $last\ (tl\ w) = last\ w$
⟨*proof*⟩

lemma *hd-middle-last*: **assumes** $1 < |w|$
shows $[hd\ w] \cdot butlast\ (tl\ w) \cdot [last\ w] = w$
⟨*proof*⟩

lemma *hd-pref*[*intro*]: $w \neq \varepsilon \Longrightarrow [hd\ w] \leq p\ w$
⟨*proof*⟩

lemma *pref-hd-pref*[*intro*]: $u \cdot v = w \Longrightarrow u \neq w \Longrightarrow\ u \cdot [hd\ v] \leq p\ w$
⟨*proof*⟩

lemma *pref-hd-pref'*[*intro*]: $u \cdot v = w \Longrightarrow v \neq \varepsilon \Longrightarrow\ u \cdot [hd\ v] \leq p\ w$
⟨*proof*⟩

lemma *Nil-take-Nil* [*simp*]: $u = \varepsilon \Longrightarrow take\ p\ u = \varepsilon$
⟨*proof*⟩

lemma *add-nth*: **assumes** $n < |w|$ **shows** $(take\ n\ w) \cdot [w!n] \leq p\ w$
⟨*proof*⟩

lemma *hd-pref'*: **assumes** $w \neq \varepsilon$ **shows** $[w!0] \leq p\ w$
⟨*proof*⟩

lemma *sub-lists-mono*: $A \subseteq B \Longrightarrow x \in lists\ A \Longrightarrow x \in lists\ B$
⟨*proof*⟩

lemma *lists-hd-in-set*[*simp*]: $us \neq \varepsilon \Longrightarrow us \in lists\ Q \Longrightarrow hd\ us \in Q$
⟨*proof*⟩

lemma *lists-last-in-set*[*simp*]: $us \neq \varepsilon \Longrightarrow us \in lists\ Q \Longrightarrow last\ us \in Q$
⟨*proof*⟩

lemma *replicate-in-lists*: $replicate\ k\ z \in lists\ \{z\}$
⟨*proof*⟩

lemma *tl-in-lists*: **assumes** $us \in lists\ A$ **shows** $tl\ us \in lists\ A$
⟨*proof*⟩

lemmas *lists-butlast* = *tl-in-lists*[*reversed*]

lemma *tl-set*: $x \in set\ (tl\ a) \Longrightarrow x \in set\ a$
⟨*proof*⟩

lemma *drop-take-inv*: $n \leq |u| \Longrightarrow drop\ n\ (take\ n\ u \cdot w) = w$
⟨*proof*⟩

lemma *split-list-long*: **assumes** $1 < |us|$ **and** $u \in set\ us$
obtains *xs ys* **where** $us = xs \cdot [u] \cdot ys$ **and** $xs \cdot ys \neq \varepsilon$
⟨*proof*⟩

lemma *flatten-lists*: $G \subseteq lists\ B \Longrightarrow xs \in lists\ G \Longrightarrow concat\ xs \in lists\ B$
⟨*proof*⟩

lemma *concat-map-sing-ident*: $concat\ (map\ (\lambda\ x.\ [x])\ xs) = xs$
⟨*proof*⟩

lemma *hd-concat-tl*: **assumes** $ws \neq \varepsilon$ **shows** $hd\ ws \cdot concat\ (tl\ ws) = concat\ ws$
⟨*proof*⟩

lemma *concat-butlast-last*: **assumes** *nemp*: $ws \neq \varepsilon$ **shows** $concat\ (butlast\ ws) \cdot last\ ws = concat\ ws$
⟨*proof*⟩

lemma *spref-butlast-pref*: **assumes** $u \leq p\ v$ **and** $u \neq v$ **shows** $u \leq p\ butlast\ v$
⟨*proof*⟩

lemma *last-concat*: $xs \neq \varepsilon \Longrightarrow last\ xs \neq \varepsilon \Longrightarrow last\ (concat\ xs) = last\ (last\ xs)$
⟨*proof*⟩

lemma *concat-last-suf*: $ws \neq \varepsilon \Longrightarrow last\ ws \leq s\ concat\ ws$
⟨*proof*⟩

lemma *concat-hd-pref*: $ws \neq \varepsilon \Longrightarrow hd\ ws \leq p\ concat\ ws$
⟨*proof*⟩

lemma *set-nemp-concat-nemp*: **assumes** $ws \neq \varepsilon$ **and** $\varepsilon \notin set\ ws$ **shows** $concat\ ws \neq \varepsilon$
⟨*proof*⟩

lemmas *takedrop* = *append-take-drop-id*

lemma *drop-neq* [*intro*]: $w \neq \varepsilon \Longrightarrow 0 < n \Longrightarrow drop\ n\ w \neq w$
⟨*proof*⟩

lemma *suf-drop-conv*: $u \leq s\ w \longleftrightarrow drop\ (|w| - |u|)\ w = u$
⟨*proof*⟩

lemma *comm-rev-iff*: $rev\ u \cdot rev\ v = rev\ v \cdot rev\ u \longleftrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *rev-induct2*:
$[\![\ P\ []\ []; \bigwedge x\ xs.\ P\ (xs{\cdot}[x])\ []; \bigwedge y\ ys.\ P\ []\ (ys{\cdot}[y]); \bigwedge x\ xs\ y\ ys.\ P\ xs\ ys \Longrightarrow P\ (xs{\cdot}[x])\ (ys{\cdot}[y])\ ]\!] \Longrightarrow P\ xs\ ys$
⟨*proof*⟩

lemma *fin-bin*: *finite* $\{x,y\}$
⟨*proof*⟩

lemma *rev-rev-image-eq* [*reversal-rule*]: $rev\ `\ rev\ `\ X = X$
⟨*proof*⟩

lemma *last-take-conv-nth*: **assumes** $n < length\ xs$ **shows** $last\ (take\ (Suc\ n)\ xs) = xs!n$
⟨*proof*⟩

lemma *inj-map-inv*: $inj\text{-}on\ f\ A \Longrightarrow u \in lists\ A \Longrightarrow u = map\ (the\text{-}inv\text{-}into\ A\ f)$

(*map f u*)
⟨*proof*⟩

lemma *last-sing*[*simp*]: *last* [*c*] = *c*
⟨*proof*⟩

lemma *hd-hdE*: (*u* = *ε* ⟹ *thesis*) ⟹ (*u* = [*hd u*] ⟹ *thesis*) ⟹ (*u* = [*hd u*, *hd* (*tl u*)] · *tl* (*tl u*) ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩

lemma *same-sing-pref*: *u* · [*a*] ≤*p v* ⟹ *u* · [*b*] ≤*p v* ⟹ *a* = *b*
⟨*proof*⟩

lemma *compow-Suc*: (*f*^^(*Suc k*)) *w* = *f* ((*f*^^*k*) *w*)
⟨*proof*⟩

lemma *compow-Suc'*: (*f*^^(*Suc k*)) *w* = (*f*^^*k*) (*f w*)
⟨*proof*⟩

2.2.1 General facts

lemma *doubleton-neq-eq-iff* [*simp*]: *x* ≠ *y* ⟹ {*a*,*b*} = {*x*,*y*} ⟷ *a* ≠ *b* ∧ *a* ∈ {*x*,*y*} ∧ *b* ∈ {*x*,*y*}
⟨*proof*⟩

lemma *two-elem-sub*: *x* ∈ *A* ⟹ *y* ∈ *A* ⟹ {*x*,*y*} ⊆ *A*
⟨*proof*⟩

thm *fun.inj-map*[*THEN injD*]

lemma *inj-comp*: **assumes** *inj* (*f* :: *'a list* ⇒ *'b list*) **shows** (*g* *w* = *h w* ⟷ (*f* ∘ *g*) *w* = (*f* ∘ *h*) *w*)
⟨*proof*⟩

lemma *inj-comp-eq*: **assumes** *inj* (*f* :: *'a list* ⇒ *'b list*) **shows** (*g* = *h* ⟷ *f* ∘ *g* = *f* ∘ *h*)
⟨*proof*⟩

lemma *two-elem-cases*[*elim*!]: **assumes** *u* ∈ {*x*, *y*} **obtains** (*fst*) *u* = *x* | (*snd*) *u* = *y*
⟨*proof*⟩

lemma *two-elem-cases2*[*elim*]: **assumes** *u* ∈ {*x*, *y*} *v* ∈ {*x*,*y*} *u* ≠ *v*
shows (*u* = *x* ⟹ *v* = *y* ⟹ *thesis*) ⟹ (*u* = *y* ⟹ *v* = *x* ⟹ *thesis*) ⟹ *thesis*
⟨*proof*⟩

lemma *two-elemP*: *u* ∈ {*x*, *y*} ⟹ *P x* ⟹ *P y* ⟹ *P u*
⟨*proof*⟩

lemma *pairs-extensional*: $(\bigwedge r\ s.\ P\ r\ s \longleftrightarrow (\exists\ a\ b.\ Q\ a\ b \wedge r = fa\ a \wedge s = fb\ b)) \Longrightarrow \{(r,s).\ P\ r\ s\} = \{(fa\ a,\ fb\ b) \mid a\ b.\ Q\ a\ b\}$
⟨*proof*⟩

lemma *pairs-extensional'*: $(\bigwedge r\ s.\ P\ r\ s \longleftrightarrow (\exists\ t.\ Q\ t \wedge r = fa\ t \wedge s = fb\ t)) \Longrightarrow \{(r,s).\ P\ r\ s\} = \{(fa\ t,\ fb\ t) \mid t.\ Q\ t\}$
⟨*proof*⟩

lemma *doubleton-subset-cases*: **assumes** $A \subseteq \{x,y\}$
obtains $A = \{\} \mid A = \{x\} \mid A = \{y\} \mid A = \{x,y\}$
⟨*proof*⟩

2.2.2 Map injective function

lemma *map-pref-conv* [*reversal-rule*]: **assumes** *inj f* **shows** $map\ f\ u \leq p\ map\ f\ v \longleftrightarrow u \leq p\ v$
⟨*proof*⟩

lemma *map-suf-conv* [*reversal-rule*]: **assumes** *inj f* **shows** $map\ f\ u \leq s\ map\ f\ v \longleftrightarrow u \leq s\ v$
⟨*proof*⟩

lemma *map-fac-conv* [*reversal-rule*]: **assumes** *inj f* **shows** $map\ f\ u \leq f\ map\ f\ v \longleftrightarrow u \leq f\ v$
⟨*proof*⟩

lemma *map-lcp-conv*: **assumes** *inj f* **shows** $(map\ f\ xs) \wedge_p (map\ f\ ys) = map\ f\ (xs \wedge_p ys)$
⟨*proof*⟩

2.2.3 Orderings on lists: prefix, suffix, factor

lemmas *self-pref* = *prefix-order.refl* **and**
pref-antisym = *prefix-order.antisym* **and**
pref-trans = *prefix-order.trans* **and**
spref-trans = *prefix-order.less-trans* **and**
suf-trans = *suffix-order.trans* **and**
fac-trans[*intro*] = *sublist-order.order.trans*

2.2.4 On the empty word

lemma *nemp-elem-setI*[*intro*]: $u \in S \Longrightarrow u \neq \varepsilon \Longrightarrow u \in S - \{\varepsilon\}$
⟨*proof*⟩

lemma *emp-concat-emp*: $us \in lists\ (S - \{\varepsilon\}) \Longrightarrow concat\ us = \varepsilon \Longrightarrow us = \varepsilon$
⟨*proof*⟩

lemma *take-nemp*: $w \neq \varepsilon \Longrightarrow 0 < n \Longrightarrow take\ n\ w \neq \varepsilon$
⟨*proof*⟩

lemma *pref-nemp* [*intro*]: $u \neq \varepsilon \Longrightarrow u \cdot v \neq \varepsilon$
⟨*proof*⟩

lemma *suf-nemp* [*intro*]: $v \neq \varepsilon \Longrightarrow u \cdot v \neq \varepsilon$
⟨*proof*⟩

lemma *pref-of-emp*: $u \cdot v = \varepsilon \Longrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *suf-of-emp*: $u \cdot v = \varepsilon \Longrightarrow v = \varepsilon$
⟨*proof*⟩

lemma *nemp-comm*: $(u \neq \varepsilon \Longrightarrow v \neq \varepsilon \Longrightarrow u \neq v \Longrightarrow u \cdot v = v \cdot u) \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *non-triv-comm* [*intro*]: $(u \neq \varepsilon \Longrightarrow v \neq \varepsilon \Longrightarrow u \neq v \Longrightarrow u \cdot v = v \cdot u) \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *split-list'*: $a \in set\ ws \Longrightarrow \exists\, p\ s.\ ws = p \cdot [a] \cdot s$
⟨*proof*⟩

lemma *split-listE*: **assumes** $a \in set\ w$
obtains $p\ s$ **where** $w = p \cdot [a] \cdot s$
⟨*proof*⟩

2.2.5 Counting letters

declare *count-list-rev* [*reversal-rule*]

lemma *count-list-map-conv* [*reversal-rule*]:
assumes *inj f* **shows** *count-list* (*map f ws*) (*f a*) = *count-list ws a*
⟨*proof*⟩

2.2.6 Set inspection method

This section defines a simple method that splits a goal into subgoals by enumerating all possibilites for x in a premise such as $x \in \{a, b, c\}$. See the demonstrations below.

method *set-inspection* = (
(*unfold insert-iff*),
(*elim disjE emptyE*),
(*simp-all only*: *singleton-iff refl True-implies-equals*)
)

lemma $u \in \{x,y\} \Longrightarrow P\ u$
⟨*proof*⟩

lemma $\bigwedge u.\ u \in \{x,y\} \Longrightarrow u = x \vee u = y$
⟨*proof*⟩

2.3 Length and its properties

lemmas *lenarg* = *arg-cong*[*of* - - *length*] **and**
lenmorph = *length-append*

lemma *lenarg-not*: $|u| \neq |v| \Longrightarrow u \neq v$
⟨*proof*⟩

lemma *len-less-neq*: $|u| < |v| \Longrightarrow u \neq v$
⟨*proof*⟩

lemmas *len-nemp-conv* = *length-greater-0-conv*

lemma *npos-len*: $|u| \leq 0 \Longrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *nemp-len*[*intro*]: $w \neq \varepsilon \Longrightarrow 0 < |w|$
⟨*proof*⟩

lemma *nemp-len-not0* [*intro*]: $w \neq \varepsilon \Longrightarrow |w| \neq 0$
⟨*proof*⟩

lemma *nemp-le-len*: $r \neq \varepsilon \Longrightarrow 1 \leq |r|$
⟨*proof*⟩

lemma *nemp-spref-len*: **assumes** $u \neq \varepsilon$ $u <p v$ **shows** $1 < |v|$
⟨*proof*⟩

lemma *nemp-ssuf-len*: **assumes** $u \neq \varepsilon$ $u <s v$ **shows** $1 < |v|$
⟨*proof*⟩

lemma *tl-emp* [*intro*]: $|w| \leq 1 \Longrightarrow tl\ w = \varepsilon$
⟨*proof*⟩

lemma *butlast-emp* [*intro*]: $|w| \leq 1 \Longrightarrow butlast\ w = \varepsilon$
⟨*proof*⟩

lemma *butlast-tl-emp*[*intro*]: $|w| \leq 2 \Longrightarrow butlast(tl\ w) = \varepsilon$
⟨*proof*⟩

lemma *tl-butlast-emp*[*intro*]: $|w| \leq 2 \Longrightarrow tl(butlast\ w) = \varepsilon$
⟨*proof*⟩

lemma *swap-len*: $|u \cdot v| = |v \cdot u|$
⟨*proof*⟩

lemma *len-after-drop*: $p + q \leq |w| \Longrightarrow \ q \leq |drop\ p\ w|$
⟨*proof*⟩

lemma *short-take-append*: $n \leq |u| \Longrightarrow take\ n\ (u \cdot v) = take\ n\ u$
⟨*proof*⟩

lemma *sing-word*: $|us| = 1 \Longrightarrow [hd\ us] = us$
⟨*proof*⟩

lemma *sing-word-concat*: **assumes** $|us| = 1$ **shows** $[concat\ us] = us$
⟨*proof*⟩

lemma *len-one-concat-in*: $ws \in lists\ A \Longrightarrow |ws| = 1 \Longrightarrow concat\ ws \in A$
⟨*proof*⟩

lemma *concat-nemp*: $concat\ us \neq \varepsilon \Longrightarrow us \neq \varepsilon$
⟨*proof*⟩

lemma *sing-len*: $|[a]| = 1$
⟨*proof*⟩

lemmas *pref-len* = *prefix-length-le* **and**
suf-len = *suffix-length-le*

lemmas *spref-len* = *prefix-length-less* **and**
ssuf-len = *suffix-length-less*

lemma *pref-len′*: $|u| \leq |u \cdot z|$
⟨*proof*⟩

lemma *suf-len′*: $|u| \leq |z \cdot u|$
⟨*proof*⟩

lemma *fac-len*: $u \leq f\ v \Longrightarrow |u| \leq |v|$
⟨*proof*⟩

lemma *fac-len′*: $|w| \leq |u \cdot w \cdot v|$
⟨*proof*⟩

lemma *fac-len-eq*: $u \leq f\ v \Longrightarrow |u| = |v| \Longrightarrow u = v$
⟨*proof*⟩

thm *length-take*

lemma *len-take1*: $|take\ p\ w| \leq p$
⟨*proof*⟩

lemma *len-take2*: $|take\ p\ w| \leq |w|$

⟨*proof*⟩

lemma *drop-len*: $|u \cdot w| \leq |u \cdot v \cdot w|$
⟨*proof*⟩

lemma *drop-pref*: $drop\ |u|\ (u \cdot w) = w$
⟨*proof*⟩

lemma *take-len*: $p \leq |w| \Longrightarrow |take\ p\ w| = p$
⟨*proof*⟩

lemma *eq-conjug-len*: $p \cdot x = x \cdot s \Longrightarrow |p| = |s|$
⟨*proof*⟩

lemma *take-nemp-len*: $u \neq \varepsilon \Longrightarrow r \neq \varepsilon \Longrightarrow take\ |r|\ u \neq \varepsilon$
⟨*proof*⟩

lemma $r \neq \varepsilon \Longrightarrow w \neq \varepsilon \Longrightarrow drop\ |r|\ w \neq w$
⟨*proof*⟩

lemma *emp-len*: $w = \varepsilon \Longrightarrow |w| = 0$
⟨*proof*⟩

lemma *take-self*: $take\ |w|\ w = w$
⟨*proof*⟩

lemma *len-le-concat*: $\varepsilon \notin set\ ws \Longrightarrow |ws| \leq |concat\ ws|$
⟨*proof*⟩

lemma *eq-len-iff*: **assumes** *eq*: $x \cdot y = u \cdot v$ **shows** $|x| \leq |u| \longleftrightarrow |v| \leq |y|$
⟨*proof*⟩

lemma *eq-len-iff-less*: **assumes** *eq*: $x \cdot y = u \cdot v$ **shows** $|x| < |u| \longleftrightarrow |v| < |y|$
⟨*proof*⟩

lemma *Suc-len-nemp*: $|w| = Suc\ n \Longrightarrow w \neq \varepsilon$
⟨*proof*⟩

lemma *same-sufix-nil*: **assumes** $z \cdot u \leq p\ u$ **shows** $z = \varepsilon$
⟨*proof*⟩

lemma *count-list-gr-0-iff*: $0 < count\text{-}list\ u\ a \longleftrightarrow a \in set\ u$
⟨*proof*⟩

lemma *bin-len-concat*: **assumes** $u \neq v\ ws \in lists\ \{u,v\}$

shows $|concat\ ws| = count\text{-}list\ ws\ u * |u| + count\text{-}list\ ws\ v * |v|$
⟨*proof*⟩

2.4 List inspection method

In this section we define a proof method, named list_inspection, which splits the goal into subgoals which enumerate possibilities on lists with fixed length and given alphabet. More specifically, it looks for a premise of the form such as $|w| = 2 \wedge w \in lists\ \{x, y, z\}$ or $|w| \leq 2 \wedge w \in lists\ \{x, y, z\}$ and substitutes the goal by the goals listing all possibilities for the word w, see demonstrations after the method definition.

context
begin

First, we define an elementary lemma used for unfolding the premise. Since it is very specific, we keep it private.

private lemma *hd-tl-len-list-iff*: $|w| = Suc\ n \wedge w \in lists\ A \longleftrightarrow hd\ w \in A \wedge\ w = hd\ w\ \#\ tl\ w \wedge |tl\ w| = n \wedge tl\ w \in lists\ A$ (**is** *?L = ?R*)
⟨*proof*⟩

We define a list of lemmas used for the main unfolding step.

private lemmas *len-list-word-dec* =
numeral-nat hd-tl-len-list-iff
insert-iff empty-iff simp-thms length-0-conv

The method itself accepts an argument called ‘add‘, which is supplied to the method simp_all to solve some simple cases, and thus lower the number of produced goals on the fly.

method *list-inspection0* **uses** *add* =
((*match* **premises in** *len*[*thin*]: $|w| \leq k$ **and** *list*[*thin*]: $w \in lists\ A$ **for** *w k A* ⇒
‹*insert conjI*[*OF len list*]›)+) *?*,
(*unfold numeral-nat le-Suc-eq le-0-eq*), — unfold numeral and e.g. $k \leq (2::'a)$
(*unfold conj-ac*(*1*)[*of* $w \in lists\ A\ |w| \leq k$ **for** *w A k*]
conj-disj-distribR[**where** *?R* = $w \in lists\ A$ **for** *w A*]) *?*,
((*match* **premises in** *len*[*thin*]: $|w| = k$ **and** *list*[*thin*]: $w \in lists\ A$ **for** *w k A* ⇒
‹*insert conjI*[*OF len list*]›)+) *?*,
— transform into the conjuction such as $|w| = 2 \wedge w \in lists\ \{x, y, z\}$
(*unfold conj-ac*(*1*)[*of* $w \in lists\ A\ |w| = k$ **for** *w A k*] *len-list-word-dec*), — unfold w
(*elim disjE conjE*), — split into all cases
(*simp-all only*: *singleton-iff lists.Nil list.sel refl True-implies-equals*) *?*, — replace w everywhere
(*simp-all only*: *add empty-iff lists.Nil bool-simps*) *?* — solve simple cases

method *list-inspection* = (*insert method-facts*, *use nothing* **in** *list-inspection0*)

List inspection demonstrations

The required premise in the form of conjuction. First, inequality bound on the length, second, equality bound.

lemma $|w| = 2 \wedge w \in \mathit{lists}\ \{x,y,z\} \Longrightarrow P\ w$
⟨*proof*⟩

The required premise as 2 separate premises.

lemma $|w| \leq 2 \Longrightarrow w \in \mathit{lists}\ \{x,y,z\} \Longrightarrow P\ w$
⟨*proof*⟩

The required premise as 2 separate assumptions.

lemma assumes $|w| \leq 2$ **and** $w \in \mathit{lists}\ \{x,y,z\}$ **shows** $P\ w$
⟨*proof*⟩

lemma $w \leq p\ w \Longrightarrow |w| \leq 2 \Longrightarrow w \in \mathit{lists}\ \{a,b\} \Longrightarrow \mathit{hd}\ w = a \Longrightarrow w \neq \varepsilon \Longrightarrow w = [a,\ b] \vee w = [a,\ a] \vee w = [a]$
⟨*proof*⟩

lemma $w \leq p\ w \Longrightarrow |w| = 2 \Longrightarrow w \in \mathit{lists}\ \{a,b\} \Longrightarrow \mathit{hd}\ w = a \Longrightarrow w = [a,\ b] \vee w = [a,\ a]$
⟨*proof*⟩

lemma $w \leq p\ w \Longrightarrow |w| = 2 \wedge w \in \mathit{lists}\ \{a,b\} \Longrightarrow \mathit{hd}\ w = a \Longrightarrow w = [a,\ b] \vee w = [a,\ a]$
⟨*proof*⟩

lemma $w \leq p\ w \Longrightarrow w \in \mathit{lists}\ \{a,b\} \wedge |w| = 2 \Longrightarrow \mathit{hd}\ w = a \Longrightarrow w = [a,\ b] \vee w = [a,\ a]$
⟨*proof*⟩

end

2.5 Prefix and prefix comparability properties

lemmas *pref-emp* = *prefix-bot.extremum-uniqueI*

lemma *triv-pref*: $r \leq p\ r \cdot s$
⟨*proof*⟩

lemma *triv-spref*: $s \neq \varepsilon \Longrightarrow r <p\ r \cdot s$
⟨*proof*⟩

lemma *pref-cancel*: $z \cdot u \leq p\ z \cdot v \Longrightarrow u \leq p\ v$
⟨*proof*⟩

lemma *pref-cancel'* [*intro*]: $u \leq_p v \Longrightarrow z \cdot u \leq_p z \cdot v$
⟨*proof*⟩

lemma *spref-cancel*: $z \cdot u <_p z \cdot v \Longrightarrow u <_p v$
⟨*proof*⟩

lemma *spref-of-nemp* [*intro*]: $u <_p w \Longrightarrow w \neq \varepsilon$
⟨*proof*⟩

lemma *spref-cancel'* [*intro*]: $u <_p v \Longrightarrow z \cdot u <_p z \cdot v$
⟨*proof*⟩

lemmas *pref-cancel-conv* = *same-prefix-prefix* **and**
suf-cancel-conv = *same-suffix-suffix* — provided by Sublist.thy

lemma *pref-cancel-hd-conv*: $a \# u \leq_p a \# v \longleftrightarrow u \leq_p v$
⟨*proof*⟩

lemma *spref-cancel-conv*: $z \cdot x <_p z \cdot y \longleftrightarrow x <_p y$
⟨*proof*⟩

lemma *spref-snoc-iff* [*simp*]: $u <_p v \cdot [a] \longleftrightarrow u \leq_p v$
⟨*proof*⟩

lemma *butlast-not-pref*[*intro*, *simp*] : **assumes** $u \neq \varepsilon$ **shows** $\neg\ u \leq_p butlast\ u$
⟨*proof*⟩

lemma *spref-two-lettersE*: **assumes** $p <_p [a,b]$ **obtains** $p = \varepsilon \mid p = [a]$
⟨*proof*⟩

lemmas *pref-ext*[*intro*] = *prefix-prefix* — provided by Sublist.thy

lemmas *pref-extD* = *append-prefixD* **and**
suf-extD = *suffix-appendD*

lemma *spref-extD*: $x \cdot y <_p z \Longrightarrow x <_p z$
⟨*proof*⟩

lemma *spref-ext*: $r <_p u \Longrightarrow r <_p u \cdot v$
⟨*proof*⟩

lemma *pref-ext-nemp*: $r \leq_p u \Longrightarrow v \neq \varepsilon \Longrightarrow r <_p u \cdot v$
⟨*proof*⟩

lemma *pref-take*: $p \leq_p w \Longrightarrow take\ |p|\ w = p$
⟨*proof*⟩

lemma *pref-take-conv*: $take\ (|r|)\ w = r \longleftrightarrow r \leq_p w$

⟨*proof*⟩

lemma *le-suf-drop*: **assumes** $i \leq j$ **shows** *drop* $j\ w \leq_s$ *drop* $i\ w$
⟨*proof*⟩

lemma *spref-take*: $p <_p w \Longrightarrow$ *take* $|p|\ w = p$
⟨*proof*⟩

lemma *pref-same-len*: $u \leq_p v \Longrightarrow |u| = |v| \Longrightarrow u = v$
⟨*proof*⟩

lemma *pref-same-len'*: $u \cdot z \leq_p v \cdot w \Longrightarrow |u| = |v| \Longrightarrow u = v$
⟨*proof*⟩

lemma *pref-comp-eq*: $u \bowtie v \Longrightarrow |u| = |v| \Longrightarrow u = v$
⟨*proof*⟩

lemma *ruler-eq-len*: $u \leq_p w \Longrightarrow v \leq_p w \Longrightarrow |u| = |v| \Longrightarrow u = v$
⟨*proof*⟩

lemma *pref-prod-eq*: $u \leq_p v \cdot z \Longrightarrow |u| = |v| \Longrightarrow u = v$
⟨*proof*⟩

lemmas *pref-comm-eq* = *pref-same-len*[*OF - swap-len*] **and**
pref-comm-eq' = *pref-prod-eq*[*OF - swap-len*, *unfolded rassoc*]

lemma *pref-comm-eq-conv*: $u \cdot v \leq_p v \cdot u \longleftrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *add-nth-pref*: **assumes** $u <_p w$ **shows** $u \cdot [w!|u|] \leq_p w$
⟨*proof*⟩

lemma *index-pref*: $|u| \leq |w| \Longrightarrow (\forall\ i < |u|.\ \ u!i = w!i) \Longrightarrow u \leq_p w$
⟨*proof*⟩

lemma *pref-index*: **assumes** $u \leq_p w\ i < |u|$ **shows** $u!i = w!i$
⟨*proof*⟩

lemma *pref-drop*: $u \leq_p v \Longrightarrow$ *drop* $p\ u \leq_p$ *drop* $p\ v$
⟨*proof*⟩

lemma *pref-prod-short*: $u \leq_p v \cdot w \Longrightarrow |u| < |v| \Longrightarrow u <_p v$
⟨*proof*⟩

2.5.1 Prefix comparability

lemma *pref-comp-sym*[*sym*]: $u \bowtie v \Longrightarrow v \bowtie u$
⟨*proof*⟩

lemma *not-pref-comp-sym*[*sym*]: $\neg\ u \bowtie v \Longrightarrow \neg\ v \bowtie u$
⟨*proof*⟩

lemma *pref-comp-sym-iff*: $u \bowtie v \longleftrightarrow v \bowtie u$
⟨*proof*⟩

lemmas *ruler-le* = *prefix-length-prefix* **and**
ruler = *prefix-same-cases* **and**
ruler′ = *prefix-same-cases*[*folded prefix-comparable-def*]

lemma *ruler-prefs*: **assumes** $L = R$ $|u| \leq |v|$ $u \leq_p L$ $v \leq_p R$
shows $u \leq_p v$
⟨*proof*⟩

lemmas *ruler-sufs* = *ruler-prefs*[*reversed*]

lemma *ruler-eq*: $u \cdot x = v \cdot y \Longrightarrow u \leq_p v \vee v \leq_p u$
⟨*proof*⟩

lemma *ruler-eq′*: $u \cdot x = v \cdot y \Longrightarrow u \leq_p v \vee v <_p u$
⟨*proof*⟩

lemmas *ruler-eqE* = *ruler-eq*[*THEN disjE*] **and**
ruler-eqE′ = *ruler-eq′*[*THEN disjE*] **and**
ruler-pref = *ruler*[*OF append-prefixD triv-pref*] **and**
ruler′[*THEN pref-comp-eq*]
lemmas *ruler-prefE* = *ruler-pref*[*THEN disjE*]

lemma *ruler-comp*: $u \leq_p v \Longrightarrow u' \leq_p v' \Longrightarrow v \bowtie v' \Longrightarrow u \bowtie u'$
⟨*proof*⟩

lemma *ruler-pref′*: $w \leq_p v{\cdot}z \Longrightarrow w \leq_p v \vee v \leq_p w$
⟨*proof*⟩

lemma *ruler-pref″*: $w \leq_p v{\cdot}z \Longrightarrow w \bowtie v$
⟨*proof*⟩

lemma *pref-prod-pref-short*: $u \leq_p z \cdot w \Longrightarrow v \leq_p w \Longrightarrow |u| \leq |z \cdot v| \Longrightarrow u \leq_p z \cdot v$
⟨*proof*⟩

lemma *pref-prod-pref*: $u \leq_p z \cdot w \Longrightarrow u \leq_p w \Longrightarrow\ u \leq_p z \cdot u$
⟨*proof*⟩

lemma *pref-prod-pref′*: **assumes** $u \leq_p z{\cdot}u{\cdot}w$ **shows** $u \leq_p z{\cdot}u$
⟨*proof*⟩

lemma *pref-prod-long*: $u \leq_p v \cdot w \Longrightarrow |v| \leq |u| \Longrightarrow v \leq_p u$

⟨*proof*⟩

lemmas *pref-prod-long-ext* = *pref-prod-long*[*OF append-prefixD*]

lemma *pref-prod-long-less*: **assumes** $u \leq_p v \cdot w$ **and** $|v| < |u|$ **shows** $v <_p u$
⟨*proof*⟩

lemma *ruler-less*: $ps \leq_p xs \Longrightarrow qs \leq_p xs \Longrightarrow |ps| < |qs| \Longrightarrow ps <_p qs$
⟨*proof*⟩

lemma *ruler-le-neq*: $ps \leq_p xs \Longrightarrow qs \leq_p xs \Longrightarrow |ps| \leq |qs| \Longrightarrow ps \neq qs \Longrightarrow ps <_p qs$
⟨*proof*⟩

lemma *pref-keeps-per-root*: $u \leq_p r \cdot u \Longrightarrow v \leq_p u \Longrightarrow v \leq_p r \cdot v$
⟨*proof*⟩

lemma *pref-keeps-per-root'*: $u <_p r \cdot u \Longrightarrow v \leq_p u \Longrightarrow v <_p r \cdot v$
⟨*proof*⟩

lemma *per-root-pref-sing*: $w <_p r \cdot w \Longrightarrow u \cdot [a] \leq_p w \Longrightarrow u \cdot [a] \leq_p r \cdot u$
⟨*proof*⟩

lemma *pref-prolong*: $w \leq_p z \cdot r \Longrightarrow r \leq_p s \Longrightarrow w \leq_p z \cdot s$
⟨*proof*⟩

lemma *spref--pref-prolong*: $w <_p z \cdot r \Longrightarrow r \leq_p s \Longrightarrow w <_p z \cdot s$
⟨*proof*⟩

lemma *pref-spref-prolong*: $w \leq_p z \cdot r \Longrightarrow r <_p s \Longrightarrow w <_p z \cdot s$
⟨*proof*⟩

lemma *spref-spref-prolong*: $w <_p z \cdot r \Longrightarrow r <_p s \Longrightarrow w <_p z \cdot s$
⟨*proof*⟩

lemmas *pref-shorten* = *pref-trans*[*OF pref-cancel'*]

lemma *pref-prolong'*: $u \leq_p w \cdot z \Longrightarrow v \cdot u \leq_p z \Longrightarrow u \leq_p w \cdot v \cdot u$
⟨*proof*⟩

lemma *pref-prolong-per-root*: $u \leq_p r \cdot s \Longrightarrow s \leq_p r \cdot s \Longrightarrow u \leq_p r \cdot u$
⟨*proof*⟩

lemma *pref-prod-le*[*intro*]: $u \leq_p v \cdot w \Longrightarrow |u| \leq |v| \Longrightarrow u \leq_p v$
⟨*proof*⟩

lemma *prod-pref-prod-le*: $u \cdot v \leq_p x \cdot y \Longrightarrow |u| \leq |x| \Longrightarrow u \leq_p x$
⟨*proof*⟩

lemma *pref-cancel-right-len*: $u \cdot z \leq_p v \cdot z' \Longrightarrow |z| = |z'| \Longrightarrow u \leq_p v$
⟨*proof*⟩

lemma *spref-cancel-right-len*: **assumes** $u \cdot z <_p v \cdot z'$ $|z| = |z'|$
shows $u <_p v$
⟨*proof*⟩

lemmas *pref-cancel-right* = *pref-cancel-right-len*[*OF* - *refl*] **and**
spref-cancel-right = *spref-cancel-right-len*[*OF* - *refl*]

lemma *eq-le-pref*[*elim*]: $x \cdot y = u \cdot v \Longrightarrow |x| \leq |u| \Longrightarrow x \leq_p u$
⟨*proof*⟩

lemma *pref-less-spref*: $u \cdot w \leq_p v \cdot z \Longrightarrow |u| < |v| \Longrightarrow u <_p v$
⟨*proof*⟩

lemma *pref-prod-cancel*: **assumes** $u \leq_p p \cdot w \cdot q$ **and** $|p| \leq |u|$ **and** $|u| \leq |p \cdot w|$
obtains r **where** $p \cdot r = u$ **and** $r \leq_p w$
⟨*proof*⟩

lemma *pref-prod-cancel'*: **assumes** $u \leq_p p \cdot w \cdot q$ **and** $|p| < |u|$ **and** $|u| \leq |p \cdot w|$
obtains r **where** $p \cdot r = u$ **and** $r \leq_p w$ **and** $r \neq \varepsilon$
⟨*proof*⟩

lemma *non-comp-parallel*: $\neg\ u \bowtie v \longleftrightarrow u \parallel v$
⟨*proof*⟩

lemma *comp-refl*: $u \bowtie u$
⟨*proof*⟩

lemma *incomp-cancel*: $\neg\ p \cdot u \bowtie p \cdot v \Longrightarrow \neg\ u \bowtie v$
⟨*proof*⟩

lemma *comm-ruler*: $r \cdot s \leq_p w1 \Longrightarrow s \cdot r \leq_p w2 \Longrightarrow w1 \bowtie w2 \Longrightarrow r \cdot s = s \cdot r$
⟨*proof*⟩

lemma *comm-comp-eq*: $r \cdot s \bowtie s \cdot r \Longrightarrow r \cdot s = s \cdot r$
⟨*proof*⟩

lemma *pref-share-take*: $u \leq_p v \Longrightarrow q \leq |u| \Longrightarrow take\ q\ u = take\ q\ v$
⟨*proof*⟩

lemma *pref-prod-longer*: $u \leq_p z \cdot w \Longrightarrow v \leq_p w \Longrightarrow |z \cdot v| \leq |u| \Longrightarrow z \cdot v \leq_p u$
⟨*proof*⟩

lemma *pref-comp-not-pref*: $u \bowtie v \Longrightarrow \neg\ v \leq p\ u \Longrightarrow u <p\ v$
⟨*proof*⟩

lemma *pref-comp-not-spref*: $u \bowtie v \Longrightarrow \neg\ u <p\ v \Longrightarrow v \leq p\ u$
⟨*proof*⟩

lemma *hd-prod*: $u \neq \varepsilon \Longrightarrow (u \cdot v)!0 = u!0$
⟨*proof*⟩

lemma *distinct-first*: **assumes** $w \neq \varepsilon\ z \neq \varepsilon\ w!0 \neq z!0$ **shows** $w \cdot w' \neq z \cdot z'$
⟨*proof*⟩

lemmas *last-no-split* = *prefix-snoc*

lemma *last-no-split'*: $u <p\ w \Longrightarrow w \leq p\ u \cdot [a] \Longrightarrow w = u \cdot [a]$
⟨*proof*⟩

lemma *comp-shorter*: $v \bowtie w \Longrightarrow |v| \leq |w| \Longrightarrow v \leq p\ w$
⟨*proof*⟩

lemma *comp-shorter-conv*: $|u| \leq |v| \Longrightarrow u \bowtie v \longleftrightarrow u \leq p\ v$
⟨*proof*⟩

lemma *pref-comp-len-trans*: $w \leq p\ v \Longrightarrow u \bowtie v \Longrightarrow |w| \leq |u| \Longrightarrow w \leq p\ u$
⟨*proof*⟩

lemma *comp-cancel*: $z \cdot w1 \bowtie z \cdot w2 \longleftrightarrow w1 \bowtie w2$
⟨*proof*⟩

lemma *emp-pref*: $\varepsilon \leq p\ u$
⟨*proof*⟩

lemma *emp-spref*: $u \neq \varepsilon \Longrightarrow \varepsilon <p\ u$
⟨*proof*⟩

lemma *long-pref*: $u \leq p\ v \Longrightarrow |v| \leq |u| \Longrightarrow u = v$
⟨*proof*⟩

lemma *not-comp-ext*: $\neg\ w1 \bowtie\ w2 \Longrightarrow \neg\ w1 \cdot z \bowtie w2 \cdot z'$
⟨*proof*⟩

lemma *mismatch-incopm*: $|u| = |v| \Longrightarrow x \neq y \Longrightarrow \neg\ u \cdot [x] \bowtie v \cdot [y]$
⟨*proof*⟩

lemma *comp-prefs-comp*: $u \cdot z \bowtie v \cdot w \Longrightarrow u \bowtie v$
⟨*proof*⟩

lemma *comp-hd-eq*: $u \bowtie v \Longrightarrow u \neq \varepsilon \Longrightarrow v \neq \varepsilon \Longrightarrow hd\ u = hd\ v$
⟨*proof*⟩

lemma *pref-hd-eq'*: $p \le_p u \Longrightarrow p \le_p v \Longrightarrow p \neq \varepsilon \Longrightarrow hd\ u = hd\ v$
⟨*proof*⟩

lemma *pref-hd-eq*: $u \le_p v \Longrightarrow u \neq \varepsilon \Longrightarrow hd\ u = hd\ v$
⟨*proof*⟩

lemma *sing-pref-hd*: $[a] \le_p v \Longrightarrow hd\ v = a$
⟨*proof*⟩

lemma *suf-last-eq*: $p \le_s u \Longrightarrow p \le_s v \Longrightarrow p \neq \varepsilon \Longrightarrow last\ u = last\ v$
⟨*proof*⟩

lemma *comp-hd-eq'*: $u \cdot r \bowtie v \cdot s \Longrightarrow u \neq \varepsilon \Longrightarrow v \neq \varepsilon \Longrightarrow hd\ u = hd\ v$
⟨*proof*⟩

2.5.2 Minimal and maximal prefix with a given property

lemma *le-take-pref*: **assumes** $k \le n$ **shows** $take\ k\ ws \le_p take\ n\ ws$
⟨*proof*⟩

lemma *min-pref*: **assumes** $u \le_p w$ **and** $P\ u$
obtains v **where** $v \le_p w$ **and** $P\ v$ **and** $\bigwedge y.\ y \le_p w \Longrightarrow P\ y \Longrightarrow v \le_p y$
⟨*proof*⟩

lemma *min-pref'*: **assumes** $u \le_p w$ **and** $P\ u$
obtains v **where** $v \le_p w$ **and** $P\ v$ **and** $\bigwedge y.\ y \le_p v \Longrightarrow P\ y \Longrightarrow y = v$
⟨*proof*⟩

lemma *max-pref*: **assumes** $u \le_p w$ **and** $P\ u$
obtains v **where** $v \le_p w$ **and** $P\ v$ **and** $\bigwedge y.\ y \le_p w \Longrightarrow P\ y \Longrightarrow y \le_p v$
⟨*proof*⟩

2.6 Suffix and suffix comparability properties

lemmas *suf-emp* = *suffix-bot.extremum-uniqueI*

lemma *triv-suf*: $u \le_s v \cdot u$
⟨*proof*⟩

lemma *emp-ssuf*: $u \neq \varepsilon \Longrightarrow \varepsilon <_s u$
⟨*proof*⟩

lemma *suf-cancel*: $u{\cdot}v \le_s w{\cdot}v \Longrightarrow u \le_s w$
⟨*proof*⟩

lemma *suf-cancel'* [*intro*]: $u \le_s w \Longrightarrow u{\cdot}v \le_s w{\cdot}v$
⟨*proof*⟩

lemma *ssuf-cancel′* [*intro*]: $u <s w \Longrightarrow u{\cdot}v <s w{\cdot}v$
⟨*proof*⟩

lemma *ssuf-cancel-conv*: $x \cdot z <s y \cdot z \longleftrightarrow x <s y$
⟨*proof*⟩

Straightforward relations of suffix and prefix follow.

lemmas *suf-rev-pref-iff* = *suffix-to-prefix* — provided by Sublist.thy

lemmas *ssuf-rev-pref-iff* = *strict-suffix-to-prefix* — provided by Sublist.thy

lemma *pref-rev-suf-iff*: $u \leq p\ v \longleftrightarrow rev\ u \leq s\ rev\ v$
⟨*proof*⟩

lemma *spref-rev-suf-iff*: $s <p\ w \longleftrightarrow rev\ s <s\ rev\ w$
⟨*proof*⟩

lemmas [*reversal-rule*] =
suf-rev-pref-iff[*symmetric*]
pref-rev-suf-iff[*symmetric*]
ssuf-rev-pref-iff[*symmetric*]
spref-rev-suf-iff[*symmetric*]

lemmas *sufE* = *prefixE*[*reversed*] **and**
prefE = *prefixE* **and**
ssuf-exE[*elim*] = *spref-exE*[*reversed*]

lemmas *suf-prod-long-ext* = *pref-prod-long-ext*[*reversed*]

lemmas *suf-prolong-per-root* = *pref-prolong-per-root*[*reversed*]

lemmas *suf-ext* = *suffix-appendI* — provided by Sublist.thy

lemmas *ssuf-ext* = *spref-ext*[*reversed*] **and**
ssuf-extD = *spref-extD*[*reversed*] **and**
suf-ext-nem = *pref-ext-nemp*[*reversed*] **and**
suf-same-len = *pref-same-len*[*reversed*] **and**
suf-take = *pref-drop*[*reversed*] **and**
suf-share-take = *pref-share-take*[*reversed*] **and**
long-suf = *long-pref*[*reversed*] **and**
strict-suffixE′ = *strict-prefixE′*[*reversed*] **and**
ssuf-tl-suf = *spref-butlast-pref*[*reversed*]

lemma *ssuf-Cons-iff* [*simp*]: $u <s a \# v \longleftrightarrow u \leq s\ v$
⟨*proof*⟩

lemma *ssuf-induct* [*case-names ssuf*]:
assumes $\bigwedge u.\ (\bigwedge v.\ v <s u \Longrightarrow P\ v) \Longrightarrow P\ u$

shows $P\ u$
⟨*proof*⟩

2.6.1 Suffix comparability

lemma *eq-le-suf*[*elim*]: **assumes** $x \cdot y = u \cdot v\ |x| \leq |u|$ **shows** $v \leq_s y$
⟨*proof*⟩

lemmas *eq-le-suf'*[*elim*] = *eq-le-pref*[*reversed*]

lemma *eq-le-suf''*[*elim*]: **assumes** $v \cdot u = y \cdot x\ |x| \leq |u|$ **shows** $x \leq_s u$
⟨*proof*⟩

lemma *pref-comp-rev-suf-comp*[*reversal-rule*]: $(rev\ w) \bowtie_s (rev\ v) \longleftrightarrow w \bowtie v$
⟨*proof*⟩

lemma *suf-comp-rev-pref-comp*[*reversal-rule*]: $(rev\ w) \bowtie (rev\ v) \longleftrightarrow w \bowtie_s v$
⟨*proof*⟩

lemmas *suf-ruler-le* = *suffix-length-suffix* — provided by Sublist.thy, same as ruler_le[reversed]

lemmas *suf-ruler* = *suffix-same-cases* — provided by Sublist.thy, same as ruler[reversed]

lemmas *suf-ruler-eq-len* = *ruler-eq-len*[*reversed*] **and**
suf-ruler-comp = *ruler-comp*[*reversed*] **and**
ruler-suf = *ruler-pref*[*reversed*] **and**
ruler-suf' = *ruler-pref'*[*reversed*] **and**
ruler-suf'' = *ruler-pref''*[*reversed*] **and**
suf-prod-le = *pref-prod-le*[*reversed*] **and**
prod-suf-prod-le = *prod-pref-prod-le*[*reversed*] **and**
suf-prod-eq = *pref-prod-eq*[*reversed*] **and**
suf-prod-cancel = *pref-prod-cancel*[*reversed*] **and**
suf-prod-cancel' = *pref-prod-cancel'*[*reversed*] **and**
suf-prod-suf-short = *pref-prod-pref-short*[*reversed*] **and**
suf-prod-suf = *pref-prod-pref*[*reversed*] **and**
suf-prod-suf' = *pref-prod-pref'*[*reversed*, *unfolded rassoc*] **and**
suf-prolong = *pref-prolong*[*reversed*] **and**
suf-prolong' = *pref-prolong'*[*reversed*, *unfolded rassoc*] **and**
suf-prod-long = *pref-prod-long*[*reversed*] **and**
suf-prod-long-less = *pref-prod-long-less*[*reversed*] **and**
suf-prod-longer = *pref-prod-longer*[*reversed*] **and**
suf-keeps-root = *pref-keeps-per-root*[*reversed*] **and**
comm-suf-ruler = *comm-ruler*[*reversed*] **and**
suf-ruler-less = *ruler-less*[*reversed*] **and**
suf-ruler-le-neq = *ruler-le-neq*[*reversed*]

lemmas *comp-sufs-comp* = *comp-prefs-comp*[*reversed*] **and**

suf-comp-not-suf = *pref-comp-not-pref*[*reversed*] **and**
suf-comp-not-ssuf = *pref-comp-not-spref*[*reversed*] **and**
suf-comp-cancel = *comp-cancel*[*reversed*] **and**
suf-not-comp-ext = *not-comp-ext*[*reversed*] **and**
mismatch-suf-incopm = *mismatch-incopm*[*reversed*] **and**
suf-comp-sym[*sym*] = *pref-comp-sym*[*reversed*] **and**
suf-comp-refl = *comp-refl*[*reversed*]

lemma *suf-comp-or*: $u \bowtie_s v \longleftrightarrow u \leq s\ v \vee v \leq s\ u$
⟨*proof*⟩

lemma *comm-comp-eq-conv*: $r \cdot s \bowtie s \cdot r \longleftrightarrow r \cdot s = s \cdot r$
⟨*proof*⟩

lemma *comm-comp-eq-conv-suf*: $r \cdot s \bowtie_s s \cdot r \longleftrightarrow r \cdot s = s \cdot r$
⟨*proof*⟩

lemma *suf-comp-last-eq*: **assumes** $u \bowtie_s v$ $u \neq \varepsilon$ $v \neq \varepsilon$
shows *last* u = *last* v
⟨*proof*⟩

lemma *suf-comp-last-eq'*: $r \cdot u \bowtie_s s \cdot v \Longrightarrow u \neq \varepsilon \Longrightarrow v \neq \varepsilon \Longrightarrow$ *last* u = *last* v
⟨*proof*⟩

2.7 Left and Right Quotient

A useful function of left quotient is given. Note that the function is sometimes undefined.

definition *left-quotient*:: *'a list* ⇒ *'a list* ⇒ *'a list* ((-$^{-1>}$)(-) [*75*,*74*] *74*)
where *left-quotient u v* = *drop* |*u*| *v*
notation (*latex* **output**) *left-quotient* (- $^{-1}$ · -)

Analogously, we define the right quotient.

definition *right-quotient* :: *'a list* ⇒ *'a list* ⇒ *'a list* ((-)($^{<-1}$-) [*76*,*77*] *76*)
where *right-quotient u v* = *rev* ((*rev v*)$^{-1>}$(*rev u*))
notation (*latex* **output**) *right-quotient* (- · - $^{-1}$)

lemmas *lq-def* = *left-quotient-def* **and**
rq-def = *right-quotient-def*

Priorities of these operations are as follows:

lemma $u^{<-1}v^{<-1}w = (u^{<-1}v)^{<-1}w$
⟨*proof*⟩

lemma $u^{-1>}v^{-1>}w = u^{-1>}(v^{-1>}w)$
⟨*proof*⟩

lemma $u^{-1>}v^{<-1}w = u^{-1>}(v^{<-1}w)$
⟨*proof*⟩

lemma $r \cdot u^{-1>}w^{<-1}v \cdot s = r \cdot (u^{-1>}w^{<-1}v) \cdot s$
⟨*proof*⟩

lemma *rq-rev-lq*[*reversal-rule*]: $(rev\ v)^{<-1}(rev\ u) = rev\ (u^{-1>}v)$
⟨*proof*⟩

lemma *lq-rev-rq*[*reversal-rule*]: $(rev\ v)^{-1>}rev\ u = rev\ (u^{<-1}v)$
⟨*proof*⟩

2.7.1 Left Quotient

lemma *lqI*: $u \cdot z = v \Longrightarrow u^{-1>}v = z$
⟨*proof*⟩

lemma *lq-triv*[*simp*]: $u^{-1>}(u \cdot z) = z$
⟨*proof*⟩

lemma *lq-triv'*[*simp*]: $u \cdot u^{-1>}(u \cdot z) = u \cdot z$
⟨*proof*⟩

lemma *append-lq*: **assumes** $u{\cdot}v \leq p\ w$ **shows** $(u{\cdot}v)^{-1>}w = v^{-1>}(u^{-1>}w)$
⟨*proof*⟩

lemma *lq-self*[*simp*]: $u^{-1>}u = \varepsilon$
⟨*proof*⟩

lemma *lq-emp*[*simp*]: $\varepsilon^{-1>}u = u$
⟨*proof*⟩

lemma *lq-pref*[*simp*]: $u \leq p\ v \Longrightarrow u \cdot (u^{-1>}v) = v$
⟨*proof*⟩

lemmas *lq-spref* = *lq-pref*[*OF sprefD1*]

lemma *lq-pref-conv*: $|u| \leq |v| \Longrightarrow u \leq p\ v \longleftrightarrow u \cdot u^{-1>}v = v$
⟨*proof*⟩

lemma *lq-len*: $|u^{-1>}v| = |v| - |u|$
⟨*proof*⟩

lemmas *lcp-lq* = *lq-pref*[*OF longest-common-prefix-prefix1*] *lq-pref*[*OF longest-common-prefix-prefix2*]

lemma *lq-pref-cancel*: $u \leq p\ v \Longrightarrow v \cdot r = u \cdot s \Longrightarrow (u^{-1>}v) \cdot r = s$
⟨*proof*⟩

lemma *lq-the*: **assumes** $u \leq p\ v$

shows $(THE\ z.\ u \cdot z = v) = (u^{-1>}v)$
⟨*proof*⟩

lemma *lq-same-len*: $|u| = |v| \Longrightarrow u^{-1>}v = \varepsilon$
⟨*proof*⟩

lemma *lq-assoc*: $|u| \leq |v| \Longrightarrow (u^{-1>}v)^{-1>}w = v^{-1>}(u \cdot w)$
⟨*proof*⟩

lemma *lq-assoc'*: $(u \cdot w)^{-1>}v = w^{-1>}(u^{-1>}v)$
⟨*proof*⟩

lemma *lq-reassoc*: $u \leq p\ v \Longrightarrow (u^{-1>}v)\cdot w = u^{-1>}(v\cdot w)$
⟨*proof*⟩

lemma *lq-trans*: $u \leq p\ v \Longrightarrow v \leq p\ w \Longrightarrow (u^{-1>}v) \cdot (v^{-1>}w) = u^{-1>}w$
⟨*proof*⟩

lemma *lq-rq-reassoc-suf*: **assumes** $u \leq p\ z\ u \leq s\ w$ **shows** $w\cdot u^{-1>}z = w^{<-1}u \cdot z$
⟨*proof*⟩

lemma *lq-ne*: **assumes** $u \neq \varepsilon$ **shows** $p^{-1>}(u\cdot p) \neq \varepsilon$
⟨*proof*⟩

lemma *lq-spref-nemp*: $u <p\ v \Longrightarrow u^{-1>}v \neq \varepsilon$
⟨*proof*⟩

lemma *lq-is-suf*[*simp*]: $r^{-1>}s \leq s\ s$
⟨*proof*⟩

lemma *lq-short-len*: $r \leq p\ s \Longrightarrow |r| + |r^{-1>}s| = |s|$
⟨*proof*⟩

lemma *pref-lq*: $v \leq p\ w \Longrightarrow u^{-1>}v \leq p\ u^{-1>}w$
⟨*proof*⟩

lemma *spref-lq*: $u \leq p\ v \Longrightarrow v <p\ w \Longrightarrow u^{-1>}v <p\ u^{-1>}w$
⟨*proof*⟩

lemma *pref-gcd-lq*: **assumes** $u \leq p\ v$ **shows** $(gcd\ |u|\ |u^{-1>}v|) = gcd\ |u|\ |v|$
⟨*proof*⟩

lemma *conjug-lq*: $x \cdot z = z \cdot y \Longrightarrow y = z^{-1>}(x \cdot z)$
⟨*proof*⟩

lemma *conjug-emp-emp*: $p \leq p\ u \cdot p \Longrightarrow p^{-1>}(u \cdot p) = \varepsilon \Longrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *hd-lq-conv-nth*: **assumes** $u <p\ v$ **shows** $hd(u^{-1>}v) = v!|u|$

⟨*proof*⟩

lemma *concat-morph-lq*: $us \leq p\ ws \Longrightarrow concat\ (us^{-1>} ws) = (concat\ us)^{-1>} (concat\ ws)$
⟨*proof*⟩

lemma *pref-suf-len-split*: **assumes** $p \leq p\ u\ s \leq s\ u\ |p \cdot s| = |u|$
shows $p \cdot s = u$
⟨*proof*⟩

lemma *pref-cancel-lq*: **assumes** $u \leq p\ x \cdot y$
shows $x^{-1>} u \leq p\ y$
⟨*proof*⟩

lemma *pref-cancel-lq-ext*: **assumes** $u \cdot v \leq p\ x \cdot y$ **and** $|x| \leq |u|$ **shows** $x^{-1>} u \cdot v \leq p\ y$
⟨*proof*⟩

lemma *pref-cancel-lq-ext'*: **assumes** $u \cdot v \leq p\ x \cdot y$ **and** $|u| \leq |x|$ **shows** $v \leq p\ u^{-1>} x \cdot y$
⟨*proof*⟩

lemmas *pref-cancel-lq-ext-pref*[*intro*] = *pref-cancel-lq-ext*[*OF - prefix-length-le*] **and**
pref-cancel-lq-ext-pref'[*intro*] = *pref-cancel-lq-ext'*[*OF - prefix-length-le*]

lemma *empty-lq-eq*: $r \leq p\ z \Longrightarrow r^{-1>} z = \varepsilon \Longrightarrow r = z$
⟨*proof*⟩

lemma *le-if-then-lq*: $|u| \leq |v| \Longrightarrow (if\ |v| \leq |u|\ then\ v^{-1>} u\ \ else\ u^{-1>} v) = u^{-1>} v$
⟨*proof*⟩

lemma *append-comp-lq*: $u \cdot v \bowtie w \Longrightarrow v \bowtie u^{-1>} w$
⟨*proof*⟩

2.7.2 Right quotient

lemmas *rqI* = *lqI*[*reversed*] **and**
rq-triv[*simp*] = *lq-triv*[*reversed*] **and**
rq-triv'[*simp*] = *lq-triv'*[*reversed*] **and**
rq-self[*simp*] = *lq-self*[*reversed*] **and**
rq-emp[*simp*] = *lq-emp*[*reversed*] **and**
rq-suf[*simp*] = *lq-pref*[*reversed*] **and**
rq-ssuf[*simp*] = *lq-spref*[*reversed*] **and**
rq-ssuf-nemp = *lq-spref-nemp*[*reversed*] **and**
rq-reassoc = *lq-reassoc*[*reversed*] **and**
rq-len = *lq-len*[*reversed*] **and**
rq-trans = *lq-trans*[*reversed*] **and**
rq-lq-reassoc-suf = *lq-rq-reassoc-suf*[*reversed*] **and**
rq-ne = *lq-ne*[*reversed*] **and**

rq-is-pref[*simp*] = *lq-is-suf*[*reversed*] **and**
suf-rq = *pref-lq*[*reversed*] **and**
ssuf-rq = *spref-lq*[*reversed*] **and**
conjug-rq = *conjug-lq*[*reversed*] **and**
conjug-emp-emp′ = *conjug-emp-emp*[*reversed*] **and**
rq-take = *lq-def*[*reversed*] **and**
empty-rq-eq = *empty-lq-eq*[*reversed*] **and**
append-rq = *append-lq*[*reversed*] **and**
rq-same-len = *lq-same-len*[*reversed*] **and**
rq-assoc = *lq-assoc*[*reversed*] **and**
rq-assoc′ = *lq-assoc′*[*reversed*] **and**
le-if-then-rq = *le-if-then-lq*[*reversed*] **and**
append-comp-rq = *append-comp-lq*[*reversed*]

2.7.3 Left and right quotients combined

lemma *pref-lq-rq-id*: $p \leq p\ w \Longrightarrow w^{<-1}(p^{-1>}w) = p$
⟨*proof*⟩

lemmas *suf-rq-lq-id* = *pref-lq-rq-id*[*reversed*]

lemma *rev-lq′*: $r \leq p\ s \Longrightarrow rev\ (r^{-1>}s) = (rev\ s)^{<-1}(rev\ r)$
⟨*proof*⟩

lemma *pref-rq-suf-lq*: $s \leq s\ u \Longrightarrow r \leq p\ (u^{<-1}s) \Longrightarrow s \leq s\ (r^{-1>}u)$
⟨*proof*⟩

lemmas *suf-lq-pref-rq* = *pref-rq-suf-lq*[*reversed*]

lemma $w{\cdot}s = v \Longrightarrow v^{<-1}s = w$ ⟨*proof*⟩

lemma *lq-rq-assoc*: $s \leq s\ u \Longrightarrow r \leq p\ (u^{<-1}s) \Longrightarrow (r^{-1>}u)^{<-1}s = r^{-1>}(u^{<-1}s)$
⟨*proof*⟩

lemmas *rq-lq-assoc* = *lq-rq-assoc*[*reversed*]

lemma *lq-prod*: $u \leq p\ v{\cdot}u \Longrightarrow u \leq p\ w \Longrightarrow\ u^{-1>}(v{\cdot}u){\cdot}u^{-1>}w = u^{-1>}(v{\cdot}w)$
⟨*proof*⟩

lemmas *rq-prod* = *lq-prod*[*reversed*]

lemma *pref-suf-mid*: **assumes** $p{\cdot}w{\cdot}s = p'{\cdot}v{\cdot}s'$ **and** $p \leq p\ p'$ **and** $s \leq s\ s'$
shows $v \leq f\ w$
⟨*proof*⟩

2.8 Equidivisibility

Equidivisibility is the following property: if

$$xy = uv,$$

then there exists a word t such that $xt = u$ and $ty = v$, or $ut = x$ and $y = tv$. For monoids over words, this property is equivalent to the freeness of the monoid. As the monoid of all words is free, we can prove that it is equidivisible. Related lemmas based on this property follow.

thm *append-eq-conv-conj*[*folded left-quotient-def*]

lemma *eqd-pref*: $x \cdot y = u \cdot v \Longrightarrow |x| \leq |u| \Longrightarrow x \cdot (x^{-1>} u) = u \wedge (x^{-1>} u) \cdot v = y$
⟨*proof*⟩

lemma *eqd*: $x \cdot y = u \cdot v \Longrightarrow |x| \leq |u| \Longrightarrow \exists\ t.\ x \cdot t = u \wedge t \cdot v = y$
⟨*proof*⟩

lemma *eqd-or*: $x \cdot y = u \cdot v \Longrightarrow \exists\ t.\ x \cdot t = u \wedge t \cdot v = y \vee u \cdot t = x \wedge t \cdot y = v$
⟨*proof*⟩

lemma *eqdE*: **assumes** $x \cdot y = u \cdot v$ **and** $|x| \leq |u|$
obtains t **where** $x \cdot t = u$ **and** $t \cdot v = y$
⟨*proof*⟩

lemma *eqd-less*: **assumes** $x \cdot y = u \cdot v$ **and** $|x| < |u|$
shows $\exists\ t.\ x \cdot t = u \wedge t \cdot v = y \wedge t \neq \varepsilon$
⟨*proof*⟩

lemma *eqd-lessE*: **assumes** $x \cdot y = u \cdot v$ **and** $|x| < |u|$
obtains t **where** $x \cdot t = u$ **and** $t \cdot v = y$ **and** $t \neq \varepsilon$
⟨*proof*⟩

lemma *eqdE'*: **assumes** $x \cdot y = u \cdot v$ **and** $|v| \leq |y|$
obtains t **where** $x \cdot t = u$ **and** $t \cdot v = y$
⟨*proof*⟩

thm *long-pref*

lemma *eqd-pref-suf-iff*: **assumes** $x \cdot y = u \cdot v$ **shows** $x \leq p\ u \longleftrightarrow v \leq s\ y$
⟨*proof*⟩

lemma *eqd-spref-ssuf-iff*: **assumes** $x \cdot y = u \cdot v$ **shows** $x <p\ u \longleftrightarrow v <s\ y$
⟨*proof*⟩

lemma *eqd-pref1*: $x \cdot y = u \cdot v \Longrightarrow |x| \leq |u| \Longrightarrow x \cdot (x^{-1>} u) = u$
⟨*proof*⟩

lemma *eqd-pref2*: $x \cdot y = u \cdot v \Longrightarrow |x| \leq |u| \Longrightarrow (x^{-1>}u) \cdot v = y$
⟨*proof*⟩

lemma *eqd-eq*: **assumes** $x \cdot y = u \cdot v$ $|x| = |u|$ **shows** $x = u$ $y = v$
⟨*proof*⟩

lemma *eqd-eq-suf*: $x \cdot y = u \cdot v \Longrightarrow |y| = |v| \Longrightarrow x = u \wedge y = v$
⟨*proof*⟩

lemma *eqd-comp*: **assumes** $x \cdot y = u \cdot v$ **shows** $x \bowtie u$
⟨*proof*⟩

lemma *eqd-linear-cases* [*elim*]: **assumes** $x \cdot y = u \cdot v$ **and**
$\bigwedge t.\ t \neq \varepsilon \Longrightarrow x \cdot t = u \Longrightarrow t \cdot v = y \Longrightarrow$ *thesis* **and**
$\bigwedge t.\ t \neq \varepsilon \Longrightarrow u \cdot t = x \Longrightarrow t \cdot y = v \Longrightarrow$ *thesis* **and**
$x = u \Longrightarrow y = v \Longrightarrow$ *thesis*
shows *thesis*
⟨*proof*⟩

lemmas *eqd-comp'* = *eqd-comp*[*reversed*]

— not equal to eqd_pref1[reversed]
lemma *eqd-suf1*: $x \cdot y = u \cdot v \Longrightarrow |x| \leq |u| \Longrightarrow (y^{<-1}v) \cdot v = y$
⟨*proof*⟩
lemma *eqd-suf2*: **assumes** $x \cdot y = u \cdot v$ $|x| \leq |u|$ **shows** $x \cdot (y^{<-1}v) = u$
⟨*proof*⟩
lemma *eqd-suf*: **assumes** $x \cdot y = u \cdot v$ **and** $|x| \leq |u|$
shows $(y^{<-1}v) \cdot v = y \wedge x \cdot (y^{<-1}v) = u$
⟨*proof*⟩

lemma *eqd-exchange-aux*:
assumes $u \cdot v = x \cdot y$ **and** $u \cdot v' = x \cdot y'$ **and** $u' \cdot v = x' \cdot y$ **and** $|u| \leq |x|$
shows $u' \cdot v' = x' \cdot y'$
⟨*proof*⟩

lemma *eqd-exchange*:
assumes $u \cdot v = x \cdot y$ **and** $u \cdot v' = x \cdot y'$ **and** $u' \cdot v = x' \cdot y$
shows $u' \cdot v' = x' \cdot y'$
⟨*proof*⟩

hide-fact *eqd-exchange-aux*

2.9 Longest common prefix

lemmas *lcp-simps* = *longest-common-prefix.simps* — provided by Sublist.thy

lemmas *lcp-sym* = *lcp.commute*

— provided by Sublist.thy

lemmas *lcp-pref* = *longest-common-prefix-prefix1*
lemmas *lcp-pref′* = *longest-common-prefix-prefix2*
lemmas *pref-pref-lcp*[*intro*] = *longest-common-prefix-max-prefix*

lemma *lcp-pref-ext*: $u \leq p\ v \Longrightarrow u \leq p\ (u \cdot w) \wedge_p (v \cdot z)$
⟨*proof*⟩

lemma *pref-non-pref-lcp-pref*: **assumes** $u \leq p\ w$ **and** $\neg\ u \leq p\ z$ **shows** $w \wedge_p z < p$ u
⟨*proof*⟩

lemmas *lcp-take* = *pref-take*[*OF lcp-pref*] **and**
lcp-take′ = *pref-take*[*OF lcp-pref′*]

lemma *lcp-take-eq*: *take* $(|u \wedge_p v|)\ u$ = *take* $(|u \wedge_p v|)\ v$
⟨*proof*⟩

lemma *lcp-pref-conv*: $u \wedge_p v = u \longleftrightarrow u \leq p\ v$
⟨*proof*⟩

lemma *lcp-pref-conv′*: $u \wedge_p v = v \longleftrightarrow v \leq p\ u$
⟨*proof*⟩

lemmas *lcp-left-idemp*[*simp*] = *lcp-pref*[*folded lcp-pref-conv′*] **and**
lcp-right-idemp[*simp*] = *lcp-pref′*[*folded lcp-pref-conv*] **and**
lcp-left-idemp′[*simp*] = *lcp-pref′*[*folded lcp-pref-conv′*] **and**
lcp-right-idemp′[*simp*] = *lcp-pref*[*folded lcp-pref-conv*]

lemma *lcp-per-root*: $r \cdot s \wedge_p s \cdot r \leq p\ r \cdot (r \cdot s \wedge_p s \cdot r)$
⟨*proof*⟩

lemma *lcp-per-root′*: $r \cdot s \wedge_p s \cdot r \leq p\ s \cdot (r \cdot s \wedge_p s \cdot r)$
⟨*proof*⟩

lemma *pref-lcp-pref*: $w \leq p\ u \wedge_p v \Longrightarrow w \leq p\ u$
⟨*proof*⟩

lemma *pref-lcp-pref′*: $w \leq p\ u \wedge_p v \Longrightarrow w \leq p\ v$
⟨*proof*⟩

lemmas *lcp-self* = *lcp.idem*

lemma *lcp-eq-len*: $|u| = |u \wedge_p v| \Longrightarrow u = u \wedge_p v$
⟨*proof*⟩

lemma *lcp-len*: $|u| \leq |u \wedge_p v| \Longrightarrow u \leq p\ v$
⟨*proof*⟩

lemma *lcp-len′*: $\neg\ u \leq p\ v \Longrightarrow |u \wedge_p v| < |u|$

⟨*proof*⟩

lemma *incomp-lcp-len*: $\neg\ u \bowtie v \Longrightarrow |u \wedge_p v| < min\ |u|\ |v|$
⟨*proof*⟩

lemma *lcp-ext-right-conv*: $\neg\ r \bowtie r' \Longrightarrow (r \cdot u) \wedge_p (r' \cdot v) = r \wedge_p r'$
⟨*proof*⟩

lemma *lcp-ext-right* [*case-names comp non-comp*]: **obtains** $r \bowtie r' \mid (r \cdot u) \wedge_p (r' \cdot v) = r \wedge_p r'$
⟨*proof*⟩

lemma *lcp-same-len*: $|u| = |v| \Longrightarrow u \neq v \Longrightarrow u \cdot w \wedge_p v \cdot w' = u \wedge_p v$
⟨*proof*⟩

lemma *lcp-mismatch*: $|u \wedge_p v| < |u| \Longrightarrow |u \wedge_p v| < |v| \Longrightarrow u!\ |u \wedge_p v| \neq v!\ |u \wedge_p v|$
⟨*proof*⟩

lemma *lcp-mismatch'*: $\neg\ u \bowtie v \Longrightarrow u!\ |u \wedge_p v| \neq v!\ |u \wedge_p v|$
⟨*proof*⟩

lemma *lcp-mismatchE*: **assumes** $\neg\ us \bowtie vs$
obtains *us' vs'*
where $(us \wedge_p vs) \cdot us' = us$ **and** $(us \wedge_p vs) \cdot vs' = vs$ **and**
$us' \neq \varepsilon$ **and** $vs' \neq \varepsilon$ **and** $hd\ us' \neq hd\ vs'$
⟨*proof*⟩

lemma *lcp-mismatch-lq*: **assumes** $\neg\ u \bowtie v$
shows
$(u \wedge_p v)^{-1>} u \neq \varepsilon$ **and**
$(u \wedge_p v)^{-1>} v \neq \varepsilon$ **and**
$hd\ ((u \wedge_p v)^{-1>} u) \neq hd\ ((u \wedge_p v)^{-1>} v)$
⟨*proof*⟩

lemma *lcp-ext-left*: $(z \cdot u) \wedge_p (z \cdot v) = z \cdot (u \wedge_p v)$
⟨*proof*⟩

lemma *lcp-first-letters*: $u!0 \neq v!0 \Longrightarrow u \wedge_p v = \varepsilon$
⟨*proof*⟩

lemma *lcp-first-mismatch*: $a \neq b \Longrightarrow w \cdot [a] \cdot u \wedge_p w \cdot [b] \cdot v\ = w$
⟨*proof*⟩

lemma *lcp-first-mismatch'*: $a \neq b \Longrightarrow u \cdot [a] \wedge_p u \cdot [b] = u$
⟨*proof*⟩

lemma *lcp-mismatch-eq-len*: **assumes** $|u| = |v|\ x \neq y$ **shows** $u \cdot [x] \wedge_p v \cdot [y] = u \wedge_p v$

⟨*proof*⟩

lemma *lcp-first-mismatch-pref*: **assumes** $p \cdot [a] \leq p\ u$ **and** $p \cdot [b] \leq p\ v$ **and** $a \neq b$
shows $u \wedge_p v = p$
⟨*proof*⟩

lemma *lcp-append-monotone*: $u \wedge_p x \leq p\ (u \cdot v) \wedge_p (x \cdot y)$
⟨*proof*⟩

lemma *lcp-distinct-hd*: $hd\ u \neq hd\ v \Longrightarrow u \wedge_p v = \varepsilon$
⟨*proof*⟩

lemma *nemp-lcp-distinct-hd*: **assumes** $u \neq \varepsilon$ **and** $v \neq \varepsilon$ **and** $u \wedge_p v = \varepsilon$
shows $hd\ u \neq hd\ v$
⟨*proof*⟩

lemma *lcp-lenI*: **assumes** $i < min\ |u|\ |v|$ **and** $take\ i\ u = take\ i\ v$ **and** $u!i \neq v!i$
shows $i = |u \wedge_p v|$
⟨*proof*⟩

lemma *lcp-prefs*: $|u \cdot w \wedge_p v \cdot w'| < |u| \Longrightarrow |u \cdot w \wedge_p v \cdot w'| < |v| \Longrightarrow u \wedge_p v = u \cdot w \wedge_p v \cdot w'$
⟨*proof*⟩

lemma *lcp-extend-eq*: **assumes** $u \leq p\ v$ **and** $u' \leq p\ v'$ **and**
$|v \wedge_p v'| \leq |u|$ **and** $|v \wedge_p v'| \leq |u'|$
shows $u \wedge_p u' = v \wedge_p v'$
⟨*proof*⟩

lemma *long-lcp-same*: **assumes** $\neg\ (u \wedge_p v \leq p\ w)$ **shows** $u \wedge_p w = v \wedge_p w$
⟨*proof*⟩

lemma *long-lcp-sameE*: **obtains** $u \wedge_p v \leq p\ w \mid u \wedge_p w = v \wedge_p w$
⟨*proof*⟩

lemma *ruler-spref-lcp*: **assumes** $u \wedge_p w <_p v \wedge_p w$
shows $u \wedge_p v = u \wedge_p w$
⟨*proof*⟩

2.9.1 Longest common prefix and prefix comparability

lemma *lexord-cancel-right*: $(u \cdot z,\ v \cdot w) \in lexord\ r \Longrightarrow \neg\ u \bowtie v \Longrightarrow (u,v) \in lexord\ r$
⟨*proof*⟩

lemma *lcp-rulersE*: **assumes** $r \leq p\ s$ **and** $r' \leq p\ s'$ **obtains** $r \bowtie r' \mid s \wedge_p s' = r \wedge_p r'$
⟨*proof*⟩

lemma *lcp-rulers*: $r \leq p\ s \Longrightarrow r' \leq p\ s' \Longrightarrow (r \bowtie r' \vee\ \ s \wedge_p s' = r \wedge_p r')$
⟨*proof*⟩

lemma *lcp-rulers'*: $w \leq p\ r \Longrightarrow w' \leq p\ s \Longrightarrow \neg\ w \bowtie w' \Longrightarrow (r \wedge_p s) = w \wedge_p w'$
⟨*proof*⟩

lemma *lcp-ruler*: $r \bowtie w1 \Longrightarrow r \bowtie w2 \Longrightarrow \neg\ w1 \bowtie w2 \Longrightarrow r \leq p\ w1 \wedge_p w2$
⟨*proof*⟩

lemma *comp-monotone*: $w \bowtie r\ \Longrightarrow u \leq p\ w \Longrightarrow u \bowtie r$
⟨*proof*⟩

lemma *comp-monotone'*: $w \bowtie r\ \Longrightarrow w \wedge_p w' \bowtie r$
⟨*proof*⟩

lemma *double-ruler-aux*: **assumes** $w \bowtie r$ **and** $w' \bowtie r'$ **and** $\neg\ r \bowtie r'$ **and** $|w| \leq |w'|$
shows $w \wedge_p w' = take\ |w|\ (r \wedge_p r')$
⟨*proof*⟩

lemma *double-ruler*: **assumes** $w \bowtie r$ **and** $w' \bowtie r'$ **and** $\neg\ r \bowtie r'$
shows $w \wedge_p w' = take\ (min\ |w|\ |w'|)\ (r \wedge_p r')$
⟨*proof*⟩

hide-fact *double-ruler-aux*

lemmas *pref-lcp-iff* = *lcp.bounded-iff*

lemma *pref-comp-ruler*: **assumes** $w \bowtie u \cdot [x]$ **and** $w \bowtie v \cdot [y]$ **and** $x \neq y$ **and** $|u| = |v|$
shows $w \leq p\ u \wedge w \leq p\ v$
⟨*proof*⟩

lemma *comp-per-partes*:
shows $(u \bowtie w \wedge v \bowtie u^{-1>} w) \longleftrightarrow u \cdot v \bowtie w$
⟨*proof*⟩

lemmas *scomp-per-partes* = *comp-per-partes*[*reversed*]

2.9.2 Longest common suffix

definition *longest-common-suffix* (- $\wedge_s$ - [*61,62*] *64*)
where
longest-common-suffix u v ≡ *rev* (*rev u* $\wedge_p$ *rev v*)

lemma *lcs-lcp* [*reversal-rule*]: *rev u* $\wedge_p$ *rev v* = *rev* (*u* $\wedge_s$ *v*)
⟨*proof*⟩

lemmas *lcs-simp* = *lcp-simps*[*reversed*] **and**

lcs-sym = *lcp-sym*[*reversed*] **and**
lcs-suf = *lcp-pref*[*reversed*] **and**
lcs-suf' = *lcp-pref'*[*reversed*] **and**
suf-suf-lcs = *pref-pref-lcp*[*reversed*] **and**
suf-non-suf-lcs-suf = *pref-non-pref-lcp-pref*[*reversed*] **and**
lcs-drop-eq = *lcp-take-eq*[*reversed*] **and**
lcs-take = *lcp-take*[*reversed*] **and**
lcs-take' = *lcp-take'*[*reversed*] **and**
lcs-suf-conv = *lcp-pref-conv*[*reversed*] **and**
lcs-suf-conv' = *lcp-pref-conv'*[*reversed*] **and**
lcs-per-root = *lcp-per-root*[*reversed*] **and**
lcs-per-root' = *lcp-per-root'*[*reversed*] **and**
suf-lcs-suf = *pref-lcp-pref*[*reversed*] **and**
suf-lcs-suf' = *pref-lcp-pref'*[*reversed*] **and**
lcs-self[*simp*] = *lcp-self*[*reversed*] **and**
lcs-eq-len = *lcp-eq-len*[*reversed*] **and**
lcs-len = *lcp-len*[*reversed*] **and**
lcs-len' = *lcp-len'*[*reversed*] **and**
suf-incomp-lcs-len = *incomp-lcp-len*[*reversed*] **and**
lcs-ext-left-conv = *lcp-ext-right-conv*[*reversed*] **and**
lcs-ext-left [*case-names comp non-comp*] = *lcp-ext-right*[*reversed*] **and**
lcs-same-len = *lcp-same-len*[*reversed*] **and**
lcs-mismatch = *lcp-mismatch*[*reversed*] **and**
lcs-mismatch' = *lcp-mismatch'*[*reversed*] **and**
lcs-mismatchE = *lcp-mismatchE*[*reversed*] **and**
lcs-mismatch-rq = *lcp-mismatch-lq*[*reversed*] **and**
lcs-ext-right = *lcp-ext-left*[*reversed*] **and**
lcs-first-mismatch = *lcp-first-mismatch*[*reversed*, *unfolded rassoc*] **and**
lcs-first-mismatch' = *lcp-first-mismatch'*[*reversed*, *unfolded rassoc*] **and**
lcs-mismatch-eq-len = *lcp-mismatch-eq-len*[*reversed*] **and**
lcs-first-mismatch-suf = *lcp-first-mismatch-pref*[*reversed*] **and**
lcs-rulers = *lcp-rulers*[*reversed*] **and**
lcs-rulers' = *lcp-rulers'*[*reversed*] **and**
suf-suf-lcs' = *lcp.mono*[*reversed*] **and**
lcs-distinct-last = *lcp-distinct-hd*[*reversed*] **and**
lcs-lenI = *lcp-lenI*[*reversed*] **and**
lcs-sufs = *lcp-prefs*[*reversed*]

lemmas *lcs-ruler* = *lcp-ruler*[*reversed*] **and**
suf-comp-monotone = *comp-monotone*[*reversed*] **and**
suf-comp-monotone' = *comp-monotone'*[*reversed*] **and**
double-ruler-suf = *double-ruler*[*reversed*] **and**
suf-lcs-iff = *pref-lcp-iff*[*reversed*] **and**
suf-comp-ruler = *pref-comp-ruler*[*reversed*]

2.10 Mismatch

The first pair of letters on which two words/lists disagree

In the following definition, $\varepsilon!0$ has a role of an end marker of each word. It also allows the alternative definition which follows

function *mismatch-pair* :: *'a list* ⇒ *'a list* ⇒ (*'a* × *'a*) **where**
mismatch-pair ε *v* = (ε!0, *v*!0) |
mismatch-pair v ε = (*v*!0, ε!0) |
mismatch-pair (*a*#*u*) (*b*#*v*) = (*if a*=*b then mismatch-pair u v else* (*a*,*b*))
⟨*proof*⟩
termination
⟨*proof*⟩

Alternatively, mismatch pair may be defined using the longest common prefix as follows.

lemma *mismatch-pair-lcp*: *mismatch-pair u v* = (*u*!|*u*∧$_p$*v*|,*v*!|*u*∧$_p$*v*|)
⟨*proof*⟩

For incomparable words the pair is out of diagonal.

lemma *incomp-neq*: ¬ *u* ⋈ *v* ⟹ (*mismatch-pair u v*) ∉ *Id*
⟨*proof*⟩

lemma *mismatch-ext-left*: ¬ *u* ⋈ *v* ⟹ *mismatch-pair u v* = *mismatch-pair* (*p*·*u*) (*p*·*v*)
⟨*proof*⟩

lemma *mismatch-ext-right*: **assumes** ¬ *u* ⋈ *v*
shows *mismatch-pair u v* = *mismatch-pair* (*u*·*z*) (*v*·*w*)
⟨*proof*⟩

lemma *mismatchI*: ¬ *u* ⋈ *v* ⟹ *i* < *min* |*u*| |*v*| ⟹ *take i u* = *take i v* ⟹ *u*!*i* ≠ *v*!*i*
⟹ *mismatch-pair u v* = (*u*!*i*,*v*!*i*)
⟨*proof*⟩

For incomparable words, the mismatch letters work in a similar way as the lexicographic order

lemma *mismatch-lexord*: **assumes** ¬ *u* ⋈ *v* **and** *mismatch-pair u v* ∈ *r*
shows (*u*,*v*) ∈ *lexord r*
⟨*proof*⟩

However, the equivalence requires r to be irreflexive. (Due to the definition of lexord which is designed for irreflexive relations.)

lemma *lexord-mismatch*: **assumes** ¬ *u* ⋈ *v* **and** *irrefl r*
shows *mismatch-pair u v* ∈ *r* ⟷ (*u*,*v*) ∈ *lexord r*
⟨*proof*⟩

2.11 Factor properties

lemmas [*simp*] = *sublist-Cons-right*

lemma *rev-fac*[*reversal-rule*]: $rev\ u \leq f\ rev\ v \longleftrightarrow u \leq f\ v$
⟨*proof*⟩

lemma *fac-pref*: $u \leq f\ v \equiv \exists\ p.\ p \cdot u \leq p\ v$
⟨*proof*⟩

lemma *fac-pref-suf*: $u \leq f\ v \Longrightarrow \exists\ p.\ p \leq p\ v \wedge u \leq s\ p$
⟨*proof*⟩

lemma *pref-suf-fac*: $r \leq p\ v \Longrightarrow u \leq s\ r \Longrightarrow u \leq f\ v$
⟨*proof*⟩

lemmas
fac-suf = *fac-pref*[*reversed*] **and**
fac-suf-pref = *fac-pref-suf*[*reversed*] **and**
suf-pref-fac = *pref-suf-fac*[*reversed*]

lemma *suf-pref-eq*: $s \leq s\ p \Longrightarrow p \leq p\ s \Longrightarrow p = s$
⟨*proof*⟩

lemma *fac-triv*: $p \cdot x \cdot q = x \Longrightarrow p = \varepsilon$
⟨*proof*⟩

lemma *fac-triv'*: $p \cdot x \cdot q = x \Longrightarrow q = \varepsilon$
⟨*proof*⟩

lemmas
suf-fac = *suffix-imp-sublist* **and**
pref-fac = *prefix-imp-sublist*

lemma *fac-ConsE*: **assumes** $u \leq f\ (a \# v)$
obtains $u \leq p\ (a \# v) \mid u \leq f\ v$
⟨*proof*⟩

lemmas
fac-snocE = *fac-ConsE*[*reversed*]

lemma *fac-elim-suf*: **assumes** $f \leq f\ m \cdot s\ \neg\ f \leq f\ s$
shows $f \leq f\ m \cdot (take\ (|f| - 1)\ s)$
⟨*proof*⟩

lemmas *fac-elim-pref* = *fac-elim-suf*[*reversed*]

lemma *fac-elim*: **assumes** $f \leq f\ p \cdot m \cdot s$ **and** $\neg\ f \leq f\ p$ **and** $\neg\ f \leq f\ s$
shows $f \leq f\ (drop\ (|p| - (|f| - 1))\ p) \cdot m \cdot (take\ (|f| - 1)\ s)$
⟨*proof*⟩

lemma *fac-ext-pref*: $u \leq f\ w \Longrightarrow u \leq f\ p \cdot w$

⟨*proof*⟩

lemma *fac-ext-suf*: $u \leq f\ w \Longrightarrow u \leq f\ w \cdot s$
⟨*proof*⟩

lemma *fac-ext*: $u \leq f\ w \Longrightarrow u \leq f\ p \cdot w \cdot s$
⟨*proof*⟩

lemma *fac-ext-hd*:$u \leq f\ w \Longrightarrow u \leq f\ a\#w$
⟨*proof*⟩

lemma *card-switch-fac*: **assumes** $2 \leq card\ (set\ ws)$
obtains $c\ d$ **where** $c \neq d$ **and** $[c,d] \leq f\ ws$
⟨*proof*⟩

lemma *fac-overlap-len*: **assumes** $u \leq f\ x \cdot y \cdot z$ **and** $|u| \leq |y|$
shows $u \leq f\ x \cdot y \vee u \leq f\ y \cdot z$
⟨*proof*⟩

2.12 Power and its properties

The core facts about iterated concatenation of a list are given in the AFP theory *List-Power.List-Power*. Word powers are an important research topic in Combinatorics on Words. We adopt a specific notation for the word power.

abbreviation *listpow* :: *'a list* ⇒ *nat* ⇒ *'a list* (**infixr** ‹$^{@}$› *80*)
where $f\ ^{@}\ n \equiv compow\ n\ f$

some simplified names

lemmas
pow-zero = *pow-list-zero* **and**
pow-Suc = *pow-list-Suc* **and**
pow-Suc2 = *pow-list-Suc2* **and**
pow-comm = *pow-list-comm* **and**
pow-add = *pow-list-add* **and**
pow-mult = *pow-list-mult* **and**
comm-pow-comm = *pow-list-commuting-commutes* **and**
rev-pow = *rev-pow-list* **and**
pow-len = *length-pow-list* **and**
eq-pow-exp = *eq-pow-list-iff-eq-exp*

lemma *pow-pos*: $0 < k \Longrightarrow a^{@}k = a \cdot a^{@}(k-1)$
⟨*proof*⟩

lemma *pow-pos2*:$0 < k \Longrightarrow a^{@}k = a^{@}(k-1) \cdot a$
⟨*proof*⟩

lemma *pow-diff*: $k < n \Longrightarrow a^{@}(n - k) = a \cdot a^{@}(n{-}k{-}1)$
⟨*proof*⟩

lemma *pow-list-3*: $u^{@}3 = u{\cdot}u{\cdot}u$
⟨*proof*⟩

named-theorems *exp-simps*
lemmas [*exp-simps*] = *pow-list-zero pow-list-one pow-list-Nil numeral-nat less-eq-Suc-le neq0-conv pow-list-mult*[*symmetric*]

named-theorems *cow-simps*
lemmas [*cow-simps*] = *emp-simps exp-simps*

— more power properties

lemma *nemp-pref-nemp*: $u \leq p\ v \Longrightarrow u \neq \varepsilon \Longrightarrow v \neq \varepsilon$
⟨*proof*⟩

lemma *pref-concat-emp*[*intro*]: $us \leq p\ vs \Longrightarrow concat\ vs = \varepsilon \Longrightarrow concat\ us = \varepsilon$
⟨*proof*⟩

lemma *nemp-exp-pos*: $w \neq \varepsilon \Longrightarrow r^{@}k = w \Longrightarrow 0 < k$
⟨*proof*⟩

lemma *nemp-pow-nemp*: $t^{@}m \neq \varepsilon \Longrightarrow t \neq \varepsilon$
⟨*proof*⟩

lemma *emp-pow-emp*: $t = \varepsilon \Longrightarrow t^{@}m = \varepsilon$
⟨*proof*⟩

lemma *list-pow-emp* [*intro*]: $t = \varepsilon \vee m = 0 \Longrightarrow t^{@}m = \varepsilon$
⟨*proof*⟩

lemma *list-pow-emp-iff* [*simp*]: $t^{@}m = \varepsilon \longleftrightarrow t = \varepsilon \vee m = 0$
⟨*proof*⟩

lemma *hd-nemp-pow*[*intro*]: $v^{@}k \neq \varepsilon \Longrightarrow hd\ (v^{@}k) = hd\ v$
⟨*proof*⟩

lemma *hd-pref-pow*[*intro*]: **assumes** $u \leq p\ v^{@}k$ **and** $u \neq \varepsilon$

shows $hd\ u = hd\ v$
⟨*proof*⟩

lemma *pop-pow*: $m \leq k \Longrightarrow u^{@} m \cdot u^{@}(k{-}m) = u^{@} k$
⟨*proof*⟩

lemma *pop-pow-2*: $1 < k \Longrightarrow u \cdot u \cdot u^{@}(k{-}2) = u^{@} k$
⟨*proof*⟩

lemma *pop-pow-cancel*: $u^{@} k \cdot v = u^{@} m \cdot w \Longrightarrow m \leq k \Longrightarrow u^{@}(k{-}m) \cdot v = w$
⟨*proof*⟩

lemma *pows-comm*: $t^{@} k \cdot t^{@} m = t^{@} m \cdot t^{@} k$
⟨*proof*⟩

lemma *comm-pows-comm*: **assumes** $r \cdot u = u \cdot r$ **shows** $r^{@} m \cdot u^{@} k = u^{@} k \cdot r^{@} m$
⟨*proof*⟩

lemma *pows-comp*: $x^{@} i \bowtie x^{@} j$
⟨*proof*⟩

lemmas *pows-suf-comp* = *pows-comp*[*reversed*, *folded rev-pow suffix-comparable-def*]

lemmas [*reversal-rule*] = *rev-pow*[*symmetric*]

lemmas *pow-eq-if-list′* = *pow-list-eq-if*[*reversed*] **and**
pop-pow-list-one′ = *pow-pos*[*reversed*] **and**
pop-pow′ = *pop-pow*[*reversed*] **and**
pop-pow-cancel′ = *pop-pow-cancel*[*reversed*]

lemma *emp-pow-pos-emp* [*intro*]: **assumes** $v^{@} j = \varepsilon\ 0 < j$ **shows** $v = \varepsilon$
⟨*proof*⟩

lemma *pow-nemp-pos-nemp*[*intro*]: $w = u\ ^{@}\ k \Longrightarrow u \neq \varepsilon \Longrightarrow 0 < k \Longrightarrow w \neq \varepsilon$
⟨*proof*⟩

lemma *nemp-emp-pow*: **assumes** $u \neq \varepsilon$ **shows** $u^{@} m = \varepsilon \longleftrightarrow m = 0$
⟨*proof*⟩

lemma *nemp-pow-nemp-pos-conv*: **assumes** $u \neq \varepsilon$ **shows** $u^{@} m \neq \varepsilon \longleftrightarrow 0 < m$
⟨*proof*⟩

lemma *nemp-Suc-pow-nemp*: $u \neq \varepsilon \Longrightarrow u^{@} Suc\ k \neq \varepsilon$
⟨*proof*⟩

lemma *Suc-pow-eq-eq*[*elim*]: $u^{@} Suc\ k = v^{@} Suc\ k \Longrightarrow u = v$

⟨*proof*⟩

lemmas [*reversal-rule*] = *map-pow-list*[*symmetric*]

named-theorems *concat-simps*
lemmas [*concat-simps*] = *concat-morph concat-sing′ concat-pow-list concat.simps emp-simps*

lemma *pow-sing-nemp* [*elim*]: $w = [a] \,^{@}\, k \Longrightarrow 0 < k \Longrightarrow w \neq \varepsilon$
⟨*proof*⟩

lemma *nemp-pow-emp*[*intro*]: $u \neq \varepsilon \Longrightarrow u^{@}n = \varepsilon \Longrightarrow n = 0$
⟨*proof*⟩

lemma *long-pow*: $r \neq \varepsilon \Longrightarrow m \leq |r^{@}m|$
⟨*proof*⟩

lemma *long-pow-exp′*: $r \neq \varepsilon \Longrightarrow m < |r^{@}(Suc\ m)|$
⟨*proof*⟩

lemma *long-pow-expE*: **assumes** $r \neq \varepsilon$ **obtains** n **where** $m \leq |r^{@}Suc\ n|$
⟨*proof*⟩

lemma *pref-pow-ext*: $x \leq p\ r^{@}k \Longrightarrow x \leq p\ r^{@}Suc\ k$
⟨*proof*⟩

lemma *pref-pow-ext′*: $u \leq p\ r^{@}k \Longrightarrow u \leq p\ r \cdot r^{@}k$
⟨*proof*⟩

lemma *pref-pow-root-ext*: $x \leq p\ r^{@}k \Longrightarrow r \cdot x \leq p\ r^{@}Suc\ k$
⟨*proof*⟩

lemma *pref-pow-root*: $u \leq p\ r^{@}k \Longrightarrow u \leq p\ r \cdot u$
⟨*proof*⟩

lemma *le-exps-pref* [*intro*]: $k \leq l \Longrightarrow r^{@}k \leq p\ r^{@}l$
⟨*proof*⟩

lemmas *less-exps-pref* = *le-exps-pref*[*OF less-imp-le*]

lemma *pref-exp-le*: **assumes** $u \neq \varepsilon\ u^{@}m \leq p\ u^{@}n$ **shows** $m \leq n$
⟨*proof*⟩

lemma *sing-exp-pref-iff*: **assumes** $a \neq b$

shows $[a]^{@}i \leq p\ [a]^{@}k{\cdot}[b] \cdot w \longleftrightarrow i \leq k$
⟨*proof*⟩

lemmas
suf-pow-ext = *pref-pow-ext*[*reversed*] **and**
suf-pow-ext′= *pref-pow-ext′*[*reversed*] **and**
suf-pow-root-ext = *pref-pow-root-ext*[*reversed*] **and**
suf-prod-root = *pref-pow-root*[*reversed*] **and**
suf-exps-pow = *le-exps-pref*[*reversed*] **and**
suf-exp-le = *pref-exp-le*[*reversed*] **and**
sing-exp-suf-iff = *sing-exp-pref-iff*[*reversed*]

lemma *comm-common-pow-list-iff*: **assumes** $r \cdot u = u \cdot r$ **shows** $r^{@}|u| = u^{@}|r|$
⟨*proof*⟩

lemma *concat-morph-power*: $xs \in lists\ B \Longrightarrow xs = ts^{@}k \Longrightarrow concat\ ts^{@}k = concat\ xs$
⟨*proof*⟩

lemma *per-exp-pref*: $u \leq p\ r \cdot u \Longrightarrow u \leq p\ r^{@}k \cdot u$
⟨*proof*⟩

lemmas
per-exp-suf = *per-exp-pref*[*reversed*]

lemma *hd-sing-pow*: $k \neq 0 \Longrightarrow hd\ ([a]^{@}k) = a$
⟨*proof*⟩

lemma *sing-pref-comp-mismatch*:
assumes $b \neq a$ **and** $c \neq a$ **and** $[a]^{@}k \cdot [b] \bowtie [a]^{@}l \cdot [c]$
shows $k = l \wedge b = c$
⟨*proof*⟩

lemma *sing-pref-comp-lcp*: **assumes** $r \neq s$ **and** $a \neq b$ **and** $a \neq c$
shows $[a]^{@}r \cdot [b] \cdot u \wedge_p [a]^{@}s \cdot [c] \cdot v = [a]^{@}(min\ r\ s)$
⟨*proof*⟩

lemmas *sing-suf-comp-mismatch* = *sing-pref-comp-mismatch*[*reversed*]

lemma *exp-pref-cancel*: **assumes** $t^{@}m \cdot y = t^{@}k$ **shows** $y = t^{@}(k - m)$
⟨*proof*⟩

lemmas *exp-suf-cancel* = *exp-pref-cancel*[*reversed*]

lemma *index-pow-mod*: $i < |r^{@}k| \Longrightarrow (r^{@}k)!i = r!(i\ mod\ |r|)$
⟨*proof*⟩

lemma *sing-pow-len* [*simp*]: $|[r]^{@}l| = l$
⟨*proof*⟩

lemma *take-sing-pow*: $k \leq l \Longrightarrow take\ k\ ([r]^{@}l) = [r]^{@}k$
⟨*proof*⟩

lemma *concat-take-sing*: **assumes** $k \leq l$
shows $concat\ (take\ k\ ([r]^{@}l)) = r^{@}k$
⟨*proof*⟩

lemma *sing-set-word*: $set\ w \subseteq \{a\} \Longrightarrow w = [a]^{@}|w|$
⟨*proof*⟩

lemma *sing-set-concat*: $set\ w \subseteq \{a\} \Longrightarrow concat\ w = a^{@}|w|$
⟨*proof*⟩

lemma *sing-set-word-hd*: $set\ w \subseteq \{a\} \Longrightarrow w = [hd\ w]^{@}|w|$
⟨*proof*⟩

lemma *sing-set-wordE*[*elim*]: **assumes** $set\ w \subseteq \{a\}$ **obtains** k **where** $w = [a]^{@}k$
⟨*proof*⟩

lemma *sing-set-wordE'*[*elim*]: **assumes** $set\ w = \{a\}$ **obtains** k **where** $w = [a]^{@}k$ **and** $0 < k$
⟨*proof*⟩

lemma *conjug-pow*: $x \cdot z = z \cdot y \Longrightarrow x^{@}k \cdot z = z \cdot y^{@}k$
⟨*proof*⟩

lemma *lq-conjug-pow*: **assumes** $p \leq p\ x \cdot p$ **shows** $p^{-1>}(x^{@}k \cdot p) = (p^{-1>}(x \cdot p))^{@}k$
⟨*proof*⟩

lemmas *rq-conjug-pow* = *lq-conjug-pow*[*reversed*]

lemma *pow-pref-root-one*: **assumes** $0 < k$ **and** $r \neq \varepsilon$ **and** $r^{@}k \leq p\ r$
shows $k = 1$
⟨*proof*⟩

lemma *comp-pows-pref*: **assumes** $v \neq \varepsilon$ **and** $(u \cdot v)^{@}k \cdot u \leq p\ (u \cdot v)^{@}m$ **shows** $k \leq m$
⟨*proof*⟩

lemma *comp-pows-pref'*: **assumes** $v \neq \varepsilon$ **and** $(u \cdot v)^{@}k \leq p\ (u \cdot v)^{@}m \cdot u$ **shows** $k \leq m$
⟨*proof*⟩

lemma *comp-pows-not-pref*: $\neg\ (u \cdot v)^{@}k \cdot u \leq p\ (u \cdot v)^{@}m \Longrightarrow m \leq k$
⟨*proof*⟩

lemma *comp-pows-spref*: $u^{@}k <p\ u^{@}m \Longrightarrow k < m$
⟨*proof*⟩

lemma *comp-pows-spref-ext*: $(u \cdot v)^{@}k \cdot u <p\ (u \cdot v)^{@}m \Longrightarrow k < m$
⟨*proof*⟩

lemma *comp-pows-pref-zero*:$(u \cdot v)^{@}k <p\ u \Longrightarrow k = 0$
⟨*proof*⟩

lemma *comp-pows-spref'*: $(u \cdot v)^{@}k <p\ (u \cdot v)^{@}m \cdot u \Longrightarrow k < Suc\ m$
⟨*proof*⟩

lemmas *comp-pows-suf* = *comp-pows-pref*[*reversed*] **and**
comp-pows-suf' = *comp-pows-pref'*[*reversed*] **and**
comp-pows-not-suf = *comp-pows-not-pref*[*reversed*] **and**
comp-pows-ssuf = *comp-pows-spref*[*reversed*] **and**
comp-pows-ssuf-ext = *comp-pows-spref-ext*[*reversed*] **and**
comp-pows-suf-zero = *comp-pows-pref-zero*[*reversed*] **and**
comp-pows-ssuf' = *comp-pows-spref'*[*reversed*]

2.12.1 Comparison

named-theorems *shifts*
lemma *shift-pow*[*shifts*]: $(u{\cdot}v)^{@}k{\cdot}u = u{\cdot}(v{\cdot}u)^{@}k$
⟨*proof*⟩
lemma[*shifts*]: $(u \cdot v)^{@}k \cdot u \cdot z = u \cdot (v \cdot u)^{@}k \cdot z$
⟨*proof*⟩
lemma[*shifts*]: $u^{@}k \cdot u \cdot z = u \cdot u^{@}k \cdot z$
⟨*proof*⟩
lemma[*shifts*]: $r^{@}k \leq p\ r \cdot r^{@}k$
⟨*proof*⟩
lemma [*shifts*]: $r^{@}k \leq p\ r \cdot r^{@}k \cdot z$
⟨*proof*⟩
lemma [*shifts*]: $(r \cdot q)^{@}k \leq p\ r \cdot q \cdot\ (r \cdot q)^{@}k \cdot z$
⟨*proof*⟩
lemma [*shifts*]: $(r \cdot q)^{@}k \leq p\ r \cdot q \cdot\ (r \cdot q)^{@}k$
⟨*proof*⟩
lemma[*shifts*]: $r^{@}k \cdot u \leq p\ r \cdot r^{@}k \cdot v \longleftrightarrow u \leq p\ r \cdot v$
⟨*proof*⟩
lemma[*shifts*]: $u \cdot u^{@}k \cdot z = u^{@}k \cdot w \longleftrightarrow u \cdot z = w$
⟨*proof*⟩
lemma[*shifts*]: $(r \cdot q)^{@}k \cdot u \leq p\ r \cdot q\ \cdot (r \cdot q)^{@}k \cdot v \longleftrightarrow u \leq p\ r \cdot q \cdot v$
⟨*proof*⟩
lemma[*shifts*]: $(r \cdot q)^{@}k \cdot u = r \cdot q\ \cdot (r \cdot q)^{@}k \cdot v \longleftrightarrow u = r \cdot q \cdot v$
⟨*proof*⟩

lemma[*shifts*]: $r \cdot q \ \cdot (r \cdot q)^{@}k \cdot v = (r \cdot q)^{@}k \cdot u \longleftrightarrow r \cdot q \cdot v = u$
⟨*proof*⟩
lemma *shifts-spec* [*shifts*]: $(u^{@}k \cdot v)^{@}l \cdot u \cdot u^{@}k \cdot z = u^{@}k \cdot (v \cdot u^{@}k)^{@}l \cdot u \cdot z$
⟨*proof*⟩
lemmas [*shifts*] = *shifts-spec*[*of* - - $r \cdot q$, *unfolded rassoc*] **for** r q
lemmas [*shifts*] = *shifts-spec*[*of* - - $r \cdot q$ - ε , *unfolded rassoc emp-simps*] **for** r q
lemmas [*shifts*] = *shifts-spec*[*of* - - $r \cdot q$ $r \cdot q$, *unfolded rassoc*] **for** r q
lemmas [*shifts*] = *shifts-spec*[*of* - - $r \cdot q$ $r \cdot q$ ε , *unfolded rassoc emp-simps*] **for** r q
lemma[*shifts*]: $(u \cdot (v \cdot u)^{@}k)^{@}j \cdot (u \cdot v)^{@}k = (u \cdot v)^{@}k \cdot (u \cdot (u \cdot v)^{@}k)^{@}j$
⟨*proof*⟩
lemma[*shifts*]: $(u \cdot (v \cdot u)^{@}k \cdot z)^{@}j \cdot (u \cdot v)^{@}k = (u \cdot v)^{@}k \cdot (u \cdot z \cdot (u \cdot v)^{@}k)^{@}j$
⟨*proof*⟩
lemmas[*shifts*] = *pow-comm cancel rassoc pow-Suc pref-cancel-conv suf-cancel-conv pow-add cancel-right numeral-nat pow-zero emp-simps*
lemmas[*shifts*] = *less-eq-Suc-le*
lemmas[*shifts*] = *neq0-conv*
lemma *shifts-hd-hd* [*shifts*]: $a\#b\#v = [a] \cdot b\#v$
⟨*proof*⟩
lemmas [*shifts*] = *shifts-hd-hd*[*of* - - ε]
lemma[*shifts*]: $n \leq k \Longrightarrow x^{@}k = x^{@}(n + (k - n))$
⟨*proof*⟩
lemma[*shifts*]: $n < k \Longrightarrow x^{@}k = x^{@}(n + (k - n))$
⟨*proof*⟩
lemmas[*shifts*] = *cancel cancel-right pref-cancel-conv suf-cancel-conv triv-pref*
lemmas[*shifts*] = *pow-diff*
lemmas[*shifts*] = *pows-comm*
lemma[*shifts*]: $m < l \Longrightarrow x^{@}m \cdot w \leq p\ x^{@}l \cdot w' \longleftrightarrow w \leq p\ x^{@}(l - m) \cdot w'$
⟨*proof*⟩
lemma[*shifts*]: $x \cdot x^{@}m \cdot w \leq p\ x^{@}m \cdot w' \longleftrightarrow x \cdot w \leq p\ \ w'$
⟨*proof*⟩

lemmas *shifts-rev* = *shifts*[*reversed*]

lemmas *shift-simps* = *shifts shifts*[*reversed*]

method *comparison* = ((*simp only*: *shifts*; *fail*) | (*simp only*: *shifts-rev*; *fail*))

lemma *take-root*: $0 < k \Longrightarrow take\ |ws|\ (ws\ ^{@}\ k) = ws$
⟨*proof*⟩

2.13 Rotation

lemma *rotate-root-self*: $rotate\ |r|\ (r^{@}k) = r^{@}k$
⟨*proof*⟩

lemma *rotate-pow-self*: $rotate\ (l*|u|)\ (u^{@}k) = u^{@}k$
⟨*proof*⟩

lemma *rotate-pow-mod*: *rotate n* $(u^{@}k)$ = *rotate* $(n \bmod |u|)$ $(u^{@}k)$
⟨*proof*⟩

lemma *rotate-conj-pow*: *rotate* $|u|$ $((u{\cdot}v)^{@}k) = (v{\cdot}u)^{@}k$
⟨*proof*⟩

lemmas *rotate-pow-comm-two* = *rotate-pow-list-swap*[*of - 2*, *unfolded pow-list-2*]

lemma *rotate-back*: *rotate* $(|u| - n \bmod |u|)$ (*rotate n u*) = *u*
⟨*proof*⟩

lemma *rotate-backE*: **obtains** *m* **where** *rotate m* (*rotate n u*) = *u*
⟨*proof*⟩

lemma *rotate-back′*: **assumes** *rotate m w* = *rotate n w*
shows *rotate* $(m-n)$ *w* = *w*
⟨*proof*⟩

lemma *rotate-class-rotate′*: $(\exists\, n.$ *rotate n w* = *u*$) \longleftrightarrow (\exists\, n.$ *rotate n* (*rotate l w*) = *u*)
⟨*proof*⟩

lemma *rotate-class-rotate*: $\{u\ .\ \exists\, n.$ *rotate n w* = *u*$\}$ = $\{u\ .\ \exists\, n.$ *rotate n* (*rotate l w*) = *u*$\}$
⟨*proof*⟩

lemma *rotate-comp-eq*:*w* $\bowtie$ *rotate n w* $\Longrightarrow$ *rotate n w* = *w*
⟨*proof*⟩

corollary *mismatch-iff-lexord*: **assumes** *rotate n w* $\neq$ *w* **and** *irrefl r*
shows *mismatch-pair w* (*rotate n w*) $\in$ *r* $\longleftrightarrow$ (*w*,*rotate n w*) $\in$ *lexord r*
⟨*proof*⟩

2.14 Lists of words and their concatenation

The helpful lemmas of this section deal with concatenation of a list of words *concat*. The main objective is to cover elementary facts needed to study factorizations of words.

lemma *concat-take-is-prefix*: *concat*(*take n ws*) $\leq p$ *concat ws*
⟨*proof*⟩

lemma *concat-take-Suc*: **assumes** *j* < $|ws|$ **shows** *concat*(*take j ws*) · *ws*!*j* = *concat*(*take* (*Suc j*) *ws*)
⟨*proof*⟩

lemma *pref-mod-list′*: **assumes** *u* $<p$ *concat ws*

obtains $j\ r$ **where** $j < |ws|$ **and** $r <p\ ws!j$ **and** $concat\ (take\ j\ ws) \cdot r = u$
⟨*proof*⟩

lemma *pref-mod-list*: **assumes** $u \leq p\ concat\ ws\ u \neq \varepsilon$
obtains $ps\ p\ s$ **where** $ps \leq p\ ws$ **and** $p \neq \varepsilon$ **and** $p \cdot s = last\ ps$
$concat\ ps = u \cdot s$ **and** $\bigwedge y.\ y \leq p\ ws \Longrightarrow u \leq p\ concat\ y \Longrightarrow ps \leq p\ y$
⟨*proof*⟩

lemma *pref-mod-pow*: **assumes** $u \leq p\ w^{@}l$ **and** $w \neq \varepsilon$
obtains $k\ z$ **where** $k \leq l$ **and** $z <p\ w$ **and** $w^{@}k \cdot z = u$
⟨*proof*⟩

lemma *pref-mod-pow'*: **assumes** $u <p\ w^{@}l$
obtains $k\ z$ **where** $k < l$ **and** $z <p\ w$ **and** $w^{@}k \cdot z = u$
⟨*proof*⟩

lemma *split-pow*: **assumes** $u \cdot v = w^{@}k\ 0 < k\ v \neq \varepsilon$
obtains $p\ s\ i\ j$ **where** $w = p \cdot s\ s \neq \varepsilon\ u = (p \cdot s)^{@}i \cdot p\ v = (s \cdot p)^{@}j \cdot s\ k = i + j + 1$
⟨*proof*⟩

lemma *del-emp-concat* [*simp*]: $concat\ (filter\ (\lambda x.\ x \neq \varepsilon)\ us) = concat\ us$
⟨*proof*⟩

lemma *lists-minus*: $us \in lists\ (C - A) \Longrightarrow us \in lists\ C$
⟨*proof*⟩

lemma *lists-minus'*: $us \in lists\ C \Longrightarrow (filter\ (\lambda x.\ x \neq \varepsilon)\ us) \in lists\ (C - \{\varepsilon\})$
⟨*proof*⟩

lemma *pref-concat-pref*[*intro*]: $us \leq p\ ws \Longrightarrow concat\ us \leq p\ concat\ ws$
⟨*proof*⟩

lemmas *suf-concat-suf* = *pref-concat-pref*[*reversed*]

lemma *concat-mono-fac*: $us \leq f\ ws \Longrightarrow concat\ us \leq f\ concat\ ws$
⟨*proof*⟩

lemma *ruler-concat-less*: **assumes** $us \leq p\ ws$ **and** $vs \leq p\ ws$ **and** $|concat\ us| < |concat\ vs|$
shows $us <p\ vs$
⟨*proof*⟩

lemma *concat-take-mono-strict*: **assumes** $concat\ (take\ i\ ws) <p\ concat\ (take\ j\ ws)$
shows $take\ i\ ws <p\ take\ j\ ws$
⟨*proof*⟩

lemma *take-pp-less*: **assumes** $take\ k\ ws <p\ take\ n\ ws$ **shows** $k < n$
⟨*proof*⟩

lemma *concat-pp-less*: **assumes** *concat* (*take k ws*) <*p concat* (*take n ws*) **shows** $k < n$
⟨*proof*⟩

lemma *take-le-take*: $j \leq k \Longrightarrow$ *take j* (*take k xs*) = *take j xs*
⟨*proof*⟩

lemma *count-in*: *count-list ws a* ≠ *0* ⟹ *a* ∈ *set ws*
⟨*proof*⟩

lemma *count-in-conv*: *count-list w a* ≠ *0* ⟷ *a* ∈ *set w*
⟨*proof*⟩

lemma *two-in-set-concat-len*: **assumes** $u \neq v$ **and** $\{u,v\} \subseteq$ *set ws*
shows $|u| + |v| \leq |$*concat ws*$|$
⟨*proof*⟩

2.15 Commutation

The solution of the easiest nontrivial word equation, $x \cdot y = y \cdot x$, is contained AFP theory *List-Power.List-Power* as $(u \cdot v = v \cdot u) = (\exists\, r\ k\ m.\ u = r\ ^{@}\ k \wedge v = r\ ^{@}\ m)$.

lemmas *comm* = *comm-append-pow-list-iff*

lemma *commE*[*elim*]: **assumes** $x \cdot y = y \cdot x$
obtains *t m k* **where** $x = t^{@}k$ **and** $y = t^{@}m$ **and** $t \neq \varepsilon$
⟨*proof*⟩

lemma *comm-nemp-eqE*: **assumes** $u \cdot v = v \cdot u$ $u \neq \varepsilon$ $v \neq \varepsilon$
obtains *k m* **where** $u^{@}k = v^{@}m$ $0 < k$ $0 < m$
⟨*proof*⟩

lemma *comm-prod*[*intro*]: **assumes** $r{\cdot}u = u{\cdot}r$ **and** $r{\cdot}v = v{\cdot}r$
shows $r{\cdot}(u{\cdot}v) = (u{\cdot}v){\cdot}r$
⟨*proof*⟩

lemma *comm-cancel-pref*: **assumes** $r \cdot u = u \cdot r$ $r \cdot u \cdot v = u \cdot v \cdot r$
shows $r \cdot v = v \cdot r$
⟨*proof*⟩

lemma *comm-cancel-suf*: **assumes** $r \cdot v = v \cdot r$ $r \cdot u \cdot v = u \cdot v \cdot r$
shows $r \cdot u = u \cdot r$
⟨*proof*⟩

lemma *LS-comm*:
assumes $y\ ^{@}\ k \cdot x = z\ ^{@}\ l$
and $z \cdot y = y \cdot z$

shows $x \cdot y = y \cdot x$
⟨*proof*⟩

2.16 Periods

periodicity is probably the most studied property of words. It captures the fact that a word overlaps with itself. Another possible point of view is that the periodic word is a prefix of an (infinite) power of some nonempty word, which can be called its period word. Both these points of view are expressed by the following definition.

2.16.1 periodic root

lemma $u <p\ r \cdot u \longleftrightarrow u \leq p\ r \cdot u \wedge r \neq \varepsilon$
⟨*proof*⟩

lemma *per-rootI*[*intro*]: $u \leq p\ r \cdot u \Longrightarrow r \neq \varepsilon \Longrightarrow u <p\ r \cdot u$
⟨*proof*⟩

lemma *per-rootI'*[*intro*]: **assumes** $u \leq p\ r^{@}k$ **and** $r \neq \varepsilon$ **shows** $u <p\ r \cdot u$
⟨*proof*⟩

lemma *per-root-nemp*[*dest*]: $u <p\ r \cdot u \Longrightarrow r \neq \varepsilon$
⟨*proof*⟩

Empty word is not a periodic root but it has all nonempty periodic roots.

Any nonempty word is its own periodic root.

lemmas *root-self* = *triv-spref*

"Short roots are prefixes"

lemma $w <p\ r \cdot u \Longrightarrow |r| \leq |w| \Longrightarrow r \leq p\ w$
⟨*proof*⟩

periodic words are prefixes of the power of the root, which motivates the notation

lemma *pref-pow-ext-take*: **assumes** $u \leq p\ r^{@}k$ **shows** $u \leq p$ *take* $|r|\ u \cdot r^{@}k$
⟨*proof*⟩

lemma *pref-pow-take*: **assumes** $u \leq p\ r^{@}k$ **shows** $u \leq p$ *take* $|r|\ u \cdot u$
⟨*proof*⟩

lemma *per-root-powE*: **assumes** $u \leq p\ r \cdot u\ r \neq \varepsilon$
obtains k **where** $u <p\ r^{@}k$ **and** $0 < k$
⟨*proof*⟩

thm *per-rootI per-rootI'*

lemma *per-root-modE'* [*elim*]: **assumes** $u \leq_p r \cdot u$ $r \neq \varepsilon$
obtains p **where** $p <_p r$ **and** $r^{@}(|u| \ div \ |r|) \cdot p = u$
⟨*proof*⟩

lemma *per-root-modE* [*elim*]: **assumes** $u \leq_p r \cdot u$ $r \neq \varepsilon$
obtains n p s **where** $p \cdot s = r$ **and** $r^{@}n \cdot p = u$ **and** $s \neq \varepsilon$
⟨*proof*⟩

lemma *nemp-per-root-conv*: $r \neq \varepsilon \Longrightarrow u <_p r \cdot u \longleftrightarrow u \leq_p r \cdot u$
⟨*proof*⟩

lemma *root-ruler*: **assumes** $w \leq_p u \cdot w$ $v \leq_p u \cdot v$ $u \neq \varepsilon$ $v \neq \varepsilon$
shows $w \bowtie v$
⟨*proof*⟩

lemmas *same-len-nemp-root-eq* = *root-ruler*[*THEN pref-comp-eq*]

lemma *per-root-add-exp*: **assumes** $u <_p r \cdot u$ $0 < m$ **shows** $u <_p r^{@}m \cdot u$
⟨*proof*⟩

theorem *per-root-pow-conv*: $x <_p r \cdot x \longleftrightarrow (\exists\ k.\ x \leq_p r^{@}k) \wedge r \neq \varepsilon$
⟨*proof*⟩

lemma [*intro*]: $r = \varepsilon \Longrightarrow u \leq_p r \Longrightarrow u = \varepsilon$
⟨*proof*⟩

lemma [*intro*]: $u = \varepsilon \Longrightarrow u \leq_p w$
⟨*proof*⟩

lemma *per-root-exp'*: **assumes** $x \leq_p r^{@}k$ **shows** $x \leq_p r^{@}|x|$
⟨*proof*⟩

lemma *per-root-exp*: **assumes** $x <_p r \cdot x$ **shows** $x \leq_p r^{@}|x|$
⟨*proof*⟩

lemma *per-root-drop-exp*: $u <_p (r^{@}m) \cdot u \ \Longrightarrow u <_p r \cdot u$
⟨*proof*⟩

lemma *per-root-exp-conv*: $u <_p (r^{@}Suc\ m) \cdot u \ \longleftrightarrow u <_p r \cdot u$
⟨*proof*⟩

lemma *per-root-sq-drop*[*simp*]: $x \leq_p y \cdot y \cdot x \Longrightarrow x \leq_p y \cdot x$
⟨*proof*⟩

lemmas *per-root-sq-drop-suf* = *per-root-sq-drop*[*reversed*, *unfolded rassoc*]

lemma *per-drop-exp*: $0 < k \Longrightarrow x \leq_p r^{@}k \cdot x \Longrightarrow x \leq_p \ r \cdot x$

⟨*proof*⟩

lemmas *per-drop-exp-rev* = *per-drop-exp*[*reversed*]

corollary *comm-drop-exp*: **assumes** $0 < m$ **and** $u \cdot r^{@}m = r^{@}m' \cdot u$ **shows** $r \cdot u = u \cdot r$
⟨*proof*⟩

lemma *comm-drop-exp'*: **assumes** $u^{@}k \cdot v = v \cdot u^{@}k'$ $0 < k'$ **shows** $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *comm-drop-exps*: $0 < k \Longrightarrow$
$0 < m \Longrightarrow u \,^{@}\, k \cdot v \,^{@}\, m = v \,^{@}\, m \cdot u \,^{@}\, k \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *comm-pow-roots*:
assumes $0 < m$ **and** $0 < k$
shows $u^{@}m \cdot v^{@}k = v^{@}k \cdot u^{@}m \longleftrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *pow-comm-comm'*: **assumes** *comm*: $u^{@}(Suc\ k) = v^{@}(Suc\ l)$ **shows** $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *comm-trans*: **assumes** *uv*: $u{\cdot}v = v{\cdot}u$ **and** *vw*: $w{\cdot}v = v{\cdot}w$ **and** *nemp*: $v \neq \varepsilon$ **shows** $u \cdot w = w \cdot u$
⟨*proof*⟩

lemma *drop-per-pref*: **assumes** $w \leq p\ u \cdot w$ **shows** $drop\ |u|\ w \leq p\ w$
⟨*proof*⟩

Note that $[\![w <_p u \cdot w;\ u <_p t \cdot u]\!] \Longrightarrow w <_p t \cdot w$ does not hold.

lemma *per-root-same-prefix*:$w <p\ r \cdot w \Longrightarrow w' \leq p\ r \cdot w' \Longrightarrow w \bowtie w'$
⟨*proof*⟩

lemma *take-after-drop*: $|u| + q \leq |w| \Longrightarrow w \leq p\ u \cdot w \Longrightarrow take\ q\ (drop\ |u|\ w) = take\ q\ w$
⟨*proof*⟩

The following lemmas are a weak version of the periodicity lemma

lemma *two-pers*:
assumes *pu*: $w \leq p\ u \cdot w$ **and** *pv*: $w \leq p\ v \cdot w$ **and** *len*: $|u| + |v| \leq |w|$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

2.16.2 Maximal root-prefix

lemma *max-root-mismatch*: **assumes** $u \cdot [a] <p\ r \cdot u \cdot [a]$ **and** $u \cdot [b] \leq p\ w$ **and** $a \neq b$
shows $w \wedge_p r \cdot w = u$
⟨*proof*⟩

lemma *max-pref-per-root*: $u \wedge_p r \cdot u \leq p\ r \cdot (u \wedge_p r \cdot u)$
⟨*proof*⟩

lemma *max-pref-pref*:
assumes $r \neq \varepsilon$
shows $u \wedge_p r \cdot u \leq p\ r^{@}|u \wedge_p r \cdot u|$
⟨*proof*⟩

lemma *max-pref-lcp-root-pow*: **assumes** $r \neq \varepsilon$ **and** $|u \wedge_p r \cdot u| \leq k$
shows $u \wedge_p r \cdot u = u \wedge_p r^{@}k$ (**is** *?max* $= u \wedge_p r^{@}k$)
⟨*proof*⟩

lemma *max-pref-shorter-lcp*: **assumes** $u \wedge_p r \cdot u <p\ v \wedge_p r \cdot v$
shows $u \wedge_p v = u \wedge_p r \cdot u$
⟨*proof*⟩

find-theorems *?u* $\wedge_p$ *?r* $\cdot$ *?u*

2.16.3 Period - numeric

Definition of a period as the length of the periodic root is often offered as the basic one. From our point of view, it is secondary, and less convenient for reasoning.

definition *period* :: *'a list* ⇒ *nat* ⇒ *bool*
where [*simp*]: *period w n* ≡ $w \leq p$ (*take n w*) $\cdot$ *w*

lemma *periodI*: $w \leq p$ (*take n w*) $\cdot$ $w \Longrightarrow$ *period w n*
⟨*proof*⟩

lemma *periodI-pref*: **assumes** $|r| = n$ $w \leq p\ r \cdot w$
shows *period w n*
⟨*proof*⟩

lemma *periodI-pref'*: **assumes** $w \leq p\ r \cdot w$
shows *period w* $|r|$
⟨*proof*⟩

The numeric definition respects the following convention about empty words and empty periods.

lemma *emp-all-periods*: *period* ε *n*
⟨*proof*⟩

lemma *zero-period*: *period w 0*
⟨*proof*⟩

lemma *periodD*: *period w n* $\Longrightarrow$ $w \leq p$ *take n w* $\cdot$ *w*
⟨*proof*⟩

A nonempty word has all "long" periods

lemma *all-long-pers*: $|w| \leq n \Longrightarrow$ *period w n*
⟨*proof*⟩

lemma *len-is-per*: *period w* $|w|$
⟨*proof*⟩

Period is preserved by reversal and taking factors

lemma *period-rev*: **assumes** *period w p* **shows** *period* (*rev w*) *p*
⟨*proof*⟩

lemma *period-rev-conv* [*reversal-rule*]: *period* (*rev w*) *n* $\longleftrightarrow$ *period w n*
⟨*proof*⟩

lemma *period-pref-period*: **assumes** *period w n v* $\leq p$ *w*
shows *period v n*
⟨*proof*⟩

lemmas *period-suf-period* = *period-pref-period*[*reversed*] **and**
periodI-suf = *periodI-pref*[*reversed*] **and**
periodI-suf′ = *periodI-pref′*[*reversed*]

lemma *period-fac-period*: **assumes** *period* (*u*·*w*·*v*) *p*
shows *period w p*
⟨*proof*⟩

lemma *period-fac-period′*: *period v p* $\Longrightarrow$ $u \leq f$ *v* $\Longrightarrow$ *period u p*
⟨*proof*⟩

The standard numeric definition of a period uses indeces.

lemma *period-indeces*: **assumes** *period w n* **and** $i + n < |w|$ **shows** $w!i = w!(i+n)$
⟨*proof*⟩

lemma *period-mod*: *period w n* $\Longrightarrow i < |w| \Longrightarrow w!i = w!(i \bmod n)$
⟨*proof*⟩

lemma *indeces-period*:

assumes *forall*: $\bigwedge i.\ i + n < |w| \Longrightarrow w!i = w!(i+n)$
shows *period w n*
⟨*proof*⟩

Alternative proof of *period ?w ?p* $\Longrightarrow$ *period* (*rev ?w*) *?p* using indeces

lemma assumes *period w p* **shows** *period* (*rev w*) *p*
⟨*proof*⟩

lemma *pow-per*[*intro*]: *period* ($y^{@}k$) $|y|$
⟨*proof*⟩

lemma *per-fac*: **assumes** $w \leq_f y^{@}k$ **shows** *period w* $|y|$
⟨*proof*⟩

lemma *per-drop* [*intro*]: **assumes** *period w n* **shows** *drop n w* $\leq_p$ *w*
⟨*proof*⟩

Two more characterizations of a period

theorem *per-shift*:
shows *period w n* $\longleftrightarrow$ *drop n w* $\leq_p$ *w*
⟨*proof*⟩

lemma *rotate-per-root*: **assumes** *w* = *rotate n w*
shows *period w n*
⟨*proof*⟩

Various lemmas on periods

lemma *ext-per-left*: **assumes** *period w p* **and** $p \leq |w|$
shows *period* (*take p w* · *w*) *p*
⟨*proof*⟩

lemma *ext-per-left-power*: *period w p* $\Longrightarrow p \leq |w| \Longrightarrow$ *period* ((*take p w*)$^{@}k$ · *w*) *p*
⟨*proof*⟩

lemma *take-several-pers*: **assumes** *period w n* **and** $m*n \leq |w|$
shows (*take n w*)$^{@}m$ = *take* (*m*∗*n*) *w*
⟨*proof*⟩

lemma *per-div*: **assumes** *n dvd* $|w|$ **and** *period w n*
shows (*take n w*)$^{@}(|w|$ *div n*) = *w*
⟨*proof*⟩

lemma *per-mult*: **assumes** *period w n* **shows** *period w* (*m*∗*n*)

⟨*proof*⟩

theorem *two-periods*:
assumes *period w p period w q* $p + q \leq |w|$
shows *period w* (*gcd p q*)
⟨*proof*⟩

lemma *index-mod-per-root*: **assumes** $r \neq \varepsilon$ **and** i: $\forall\ i < |w|.\ w!i = r!(i \bmod |r|)$
shows $w <_p r \cdot w$
⟨*proof*⟩

lemma *index-pref-pow-mod*: $w \leq_p r^{@}k \Longrightarrow i < |w| \Longrightarrow\ w!i = r!(i \bmod |r|\ )$
⟨*proof*⟩

lemma *index-per-root-mod*: **assumes** $w \leq_p r \cdot w\ r \neq \varepsilon\ i < |w|$
shows $w!i = r!(i \bmod |r|)$
⟨*proof*⟩

lemma *prepend-periods*: **assumes** *p dvd n* $n \leq |w|$ *period w p*
shows *period* ((*take n w*) · *w*) *p*
⟨*proof*⟩

2.16.4 Period: overview

notepad
begin
⟨*proof*⟩
end

2.16.5 Singleton and its power

lemma *concat-len-one*: **assumes** $|us| = 1$ **shows** *concat us* = *hd us*
⟨*proof*⟩

lemma *sing-pow-hd-tl*: *set* (*c* # *w*) ⊆ {*a*} ⟷ *c* = *a* ∧ *set w* ⊆ {*a*}
⟨*proof*⟩

lemma *pref-sing-pow*: **assumes** $w \leq_p [a]^{@}m$ **shows** $w = [a]^{@}|w|$
⟨*proof*⟩

lemmas *suf-sing-pow* = *pref-sing-pow*[*reversed*]

lemma *sing-pow-palindrom*: **assumes** $w = [a]^{@}k$ **shows** *rev w* = *w*
⟨*proof*⟩

lemma *sing-pow-palindrom'*[*simp*]: *rev* $[a]^{@}k = [a]^{@}k$
⟨*proof*⟩

lemma *sing-fac-pow*: **assumes** *set w* ⊆ {*a*} **and** $v \leq_f w$ **shows** *set v* ⊆ {*a*}

⟨*proof*⟩

lemma *sing-pow-fac*: **assumes** $a \neq b$ **and** $set\ w \subseteq \{a\}$ **shows** $\neg\ ([b] \leq f\ w)$
⟨*proof*⟩

lemma *all-set-sing-pow*: $(\forall\ b.\ b \in set\ w \longrightarrow b = a) \longleftrightarrow set\ w \subseteq \{a\}$
⟨*proof*⟩

lemma *sing-fac-set*: $[a] \leq f\ x \Longrightarrow a \in set\ x$
⟨*proof*⟩

lemma *set-sing-pow-hd* [*simp*]: **assumes** $0 < k$ **shows** $a \in set\ ([a]^{@}k)$
⟨*proof*⟩

lemma *neq-set-not-root*: $a \neq b \Longrightarrow b \in set\ w \Longrightarrow \neg\ set\ w \subseteq \{a\}$
⟨*proof*⟩

lemma *sing-pow-set-Suc*[*simp*]: $set\ ([a]^{@}Suc\ k) = \{a\}$
⟨*proof*⟩

lemma *per-one*: $set\ w \subseteq \{a\} \longleftrightarrow w <p\ [a] \cdot w$
⟨*proof*⟩

lemma *per-one'*: **assumes** $set\ r \subseteq \{a\}\ w <p\ r \cdot w$
shows $set\ w \subseteq \{a\}$
⟨*proof*⟩

thm *pref-sing-pow*

lemma *pref-sing-pow'*: $w \leq p\ [a]^{@}m \Longrightarrow set\ w \subseteq \{a\}$
⟨*proof*⟩

lemma *sing-pow-set'*: $set\ ([a]\ ^{@}\ k) \subseteq \{a\}$
⟨*proof*⟩

lemma *sing-pow-set-sub* [*elim*]: $x = [a]\ ^{@}\ k \Longrightarrow set\ x \subseteq \{a\}$
⟨*proof*⟩

lemma *set-sing-nemp-eq* [*intro*]: $set\ w \subseteq \{a\} \Longrightarrow w \neq \varepsilon \Longrightarrow set\ w = \{a\}$
⟨*proof*⟩

lemma *sing-pow-set-pos-eq* [*intro*]: $w = [a]\ ^{@}\ k \Longrightarrow 0 < k \Longrightarrow set\ w = \{a\}$
⟨*proof*⟩

lemma *unique-letter-fac-expE*: **assumes** $w \leq f\ [a]^{@}k$
obtains m **where** $w = [a]^{@}m$
⟨*proof*⟩

lemma *neq-in-set-not-pow*: **assumes** $a \neq b\ b \in set\ x$ **shows** $x \neq [a]^{@}k$

⟨*proof*⟩

lemma *sing-pow-card-set-Suc*: **assumes** $c = [a]^{@}Suc\ k$ **shows** $card(set\ c) = 1$
⟨*proof*⟩

lemma *sing-pow-card-set*: **assumes** $k \neq 0$ **and** $c = [a]^{@}k$ **shows** $card(set\ c) = 1$
⟨*proof*⟩

lemma *takeWhile-eq-set-conv*[*simp*]: $takeWhile(\lambda\ x.\ x = a)\ w = w \longleftrightarrow set\ w \subseteq \{a\}$
⟨*proof*⟩

lemma *dropWhile-distinct*: **assumes** $\neg\ set\ w \subseteq \{a\}$
shows $takeWhile(\lambda\ x.\ x = a)\ w\ \cdot [hd\ (dropWhile\ (\lambda x.\ x = a)\ w)] \leq p\ w$
⟨*proof*⟩

lemma *takeWhile-subset*: $set\ (takeWhile\ (\lambda\ x.\ x = a)\ w) \subseteq \{a\}$
⟨*proof*⟩

lemma *takeWhile-sing-pow*: $takeWhile\ (\lambda\ x.\ x = a)\ w = w \longleftrightarrow w = [a]^{@}|w|$
⟨*proof*⟩

lemma *letter-pref-exp-pref*: $[a]^{@}|takeWhile\ (\lambda\ x.\ x = a)\ u| \cdot dropWhile\ (\lambda x.\ x = a)\ u = u$
⟨*proof*⟩

lemma *letter-pref-exp-mismatch*: $u = [a]^{@}|takeWhile\ (\lambda\ x.\ x = a)\ u| \cdot v \Longrightarrow v \neq \varepsilon \Longrightarrow hd\ v \neq a$
⟨*proof*⟩

lemma *dropWhile-sing-pow*: $dropWhile\ (\lambda\ x.\ x = a)\ w = \varepsilon \longleftrightarrow w = [a]^{@}|w|$
⟨*proof*⟩

lemma *sing-exp-len*: $u = [a]^{@}m \Longrightarrow u = [a]^{@}|u|$
⟨*proof*⟩

lemma *nemp-takeWhile-hd*: $us \neq \varepsilon \Longrightarrow hd\ (takeWhile\ (\lambda\ a.\ a = hd\ us)\ us) = hd\ us$
⟨*proof*⟩

lemma $w = [a]^{@}|w| \longleftrightarrow set\ w \subseteq \{a\}$
⟨*proof*⟩

lemma *distinct-letter-in*: **assumes** $\neg\ set\ w \subseteq \{a\}$
obtains $m\ b\ q$ **where** $[a]^{@}m \cdot [b] \cdot q = w$ **and** $b \neq a$
⟨*proof*⟩

lemma *distinct-letter-in-hd*: **assumes** $\neg\ set\ w \subseteq \{hd\ w\}$

obtains $m\ b\ q$ **where** $[hd\ w]^{@}m \cdot [b] \cdot q = w$ **and** $b \neq hd\ w$ **and** $m \neq 0$
⟨*proof*⟩

lemma *distinct-letter-in-hd′*: **assumes** $\neg$ *set* $w \subseteq \{hd\ w\}$
obtains $m\ b\ q$ **where** $[hd\ w]^{@}Suc\ m \cdot [b] \cdot q = w$ **and** $b \neq hd\ w$
⟨*proof*⟩

lemma *distinct-letter-in-suf*: **assumes** $\neg$ *set* $w \subseteq \{a\}$
obtains $m\ b$ **where** $[b] \cdot [a]^{@}m \ \leq s\ w$ **and** $b \neq a$
⟨*proof*⟩

lemma *per-sing-one*: **assumes** $w \leq p\ [a] \cdot w$ **shows** *period* $w\ 1$
⟨*proof*⟩

lemma *sing-pow-exp*: *set* $w \subseteq \{a\} \longleftrightarrow w = [a]^{@}|w|$
⟨*proof*⟩

lemma *sing-power′*: **assumes** *set* $w \subseteq \{a\}$ **and** $i < |w|$ **shows** $w\ !\ i = a$
⟨*proof*⟩

lemma *sing-set-palindrom*: *set* $w \subseteq \{a\} \Longrightarrow rev\ w = w$
⟨*proof*⟩

lemma *lcp-letter-power*:
assumes $w \neq \varepsilon$ **and** *set* $w \subseteq \{a\}$ **and** $[a]^{@}m \cdot [b] \leq p\ z$ **and** $a \neq b$
shows $w \cdot z \wedge_p z \cdot w = [a]^{@}m$
⟨*proof*⟩

2.17 Border

A non-empty word $x \neq w$ is a *border* of a word w if it is both its prefix and suffix. This elementary property captures how much the word w overlaps with itself, and it is in the obvious way related to a period of w. However, in many cases it is much easier to reason about borders than about periods.

definition *border* :: *'a list* ⇒ *'a list* ⇒ *bool* (- ≤*b* - [*51*,*51*] *60*)
where [*simp*]: *border* $x\ w = (x \leq p\ w \wedge x \leq s\ w \wedge x \neq w \wedge x \neq \varepsilon)$

definition *bordered* :: *'a list* ⇒ *bool*
where [*simp*]: *bordered* $w = (\exists\, b.\ b \leq b\ w)$

lemma *borderI*[*intro*]: $x \leq p\ w \Longrightarrow x \leq s\ w \Longrightarrow x \neq w \Longrightarrow x \neq \varepsilon \Longrightarrow x \leq b\ w$
⟨*proof*⟩

lemma *borderD-pref*: $x \leq b\ w \Longrightarrow x \leq p\ w$
⟨*proof*⟩

lemma *borderD-spref*: $x \leq b\ w \Longrightarrow x <p\ w$
⟨*proof*⟩

lemma *border-sprefE* [*elim*]: **assumes** $b \leq_b w$ **obtains** r **where** $b \cdot r = w$ **and** $r \neq \varepsilon$
⟨*proof*⟩

lemma *borderD-suf*: $x \leq_b w \Longrightarrow x \leq_s w$
⟨*proof*⟩

lemma *borderD-ssuf*: $x \leq_b w \Longrightarrow x <_s w$
⟨*proof*⟩

lemma *borderD-nemp*: $x \leq_b w \Longrightarrow x \neq \varepsilon$
⟨*proof*⟩

lemma *border-ssufE* [*elim*]: **assumes** $b \leq_b w$ **obtains** r **where** $r \cdot b = w$ **and** $r \neq \varepsilon$
⟨*proof*⟩

lemma *borderD-neq*: $x \leq_b w \Longrightarrow x \neq w$
⟨*proof*⟩

lemma *borderedI*: $u \leq_b w \Longrightarrow$ *bordered* w
⟨*proof*⟩

lemma *border-lq-nemp*: **assumes** $x \leq_b w$ **shows** $x^{-1>}w \neq \varepsilon$
⟨*proof*⟩

lemma *border-rq-nemp*: **assumes** $x \leq_b w$ **shows** $w^{<-1}x \neq \varepsilon$
⟨*proof*⟩

lemma *border-trans*[*trans*]: **assumes** $t \leq_b x$ $x \leq_b w$
shows $t \leq_b w$
⟨*proof*⟩

lemma *border-rev-conv*[*reversal-rule*]: *rev* $x \leq_b$ *rev* $w \longleftrightarrow x \leq_b w$
⟨*proof*⟩

lemma *border-lq-comp*: $x \leq_b w \Longrightarrow (w^{<-1}x) \bowtie x$
⟨*proof*⟩

lemmas *border-lq-suf-comp* = *border-lq-comp*[*reversed*]

2.17.1 The shortest border

lemma *border-len*: **assumes** $x \leq_b w$
shows $1 < |w|$ **and** $0 < |x|$ **and** $|x| < |w|$
⟨*proof*⟩

lemma *borders-compare*: **assumes** $x \leq_b w$ **and** $x' \leq_b w$ **and** $|x'| < |x|$

shows $x' \leq_b x$
⟨*proof*⟩

lemma *unbordered-border*:
$bordered\ w \Longrightarrow \exists\ x.\ x \leq_b w \wedge \neg\ bordered\ x$
⟨*proof*⟩

lemma *unbordered-border-shortest*: $x \leq_b w \Longrightarrow \neg\ bordered\ x \Longrightarrow\ y \leq_b w \Longrightarrow |x| \leq |y|$
⟨*proof*⟩

lemma *long-border-bordered*: **assumes** *long*: $|w| < |x| + |x|$ **and** *border*: $x \leq_b w$
shows $(w^{<-1}x)^{-1>}x \leq_b x$
⟨*proof*⟩

thm *long-border-bordered*[*reversed*]

lemma *border-short-dec*: **assumes** *border*: $x \leq_b w$ **and** *short*: $|x| + |x| \leq |w|$
shows $x \cdot x^{-1>}(w^{<-1}x) \cdot x = w$
⟨*proof*⟩

lemma *bordered-dec*: **assumes** *bordered* w
obtains $u\ v$ **where** $u{\cdot}v{\cdot}u = w$ **and** $u \neq \varepsilon$
⟨*proof*⟩

lemma *emp-not-bordered*: $\neg\ bordered\ \varepsilon$
⟨*proof*⟩

lemma *bordered-nemp*: $bordered\ w \Longrightarrow w \neq \varepsilon$
⟨*proof*⟩

lemma *sing-not-bordered*: $\neg\ bordered\ [a]$
⟨*proof*⟩

2.17.2 Relation to period and conjugation

lemma *border-conjug-eq*: $x \leq_b w \Longrightarrow (w^{<-1}x) \cdot w = w \cdot (x^{-1>}w)$
⟨*proof*⟩

lemma *border-per-root*: $x \leq_b w \Longrightarrow w \leq_p (w^{<-1}x) \cdot w$
⟨*proof*⟩

lemma *per-root-border*: **assumes** $|r| < |w|$ **and** $r \neq \varepsilon$ **and** $w \leq_p r \cdot w$
shows $r^{-1>}w \leq_b w$
⟨*proof*⟩

lemma *pref-suf-neq-per*: **assumes** $x \leq_p w$ **and** $x \leq_s w$ **and** $x \neq w$ **shows** *period* w $(|w|{-}|x|)$
⟨*proof*⟩

lemma *border-per*: $x \leq_b w \Longrightarrow$ *period* w $(|w|-|x|)$
⟨*proof*⟩

lemma *per-border*: **assumes** $n < |w|$ **and** $n \neq 0$ **and** *period* w n
shows *take* $(|w| - n)$ w $\leq_b w$
⟨*proof*⟩

2.18 The longest border and the shortest period

2.18.1 The longest border

definition *max-borderP* :: *'a list* ⇒ *'a list* ⇒ *bool* **where**
max-borderP u w = $(u \leq_p w \wedge u \leq_s w \wedge (u = w \longrightarrow w = \varepsilon) \wedge (\forall\ v.\ v \leq_b w \longrightarrow\ v \leq_p u))$

lemma *max-borderP-emp-emp*: *max-borderP* ε ε
⟨*proof*⟩

lemma *max-borderP-exE*: **obtains** u **where** *max-borderP* u w
⟨*proof*⟩

lemma *max-borderP-of-nemp*: *max-borderP* u $\varepsilon \Longrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *max-borderP-D-neq*: $w \neq \varepsilon \Longrightarrow$ *max-borderP* u $w \Longrightarrow u \neq w$
⟨*proof*⟩

lemma *max-borderP-D-pref*: *max-borderP* u $w \Longrightarrow u \leq_p w$
⟨*proof*⟩

lemma *max-borderP-D-suf*: *max-borderP* u $w \Longrightarrow u \leq_s w$
⟨*proof*⟩

lemma *max-borderP-D-max*: *max-borderP* u $w \Longrightarrow v \leq_b w \Longrightarrow\ v \leq_p u$
⟨*proof*⟩

lemma *max-borderP-D-max'*: *max-borderP* u $w \Longrightarrow v \leq_b w \Longrightarrow v \leq_s u$
⟨*proof*⟩

lemma *unbordered-max-border-emp*: ¬ *bordered* $w \Longrightarrow$ *max-borderP* u $w \Longrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *bordered-max-border-nemp*: *bordered* $w \Longrightarrow$ *max-borderP* u $w \Longrightarrow u \neq \varepsilon$
⟨*proof*⟩

lemma *max-borderP-border*: *max-borderP* u $w \Longrightarrow u \neq \varepsilon \Longrightarrow u \leq_b w$
⟨*proof*⟩

lemma *max-borderP-rev*: *max-borderP* (*rev u*) (*rev w*) $\Longrightarrow$ *max-borderP u w*
⟨*proof*⟩

lemma *max-borderP-rev-conv*: *max-borderP* (*rev u*) (*rev w*) $\longleftrightarrow$ *max-borderP u w*
⟨*proof*⟩

definition *max-border* :: *'a list* ⇒ *'a list* **where**
max-border w = (*THE u*. (*max-borderP u w*))

lemma *max-border-unique*: **assumes** *max-borderP u w max-borderP v w*
shows *u* = *v*
⟨*proof*⟩

lemma *max-border-ex*: *max-borderP* (*max-border w*) *w*
⟨*proof*⟩

lemma *max-borderP-max-border*: *max-borderP u w* $\Longrightarrow$ *max-border w* = *u*
⟨*proof*⟩

lemma *max-border-len-rev*: |*max-border u*| = |*max-border* (*rev u*)|
⟨*proof*⟩

lemma *max-border-border*: **assumes** *bordered w* **shows** *max-border w* $\leq_b$ *w*
⟨*proof*⟩

theorem *max-border-border'*: *max-border w* $\neq \varepsilon \Longrightarrow$ *max-border w* $\leq_b$ *w*
⟨*proof*⟩

lemma *max-border-sing-emp*: *max-border* [*a*] = ε
⟨*proof*⟩

lemma *max-border-suf*: *max-border w* $\leq_s$ *w*
⟨*proof*⟩

lemma *max-border-nemp-neq*: *w* $\neq \varepsilon \Longrightarrow$ *max-border w* $\neq$ *w*
⟨*proof*⟩

lemma *max-borderI*: **assumes** *u* $\neq$ *w* **and** *u* $\leq_p$ *w* **and** *u* $\leq_s$ *w* **and** $\forall$ *v*. *v* $\leq_b$ *w* $\longrightarrow$ *v* $\leq_p$ *u*
shows *max-border w* = *u*
⟨*proof*⟩

lemma *max-border-less-len*: **assumes** *w* $\neq \varepsilon$ **shows** |*max-border w*| < |*w*|
⟨*proof*⟩

theorem *max-border-max-pref*: **assumes** *u* $\leq_b$ *w* **shows** *u* $\leq_p$ *max-border w*
⟨*proof*⟩

theorem *max-border-max-suf*: **assumes** $u \leq_b w$ **shows** $u \leq_s$ *max-border* w
⟨*proof*⟩

lemma *bordered-max-bord-nemp-conv*[*code*]: *bordered* $w \longleftrightarrow$ *max-border* $w \neq \varepsilon$
⟨*proof*⟩

lemma *max-bord-take*: *max-border* w = *take* $|$*max-border* $w|$ w
⟨*proof*⟩

2.18.2 The shortest period

definition *min-period-root* :: $'a$ *list* ⇒ $'a$ *list* (π) **where**
min-period-root w = *take* (*LEAST* n. $0 < n \wedge$ *period* w n) w

definition *min-period* :: $'a$ *list* ⇒ *nat* **where**
min-period $w = |\pi\ w|$

lemma *min-per-emp*[*simp*]: $\pi\ \varepsilon = \varepsilon$
⟨*proof*⟩

lemma *min-per-zero*[*simp*]: *min-period* ε = *0*
⟨*proof*⟩

lemma *min-per-per*: *period* w (*min-period* w) **and**
min-per-pos: $w \neq \varepsilon \Longrightarrow$ *0* $<$ *min-period* w
⟨*proof*⟩

lemma $n \leq |w| \Longrightarrow |$*take* n $w|$ = n
⟨*proof*⟩

lemma $u \leq_p w \Longrightarrow$ *take* $|u|$ w = u
⟨*proof*⟩

lemma *min-per-root-take-eq* : *take* $|\pi\ w|$ $w = \pi\ w$
⟨*proof*⟩

lemma *min-per-len*: *min-period* $w \leq |w|$
⟨*proof*⟩

lemmas *min-per-root-len* = *min-per-len*[*unfolded min-period-def*]

lemma *min-per-sing*: *min-period* [a] = *1*
⟨*proof*⟩

lemma *min-per-root-per*: **assumes** $w \neq \varepsilon$ **shows** $w \leq_p (\pi\ w) \cdot w$
⟨*proof*⟩

lemma *min-per-root-per-root*: **assumes** $w \neq \varepsilon$ **shows** $w <_p (\pi\ w) \cdot w$
⟨*proof*⟩

lemma *min-per-pref*: $\pi\ w \leq_p w$
⟨*proof*⟩

lemma *min-per-nemp*: $w \neq \varepsilon \Longrightarrow \pi\ w \neq \varepsilon$
⟨*proof*⟩

lemma *min-per-min*: **assumes** $w <_p r \cdot w$ **shows** $\pi\ w \leq_p r$
⟨*proof*⟩

lemma *lq-min-per-pref*: $\pi\ w^{-1>} w \leq_p w$
⟨*proof*⟩

lemma *max-bord-emp*: *max-border* $\varepsilon = \varepsilon$
⟨*proof*⟩

theorem *min-per-max-border*: $\pi\ w \cdot$ *max-border* $w = w$
⟨*proof*⟩

lemma *min-per-len-diff*: *min-period* $w = |w| - |$*max-border* $w|$
⟨*proof*⟩

lemma *min-per-root-take* [*code*]: $\pi\ w =$ *take* $(|w| - |$*max-border* $w|)\ w$
⟨*proof*⟩

2.19 Primitive words

If a word w is not a non-trivial power of some other word, we say it is primitive.

definition *primitive* :: *'a list* ⇒ *bool*
where *primitive* $u = (\forall\ r\ k.\ r^{@}k = u \longrightarrow k = 1)$

lemma *emp-not-prim*[*simp*]: ¬ *primitive* ε
⟨*proof*⟩

lemma *primI*[*intro*]: $(\bigwedge\ r\ k.\ r^{@}k = u \Longrightarrow k = 1) \Longrightarrow$ *primitive* u
⟨*proof*⟩

lemma *prim-nemp*: *primitive* $u \Longrightarrow u \neq \varepsilon$
⟨*proof*⟩

lemma *prim-exp-one*: *primitive* $u \Longrightarrow r^{@}k = u \Longrightarrow k = 1$
⟨*proof*⟩

lemma *pow-nemp-imprim*[*intro*]: $2 \leq k \Longrightarrow$ ¬ *primitive* $(u^{@}k)$
⟨*proof*⟩

lemma *pow-not-prim*: ¬ *primitive* $(u^{@}$*Suc*$($*Suc* $k))$

⟨*proof*⟩

lemma *pow-non-prim*: $k \neq 1 \Longrightarrow \neg$ *primitive* $(w^{@}k)$
⟨*proof*⟩

lemma *prim-exp-eq*: *primitive* $u \Longrightarrow r^{@}k = u \Longrightarrow u = r$
⟨*proof*⟩

lemma *prim-per-div*: **assumes** *primitive* v **and** $n \neq 0$ **and** $n \leq |v|$ **and** *period* v $(gcd\ |v|\ n)$
shows $n = |v|$
⟨*proof*⟩

lemma *prim-comm-exp*[*elim*]: **assumes** *primitive* r **and** $u \cdot r = r \cdot u$
obtains k **where** $r^{@}k = u$
⟨*proof*⟩

lemma *set-singleton-iff*: *card* (*set* u) $= 1 \longleftrightarrow$ *set* $u = \{hd\ u\}$
⟨*proof*⟩

lemma *set-empty-iff*: *card* (*set* u) $= 0 \longleftrightarrow$ *set* $u = \{\}$
⟨*proof*⟩

lemma *set-large-iff*: $1 <$ *card* (*set* u) $\longleftrightarrow \neg$ *set* $u \subseteq \{hd\ u\}$
⟨*proof*⟩

lemma *prim-card-set*: **assumes** *primitive* u **and** $|u| \neq 1$ **shows** $1 <$ *card* (*set* u)
⟨*proof*⟩

lemma *comm-not-prim*: **assumes** $u \neq \varepsilon$ $v \neq \varepsilon$ $u \cdot v = v \cdot u$ **shows** $\neg$ *primitive* $(u \cdot v)$
⟨*proof*⟩

lemma *prim-rotate-conv*: *primitive* $w \longleftrightarrow$ *primitive* (*rotate* n w)
⟨*proof*⟩

lemma *non-prim*: **assumes** $\neg$ *primitive* w **and** $w \neq \varepsilon$
obtains r k **where** $r \neq \varepsilon$ **and** $1 < k$ **and** $r^{@}k = w$ **and** $w \neq r$
⟨*proof*⟩

lemma *prim-no-rotate*: **assumes** *primitive* w **and** $0 < n$ **and** $n < |w|$
shows *rotate* n $w \neq w$
⟨*proof*⟩

lemma *no-rotate-prim*: **assumes** $w \neq \varepsilon$ **and** $\bigwedge n.\ 0 < n \Longrightarrow n < |w| \Longrightarrow$ *rotate* n $w \neq w$
shows *primitive* w

⟨*proof*⟩

corollary *prim-iff-rotate*: **assumes** $w \neq \varepsilon$ **shows**
primitive $w \longleftrightarrow (\forall\ n.\ 0 < n \wedge n < |w| \longrightarrow \mathit{rotate}\ n\ w \neq w)$
⟨*proof*⟩

lemma *prim-sing*: *primitive* $[a]$
⟨*proof*⟩

lemma *sing-pow-conv* [*simp*]: $[u] = t^{@}k \longleftrightarrow t = [u] \wedge k = 1$
⟨*proof*⟩

lemma *prim-rev-iff*[*reversal-rule*]: *primitive* $(\mathit{rev}\ u) \longleftrightarrow \mathit{primitive}\ u$
⟨*proof*⟩

lemma *prim-map-prim*: *primitive* $(\mathit{map}\ f\ ws) \Longrightarrow \mathit{primitive}\ ws$
⟨*proof*⟩

lemma *inj-map-prim*: **assumes** *inj-on* $f\ A$ **and** $u \in \mathit{lists}\ A$ **and**
primitive u
shows *primitive* $(\mathit{map}\ f\ u)$
⟨*proof*⟩

lemma *prim-map-iff* [*reversal-rule*]:
assumes *inj* f **shows** *primitive* $(\mathit{map}\ f\ ws) = \mathit{primitive}\ (ws)$
⟨*proof*⟩

lemma *prim-concat-prim*: *primitive* $(\mathit{concat}\ ws) \Longrightarrow \mathit{primitive}\ ws$
⟨*proof*⟩

lemma *eq-append-not-prim*: $x = y \Longrightarrow \neg\ \mathit{primitive}\ (x \cdot y)$
⟨*proof*⟩

2.20 Primitive root

Given a non-empty word w which is not primitive, it is natural to look for the shortest u such that $w = u^k$. Such a word is primitive, and it is the primitive root of w.

definition *primitive-root* :: *'a list* ⇒ *'a list* (ϱ) **where**
primitive-root $x = (\mathit{if}\ x \neq \varepsilon\ \mathit{then}\ (\mathit{THE}\ r.\ \mathit{primitive}\ r \wedge (\exists\ k.\ x = r^{@}k))\ \mathit{else}\ \varepsilon)$

definition *primitive-root-exp* :: *'a list* ⇒ *nat* (e_ϱ) **where**
primitive-root-exp $x = (\mathit{if}\ x = \varepsilon\ \mathit{then}\ 0\ \mathit{else}\ (\mathit{THE}\ k.\ x = (\varrho\ x)^{@}k))$

lemma *primroot-emp*[*simp*]: $\varrho\ \varepsilon = \varepsilon$
⟨*proof*⟩

lemma *comm-prim*: **assumes** *primitive* r **and** *primitive* s **and** $r{\cdot}s = s{\cdot}r$

shows $r = s$
⟨*proof*⟩

lemma *primroot-ex*: **assumes** $x \neq \varepsilon$ **shows** $\exists\ r\ k.\ primitive\ r \wedge k \neq 0 \wedge x = r^{@}k$
⟨*proof*⟩

lemma *primroot-exE*: **assumes**$x \neq \varepsilon$
obtains $r\ k$ **where** *primitive* r **and** $0 < k$ **and** $x = r^{@}k$
⟨*proof*⟩

Uniqueness of the primitive root follows from the following lemma

lemma *primroot-unique*: **assumes** $u \neq \varepsilon$ **and** *primitive* r **and** $u = r^{@}k$ **shows** $\varrho\ u = r$
⟨*proof*⟩

lemma *primroot-unique'*: **assumes** $0 < k$ *primitive* r **and** $u = r^{@}k$ **shows** $\varrho\ u = r$
⟨*proof*⟩

lemma *prim-primroot*[*intro*]: **assumes** *primitive* x **shows** $\varrho\ x = x$
⟨*proof*⟩

lemma *primroot-exp-unique*: **assumes** $u \neq \varepsilon$ **and** $(\varrho\ u)^{@}k = u$ **shows** $e_{\varrho}\ u = k$
⟨*proof*⟩

lemma *primroot-prim*[*intro*]: $x \neq \varepsilon \Longrightarrow primitive\ (\varrho\ x)$
⟨*proof*⟩

Existence and uniqueness of the primitive root justifies the function ϱ: it indeed yields the primitive root of a nonempty word.

lemma *primroot-expE*: **obtains** k **where** $(\varrho\ x)^{@}k = x$ **and** $0 < k$
⟨*proof*⟩

lemma *primroot-exp-eq* [*simp*]: $(\varrho\ u)^{@}(e_{\varrho}\ u) = u$
⟨*proof*⟩

lemma *prim-pow-exp-one*: $primitive\ (u^{@}i) \Longrightarrow i = 1$
⟨*proof*⟩

lemma *prim-exp-emp*[*simp*]: $e_{\varrho}\ \varepsilon = 0$
⟨*proof*⟩

lemma *prim-exp-zero-iff*[*simp*]: $e_{\varrho}\ x = 0 \longleftrightarrow x = \varepsilon$
⟨*proof*⟩

lemma *prim-exp-one-iff*: $primitive\ x \longleftrightarrow e_{\varrho}\ x = 1$
⟨*proof*⟩

lemma *nemp-not-prim-exp*: $x \neq \varepsilon \Longrightarrow \neg$ *primitive* $x \longleftrightarrow 1 < e_\varrho\ x$
⟨*proof*⟩

lemma *not-prim-pref-primroots*: $\neg$ *primitive* $x \longleftrightarrow \varrho\ x \cdot \varrho\ x \leq p\ x$
⟨*proof*⟩

lemma *zero-prim-exp-eq*[*intro*]: $n = 0 \Longrightarrow e_\varrho(r^{@}n) = n$
⟨*proof*⟩

lemma *primroot-exp-len*:
shows $e_\varrho\ w * |\varrho\ w| = |w|$
⟨*proof*⟩

lemma *primroot-exp-nemp* [*intro*]: $u \neq \varepsilon \Longrightarrow 0 < e_\varrho\ u$
⟨*proof*⟩

lemma *primroot-exp-zero* [*intro*]: $e_\varrho\ u = 0 \Longrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *primroot-exp-zero-conv* [*simp*]: $e_\varrho\ u = 0 \longleftrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *pop-primroot*: $u = \varrho\ u \cdot (\varrho\ u)^{@}(e_\varrho\ u - 1)$
⟨*proof*⟩

lemma *pop-primroot'*: $u = (\varrho\ u)^{@}(e_\varrho\ u - 1) \cdot \varrho\ u$
⟨*proof*⟩

lemma *primroot-exp-minus-Suc-eq* [*simp*]: $\varrho\ x\ ^{@}\ (e_\varrho\ x - Suc\ 0) \cdot \varrho\ x = x$
⟨*proof*⟩

lemma *primroot-nemp*[*intro*!]: $x \neq \varepsilon \Longrightarrow \varrho\ x \neq \varepsilon$
⟨*proof*⟩

lemma *primroot-idemp*[*simp*]: $\varrho\ (\varrho\ x) = \varrho\ x$
⟨*proof*⟩

lemma *prim-primroot-conv*: **assumes** $w \neq \varepsilon$ **shows** *primitive* $w \longleftrightarrow \varrho\ w = w$
⟨*proof*⟩

lemma *not-prim-primroot-expE*: **assumes** $\neg$ *primitive* w
obtains k **where** $\varrho\ w\ ^{@}k = w$ **and** $2 \leq k$
⟨*proof*⟩

lemma *not-prim-expE*: **assumes** $\neg$ *primitive* x **and** $x \neq \varepsilon$
obtains $r\ k$ **where** *primitive* r **and** $2 \leq k$ **and** $r^{@}k = x$

⟨*proof*⟩

lemma *primroot-len-le*: $u \neq \varepsilon \Longrightarrow |\varrho\ u| \leq |u|$
⟨*proof*⟩

lemma *primroot-pref*: $\varrho\ u \leq p\ u$
⟨*proof*⟩

lemma *primroot-take*: **assumes** $u \neq \varepsilon$ **shows** $\varrho\ u = (take\ (\ |\varrho\ u|\ )\ u)$
⟨*proof*⟩

lemma *primroot-rotate-comm*: **assumes** $w \neq \varepsilon$ **shows** $\varrho\ (rotate\ n\ w) = rotate\ n\ (\varrho\ w)$
⟨*proof*⟩

lemma *primroot-rotate*: $\varrho\ w = r \longleftrightarrow \varrho\ (rotate\ (k{*}|r|)\ w) = r$ (**is** *?L* $\longleftrightarrow$ *?R*)
⟨*proof*⟩

lemma *primrootI*[*intro*]: **assumes** *pow*: $u = r^{@}(Suc\ k)$ **and** *primitive r* **shows** $\varrho\ u = r$
⟨*proof*⟩

lemma *sing-primroot*[*simp*]: $\varrho\ [a] = [a]$
⟨*proof*⟩

lemma *short-primroot*: **assumes** $u \neq \varepsilon$ $\neg$ *primitive u* **shows** $|\varrho\ u| < |u|$
⟨*proof*⟩

lemma *prim-primroot-cases*: **obtains** $u = \varepsilon \mid$ *primitive u* $\mid |\varrho\ u| < |u|$
⟨*proof*⟩

We also have the standard characterization of commutation for nonempty words.

theorem *comm-primroots*: **assumes** $u \neq \varepsilon$ **and** $v \neq \varepsilon$ **shows** $u \cdot v = v \cdot u \longleftrightarrow \varrho\ u = \varrho\ v$
⟨*proof*⟩

lemma *comm-primroots'*: $u \neq \varepsilon \Longrightarrow v \neq \varepsilon \Longrightarrow u \cdot v = v \cdot u \Longrightarrow \varrho\ u = \varrho\ v$
⟨*proof*⟩

lemma *primroot-add-exp*: **assumes** $0 < l$ **shows** $\varrho\ (x^{@}l) = \varrho\ x$
⟨*proof*⟩

lemma *same-primroots-comm*: $\varrho\ x = \varrho\ y \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *pow-primroot*: **assumes** $x \neq \varepsilon$ **shows** $\varrho\ (x^{@}Suc\ k) = \varrho\ x$
⟨*proof*⟩

lemma *comm-primroot-exp*: **assumes** $v \neq \varepsilon$ **and** $u \cdot v = v \cdot u$
obtains n **where** $(\varrho\ v)^{@} n = u$
⟨*proof*⟩

lemma *primE*: **obtains** t **where** *primitive* t
⟨*proof*⟩

lemma *comm-primrootE'*: **assumes** $x \cdot y = y \cdot x$
obtains $t\ m\ k$ **where** $x = t^{@} k$ **and** $y = t^{@} m$ **and** *primitive* t
⟨*proof*⟩

lemma *comm-nemp-pows-posE*: **assumes** $x \cdot y = y \cdot x$ **and** $x \neq \varepsilon$ **and** $y \neq \varepsilon$
obtains $t\ k\ m$ **where** $x = t^{@} k$ **and** $y = t^{@} m$ **and** $0 < k$ **and** $0 < m$ **and** *primitive* t
⟨*proof*⟩

lemma *comm-primroot-conv*: $u \cdot v = v \cdot u \longleftrightarrow u \cdot \varrho\ v = \varrho\ v \cdot u$
⟨*proof*⟩

lemma *comm-primroot* [*simp*, *intro*]: $u \cdot \varrho\ u = \varrho\ u \cdot u$
⟨*proof*⟩

lemma *comm-primroot-conv'*: **shows** $u \cdot v = v \cdot u \longleftrightarrow \varrho\ u \cdot \varrho\ v = \varrho\ v \cdot \varrho\ u$
⟨*proof*⟩

lemma *comm-primrootI* [*intro*]: $\varrho\ u \cdot \varrho\ v = \varrho\ v \cdot \varrho\ u \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *per-root-primroot*: $w <p\ r \cdot w \Longrightarrow\ w <p\ \varrho\ r \cdot w$
⟨*proof*⟩

lemma *per-root-primroot'*: $w <p\ \varrho\ r \cdot w \Longrightarrow\ w <p\ r \cdot w$
⟨*proof*⟩

lemma *per-root-primroot-iff*: $w <p\ \varrho\ r \cdot w \longleftrightarrow\ w <p\ r \cdot w$
⟨*proof*⟩

lemma *per-root-primroot-iff'*: $x \leq p\ \varrho\ u \cdot x \longleftrightarrow x \leq p\ u \cdot x$
⟨*proof*⟩

lemma *primroot-per-root*: $r \neq \varepsilon \Longrightarrow\ r <p\ \varrho\ r \cdot r$
⟨*proof*⟩

lemma *prim-comm-short-emp*: **assumes** *primitive* p **and** $u \cdot p = p \cdot u$ **and** $|u| < |p|$
shows $u = \varepsilon$
⟨*proof*⟩

lemma *primroot-rev*[*reversal-rule*]: **shows** $\varrho\ (rev\ u) = rev\ (\varrho\ u)$
⟨*proof*⟩

lemmas *primroot-suf* = *primroot-pref*[*reversed*]

lemma *per-le-prim-iff*:
assumes $u \leq p\ p \cdot u$ **and** $p \neq \varepsilon$ **and** $2 * |p| \leq |u|$
shows *primitive* $u \longleftrightarrow u \cdot p \neq p \cdot u$
⟨*proof*⟩

lemma *per-root-mod-primE* [*elim*]: **assumes** $u <p\ r \cdot u$
obtains $n\ p\ s$ **where** $p \cdot s = \varrho\ r$ **and** $(p{\cdot}s)^{@}n \cdot p = u$ **and** $s \neq \varepsilon$
⟨*proof*⟩

lemmas[*shifts*] = *pows-comm*

lemma *per-root-shift*: **assumes** $x = (r \cdot s)^{@}k \cdot r\ \ y = (r \cdot s)^{@}m$
shows $y \cdot x = x \cdot (s \cdot r)^{@}m$
⟨*proof*⟩

lemma *root-comm-root*: **assumes** $x \leq p\ u \cdot x$ **and** $v \cdot u = u \cdot v$ **and** $u \neq \varepsilon$
shows $x \leq p\ v \cdot x$
⟨*proof*⟩

2.20.1 Primitivity and the shortest period

lemma *min-per-primitive*: **assumes** $w \neq \varepsilon$ **shows** *primitive* $(\pi\ w)$
⟨*proof*⟩

lemma *min-per-short-primroot*: **assumes** $w \neq \varepsilon$ **and** $(\varrho\ w)^{@}k = w$ **and** $k \neq 1$
shows $\pi\ w = \varrho\ w$
⟨*proof*⟩

lemma *primitive-iff-per*: *primitive* $w \longleftrightarrow w \neq \varepsilon \wedge (\pi\ w = w \vee \pi\ w \cdot w \neq w \cdot \pi\ w)$
⟨*proof*⟩

2.21 Conjugation

Two words x and y are conjugated if one is a rotation of the other. Or, equivalently, there exists z such that

$$xz = zy.$$

definition *conjugate* (**infix** $\sim$ *51*)
where $u \sim v \equiv \exists r\ s.\ r \cdot s = u \wedge s \cdot r = v$

lemma *conjugE* [*elim*]:
assumes $u \sim v$
obtains $r\ s$ **where** $r \cdot s = u$ **and** $s \cdot r = v$

⟨*proof*⟩

lemma *conjugE-nemp*[*elim*]:
assumes $u \sim v$ **and** $u \neq \varepsilon$
obtains $r\ s$ **where** $r \cdot s = u$ **and** $s \cdot r = v$ **and** $s \neq \varepsilon$
⟨*proof*⟩

lemma *conjugE1* [*elim*]:
assumes $u \sim v$
obtains r **where** $u \cdot r = r \cdot v$
⟨*proof*⟩

lemma *conjug-rev-conv* [*reversal-rule*]: $rev\ u \sim rev\ v \longleftrightarrow u \sim v$
⟨*proof*⟩

lemma *conjug-rotate-iff*: $u \sim v \longleftrightarrow (\exists\ n.\ v = rotate\ n\ u)$
⟨*proof*⟩

lemma *rotate-conjug*: $w \sim rotate\ n\ w$
⟨*proof*⟩

lemma *conjug-rotate-iff-le*:
shows $u \sim v \longleftrightarrow (\exists\ n \leq |u| - 1.\ v = rotate\ n\ u)$
⟨*proof*⟩

lemma *conjugI* [*intro*]: $r \cdot s = u \Longrightarrow s \cdot r = v \Longrightarrow u \sim v$
⟨*proof*⟩

lemma *conjugI'* [*intro*!]: $r \cdot s \sim s \cdot r$
⟨*proof*⟩

lemma *conjug-refl*: $u \sim u$
⟨*proof*⟩

lemma *conjug-sym*[*sym*]: $u \sim v \Longrightarrow v \sim u$
⟨*proof*⟩

lemma *conjug-swap*: $u \sim v \longleftrightarrow v \sim u$
⟨*proof*⟩

lemma *conjug-nemp-iff*: $u \sim v \Longrightarrow u = \varepsilon \longleftrightarrow v = \varepsilon$
⟨*proof*⟩

lemma *conjug-len*: $u \sim v \Longrightarrow |u| = |v|$
⟨*proof*⟩

lemma *pow-conjug*:
assumes *eq*: $t^{@}i \cdot r \cdot u = t^{@}k$ **and** t: $r \cdot s = t$
shows $u \cdot t^{@}i \cdot r = (s \cdot r)^{@}k$

⟨*proof*⟩

lemma *conjug-set*: **assumes** $u \sim v$ **shows** *set* u = *set* v
⟨*proof*⟩

lemma *conjug-concat-conjug*: $xs \sim ys \Longrightarrow$ *concat* $xs \sim$ *concat* ys
⟨*proof*⟩

The solution of the equation

$$xz = zy$$

is given by the next lemma.

lemma *conjug-eqE* [*elim*, *consumes 2*]:
assumes *eq*: $x \cdot z = z \cdot y$ **and** $x \neq \varepsilon$
obtains u v k **where** $u \cdot v = x$ **and** $v \cdot u = y$ **and** $(u \cdot v)^{@}k \cdot u = z$ **and** $v \neq \varepsilon$
⟨*proof*⟩

theorem *conjugation*: **assumes** $x{\cdot}z = z{\cdot}y$ **and** $x \neq \varepsilon$
shows $\exists\ u\ v\ k.\ u \cdot v = x \wedge v \cdot u\ = y \wedge (u \cdot v)^{@}k \cdot u = z$
⟨*proof*⟩

lemma *conjug-eqE′* [*elim*]:
assumes *eq*: $x \cdot z = z \cdot y$
obtains u v k i **where** $(u \cdot v)^{@}i = x$ **and** $(v \cdot u)^{@}i = y$ **and** $(u \cdot v)^{@}k \cdot u = z$
⟨*proof*⟩

lemma *conjug-eq-primrootE′* [*elim*, *consumes 2*]:
assumes *eq*: $x \cdot z = z \cdot y$ **and** $x \neq \varepsilon$
obtains r s i n **where**
$(r \cdot s)^{@}i = x$ **and**
$(s \cdot r)^{@}i = y$ **and**
$(r \cdot s)^{@}n \cdot r = z$ **and**
$s \neq \varepsilon$ **and** $0 < i$ **and** *primitive* $(r \cdot s)$
⟨*proof*⟩

lemma *conjugI1* [*intro*]:
assumes *eq*: $u \cdot r = r \cdot v$
shows $u \sim v$
⟨*proof*⟩

lemma *pow-conjug-conjug-conv*: **assumes** $0 < k$ **shows** $u^{@}k \sim v^{@}k \longleftrightarrow u \sim v$
⟨*proof*⟩

lemma *conjug-trans* [*trans*]:
assumes *uv*: $u \sim v$ **and** *vw*: $v \sim w$
shows $u \sim w$
⟨*proof*⟩

lemma *conjug-trans′*: **assumes** *uv′*: $u \cdot r = r \cdot v$ **and** *vw′*: $v \cdot s = s \cdot w$ **shows** $u \cdot (r \cdot s) = (r \cdot s) \cdot w$

⟨*proof*⟩

lemma *root-conjugE*:
assumes $x \leq p\ r \cdot x$
obtains $u\ v\ k\ i$ **where** $(u \cdot v)^{@} i = r$ **and** $(u \cdot v)^{@} k \cdot u = x$ **and** $(v \cdot u)^{@}\ i = x^{-1>}(r \cdot x)$
⟨*proof*⟩

lemmas *suf-root-conjugE* = *root-conjugE*[*reversed*]

Of course, conjugacy is an equivalence relation.

lemma *conjug-equiv*: *equivp* $(\sim)$
⟨*proof*⟩

lemma *append-split-mod-primroot*:
obtains $r\ s\ i\ j$ **where** $(r{\cdot}s)^{@} i{\cdot}r = x\ (s{\cdot}r)^{@} j{\cdot}s = y\ \ r{\cdot}s = \varrho\ (x{\cdot}y)\ s{\cdot}r = x^{-1>}(\varrho\ (x{\cdot}y) \cdot x)$
⟨*proof*⟩

lemma *rotate-fac-pref*: **assumes** $u \leq f\ w$
obtains w' **where** $w' \sim w$ **and** $u \leq p\ w'$
⟨*proof*⟩

lemma *rotate-into-pos-sq*: **assumes** $s{\cdot}p \leq f\ w{\cdot}w$ **and** $|s| \leq |w|$ **and** $|p| \leq |w|$
obtains w' **where** $w \sim w'\ p \leq p\ w'\ s \leq s\ w'$
⟨*proof*⟩

lemma *rotate-into-pref-sq*: **assumes** $p \leq f\ w{\cdot}w$ **and** $|p| \leq |w|$
obtains w' **where** $w \sim w'\ p \leq p\ w'$
⟨*proof*⟩

lemmas *rotate-into-suf-sq* = *rotate-into-pref-sq*[*reversed*]

lemma *rotate-into-pos*: **assumes** $s{\cdot}p \leq f\ w$
obtains w' **where** $w \sim w'\ p \leq p\ w'\ s \leq s\ w'$
⟨*proof*⟩

lemma *rotate-into-pos-conjug*: **assumes** $w \sim v$ **and** $s{\cdot}p \leq f\ v$
obtains w' **where** $w \sim w'\ p \leq p\ w'\ s \leq s\ w'$
⟨*proof*⟩

lemma *nconjug-neq*: $\neg\ u \sim v \Longrightarrow u \neq v$
⟨*proof*⟩

lemma *prim-conjug*:
assumes *prim*: *primitive u* **and** *conjug*: $u \sim v$
shows *primitive v*
⟨*proof*⟩

lemma *conjug-prim-iff*: **assumes** $u \sim v$ **shows** *primitive* $u =$ *primitive* v
⟨*proof*⟩

lemmas *conjug-prim-iff'* = *conjug-prim-iff*[*OF conjugI'*]

lemmas *conjug-concat-prim-iff* = *conjug-concat-conjug*[*THEN conjug-prim-iff*]

lemma *conjug-eq-primrootE* [*elim*, *consumes 2*]:
assumes *eq*: $x \cdot z = z \cdot y$ **and** $x \neq \varepsilon$
obtains $r\ s\ i\ n$ **where**
$(r \cdot s)^{@} i = x$ **and**
$(s \cdot r)^{@} i = y$ **and**
$(r \cdot s)^{@} n \cdot r = z$ **and**
$s \neq \varepsilon$ **and** $0 < i$ **and** *primitive* $(r \cdot s)$
and *primitive* $(s \cdot r)$
⟨*proof*⟩

lemma *conjug-primrootsE*: **assumes** $\varrho\ p \sim \varrho\ q$
obtains $r\ s\ k\ l$ **where** $p = (r \cdot s)^{@} k$ **and** $q = (s \cdot r)^{@} l$ **and** *primitive* $(r \cdot s)$
⟨*proof*⟩

lemma *root-conjug*: $u \leq p\ r \cdot u \Longrightarrow u^{-1>}(r \cdot u) \sim r$
⟨*proof*⟩

lemmas *conjug-prim-iff-pref* = *conjug-prim-iff*[*OF root-conjug*]

lemma *conjug-primroot-word*:
assumes *conjug*: $u \cdot t = t \cdot v$
shows $(\varrho\ u) \cdot t = t \cdot (\varrho\ v)$
⟨*proof*⟩

lemma *conjug-primroot*:
assumes $u \sim v$
shows $\varrho\ u \sim \varrho\ v$
⟨*proof*⟩

lemma *conjug-primroots-nemp*: **assumes** $x \cdot y \neq y \cdot x$ **and** $r \cdot s = \varrho\ (x \cdot y)$ **and** $s \cdot r = \varrho\ (y \cdot x)$
shows $r \neq \varepsilon$ **and** $s \neq \varepsilon$
⟨*proof*⟩

lemma *conjugE-primrootsE*[*elim*]: **assumes** $x \cdot y \neq y \cdot x$
obtains $r\ s$ **where** $r \cdot s = \varrho\ (x \cdot y)$ **and** $s \cdot r = \varrho\ (y \cdot x)$ **and** $r \neq \varepsilon$ **and** $s \neq \varepsilon$
⟨*proof*⟩

lemma *conjug-add-exp*: $u \sim v \Longrightarrow u^{@} k \sim v^{@} k$
⟨*proof*⟩

lemma *conjug-primroot-iff*:
  assumes *len*: $|u| = |v|$
  shows $\varrho\ u\ \sim \varrho\ v \longleftrightarrow u \sim v$
⟨*proof*⟩

lemma *two-conjugs-aux*: **assumes** $u{\cdot}v = x{\cdot}y$ **and** $v{\cdot}u = y{\cdot}x$ **and** $u \neq \varepsilon$ **and** $u \neq x$ **and** $|u| \leq |x|$
  obtains $r\ s\ k\ l\ m\ n$ **where**
    $u = (s \cdot r)^{@}k \cdot s$ **and** $v = (r \cdot s)^{@}l \cdot r$ **and**
    $x = (s \cdot r)^{@}m \cdot s$ **and** $y = (r \cdot s)^{@}n \cdot r$ **and**
    primitive $(r \cdot s)$ **and** *primitive* $(s \cdot r)$
⟨*proof*⟩

lemma *two-conjugs*: **assumes** $u{\cdot}v = x{\cdot}y$ **and** $v{\cdot}u = y{\cdot}x$ **and** $u \neq \varepsilon$ **and** $x \neq \varepsilon$ **and** $u \neq x$
  obtains $r\ s\ k\ l\ m\ n$ **where**
    $u = (s \cdot r)^{@}k \cdot s$ **and** $v = (r \cdot s)^{@}l \cdot r$ **and**
    $x = (s \cdot r)^{@}m \cdot s$ **and** $y = (r \cdot s)^{@}n \cdot r$ **and**
    primitive $(r \cdot s)$ **and** *primitive* $(s \cdot r)$
  ⟨*proof*⟩

lemma *fac-pow-pref-conjug*:
  assumes $u \leq_f t^{@}k$
  obtains t' **where** $t \sim t'$ **and** $u \leq_p t'^{@}k$
⟨*proof*⟩

lemmas *fac-pow-suf-conjug* = *fac-pow-pref-conjug*[*reversed*]

lemma *fac-pow-len-conjug*[*intro*]: **assumes** $|u| = |v|$ **and** $u \leq_f v^{@}k$ **shows** $v \sim u$
⟨*proof*⟩

lemma *conjug-fac-sq*:
  $u \sim v \Longrightarrow u \leq_f v \cdot v$
  ⟨*proof*⟩

lemma *conjug-fac-pow-conv*: **assumes** $|u| = |v|$ **and** $2 \leq k$
  shows $u \sim v \longleftrightarrow u \leq_f v^{@}k$
⟨*proof*⟩

lemma *conjug-fac-Suc*: **assumes** $t \sim v$
  shows $t^{@}k \leq_f v^{@}Suc\ k$
⟨*proof*⟩

lemma *fac-pow-conjug*: **assumes** $u \leq_f v^{@}k$ **and** $t \sim v$
  shows $u \leq_f t^{@}Suc\ k$
⟨*proof*⟩

lemma *border-conjug*: $x \leq_b w \Longrightarrow w^{<-1}x \sim x^{-1>}w$
  ⟨*proof*⟩

lemma *count-list-conjug*: **assumes** $u \sim v$ **shows** *count-list* $u\ a$ = *count-list* $v\ a$
⟨*proof*⟩

lemma *conjug-in-lists*: $us \sim vs \Longrightarrow vs \in lists\ A \Longrightarrow us \in lists\ A$
⟨*proof*⟩

lemma *conjug-in-lists'*: $us \sim vs \Longrightarrow us \in lists\ A \Longrightarrow vs \in lists\ A$
⟨*proof*⟩

lemma *conjug-in-lists-iff*: $us \sim vs \Longrightarrow us \in lists\ A \longleftrightarrow vs \in lists\ A$
⟨*proof*⟩

lemma *prim-conjug-unique*: **assumes** *primitive* $(u \cdot v)$ **and** $u \cdot v = r \cdot s$ **and** $v \cdot u = s \cdot r$ **and** $u \cdot v \neq v \cdot u$
shows $u = r$ **and** $v = s$
⟨*proof*⟩

lemma *prim-conjugE*[*elim*, *consumes 3*]: **assumes** $(u \cdot v) \cdot z = z \cdot (v \cdot u)$ **and** *primitive* $(u \cdot v)$ **and** $v \neq \varepsilon$
obtains k **where** $(u \cdot v)^{@}k \cdot u = z$
⟨*proof*⟩

lemma *prim-conjugE'*[*elim*, *consumes 3*]: **assumes** $(r \cdot s) \cdot z = z \cdot (s \cdot r)$ **and** *primitive* $(r \cdot s)$ **and** $z \neq \varepsilon$
obtains k **where** $(r \cdot s)^{@}k \cdot r = z$
⟨*proof*⟩

lemma *conjug-primroots-unique*: **assumes** $x \cdot y \neq y \cdot x$ **and**
$r \cdot s = \varrho\ (x \cdot y)$ **and** $s \cdot r = \varrho\ (y \cdot x)$ **and**
$r' \cdot s' = \varrho\ (x \cdot y)$ **and** $s' \cdot r' = \varrho\ (y \cdot x)$
shows $r = r'$ **and** $s = s'$
⟨*proof*⟩

lemma *prim-conjug-pref*: **assumes** *primitive* $(s \cdot r)$ **and** $u \cdot r \cdot s \leq p\ (s \cdot r)^{@}n$ **and** $r \neq \varepsilon$
obtains n **where** $(s \cdot r)^{@}n \cdot s = u$
⟨*proof*⟩

lemma *fac-per-conjug*: **assumes** *period* $w\ n$ **and** $v \leq f\ w$ **and** $|v| = n$
shows $v \sim take\ n\ w$
⟨*proof*⟩

lemma *fac-pers-conjug*: **assumes** *period* $w\ n$ **and** $v \leq f\ w$ **and** $|v| = n$ **and** $u \leq f\ w$ **and** $|u| = n$
shows $v \sim u$
⟨*proof*⟩

lemma *conjug-pow-powE*: **assumes** $w \sim r^{@}k$ **obtains** s **where** $w = s^{@}k$
⟨*proof*⟩

lemma *find-second-letter*: **assumes** $a \neq b$ **and** $set\ ws = \{a,b\}$
shows $dropWhile\ (\lambda\ c.\ c = a)\ ws \neq \varepsilon$ **and** $hd\ (dropWhile\ (\lambda\ c.\ c = a)\ ws) = b$
⟨*proof*⟩

lemma *fac-conjuq-sq*:
assumes $u \sim v$ **and** $|w| \leq |u|$ **and** $w \leq f\ u \cdot u$
shows $w \leq f\ v \cdot v$
⟨*proof*⟩

lemma *fac-conjuq-sq-iff*:
assumes $u \sim v$ **shows** $|w| \leq |u| \Longrightarrow w \leq f\ u \cdot u \longleftrightarrow w \leq f\ v \cdot v$
⟨*proof*⟩

lemma *map-conjug*:
$u \sim v \Longrightarrow map\ f\ u \sim map\ f\ v$
⟨*proof*⟩

lemma *concat-map-conjug*:
$u \sim v \Longrightarrow concat\ (map\ f\ u) \sim concat\ (map\ f\ v)$
⟨*proof*⟩

lemma *map-conjug-iff* [*reversal-rule*]:
assumes $inj\ f$ **shows** $map\ f\ u \sim map\ f\ v \longleftrightarrow u \sim v$
⟨*proof*⟩

lemma *set-conjug*: **assumes** $w \neq \varepsilon$
shows $\{w'.\ w \sim w'\} = \{rotate\ n\ w \mid n.\ n < |\varrho\ w|\}$
⟨*proof*⟩

lemma *card-conjug*: **assumes** $w \neq \varepsilon$
shows $card\ \{w'.\ w \sim w'\} = |\varrho\ w|$
⟨*proof*⟩

lemma *finite-Bex-conjug*: **assumes** *finite A*
shows *finite* $\{r.\ Bex\ A\ (conjugate\ r)\}$
⟨*proof*⟩

2.21.1 Enumerating conjugates

definition *bounded-conjug*
where $bounded\text{-}conjug\ w'\ w\ k \equiv (\exists\ n \leq k.\ w = rotate\ n\ w')$

named-theorems *bounded-conjug*

lemma[*bounded-conjug*]: $bounded\text{-}conjug\ w'\ w\ 0 \longleftrightarrow w = w'$
⟨*proof*⟩

lemma[*bounded-conjug*]: *bounded-conjug* w' w (*Suc* k) $\longleftrightarrow$ *bounded-conjug* w' w k $\vee$ w = *rotate* (*Suc* k) w'
⟨*proof*⟩

lemma[*bounded-conjug*]: $w' \sim w \longleftrightarrow$ *bounded-conjug* w w' ($|w|-1$)
⟨*proof*⟩

lemma $w \sim [a,b,c] \longleftrightarrow w = [a,b,c] \vee w = [b,c,a] \vee w = [c,a,b]$
⟨*proof*⟩

2.21.2 General lemmas using conjugation

lemma *switch-fac*: **assumes** $x \neq y$ **and** *set* $ws = \{x,y\}$ **shows** $[x,y] \leq f\ ws \cdot ws$
⟨*proof*⟩

lemma *imprim-ext-pref-comm*: **assumes** ¬ *primitive* ($u \cdot v$) **and** ¬ *primitive* ($u \cdot v \cdot u$)
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *imprim-ext-suf-comm*:
¬ *primitive* ($u \cdot v$) $\Longrightarrow$ ¬ *primitive* ($u \cdot v \cdot v$) $\Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *prim-xyky*: **assumes** $2 \leq k$ **and** ¬ *primitive* $((x \cdot y)^{@}k \cdot y)$ **shows** $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *fac-pow-div*: **assumes** $u \leq f\ w^{@}l$ *primitive* w
shows $w^{@}((|u|\ div\ |w|) - 1) \leq f\ u$
⟨*proof*⟩

2.22 Element of lists: a method for testing if a word is in lists A

lemma *append-in-lists*[*simp*, *intro*]: $u \in$ *lists* $A \Longrightarrow v \in$ *lists* $A \Longrightarrow u \cdot v \in$ *lists* A
⟨*proof*⟩

lemma *pref-in-lists*: $u \leq p\ v \Longrightarrow v \in$ *lists* $A \Longrightarrow u \in$ *lists* A
⟨*proof*⟩

lemmas *suf-in-lists* = *pref-in-lists*[*reversed*]

lemma *fac-in-lists*: $ws \in$ *lists* $S \Longrightarrow vs \leq f\ ws \Longrightarrow vs \in$ *lists* S
⟨*proof*⟩

lemma *lq-in-lists*: $v \in$ *lists* $A \Longrightarrow u^{-1>}v \in$ *lists* A

⟨*proof*⟩

lemmas *rq-in-lists* = *lq-in-lists*[*reversed*]

lemma *take-in-lists*: $w \in \mathit{lists}\ A \Longrightarrow \mathit{take}\ j\ w \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *drop-in-lists*: $w \in \mathit{lists}\ A \Longrightarrow \mathit{drop}\ j\ w \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *lcp-in-lists*: $u \in \mathit{lists}\ A \Longrightarrow\ u \wedge_p v \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *lcp-in-lists'*: $v \in \mathit{lists}\ A \Longrightarrow\ u \wedge_p v \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *append-in-lists-dest*[*elim*]: $u \cdot v \in \mathit{lists}\ A \Longrightarrow u \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *append-in-lists-dest'*: $u \cdot v \in \mathit{lists}\ A \Longrightarrow v \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *pow-in-lists*: $u \in \mathit{lists}\ A \Longrightarrow u^{@}k \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *takeWhile-in-list*: $u \in \mathit{lists}\ A \Longrightarrow \mathit{takeWhile}\ P\ u \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *rev-in-lists*: $u \in \mathit{lists}\ A \Longrightarrow \mathit{rev}\ u \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *append-in-lists-dest1* [*elim*]: $u \cdot v = w \Longrightarrow w \in \mathit{lists}\ A \Longrightarrow u \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *append-in-lists-dest2*: $u \cdot v = w \Longrightarrow w \in \mathit{lists}\ A \Longrightarrow v \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *pow-in-lists-dest1*: $u \cdot v = w^{@}n \Longrightarrow w \in \mathit{lists}\ A \Longrightarrow u \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *pow-in-lists-dest1-sym*: $w^{@}n = u \cdot v \Longrightarrow w \in \mathit{lists}\ A \Longrightarrow u \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *pow-in-lists-dest2*: $u \cdot v = w^{@}n \Longrightarrow w \in \mathit{lists}\ A \Longrightarrow v \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *pow-in-lists-dest2-sym*: $w^{@}n = u \cdot v \Longrightarrow w \in \mathit{lists}\ A \Longrightarrow v \in \mathit{lists}\ A$
⟨*proof*⟩

lemma *per-in-lists*: $w <p\ r \cdot w \Longrightarrow r \in lists\ A \Longrightarrow w \in lists\ A$
⟨*proof*⟩

lemma *nth-in-lists*: $j < |w| \Longrightarrow w \in lists\ A \Longrightarrow w\ !\ j \in A$
⟨*proof*⟩

method *inlists* =
(*insert method-facts*, *use nothing* **in** ‹
((*elim suf-in-lists* | *elim pref-in-lists*[*elim-format*] | *rule lcp-in-lists* | *rule drop-in-lists*
|
rule lq-in-lists | *rule rq-in-lists* | *rule lists-butlast* | *rule tl-in-lists* |
rule take-in-lists | *intro lq-in-lists* | *rule nth-in-lists* |
rule append-in-lists | *elim conjug-in-lists* | *rule pow-in-lists* | *rule takeWhile-in-list*
| *elim append-in-lists-dest1* | *elim append-in-lists-dest2*
| *elim pow-in-lists-dest2* | *elim pow-in-lists-dest2-sym*
| *elim pow-in-lists-dest1* | *elim pow-in-lists-dest1-sym*)
| (*simp* | *fact*))+›)

2.23 Reversed mappings

definition *rev-map* :: $('a\ list \Rightarrow 'b\ list) \Rightarrow ('a\ list \Rightarrow 'b\ list)$ **where**
rev-map f = *rev* ∘ *f* ∘ *rev*

lemma *rev-map-idemp*[*simp*]: *rev-map* (*rev-map f*) = *f*
⟨*proof*⟩

lemma *rev-map-arg*: *rev-map f u* = *rev* (*f* (*rev u*))
⟨*proof*⟩

lemma *rev-map-arg'*: *rev* ((*rev-map f*) *w*) = *f* (*rev w*)
⟨*proof*⟩

lemmas *rev-map-arg-rev*[*reversal-rule*] = *rev-map-arg*[*reversed add*: *rev-rev-ident*]

lemma *rev-map-sing*: *rev-map f* [*a*] = *rev* (*f* [*a*])
⟨*proof*⟩

lemma *rev-maps-eq-iff*[*simp*]: *rev-map g* = *rev-map h* ⟷ *g* = *h*
⟨*proof*⟩

lemma *rev-map-funpow*[*reversal-rule*]: (*rev-map* (*f*::'*a list* ⇒'*a list*)) ^^ *k* = *rev-map* (*f* ^^ *k*)
⟨*proof*⟩

2.24 Overlapping powers, periods, prefixes and suffixes

lemma *pref-suf-overlapE*: **assumes** $p \leq_p w$ **and** $s \leq_s w$ **and** $|w| \leq |p| + |s|$
obtains *p1 u s1* **where** $p1 \cdot u \cdot s1 = w$ **and** $p1 \cdot u = p$ **and** $u \cdot s1 = s$
⟨*proof*⟩

lemma *mid-sq*: **assumes** $p{\cdot}x{\cdot}q{=}x{\cdot}x$ **shows** $x{\cdot}p{=}p{\cdot}x$ **and** $x{\cdot}q{=}q{\cdot}x$
⟨*proof*⟩

lemma *mid-sq′*: **assumes** $p{\cdot}x{\cdot}q{=}x{\cdot}x$ **shows** $q \cdot p = x$ **and** $p \cdot q = x$
⟨*proof*⟩

lemma *mid-sq-pref*: $p \cdot u \leq_p u \cdot u \Longrightarrow p \cdot u = u \cdot p$
⟨*proof*⟩

lemmas *mid-sq-suf* = *mid-sq-pref*[*reversed*]

lemma *mid-sq-pref-suf*: **assumes** $p{\cdot}x{\cdot}q{=}x{\cdot}x$ **shows** $p \leq_p x$ **and** $p \leq_s x$ **and** $q \leq_p x$ **and** $q \leq_s x$
⟨*proof*⟩

lemma *mid-sq-primroot*: **assumes** $p \cdot x <_p x \cdot x$
shows $x \cdot x = p \cdot x \cdot (\varrho\ x)^{@}(e_\varrho\ ((p \cdot x)^{-1>}(x \cdot x)))$ (**is** $x \cdot x = p \cdot x \cdot (\varrho\ x)^{@}e_\varrho\ ?z$)
$0 < e_\varrho\ ((p \cdot x)^{-1>}(x \cdot x))$ (**is** $0 < e_\varrho\ ?z$)
⟨*proof*⟩

lemma *mid-pow*: **assumes** $p{\cdot}x^{@}(Suc\ l){\cdot}q = x^{@}k$
shows $x{\cdot}p{=}p{\cdot}x$ **and** $x{\cdot}q{=}q{\cdot}x$
⟨*proof*⟩

lemma *root-suf-comm*: **assumes** $x \leq_p r \cdot x$ **and** $r \leq_s r \cdot x$ **shows** $r \cdot x = x \cdot r$
⟨*proof*⟩

lemma *pref-marker*: **assumes** $w \leq_p v \cdot w$ **and** $u \cdot v \leq_p w$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *pref-marker-ext*: **assumes** $|x| \leq |y|$ **and** $v \neq \varepsilon$ **and** $y \cdot v \leq_p x \cdot v^{@}k$
obtains n **where** $y = x \cdot (\varrho\ v)^{@}n$
⟨*proof*⟩

lemma *pref-marker-sq*: $p \cdot x \leq_p x \cdot x \Longrightarrow p \cdot x = x \cdot p$
⟨*proof*⟩

lemmas *suf-marker-sq* = *pref-marker-sq*[*reversed*]

lemma *pref-marker-conjug*: **assumes** $w \neq \varepsilon$ **and** $w \cdot r \cdot s \leq_p s \cdot (r \cdot s)^{@}m$ **and**

primitive $(r \cdot s)$
obtains n **where** $w = s \cdot (r \cdot s)^{@} n$
⟨*proof*⟩

lemmas *pref-marker-reversed* = *pref-marker*[*reversed*]

lemma *suf-marker-per-root*: **assumes** $w \leq p\ v \cdot w$ **and** $p \cdot v \cdot u \leq p\ w$
shows $u \leq p\ v \cdot u$
⟨*proof*⟩

lemma *suf-marker-per-root'*: **assumes** $w \leq p\ v \cdot w$ **and** $p \cdot v \cdot u \leq p\ w$ **and** $v \neq \varepsilon$
shows $u \leq p\ p \cdot u$
⟨*proof*⟩

lemma *marker-fac-pref*: **assumes** $u \leq f\ r^{@} k$ **and** $r \leq p\ u$ **shows** $u \leq p\ r^{@} k$
⟨*proof*⟩

lemma *marker-fac-pref-len*: **assumes** $u \leq f\ r^{@} k$ **and** $t \leq p\ u$ **and** $|t| = |r|$
shows $u \leq p\ t^{@} k$
⟨*proof*⟩

lemma *pref-comm-cancel*: $u \cdot z \leq p\ v \Longrightarrow u \cdot v = v \cdot u \Longrightarrow z \leq p\ v$
⟨*proof*⟩

lemma *marker-fac-pref-ext*: **assumes** $w \leq f\ v^{@} k\ v = p \cdot s\ s \cdot v \bowtie w\ |v| \leq |w|$
shows $p \cdot w \leq p\ v \cdot p \cdot w$
⟨*proof*⟩

lemma *root-suf-comm'*: $x \leq p\ r \cdot x \Longrightarrow r \leq s\ x \Longrightarrow r \cdot x = x \cdot r$
⟨*proof*⟩

lemmas *suf-root-pref-comm* = *root-suf-comm'*[*reversed*]

lemma *mid-marker-root*: **assumes** $u \cdot v \leq p\ r^{@} k\ \ r \leq s\ u$ **shows** $v \leq p\ r \cdot v$
⟨*proof*⟩

lemmas *mid-marker-root'* = *mid-marker-root*[*reversed*]

lemma *marker-pref-suf-fac*: **assumes** $u \leq p\ v$ **and** $u \leq s\ v$ **and** $v \leq f\ u^{@} k$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *pref-suf-per-fac-comm*:
assumes $v \leq p\ u \cdot v$ **and** $v \leq s\ v \cdot u$ **and** $u \leq f\ v^{@} k$ **shows** $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *mid-long-pow*: **assumes** *eq*: $y^{@} m = u \cdot x^{@} (Suc\ k) \cdot v$ **and** $|y| \leq |x^{@} k|$
shows $(u \cdot v) \cdot y = y \cdot (u \cdot v)$ **and** $(u \cdot x^{@} l \cdot v) \cdot y = y \cdot (u \cdot x^{@} l \cdot v)$ **and**
$(u^{-1>}(y \cdot u)) \cdot x = x \cdot (u^{-1>}(y \cdot u))$

⟨*proof*⟩

lemma *mid-pow-pref-suf'*: **assumes** $s{\cdot}w^{@}(Suc\ l){\cdot}p \leq_f w^{@}k$ **shows** $p \leq_p w^{@}k$ **and** $s \leq_s w^{@}k$
⟨*proof*⟩

lemma *mid-pow-pref-suf*: **assumes** $s{\cdot}w{\cdot}p \leq_f w^{@}k$ **shows** $p \leq_p w^{@}k$ **and** $s \leq_s w^{@}k$
⟨*proof*⟩

lemma *fac-marker-pref*: $y \cdot x \leq_f y^{@}k \Longrightarrow x \leq_p y \cdot x$
⟨*proof*⟩

lemmas *fac-marker-suf* = *fac-marker-pref*[*reversed*]

lemma *fac-two-markers*: **assumes** $u \cdot v \cdot u \leq_f u^{@}k$ **shows** $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *prim-overlap-sqE* [*consumes 2*]:
assumes *prim*: *primitive r* **and** *eq*: $p \cdot r \cdot q = r \cdot r$
obtains (*pref-emp*) $p = \varepsilon$ | (*suff-emp*) $q = \varepsilon$
⟨*proof*⟩

lemma *prim-overlap-sqE'* [*consumes 2*]:
assumes *prim*: *primitive r* **and** *eq*: $p \cdot r \cdot q = r \cdot r$
obtains (*pref-emp*) $p = \varepsilon$ | (*suff-emp*) $p = r$
⟨*proof*⟩

lemma *prim-overlap-sq-prefE*:
assumes *primitive r* **and** $p \cdot r \leq_p r \cdot r$
obtains (*pref-emp*) $p = \varepsilon$ | (*suff-emp*) $p = r$
⟨*proof*⟩

lemma *prim-overlap-sq*:
assumes *prim*: *primitive r* **and** *eq*: $p \cdot r \cdot q = r \cdot r$
shows $p = \varepsilon \vee q = \varepsilon$
⟨*proof*⟩

lemma *prim-overlap-sq'*:
assumes *prim*: *primitive r* **and** *pref*: $p \cdot r \leq_p r \cdot r$ **and** *len*: $|p| < |r|$
shows $p = \varepsilon$
⟨*proof*⟩

lemma *zxy-fac-xyzx*: **assumes** *nemp*: $y \neq \varepsilon\ z \neq \varepsilon$ **and** *fac*: $z \cdot x \cdot y \leq_f x \cdot y \cdot z \cdot x$
shows ¬ *primitive* $(z{\cdot}x{\cdot}y)$
⟨*proof*⟩

lemma *prim-overlap-pow*[*elim*]:

assumes *prim*: *primitive r* **and** *pref*: $u \cdot r \leq p\ r^{@}k$
obtains *i* **where** $u = r^{@}i$ **and** $i < k$
⟨*proof*⟩

lemma *prim-overlap-pow'*:
assumes *prim*: *primitive r* **and** *pref*: $u \cdot r \leq p\ r^{@}k$ **and** *less*: $|u| < |r|$
shows $u = \varepsilon$
⟨*proof*⟩

lemma *prim-sqs-overlap*:
assumes *prim*: *primitive r* **and** *comp*: $u \cdot r \cdot r \bowtie v \cdot r \cdot r$
and *len-u*: $|u| < |v| + |r|$ **and** *len-v*: $|v| < |u| + |r|$
shows $u = v$
⟨*proof*⟩

lemma *drop-pref-prim*: **assumes** $Suc\ n < |w|$ **and** $w \leq p\ drop\ (Suc\ n)\ (w \cdot w)$ **and** *primitive w*
shows *False*
⟨*proof*⟩

lemma *root-suf-conjug*: **assumes** *primitive* $(s \cdot r)$ **and** $y \leq p\ (s \cdot r) \cdot y$ **and** $y \leq s\ y \cdot (r \cdot s)$ **and** $|s \cdot r| \leq |y|$
obtains *l* **where** $y = (s \cdot r)^{@}l \cdot s$
⟨*proof*⟩

lemma *root-suf-pow-comm*: **assumes** $x \leq p\ r \cdot x$ **and** $r \leq s\ x^{@}(Suc\ k)$ **shows** $r \cdot x = x \cdot r$
⟨*proof*⟩

lemma *suf-pow-short-suf*: $r \leq s\ x^{@}k \Longrightarrow |x| \leq |r| \Longrightarrow x \leq s\ r$
⟨*proof*⟩

thm *suf-pow-short-suf*[*reversed*]

lemma *sq-short-per*: **assumes** $|u| \leq |v|$ **and** $v{\cdot}v \leq p\ u{\cdot}(v{\cdot}v)$
shows $u{\cdot}v = v{\cdot}u$
⟨*proof*⟩

lemma *fac-marker*: **assumes** $w \leq p\ u{\cdot}w$ **and** $u{\cdot}v{\cdot}u \leq f\ w$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma $4 = Suc(Suc(Suc(Suc\ 0)))$
⟨*proof*⟩

lemma *xyxy-conj-yxxy*: **assumes** $x \cdot y \cdot x \cdot y \sim y \cdot x \cdot x \cdot y$
shows $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *per-glue*: **assumes** *period u n* **and** *period v n* **and** $u \leq_p w$ **and** $v \leq_s w$ **and**
$|w| + n \leq |u| + |v|$
shows *period w n*
⟨*proof*⟩

lemma *per-glue-facs*: **assumes** $u \cdot z \leq_f w^{@} k$ **and** $z \cdot v \leq_f w^{@} m$ **and** $|w| \leq |z|$
obtains *l* **where** $u \cdot z \cdot v \leq_f w^{@} l$
⟨*proof*⟩

lemma *per-fac-pow-fac*: **assumes** *period w n* **and** $v \leq_f w$ **and** $|v| = n$ **and** $n \neq 0$
obtains *k* **where** $w \leq_f v^{@} k$
⟨*proof*⟩

lemma *refine-per*: **assumes** *period w n* **and** $v \leq_f w$ **and** $n \leq |v|$ **and** *period v k* **and** *k dvd n* **and** $n \neq 0$
shows *period w k*
⟨*proof*⟩

lemma *xy-per-comp*: **assumes** $x{\cdot}y \leq_p q{\cdot}x{\cdot}y$
and $q \neq \varepsilon$ **and** $q \bowtie y$
shows $x \bowtie y$
⟨*proof*⟩

lemma *prim-xyxyy*: $x \cdot y \neq y \cdot x \Longrightarrow$ *primitive* $(x \cdot y \cdot x \cdot y \cdot y)$
⟨*proof*⟩

lemma *prim-min-per-suf-eq*: **assumes** *primitive x* **and** $\pi\ x \leq_s x$ **shows** $\pi\ x = x$
⟨*proof*⟩

lemma *primroot-code*[*code*]: $\varrho\ x =$ (*if* $x \neq \varepsilon$ *then* (*if* $\pi\ x \leq_s x$ *then* $\pi\ x$ *else* x) *else Code.abort* (*STR ''Empty word has no primitive root.''*) (λ-. ($\varrho\ x$)))
⟨*proof*⟩

lemma *per-lemma-pref-suf*: **assumes** $w <_p p \cdot w$ **and** $w <_s w \cdot q$ **and**
fw: $|p| + |q| \leq |w|$
obtains *r s k l m* **where** $p = (r \cdot s)^{@} k$ **and** $q = (s \cdot r)^{@} l$ **and** $w = (r \cdot s)^{@} m \cdot r$ **and** *primitive* $(r{\cdot}s)$
⟨*proof*⟩

lemma *fac-two-conjug-primrootE*:
assumes *facs*: $w \leq_f p^{@} k$ $w \leq_f q^{@} l$ **and** *nemps*: $p \neq \varepsilon$ $q \neq \varepsilon$ **and** *len*: $|p| + |q| \leq |w|$
obtains *r s m* **where** $\varrho\ p \sim r \cdot s$ **and** $\varrho\ q \sim r \cdot s$ **and** $w = (r \cdot s)^{@} m \cdot r$ **and** *primitive* $(r{\cdot}s)$
⟨*proof*⟩

corollary *fac-two-conjug-primroot*:
  assumes *facs*: $u \leq f\ r^{@}k\ u \leq f\ s^{@}l$ **and** *len*: $|r| + |s| \leq |u|$
  shows $\varrho\ r \sim \varrho\ s$
⟨*proof*⟩

lemma *fac-two-prim-conjug*:
 assumes $w \leq f\ u^{@}n\ w \leq f\ v^{@}m$ *primitive u primitive v* $|u| + |v| \leq |w|$
 shows $u \sim v$
 ⟨*proof*⟩

lemma *fac-pow-conjug-primroot*: **assumes** $u^{@}k \leq f\ v^{@}l$ **and** $|u^{@}k| \geq 2*|v|$ **and** $2 \leq k$ **and** $u \neq \varepsilon$
 shows $\varrho\ u \sim \varrho\ v$
⟨*proof*⟩

2.25 Testing primitivity

This section defines a proof method used to prove that a word is primitive.

lemma *primitive-iff* [*code*]: *primitive* $w \longleftrightarrow \neg\ w \leq f\ tl\ w \cdot butlast\ w$
⟨*proof*⟩

method *primitivity-inspection* = (*insert method-facts*, *use nothing* **in**
  ‹*simp add*: *primitive-iff pow-pos*›)

— This is out of scope of the method, and has to be proved separately
lemma *alternate-prim*: **assumes** $x \neq y$ **shows** *primitive* $([x,y]^{@}n \cdot [x])$
⟨*proof*⟩

2.26 Factor interpretation

definition *factor-interpretation* :: *'a list* ⇒ *'a list* ⇒ *'a list* ⇒ *'a list list* ⇒ *bool*
(- - - $\sim_{\mathcal{I}}$ - [*52*,*52*,*52*,*52*] *60*)
 where *factor-interpretation p u s ws* = ($p <p\ hd\ ws \wedge s <s\ last\ ws \wedge p \cdot u \cdot s = concat\ ws$)

lemma *fac-interp-nemp*: $u \neq \varepsilon \Longrightarrow p\ u\ s \sim_{\mathcal{I}} ws \Longrightarrow ws \neq \varepsilon$
 ⟨*proof*⟩

lemma *fac-interpD*: **assumes** $p\ u\ s \sim_{\mathcal{I}} ws$
 shows $p <p\ hd\ ws$ **and** $s <s\ last\ ws$ **and** $p \cdot u \cdot s = concat\ ws$
 ⟨*proof*⟩

lemma *fac-interpI*[*intro*]:
 $p <p\ hd\ ws \Longrightarrow s <s\ last\ ws \Longrightarrow p \cdot u \cdot s = concat\ ws \Longrightarrow p\ u\ s \sim_{\mathcal{I}} ws$
 ⟨*proof*⟩
lemma *fac-fac-interpE*: **assumes** $pu \cdot u \cdot su = concat\ ws$ **and** $u \neq \varepsilon$
 obtains *ps ss p s vs* **where** $p\ u\ s \sim_{\mathcal{I}} vs$ **and** $ps \cdot vs \cdot ss = ws$ **and** $concat\ ps \cdot p = pu$ **and**

$s \cdot concat\ ss = su$
⟨*proof*⟩

lemma *pref-interpE*: **assumes** $p\ u\ s \sim_{\mathcal{I}} ws\ v \leq p\ u\ v \neq \varepsilon$
obtains *vs s'* **where** $p\ v\ s' \sim_{\mathcal{I}} vs\ vs \leq p\ ws$
⟨*proof*⟩

lemma *fac-fac-interpE'*: **assumes** $u \leq f\ concat\ ws$ **and** $u \neq \varepsilon$
obtains *p s vs* **where** $p\ u\ s \sim_{\mathcal{I}} vs$ **and** $vs \leq f\ ws$
⟨*proof*⟩

lemma *fac-pow-longE*: **assumes** $w \leq f\ v^{@}k$ **and** $|v| \leq |w|$
obtains *m v1 v2* **where** $v1 \leq s\ v\ v2 \leq p\ v\ w = v1 \cdot v^{@}m \cdot v2$
⟨*proof*⟩

lemma *fac-interp-inner*: **assumes** $u \neq \varepsilon$ **and** $p\ u\ s \sim_{\mathcal{I}} ws$ **and** $1 < |ws|$
shows $p^{-1>}(hd\ ws) \cdot concat(butlast\ (tl\ ws)) \cdot (last\ ws)^{<-1} s = u$
⟨*proof*⟩

lemma *fac-interp-inner-len*: **assumes** $u \neq \varepsilon$ **and** $p\ u\ s \sim_{\mathcal{I}} ws$
shows $|concat(butlast\ (tl\ ws))| < |u|$
⟨*proof*⟩

lemma *rev-in-set-map-rev-conv*: $rev\ u \in set\ (map\ rev\ ws) \longleftrightarrow u \in set\ ws$
⟨*proof*⟩

lemma *rev-fac-interp*: **assumes** $p\ u\ s \sim_{\mathcal{I}} ws$ **shows** $(rev\ s)\ (rev\ u)\ (rev\ p) \sim_{\mathcal{I}} rev\ (map\ rev\ ws)$
⟨*proof*⟩

lemma *rev-fac-interp-iff* [*reversal-rule*]: $(rev\ s)\ (rev\ u)\ (rev\ p) \sim_{\mathcal{I}} rev\ (map\ rev\ ws) \longleftrightarrow p\ u\ s \sim_{\mathcal{I}} ws$
⟨*proof*⟩

lemma *fac-interp-mid-fac*: **assumes** $p\ u\ s \sim_{\mathcal{I}} ws$
shows $concat\ (butlast\ (tl\ ws)) \leq f\ u$
⟨*proof*⟩

definition *disjoint-interpretation* :: *'a list* ⇒ *'a list list* ⇒ *'a list* ⇒ *'a list list* ⇒ *bool* (- - - $\sim_{\mathcal{D}}$ - [*51,51,51,51*] *60*)
where $p\ us\ s \sim_{\mathcal{D}} ws \equiv p\ (concat\ us)\ s \sim_{\mathcal{I}} ws \wedge$
$(\forall\ u\ v.\ u \leq p\ us \wedge v \leq p\ ws \longrightarrow p \cdot concat\ u \neq concat\ v)$

lemma *disj-interpI*: $p\ (concat\ us)\ s \sim_{\mathcal{I}} ws \Longrightarrow$
$(\forall\ u\ v.\ u \leq p\ us \wedge v \leq p\ ws \longrightarrow p \cdot concat\ u \neq concat\ v) \Longrightarrow p\ us\ s \sim_{\mathcal{D}} ws$
⟨*proof*⟩

lemma *disj-interpI'*[*intro*]: $p\ (concat\ us)\ s \sim_{\mathcal{I}} ws \Longrightarrow$
$(\bigwedge u\ v.\ u \leq p\ us \Longrightarrow v \leq p\ ws \Longrightarrow\ p \cdot concat\ u \neq concat\ v) \Longrightarrow p\ us\ s \sim_{\mathcal{D}} ws$
⟨*proof*⟩

lemma *disj-interpD0*: $p\ us\ s \sim_{\mathcal{D}} ws \Longrightarrow p\ (concat\ us)\ s \sim_{\mathcal{I}} ws$
⟨*proof*⟩

lemmas *disj-interpD* = *fac-interpD*[*OF disj-interpD0*]

lemma *disj-interpD1*: **assumes** $p\ us\ s \sim_{\mathcal{D}} ws$ **and** $us' \leq p\ us$ **and** $ws' \leq p\ ws$
shows $p \cdot concat\ us' \neq concat\ ws'$
⟨*proof*⟩

lemma *disj-interp-nemp*: **assumes** $p\ us\ s \sim_{\mathcal{D}} ws$
shows $p \neq \varepsilon$ **and** $s \neq \varepsilon$ **and** $ws \neq \varepsilon$
⟨*proof*⟩

lemma *non-disjoint-interpE*: **assumes** $p\ (concat\ us)\ s \sim_{\mathcal{I}} ws$ **and**
$\neg\ p\ us\ s \sim_{\mathcal{D}} ws$
obtains *ws1 ws2 us1 us2* **where** $us1 \cdot us2 = us\ ws1 \cdot ws2 = ws$
$p \cdot concat\ us1 = concat\ ws1\ concat\ us2 \cdot s = concat\ ws2$
⟨*proof*⟩

lemma *emp-disj-interp*: **assumes** $p \neq \varepsilon\ p <_p u$
shows $p\ \varepsilon\ (p^{-1>}u) \sim_{\mathcal{D}} [u]$
⟨*proof*⟩

lemma *emp-fac-interp*: $p \neq \varepsilon \Longrightarrow s \neq \varepsilon \Longrightarrow p\ \varepsilon\ s \sim_{\mathcal{I}} [p \cdot s]$
⟨*proof*⟩

lemma *emp-disj-interp'*:
assumes $1 < |w|$ **shows** $[hd\ w]\ \varepsilon\ (tl\ w) \sim_{\mathcal{D}} [w]$
⟨*proof*⟩

lemma *pref-disj-interpE*: **assumes** $p\ us\ s \sim_{\mathcal{D}} ws\ vs \leq p\ us$
obtains $ws'\ s'$ **where** $p\ vs\ s' \sim_{\mathcal{D}} ws'\ ws' \leq p\ ws$
⟨*proof*⟩

lemma *rev-disj-interp*: **assumes** $p\ us\ s \sim_{\mathcal{D}} ws$ **shows** $(rev\ s)\ (rev\ (map\ rev\ us))$
$(rev\ p) \sim_{\mathcal{D}} rev\ (map\ rev\ ws)$
⟨*proof*⟩

lemma *rev-disj-interp-iff* [*reversal-rule*]: $(rev\ s)\ (rev\ (map\ rev\ us))\ (rev\ p) \sim_{\mathcal{D}} rev$
$(map\ rev\ ws) \longleftrightarrow p\ us\ s \sim_{\mathcal{D}} ws$
⟨*proof*⟩

lemma *disj-interp-split*:
assumes *disj*: $p\ us\ s \sim_{\mathcal{D}} ws$ **and** $us = us_1 \cdot us_2$
obtains $ws_1\ ws_2\ p'\ s'$ **where** $p\ us_1\ s' \sim_{\mathcal{D}} ws_1 \cdot \lceil p' \cdot s' \rceil\ p'\ us_2\ s \sim_{\mathcal{D}} \lceil p' \cdot s' \rceil \cdot ws_2$

$ws = ws_1 \cdot [p' \cdot s'] \cdot ws_2$
⟨*proof*⟩

corollary *disj-interp-fac*:
assumes *disj*: $p\ us\ s \sim_{\mathcal{D}} ws$ **and** $us = us_1 \cdot us_2 \cdot us_3$
obtains $ws_1\ ws_2\ ws_3\ p'\ s'$ **where** $p'\ us_2\ s' \sim_{\mathcal{D}} ws_2$
and $p \cdot concat\ us_1 = concat\ ws_1 \cdot p'\ concat\ us_3 \cdot s = s' \cdot concat\ ws_3$
and $ws = ws_1 \cdot ws_2 \cdot ws_3$
⟨*proof*⟩

lemma *disj-interp-alternate*:
assumes *disj*: $p\ us\ s \sim_{\mathcal{D}} ws$ **and** $1 < |ws|$
obtains $us_1\ us_2\ us'\ p'\ s'$ **where** $p'\ (butlast\ (tl\ ws))\ \ s' \sim_{\mathcal{D}} us'$
$hd\ ws = p \cdot concat\ us_1 \cdot p'\ last\ ws = s' \cdot concat\ us_2 \cdot s\ us_1 \cdot us' \cdot us_2 = us$
⟨*proof*⟩

lemma *disj-interp-long*: **assumes** $p\ us\ s \sim_{\mathcal{D}} ws$
shows $1 < |hd\ ws|\ 1 < |last\ ws|$
⟨*proof*⟩

end

theory *Border-Array*

imports
CoWBasic

begin

2.26.1 Auxiliary lemmas on suffix and border extension

lemma *border-ConsD*: **assumes** $b\#x \leq_b a\#w$
shows $a = b$ **and**
$x \neq \varepsilon \Longrightarrow x \leq_b w$ **and**
border-ConsD-neq: $x \neq w$ **and**
border-ConsD-pref: $x \leq_p w$ **and**
border-ConsD-suf: $x \leq_s w$
⟨*proof*⟩

lemma *ext-suf-Cons*:
$Suc\ i + |u| = |w| \Longrightarrow u \leq_s w \Longrightarrow (w!i)\#u \leq_s (w!i)\#w$
⟨*proof*⟩

lemma *ext-suf-Cons-take-drop*: **assumes** $take\ k\ (drop\ (Suc\ i)\ w) \leq_s drop\ (Suc\ i)\ w$ **and** $w\ !\ i = w\ !\ (|w| - Suc\ k)$
shows $take\ (Suc\ k)\ (drop\ i\ w) \leq_s drop\ i\ w$
⟨*proof*⟩

lemma *ext-border-Cons*:
Suc i + |*u*| = |*w*| ⟹ *u* ≤*b w* ⟹ (*w*!*i*)#*u* ≤*b* (*w*!*i*)#*w*
⟨*proof*⟩

lemma *border-add-Cons-len*: **assumes** *max-borderP u w* **and** *v* ≤*b* (*a*#*w*) **shows** |*v*| ≤ *Suc* |*u*|
⟨*proof*⟩

2.27 Computing the Border Array

The computation is a special case of the Knuth-Morris-Pratt algorithm.

- KMP w arr bord pos
- w: processed word does not change; it is processed starting from the last letter
- pos: actually examined pos-th letter; that is, it is w!(pos-1)
- arr: already calculated suffix-border-array of w; that is, the length of array is (|w| - pos) and arr!(|w| - pos - bord) is the max border length of the suffix of w of length bord
- bord: length of the current max border length candidate to see whether it can be extended we compare: w!(pos-1) ?= w!(|w| - (Suc bord)); (Suc bord) is the length of the max border if the comparison is succesful
- if the comparison fails we move to the max border of the suffix of length bord; its max border length is stored in arr!(|w| - pos - bord)
- if bord was 0 and the comparison failed, the word is unbordered

fun *KMP-arr* :: *'a list* ⇒ *nat list* ⇒ *nat* ⇒ *nat* ⇒ *nat list*
where *KMP-arr - arr - 0 = arr* |
KMP-arr w arr bord (*Suc i*) =
(*if w*!*i* = *w*!(|*w*| − (*Suc bord*))
then (*Suc bord*) # *arr*
else (*if bord* = *0*
then *0*#*arr*
else (*if* (*arr*!(|*w*| − (*Suc i*) − *bord*)) < *bord* — always True, for sake of termination
then arr
else *undefined*#*arr* — else: dummy termination condition
)
)
)

```
fun KMP-bord :: 'a list ⇒ nat list ⇒ nat ⇒ nat ⇒ nat
  where     KMP-bord - - bord 0 = bord |
    KMP-bord w arr bord (Suc i) =
          (if w!i = w!(|w| − (Suc bord))
          then Suc bord
          else (if bord = 0
                then  0
                else (if (arr!(|w| − (Suc i) − bord)) < bord — always True, for sake
of termination
                      then arr!(|w| − (Suc i) − bord)
                      else  0 — dummy termination condition
                     )
                )
          )

fun KMP-pos :: 'a list ⇒ nat list ⇒ nat ⇒ nat ⇒ nat
  where
    KMP-pos - - - 0 = 0 |
    KMP-pos w arr bord (Suc i) =
          (if w!i = w!(|w| − (Suc bord))
          then i
          else (if bord = 0
                then  i
                else (if (arr!(|w| − (Suc i) − bord)) < bord — always True, for sake
of termination
                      then Suc i
                      else  i — else: dummy termination condition
                     )
                )
          )

thm prod-cases
    nat.exhaust
    prod.exhaust
    prod-cases3

function KMP :: 'a list ⇒ nat list ⇒ nat ⇒ nat ⇒ nat list where
  KMP w arr bord 0 = arr  |
  KMP w arr bord (Suc i) = KMP w (KMP-arr w arr bord (Suc i)) (KMP-bord w
arr bord (Suc i)) (KMP-pos w arr bord (Suc i))
  ⟨proof⟩
termination
  ⟨proof⟩

lemma KMP-len: |KMP w arr bord pos| = |arr| + pos
  ⟨proof⟩

value[nbe] KMP [a] [0] 0 0
```

value *KMP* [*0*::*nat*] [*0*] *0 0*
value *KMP* [*5*,*4*,*5*,*3*,*5*,*5*::*nat*] [*0*] *0 5*
value *KMP* [*5*,*4*::*nat*,*5*,*3*,*5*,*5*] [*1*,*0*] *1 4*
value *KMP* [*0*,*1*,*1*,*0*::*nat*,*0*,*0*,*1*,*1*,*1*] [*0*] *0 8*
value *KMP* [*0*::*nat*,*1*] [*0*] *0 1*

2.27.1 Verification of the computation

definition *KMP-valid* :: *'a list* ⇒ *nat list* ⇒ *nat* ⇒ *nat* ⇒ *bool*
where *KMP-valid w arr bord pos* = (|*arr*| + *pos* = |*w*| ∧
— bord is the length of a border of (drop pos w), or 0
pos + *bord* < |*w*| ∧
take bord (*drop pos w*) ≤*p* (*drop pos w*) ∧
take bord (*drop pos w*) ≤*s* (*drop pos w*) ∧
— ... and no longer border can be extended
(∀ *v*. *v* ≤*b* *w*!(*pos* − *1*)#(*drop pos w*) ⟶ |*v*| ≤ *Suc bord*) ∧
— the array gives maximal border lengths of corresponding suffixes
(∀ *k* < |*arr*|. *max-borderP* (*take* (*arr*!*k*) (*drop* (*pos* + *k*) *w*)) (*drop* (*pos* + *k*) *w*))
)

lemma *KMP-valid w arr bord pos* ⟹ *w* ≠ *ε*
⟨*proof*⟩

lemma *KMP-valid-base*: **assumes** *w* ≠ *ε* **shows** *KMP-valid w* [*0*] *0* (|*w*|−*1*)
⟨*proof*⟩

lemma *KMP-valid-step*: **assumes** *KMP-valid w arr bord* (*Suc i*)
shows *KMP-valid w* (*KMP-arr w arr bord* (*Suc i*)) (*KMP-bord w arr bord* (*Suc i*)) (*KMP-pos w arr bord* (*Suc i*))
⟨*proof*⟩

lemma *KMP-valid-max*: **assumes** *KMP-valid w arr bord pos k* < |*w*|
shows *max-borderP* (*take* ((*KMP w arr bord pos*)!*k*) (*drop k w*)) (*drop k w*)
⟨*proof*⟩

2.28 Border array

fun *border-array* :: *'a list* ⇒ *nat list* **where**
border-array ε = *ε*
| *border-array* (*a*#*w*) = *rev* (*KMP* (*rev* (*a*#*w*)) [*0*] *0* (|*a*#*w*|−*1*))

lemma *border-array-len*: |*border-array w*| = |*w*|
⟨*proof*⟩

theorem *bord-array*: **assumes** *Suc k* ≤ |*w*| **shows** (*border-array w*)!*k* = |*max-border*

(*take* (*Suc k*) *w*)|
⟨*proof*⟩

lemma *max-border-comp* [*code*]: *max-border w* = *take* ((*border-array w*)!(|*w*|−*1*))
w
⟨*proof*⟩

value[*nbe*] *primitive* [*a*,*b*,*a*]

value *primitive* [*0*,*1*,*0*::*nat*]

value *border-array* [*5*,*4*,*5*,*3*,*5*,*5*,*5*,*4*,*5*::*nat*]

value *primitive* [*5*,*4*,*5*,*3*,*5*,*5*,*5*,*4*,*5*::*nat*]

value *primitive* [*5*,*4*,*5*,*3*,*5*,*5*,*5*,*4*,*5*::*nat*]

value[*nbe*] *bordered* []

value *border-array* [*0*,*1*,*1*,*0*,*0*,*0*,*1*,*1*,*1*,*1*,*1*,*0*,*0*,*0*,*1*,*1*,*1*,*0*,*1*,*1*,*0*,*0*,*0*,*1*,*1*,*1*,*1*,*1*,*1*,*0*,*0*,*0*,*1*,*1*,*1*,*1*,*0*,*1*,*1*,*0*,*0*,*0*,*1*,*1*,*1*,*1*,*0*,*1*,*1*,*0*,

value[*nbe*] *border-array* *ε*

value *border-array* [*1*,*0*,*1*,*0*,*1*,*1*,*0*,*0*::*nat*]

value *max-border* [*1*,*0*,*1*,*0*,*1*,*1*,*0*,*0*,*1*,*0*,*1*,*1*,*0*,*1*,*0*,*1*,*1*,*0*,*0*,*1*,*0*,*1*,*1*,*0*,*1*,*0*,*1*,*1*,*0*,*0*,*1*,*0*,*1*,*0*,*0*,*1*::*nat*]

thm *max-border-comp* — code for *max-border*, based on *border-array*

value *bordered* [*1*,*0*::*nat*,*1*,*0*,*1*,*1*,*0*,*1*]

value *π* [*1*::*nat*,*0*,*1*,*0*,*1*,*1*,*0*,*1*]

thm *min-per-root-take* — code for *π*, based on *max-border*

value |*π* [*1*::*nat*,*0*,*1*,*0*,*1*,*1*,*0*,*1*]|

value *ϱ* [*1*::*nat*,*0*,*1*,*1*,*0*,*1*,*1*,*0*,*1*]

thm *primroot-code* — code for *ϱ*, based on *π*

value *ϱ* [*1*::*nat*,*0*,*1*,*1*,*0*,*1*,*1*,*0*]

value[*nbe*] *π* *ε*

end

theory *Submonoids*
  imports *CoWBasic HOL.Hull*
begin

Chapter 3

Submonoids of a free monoid

This chapter deals with properties of submonoids of a free monoid, that is, with monoids of words. See more in Chapter 1 of [4].

3.1 Generated monoid (also called hull)

First, we define the hull of a set of words, that is, the monoid generated by them.

definition *word-monoid* **where**
word-monoid M ≡ (∀ *w1 w2*. *w1* ∈ *M* ⟶ *w2* ∈ *M* ⟶ *w1* · *w2* ∈ *M*) ∧ *ε* ∈ *M*

lemma *word-monoidI*[*intro*]: **assumes** ⋀ *w1 w2*. *w1* ∈ *M* ⟹ *w2* ∈ *M* ⟹ *w1* · *w2* ∈ *M ε* ∈ *M*
shows *word-monoid M*
⟨*proof*⟩

inductive-set *hull* :: *'a list set* ⇒ *'a list set* (⟨-⟩)
for *G* **where**
emp-in[*simp*]: *ε* ∈ ⟨*G*⟩ |
prod-cl: *w1* ∈ *G* ⟹ *w2* ∈ ⟨*G*⟩ ⟹ *w1* · *w2* ∈ ⟨*G*⟩

lemmas [*intro*] = *hull.intros*

lemma *hull-closed*[*intro*]: *w1* ∈ ⟨*G*⟩ ⟹ *w2* ∈ ⟨*G*⟩ ⟹ *w1* · *w2* ∈ ⟨*G*⟩
⟨*proof*⟩

lemma *gen-in* [*intro*]: *w* ∈ *G* ⟹ *w* ∈ ⟨*G*⟩
⟨*proof*⟩

lemma *gen-monoid-monoid*: *word-monoid* ⟨*G*⟩
⟨*proof*⟩

lemma *gen-monoid-hull*: *word-monoid hull G* = ⟨*G*⟩

⟨*proof*⟩

lemma *gen-monoid-induct*: **assumes** $x \in \langle G \rangle$ $P\ \varepsilon$ $\bigwedge w.\ w \in G \Longrightarrow P\ w$
$\bigwedge w1\ w2.\ w1 \in \langle G \rangle \Longrightarrow P\ w1 \Longrightarrow w2 \in \langle G \rangle \Longrightarrow P\ w2 \Longrightarrow P\ (w1 \cdot w2)$ **shows** $P\ x$
⟨*proof*⟩

lemma *genset-sub*[*simp*]: $G \subseteq \langle G \rangle$
⟨*proof*⟩

lemma *lists-sub*: $ws \in lists\ G \Longrightarrow ws \in lists\ \langle G \rangle$
⟨*proof*⟩

lemma *in-lists-conv-set-subset*: $set\ ws \subseteq G \longleftrightarrow ws \in lists\ G$
⟨*proof*⟩

lemma *concat-in-hull* [*intro*]:
assumes $set\ ws \subseteq G$
shows $concat\ ws \in \langle G \rangle$
⟨*proof*⟩

lemma *concat-in-hull'* [*intro*]:
assumes $ws \in lists\ G$
shows $concat\ ws \in \langle G \rangle$
⟨*proof*⟩

lemma *hull-concat-lists0*:
$w \in \langle G \rangle \Longrightarrow \exists\ ws \in lists\ G.\ concat\ ws = w$ (**is** $w \in \langle G \rangle \Longrightarrow ?P\ w$)
⟨*proof*⟩

lemma *hull-concat-listsE*[*elim*]: **assumes** $w \in \langle G \rangle$
obtains *ws* **where** $ws \in lists\ G$ **and** $concat\ ws = w$
⟨*proof*⟩

lemma *hull-concat-lists*: $\langle G \rangle = concat\ `\ lists\ G$
⟨*proof*⟩

lemma *hull-concat-lists'*: $\langle G \rangle = \{concat\ xs \mid xs.\ xs \in lists\ G\}$
⟨*proof*⟩

lemma *hull-mono*: $A \subseteq B \Longrightarrow \langle A \rangle \subseteq \langle B \rangle$
⟨*proof*⟩

lemma *hull-concat-lists-nempE*[*elim*]: **assumes** $w \in \langle G \rangle$
obtains *ws* **where** $ws \in lists\ (G - \{\varepsilon\})$ **and** $concat\ ws = w$
⟨*proof*⟩

lemma *hull-nemp-eq-hull*[*simp*]: $\langle G - \{\varepsilon\} \rangle = \langle G \rangle$
⟨*proof*⟩

lemma *emp-gen-set*: $\langle\{\}\rangle = \{\varepsilon\}$
⟨*proof*⟩

lemma *concat-lists-minus*[*simp*]: *concat* ‘ *lists* $(G - \{\varepsilon\}) =$ *concat* ‘ *lists* G
⟨*proof*⟩

lemma *hull-drop-one*: $\langle G - \{\varepsilon\}\rangle = \langle G\rangle$
⟨*proof*⟩

lemma *sing-gen-power*: $u \in \langle\{x\}\rangle \Longrightarrow \exists\, k.\ u = x^{@}k$
⟨*proof*⟩

lemma *pow-sing-gen*[*simp*]: $x^{@}k \in \langle\{x\}\rangle$
⟨*proof*⟩

lemma *sing-gen-pow-ex-conv*: $u \in \langle\{x\}\rangle \longleftrightarrow (\exists\, k.\ u = x^{@}k)$
⟨*proof*⟩

lemma *sing-gen-primroot-gen*: **assumes** $w \in \langle\{x\}\rangle$
shows $w \in \langle\{\varrho\ x\}\rangle$
⟨*proof*⟩

lemma *sing-gen-primroot-eq*:
assumes $w \in \langle\{x\}\rangle$ $w \neq \varepsilon$
shows $\varrho\ w = \varrho\ x$
⟨*proof*⟩

lemma *sing-gen-pow-conv*: $w \in \langle\{\varrho\ x\}\rangle \longleftrightarrow w = \varrho\ x^{@}e_{\varrho}\ w$
⟨*proof*⟩

lemma *two-primroots-comm*: **assumes** $w \neq \varepsilon$ **and** $w \in \langle\{\varrho\ x\}\rangle$ **and** $w \in \langle\{\varrho\ y\}\rangle$
shows $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *comm-rootI*: $x \in \langle\{r\}\rangle \Longrightarrow y \in \langle\{r\}\rangle \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *sing-genE*[*elim*]:
assumes $u \in \langle\{x\}\rangle$
obtains k **where** $x^{@}k = u$
⟨*proof*⟩

lemma *lists-gen-to-hull*: *us* ∈ *lists* $(G - \{\varepsilon\}) \Longrightarrow$ *us* ∈ *lists* $(\langle G\rangle - \{\varepsilon\})$

⟨*proof*⟩

lemma *rev-hull*: *rev*‘⟨*G*⟩ = ⟨*rev*‘*G*⟩
⟨*proof*⟩

lemma *power-in*[*intro*]: $x \in \langle G\rangle \Longrightarrow x^{@}k \in \langle G\rangle$
⟨*proof*⟩

lemma *hull-closed-lists*: *us* ∈ *lists* ⟨*G*⟩ ⟹ *concat us* ∈ ⟨*G*⟩
⟨*proof*⟩

lemma *hull-I* [*intro*]:
$\varepsilon \in H \Longrightarrow (\bigwedge x\ y.\ x \in H \Longrightarrow y \in H \Longrightarrow x \cdot y \in H) \Longrightarrow \langle H\rangle = H$
⟨*proof*⟩

lemma *gen-idemp*: ⟨⟨*G*⟩⟩ = ⟨*G*⟩
⟨*proof*⟩

lemma *hull-mono*′[*intro*]: *A* ⊆ ⟨*B*⟩ ⟹ ⟨*A*⟩ ⊆ ⟨*B*⟩
⟨*proof*⟩

lemma *hull-conjug* [*elim*]: *w* ∈ ⟨{*r*·*s*,*s*·*r*}⟩ ⟹ *w* ∈ ⟨{*r*,*s*}⟩
⟨*proof*⟩

lemma *root-divisor*: **assumes** *period w k* **and** *k dvd* |*w*| **shows** *w* ∈ ⟨{*take k w*}⟩
⟨*proof*⟩

Intersection of hulls is a hull.

lemma *hulls-inter*: ⟨⋂ {⟨*G*⟩ | *G*. *G* ∈ *S*}⟩ = ⋂ {⟨*G*⟩ | *G*. *G* ∈ *S*}
⟨*proof*⟩

theorem *hull-minimal*: ⟨*G*⟩ = ⋂ {*S* . *G* ⊆ *S* ∧ ⟨*S*⟩ = *S*}
⟨*proof*⟩

lemma *all-gen-comm-hull-comm*: **assumes** $\bigwedge x\ y.\ x \in G \Longrightarrow y \in G \Longrightarrow x \cdot y = y \cdot x$
u ∈ ⟨*G*⟩ *v* ∈ ⟨*G*⟩
shows *u* · *v* = *v* · *u*
⟨*proof*⟩

lemma *bin-comm-hull-comm*: **assumes** *x* · *y* = *y* · *x u* ∈ ⟨{*x*,*y*}⟩ *v* ∈ ⟨{*x*,*y*}⟩
shows *u* · *v* = *v* · *u*
⟨*proof*⟩

lemma[*reversal-rule*]: *rev* ‘ ⟨{*rev u*, *rev v*}⟩ = ⟨{*u*,*v*}⟩
⟨*proof*⟩

lemma[*reversal-rule*]: $rev\ w \in \langle rev\ `\ G\rangle \equiv w \in \langle G\rangle$
⟨*proof*⟩

3.2 Factorization into generators

We define a decomposition (or a factorization) of a into elements of a given generating set. Such a decomposition is well defined only if the decomposed word is an element of the hull. Even int that case, however, the decomposition need not be unique.

definition *decompose* :: *'a list set* ⇒ *'a list* ⇒ *'a list list* (*Dec - - [55,55] 56*)
where
decompose G u = (*SOME us. us* ∈ *lists* (*G* − {*ε*}) ∧ *concat us* = *u*)

lemma *dec-ex*: **assumes** $u \in \langle G\rangle$ **shows** ∃ *us*. (*us* ∈ *lists* (*G* − {*ε*}) ∧ *concat us* = *u*)
⟨*proof*⟩

lemma *dec-in-lists'*: $u \in \langle G\rangle \Longrightarrow (Dec\ G\ u) \in lists\ (G - \{\varepsilon\})$
⟨*proof*⟩

lemma *concat-dec*[*simp*, *intro*] : $u \in \langle G\rangle \Longrightarrow concat\ (Dec\ G\ u) = u$
⟨*proof*⟩

lemma *dec-emp* [*simp*]: $Dec\ G\ \varepsilon = \varepsilon$
⟨*proof*⟩

lemma *dec-nemp*: $u \in \langle G\rangle - \{\varepsilon\} \Longrightarrow Dec\ G\ u \neq \varepsilon$
⟨*proof*⟩

lemma *dec-nemp'*[*simp*, *intro*]: $u \neq \varepsilon \Longrightarrow u \in \langle G\rangle \Longrightarrow Dec\ G\ u \neq \varepsilon$
⟨*proof*⟩

lemma *dec-eq-emp-iff* [*simp*]: **assumes** $u \in \langle G\rangle$ **shows** $Dec\ G\ u = \varepsilon \longleftrightarrow u = \varepsilon$
⟨*proof*⟩

lemma *dec-in-lists*[*simp*]: $u \in \langle G\rangle \Longrightarrow Dec\ G\ u \in lists\ G$
⟨*proof*⟩

lemma *set-dec-sub*: $x \in \langle G\rangle \Longrightarrow set\ (Dec\ G\ x) \subseteq G$
⟨*proof*⟩

lemma *dec-hd*: $u \neq \varepsilon \Longrightarrow u \in \langle G\rangle \Longrightarrow hd\ (Dec\ G\ u) \in G$
⟨*proof*⟩

lemma *non-gen-dec*: **assumes** $u \in \langle G\rangle\ u \notin G$ **shows** $Dec\ G\ u \neq [u]$
⟨*proof*⟩

lemma *fac-fac-interpE-dec*: **assumes** $w \in \langle G\rangle\ \ u \leq f\ w\ u \neq \varepsilon$

obtains $p\ s\ ws$ **where** $ws \in lists\ (G - \{\varepsilon\})\ p\ u\ s \sim_{\mathcal{I}} ws\ ws \leq f\ Dec\ G\ w$
⟨*proof*⟩

lemma *set-len-mod-concat*: **assumes** $\forall\ x \in set\ us.\ |x|\ mod\ n = 0\ 0 < n$
shows $|concat\ us|\ mod\ n = 0$
⟨*proof*⟩

lemma *hull-len-mod-concat*: **assumes** $\forall\ x \in G.\ |x|\ mod\ n = 0\ 0 < n\ w \in \langle G \rangle$
shows $|w|\ mod\ n = 0$
⟨*proof*⟩

3.2.1 Refinement into a specific decomposition

We extend the decomposition to lists of words. This can be seen as a refinement of a previous decomposition of some word.

definition *refine* :: *'a list set* ⇒ *'a list list* ⇒ *'a list list* (*Ref - - [55,56] 56*) **where**
refine G us = concat(*map* (*decompose G*) *us*)

lemma *ref-morph*: $us \in lists\ \langle G \rangle \Longrightarrow vs \in lists\ \langle G \rangle \Longrightarrow refine\ G\ (us \cdot vs) = refine\ G\ us \cdot refine\ G\ vs$
⟨*proof*⟩

lemma *ref-morph-plus*: $us \in lists\ (\langle G \rangle - \{\varepsilon\}) \Longrightarrow vs \in lists\ (\langle G \rangle - \{\varepsilon\}) \Longrightarrow refine\ G\ (us \cdot vs) = refine\ G\ us \cdot refine\ G\ vs$
⟨*proof*⟩

lemma *ref-pref-mono*: $ws \in lists\ \langle G \rangle \Longrightarrow us \leq p\ ws \Longrightarrow Ref\ G\ us \leq p\ Ref\ G\ ws$
⟨*proof*⟩

lemma *ref-suf-mono*: $ws \in lists\ \langle G \rangle \Longrightarrow us \leq s\ ws \Longrightarrow (Ref\ G\ us) \leq s\ Ref\ G\ ws$
⟨*proof*⟩

lemma *ref-fac-mono*: $ws \in lists\ \langle G \rangle \Longrightarrow us \leq f\ ws \Longrightarrow (Ref\ G\ us) \leq f\ Ref\ G\ ws$
⟨*proof*⟩

lemma *ref-pop-hd*: $us \neq \varepsilon \Longrightarrow us \in lists\ \langle G \rangle \Longrightarrow refine\ G\ us = decompose\ G\ (hd\ us) \cdot refine\ G\ (tl\ us)$
⟨*proof*⟩

lemma *ref-in*: $us \in lists\ \langle G \rangle \Longrightarrow (Ref\ G\ us) \in lists\ (G - \{\varepsilon\})$
⟨*proof*⟩

lemma *ref-in'*[*intro*]: $us \in lists\ \langle G \rangle \Longrightarrow (Ref\ G\ us) \in lists\ G$
⟨*proof*⟩

lemma *concat-ref*: $us \in lists\ \langle G \rangle \Longrightarrow concat\ (Ref\ G\ us) = concat\ us$
⟨*proof*⟩

lemma *ref-gen*: $us \in lists\ B \Longrightarrow B \subseteq \langle G \rangle \Longrightarrow Ref\ G\ us \in \langle decompose\ G\ `\ B \rangle$

⟨*proof*⟩

lemma *ref-set*: *ws* ∈ *lists* ⟨*G*⟩ ⟹ *set* (*Ref G ws*) = ⋃ (*set*‘(*decompose G*)‘*set ws*)
⟨*proof*⟩

lemma *emp-ref*: **assumes** *us* ∈ *lists* (⟨*G*⟩ − {*ε*}) **and** *Ref G us* = *ε* **shows** *us* = *ε*
⟨*proof*⟩

lemma *sing-ref-sing*:
assumes *us* ∈ *lists* (⟨*G*⟩ − {*ε*}) **and** *refine G us* = [*b*]
shows *us* = [*b*]
⟨*proof*⟩

lemma *ref-ex*: **assumes** *Q* ⊆ ⟨*G*⟩ **and** *us* ∈ *lists Q*
shows *Ref G us* ∈ *lists* (*G* − {*ε*}) **and** *concat* (*Ref G us*) = *concat us*
⟨*proof*⟩

lemma *ref-emp* [*simp*]: *Ref G ε* = *ε*
⟨*proof*⟩

3.3 Basis

An important property of monoids of words is that they have a unique minimal generating set. Which is the set consisting of indecomposable elements.

Indecomposable element is an element that is not generated by other ones.

definition *ungenerated* :: ′*a list* ⇒ ′*a list set* ⇒ *bool* (- ∈*U* - [*51*,*51*] *50*) **where**
ungenerated b G ≡ *b* ∈ *G* ∧ *b* ∉ ⟨*G* − {*b*}⟩

lemma *ungen-nemp*[*simp*]: *b* ∈*U G* ⟹ *b* ≠ *ε*
⟨*proof*⟩

lemma *ungen-in*[*intro*]: *ungenerated b G* ⟹ *b* ∈ *G* **and**
ungenD[*intro*]: *ungenerated b G* ⟹ *b* ∉ ⟨*G* − {*b*}⟩
⟨*proof*⟩

An ungenerated element has no nontrivial decomposition

lemma *ungen-dec-triv*: **assumes** *u* ∈ ⟨*G*⟩ *v* ∈ ⟨*G*⟩ *u* · *v* ∈*U* ⟨*G*⟩ **shows** *u* = *ε* ∨ *v* = *ε*
⟨*proof*⟩

lemma *ungen-dec-triv*′: **assumes** *us* ∈ *lists* (⟨*G*⟩ − {*ε*}) *concat us* ∈*U* ⟨*G*⟩ **shows** |*us*| = *1*
⟨*proof*⟩

Conversely, any nonempty element that is not ungenerated is a product of at least two shorter elements

lemma *gen-elem-list*: **assumes** $u \in \langle G - \{u\}\rangle$ $u \neq \varepsilon$
obtains *us* **where** $us \in lists\ (G - \{u\} - \{\varepsilon\})$ $concat\ us = u$ $1 < |us|$
$\bigwedge c.\ c \in set\ us \Longrightarrow |c| < |u|$
⟨*proof*⟩

lemma *gen-elem-dec*: **assumes** $b \in \langle G - \{b\}\rangle$ $b \neq \varepsilon$
obtains $u\ v$ **where** $u \in \langle G\rangle$ $u \neq \varepsilon$ $v \in \langle G\rangle$ $v \neq \varepsilon$ $u \cdot v = b$
⟨*proof*⟩

This yields several criteria for being ungenerated

lemma *ungeneratedI*:
assumes $b \in G$ **and** $b \neq \varepsilon$
and *all*: $\bigwedge u\ v.\ u \neq \varepsilon \Longrightarrow\ u \in \langle G\rangle \Longrightarrow\ v \neq \varepsilon \Longrightarrow v \in \langle G\rangle \Longrightarrow u \cdot v \neq b$
shows $b \in\!U\ G$
⟨*proof*⟩

lemma *ungeneratedI'*:
assumes $b \in G$ **and** $b \neq \varepsilon$
and *all*: $\bigwedge us.\ us \in lists\ (G - \{b\} - \{\varepsilon\}) \Longrightarrow concat\ us = b \Longrightarrow |us| \leq 1$
shows $b \in\!U\ G$
⟨*proof*⟩

lemma *ungenerated-shortest*:
assumes $b \in G$ **and** $b \neq \varepsilon$
and *all*: $\bigwedge c.\ c \in G - \{\varepsilon\} \Longrightarrow |b| \leq |c|$
shows $b \in\!U\ G$
⟨*proof*⟩

lemma *ungenerated-sing*:
assumes $[a] \in G$
shows $[a] \in\!U\ G$
⟨*proof*⟩

lemma *ungen-hull-ungen*: $b \in\!U\ \langle G\rangle \longleftrightarrow b \in\!U\ G$
⟨*proof*⟩

The *basis* is the set of all ungenerated elements.

definition *basis* :: *'a list set* ⇒ *'a list set* ($\mathfrak{B}$ - [*51*]) **where**
basis $G = \{x.\ x \in\!U\ G\}$

lemma *basisD*: $x \in \mathfrak{B}\ G \Longrightarrow x \in\!U\ G$
⟨*proof*⟩

lemma *basis-in*: $x \in \mathfrak{B}\ G \Longrightarrow x \in G$
⟨*proof*⟩

lemma *emp-not-basis*: $x \in \mathfrak{B}\ G \Longrightarrow x \neq \varepsilon$
⟨*proof*⟩

lemma *basis-sub-gen*: $\mathfrak{B}\ G \subseteq G$
⟨*proof*⟩

The basis generates the generating set and therefore also the whole monoid

lemma *gen-sub-basis*: $G \subseteq \langle \mathfrak{B}\ G \rangle$
⟨*proof*⟩

lemma *basis-concat-listsE*:
assumes $w \in G$
obtains *ws* **where** $ws \in lists\ \mathfrak{B}\ G$ **and** $concat\ ws = w$
⟨*proof*⟩

theorem *basis-gen-hull*: $\langle \mathfrak{B}\ G \rangle = \langle G \rangle$
⟨*proof*⟩

theorem *basis-of-hull*: $\mathfrak{B}\ \langle G \rangle = \mathfrak{B}\ G$
⟨*proof*⟩

lemma *basis-gen-hull'*: $\langle \mathfrak{B}\ \langle G \rangle \rangle = \langle G \rangle$
⟨*proof*⟩

lemma *basis-hull-sub*: $\mathfrak{B}\ \langle G \rangle \subseteq G$
⟨*proof*⟩

The basis is the smallest generating set.

theorem *gen-basis-sub*: $\langle S \rangle = \langle G \rangle \Longrightarrow \mathfrak{B}\ G \subseteq S$
⟨*proof*⟩

lemma *basis-min-gen*: $S \subseteq \mathfrak{B}\ G \Longrightarrow \langle S \rangle = G \Longrightarrow S = \mathfrak{B}\ G$
⟨*proof*⟩

lemma *basisI*: $(\bigwedge B.\ \langle B \rangle = \langle C \rangle \Longrightarrow C \subseteq B) \Longrightarrow \mathfrak{B}\ \langle C \rangle = C$
⟨*proof*⟩

An arbitrary set between basis and the hull is generating...

lemma *gen-sets*: **assumes** $\mathfrak{B}\ G \subseteq S$ **and** $S \subseteq \langle G \rangle$ **shows** $\langle S \rangle = \langle G \rangle$
⟨*proof*⟩

... and has the same basis

lemma *basis-sets*: $\mathfrak{B}\ G \subseteq S \Longrightarrow S \subseteq \langle G \rangle \Longrightarrow \mathfrak{B}\ G = \mathfrak{B}\ S$
⟨*proof*⟩

3.4 Code

locale *nemp-words* =
fixes G
assumes *emp-not-in*: $\varepsilon \notin G$

begin
lemma *drop-empD*: $G - \{\varepsilon\} = G$
⟨*proof*⟩

lemmas *emp-concat-emp'* = *emp-concat-emp*[*of* - *G*, *unfolded drop-empD*]

thm *disjE*[*OF ruler*[*OF take-is-prefix take-is-prefix*]]

lemma *nemp*: $x \in G \Longrightarrow x \neq \varepsilon$
⟨*proof*⟩

lemma *concat-eq-emp-conv* [*simp*]: $us \in lists\ G \Longrightarrow concat\ us = \varepsilon \longleftrightarrow us = \varepsilon$
⟨*proof*⟩

lemma *root-dec-inj-on*: *inj-on* $(\lambda\ x.\ [\varrho\ x]^{@}(e_\varrho\ x))\ G$
⟨*proof*⟩

lemma *concat-len-ruler*: **assumes** $ws \in lists\ G\ us \leq p\ ws\ vs \leq p\ ws\ |concat\ us| \leq |concat\ vs|$
shows $us \leq p\ vs$
⟨*proof*⟩

end

lemma *concat-emp*:
$\varepsilon \notin G \Longrightarrow us \in lists\ G \Longrightarrow concat\ us = \varepsilon \Longrightarrow us = \varepsilon$
⟨*proof*⟩

A basis freely generating its hull is called a *code*. By definition, this means that generated elements have unique factorizations into the elements of the code.

locale *code* =
fixes $\mathcal{C}$
assumes *is-code*: $xs \in lists\ \mathcal{C} \Longrightarrow ys \in lists\ \mathcal{C} \Longrightarrow concat\ xs = concat\ ys \Longrightarrow xs = ys$
begin

lemma *code-comm-eq*: $x \in \mathcal{C} \Longrightarrow y \in \mathcal{C} \Longrightarrow x \cdot y = y \cdot x \Longrightarrow x = y$
⟨*proof*⟩

lemma *emp-not-in*: $\varepsilon \notin \mathcal{C}$
⟨*proof*⟩

lemma *nemp*: $u \in \mathcal{C} \Longrightarrow u \neq \varepsilon$
⟨*proof*⟩

sublocale *nemp-words* $\mathcal{C}$

⟨*proof*⟩

lemma *code-elem-dec*: $us \in lists\ \mathcal{C} \Longrightarrow concat\ us = c \Longrightarrow c \in \mathcal{C} \Longrightarrow us = [c]$
⟨*proof*⟩

lemma *code-ungen*: **assumes** $c \in \mathcal{C}$ **shows** $c \in U\ \mathcal{C}$
⟨*proof*⟩

lemma *code-is-basis*: $\mathfrak{B}\ \mathcal{C} = \mathcal{C}$
⟨*proof*⟩

lemma *code-unique-dec'*: $us \in lists\ \mathcal{C} \Longrightarrow Dec\ \mathcal{C}\ (concat\ us) = us$
⟨*proof*⟩

lemma *code-unique-dec* [*intro*!]: $us \in lists\ \mathcal{C} \Longrightarrow concat\ us = u \Longrightarrow\ Dec\ \mathcal{C}\ u = us$
⟨*proof*⟩

lemma *triv-refine*[*intro*!] : $us \in lists\ \mathcal{C} \Longrightarrow concat\ us = u \Longrightarrow\ Ref\ \mathcal{C}\ [u] = us$
⟨*proof*⟩

lemma *code-unique-ref*: $us \in lists\ \langle\mathcal{C}\rangle \Longrightarrow refine\ \mathcal{C}\ us = decompose\ \mathcal{C}\ (concat\ us)$
⟨*proof*⟩

lemma *refI* [*intro*]: $us \in lists\ \langle\mathcal{C}\rangle \Longrightarrow vs \in lists\ \mathcal{C} \Longrightarrow concat\ vs = concat\ us \Longrightarrow$
$Ref\ \mathcal{C}\ us = vs$
⟨*proof*⟩

lemma *code-dec-morph*: **assumes** $x \in \langle\mathcal{C}\rangle\ y \in \langle\mathcal{C}\rangle$
shows $(Dec\ \mathcal{C}\ x) \cdot (Dec\ \mathcal{C}\ y) = Dec\ \mathcal{C}\ (x{\cdot}y)$
⟨*proof*⟩

lemma *dec-pow*: $rs \in \langle\mathcal{C}\rangle \Longrightarrow Dec\ \mathcal{C}\ (rs^{@}k) = (Dec\ \mathcal{C}\ rs)^{@}k$
⟨*proof*⟩

lemma *code-el-dec*: $c \in \mathcal{C} \Longrightarrow decompose\ \mathcal{C}\ c = [c]$
⟨*proof*⟩

lemma *code-ref-list*: $us \in lists\ \mathcal{C} \Longrightarrow refine\ \mathcal{C}\ us = us$
⟨*proof*⟩

lemma *code-ref-gen*: **assumes** $G \subseteq \langle\mathcal{C}\rangle\ u \in \langle G\rangle$
shows $Dec\ \mathcal{C}\ u \in \langle decompose\ \mathcal{C}\ `\ G\rangle$
⟨*proof*⟩

find-theorems $\varrho\ ?x\ ^{@}\ ?k = ?x\ 0 < ?k$

lemma *code-rev-code*: $code\ (rev\ `\ \mathcal{C})$
⟨*proof*⟩

lemma *dec-rev* [*simp*, *reversal-rule*]:
$u \in \langle \mathcal{C} \rangle \Longrightarrow Dec\ rev\ `\ \mathcal{C}\ (rev\ u) = rev\ (map\ rev\ (Dec\ \mathcal{C}\ u))$
⟨*proof*⟩

lemma *elem-comm-sing-set*: **assumes** $ws \in lists\ \mathcal{C}$ **and** $ws \neq \varepsilon$ **and** $u \in \mathcal{C}$ **and** $concat\ ws \cdot u = u \cdot concat\ ws$
shows $set\ ws = \{u\}$
⟨*proof*⟩

lemma *pure-code-pres-prim*: **assumes** *pure*: $\forall\, u \in \langle \mathcal{C} \rangle.\ \varrho\ u \in \langle \mathcal{C} \rangle$ **and** $w \in \langle \mathcal{C} \rangle$ **and** $primitive\ (Dec\ \mathcal{C}\ w)$
shows $primitive\ w$
⟨*proof*⟩

lemma *inj-on-dec*: $inj\text{-}on\ (decompose\ \mathcal{C})\ \langle \mathcal{C} \rangle$
⟨*proof*⟩

lemma *ref-disj-interp*: **assumes** $vs \in lists\ \langle \mathcal{C} \rangle\ p\ Ref\ \mathcal{C}\ vs\ s \sim_{\mathcal{D}} ws$
shows $p\ vs\ s \sim_{\mathcal{D}} ws$
⟨*proof*⟩

end — end context code

lemma *emp-is-code*: $code\ \{\}$
⟨*proof*⟩

lemma *code-rev-code-iff* [*reversal-rule*]: $code\ (rev\ `\ C) \longleftrightarrow code\ C$
⟨*proof*⟩

lemma *code-induct-hd*: **assumes** $\varepsilon \notin C$ **and**
$\bigwedge xs\ ys.\ xs \in lists\ C \Longrightarrow ys \in lists\ C \Longrightarrow concat\ xs = concat\ ys \Longrightarrow hd\ xs = hd\ ys$
shows $code\ C$
⟨*proof*⟩

lemma *ref-set-primroot*: **assumes** $ws \in lists\ (G - \{\varepsilon\})$ **and** $code\ (\varrho `G)$
shows $set\ (Ref\ \varrho `G\ ws) = \varrho `(set\ ws)$
⟨*proof*⟩

3.5 Prefix code

locale *prefix-code* =
fixes $\mathcal{C}$
assumes
emp-not-in: $\varepsilon \notin \mathcal{C}$ **and**
pref-free: $u \in \mathcal{C} \Longrightarrow v \in \mathcal{C} \Longrightarrow u \leq p\ v \Longrightarrow u = v$

begin

lemma *nemp*: $u \in \mathcal{C} \Longrightarrow u \neq \varepsilon$
⟨*proof*⟩

lemma *concat-pref-concat*:
assumes $us \in lists\ \mathcal{C}$ $vs \in lists\ \mathcal{C}$ $concat\ us \leq p\ concat\ vs$
shows $us \leq p\ vs$
⟨*proof*⟩

lemma *concat-pref-concat-conv*:
assumes $us \in lists\ \mathcal{C}$ $vs \in lists\ \mathcal{C}$
shows $concat\ us \leq p\ concat\ vs \longleftrightarrow us \leq p\ vs$
⟨*proof*⟩

sublocale *code*
⟨*proof*⟩

lemmas *is-code* = *is-code* **and**
code = *code-axioms*

lemma *dec-pref-unique*:
$w \in \langle \mathcal{C} \rangle \Longrightarrow p \in \langle \mathcal{C} \rangle \Longrightarrow p \leq p\ w \Longrightarrow Dec\ \mathcal{C}\ p \leq p\ Dec\ \mathcal{C}\ w$
⟨*proof*⟩

lemma *concat-suf-eq*: **assumes**
$us \in lists\ \mathcal{C}$ $ws \in lists\ \mathcal{C}$ **and**
$concat\ us \cdot s = concat\ ws$ **and** $s <s last\ ws$
shows $us = ws$ **and** $s = \varepsilon$
⟨*proof*⟩

end

thm *prefix-code.concat-suf-eq*[*reversed*]

3.5.1 Suffix code

locale *suffix-code* = *prefix-code* (*rev* ‘ $\mathcal{C}$) **for** $\mathcal{C}$
begin

thm *dec-rev*
code

sublocale *code*
⟨*proof*⟩

lemmas *concat-suf-concat* = *concat-pref-concat*[*reversed*] **and**
concat-suf-concat-conv = *concat-pref-concat-conv*[*reversed*] **and**
nemp = *nemp*[*reversed*] **and**
suf-free = *pref-free*[*reversed*] **and**
dec-suf-unique = *dec-pref-unique*[*reversed*]

lemma *concat-pref-eq*: **assumes**
us ∈ *lists* $\mathcal{C}$ *ws* ∈ *lists* $\mathcal{C}$ **and**
p · *concat us* = *concat ws* **and** *p* <*p* *hd ws*
shows *us* = *ws* **and** *p* = ε
⟨*proof*⟩

thm *is-code*
code-axioms
code

end

3.5.2 Bifix code

locale *bifix-code* = *prefix-code* + *suf*: *suffix-code*
begin

lemma *joint-interp-triv*: **assumes**
us ∈ *lists* $\mathcal{C}$ *ws* ∈ *lists* $\mathcal{C}$ **and**
interp: *p* (*concat us*) *s* $\sim_{\mathcal{I}}$ *ws* **and**
joint: ¬ *p us s* $\sim_{\mathcal{D}}$ *ws*
shows *p* = ε **and** *s* = ε **and** *us* = *ws*
⟨*proof*⟩

end

3.6 Marked code

locale *marked-code* =
fixes $\mathcal{C}$
assumes
emp-not-in: $\varepsilon \notin \mathcal{C}$ **and**
marked: *u* ∈ $\mathcal{C}$ ⟹ *v* ∈ $\mathcal{C}$ ⟹ *hd u* = *hd v* ⟹ *u* = *v*

begin

lemma *nemp*: *u* ∈ $\mathcal{C}$ ⟹ *u* ≠ ε
⟨*proof*⟩

sublocale *prefix-code*
⟨*proof*⟩

lemma *marked-concat-lcp*: *us* ∈ *lists* $\mathcal{C}$ ⟹ *vs* ∈ *lists* $\mathcal{C}$ ⟹ *concat* (*us* $\wedge_p$ *vs*) = (*concat us*) $\wedge_p$ (*concat vs*)
⟨*proof*⟩

lemma *hd-concat-hd*: **assumes** *xs* ∈ *lists* $\mathcal{C}$ **and** *ys* ∈ *lists* $\mathcal{C}$ **and** *xs* ≠ ε **and** *ys*

$\neq \varepsilon$ **and**
  $hd\ (concat\ xs) = hd\ (concat\ ys)$
shows $hd\ xs = hd\ ys$
⟨*proof*⟩

end

3.7 Non-overlapping code

locale *non-overlapping* =
  fixes $\mathcal{C}$
  assumes
    emp-not-in: $\varepsilon \notin \mathcal{C}$ **and**
    no-overlap: $u \in \mathcal{C} \Longrightarrow v \in \mathcal{C} \Longrightarrow z \leq p\ u \Longrightarrow z \leq s\ v \Longrightarrow z \neq \varepsilon \Longrightarrow u = v$ **and**
    no-fac: $u \in \mathcal{C} \Longrightarrow v \in \mathcal{C} \Longrightarrow u \leq f\ v \Longrightarrow\ u = v$
begin

lemma *nemp*: $u \in \mathcal{C}\ \Longrightarrow u \neq \varepsilon$
  ⟨*proof*⟩

sublocale *prefix-code*
  ⟨*proof*⟩

lemma *rev-non-overlapping*: *non-overlapping* $(rev\ `\ \mathcal{C})$
⟨*proof*⟩

sublocale *suf*: *suffix-code* $\mathcal{C}$
⟨*proof*⟩

lemma *overlap-concat-last*: **assumes** $u \in \mathcal{C}$ **and** $vs \in lists\ \mathcal{C}$ **and** $vs \neq \varepsilon$ **and**
    $r \neq \varepsilon$ **and** $r \leq p\ u$ **and** $r \leq s\ p \cdot concat\ vs$
  shows $u = last\ vs$
⟨*proof*⟩

lemma *overlap-concat-hd*: **assumes** $u \in \mathcal{C}$ **and** $vs \in lists\ \mathcal{C}$ **and** $vs \neq \varepsilon$ **and** $r \neq \varepsilon$ **and** $r \leq s\ u$ **and** $r \leq p\ concat\ vs \cdot s$
  shows $u = hd\ vs$
⟨*proof*⟩

lemma *fac-concat-fac*:
  assumes $us \in lists\ \mathcal{C}$ $vs \in lists\ \mathcal{C}$
    and $1 < card\ (set\ us)$
    and $concat\ vs = p \cdot concat\ us \cdot s$
  obtains *ps ss* **where** $concat\ ps = p$ **and** $concat\ ss = s$ **and** $ps \cdot us \cdot ss = vs$
⟨*proof*⟩

theorem *prim-morph*:
  assumes $ws \in lists\ \mathcal{C}$
    and $|ws| \neq 1$

and *primitive ws*
shows *primitive* (*concat ws*)
⟨*proof*⟩

lemma *prim-concat-conv*:
assumes *ws* ∈ *lists* $\mathcal{C}$
and |*ws*| ≠ *1*
shows *primitive* (*concat ws*) ⟷ *primitive ws*
⟨*proof*⟩

end

3.8 Binary code

We pay a special attention to two element codes. In particular, we show that two words form a code if and only if they do not commute. This means that two words either commute, or do not satisfy any nontrivial relation.

definition *bin-lcp* **where** *bin-lcp x y* = *x*·*y* ∧$_p$ *y*·*x*
definition *bin-lcs* **where** *bin-lcs x y* = *x*·*y* ∧$_s$ *y*·*x*

definition *bin-mismatch* **where** *bin-mismatch x y* = (*x*·*y*)!|*bin-lcp x y*|
definition *bin-mismatch-suf* **where** *bin-mismatch-suf x y* = *bin-mismatch* (*rev y*) (*rev x*)

value[*nbe*] [*0*::*nat*,*1*,*0*]!*3*

lemma *bin-lcs-rev*: *bin-lcs x y* = *rev* (*bin-lcp* (*rev x*) (*rev y*))
⟨*proof*⟩

lemma *bin-lcp-sym*: *bin-lcp x y* = *bin-lcp y x*
⟨*proof*⟩

lemma *bin-mismatch-comm*: (*bin-mismatch x y* = *bin-mismatch y x*) ⟷ (*x* · *y* = *y* · *x*)
⟨*proof*⟩

lemma *bin-lcp-pref-fst-snd*: *bin-lcp x y* ≤*p x* · *y*
⟨*proof*⟩

lemma *bin-lcp-pref-snd-fst*: *bin-lcp x y* ≤*p y* · *x*
⟨*proof*⟩

lemma *bin-lcp-bin-lcs* [*reversal-rule*]: *bin-lcp* (*rev x*) (*rev y*) = *rev* (*bin-lcs x y*)
⟨*proof*⟩

lemmas *bin-lcs-sym* = *bin-lcp-sym*[*reversed*]

lemma *bin-lcp-len*: *x* · *y* ≠ *y* · *x* ⟹ |*bin-lcp x y*| < |*x* · *y*|

⟨*proof*⟩

lemmas *bin-lcs-len* = *bin-lcp-len*[*reversed*]

lemma *bin-mismatch-pref-suf'*[*reversal-rule*]:
bin-mismatch (*rev y*) (*rev x*) = *bin-mismatch-suf x y*
⟨*proof*⟩

3.8.1 Binary code locale

locale *binary-code* =
fixes u_0 u_1
assumes *non-comm*: $u_0 \cdot u_1 \neq u_1 \cdot u_0$

begin

A crucial property of two element codes is the constant decoding delay given by the word α, which is a prefix of any generating word (sufficiently long), while the letter immediately after this common prefix indicates the first element of the decomposition.

definition *uu* **where** *uu a* = (*if a then* u_0 *else* u_1)

lemma *bin-code-set-bool*: {*uu a*,*uu* (¬ *a*)} = {u_0,u_1}
⟨*proof*⟩

lemma *bin-code-set-bool'*: {*uu a*,*uu* (¬ *a*)} = {u_1,u_0}
⟨*proof*⟩

lemma *bin-code-swap*: *binary-code* u_1 u_0
⟨*proof*⟩

lemma *bin-code-bool*: *binary-code* (*uu a*) (*uu* (¬ *a*))
⟨*proof*⟩

lemma *bin-code-neq*: $u_0 \neq u_1$
⟨*proof*⟩

lemma *bin-code-neq-bool*: *uu a* ≠ *uu* (¬ *a*)
⟨*proof*⟩

lemma *bin-fst-nemp*: $u_0 \neq \varepsilon$ **and** *bin-snd-nemp*: $u_1 \neq \varepsilon$ **and** *bin-nemp-bool*: *uu a* ≠ ε
⟨*proof*⟩

lemma *bin-not-comp*: ¬ $u_0 \cdot u_1 \bowtie u_1 \cdot u_0$
⟨*proof*⟩

lemma *bin-not-comp-bool*: ¬ (*uu a* · *uu* (¬ *a*) ⋈ *uu* (¬ *a*) · *uu a*)
⟨*proof*⟩

lemma *bin-not-comp-suf*: $\neg\ u_0 \cdot u_1 \bowtie_s u_1 \cdot u_0$
⟨*proof*⟩

lemma *bin-not-comp-suf-bool*: $\neg\ (uu\ a \cdot uu\ (\neg\ a) \bowtie_s uu\ (\neg\ a) \cdot uu\ a)$
⟨*proof*⟩

lemma *bin-mismatch-neq*: *bin-mismatch* $u_0\ u_1 \neq$ *bin-mismatch* $u_1\ u_0$
⟨*proof*⟩

abbreviation *bin-code-lcp* (α) **where** *bin-code-lcp* $\equiv$ *bin-lcp* $u_0\ u_1$
abbreviation *bin-code-lcs* **where** *bin-code-lcs* $\equiv$ *bin-lcs* $u_0\ u_1$
abbreviation *bin-code-mismatch-fst* (c_0) **where** *bin-code-mismatch-fst* $\equiv$ *bin-mismatch* $u_0\ u_1$
abbreviation *bin-code-mismatch-snd* (c_1) **where** *bin-code-mismatch-snd* $\equiv$ *bin-mismatch* $u_1\ u_0$

definition *cc* **where** *cc a* = (*if a then* c_0 *else* c_1)

lemmas *bin-lcp-swap* = *bin-lcp-sym*[*of* $u_0\ u_1$, *symmetric*] **and**
bin-lcp-pref = *bin-lcp-pref-fst-snd*[*of* $u_0\ u_1$] **and**
bin-lcp-pref′ = *bin-lcp-pref-snd-fst*[*of* $u_0\ u_1$] **and**
bin-lcp-short = *bin-lcp-len*[*OF non-comm*, *unfolded lenmorph*]

lemmas *bin-code-simps* = *cc-def uu-def if-True if-False bool-simps*

lemma *bin-lcp-bool*: *bin-lcp* (*uu a*) (*uu* ($\neg$ *a*)) = *bin-code-lcp*
⟨*proof*⟩

lemma *bin-lcp-spref*: $\alpha <p\ u_0 \cdot u_1$
⟨*proof*⟩

lemma *bin-lcp-spref′*: $\alpha <p\ u_1 \cdot u_0$
⟨*proof*⟩

lemma *bin-lcp-spref-bool*: $\alpha <p\ uu\ a \cdot uu\ (\neg\ a)$
⟨*proof*⟩

lemma *bin-mismatch-bool′*: $\alpha \cdot [cc\ a] \leq p\ uu\ a \cdot uu\ (\neg\ a)$
⟨*proof*⟩

lemma *bin-mismatch-bool*: $\alpha \cdot [cc\ a] \leq p\ uu\ a \cdot \alpha$
⟨*proof*⟩

lemmas *bin-fst-mismatch* = *bin-mismatch-bool*[*of True*, *unfolded bin-code-simps*]
and
bin-fst-mismatch′ = *bin-mismatch-bool′*[*of True*, *unfolded bin-code-simps*] **and**
bin-snd-mismatch = *bin-mismatch-bool*[*of False*, *unfolded bin-code-simps*] **and**

bin-snd-mismatch′ = bin-mismatch-bool′[*of False, unfolded bin-code-simps*]

lemma *bin-lcp-pref-all*: $xs \in lists\ \{u_0,u_1\} \Longrightarrow \alpha \leq p\ concat\ xs \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-pref-all-hull*: $w \in \langle\{u_0,u_1\}\rangle \Longrightarrow \alpha \leq p\ w \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-mismatch-pref-all-bool*: **assumes** $q \leq p\ w$ **and** $w \in \langle\{uu\ b, uu\ (\neg\ b)\}\rangle$ **and** $|\alpha| < |uu\ a \cdot q|$
shows $\alpha \cdot [cc\ a] \leq p\ uu\ a \cdot q$
⟨*proof*⟩

lemmas *bin-lcp-mismatch-pref-all-fst = bin-lcp-mismatch-pref-all-bool*[*of - - True True, unfolded bin-code-simps*] **and**
bin-lcp-mismatch-pref-all-snd = bin-lcp-mismatch-pref-all-bool[*of - - True False, unfolded bin-code-simps*]

lemma *bin-lcp-pref-all-len*: **assumes** $q \leq p\ w$ **and** $w \in \langle\{u_0,u_1\}\rangle$ **and** $|\alpha| \leq |q|$
shows $\alpha \leq p\ q$
⟨*proof*⟩

lemma *bin-mismatch-all-bool*: **assumes** $xs \in lists\ \{uu\ b, uu\ (\neg\ b)\}$ **shows** $\alpha \cdot [cc\ a] \leq p\ (uu\ a) \cdot concat\ xs \cdot \alpha$
⟨*proof*⟩

lemmas *bin-fst-mismatch-all = bin-mismatch-all-bool*[*of - True True, unfolded bin-code-simps*] **and**
bin-snd-mismatch-all = bin-mismatch-all-bool[*of - True False, unfolded bin-code-simps*]

lemma *bin-mismatch-all-hull-bool*: **assumes** $w \in \langle\{uu\ b, uu\ (\neg\ b)\}\rangle$ **shows** $\alpha \cdot [cc\ a] \leq p\ uu\ a \cdot w \cdot \alpha$
⟨*proof*⟩

lemmas *bin-fst-mismatch-all-hull = bin-mismatch-all-hull-bool*[*of - True True, unfolded bin-code-simps*] **and**
bin-snd-mismatch-all-hull = bin-mismatch-all-hull-bool[*of - True False, unfolded bin-code-simps*]

lemma *bin-mismatch-all-len-bool*: **assumes** $q \leq p\ uu\ a \cdot w$ **and** $w \in \langle\{uu\ b, uu\ (\neg\ b)\}\rangle$ **and** $|\alpha| < |q|$
shows $\alpha \cdot [cc\ a] \leq p\ q$
⟨*proof*⟩

lemmas *bin-fst-mismatch-all-len = bin-mismatch-all-len-bool*[*of - True - True, unfolded bin-code-simps*] **and**
bin-snd-mismatch-all-len = bin-mismatch-all-len-bool[*of - False - True, unfolded bin-code-simps*]

lemma *bin-code-delay*: **assumes** $|\alpha| \leq |q_0|$ **and** $|\alpha| \leq |q_1|$ **and**
$q_0 \leq p\ u_0 \cdot w_0$ **and** $q_1 \leq p\ u_1 \cdot w_1$ **and**
$w_0 \in \langle\{u_0, u_1\}\rangle$ **and** $w_1 \in \langle\{u_0, u_1\}\rangle$
shows $q_0 \wedge_p q_1 = \alpha$
⟨*proof*⟩

lemma *hd-lq-mismatch-fst*: $hd\ (\alpha^{-1>}(u_0 \cdot \alpha)) = c_0$
⟨*proof*⟩

lemma *hd-lq-mismatch-snd*: $hd\ (\alpha^{-1>}(u_1 \cdot \alpha)) = c_1$
⟨*proof*⟩

lemma *hds-bin-mismatch-neq*: $hd\ (\alpha^{-1>}(u_0 \cdot \alpha)) \neq hd\ (\alpha^{-1>}(u_1 \cdot \alpha))$
⟨*proof*⟩

lemma *bin-lcp-fst-pow-pref*: **assumes** $0 < k$ **shows** $\alpha \cdot [c_0] \leq p\ {u_0}^{@}k \cdot u_1 \cdot z$
⟨*proof*⟩

lemmas *bin-lcp-snd-pow-pref* = *binary-code.bin-lcp-fst-pow-pref*[*OF bin-code-swap*, *unfolded bin-lcp-swap*]

lemma *bin-lcp-fst-lcp*: $\alpha \leq p\ u_0 \cdot \alpha$ **and** *bin-lcp-snd-lcp*: $\alpha \leq p\ u_1 \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-pref-all-set*: **assumes** $set\ ws = \{u_0,u_1\}$
shows $\alpha \leq p\ concat\ ws$
⟨*proof*⟩

lemma *bin-lcp-conjug-morph*:
assumes $u \in \langle\{u_0,u_1\}\rangle$ **and** $v \in \langle\{u_0,u_1\}\rangle$
shows $\alpha^{-1>}(u \cdot \alpha) \cdot \alpha^{-1>}(v \cdot \alpha) = \alpha^{-1>}((u \cdot v) \cdot \alpha)$
⟨*proof*⟩

lemma *lcp-bin-conjug-prim-iff*:
$set\ ws = \{u_0,u_1\} \Longrightarrow primitive\ (\alpha^{-1>}(concat\ ws) \cdot \alpha) \longleftrightarrow primitive\ (concat\ ws)$
⟨*proof*⟩

lemma *bin-lcp-conjug-inj-on*: $inj\text{-}on\ (\lambda u.\ \alpha^{-1>}(u \cdot \alpha))\ \langle\{u_0,u_1\}\rangle$
⟨*proof*⟩

lemma *bin-code-lcp-marked*: **assumes** $us \in lists\ \{u_0,u_1\}$ **and** $vs \in lists\ \{u_0,u_1\}$ **and** $hd\ us \neq hd\ vs$
shows $concat\ us \cdot \alpha \wedge_p concat\ vs \cdot \alpha = \alpha$
⟨*proof*⟩
lemma **assumes** $us \in lists\ \{u_0,u_1\}$ **and** $vs \in lists\ \{u_0,u_1\}$ **and** $hd\ us \neq hd\ vs$
shows $concat\ us \cdot \alpha \wedge_p concat\ vs \cdot \alpha = \alpha$
⟨*proof*⟩

lemma *bin-code-lcp-concat*: **assumes** $us \in lists\ \{u_0,u_1\}$ **and** $vs \in lists\ \{u_0,u_1\}$ **and** $\neg\ us \bowtie vs$
shows $concat\ us \cdot \alpha \wedge_p concat\ vs \cdot \alpha = concat\ (us \wedge_p vs) \cdot \alpha$
⟨*proof*⟩

lemma *bin-code-lcp-concat'*: **assumes** $us \in lists\ \{u_0,u_1\}$ **and** $vs \in lists\ \{u_0,u_1\}$ **and** $\neg\ concat\ us \bowtie concat\ vs$
shows $concat\ us \wedge_p concat\ vs = concat\ (us \wedge_p vs) \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-pows*: $0 < k \Longrightarrow 0 < l \Longrightarrow {u_0}^{@}k \cdot u_1 \cdot z \wedge_p {u_1}^{@}l \cdot u_0 \cdot z' = \alpha$
⟨*proof*⟩

theorem *bin-code*: **assumes** $us \in lists\ \{u_0,u_1\}$ **and** $vs \in lists\ \{u_0,u_1\}$ **and** $concat\ us = concat\ vs$
shows $us = vs$
⟨*proof*⟩

lemma *code-bin-roots*: *binary-code* $(\varrho\ u_0)\ (\varrho\ u_1)$
⟨*proof*⟩

sublocale *code* $\{u_0,u_1\}$
⟨*proof*⟩

lemma *primroot-dec*: $(Dec\ \{\varrho\ u_0,\ \varrho\ u_1\}\ u_0) = [\varrho\ u_0]^{@}e_\varrho\ u_0\ (Dec\ \{\varrho\ u_0,\ \varrho\ u_1\}\ u_1) = [\varrho\ u_1]^{@}e_\varrho\ u_1$
⟨*proof*⟩

lemma *bin-code-prefs*: **assumes** $w \in \langle\{u_0,u_1\}\rangle$ **and** $p \leq p\ w\ w' \in \langle\{u_0,u_1\}\rangle$ **and** $|u_1| \leq |p|$
shows $\neg\ u_0 \cdot\ \ p \leq p\ u_1 \cdot w'$
⟨*proof*⟩

lemma *bin-code-rev*: *binary-code* $(rev\ u_0)\ (rev\ u_1)$
⟨*proof*⟩

lemma *bin-lcp-pows-lcp*: $0 < k \Longrightarrow 0 < l \Longrightarrow {u_0}^{@}k \cdot {u_1}^{@}l \wedge_p {u_1}^{@}l \cdot {u_0}^{@}k = u_0 \cdot u_1 \wedge_p u_1 \cdot u_0$
⟨*proof*⟩

lemma *bin-mismatch*: $u_0 \cdot \alpha \wedge_p u_1 \cdot \alpha = \alpha$
⟨*proof*⟩

lemma *not-comp-bin-fst-snd*: $\neg\ u_0 \cdot \alpha \bowtie u_1 \cdot\ \alpha$
⟨*proof*⟩

theorem *bin-bounded-delay*: **assumes** $z \leq p\ u_0 \cdot w_0$ **and** $z \leq p\ u_1 \cdot w_1$
and $w_0 \in \langle\{u_0,u_1\}\rangle$ **and** $w_1 \in \langle\{u_0,u_1\}\rangle$

shows $|z| \leq |\alpha|$
⟨*proof*⟩

thm *binary-code.bin-lcp-pows-lcp*

lemma *prim-roots-lcp*: *bin-lcp* $(\varrho\ u_0)\ (\varrho\ u_1) = \alpha$
⟨*proof*⟩

lemma *bin-roots-decompose*:
$Dec\ \{\varrho\ u_0,\ u_1\}\ u_0 = [\varrho\ u_0]^{@} e_\varrho\ u_0$
$Dec\ \{\varrho\ u_0,\ u_1\}\ u_1 = [u_1]$
$Dec\ \{u_0,\ \varrho\ u_1\}\ u_1 = [\varrho\ u_1]^{@} e_\varrho\ u_1$
$Dec\ \{u_0,\ \varrho\ u_1\}\ u_0 = [u_0]$
$Dec\ \{u_0,u_1\}\ u_0 = [u_0]$
$Dec\ \{u_0,u_1\}\ u_1 = [u_1]$
⟨*proof*⟩

lemma *ref-fst-sq*: *Ref* $\{\varrho\ u_0,\ u_1\}[u_0,u_0] = [\varrho\ u_0]^{@}(e_\varrho\ u_0 * 2)$
⟨*proof*⟩

lemma *ref-fst-pow*: *Ref* $\{\varrho\ u_0,\ u_1\}[u_0]^{@}k = [\varrho\ u_0]^{@}(e_\varrho\ u_0 * k)$
⟨*proof*⟩

lemma *bin-code-concat-len*: **assumes** $ws \in lists\ \{u_0,u_1\}$
shows $|concat\ ws| = count\text{-}list\ ws\ u_0 * |u_0| + count\text{-}list\ ws\ u_1 * |u_1|$
⟨*proof*⟩

Maximal r-prefixes

lemma *bin-lcp-per-root-max-pref-short*: **assumes** $\alpha <p\ u_0 \cdot u_1 \wedge_p r \cdot u_0 \cdot u_1$ **and** $r \neq \varepsilon$ **and** $q \leq p\ w$ **and** $w \in \langle\{u_0,\ u_1\}\rangle$
shows $u_1 \cdot q \wedge_p r \cdot u_1 \cdot q = take\ |u_1 \cdot q|\ \alpha$
⟨*proof*⟩

lemma *bin-per-root-max-pref-short*: **assumes** $(u_0 \cdot u_1) <p\ r \cdot u_0 \cdot u_1$ **and** $q \leq p\ w$ **and** $w \in \langle\{u_0,\ u_1\}\rangle$
shows $u_1 \cdot q \wedge_p r \cdot u_1 \cdot q = take\ |u_1 \cdot q|\ \alpha$
⟨*proof*⟩

lemma *bin-root-max-pref-long*: **assumes** $r \cdot u_0 \cdot u_1 = u_0 \cdot u_1 \cdot r$ **and** $q \leq p\ w$ **and** $w \in \langle\{u_0,\ u_1\}\rangle$ **and** $|\alpha| \leq |q|$
shows $u_0 \cdot \alpha \leq p\ u_0 \cdot q \wedge_p r \cdot u_0 \cdot q$
⟨*proof*⟩

lemma *per-root-lcp-per-root*: $u_0 \cdot u_1 <p\ r \cdot u_0 \cdot u_1 \Longrightarrow \alpha \cdot [c_0] \leq p\ r \cdot \alpha$
⟨*proof*⟩

lemma *per-root-bin-fst-snd-lcp*: **assumes** $u_0 \cdot u_1 <p\ r \cdot u_0 \cdot u_1$ **and**
$q \leq p\ w$ **and** $w \in \langle\{u_0,u_1\}\rangle$ **and** $|\alpha| < |u_1 \cdot q|$

$q' \leq_p w'$ **and** $w' \in \langle\{u_0,u_1\}\rangle$ **and** $|\alpha| \leq |q'|$
shows $u_1 \cdot q \wedge_p r \cdot q' = \alpha$
⟨*proof*⟩

end

lemmas allowing to translate properties of binary code to its roots

named-theorems *bin-code-primroots*
lemma *bin-lcp-eq-primroots* [*bin-code-primroots*]: **assumes** $x \cdot y \neq y \cdot x$
shows *bin-lcp* (ϱ *x*) (ϱ *y*) = *bin-lcp x y*
⟨*proof*⟩

lemma *bin-lcs-eq-primroots* [*bin-code-primroots*]: **assumes** $x \cdot y \neq y \cdot x$
shows *bin-lcs* (ϱ *x*) (ϱ *y*) = *bin-lcs x y*
⟨*proof*⟩

lemma *bin-mismatch-fst-eq-primroots* [*bin-code-primroots*]: **assumes** $x \cdot y \neq y \cdot x$
shows *bin-mismatch* (ϱ *x*) (ϱ *y*) = *bin-mismatch x y*
⟨*proof*⟩

lemmas *bin-mismatch-snd-eq-primroots*[*bin-code-primroots*] = *bin-mismatch-fst-eq-primroots*[*OF not-sym*] **and**
bin-mismatch-suf-fst-eq-primroots[*bin-code-primroots*] = *bin-mismatch-fst-eq-primroots*[*reversed*]
and
bin-mismatch-suf-snd-primroots[*bin-code-primroots*] = *bin-mismatch-fst-eq-primroots*[*reversed*, *OF not-sym*]

lemma *bin-lcp-eq-primroots'* [*bin-code-primroots*]: $x \cdot y \neq y \cdot x \Longrightarrow \varrho\ x \cdot \varrho\ y \wedge_p \varrho\ y \cdot \varrho\ x = x \cdot y \wedge_p y \cdot x$
⟨*proof*⟩

lemmas *no-comm-bin-code* = *binary-code.bin-code*[*unfolded binary-code-def*]

theorem *bin-code-code*[*intro*]: **assumes** $u \cdot v \neq v \cdot u$ **shows** *code* {*u*, *v*}
⟨*proof*⟩

lemma *code-bin-code*: $u \neq v \Longrightarrow$ *code* {*u*,*v*} $\Longrightarrow u \cdot v \neq v \cdot u$
⟨*proof*⟩

lemma *lcp-roots-lcp*: $x \cdot y \neq y \cdot x \Longrightarrow x \cdot y \wedge_p y \cdot x = \varrho\ x \cdot \varrho\ y \wedge_p \varrho\ y \cdot \varrho\ x$
⟨*proof*⟩

lemma *sing-gen-primroot* [*simp*]: $u \in \langle\{\varrho\ u\}\rangle$
⟨*proof*⟩

lemma *sing-gen-pref-cancel* [*elim*]: $u \cdot v \in \langle\{r\}\rangle \Longrightarrow u \in \langle\{r\}\rangle \Longrightarrow v \in \langle\{r\}\rangle$
⟨*proof*⟩

lemma *sing-gen-suf-cancel* [*elim*]: $u \cdot v \in \langle\{r\}\rangle \Longrightarrow v \in \langle\{r\}\rangle \Longrightarrow u \in \langle\{r\}\rangle$
⟨*proof*⟩

lemma *prim-comm-root*[*elim*]: **assumes** *primitive* r **and** $u \cdot r = r \cdot u$ **shows** $u \in \langle\{r\}\rangle$
⟨*proof*⟩

lemma *prim-root-drop-exp*[*elim*]: **assumes** $u^{@}k \in \langle\{r\}\rangle$ **and** $0 < k$ **and** *primitive* r
shows $u \in \langle\{r\}\rangle$
⟨*proof*⟩

lemma *per-root-trans*[*intro*]: **assumes** $w <p\ u \cdot w$ **and** $u \in \langle\{t\}\rangle$ **shows** $w <p\ t \cdot w$
⟨*proof*⟩

lemma *per-root-trans'*[*intro*]: $w \leq p\ u \cdot w \Longrightarrow u \in \langle\{r\}\rangle \Longrightarrow u \neq \varepsilon \Longrightarrow w \leq p\ r \cdot w$
⟨*proof*⟩

lemmas *per-root-trans-suf'*[*intro*] = *per-root-trans'*[*reversed*]

lemma *per-root-pref*: $w \neq \varepsilon \Longrightarrow w \in \langle\{r\}\rangle \Longrightarrow r \leq p\ w$
⟨*proof*⟩

lemmas *per-root-suf* = *per-root-pref*[*reversed*]

lemma *comm-primrootE*: **assumes** $x \cdot y = y \cdot x$
obtains t **where** $x \in \langle\{t\}\rangle$ **and** $y \in \langle\{t\}\rangle$ **and** *primitive* t
⟨*proof*⟩

lemma *root-trans*[*trans*]: $[\![v \in \langle\{u\}\rangle;\ u \in \langle\{t\}\rangle]\!] \Longrightarrow v \in \langle\{t\}\rangle$
⟨*proof*⟩

lemma *root-rev-iff*[*reversal-rule*]: $((rev\ u) \in \langle\{rev\ t\}\rangle) \longleftrightarrow (u \in \langle\{t\}\rangle)$
⟨*proof*⟩

3.9 Two words hull (not necessarily a code)

lemma *bin-lists-len-count*: **assumes** $x \neq y$ **and** $ws \in lists\ \{x,y\}$ **shows**
count-list ws x + *count-list ws y* = $|ws|$
⟨*proof*⟩

lemma *two-elem-first-block*: **assumes** $w \in \langle\{u,v\}\rangle$

obtains m **where** $u^{@}m \cdot v \leq_p w \vee w = u^{@}m$
⟨*proof*⟩

lemmas *two-elem-last-block* = *two-elem-first-block*[*reversed*]

lemma *two-elem-pref*: **assumes** $v \leq_p u \cdot p$ **and** $p \in \langle\{u,v\}\rangle$
shows $v \leq_p u \cdot v$
⟨*proof*⟩

lemmas *two-elem-suf* = *two-elem-pref*[*reversed*]

lemma *gen-drop-exp*: **assumes** $p \in \langle\{u,v^{@}(Suc\ k)\}\rangle$ **shows** $p \in \langle\{u,v\}\rangle$
⟨*proof*⟩

lemma *gen-drop-exp-pos*: **assumes** $p \in \langle\{u,v^{@}k\}\rangle$ $0 < k$ **shows** $p \in \langle\{u,v\}\rangle$
⟨*proof*⟩

lemma *gen-prim*: $p \in \langle\{u,v\}\rangle \Longrightarrow p \in \langle\{u,\varrho\ v\}\rangle$
⟨*proof*⟩

lemma *roots-hull*: **assumes** $w \in \langle\{u^{@}k,v^{@}m\}\rangle$ **shows** $w \in \langle\{u,v\}\rangle$
⟨*proof*⟩

lemma *in-hull-primroots*: $w \in \langle\{x,\ y\}\rangle \Longrightarrow w \in \langle\{\varrho\ x,\ \varrho\ y\}\rangle$
⟨*proof*⟩

lemma *roots-hull-sub*: $\langle\{u^{@}k,v^{@}m\}\rangle \subseteq \langle\{u,v\}\rangle$
⟨*proof*⟩

lemma *primroot-gen*[*simp*,*intro*]: $v \in \langle\{u,\ \varrho\ v\}\rangle$
⟨*proof*⟩

lemma *primroot-gen'*[*simp*, *intro*]: $u \in \langle\{\varrho\ u,\ v\}\rangle$
⟨*proof*⟩

lemma *lists-lists-gen-primroots* [*intro*]: $u \in$ *lists* $\{x,\ y\} \Longrightarrow u \in$ *lists* $\langle\{\varrho\ x,\ \varrho\ y\}\rangle$
⟨*proof*⟩

lemma *dec-primroot-bin-sing* [*intro*]: **assumes** $a \in \{x,y\}$ $x \cdot y \neq y \cdot x$
shows *Dec* $\{\varrho\ x,\ \varrho\ y\}$ $a = [\varrho\ a]^{@}e_{\varrho}\ a$
⟨*proof*⟩

lemma *ref-primroot-bin-sing*: **assumes** $a \in \{x,y\}$ $x \cdot y \neq y \cdot x$
shows *Ref* $\{\varrho\ x,\ \varrho\ y\}$ $[a] = [\varrho\ a]^{@}e_{\varrho}\ a$
⟨*proof*⟩

lemma *lists-sub-mono-gen*: $ws \in$ *lists* $S \Longrightarrow S \subseteq \langle G\rangle \Longrightarrow ws \in$ *lists* $\langle G\rangle$
⟨*proof*⟩

3.9.1 Binary Mismatch tools

definition *bin-mismatch-hard* :: *'a list* ⇒ *'a list* ⇒ *'a list* ⇒ *bool* **where**
bin-mismatch-hard x y w ≡ ∃ *k*. *ϱ x* · (*ϱ x*)^@*k* · *ϱ y* ≤*p w*

lemma *bin-mismatch-hard-def'*: **assumes** *x* · *y* ≠ *y* · *x bin-mismatch-hard x y w*
shows *bin-lcp x y* · [*bin-mismatch x y*] ≤*p w*
⟨*proof*⟩

definition *bin-mismatch-pref* :: *'a list* ⇒ *'a list* ⇒ *'a list* ⇒ *bool* **where**
bin-mismatch-pref x y w ≡ ∃ *k*. *ϱ x*^@*k* · *ϱ y* ≤*p w*

lemma *bm-pref-letter*: **assumes** *x* · *y* ≠ *y* · *x* **and** *bin-mismatch-pref x y* (*w* · *ϱ y*)
shows *bin-lcp x y* · [*bin-mismatch x y*] ≤*p x* · *w* · *bin-lcp x y* (**is** *?α* · [*?c*] ≤*p x* · *w* · *?α*)
⟨*proof*⟩
named-theorems *bm-elims*

lemma *bm-eq1* [*bm-elims*]: **assumes** *x* · *w1* = *y* · *w2* **and** *bin-mismatch-pref x y* (*w1* · *ϱ y*) **and** *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemma *bm-eq2* [*bm-elims*]: **assumes** *ϱ x* · *w1* = *y* · *w2* **and** *bin-mismatch-pref x y* (*w1* · *ϱ y*) **and** *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemma *bm-eq3* [*bm-elims*]: **assumes** *x* · *w1* = *ϱ y* · *w2* **and** *bin-mismatch-pref x y* (*w1* · *ϱ y*) **and** *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemma *bm-eq4* [*bm-elims*]: **assumes** *ϱ x* · *w1* = *ϱ y* · *w2* **and** *bin-mismatch-pref x y* (*w1* · *ϱ y*) **and** *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemma *bm-hard-lcp* [*bm-elims*]: **assumes** *x* · *y* ≠ *y* · *x* **and** *bin-mismatch-hard x y w1* **and** *bin-mismatch-hard y x w2*
shows *w1* ∧*p w2* = *x* · *y* ∧*p y* · *x*
⟨*proof*⟩

lemma *bin-mismatch-pref-ext*: *bin-mismatch-pref x y w1* ⟹ *bin-mismatch-pref x y* (*w1* · *z*)
⟨*proof*⟩

lemma *bm-pref1* [*bm-elims*]: **assumes** *x* · *w1* ≤*p y* · *w2* **and** *bin-mismatch-pref x y w1*

and *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemma *bm-pref2* [*bm-elims*]: **assumes** *ϱ x* · *w1* ≤*p* *y* · *w2* **and** *bin-mismatch-pref x y w1*
and *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemma *bm-pref3* [*bm-elims*]: **assumes** *x* · *w1* ≤*p* *ϱ y* · *w2* **and** *bin-mismatch-pref x y w1*
and *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemma *bm-pref4* [*bm-elims*]: **assumes** *ϱ x* · *w1* ≤*p* *ϱ y* · *w2* **and** *bin-mismatch-pref x y w1*
and *bin-mismatch-pref y x* (*w2* · *ϱ x*)
shows *x* · *y* = *y* · *x*
⟨*proof*⟩

lemmas [*bm-elims*] = *bm-elims*[*symmetric*]

— Binary mismatch predicate evaluation

named-theorems *bm-simps*
lemma *bm-mismatch-rhoI1* [*bm-simps*]: **assumes** *bin-mismatch-pref x y w* **shows** *bin-mismatch-hard x y* (*ϱ x* · *w*)
⟨*proof*⟩

lemma *bm-mismatchI1* [*bm-simps*]: **assumes** *bin-mismatch-pref x y w* **shows** *bin-mismatch-hard x y* (*x* · *w*)
⟨*proof*⟩

lemma *bm-mismatch-rhoI2′* [*bm-simps*]: **assumes** *bin-mismatch-pref x y w* **shows** *bin-mismatch-pref x y* (*ϱ x*$^{@}$*k* · *w*)
⟨*proof*⟩

lemma *bm-mismatch-rhoI2* [*bm-simps*]: **assumes** *bin-mismatch-pref x y w* **shows** *bin-mismatch-pref x y* (*ϱ x* · *w*)
⟨*proof*⟩

lemma *bm-mismatchI2* [*bm-simps*]: **assumes** *bin-mismatch-pref x y w* **shows** *bin-mismatch-pref x y* (*x* · *w*)
⟨*proof*⟩

lemma *bm-mismatchI2′* [*bm-simps*]: **assumes** *bin-mismatch-pref x y w* **shows** *bin-mismatch-pref x y* (*x*$^{@}$*k* · *w*)

⟨*proof*⟩

lemma [*bm-simps*]: *bin-mismatch-pref x y* (*y* · *v*)
⟨*proof*⟩

lemma [*bm-simps*]: *bin-mismatch-pref x y y*
⟨*proof*⟩

lemma [*bm-simps*]: **assumes** *w1* ∈ ⟨{*x*,*y*}⟩ *bin-mismatch-pref x y w*
shows *bin-mismatch-pref x y* (*w1* · *w*)
⟨*proof*⟩

lemma *bm-pref-triv-rho* [*bm-simps*]: *bin-mismatch-pref x y* (ϱ *y* · *w*)
⟨*proof*⟩

lemma *bm-pref-fst-rho* [*bm-simps*]: *bin-mismatch-pref x y* (ϱ *y*)
⟨*proof*⟩

lemma[*bm-simps*]: *x* ∈ ⟨{*x*,*y*}⟩
⟨*proof*⟩
lemma[*bm-simps*]: *y* ∈ ⟨{*x*,*y*}⟩
⟨*proof*⟩
lemma[*bm-simps*]: *w* ∈ ⟨{*x*,*y*}⟩ ⟷ *w* ∈ ⟨{*y*,*x*}⟩
⟨*proof*⟩

lemmas[*bm-simps*] = *hull-closed power-in*
lemmas [*bm-simps*] = *lcp-ext-left*

— the method setup

method *mismatch0* =
(*insert method-facts*, *use nothing* **in**
‹(
((*simp only*: *shifts bm-simps*)*?*,
(*elim bm-elims*);(*simp-all only*: *shifts bm-simps*))
)›
)

lemmas *bm-simps-rev* = *bm-simps*[*reversed*]
lemmas *bm-elims-rev* = *bm-elims*[*reversed*]

method *mismatch-rev* =
(*insert method-facts*, *use nothing* **in**
‹(
((*simp only*: *shifts-rev bm-simps-rev*)*?*,
(*elim bm-elims-rev*);(*simp-all only*: *shifts-rev bm-simps-rev*))
)›
)

method *mismatch* =
(*insert method-facts*, *use nothing* **in**
‹(*mismatch0*;*fail*|*mismatch-rev*)›
)

find-theorems *name*: *bm-elims ?x* $\wedge_p$ *?y*

Mismatch method demonstrations

lemma assumes $x \cdot y \cdot z = y \cdot y \cdot x \cdot v$ **shows** $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma assumes $y \cdot x \cdot x \cdot y \cdot z = (y \cdot x \cdot y) \cdot y \cdot x \cdot v$ **shows** $x \cdot y = y \cdot x$ — cancel
⟨*proof*⟩

lemma $y \cdot y \cdot x \cdot v = x \cdot x \cdot y \cdot z \Longrightarrow x \cdot y = y \cdot x$ — swap
⟨*proof*⟩

lemma $\varrho\ x \cdot \varrho\ x^{@}k \cdot y = y \cdot w \cdot \varrho\ x \Longrightarrow w \in \langle\{x,y\}\rangle \Longrightarrow x \cdot y = y \cdot x$ — primitive root and hull
⟨*proof*⟩

lemma $(r \cdot q) \cdot (r \cdot q)\ ^{@}\ t \cdot r \cdot (q \cdot r \cdot r \cdot q)\ ^{@}\ k = ((r \cdot q)\ ^{@}\ t \cdot r \cdot (q \cdot r \cdot r \cdot q)\ ^{@}\ k) \cdot r \cdot q \Longrightarrow$
$0 < k \Longrightarrow r \cdot q = q \cdot r$ — power
⟨*proof*⟩

lemma $((u \cdot v)\ ^{@}\ m \cdot u) \cdot v \cdot (u \cdot v)\ ^{@}\ k = (v \cdot (u \cdot v)\ ^{@}\ k) \cdot (u \cdot v)\ ^{@}\ m \cdot u \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma $w1 \in \langle\{x,y\}\rangle \Longrightarrow y \cdot x \cdot w2 \cdot z = x \cdot w1 \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $x \cdot y \cdot u \cdot x \cdot y = y \cdot v \cdot y \cdot x \Longrightarrow x \cdot y = y \cdot x$ — reverse mismatch
⟨*proof*⟩

lemma $z \cdot x \cdot y \cdot x \cdot x = v \cdot x \cdot y \cdot y \Longrightarrow y \cdot x = x \cdot y$ — reverse mismatch
⟨*proof*⟩

lemma $k \neq 0 \Longrightarrow j \neq 0 \Longrightarrow (x\ ^{@}\ j \cdot y\ ^{@}\ k) \cdot y = y^{@}k \cdot x\ ^{@}\ j \cdot y\ ^{@}\ (k - 1) \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩
lemma $\varrho\ x \cdot \varrho\ x^{@}k \cdot y \leq p\ y \cdot w \cdot \varrho\ x \Longrightarrow w \in \langle\{x,y\}\rangle \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $y \cdot x \leq p\ x^{@}k \cdot x \cdot y \cdot w \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $0 < k \Longrightarrow y \cdot x \leq_p x^{@}k \cdot y \cdot w \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $0 < k \Longrightarrow 0 < l \Longrightarrow y^{@}l \cdot x \leq_p x^{@}k \cdot y \cdot w \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $0 < k \Longrightarrow 1 < l \Longrightarrow y^{@}l \cdot x \leq_p y \cdot x^{@}k \cdot y \cdot w \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $0 < k \Longrightarrow m < l \Longrightarrow y^{@}l \cdot x \leq_p y^{@}m \cdot x^{@}k \cdot y \cdot w \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma assumes $0 < k$ $m < l$ $y^{@}l \cdot x \leq_p y^{@}m \cdot x^{@}k \cdot y \cdot w$ **shows** $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $w1 \in \langle\{x,y\}\rangle \Longrightarrow w2 \in \langle\{x,y\}\rangle \Longrightarrow x \cdot w2 \cdot y \cdot z = y \cdot w1 \cdot x \cdot v \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma assumes $x \cdot x \cdot y \cdot y \cdot y \cdot y \leq_s z \cdot y \cdot y \cdot x \cdot x$ **shows** $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $u \cdot v \ ^{@} \ k \cdot v \leq_s v \cdot w \cdot u \Longrightarrow w \in \langle\{u, v\}\rangle \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma $u \cdot \varrho\ v \ ^{@} \ k \cdot \varrho\ v \leq_s \varrho\ v \cdot w \cdot u \Longrightarrow w \in \langle\{u, v\}\rangle \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma $u \cdot v \ ^{@} \ k \cdot \varrho\ v \leq_s \varrho\ v \cdot w \cdot u \Longrightarrow w \in \langle\{u, v\}\rangle \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

lemma $2 \leq j \Longrightarrow q \cdot p \cdot q \leq_s (q \cdot p) \ ^{@} \ et' \cdot q \ ^{@} \ j \Longrightarrow p \cdot q = q \cdot p$
⟨*proof*⟩

lemma $w1 \in \langle\{x,y\}\rangle \Longrightarrow w2 \in \langle\{x,y\}\rangle \Longrightarrow x \cdot y \cdot w2 \cdot x \leq_s x \cdot w1 \cdot y \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma $w \in \langle\{x,y\}\rangle \Longrightarrow w' \in \langle\{x,y\}\rangle \Longrightarrow$ *bin-mismatch-pref* $x \ \ y\ (w \cdot w \cdot y)$
⟨*proof*⟩
lemma $x \cdot y = y^{@}k \Longrightarrow 0 < k \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma assumes $x \cdot y \neq y \cdot x$
shows $x \cdot x \cdot y \wedge_p y \cdot y \cdot x = (x \cdot y \wedge_p y \cdot x)$
⟨*proof*⟩

lemma assumes $x \cdot y \neq y \cdot x$

shows $w \cdot z \cdot x \cdot x \cdot y \wedge_p w \cdot z \cdot y \cdot y \cdot x = (w \cdot z) \cdot (x \cdot y \wedge_p y \cdot x)$
⟨*proof*⟩

lemma assumes $x \cdot y \neq y \cdot x$
shows $y \cdot y \cdot x \wedge_p x \cdot x \cdot y = (x \cdot y \wedge_p y \cdot x)$
⟨*proof*⟩

lemma assumes $x \cdot y \neq y \cdot x$
shows $x \cdot x \cdot y \wedge_s y \cdot y \cdot x = (x \cdot y \wedge_s y \cdot x)$
⟨*proof*⟩

lemma assumes $x \cdot y \neq y \cdot x$
shows $x \cdot x \cdot y \cdot z \cdot y \wedge_s y \cdot y \cdot x \cdot z \cdot y = (x \cdot y \wedge_s y \cdot x) \cdot (z \cdot y)$
⟨*proof*⟩

3.9.2 Applied mismatch

lemma assumes $x^{@}j = y^{@}k$ $0 < j$ **shows** $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *pows-comm-comm*: **assumes** $u^{@}k \cdot v^{@}m = u^{@}l \cdot v^{@}n$ $k \neq l$ **shows** $u \cdot v = v \cdot u$
⟨*proof*⟩
lemma *sq-not-pref-mesosome-cover*: **assumes** *eq*: $x \cdot x \cdot concat\ ws = p \cdot x \cdot y^{@}k \cdot x \cdot s$ **and** $ws \in lists\ \{x,y\}$ $p <p\ x\ p \neq \varepsilon$ ‹*1 < k*›
shows $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma (**in** *binary-code*) *bin-mismatch-pows*: ${u_0}^{@}Suc\ k \cdot u_1 \cdot z \neq {u_1}^{@}Suc\ l \cdot u_0 \cdot z'$
⟨*proof*⟩

3.10 Free hull

While not every set G of generators is a code, there is a unique minimal free monoid containing it, called the *free hull* of G. It can be defined inductively using the property known as the *stability condition*.

inductive-set *free-hull* :: *'a list set* ⇒ *'a list set* ($\langle\text{-}\rangle_F$)
for G **where**
$\varepsilon \in \langle G\rangle_F$
| *free-gen-in*: $w \in G \Longrightarrow w \in \langle G\rangle_F$
| *w1* $\in \langle G\rangle_F \Longrightarrow$ *w2* $\in \langle G\rangle_F \Longrightarrow$ *w1* $\cdot$ *w2* $\in \langle G\rangle_F$
| *stability*: $p \in \langle G\rangle_F \Longrightarrow q \in \langle G\rangle_F \Longrightarrow p \cdot w \in \langle G\rangle_F \Longrightarrow w \cdot q \in \langle G\rangle_F \Longrightarrow w \in \langle G\rangle_F$ — the stability condition

lemmas [*intro*] = *free-hull.intros*

The defined set indeed is a hull.

lemma *free-hull-hull*[*simp*]: $\langle \langle G \rangle_F \rangle = \langle G \rangle_F$
⟨*proof*⟩

The free hull is always (non-strictly) larger than the hull.

lemma *hull-sub-free-hull*: $\langle G \rangle \subseteq \langle G \rangle_F$
⟨*proof*⟩

On the other hand, it can be proved that the *free basis*, defined as the basis of the free hull, has a (non-strictly) smaller cardinality than the ordinary basis. (See the AFP theory Combinatorics-Words-Graph-Lemma.Graph-Lemma)

definition *free-basis* :: *'a list set* ⇒ *'a list set* ($\mathfrak{B}_F$ - [*54*] *55*)
where *free-basis* $G \equiv \mathfrak{B}\ \langle G \rangle_F$

lemma *basis-gen-hull-free*: $\langle \mathfrak{B}_F\ G \rangle = \langle G \rangle_F$
⟨*proof*⟩

lemma *genset-sub-free*: $G \subseteq \langle G \rangle_F$
⟨*proof*⟩

We have developed two points of view on freeness:

- inductive point of view: to satisfy the stability condition;
- being generated by a code.

We now show their equivalence

First, basis of a free hull is a code.

lemma *free-basis-code*[*simp*]: *code* ($\mathfrak{B}_F\ G$)
⟨*proof*⟩

lemma *gen-in-free-hull*: $x \in G \Longrightarrow x \in \langle \mathfrak{B}_F\ G \rangle$
⟨*proof*⟩

Second, a code generates its free hull.

lemma (**in** *code*) *code-gen-free-hull*: $\langle \mathcal{C} \rangle_F = \langle \mathcal{C} \rangle$
⟨*proof*⟩

That is, a code is its own free basis

lemma (**in** *code*) *code-free-basis*: $\mathcal{C} = \mathfrak{B}_F\ \mathcal{C}$
⟨*proof*⟩

This allows to use the introduction rules of the free hull to prove one of the basic characterizations of the code, called the stability condition

lemma (**in** *code*) *stability*: $p \in \langle \mathcal{C} \rangle \Longrightarrow q \in \langle \mathcal{C} \rangle \Longrightarrow p \cdot w \in \langle \mathcal{C} \rangle \Longrightarrow w \cdot q \in \langle \mathcal{C} \rangle \Longrightarrow w \in \langle \mathcal{C} \rangle$
⟨*proof*⟩

Moreover, the free hull of G is the smallest code-generated hull containing G. In other words, the term free hull is appropriate.

First, several intuitive monotonicity and closure results.

lemma *free-hull-mono*: **assumes** $G \subseteq H$ **shows** $\langle G \rangle_F \subseteq \langle H \rangle_F$
⟨*proof*⟩

lemma *free-hull-idem*: $\langle \langle G \rangle_F \rangle_F = \langle G \rangle_F$
⟨*proof*⟩

lemma *hull-gen-free-hull*: $\langle \langle G \rangle \rangle_F = \langle G \rangle_F$
⟨*proof*⟩

Code is also the free basis of its hull.

lemma (**in** *code*) *code-free-basis-hull*: $\mathcal{C} = \mathfrak{B}_F\ \langle \mathcal{C} \rangle$
⟨*proof*⟩

The minimality of the free hull easily follows.

theorem (**in** *code*) *free-hull-min*: **assumes** $G \subseteq \langle \mathcal{C} \rangle$ **shows** $\langle G \rangle_F \subseteq \langle \mathcal{C} \rangle$
⟨*proof*⟩

theorem *free-hull-inter*: $\langle G \rangle_F = \bigcap \{M.\ G \subseteq M \wedge M = \langle M \rangle_F\}$
⟨*proof*⟩

Decomposition into the free basis is a morphism.

lemma *free-basis-dec-morph*: $u \in \langle G \rangle_F \Longrightarrow v \in \langle G \rangle_F \Longrightarrow$
$Dec\ (\mathfrak{B}_F\ G)\ (u \cdot v) = (Dec\ (\mathfrak{B}_F\ G)\ u) \cdot (Dec\ (\mathfrak{B}_F\ G)\ v)$
⟨*proof*⟩

3.11 Reversing hulls and decompositions

lemma *basis-rev-commute*[*reversal-rule*]: $\mathfrak{B}\ (rev\ `\ G) = rev\ `\ (\mathfrak{B}\ G)$
⟨*proof*⟩

lemma *rev-free-hull-comm*: $\langle rev\ `\ X \rangle_F = rev\ `\ \langle X \rangle_F$
⟨*proof*⟩

lemma *free-basis-rev-commute* [*reversal-rule*]: $\mathfrak{B}_F\ rev\ `\ X = \ rev\ `\ (\mathfrak{B}_F\ X)$
⟨*proof*⟩

lemma *rev-dec*[*reversal-rule*]: **assumes** $x \in \langle X \rangle_F$ **shows** $Dec\ rev\ `\ (\mathfrak{B}_F\ X)\ (rev\ x) = map\ rev\ (rev\ (Dec\ (\mathfrak{B}_F\ X)\ x))$
⟨*proof*⟩

lemma *rev-hd-dec-last-eq*[*reversal-rule*]: **assumes** $x \in X$ **and** $x \neq \varepsilon$ **shows**
$rev\ (hd\ (Dec\ (rev\ `\ (\mathfrak{B}_F\ X))\ (rev\ x))) = last\ (Dec\ \mathfrak{B}_F\ X\ x)$
⟨*proof*⟩

lemma *rev-hd-dec-last-eq'*[*reversal-rule*]: **assumes** $x \in X$ **and** $x \neq \varepsilon$ **shows**
(*hd* (*Dec* (*rev* ‘ ($\mathfrak{B}_F$ *X*)) (*rev x*))) = *rev* (*last* (*Dec* $\mathfrak{B}_F$ *X x*))
⟨*proof*⟩

3.12 Lists as the free hull of singletons

A crucial property of free monoids of words is that they can be seen as lists over the free basis, instead as lists over the original alphabet.

abbreviation *sings* **where** *sings B* ≡ {[*b*] | *b*. *b* ∈ *B*}

term *Set.filter P A*

lemma *sings-image*: *sings B* = (λ *x*. [*x*]) ‘ *B*
⟨*proof*⟩

lemma *lists-sing-map-concat-ident*: *xs* ∈ *lists* (*sings B*) ⟹ *xs* = *map* (λ *x*. [*x*]) (*concat xs*)
⟨*proof*⟩

lemma *code-sings*: *code* (*sings B*)
⟨*proof*⟩

lemma *sings-gen-lists*: ⟨*sings B*⟩ = *lists B*
⟨*proof*⟩

lemma *sing-gen-lists*: *lists* {*x*} = ⟨{[*x*]}⟩
⟨*proof*⟩

lemma *bin-gen-lists*: *lists* {*x*, *y*} = ⟨{[*x*],[*y*]}⟩
⟨*proof*⟩

lemma *sings B* = $\mathfrak{B}_F$ (*lists B*)
⟨*proof*⟩

lemma *map-sings*: *xs* ∈ *lists B* ⟹ *map* (λ*x*. *x* # ε) *xs* ∈ *lists* (*sings B*)
⟨*proof*⟩

lemma *dec-sings*: *xs* ∈ *lists B* ⟹ *Dec* (*sings B*) *xs* = *map* (λ *x*. [*x*]) *xs*
⟨*proof*⟩

lemma *sing-lists-exp*: **assumes** *ws* ∈ *lists* {*a*}
obtains *k* **where** *ws* = $[a]^{@}k$
⟨*proof*⟩

lemma *sing-lists-exp-len*: *ws* ∈ *lists* {*a*} ⟷ $[a]^{@}|ws|$ = *ws*
⟨*proof*⟩

lemma *sing-lists-exp-count*: *ws* ∈ *lists* {*a*} ⟷ $[a]^{@}$(*count-list ws a*) = *ws*

⟨*proof*⟩

lemma *sing-set-pow-count-list*: *set ws* ⊆ {*a*} ⟷ [*a*]^@(*count-list ws a*) = *ws*
⟨*proof*⟩

lemma *count-le-length-iff*: *set ws* ⊆ {*a*} ⟷ *count-list ws a* = |*ws*|
⟨*proof*⟩

lemma *sing-set-pow*: *set ws* ⊆ {*a*} ⟷ [*a*]^@|*ws*| = *ws*
⟨*proof*⟩

lemma *count-sing-exp*[*simp*]: *count-list* ([*a*]^@*k*) *a* = *k*
⟨*proof*⟩

lemma *count-sing-exp′*[*simp*]: *count-list* ([*a*]) *a* = *1*
⟨*proof*⟩

lemma *count-sing-distinct*[*simp*]: *a* ≠ *b* ⟹ *count-list* ([*a*]^@*k*) *b* = *0*
⟨*proof*⟩

lemma *count-sing-distinct′*[*simp*]: *a* ≠ *b* ⟹ *count-list* ([*a*]) *b* = *0*
⟨*proof*⟩

lemma *sing-letter-imp-prim*: **assumes** *count-list w a* = *1* **shows** *primitive w*
⟨*proof*⟩

lemma *prim-abk*: *a* ≠ *b* ⟹ *primitive* ([*a*] · [*b*] ^@ *k*)
⟨*proof*⟩

lemma *sing-code*: *x* ≠ *ε* ⟹ *code* {*x*}
⟨*proof*⟩

lemma *sings-card*: *card A* = *card* (*sings A*)
⟨*proof*⟩

lemma *sings-finite*: *finite A* = *finite* (*sings A*)
⟨*proof*⟩

lemma *sings-conv*: *A* = *B* ⟷ *sings A* = *sings B*
⟨*proof*⟩

3.13 Various additional lemmas

3.13.1 Roots of binary set

lemma *two-roots-code*: **assumes** *x* ≠ *ε* **and** *y* ≠ *ε* **shows** *code* {*ϱ x*, *ϱ y*}
⟨*proof*⟩

lemma *primroot-in-set-dec*: **assumes** *x* ≠ *ε* **and** *y* ≠ *ε* **shows** *ϱ x* ∈ *set* (*Dec* {*ϱ*

$x,\ \varrho\ y\}\ x)$
⟨*proof*⟩

lemma *primroot-dec*: **assumes** $x \cdot y \neq y \cdot x$
shows $(Dec\ \{\varrho\ x,\ \varrho\ y\}\ x) = [\varrho\ x]^{@} e_{\varrho}\ x\ (Dec\ \{\varrho\ x,\ \varrho\ y\}\ y) = [\varrho\ y]^{@} e_{\varrho}\ y$
⟨*proof*⟩

3.13.2 Other

lemma *bin-count-one-decompose*: **assumes** $ws \in lists\ \{x,y\}$ **and** $x \neq y$ **and** *count-list ws y = 1*
obtains $k\ m$ **where** $[x]^{@} k \cdot [y] \cdot [x]^{@} m = ws$
⟨*proof*⟩

lemma *bin-count-one-conjug*: **assumes** $ws \in lists\ \{x,y\}$ **and** $x \neq y$ **and** *count-list ws y = 1*
shows $ws \sim [x]^{@}(count\text{-}list\ ws\ x) \cdot [y]$
⟨*proof*⟩

lemma *bin-prim-long-set*: **assumes** $ws \in lists\ \{x,y\}$ **and** *primitive ws* **and** $2 \leq |ws|$
shows $set\ ws = \{x,y\}$
⟨*proof*⟩

lemma *bin-prim-long-pref*: **assumes** $ws \in lists\ \{x,y\}$ **and** *primitive ws* **and** $2 \leq |ws|$
obtains ws' **where** $ws \sim ws'$ **and** $[x,y] \leq p\ ws'$
⟨*proof*⟩

end

theory *Morphisms*

imports *CoWBasic Submonoids*

begin

Chapter 4

Morphisms

4.1 One morphism

4.1.1 Morphism, core map and extension

definition *list-extension* :: $('a \Rightarrow 'b\ list) \Rightarrow ('a\ list \Rightarrow 'b\ list)$ $(\text{-}^{\mathcal{L}}\ [1000]\ 1000)$
where $t^{\mathcal{L}} \equiv (\lambda\ x.\ concat\ (map\ t\ x))$

definition *morphism-core* :: $('a\ list \Rightarrow 'b\ list) \Rightarrow ('a\ \Rightarrow 'b\ list)$ $(\text{-}^{\mathcal{C}}\ [1000]\ 1000)$
where *core-def*: $f^{\mathcal{C}} \equiv (\lambda\ x.\ f\ [x])$

lemma *core-sing*: $f^{\mathcal{C}}\ a = f\ [a]$
⟨*proof*⟩

lemma *range-map-core*: $range\ (map\ f^{\mathcal{C}}) = lists\ (range\ f^{\mathcal{C}})$
⟨*proof*⟩

lemma *map-core-lists*[*simp*]: $(map\ f^{\mathcal{C}}\ w) \in lists\ (range\ f^{\mathcal{C}})$
⟨*proof*⟩

lemma *comp-core*: $(f \circ g)^{\mathcal{C}} = f \circ g^{\mathcal{C}}$
⟨*proof*⟩

lemma *ext-core-id* [*simp*]: $f^{\mathcal{LC}} = f$
⟨*proof*⟩

locale *morphism-on* =
fixes $f :: 'a\ list \Rightarrow 'b\ list$ **and** $A :: 'a\ list\ set$
assumes *morph-on*: $u \in \langle A \rangle \Longrightarrow v \in \langle A \rangle \Longrightarrow f\ (u \cdot v) = f\ u \cdot f\ v$

begin

lemma *emp-to-emp-on*[*simp*]: $f\ \varepsilon = \varepsilon$
⟨*proof*⟩

lemma *emp-to-emp'*: $w = \varepsilon \Longrightarrow f\ w = \varepsilon$
⟨*proof*⟩

lemma *morph-concat-concat-map-on*: $ws \in lists\ \langle A\rangle \Longrightarrow f\ (concat\ ws) = concat\ (map\ f\ ws)$
⟨*proof*⟩

lemma *hull-im-hull*:
shows $\langle f\ `\ A\rangle = f\ `\ \langle A\rangle$
⟨*proof*⟩

lemma *inj-basis-to-basis*: **assumes** *inj-on f* $\langle A\rangle$
shows $f\ `\ (\mathfrak{B}\ \langle A\rangle) = \mathfrak{B}\ (f`\langle A\rangle)$
⟨*proof*⟩

lemma *inj-code-to-code*: **assumes** *inj-on f* $\langle A\rangle$ **and** *code A*
shows *code* $(f\ `\ A)$
⟨*proof*⟩

end

locale *morphism* =
fixes $f :: 'a\ list \Rightarrow 'b\ list$
assumes *morph*: $f\ (u \cdot v) = f\ u \cdot f\ v$
begin

sublocale *morphism-on f UNIV*
⟨*proof*⟩

lemmas *emp-to-emp* = *emp-to-emp-on*

lemma *pow-morph*: $f\ (x^{@}k) = (f\ x)^{@}k$
⟨*proof*⟩

lemma *rev-map-pow*: $(rev\text{-}map\ f)\ (w^{@}n) = rev\ ((f\ (rev\ w))^{@}n)$
⟨*proof*⟩

lemma *pop-hd*: $f\ (a\#u) = f\ [a] \cdot f\ u$
⟨*proof*⟩

lemma *pop-hd-nemp*: $u \neq \varepsilon \Longrightarrow f\ (u) = f\ [hd\ u] \cdot f\ (tl\ u)$
⟨*proof*⟩

lemma *pop-last-nemp*: $u \neq \varepsilon \Longrightarrow f\ (u) = f\ (butlast\ u) \cdot f\ [last\ u]$
⟨*proof*⟩

lemma *pref-mono*: $u \leq p\ v \Longrightarrow f\ u \leq p\ f\ v$
⟨*proof*⟩

lemma *suf-mono*: $u \leq s\ v \Longrightarrow f\ u \leq s\ f\ v$
⟨*proof*⟩

lemma *morph-concat-map*: $concat\ (map\ f^{\mathcal{C}}\ x) = f\ x$
⟨*proof*⟩

lemma *morph-concat-map'*: $(\lambda\ x.\ concat\ (map\ f^{\mathcal{C}}\ x)) = f$
⟨*proof*⟩

lemma *morph-to-concat*:
obtains *xs* **where** $xs \in lists\ (range\ f^{\mathcal{C}})$ **and** $f\ x = concat\ xs$
⟨*proof*⟩

lemma *range-hull*: $range\ f = \langle (range\ f^{\mathcal{C}})\rangle$
⟨*proof*⟩

lemma *im-in-hull*: $f\ w \in \langle (range\ f^{\mathcal{C}})\rangle$
⟨*proof*⟩

lemma *core-ext-id*: $f^{\mathcal{C}\mathcal{L}} = f$
⟨*proof*⟩

lemma *rev-map-morph*: $morphism\ (rev\text{-}map\ f)$
⟨*proof*⟩

lemma *morph-rev-len*: $|f\ (rev\ u)| = |f\ u|$
⟨*proof*⟩

lemma *rev-map-len*: $|rev\text{-}map\ f\ u| = |f\ u|$
⟨*proof*⟩

lemma *in-set-morph-len*: **assumes** $a \in set\ w$ **shows** $|f\ [a]| \leq |f\ w|$
⟨*proof*⟩

lemma *morph-lq-comm*: $u \leq p\ v \Longrightarrow f\ (u^{-1>}v) = (f\ u)^{-1>}(f\ v)$
⟨*proof*⟩

lemma *morph-rq-comm*: **assumes** $v \leq s\ u$
shows $f\ (u^{<-1}v) = (f\ u)^{<-1}(f\ v)$
⟨*proof*⟩

lemma *code-set-morph*: **assumes** *c*: $code\ (f^{\mathcal{C}}\ `(set\ (u \cdot v)))$ **and** *i*: $inj\text{-}on\ f^{\mathcal{C}}\ (set\ (u \cdot v))$
and $f\ u = f\ v$
shows $u = v$
⟨*proof*⟩

lemma *morph-concat-concat-map*: $f\ (concat\ ws) = concat\ (map\ f\ ws)$

⟨*proof*⟩

lemma *morph-on-all*: *morphism-on f A*
⟨*proof*⟩

lemma *noner-sings-conv*: $(\forall\ w.\ w = \varepsilon \longleftrightarrow f\ w = \varepsilon) \longleftrightarrow (\forall\ a.\ f\ [a] \neq \varepsilon)$
⟨*proof*⟩

lemma *fac-mono*: $u \leq f\ w \Longrightarrow f\ u \leq f\ f\ w$
⟨*proof*⟩

lemma *set-core-set*: $set\ (f\ w) = \bigcup\ (set\ `\ f^{\mathcal{C}}\ `\ (set\ w))$
⟨*proof*⟩

end

lemma *morph-map*: *morphism* (*map f*)
⟨*proof*⟩

lemma *list-ext-morph*: $morphism\ t^{\mathcal{L}}$
⟨*proof*⟩

lemma *ext-def-on-set*[*intro*]: $(\bigwedge\ a.\ a \in set\ u \Longrightarrow g\ a = f\ a) \Longrightarrow g^{\mathcal{L}}\ u = f^{\mathcal{L}}\ u$
⟨*proof*⟩

lemma *morph-def-on-set*: *morphism f* $\Longrightarrow$ *morphism g* $\Longrightarrow$ $(\bigwedge\ a.\ a \in set\ u \Longrightarrow g^{\mathcal{C}}\ a = f^{\mathcal{C}}\ a) \Longrightarrow g\ u = f\ u$
⟨*proof*⟩

lemma *morph-compose*: *morphism f* $\Longrightarrow$ *morphism g* $\Longrightarrow$ *morphism* $(f \circ g)$
⟨*proof*⟩

4.1.2 periodic morphism

locale *periodic-morphism* = *morphism* +
assumes *ims-comm*: $\bigwedge\ u\ v.\ f\ u \cdot f\ v = f\ v \cdot f\ u$ **and**
not-triv-emp: $\neg\ (\forall\ c.\ f\ [c] = \varepsilon)$
begin

lemma *per-morph-root-ex*:
$\exists\ r.\ \forall\ u.\ \exists\ n.\ f\ u = r^{@}n \wedge primitive\ r$
⟨*proof*⟩

definition *mroot* **where** $mroot \equiv (SOME\ r.\ (\forall\ u.\ \exists\ n.\ f\ u = r^{@}n) \wedge primitive\ r)$
definition *im-exp* :: $'a\ list \Rightarrow nat$ **where** $im\text{-}exp\ u \equiv (SOME\ n.\ f\ u = mroot^{@}n)$

lemma *per-morph-rootI*: $\forall\ u.\ \exists\ n.\ f\ u = mroot^{@}n$ **and**
per-morph-root-prim: *primitive mroot*
⟨*proof*⟩

lemma *per-morph-im-exp*:
$f\ u = mroot^{@}\, im\text{-}exp\ u$
⟨*proof*⟩

lemma *mroot-primroot*: **assumes** $f\ u \neq \varepsilon$
shows $mroot = \varrho\ (f\ u)$
⟨*proof*⟩

interpretation *mirror*: *periodic-morphism rev-map f*
⟨*proof*⟩

lemma *per-morph-rev-map*: $rev\text{-}map\ f\ u = rev\ (f\ u)$
⟨*proof*⟩

lemma *mroot-rev*: $mirror.mroot = rev\ mroot$
⟨*proof*⟩

end

4.1.3 Non-erasing morphism

locale *nonerasing-morphism* = *morphism* +
assumes *nonerasing*: $f\ w = \varepsilon \Longrightarrow w = \varepsilon$
begin

lemma *core-nemp*: $f^{\mathcal{C}}\ a \neq \varepsilon$
⟨*proof*⟩

lemma *nemp-to-nemp*: $w \neq \varepsilon \Longrightarrow f\ w \neq \varepsilon$
⟨*proof*⟩

lemma *sing-to-nemp*: $f\ [a] \neq \varepsilon$
⟨*proof*⟩

lemma *pref-morph-pref-eq*: $u \leq p\ v \Longrightarrow f\ v \leq p\ f\ u \Longrightarrow u = v$
⟨*proof*⟩

lemma *comm-eq-im-eq*:
$u \cdot v = v \cdot u \Longrightarrow f\ u = f\ v \Longrightarrow u = v$
⟨*proof*⟩

lemma *comm-eq-im-iff* :
assumes $u \cdot v = v \cdot u$
shows $f\ u = f\ v \longleftrightarrow u = v$
⟨*proof*⟩

lemma *rev-map-nonerasing*: *nonerasing-morphism* (*rev-map f*)
⟨*proof*⟩

lemma *first-of-first*: $(f\ (a\ \#\ ws))!0 = f\ [a]!0$
⟨*proof*⟩

lemma *hd-im-hd-hd*: **assumes** $u \neq \varepsilon$ **shows** $hd\ (f\ u) = hd\ (f\ [hd\ u])$
⟨*proof*⟩

lemma *ssuf-mono*: $u <_s v \Longrightarrow f\ u <_s f\ v$
⟨*proof*⟩

lemma *im-len-le*: $|u| \leq |f\ u|$
⟨*proof*⟩

lemma *im-len-eq-iff*: $|u| = |f\ u| \longleftrightarrow (\forall\ c.\ c \in set\ u \longrightarrow |f\ [c]| = 1)$
⟨*proof*⟩

lemma *im-len-less*: $a \in set\ u \Longrightarrow |f\ [a]| \neq 1 \Longrightarrow |u| < |f\ u|$
⟨*proof*⟩

end

lemma (**in** *morphism*) *nonerI*[*intro*]: **assumes** $(\bigwedge\ a.\ f^{\mathcal{C}}\ a \neq \varepsilon)$
shows *nonerasing-morphism f*
⟨*proof*⟩

lemma (**in** *morphism*) *prim-morph-noner*:
assumes *prim-morph*: $\bigwedge u.\ 2 \leq |u| \Longrightarrow primitive\ u \Longrightarrow primitive\ (f\ u)$
and *non-single-dom*: $\exists\, a\ b :: 'a.\ a \neq b$
shows *nonerasing-morphism f*
⟨*proof*⟩

4.1.4 Code morphism

The term "Code morphism" is equivalent to "injective morphism".

Note that this is not equivalent to $code\ (range\ f^{\mathcal{C}})$, since the core can be not injective.

lemma (**in** *morphism*) *code-core-range-inj*: $inj\ f \longleftrightarrow code\ (range\ f^{\mathcal{C}}) \wedge inj\ f^{\mathcal{C}}$
⟨*proof*⟩

locale *code-morphism* = *morphism f* **for** *f* +
assumes *code-morph*: *inj f*

begin

lemma *inj-core*: $inj\ f^{\mathcal{C}}$
⟨*proof*⟩

lemma *sing-im-core*: $f\ [a] \in (range\ f^{\mathcal{C}})$
⟨*proof*⟩

lemma *code-im*: *code* $(range\ f^{\mathcal{C}})$
⟨*proof*⟩

sublocale *code range* $f^{\mathcal{C}}$
⟨*proof*⟩

sublocale *nonerasing-morphism*
⟨*proof*⟩

lemma *code-morph-code*: **assumes** $f\ r = f\ s$ **shows** $r = s$
⟨*proof*⟩

lemma *code-morph-bij*: *bij-betw f UNIV* $\langle(range\ f^{\mathcal{C}})\rangle$
⟨*proof*⟩

lemma *code-morphism-rev-map*: *code-morphism* (*rev-map f*)
⟨*proof*⟩

lemma *morph-on-inj-on*:
morphism-on f A inj-on f A
⟨*proof*⟩

end

lemma (**in** *morphism*) *code-morphismI*: *inj f* $\Longrightarrow$ *code-morphism f*
⟨*proof*⟩

lemma (**in** *nonerasing-morphism*) *code-morphismI'* :
assumes *comm*: $\bigwedge u\ v.\ f\ u = f\ v \Longrightarrow u \cdot v = v \cdot u$
shows *code-morphism f*
⟨*proof*⟩

4.1.5 Prefix code morphism

locale *prefix-code-morphism* = *nonerasing-morphism* +
assumes
pref-free: $f^{\mathcal{C}}\ a \leq p\ f^{\mathcal{C}}\ b \Longrightarrow a = b$

begin

interpretation *prefrange*: *prefix-code* $(range\ f^{\mathcal{C}})$
⟨*proof*⟩

lemma *inj-core*: *inj* $f^{\mathcal{C}}$
⟨*proof*⟩

sublocale *code-morphism*
⟨*proof*⟩

lemma *pref-free-morph*: **assumes** $f\ r \leq_p f\ s$ **shows** $r \leq_p s$
⟨*proof*⟩

end

4.1.6 Marked morphism

locale *marked-morphism* = *nonerasing-morphism* +
assumes
marked-core: $hd\ (f^{\mathcal{C}}\ a) = hd\ (f^{\mathcal{C}}\ b) \Longrightarrow a = b$

begin

lemma *marked-im*: *marked-code* (*range* $f^{\mathcal{C}}$)
⟨*proof*⟩

interpretation *marked-code* (*range* $f^{\mathcal{C}}$)
⟨*proof*⟩

lemmas *marked-morph* = *marked-core*[*unfolded core-sing*]

sublocale *prefix-code-morphism*
⟨*proof*⟩

lemma *hd-im-eq-hd-eq*: **assumes** $u \neq \varepsilon$ **and** $v \neq \varepsilon$ **and** $hd\ (f\ u) = hd\ (f\ v)$
shows $hd\ u = hd\ v$
⟨*proof*⟩

lemma *marked-morph-lcp*: $f\ (r \wedge_p s) = f\ r \wedge_p f\ s$
⟨*proof*⟩

lemma *marked-inj-map*: *inj* $e \Longrightarrow$ *marked-morphism* $((map\ e) \circ f)$
⟨*proof*⟩

end

thm *morphism.nonerI*

lemma (**in** *morphism*) *marked-morphismI*:
$(\bigwedge a.\ f[a] \neq \varepsilon) \Longrightarrow (\bigwedge a\ b.\ a \neq b) \Longrightarrow hd\ (f[a]) \neq hd\ (f[b]) \Longrightarrow$ *marked-morphism* *f*
⟨*proof*⟩

4.1.7 Image length

definition *max-image-length*:: $('a\ list \Rightarrow 'b\ list) \Rightarrow nat$ $(\lceil\text{-}\rceil)$
where *max-image-length* $f = Max\ (length`(range\ f^{\mathcal{C}}))$

definition *min-image-length*::(*'a list* ⇒ *'b list*) ⇒ *nat* (⌊-⌋)
where *min-image-length f* = *Min* (*length*‘(*range* $f^{\mathcal{C}}$))

lemma *max-im-len-id*: ⌈*id*::(*'a list* ⇒ *'a list*)⌉ = *1* **and** *min-im-len-id*: ⌊*id*::(*'a list* ⇒ *'a list*)⌋ = *1*
⟨*proof*⟩

context *morphism*
begin

lemma *max-im-len-le*: *finite* (*length*‘*range* $f^{\mathcal{C}}$) ⟹ |*f z*| ≤ |*z*|∗⌈*f*⌉
⟨*proof*⟩

lemma *max-im-len-le-sing*: **assumes** *finite* (*length*‘*range* $f^{\mathcal{C}}$)
shows |*f* [*a*]| ≤ ⌈*f*⌉
⟨*proof*⟩

lemma *min-im-len-ge*: *finite* (*length*‘*range* $f^{\mathcal{C}}$) ⟹ |*z*| ∗ ⌊*f*⌋ ≤ |*f z*|
⟨*proof*⟩

lemma *max-im-len-comp-le*: **assumes** *finite-f*: *finite* (*length*‘*range* $f^{\mathcal{C}}$) **and**
finite-g: *finite* (*length*‘*range* $g^{\mathcal{C}}$) **and** *morphism g*
shows *finite* (*length* ‘ *range* $(g \circ f)^{\mathcal{C}}$) ⌈*g* ∘ *f*⌉ ≤ ⌈*f*⌉∗⌈*g*⌉
⟨*proof*⟩

lemma *max-im-len-emp*: **assumes** *finite* (*length* ‘ *range* $f^{\mathcal{C}}$)
shows ⌈*f*⌉ = *0* ⟷ (*f* = (λ*w*. *ε*))
⟨*proof*⟩

lemmas *max-im-len-le-dom* = *max-im-len-le*[*OF finite-imageI*, *OF finite-imageI*]
and
max-im-len-le-sing-dom = *max-im-len-le-sing*[*OF finite-imageI*, *OF finite-imageI*]
and
min-im-len-ge-dom = *min-im-len-ge*[*OF finite-imageI*, *OF finite-imageI*] **and**
max-im-len-comp-le-dom = *max-im-len-comp-le*[*OF finite-imageI*, *OF finite-imageI*]
and
max-im-len-emp-dom = *max-im-len-emp*[*OF finite-imageI*, *OF finite-imageI*]

lemma *Cons-im*: *f* (*x*#*xs*) = $f^{\mathcal{C}}$ *x* · *f xs*
⟨*proof*⟩

end

4.1.8 Endomorphism

locale *endomorphism* = *morphism f* **for** *f*:: *'a list* ⇒ *'a list*
begin

lemma *pow-endomorphism*: *endomorphism* $(f^{\frown\frown}k)$
⟨*proof*⟩

interpretation *pow-endm*: *endomorphism* $(f^{\frown\frown}k)$
⟨*proof*⟩

lemmas *pow-morphism* = *pow-endm.morphism-axioms* **and**
pow-morph = *pow-endm.morph* **and**
pow-emp-to-emp = *pow-endm.emp-to-emp*

lemma *pow-sets-im*: *set* w = *set* $v \Longrightarrow$ *set* $((f^{\frown\frown}k)\ w)$ = *set* $((f^{\frown\frown}k)\ v)$
⟨*proof*⟩

lemma *fin-len-ran-pow*: *finite* (*length* ‘ *range* $f^{\mathcal{C}}$) $\Longrightarrow$ *finite* (*length* ‘ *range* $(f^{\frown\frown}k)^{\mathcal{C}}$)
⟨*proof*⟩

lemma *max-im-len-pow-le*: **assumes** *finite* (*length* ‘ *range* $f^{\mathcal{C}}$) **shows** $\lceil f^{\frown\frown}k\rceil \leq \lceil f\rceil\ ^\frown k$
⟨*proof*⟩

lemma *max-im-len-pow-le′*: *finite* (*length* ‘ *range* $f^{\mathcal{C}}$) $\Longrightarrow |(f^{\frown\frown}k)\ w| \leq |w| * \lceil f\rceil\ ^\frown k$
⟨*proof*⟩

lemmas *max-im-len-pow-le-dom* = *max-im-len-pow-le*[*OF finite-imageI*, *OF finite-imageI*] **and**
max-im-len-pow-le′-dom = *max-im-len-pow-le′*[*OF finite-imageI*, *OF finite-imageI*]

lemma *funpow-nonerasing-morphism*: **assumes** *nonerasing-morphism f*
shows *nonerasing-morphism* $(f^{\frown\frown}k)$
⟨*proof*⟩

lemma *im-len-pow-mono*: **assumes** *nonerasing-morphism f* $i \leq j$
shows $(|(f^{\frown\frown}i)\ w| \leq |(f^{\frown\frown}j)\ w|)$
⟨*proof*⟩

lemma *fac-mono-pow*: $u \leq_f (f^{\frown\frown}k)\ w \Longrightarrow (f^{\frown\frown}l)\ u \leq_f (f^{\frown\frown}(l+k))\ w$
⟨*proof*⟩

lemma *rev-map-endomorph*: *endomorphism* (*rev-map f*)
⟨*proof*⟩

end

4.1.9 Decomposition into primitive roots

definition *root-refine* (Ref_ϱ - [55] 56) **where** *root-refine* = $(\lambda\ x.\ [\varrho\ x]^{@} e_\varrho\ x)^{\mathcal{L}}$

lemma *root-ref-morph*: *morphism root-refine*
⟨*proof*⟩

thm *morphism.morph-concat-concat-map*

interpretation *rr-morph*: *morphism root-refine*
⟨*proof*⟩

lemma *root-ref-def*: $Ref_\varrho\ us = concat\ (map\ (\lambda\ x.\ [\varrho\ x]^{@} e_\varrho\ x)\ us)$
⟨*proof*⟩

lemma *root-ref-refines*: $concat\ (Ref_\varrho\ ws) = concat\ ws$
⟨*proof*⟩

lemmas *root-ref-emp* = *rr-morph.emp-to-emp*

lemma $Ref_\varrho\ [\varepsilon,\varepsilon] = \varepsilon$
⟨*proof*⟩

lemma *root-ref-empty-conv*: $Ref_\varrho\ us = \varepsilon \longleftrightarrow set\ us \subseteq \{\varepsilon\}$
⟨*proof*⟩

lemma *root-ref-hd*: **assumes** $x \neq \varepsilon$
shows $hd\ (Ref_\varrho\ (x\#xs)) = \varrho\ x$
⟨*proof*⟩

lemma *root-ref-injective*:
assumes
emp-not-in: $\varepsilon \notin \mathcal{C}$ **and**
comm-eq: $\bigwedge\ x\ y.\ x \in \mathcal{C} \Longrightarrow y \in \mathcal{C} \Longrightarrow x \cdot y = y \cdot x \Longrightarrow x = y$ **and**
$us \in lists\ \mathcal{C}\ vs \in lists\ \mathcal{C}$
shows $Ref_\varrho\ us = Ref_\varrho\ vs \Longrightarrow us = vs$
⟨*proof*⟩

lemma (**in** *code*) *code-root-ref-injective*:
assumes $us \in lists\ \mathcal{C}\ vs \in lists\ \mathcal{C}$
shows $Ref_\varrho\ us = Ref_\varrho\ vs \Longrightarrow us = vs$
⟨*proof*⟩

find-theorems *set ?w* = {*?a*}

lemma (**in** *code*) *code-roots-non-overlapping*: *non-overlapping* $((\lambda\ x.\ [\varrho\ x]^{@}(e_\varrho\ x))$ ‘ $\mathcal{C})$
⟨*proof*⟩

lemma (**in** *binary-code*) *bin-roots-sings-code*: *non-overlapping* {*Dec* {ϱ u_0, ϱ u_1} u_0, *Dec* {ϱ u_0, ϱ u_1} u_1}
⟨*proof*⟩

theorem (**in** *code*) *roots-prim-morph*:
assumes *ws* ∈ *lists* $\mathcal{C}$
and |*ws*| ≠ *1*
and *primitive ws*
shows *primitive* (*Ref*$_\varrho$ *ws*)
⟨*proof*⟩

4.2 Primitivity preserving morphisms

locale *primitivity-preserving-morphism* = *nonerasing-morphism* +
assumes *prim-morph* : *2* ≤ |*u*| ⟹ *primitive u* ⟹ *primitive* (*f u*)
begin

sublocale *code-morphism*
⟨*proof*⟩

lemmas *code-morph* = *code-morph*

end

4.3 Two morphisms

Solutions and the coincidence pairs are defined for any two mappings

4.3.1 Solutions

definition *minimal-solution* :: *'a list* ⇒ (*'a list* ⇒ *'b list*) ⇒ (*'a list* ⇒ *'b list*) ⇒ *bool*
(- ∈ - =$_M$ - [*80,80,80*] *51*)
where *min-sol-def*: *minimal-solution s g h* ≡ *s* ≠ *ε* ∧ *g s* = *h s*
∧ (∀ *s'*. *s'* ≠ *ε* ∧ *s'* ≤*p s* ∧ *g s'* = *h s'* ⟶ *s'* = *s*)

lemma *min-solD*: *s* ∈ *g* =$_M$ *h* ⟹ *g s* = *h s*
⟨*proof*⟩

lemma *min-solD'*: *s* ∈ *g* =$_M$ *h* ⟹ *s* ≠ *ε*
⟨*proof*⟩

lemma *min-solD-min*: *s* ∈ *g* =$_M$ *h* ⟹ *p* ≠ *ε* ⟹ *p* ≤*p s* ⟹ *g p* = *h p* ⟹ *p* = *s*
⟨*proof*⟩

lemma *min-solI*[*intro*]: *s* ≠ *ε* ⟹ *g s* = *h s* ⟹ (⋀ *s'*. *s'*≤*p s* ⟹ *s'* ≠ *ε* ⟹ *g s'* = *h s'* ⟹ *s'* = *s*) ⟹ *s* ∈ *g* =$_M$ *h*

⟨*proof*⟩

lemma *min-sol-sym-iff*: $s \in g =_M h \longleftrightarrow s \in h =_M g$
⟨*proof*⟩

lemma *min-sol-sym*[*sym*]: $s \in g =_M h \Longrightarrow s \in h =_M g$
⟨*proof*⟩

lemma *min-sol-prefE*:
assumes $g\ r = h\ r$ **and** $r \neq \varepsilon$
obtains e **where** $e \in g =_M h$ **and** $e \leq p\ r$
⟨*proof*⟩

4.3.2 Coincidence pairs

definition *coincidence-set* :: (*'a list* ⇒ *'b list*) ⇒ (*'a list* ⇒ *'b list*) ⇒ (*'a list* × *'a list*) *set* ($\mathfrak{C}$)
where *coincidence-set g h* ≡ {(*r*,*s*). *g r* = *h s*}

lemma *coin-set-eq*: (*g* ∘ *fst*) ‘($\mathfrak{C}$ *g h*) = (*h* ∘ *snd*) ‘($\mathfrak{C}$ *g h*)
⟨*proof*⟩

lemma *coin-setD*: *pair* ∈ $\mathfrak{C}$ *g h* ⟹ *g* (*fst pair*) = *h* (*snd pair*)
⟨*proof*⟩

lemma *coin-setD-iff*: *pair* ∈ $\mathfrak{C}$ *g h* ⟷ *g* (*fst pair*) = *h* (*snd pair*)
⟨*proof*⟩

lemma *coin-set-sym*: *fst*‘($\mathfrak{C}$ *g h*) = *snd* ‘($\mathfrak{C}$ *h g*)
⟨*proof*⟩

lemma *coin-set-inter-fst*: (*g* ∘ *fst*) ‘($\mathfrak{C}$ *g h*) = *range g* ∩ *range h*
⟨*proof*⟩

lemmas *coin-set-inter-snd* = *coin-set-inter-fst*[*unfolded coin-set-eq*]

definition *minimal-coincidence* :: (*'a list* ⇒ *'b list*) ⇒ *'a list* ⇒ (*'a list* ⇒ *'b list*) ⇒ *'a list* ⇒ *bool* ((- -) =$_m$ (- -) [*80*,*81*,*80*,*81*] *51*)
where *min-coin-def*: *minimal-coincidence g r h s* ≡ $r \neq \varepsilon \wedge s \neq \varepsilon \wedge g\ r = h\ s \wedge (\forall\ r'\ s'.\ r' \leq p\ r \wedge r' \neq \varepsilon \wedge s' \leq p\ s \wedge s' \neq \varepsilon \wedge g\ r' = h\ s' \longrightarrow r' = r \wedge s' = s)$

definition *min-coincidence-set* :: (*'a list* ⇒ *'b list*) ⇒ (*'a list* ⇒ *'b list*) ⇒ (*'a list* × *'a list*) *set* ($\mathfrak{C}_m$)
where *min-coincidence-set g h* ≡ ({(*r*,*s*) . *g r* =$_m$ *h s*})

lemma *min-coin-minD*: *g r* =$_m$ *h s* ⟹ $r' \leq p\ r \Longrightarrow r' \neq \varepsilon \Longrightarrow s' \leq p\ s \Longrightarrow s' \neq \varepsilon \Longrightarrow$ *g r'* = *h s'* ⟹ $r' = r \wedge s' = s$
⟨*proof*⟩

lemma *min-coin-setD*: $p \in \mathfrak{C}_m\ g\ h \Longrightarrow g\ (fst\ p) =_m h\ (snd\ p)$
⟨*proof*⟩

lemma *min-coinD*: $g\ r =_m h\ s \Longrightarrow g\ r = h\ s$
⟨*proof*⟩

lemma *min-coinD'*: $g\ r =_m h\ s \Longrightarrow r \neq \varepsilon \wedge s \neq \varepsilon$
⟨*proof*⟩

lemma *min-coin-sub*: $\mathfrak{C}_m\ g\ h \subseteq \mathfrak{C}\ g\ h$
⟨*proof*⟩

lemma *min-coin-defI*: **assumes** $r \neq \varepsilon$ **and** $s \neq \varepsilon$ **and** $g\ r = h\ s$ **and**
$(\bigwedge r'\ s'.\ r' \leq p\ r \Longrightarrow r' \neq \varepsilon \Longrightarrow s' \leq p\ s \Longrightarrow s' \neq \varepsilon \Longrightarrow g\ r' = h\ s' \Longrightarrow r' = r \wedge s' = s)$
shows $g\ r =_m h\ s$
⟨*proof*⟩

lemma *min-coin-sym*[*sym*]: $g\ r =_m h\ s \Longrightarrow h\ s =_m g\ r$
⟨*proof*⟩

lemma *min-coin-sym-iff*: $g\ r =_m h\ s \longleftrightarrow h\ s =_m g\ r$
⟨*proof*⟩

lemma *min-coin-set-sym*: $fst`(\mathfrak{C}_m\ g\ h) = snd`(\mathfrak{C}_m\ h\ g)$
⟨*proof*⟩

4.3.3 Basics

locale *two-morphisms* = *g*: *morphism g* + *h*: *morphism h* **for** *g h* :: *'a list* ⇒ *'b list*

begin

lemma *def-on-sings*:
assumes $\bigwedge a.\ a \in set\ u \Longrightarrow g\ [a] = h\ [a]$
shows $g\ u = h\ u$
⟨*proof*⟩

lemma *def-on-sings-eq*:
assumes $\bigwedge a.\ g\ [a] = h\ [a]$
shows $g = h$
⟨*proof*⟩

lemma *ims-prefs-comp*:
assumes $u \leq p\ u'$ **and** $v \leq p\ v'$ **and** $g\ u' \bowtie h\ v'$ **shows** $g\ u \bowtie h\ v$
⟨*proof*⟩

lemma *ims-sufs-comp*:

assumes $u \leq_s u'$ **and** $v \leq_s v'$ **and** $g\ u' \bowtie_s h\ v'$ **shows** $g\ u \bowtie_s h\ v$
⟨*proof*⟩

lemma *ims-hd-eq-comp*:
assumes $u \neq \varepsilon$ **and** $g\ u = h\ u$ **shows** $g\ [hd\ u] \bowtie h\ [hd\ u]$
⟨*proof*⟩

lemma *ims-last-eq-suf-comp*:
assumes $u \neq \varepsilon$ **and** $g\ u = h\ u$ **shows** $g\ [last\ u] \bowtie_s h\ [last\ u]$
⟨*proof*⟩

lemma *len-im-le*:
assumes $(\bigwedge a.\ a \in set\ s \Longrightarrow |g\ [a]| \leq |h\ [a]|)$
shows $|g\ s| \leq |h\ s|$
⟨*proof*⟩

lemma *len-im-less*:
assumes $\bigwedge a.\ a \in set\ s \Longrightarrow |g\ [a]| \leq |h\ [a]|$ **and**
$b \in set\ s$ **and** $|g\ [b]| < |h\ [b]|$
shows $|g\ s| < |h\ s|$
⟨*proof*⟩

lemma *solution-eq-len-eq*:
assumes $g\ s = h\ s$ **and** $\bigwedge a.\ a \in set\ s \Longrightarrow |g\ [a]| = |h\ [a]|$
shows $\bigwedge a.\ a \in set\ s \Longrightarrow g\ [a] = h\ [a]$
⟨*proof*⟩

lemma *rev-maps*: *two-morphisms* (*rev-map g*) (*rev-map h*)
⟨*proof*⟩

lemma *min-solD-min-suf*: **assumes** $sol \in g =_M h$ **and** $s \neq \varepsilon$ $s \leq_s sol$ **and** $g\ s = h\ s$
shows $s = sol$
⟨*proof*⟩

lemma *min-sol-rev*[*reversal-rule*]:
assumes $s \in g =_M h$
shows $(rev\ s) \in (rev\text{-}map\ g) =_M (rev\text{-}map\ h)$
⟨*proof*⟩

lemma *coin-set-lists-concat*: $ps \in lists\ (\mathfrak{C}\ g\ h) \Longrightarrow g\ (concat\ (map\ fst\ ps)) = h\ (concat\ (map\ snd\ ps))$
⟨*proof*⟩

lemma *coin-set-hull*: $\langle snd\ `(\mathfrak{C}\ g\ h)\rangle = snd\ `(\mathfrak{C}\ g\ h)$
⟨*proof*⟩

lemma *min-sol-sufE*:
assumes $g\ r = h\ r$ **and** $r \neq \varepsilon$

obtains e **where** $e \in g =_M h$ **and** $e \leq_s r$
⟨*proof*⟩

lemma *min-sol-primitive*: **assumes** $sol \in g =_M h$ **shows** *primitive sol*
⟨*proof*⟩

lemma *prim-sol-two-sols*:
assumes $g\ u = h\ u$ **and** $\neg\ u \in g =_M h$ **and** *primitive u*
obtains *s1 s2* **where** $s1 \in g =_M h$ **and** $s2 \in g =_M h$ **and** $s1 \neq s2$
⟨*proof*⟩

lemma *prim-sols-two-sols*:
assumes $g\ r = h\ r$ **and** $g\ s = h\ s$ **and** *primitive s* **and** *primitive r* **and** $r \neq s$
obtains *s1 s2* **where** $s1 \in g =_M h$ **and** $s2 \in g =_M h$ **and** $s1 \neq s2$
⟨*proof*⟩

end

4.3.4 Factor intepretation of morphic images

context *morphism*
begin

lemma *image-fac-interp'*: **assumes** $w \leq_f f\ z\ w \neq \varepsilon$
obtains *p w-pred s* **where** *w-pred* $\leq_f z\ p\ w\ s \sim_{\mathcal{I}} (map\ f^{\mathcal{C}}\ w\text{-}pred)$
⟨*proof*⟩

lemma *image-fac-interp*: **assumes** $u \cdot w \cdot v = f\ z\ w \neq \varepsilon$
obtains *p w-pred s u-pred v-pred* **where**
$u\text{-}pred \cdot w\text{-}pred \cdot v\text{-}pred = z\ p\ w\ s \sim_{\mathcal{I}} (map\ f^{\mathcal{C}}\ w\text{-}pred)$
$u = (f\ u\text{-}pred) \cdot p\ v = s \cdot (f\ v\text{-}pred)$
⟨*proof*⟩

lemma *image-fac-interp-mid*: **assumes** $p\ w\ s \sim_{\mathcal{I}} map\ f^{\mathcal{C}}\ w\text{-}pred\ 2 \leq |w\text{-}pred|$
obtains *pw sw* **where**
$w = pw \cdot (f\ (butlast\ (tl\ w\text{-}pred))) \cdot sw\ p \cdot pw = f\ [hd\ w\text{-}pred]\ sw \cdot s = f\ [last\ w\text{-}pred]$
⟨*proof*⟩

end

4.3.5 Two nonerasing morphisms

Minimal coincidence pairs and minimal solutions make good sense for nonerasing morphisms only.

locale *two-nonerasing-morphisms* = *two-morphisms* +
g: *nonerasing-morphism g* +
h: *nonerasing-morphism h*

begin

thm *g.morph*
thm *g.emp-to-emp*

lemma *two-nonerasing-morphisms-swap*: *two-nonerasing-morphisms h g*
⟨*proof*⟩

lemma *noner-eq-emp-iff*: $g\ u = h\ v \Longrightarrow u = \varepsilon \longleftrightarrow v = \varepsilon$
⟨*proof*⟩

lemma *min-coin-rev*:
assumes $g\ r =_m h\ s$
shows $(\mathit{rev\text{-}map}\ g)\ (\mathit{rev}\ r) =_m (\mathit{rev\text{-}map}\ h)\ (\mathit{rev}\ s)$
⟨*proof*⟩

lemma *min-coin-pref-eq*:
assumes $g\ e =_m h\ f$ **and** $g\ e' = h\ f'$ **and** $e' \leq_p e$ **and** $e' \neq \varepsilon$ **and** $f' \bowtie f$
shows $e' = e$ **and** $f' = f$
⟨*proof*⟩

lemma *min-coin-prefE*:
assumes $g\ r = h\ s$ **and** $r \neq \varepsilon$
obtains $e\ f$ **where** $g\ e =_m h\ f$ **and** $e \leq_p r$ **and** $f \leq_p s$ **and** $\mathit{hd}\ e = \mathit{hd}\ r$
⟨*proof*⟩

lemma *min-coin-dec*: **assumes** $g\ e = h\ f$
obtains *ps* **where** $\mathit{concat}\ (\mathit{map}\ \mathit{fst}\ \mathit{ps}) = e$ **and** $\mathit{concat}\ (\mathit{map}\ \mathit{snd}\ \mathit{ps}) = f$ **and**
$\bigwedge p.\ p \in \mathit{set}\ \mathit{ps} \Longrightarrow g\ (\mathit{fst}\ p) =_m h\ (\mathit{snd}\ p)$
⟨*proof*⟩

lemma *min-coin-code*:
assumes $\mathit{xs} \in \mathit{lists}\ (\mathfrak{C}_m\ g\ h)$ **and** $\mathit{ys} \in \mathit{lists}\ (\mathfrak{C}_m\ g\ h)$ **and**
$\mathit{concat}\ (\mathit{map}\ \mathit{fst}\ \mathit{xs}) = \mathit{concat}\ (\mathit{map}\ \mathit{fst}\ \mathit{ys})$ **and**
$\mathit{concat}\ (\mathit{map}\ \mathit{snd}\ \mathit{xs}) = \mathit{concat}\ (\mathit{map}\ \mathit{snd}\ \mathit{ys})$
shows $\mathit{xs} = \mathit{ys}$
⟨*proof*⟩

lemma *coin-closed*: $\mathit{ps} \in \mathit{lists}\ (\mathfrak{C}\ g\ h) \Longrightarrow (\mathit{concat}\ (\mathit{map}\ \mathit{fst}\ \mathit{ps}), \mathit{concat}\ (\mathit{map}\ \mathit{snd}\ \mathit{ps})) \in \mathfrak{C}\ g\ h$
⟨*proof*⟩

lemma *min-coin-gen-snd*: $\langle \mathit{snd}\ `\ (\mathfrak{C}_m\ g\ h)\rangle = \mathit{snd}\ `(\mathfrak{C}\ g\ h)$
⟨*proof*⟩

lemma *min-coin-gen-fst*: $\langle \mathit{fst}\ `\ (\mathfrak{C}_m\ g\ h)\rangle = \mathit{fst}\ `(\mathfrak{C}\ g\ h)$
⟨*proof*⟩

lemma *min-coin-code-snd*:

assumes *code-morphism g* **shows** *code* (*snd* ‘ ($\mathfrak{C}_m$ *g h*))
⟨*proof*⟩

lemma *min-coin-code-fst*:
code-morphism h ⟹ *code* (*fst* ‘ ($\mathfrak{C}_m$ *g h*))
⟨*proof*⟩

lemma *min-coin-basis-snd*:
assumes *code-morphism g*
shows $\mathfrak{B}$ (*snd* ‘($\mathfrak{C}$ *g h*)) = *snd* ‘ ($\mathfrak{C}_m$ *g h*)
⟨*proof*⟩

lemma *min-coin-basis-fst*:
code-morphism h ⟹ $\mathfrak{B}$ (*fst* ‘($\mathfrak{C}$ *g h*)) = *fst* ‘ ($\mathfrak{C}_m$ *g h*)
⟨*proof*⟩

lemma *sol-im-len-less*: **assumes** *g u* = *h u* **and** *g* ≠ *h* **and** *set u* = *UNIV*
shows |*u*| < |*g u*|
⟨*proof*⟩

end

locale *two-code-morphisms* = *g*: *code-morphism g* + *h*: *code-morphism h*
for *g h* :: *'a list* ⇒ *'b list*

begin

sublocale *two-nonerasing-morphisms g h*
⟨*proof*⟩

lemmas *code-morphs* = *g.code-morphism-axioms h.code-morphism-axioms*

lemma *revs-two-code-morphisms*: *two-code-morphisms* (*rev-map g*) (*rev-map h*)
⟨*proof*⟩

lemma *min-coin-im-basis*:
$\mathfrak{B}$ (*h*‘ (*snd* ‘($\mathfrak{C}$ *g h*))) = *h* ‘ *snd* ‘ ($\mathfrak{C}_m$ *g h*)
$\mathfrak{B}$ (*g*‘ (*fst* ‘($\mathfrak{C}$ *g h*))) = *g* ‘ *fst* ‘ ($\mathfrak{C}_m$ *g h*)
⟨*proof*⟩

lemma *range-inter-basis-snd*:
shows $\mathfrak{B}$ (*range g* ∩ *range h*) = *h* ‘ (*snd* ‘ $\mathfrak{C}_m$ *g h*)
$\mathfrak{B}$ (*range g* ∩ *range h*) = *g* ‘ (*fst* ‘ $\mathfrak{C}_m$ *g h*)
⟨*proof*⟩

lemma *range-inter-code*:
shows *code* $\mathfrak{B}$ (*range g* ∩ *range h*)
⟨*proof*⟩

end

4.3.6 Two marked morphisms

locale *two-marked-morphisms* = *two-nonerasing-morphisms* +
  g: *marked-morphism g* + *h*: *marked-morphism h*

begin

sublocale *revs*: *two-code-morphisms g h*
⟨*proof*⟩

lemmas *ne-g* = *g.nonerasing* **and**
  ne-h = *h.nonerasing*

lemma *unique-continuation*:
$z \cdot g\ r = z' \cdot h\ s \Longrightarrow\ z \cdot g\ r' = z' \cdot h\ s' \Longrightarrow z \cdot g\ (r \wedge_p r') = z' \cdot h\ (s \wedge_p s')$
⟨*proof*⟩

lemmas *empty-sol* = *noner-eq-emp-iff*

lemma *comparable-min-sol-eq*: **assumes** $r \leq p\ r'$ **and** $g\ r =_m h\ s$ **and** $g\ r' =_m h\ s'$
shows $r = r'$ **and** $s = s'$
⟨*proof*⟩

lemma *first-letter-determines*:
assumes $g\ e =_m h\ f$ **and** $g\ e' = h\ f'$ **and** $hd\ e = hd\ e'$ **and** $e' \neq \varepsilon$
shows $e \leq p\ e'$ **and** $f \leq p\ f'$
⟨*proof*⟩

corollary *first-letter-determines'*:
assumes $g\ e =_m h\ f$ **and** $g\ e' =_m h\ f'$ **and** $hd\ e = hd\ e'$
shows $e = e'$ **and** $f = f'$
⟨*proof*⟩

lemma *first-letter-determines-sol*: **assumes** $r \in g =_M h$ **and** $s \in g =_M h$ **and** $hd\ r = hd\ s$
shows $r = s$
⟨*proof*⟩

definition *pre-block* :: *'a* ⇒ *'a list* × *'a list*
where *pre-block a* = (*THE p*. (*g* (*fst p*) =$_m$ *h* (*snd p*)) ∧ *hd* (*fst p*) = *a*)
— *pre-block a* may not be a block, if no such exists

definition *blockP* :: *'a* ⇒ *bool*
where *blockP a* ≡ *g* (*fst* (*pre-block a*)) =$_m$ *h* (*snd* (*pre-block a*)) ∧ *hd* (*fst* (*pre-block a*)) = *a*
— Predicate: the *pre-block* of the letter *a* is indeed a block

lemma *pre-blockI*: $g\ u =_m h\ v \Longrightarrow$ *pre-block* $(hd\ u) = (u,v)$
⟨*proof*⟩

lemma *blockI*: **assumes** $g\ u =_m h\ v$ **and** $hd\ u = a$
shows *blockP* a
⟨*proof*⟩

lemma *hd-im-comm-eq-aux*:
assumes $g\ w = h\ w$ **and** $w \neq \varepsilon$ **and** *comm*: $g^{\mathcal{C}}\ (hd\ w) \cdot h^{\mathcal{C}}(hd\ w) = h^{\mathcal{C}}\ (hd\ w) \cdot g^{\mathcal{C}}(hd\ w)$ **and** *len*: $|g^{\mathcal{C}}\ (hd\ w)| \leq |h^{\mathcal{C}}(hd\ w)|$
shows $g^{\mathcal{C}}\ (hd\ w) = h^{\mathcal{C}}\ (hd\ w)$
⟨*proof*⟩

lemma *hd-im-comm-eq*:
assumes $g\ w = h\ w$ **and** $w \neq \varepsilon$ **and** *comm*: $g^{\mathcal{C}}(hd\ w) \cdot h^{\mathcal{C}}(hd\ w) = h^{\mathcal{C}}(hd\ w) \cdot g^{\mathcal{C}}(hd\ w)$
shows $g^{\mathcal{C}}\ (hd\ w) = h^{\mathcal{C}}\ (hd\ w)$
⟨*proof*⟩

lemma *block-ex*: **assumes** $g\ u =_m h\ v$ **shows** *blockP* $(hd\ u)$
⟨*proof*⟩

lemma *sol-block-ex*: **assumes** $g\ u = h\ v$ **and** $u \neq \varepsilon$ **shows** *blockP* $(hd\ u)$
⟨*proof*⟩

definition *suc-fst* **where** *suc-fst* $\equiv \lambda\ x.$ *concat*(*map* $(\lambda\ y.$ *fst* (*pre-block* $y))\ x)$
definition *suc-snd* **where** *suc-snd* $\equiv \lambda\ x.$ *concat*(*map* $(\lambda\ y.$ *snd* (*pre-block* $y))\ x)$

lemma *blockP-D*: *blockP* $a \Longrightarrow g$ (*suc-fst* $[a]) =_m h$ (*suc-snd* $[a]$)
⟨*proof*⟩

lemma *blockP-D-hd*: *blockP* $a \Longrightarrow hd$ (*suc-fst* $[a]) = a$
⟨*proof*⟩

abbreviation *blocks* $\tau \equiv (\forall\ a.\ a \in set\ \tau \longrightarrow$ *blockP* $a)$

sublocale *sucs*: *two-morphisms suc-fst suc-snd*
⟨*proof*⟩

lemma *blockP-D-hd-hd*: **assumes** *blockP* a
shows $hd\ (h^{\mathcal{C}}\ (hd$ (*suc-snd* $[a]))) = hd\ (g^{\mathcal{C}}\ a)$
⟨*proof*⟩

lemma *suc-morph-sings*: **assumes** $g\ e =_m h\ f$
shows *suc-fst* $[hd\ e] = e$ **and** *suc-snd* $[hd\ e] = f$
⟨*proof*⟩

lemma *blocks-eq*: *blocks* $\tau \Longrightarrow g\ (\textit{suc-fst}\ \tau) = h\ (\textit{suc-snd}\ \tau)$
⟨*proof*⟩

lemma *suc-eq'*: **assumes** $\bigwedge a.\ \textit{blockP}\ a$ **shows** $g(\textit{suc-fst}\ w) = h(\textit{suc-snd}\ w)$
⟨*proof*⟩

lemma *suc-eq*: **assumes** *all-blocks*: $\bigwedge a.\ \textit{blockP}\ a$ **shows** $g \circ \textit{suc-fst} = h \circ \textit{suc-snd}$
⟨*proof*⟩

lemma *block-eq*: $\textit{blockP}\ a \Longrightarrow g\ (\textit{suc-fst}\ [a]) = h\ (\textit{suc-snd}\ [a])$
⟨*proof*⟩

lemma *blocks-inj-suc*: **assumes** *blocks* τ **shows** *inj-on* $\textit{suc-fst}^{\mathcal{C}}\ (\textit{set}\ \tau)$
⟨*proof*⟩

lemma *blocks-inj-suc'*: **assumes** *blocks* τ **shows** *inj-on* $\textit{suc-snd}^{\mathcal{C}}\ (\textit{set}\ \tau)$
⟨*proof*⟩

lemma *blocks-marked-code*: **assumes** *blocks* τ
shows *marked-code* $(\textit{suc-fst}^{\mathcal{C}}\ \text{`}(\textit{set}\ \tau))$
⟨*proof*⟩

lemma *blocks-marked-code'*: **assumes** *all-blocks*: $\bigwedge a.\ a \in \textit{set}\ \tau \Longrightarrow \textit{blockP}\ a$
shows *marked-code* $(\textit{suc-snd}^{\mathcal{C}}\ \text{`}(\textit{set}\ \tau))$
⟨*proof*⟩

lemma *sucs-marked-morphs*: **assumes** *all-blocks*: $\bigwedge a.\ \textit{blockP}\ a$
shows *two-marked-morphisms suc-fst suc-snd*
⟨*proof*⟩

lemma *pre-blocks-range*: $\{(e,f).\ g\ e =_m h\ f\ \} \subseteq \textit{range pre-block}$
⟨*proof*⟩

corollary *card-blocks*: **assumes** *finite* $(\textit{UNIV} :: 'a\ \textit{set})$ **shows** *card* $\{(e,f).\ g\ e =_m h\ f\ \} \leq \textit{card}\ (\textit{UNIV} :: 'a\ \textit{set})$
⟨*proof*⟩

lemma *block-decomposition*: **assumes** $g\ e = h\ f$
obtains τ **where** $\textit{suc-fst}\ \tau = e$ **and** $\textit{suc-snd}\ \tau = f$ **and** *blocks* τ
⟨*proof*⟩

lemma *block-decomposition-unique*: **assumes** $g\ e = h\ f$ **and**
$\textit{suc-fst}\ \tau = e$ **and** $\textit{suc-fst}\ \tau' = e$ **and** *blocks* τ **and** *blocks* τ' **shows** $\tau = \tau'$
⟨*proof*⟩

lemma *block-decomposition-unique'*: **assumes** $g\ e = h\ f$ **and**
$\textit{suc-snd}\ \tau = f$ **and** $\textit{suc-snd}\ \tau' = f$ **and** *blocks* τ **and** *blocks* τ'
shows $\tau = \tau'$

⟨*proof*⟩

lemma *comm-sings-block*: **assumes** $g[a] \cdot h[b] = h[b] \cdot g[a]$
obtains $m\ n$ **where** *suc-fst* $[a] = [a]^{@}Suc\ m$ **and** *suc-snd* $[a] = [b]^{@}Suc\ n$
⟨*proof*⟩

definition *sucs-encoding* **where** *sucs-encoding* $= (\lambda\ a.\ hd\ (g\ [a]))$
definition *sucs-decoding* **where** *sucs-decoding* $= (\lambda\ a.\ SOME\ c.\ hd\ (g[c]) = a)$

lemma *sucs-encoding-inv*: *sucs-decoding* $\circ$ *sucs-encoding* $=$ *id*
⟨*proof*⟩

lemma *encoding-inj*: *inj sucs-encoding*
⟨*proof*⟩

lemma *map-encoding-inj*: *inj* (*map sucs-encoding*)
⟨*proof*⟩

definition *suc-fst′* **where** *suc-fst′* $=$ (*map sucs-encoding*) $\circ$ *suc-fst*
definition *suc-snd′* **where** *suc-snd′* $=$ (*map sucs-encoding*) $\circ$ *suc-snd*

lemma *encoded-sucs-eq-conv*: *suc-fst* $w =$ *suc-snd* $w' \longleftrightarrow$ *suc-fst′* $w =$ *suc-snd′* w'
⟨*proof*⟩

lemma *encoded-sucs-eq-conv′*: *suc-fst* $=$ *suc-snd* $\longleftrightarrow$ *suc-fst′* $=$ *suc-snd′*
⟨*proof*⟩

lemma *encoded-sucs*: **assumes** $\bigwedge$ *c. blockP c* **shows** *two-marked-morphisms suc-fst′ suc-snd′*
⟨*proof*⟩

lemma *encoded-sucs-len*: $|$*suc-fst* $w| = |$*suc-fst′* $w|$ **and** $|$*suc-snd* $w| = |$*suc-snd′* $w|$
⟨*proof*⟩

end

end

theory *Periodicity-Lemma*
imports *CoWBasic*
begin

Chapter 5

The periodicity Lemma

The periodicity Lemma says that if a sufficiently long word has two periods p and q, then the period can be refined to *gcd p q*. The consequence is equivalent to the fact that the corresponding periodic roots commute. "Sufficiently long" here means at least $p + q - gcd\ p\ q$. It is also known as the Fine and Wilf theorem due to its authors [3].

If we relax the requirement to $p + q$, then the claim becomes easy, and it is proved in theory *Combinatorics-Words.CoWBasic* as *two-pers*: $[\![w \leq p\ u \cdot w;\ w \leq p\ v \cdot w;\ |u| + |v| \leq |w|]\!] \Longrightarrow u \cdot v = v \cdot u$.

theorem *per-lemma-relaxed*:
assumes *period w p* **and** *period w q* **and** $p + q \leq |w|$
shows $(take\ p\ w) \cdot (take\ q\ w) = (take\ q\ w) \cdot (take\ p\ w)$
⟨*proof*⟩

5.1 Main claim

We first formulate the claim of the periodicity lemma in terms of commutation of two periodic roots. For trivial reasons we can also drop the requirement that the roots are nonempty.

The proof is by induction which mimics the Euclidean algorithm. The step is given by the following lemma:

lemma *per-lemma-step*:
fixes $u\ v\ w :: 'a\ list$
defines $w' \equiv u \cdot w$ **and** $v' \equiv u \cdot v$
shows $w' \leq p\ u \cdot w' \wedge w' \leq p\ v' \cdot w' \longleftrightarrow\ w \leq p\ u \cdot w \wedge w \leq p\ v \cdot w$
⟨*proof*⟩

theorem *per-lemma-comm*:
assumes $w \leq p\ r \cdot w$ **and** $w \leq p\ s \cdot w$
and *len*: $|r| + |s| - (gcd\ |r|\ |s|) \leq |w|$
shows $r \cdot s = s \cdot r$

⟨*proof*⟩

We can now prove the numeric version.

theorem *per-lemma*: **assumes** *period w p* **and** *period w q* **and** *len*: $p + q - gcd\ p\ q \leq |w|$
shows *period w* (*gcd p q*)
⟨*proof*⟩

5.2 Optimality of the bound by construction of the Fine and Wilf word.

FW-word (where FW stands for Fine and Wilf) yields a word which shows the optimality of the bound in the periodicity lemma.

definition *fw-per k d* ≡ $[0..<d]^{@}k \cdot [0..<(d-1)]$
definition *fw-base k d* ≡ *fw-per k d* · [*d*] · *fw-per k d*

fun *FW-word* :: *nat* ⇒ *nat* ⇒ *nat list* **where**
FW-word-def: *FW-word p q* =
— symmetry (*if q < p then FW-word q p else*
— artificial value *if p = 0 then ε else*
— artificial value *if p dvd q then ε else*
— base case *if gcd p q = q − p then fw-base* (*p div* (*gcd p q*) −1) (*gcd p q*)
— step *else* (*take p* (*FW-word p* (*q*−*p*))) · *FW-word p* (*q*−*p*))

lemma *FW-sym*: *FW-word p q* = *FW-word q p*
⟨*proof*⟩

lemma *FW-dvd*: **assumes** *p dvd q* **shows** *FW-word p q* = *ε*
⟨*proof*⟩

lemma *upt-minus-pref*: $[i..< j-d] \leq_p [i..< j]$
⟨*proof*⟩

lemma *fw-per-pref*: *fw-per k d* ≤*p take d* (*fw-per k d*) · (*fw-per k d*)
⟨*proof*⟩

lemma *fw-per-len*: **shows** |*fw-per k d*| = (*k*+1)∗*d* − 1
⟨*proof*⟩

lemma *fw-base-len*: **assumes** 0 < *d* **shows** |*fw-base k d*| = (*k*+1)∗*d* + (*k*+1)∗*d* − 1
⟨*proof*⟩

lemma *fw-base-per1*: **assumes** 0 < *d*
shows *period* (*fw-base k d*) ((*k*+1)∗*d*)
⟨*proof*⟩

lemma *fw-base-per2*: **assumes** *0 < d*
shows *(fw-base k d) <p take ((k+1)*d + d) (fw-base k d) · (fw-base k d)*
⟨*proof*⟩

lemma *fw-base-not-per*: **assumes** *0 < d 0 < k*
shows *¬ period (fw-base k d) d* (**is** *¬ period ?w d*)
⟨*proof*⟩

lemma *fw-per-per*: **assumes** *fw-per k d ≠ ε*
shows *(fw-per k d) <p take d (fw-per k d) · (fw-per k d)*
⟨*proof*⟩

lemma *fw-per-nth*: **assumes** *i < |fw-per k d|*
shows *(fw-per k d)!i = i mod d*
⟨*proof*⟩

lemma *fw-base-nth*: **assumes** *i < |fw-base k d|*
shows *(fw-base k d)!i = (if i = |fw-per k d| then d else i mod d)*
⟨*proof*⟩

lemma *fw-base-match*: **assumes** *i < |fw-base k d| j < |fw-base k d| i ≠ j* **and**
eq: *(fw-base k d)!i = (fw-base k d)!j*
shows *i mod d = j mod d* **and** *i ≠ |fw-per k d|* **and** *j ≠ |fw-per k d|*
⟨*proof*⟩
lemma *ext-per-sum*: **assumes** *period w p* **and** *period w q* **and** *p ≤ |w|*
shows *period ((take p w) · w) (p+q)*
⟨*proof*⟩

lemma *drop-per-diff*: **assumes** *period w p* **and** *period w q* **and** *p < q* **and** *p < |w|*
shows *period (drop p w) (q−p)*
⟨*proof*⟩

theorem *fw-word*: **assumes** *¬ p dvd q ¬ q dvd p*
shows *|FW-word p q| = p + q − gcd p q − 1*
period (FW-word p q) p **and** *period (FW-word p q) q*
¬ period (FW-word p q) (gcd p q)
⟨*proof*⟩

Calculation examples

value *FW-word 0 4*
value *FW-word 3 4*
value *FW-word 4 7*
value *FW-word 3 7*
value *FW-word 5 7*
value *FW-word 5 13*
value *FW-word 4 6*
value *FW-word 12 18*
value *FW-word 6 10*

value *FW-word 10 16*

5.3 Optimality of the Fine and Wilf word.

The *FW-word* is the most general word having the two desired properties. That is, each equality of letters is forced by periods.

lemma *fw-base-mod-aux*: **assumes** $(i :: nat) < p + p - 1$ $i \neq p - 1$
shows $i \bmod p < p - 1$
⟨*proof*⟩

theorem *fw-minimal*: **assumes** *period w p* **and** *period w q* **and** $|w| = |FW\text{-}word\ p\ q|$ **and**
$i < |w|$ **and** $j < |w|$ **and** $(FW\text{-}word\ p\ q)!i = (FW\text{-}word\ p\ q)!j$
shows $w!i = w!j$
⟨*proof*⟩

theorem *fw-minimal-set*: **assumes** *period w p* **and** *period w q* **and** $|w| = |FW\text{-}word\ p\ q|$
shows $card(set\ w) \leq card(set\ (FW\text{-}word\ p\ q))$
⟨*proof*⟩

5.4 Other variants of the periodicity lemma

periodicity lemma is one of the most frequent tools in Combinatorics on words. Here are some useful variants.

Note that the following lemmas are stronger versions of $[\![?w <_p ?p \cdot ?w; <_s ?w\ (?w \cdot ?q); |?p| + |?q| \leq |?w|; \bigwedge r\ s\ k\ l\ m.\ [\![?p = (r \cdot s)\ ^{@}\ k; ?q = (s \cdot r)\ ^{@}\ l; ?w = (r \cdot s)\ ^{@}\ m \cdot r; primitive\ (r \cdot s)]\!] \Longrightarrow ?thesis]\!] \Longrightarrow ?thesis$
$[\![\leq_f ?u\ (?r\ ^{@}\ ?k); \leq_f ?u\ (?s\ ^{@}\ ?l); |?r| + |?s| \leq |?u|]\!] \Longrightarrow \varrho\ ?r \sim \varrho\ ?s$
$[\![\leq_f ?u\ (?r\ ^{@}\ ?k); \leq_f ?u\ (?s\ ^{@}\ ?l); |?r| + |?s| \leq |?u|]\!] \Longrightarrow \varrho\ ?r \sim \varrho\ ?s$
$[\![\leq_f ?w\ (?u\ ^{@}\ ?n); \leq_f ?w\ (?v\ ^{@}\ ?m); primitive\ ?u; primitive\ ?v; |?u| + |?v| \leq |?w|]\!] \Longrightarrow ?u \sim ?v$ that have a relaxed length assumption $|p| + |q| \leq |w|$ instead of $|p| + |q| - gcd\ |p|\ |q| \leq |w|$ (and which follow from the relaxed version of periodicity lemma $[\![?w \leq p\ ?u \cdot ?w; ?w \leq p\ ?v \cdot ?w; |?u| + |?v| \leq |?w|]\!] \Longrightarrow ?u \cdot ?v = ?v \cdot ?u$.

lemma *per-lemma-comm-pref*:
assumes $u \leq p\ r^{@}k$ $u \leq p\ s^{@}l$
and *len*: $|r| + |s| - gcd\ (|r|)\ (|s|) \leq |u|$
shows $r \cdot s = s \cdot r$
⟨*proof*⟩

lemma *per-lemma-pref-suf-gcd*: **assumes** $w <p\ p \cdot w$ **and** $w <s\ w \cdot q$ **and**
fw: $|p| + |q| - (gcd\ |p|\ |q|) \leq |w|$

obtains $r\ s\ k\ l\ m$ **where** $p = (r \cdot s)^{@} k$ **and** $q = (s \cdot r)^{@} l$ **and** $w = (r \cdot s)^{@} m \cdot r$ **and** *primitive* $(r{\cdot}s)$
⟨*proof*⟩

lemma *fac-two-conjug-primroot-gcd*:
assumes *facs*: $w \leq_f p^{@} k\ w \leq_f q^{@} l$ **and** *nemps*: $p \neq \varepsilon\ q \neq \varepsilon$ **and** *len*: $|p| + |q| - gcd\ (|p|)\ (|q|) \leq |w|$
obtains $r\ s\ m$ **where** $\varrho\ p \sim r \cdot s$ **and** $\varrho\ q \sim r \cdot s$ **and** $w = (r \cdot s)^{@} m \cdot r$ **and** *primitive* $(r{\cdot}s)$
⟨*proof*⟩

corollary *fac-two-conjug-primroot'-gcd*:
assumes *facs*: $u \leq_f r^{@} k\ u \leq_f s^{@} l$ **and** *nemps*: $r \neq \varepsilon\ s \neq \varepsilon$ **and** *len*: $|r| + |s| - gcd\ (|r|)\ (|s|) \leq |u|$
shows $\varrho\ r \sim \varrho\ s$
⟨*proof*⟩

lemma *fac-two-conjug-primroot''-gcd*:
assumes *facs*: $u \leq_f r^{@} k\ u \leq_f s^{@} l$ **and** $u \neq \varepsilon$ **and** *len*: $|r| + |s| - gcd\ (|r|)\ (|s|) \leq |u|$
shows $\varrho\ r \sim \varrho\ s$
⟨*proof*⟩

lemma *fac-two-prim-conjug-gcd*:
assumes $w \leq_f u^{@} n\ w \leq_f v^{@} m$ *primitive u primitive v* $|u| + |v| - gcd\ (|u|)\ (|v|) \leq |w|$
shows $u \sim v$
⟨*proof*⟩

lemma *two-pers-1*:
assumes *pu*: $w \leq_p u \cdot w$ **and** *pv*: $w \leq_p v \cdot w$ **and** *len*: $|u| + |v| - 1 \leq |w|$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

end

theory *Lyndon-Schutzenberger*
imports *Submonoids Periodicity-Lemma*

begin

Chapter 6

Lyndon-Schützenberger Equation

6.1 The original result

The Lyndon-Schützenberger equation is the following equation:

$$x^a y^b = z^c,$$

in this formalization denoted as $x \,^{@}\, a \cdot y \,^{@}\, b = z \,^{@}\, c$.

We formalize here a complete solution of this equation.

The main result, proved by Lyndon and Schützenberger is that the equation has periodic solutions only in free groups if $2 \leq a, b, c$ In this formalization we consider the equation in words only. Then the original result can be formulated as saying that all words x, y and z satisfying the equality ith $2 \leq a, b, c$ pairwise commute.

The result in free groups was first proved in [7]. For words, there are several proofs to be found in the literature (for instance [4, 2]). The presented proof is the authors' proof.

In addition, we give a full parametric solution of the equation for any a, b and c.

6.2 The original result

If x^a or y^b is sufficiently long, then the claim follows from the periodicity Lemma.

lemma *LS-per-lemma-case1*:
  assumes *eq*: $x^{@}a \cdot y^{@}b = z^{@}c$ **and** $0 < a$ **and** $0 < b$ **and** $|z| + |x| - 1 \leq |x^{@}a|$
  shows $x \cdot y = y \cdot x$ **and** $x \cdot z = z \cdot x$
⟨*proof*⟩

A weaker version will be often more convenient

lemma *LS-per-lemma-case*:
assumes *eq*: $x^{@}a{\cdot}y^{@}b = z^{@}c$ **and** $0 < a$ **and** $0 < b$ **and** $|z| + |x| \leq |x^{@}a|$
shows $x \cdot y = y \cdot x$ **and** $x \cdot z = z \cdot x$
⟨*proof*⟩

The most challenging case is when $c = 3$.

lemma *LS-core-case*:
assumes
eq: $x^{@}a \cdot y^{@}b = z^{@}c$ **and**
$2 \leq a$ **and** $2 \leq b$ **and** $2 \leq c$ **and**
$c = 3$ **and**
$b{*}|y| \leq a{*}|x|$ **and** $x \neq \varepsilon$ **and** $y \neq \varepsilon$ **and**
lenx: $a{*}|x| < |z| + |x|$ **and**
leny: $b{*}|y| < |z| + |y|$
shows $x{\cdot}y = y{\cdot}x$
⟨*proof*⟩

The main proof is by induction on the length of z. It also uses the reverse symmetry of the equation which is exploited by two interpretations of the locale *LS*. Note also that the case $|x^a| < |y^b|$ is solved by using induction on $|z| + |y^b|$ instead of just on $|z|$.

lemma *Lyndon-Schutzenberger′*:
$[\![\ x^{@}a{\cdot}y^{@}b = z^{@}c;\ \ 2 \leq a;\ \ 2 \leq b;\ 2 \leq c\ ]\!]$
$\Longrightarrow x{\cdot}y = y{\cdot}x$
⟨*proof*⟩

theorem *Lyndon-Schutzenberger*:
assumes $x^{@}a{\cdot}y^{@}b = z^{@}c$ **and** $2 \leq a$ **and** $2 \leq b$ **and** $2 \leq c$
shows $x{\cdot}y = y{\cdot}x$ **and** $x{\cdot}z = z{\cdot}x$ **and** $y{\cdot}z = z{\cdot}y$
⟨*proof*⟩
hide-fact *Lyndon-Schutzenberger′ LS-core-case*

6.2.1 Some alternative formulations.

lemma *Lyndon-Schutzenberger-conjug*: **assumes** $u \sim v$ **and** $\neg$ *primitive* $(u \cdot v)$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *Lyndon-Schutzenberger-prim*: **assumes** $\neg$ *primitive* x **and** $\neg$ *primitive* y
and $\neg$ *primitive* $(x \cdot y)$
shows $x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *Lyndon-Schutzenberger-rotate*: **assumes** $x^{@}c = r\ ^{@}\ k \cdot u^{@}b \cdot r\ ^{@}\ k'$
and $2 \leq b$ **and** $2 \leq c$ **and** $0 < k$ **and** $0 < k'$
shows $u \cdot r = r \cdot u$
⟨*proof*⟩

6.3 Parametric solution of the equation $x \,^{@}\, j \cdot y \,^{@}\, k = z \,^{@}\, l$

6.3.1 Auxiliary lemmas

lemma *xjy-imprim-len*: **assumes** $x \cdot y \neq y \cdot x$ **and** *eq*: $x^{@}j \cdot y = z^{@}l$ **and** $2 \leq j$ **and** $2 \leq l$
shows $|x^{@}j| < |y| + 2*|x|$ **and** $|z| < |x| + |y|$ **and** $|x| < |z|$ **and** $|x^{@}j| < |z| + |x|$
⟨*proof*⟩

lemma *case-j1k1*: **assumes**
eq: $x{\cdot}y = z^{@}l$ **and**
non-comm: $x \cdot y \neq y \cdot x$ **and**
l-min: $2 \leq l$
obtains $r\ q\ m\ n$ **where**
$x = (r{\cdot}q)^{@}m{\cdot}r$ **and**
$y = q{\cdot}\ (r \cdot q)^{@}n$ **and**
$z = r{\cdot}q$ **and**
$l = m + n + 1$ **and** $r{\cdot}q \neq q{\cdot}r$ **and** $|x| + |y| \geq 4$
⟨*proof*⟩

6.3.2 x is longer

We set up a locale representing the Lyndon-Schützenberger Equation with relaxed exponents and a length assumption breaking the symmetry.

locale *LS-len-le* = *binary-code x y* **for** $x\ y$ +
fixes $j\ k\ l\ z$
assumes
y-le-x: $|y| \leq |x|$
and *eq*: $x^{@}j \cdot y^{@}k = z^{@}l$
and *l-min*: $2 \leq l$
and *j-min*: $1 \leq j$
and *k-min*: $1 \leq k$
begin

lemma *jk-small*: **obtains** $j = 1 \mid k = 1$
⟨*proof*⟩

case $2 \leq j$

lemma *case-j2k1*: **assumes** $2 \leq j\ k = 1$
obtains $r\ q\ t$ **where**
$(r \cdot q)\ ^{@}\ t \cdot r = x$ **and**
$q \cdot r \cdot r \cdot q\ =\ y$ **and**
$(r \cdot q)\ ^{@}\ t \cdot r \cdot r \cdot q = z$ **and** $2 \leq t$
$j = 2$ **and** $l = 2$ **and** $r{\cdot}q \neq q{\cdot}r$ **and**
primitive x **and** *primitive y*
⟨*proof*⟩

case $j = 1$

lemma *case-j1k2-primitive*: **assumes** $j = 1$ $2 \le k$
shows *primitive* x
⟨*proof*⟩

lemma *case-j1k2-a*: **assumes** $j = 1$ $2 \le k$ $z \le_s y^{@}k$
obtains r q t **where**
$x = ((q \cdot r) \cdot (r \cdot (q \cdot r)^{@} t)^{@} (k - 1))^{@} (l - 2) \cdot$
$(((q \cdot r) \cdot (r \cdot (q \cdot r)^{@} t)^{@} (k - 2)) \cdot r) \cdot q$ **and**
$y = r \cdot (q \cdot r)^{@} t$ **and**
$z = (q \cdot r) \cdot (r \cdot (q \cdot r)^{@} t)^{@} (k - 1)$ **and** $\langle 0 < t \rangle$ **and** $r{\cdot}q \neq q{\cdot}r$
⟨*proof*⟩

lemma *case-j1k2-b*: **assumes** $j = 1$ $2 \le k$ $y^{@}k <_s z$
obtains q **where**
$x = (q{\cdot}y^{@}k)^{@}(l{-}1){\cdot}q$ **and**
$z = q{\cdot}y^{@}k$ **and**
$q{\cdot}y \neq y{\cdot}q$
⟨*proof*⟩

6.3.3 Putting things together

lemma *solution-cases*: **obtains**
$j = 2$ $k = 1$ |
$j = 1$ $2 \le k$ $z <_s y^{@}k$ |
$j = 1$ $2 \le k$ $y^{@}k <_s z$ |
$j = 1$ $k = 1$
⟨*proof*⟩

theorem *parametric-solutionE*: **obtains**
— case $x \cdot y$
r q m n **where**
$x = (r{\cdot}q)^{@}m{\cdot}r$ **and**
$y = q{\cdot}(r{\cdot}q)^{@}n$ **and**
$z = r{\cdot}q$ **and**
$l = m + n + 1$ **and** $r{\cdot}q \neq q{\cdot}r$
|
— case $x \cdot y^{@} k$ with $2 \le k$ and $<_s z\ (y^{@} k)$
r q t **where**
$x = ((q \cdot r) \cdot (r \cdot (q \cdot r)^{@} t)^{@} (k - 1))^{@} (l - 2) \cdot$
$(((q \cdot r) \cdot (r \cdot (q \cdot r)^{@} t)^{@} (k - 2)) \cdot r) \cdot q$ **and**
$y = r \cdot (q \cdot r)^{@} t$ **and**
$z = (q \cdot r) \cdot (r \cdot (q \cdot r)^{@} t)^{@} (k - 1)$ **and**
$0 < t$ **and** $r{\cdot}q \neq q{\cdot}r$
|
— case $x \cdot y^{@} k$ with $2 \le k$ and $<_s (y^{@} k)\ z$
q **where**
$x = (q{\cdot}y^{@}k)^{@}(l{-}1){\cdot}q$ **and**
$z = q{\cdot}y^{@}k$ **and**

$q{\cdot}y \neq y{\cdot}q$
|
— case $x \,^{@}\, j \cdot y$ with $2 \leq j$
$r\ q\ t$ **where**
$x = (r \cdot q) \,^{@}\, t \cdot r$ **and**
$y = q \cdot r \cdot r \cdot q$ **and**
$z = (r \cdot q) \,^{@}\, t \cdot r \cdot r \cdot q$ **and**
$j = 2$ **and** $l = 2$ **and** $2 \leq t$ **and** $r{\cdot}q \neq q{\cdot}r$ **and**
primitive x **and** *primitive* y
⟨*proof*⟩

end

Using the solution from locale *LS-len-le*, the following theorem gives the full characterization of the equation in question:

$$x^i y^j = z^\ell$$

theorem *LS-parametric-solution*:
assumes *y-le-x*: $|y| \leq |x|$
and *j-min*: $1 \leq j$ **and** *k-min*: $1 \leq k$ **and** *l-min*: $2 \leq l$
shows
$x^{@}j \cdot y^{@}k = z^{@}l$
$\longleftrightarrow$
$(\exists\, r\ m\ n\ t.$
$x = r^{@}m \wedge y = r^{@}n \wedge z = r^{@}t \wedge m{*}j{+}n{*}k{=}t{*}l)$ — Case A: x,y is not a code
$\vee\ (j = 1 \wedge k = 1) \wedge$
$(\exists\, r\ q\ m\ n.$
$x = (r{\cdot}q)^{@}m{\cdot}r \wedge y = q{\cdot}(r{\cdot}q)^{@}n \wedge z = r{\cdot}q \wedge m + n + 1 = l \wedge r{\cdot}q \neq q{\cdot}r)$ — Case B
$\vee\ (j = 1 \wedge 2 \leq k) \wedge$
$(\exists\, r\ q.$
$x = (q{\cdot}r^{@}k)^{@}(l{-}1){\cdot}q \wedge y = r \wedge z = q{\cdot}r^{@}k \wedge r{\cdot}q \neq q{\cdot}r)$ — Case C
$\vee\ (j = 1 \wedge 2 \leq k) \wedge$
$(\exists\, r\ q\ t.\ 0 < t \wedge$
$x = ((q \cdot r) \cdot (r \cdot (q \cdot r) \,^{@}\, t) \,^{@}\, (k - 1)) \,^{@}\, (l - 2){\cdot}(((q \cdot r) \cdot$
$(r \cdot (q \cdot r) \,^{@}\, t) \,^{@}\, (k - 2)) \cdot r) \cdot q$
$\wedge\ y = r \cdot (q \cdot r) \,^{@}\, t$
$\wedge\ z = (q \cdot r) \cdot (r \cdot (q \cdot r) \,^{@}\, t) \,^{@}\, (k - 1)$
$\wedge\ r{\cdot}q \neq q{\cdot}r)$ — Case D
$\vee\ (j = 2 \wedge k = 1 \wedge l = 2) \wedge$
$(\exists\, r\ q\ t.\ 2 \leq t \wedge$
$x = (r \cdot q) \,^{@}\, t \cdot r \wedge y = q \cdot r \cdot r \cdot q$
$\wedge\ z = (r \cdot q) \,^{@}\, t \cdot r \cdot r \cdot q\ \wedge r{\cdot}q \neq q{\cdot}r\ )$ — Case E

$($**is** *?eq* =
$($*?sol-per* $\vee$ ($?cond\text{-}j1k1 \wedge ?sol\text{-}j1k1$) $\vee$
$($*?cond-j1k2* $\wedge$ *?sol-j1k2-b*) $\vee$

(*?cond-j1k2* ∧ *?sol-j1k2-a*) ∨
(*?cond-j2k1l2* ∧ *?sol-j2k1l2*)))
⟨*proof*⟩

6.3.4 Uniqueness of the imprimitivity witness

In this section, we show that given a binary code $\{x, y\}$ and two imprimitive words $x \,@\, j \cdot y \,@\, k$ and $x \,@\, j' \cdot y \,@\, k'$ is possible only if the two words are equals, that is, if $j = j'$ and $k = k'$.

lemma *LS-unique-same*: **assumes** $x \cdot y \neq y \cdot x$
and $0 < j$ **and** $0 < k$ **and** ¬ *primitive*($x^{@}j \cdot y^{@}k$)
and $0 < k'$ **and** ¬ *primitive*($x^{@}j \cdot y^{@}k'$)
shows $k = k'$
⟨*proof*⟩

lemma *LS-unique-distinct-le*: **assumes** $x \cdot y \neq y \cdot x$
and $2 \leq j$ **and** ¬ *primitive*($x^{@}j \cdot y$)
and $2 \leq k$ **and** ¬ *primitive*($x \cdot y^{@}k$)
and $|y| \leq |x|$
shows *False*
⟨*proof*⟩

lemma *LS-unique-distinct*: **assumes** $x \cdot y \neq y \cdot x$
and $2 \leq j$ **and** ¬ *primitive*($x^{@}j \cdot y$)
and $2 \leq k$ **and** ¬ *primitive*($x \cdot y^{@}k$)
shows *False*
⟨*proof*⟩

lemma *LS-unique'*: **assumes** $x \cdot y \neq y \cdot x$
and $0 < j$ **and** $0 < k$ **and** ¬ *primitive*($x^{@}j \cdot y^{@}k$)
and $0 < j'$ **and** $0 < k'$ **and** ¬ *primitive*($x^{@}j' \cdot y^{@}k'$)
shows $k = k'$
⟨*proof*⟩

lemma *LS-unique*: **assumes** $x \cdot y \neq y \cdot x$
and $0 < j$ **and** $0 < k$ **and** ¬ *primitive*($x^{@}j \cdot y^{@}k$)
and $0 < j'$ **and** $0 < k'$ **and** ¬ *primitive*($x^{@}j' \cdot y^{@}k'$)
shows $j = j'$ **and** $k = k'$
⟨*proof*⟩

6.4 The bound on the exponent in Lyndon-Schützenberger equation

lemma (**in** *LS-len-le*) *case-j1k2-exp-le*:
assumes $j = 1$ $2 \leq k$
shows $k*|y| + 4 \leq |x| + 2*|y|$
⟨*proof*⟩

lemma (**in** *LS-len-le*) *case-j2k1-exp-le*:
assumes $2 \leq j$ $k = 1$
shows $j*|x| + 4 \leq |y| + 2*|x|$
⟨*proof*⟩

theorem *LS-exp-le-one*:
assumes *eq*: $x \cdot y \,^{@}\, k = z \,^{@}\, l$
and $2 \leq l$
and $x \cdot y \neq y \cdot x$
and $1 \leq k$
shows $k*|y| + 4 \leq |x|+2*|y|$
⟨*proof*⟩

lemma *LS-exp-le-conv-rat*:
fixes x y k::$'a$::*linordered-field*
assumes $y > 0$
shows $k * y + 4 \leq x + 2 * y \longleftrightarrow k \leq (x - 4)/y + 2$
⟨*proof*⟩

end

theory *Binary-Code-Morphisms*
imports *CoWBasic Submonoids Morphisms*

begin

Chapter 7

Binary alphabet and binary morphisms

7.1 Datatype of a binary alphabet

Basic elements for construction of binary words.

type-notation *Enum.finite-2* (*binA*)
notation *finite-2.*a_1 (*bina*)
notation *finite-2.*a_2 (*binb*)

lemmas *bin-distinct* = *Enum.finite-2.distinct*
lemmas *bin-exhaust* = *Enum.finite-2.exhaust*
lemmas *bin-induct* = *Enum.finite-2.induct*
lemmas *bin-UNIV* = *Enum.UNIV-finite-2*
lemmas *bin-eq-neq-iff* = *Enum.neq-finite-2-*a_2*-iff*
lemmas *bin-eq-neq-iff'* = *Enum.neq-finite-2-*a_1*-iff*

abbreviation *bin-word-a* :: *binA list* (𝔞) **where**
bin-word-a ≡ [*bina*]

abbreviation *bin-word-b* :: *binA list* (𝔟) **where**
bin-word-b ≡ [*binb*]

abbreviation *binUNIV* :: *binA set* **where** *binUNIV* ≡ *UNIV*

lemma *binUNIV-I* [*simp*, *intro*]: *bina* ∈ *A* ⟹ *binb* ∈ *A* ⟹ *A* = *UNIV*
⟨*proof*⟩

lemma *bin-basis-code*: *code* {𝔞,𝔟}
⟨*proof*⟩

lemma *bin-num*: *bina* = *0* *binb* = *1*
⟨*proof*⟩

lemma *binA-simps* [*simp*]: *bina* − *binb* = *binb binb* − *bina* = *binb 1* − *bina* = *binb 1* − *binb* = *bina a* − *a* = *bina 1* − (*1* − *a*) = *a*
⟨*proof*⟩

definition *bin-swap* :: *binA* ⇒ *binA* **where** *bin-swap x* ≡ *1* − *x*

lemma *bin-swap-if-then*: *1*−*x* = (*if x* = *bina then binb else bina*)
⟨*proof*⟩

definition *bin-swap-morph* **where** *bin-swap-morph* ≡ *map bin-swap*

lemma *alphabet-or*[*simp*]: *a* = *bina* ∨ *a* = *binb*
⟨*proof*⟩

lemma *bin-im-or*: *f* [*a*] = *f* 𝔞 ∨ *f* [*a*] = *f* 𝔟
⟨*proof*⟩

thm *triv-forall-equality*

lemma *binUNIV-card*: *card binUNIV* = *2*
⟨*proof*⟩

lemma *other-letter*: **obtains** *b* **where** *b* ≠ (*a* :: *binA*)
⟨*proof*⟩

lemma *alphabet-or-neq*: *x* ≠ *y* ⟹ *x* = (*a* :: *binA*) ∨ *y* = *a*
⟨*proof*⟩

lemma *binA-neq-cases*: **assumes** *neq*: *a* ≠ *b*
obtains *a* = *bina* **and** *b* = *binb* | *a* = *binb* **and** *b* = *bina*
⟨*proof*⟩

lemma *bin-neq-sym-pred*: **assumes** *a* ≠ *b* **and** *P bina binb* **and** *P binb bina* **shows** *P a b*
⟨*proof*⟩

lemma *no-third*: (*c* :: *binA*) ≠ *a* ⟹ *b* ≠ *a* ⟹ *b* = *c*
⟨*proof*⟩

lemma *two-in-bin-UNIV*: **assumes** *a* ≠ *b* **and** *a* ∈ *S* **and** *b* ∈ *S* **shows** *S* = *binUNIV*
⟨*proof*⟩

lemmas *two-in-bin-set* = *two-in-bin-UNIV*[*unfolded bin-UNIV*]

lemma *bin-not-comp-set-UNIV*: **assumes** ¬ *u* ⋈ *v* **shows** *set* (*u* · *v*) = *binUNIV*
⟨*proof*⟩

lemma *bin-basis-singletons*: {[*q*] |*q*. *q* ∈ {*bina*, *binb*}} = {𝔞,𝔟}

⟨*proof*⟩

lemma *bin-basis-generates*: $\langle\{\mathfrak{a},\mathfrak{b}\}\rangle = UNIV$
⟨*proof*⟩

lemma *a-in-bin-basis*: $[a] \in \{\mathfrak{a},\mathfrak{b}\}$
⟨*proof*⟩

lemma *lcp-zero-one-emp*: $\mathfrak{a} \wedge_p \mathfrak{b} = \varepsilon$ **and** *lcp-one-zero-emp*: $\mathfrak{b} \wedge_p \mathfrak{a} = \varepsilon$
⟨*proof*⟩

lemma *bin-neq-induct*: $(a::binA) \neq b \Longrightarrow P\ a \Longrightarrow P\ b \Longrightarrow P\ c$
⟨*proof*⟩

lemma *bin-neq-induct'*: **assumes**$(a::binA) \neq b$ **and** $P\ a$ **and** $P\ b$ **shows** $\bigwedge c.\ P\ c$
⟨*proof*⟩

lemma *neq-exhaust*: **assumes** $(a::binA) \neq b$ **obtains** $c = a \mid c = b$
⟨*proof*⟩

lemma *bin-swap-neq* [*simp*]: $1{-}(a :: binA) \neq a$
⟨*proof*⟩
lemmas *bin-swap-neq'*[*simp*] = *bin-swap-neq*[*symmetric*]

lemmas *bin-swap-induct* = *bin-neq-induct*[*OF bin-swap-neq'*]
and *bin-swap-exhaust* = *neq-exhaust*[*OF bin-swap-neq'*]

lemma *bin-swap-induct'*: $P\ (a :: binA) \Longrightarrow P\ (1{-}a) \Longrightarrow (\bigwedge c.\ P\ c)$
⟨*proof*⟩

lemma *swap-UNIV*: $\{a,\ 1{-}a\} = binUNIV$ (**is** *?P a*)
⟨*proof*⟩

lemma *bin-neq-swap'*[*intro*]: $a \neq b \Longrightarrow 1 - b = (a :: binA)$
⟨*proof*⟩

lemma *bin-neq-swap*[*intro*]: $a \neq b \Longrightarrow 1 - a = (b :: binA)$
⟨*proof*⟩

lemma *bin-neq-swap''*[*intro*]: $a \neq b \Longrightarrow b = 1{-}(a:: binA)$
⟨*proof*⟩

lemma *bin-neq-swap'''*[*intro*]: $a \neq b \Longrightarrow a = 1{-}(b:: binA)$
⟨*proof*⟩

lemma *bin-neq-iff*: $c \neq d \longleftrightarrow 1 - d = (c :: binA)$
⟨*proof*⟩

lemma *bin-neq-iff'*: $c \neq d \longleftrightarrow 1 - c = (d :: binA)$

⟨*proof*⟩

lemma *binA-neq-cases-swap*: **assumes** *neq*: $a \neq (b :: binA)$
obtains $a = c$ **and** $b = 1 - c \mid a = 1 - c$ **and** $b = c$
⟨*proof*⟩

lemma *im-swap-neq*: $f\ a = f\ b \Longrightarrow f\ bina \neq f\ binb \Longrightarrow a = b$
⟨*proof*⟩

lemma *bin-without-letter*: **assumes** $(a :: binA) \notin set\ w$
obtains k **where** $w = [1-a]^{@}k$
⟨*proof*⟩

lemma *bin-empty-iff*: $S = \{\} \longleftrightarrow (a :: binA) \notin S \wedge 1-a \notin S$
⟨*proof*⟩

lemma *bin-UNIV-iff*: $S = binUNIV \longleftrightarrow a \in S \wedge 1-a \in S$
⟨*proof*⟩

lemma *bin-UNIV-I*: $a \in S \Longrightarrow 1-a \in S \Longrightarrow S = binUNIV$
⟨*proof*⟩

lemma *bin-sing-iff*: $A = \{a :: binA\} \longleftrightarrow a \in A \wedge 1-a \notin A$
⟨*proof*⟩

lemma *set-binUNIV-iff*: $\neg\ set\ w \subseteq \{hd\ w\} \longleftrightarrow set\ w = binUNIV$
⟨*proof*⟩

lemma *bin-set-cases*: **obtains** $S = \{\} \mid S = \{a\} \mid S = \{1-a\} \mid S = binUNIV$
⟨*proof*⟩

lemma *not-UNIV-E*: **assumes** $A \neq binUNIV$ **obtains** a **where** $A \subseteq \{a\}$
⟨*proof*⟩

lemma *not-UNIV-nempE*: **assumes** $A \neq binUNIV$ **and** $A \neq \{\}$ **obtains** a **where** $A = \{a\}$
⟨*proof*⟩

lemma *bin-sing-gen-iff*: $x \in \langle\{[a]\}\rangle \longleftrightarrow 1-(a :: binA) \notin set\ x$
⟨*proof*⟩

lemma *set-hd-pow-conv*: $set\ w \subseteq \{hd\ w\} \longleftrightarrow set\ w \neq binUNIV$
⟨*proof*⟩

lemma *not-swap-eq*: $P\ a\ b \Longrightarrow (\bigwedge (c :: binA).\ \neg\ P\ c\ (1-c)) \Longrightarrow a = b$
⟨*proof*⟩

lemma *bin-distinct-letter*: **assumes** $set\ w = binUNIV$
obtains $k\ w'$ **where** $[hd\ w]^{@}Suc\ k \cdot [1-hd\ w] \cdot w' = w$

⟨*proof*⟩

lemma $P\ a \longleftrightarrow P\ (1{-}a) \Longrightarrow P\ a \Longrightarrow (\bigwedge\ (b :: binA).\ P\ b)$
⟨*proof*⟩

lemma *bin-sym-all*: $P\ (a :: binA) \longleftrightarrow P\ (1{-}a) \Longrightarrow P\ a \Longrightarrow P\ x$
⟨*proof*⟩

lemma *bin-sym-all-comm*:
$f\ [a] \cdot f\ [1{-}a] \neq f\ [1{-}a] \cdot f\ [a] \Longrightarrow f\ [b] \cdot f\ [1{-}b] \neq f\ [1{-}b] \cdot f\ [(b :: binA)]$ (**is** *?P a* $\Longrightarrow$ *?P b*)
⟨*proof*⟩

lemma *bin-sym-all-neq*:
$f\ [(a :: binA)] \neq f\ [1{-}a] \Longrightarrow f\ [b] \neq f\ [1{-}b]$ (**is** *?P a* $\Longrightarrow$ *?P b*)
⟨*proof*⟩

lemma *bin-len-count*:
fixes *w* :: *binA list*
shows $|w| =$ *count-list w a* $+$ *count-list w* $(1{-}a)$
⟨*proof*⟩

lemma *bin-len-count'*:
fixes *w* :: *binA list*
shows $|w| =$ *count-list w bina* $+$ *count-list w binb*
⟨*proof*⟩

7.2 Binary morphisms

lemma *bin-map-core-lists*: $(map\ f^{\mathcal{C}}\ w) \in lists\ \{f\ \mathfrak{a}, f\ \mathfrak{b}\}$
⟨*proof*⟩

lemma *bin-range*: $range\ f = \{f\ bina, f\ binb\}$
⟨*proof*⟩

lemma *bin-core-range*: $range\ f^{\mathcal{C}} = \{f\ \mathfrak{a}, f\ \mathfrak{b}\}$
⟨*proof*⟩

lemma *bin-core-range-swap*: $range\ f^{\mathcal{C}} = \{f\ [(a :: binA)], f\ [1{-}a]\}$ (**is** *?P a*)
⟨*proof*⟩

lemma *bin-map-core-lists-swap*: $(map\ f^{\mathcal{C}}\ w) \in lists\ \{f\ [(a :: binA)], f\ [1{-}a]\}$
⟨*proof*⟩

locale *binary-morphism* = *morphism f*
for *f* :: *binA list* $\Rightarrow$ *'a list*
begin

lemma *bin-len-count-im*:

fixes $a :: binA$
shows $|f\ w| = count\text{-}list\ w\ a * |f\ [a]| + count\text{-}list\ w\ (1-a) * |f\ [1-a]|$
⟨*proof*⟩

lemma *bin-len-count-im′*:
shows $|f\ w| = count\text{-}list\ w\ bina * |f\ \mathfrak{a}| + count\text{-}list\ w\ binb * |f\ \mathfrak{b}|$
⟨*proof*⟩

lemma *bin-neq-inj-core*: **assumes** $f\ [a] \neq f\ [1-a]$ **shows** $inj\ f^{\mathcal{C}}$
⟨*proof*⟩

lemma *bin-code-morphism-inj*: **assumes** $f\ [a] \cdot f\ [1-a] \neq f\ [1-a] \cdot f\ [a]$
shows $inj\ f$
⟨*proof*⟩

lemma *bin-code-morphismI*: $f\ [a] \cdot f\ [1-a] \neq f\ [1-a] \cdot f\ [a] \Longrightarrow code\text{-}morphism\ f$
⟨*proof*⟩

end

7.2.1 Binary periodic morphisms

locale *binary-periodic-morphism* = *periodic-morphism f*
for $f :: binA\ list \Rightarrow 'a\ list$
begin

sublocale *binary-morphism*
⟨*proof*⟩

definition *fn0* **where** $fn0 \equiv (SOME\ n.\ f\ \mathfrak{a} = mroot^{@}n)$
definition *fn1* **where** $fn1 \equiv (SOME\ n.\ f\ \mathfrak{b} = mroot^{@}n)$

lemma *bin0-im*: $f\ \mathfrak{a} = mroot^{@}fn0$
⟨*proof*⟩

lemma *bin1-im*: $f\ \mathfrak{b} = mroot^{@}fn1$
⟨*proof*⟩

lemma *sorted-image* : $f\ w = (f\ [a])^{@}(count\text{-}list\ w\ a) \cdot (f\ [1-a])^{@}(count\text{-}list\ w\ (1-a))$
⟨*proof*⟩

lemma *bin-per-morph-expI*: $f\ u = mroot^{@}((im\text{-}exp\ \mathfrak{a}) * (count\text{-}list\ u\ bina) + im\text{-}exp\ \mathfrak{b} * (count\text{-}list\ u\ binb))$
⟨*proof*⟩

end

7.3 From two words to a binary morphism

definition *bin-morph-of'* :: *'a list* ⇒ *'a list* ⇒ *binA list* ⇒ *'a list* **where** *bin-morph-of' x y u* = *concat* (*map* (λ *a*. (*case a of bina* ⇒ *x* | *binb* ⇒ *y*)) *u*)

definition *bin-morph-of* :: *'a list* ⇒ *'a list* ⇒ *binA list* ⇒ *'a list* **where** *bin-morph-of x y u* = *concat* (*map* (λ *a*. *if a* = *bina then x else y*) *u*)

lemma *case-finite-2-if-else*: *case-finite-2 x y* = (λ *a*. *if a* = *bina then x else y*)
⟨*proof*⟩

lemma *bin-morph-of-case-def*: *bin-morph-of x y u* = *concat* (*map* (λ *a*. (*case a of bina* ⇒ *x* | *binb* ⇒ *y*)) *u*)
⟨*proof*⟩

lemma *case-finiteD*: *case-finite-2* (*f* 𝔞) (*f* 𝔟) = $f^{\mathcal{C}}$
⟨*proof*⟩

lemma *case-finiteD'*: *case-finite-2* (*f* 𝔞) (*f* 𝔟) *u* = $f^{\mathcal{C}}$ *u*
⟨*proof*⟩

lemma *bin-morph-of-maps*: *bin-morph-of x y* = *List.maps* (*case-finite-2 x y*)
⟨*proof*⟩

lemma *bin-morph-ofD*: (*bin-morph-of x y*) 𝔞 = *x* (*bin-morph-of x y*) 𝔟 = *y*
⟨*proof*⟩

lemma *bin-range-swap*: *range f* = {*f* (*a*::*binA*), *f* (*1*−*a*)} (**is** *?P a*)
⟨*proof*⟩

lemma *bin-morph-of-core-range*: *range* (*bin-morph-of x y*)$^{\mathcal{C}}$ = {*x*,*y*}
⟨*proof*⟩

lemma *bin-morph-of-morph*: *morphism* (*bin-morph-of x y*)
⟨*proof*⟩

lemma *bin-morph-of-bin-morph*: *binary-morphism* (*bin-morph-of x y*)
⟨*proof*⟩

lemma *bin-morph-of-range*: *range* (*bin-morph-of x y*) = ⟨{*x*,*y*}⟩
⟨*proof*⟩

context *binary-code*
begin

lemma *code-morph-of*: *code-morphism* (*bin-morph-of* u_0 u_1)
⟨*proof*⟩

lemma *inj-morph-of*: *inj* (*bin-morph-of* u_0 u_1)

⟨*proof*⟩

end

7.4 Two binary morphism

locale *two-binary-morphisms* = *two-morphisms g h*
  for *g h* :: *binA list* ⇒ *'a list*

begin

lemma *eq-on-letters-eq*: *g* 𝔞 = *h* 𝔞 ⟹ *g* 𝔟 = *h* 𝔟 ⟹ *g* = *h*
  ⟨*proof*⟩

sublocale *g*: *binary-morphism g*
  ⟨*proof*⟩
sublocale *h*: *binary-morphism h*
  ⟨*proof*⟩

lemma *rev-morphs*: *two-binary-morphisms* (*rev-map g*) (*rev-map h*)
  ⟨*proof*⟩

lemma *solution-UNIV*:
  assumes *s* ≠ *ε* **and** *g s* = *h s* **and** ⋀*a*. *g* [*a*] ≠ *h* [*a*]
  shows *set s* = *UNIV*
⟨*proof*⟩

lemma *solution-len-im-sing-less*:
  assumes *sol*: *g s* = *h s* **and** *set*: *a* ∈ *set s* **and** *less*: |*g* [*a*]| < |*h* [*a*]|
  shows |*h* [*1*−*a*]| < |*g* [*1*−*a*]|
⟨*proof*⟩

lemma *solution-len-im-sing-le*:
  assumes *sol*: *g s* = *h s* **and** *set*: *set s* = *UNIV* **and** *less*: |*g* [*a*]| ≤ |*h* [*a*]|
  shows |*h* [*1*−*a*]| ≤ |*g* [*1*−*a*]|
⟨*proof*⟩

lemma *solution-sing-len-cases*:
  assumes *set*: *set s* = *UNIV* **and** *sol*: *g s* = *h s* **and** *g* ≠ *h*
  obtains *a* **where** |*g* [*a*]| < |*h* [*a*]| **and** |*h* [*1*−*a*]| < |*g* [*1*−*a*]|
⟨*proof*⟩

lemma *len-ims-sing-neq*:
  assumes *g s* = *h s g* ≠ *h set s* = *binUNIV*
  shows |*g* [*c*]| ≠ |*h* [*c*]|
⟨*proof*⟩

end

lemma *two-binary-morphismsI*: *binary-morphism g* $\Longrightarrow$ *binary-morphism h* $\Longrightarrow$ *two-binary-morphisms g h*
⟨*proof*⟩

7.5 Binary code morphism

7.5.1 Locale - binary code morphism

locale *binary-code-morphism* = *code-morphism f* :: *binA list* ⇒ *'a list* **for** *f*

begin

lemma *morph-bin-morph-of*: $f = \textit{bin-morph-of}\ (f\ \mathfrak{a})\ (f\ \mathfrak{b})$
⟨*proof*⟩

lemma *non-comm-morph* [*simp*]: $f\ [a] \cdot f\ [1-a] \neq f\ [1-a] \cdot f\ [a]$
⟨*proof*⟩

lemma *non-comp-morph*: $\neg\ f\ [a] \cdot f\ [1-a] \bowtie f\ [1-a] \cdot f\ [a]$
⟨*proof*⟩

lemma *swap-non-comm-morph* [*simp*, *intro*]: $a \neq b \Longrightarrow f\ [a] \cdot f\ [b] \neq f\ [b] \cdot f\ [a]$
⟨*proof*⟩

thm *bin-core-range*[*of f*]

lemma *bin-code-morph-rev-map*: *binary-code-morphism* (*rev-map f*)
⟨*proof*⟩

sublocale *swap*: *binary-code f* $\mathfrak{b}$ *f* $\mathfrak{a}$
⟨*proof*⟩

sublocale *binary-code f* $\mathfrak{a}$ *f* $\mathfrak{b}$
⟨*proof*⟩

notation *bin-code-lcp* (α) **and**
bin-code-lcs (β) **and**
bin-code-mismatch-fst (c_0) **and**
bin-code-mismatch-snd (c_1)
term *bin-lcp* (*f* $\mathfrak{a}$) (*f* $\mathfrak{b}$)
abbreviation *bin-morph-mismatch* ($\mathfrak{c}$)
where *bin-morph-mismatch a* ≡ *bin-mismatch* (*f*[*a*]) (*f*[*1*−*a*])
abbreviation *bin-morph-mismatch-suf* ($\mathfrak{d}$)
where *bin-morph-mismatch-suf a* ≡ *bin-mismatch-suf* (*f*[*1*−*a*]) (*f*[*a*])

lemma *bin-lcp-def'*: $\alpha = f\ ([a] \cdot [1-a]) \wedge_p f\ ([1-a] \cdot [a])$
⟨*proof*⟩

lemma *bin-lcp-neq*: $a \neq b \Longrightarrow \alpha = f\ ([a] \cdot [b]) \wedge_p f\ ([b] \cdot [a])$

⟨*proof*⟩

lemma *sing-im*: $f\ [a] \in \{f\ \mathfrak{a}, f\ \mathfrak{b}\}$
⟨*proof*⟩

lemma *bin-mismatch-inj*: *inj* $\mathfrak{c}$
⟨*proof*⟩

lemma *map-in-lists*: *map* $(\lambda x.\ f\ [x])\ w \in$ *lists* $\{f\ \mathfrak{a}, f\ \mathfrak{b}\}$
⟨*proof*⟩

lemma *bin-morph-lcp-short*: $|\alpha| < |f\ [a]| + |f[1-a]|$
⟨*proof*⟩

lemma *swap-not-pref-bin-lcp*: $\neg\ f([a] \cdot [1-a]) \leq p\ \alpha$
⟨*proof*⟩

thm *local.bin-mismatch-inj*

lemma *bin-mismatch-suf-inj*: *inj* $\mathfrak{d}$
⟨*proof*⟩

lemma *bin-lcp-sing*: *bin-lcp* $(f\ [a])\ (f\ [1-a]) = \alpha$
⟨*proof*⟩

lemma *bin-lcs-sing*: *bin-lcs* $(f\ [a])\ (f\ [1-a]) = \beta$
⟨*proof*⟩

lemma *bin-code-morph-sing*: *binary-code* $(f\ [a])\ (f\ [1-a])$
⟨*proof*⟩

lemma *bin-mismatch-swap-neq*: $\mathfrak{c}\ a \neq \mathfrak{c}\ (1-a)$
⟨*proof*⟩

lemma *set* $w \subseteq \{hd\ w\} \longleftrightarrow (\exists\ k.\ w = [hd\ w]^{@}k)$
⟨*proof*⟩

lemma *long-bin-lcp-hd*: **assumes** $|f\ w| \leq |\alpha|$
shows *set* $w \subseteq \{hd\ w\}$
⟨*proof*⟩

thm *nonerasing*
nonerasing-morphism.nonerasing
lemmas *nonerasing* = *nonerasing*
thm *nonerasing-morphism.nonerasing*
binary-code-morphism.nonerasing

lemma *bin-morph-lcp-mismatch-pref*:

$\alpha \cdot [\mathfrak{c}\ a] \leq_p f\ [a] \cdot \alpha$
⟨*proof*⟩

lemma $[\mathfrak{d}\ a] \cdot \beta \leq_s \beta \cdot f\ [a]$ ⟨*proof*⟩

lemma *bin-lcp-pref-all*: $\alpha \leq_p f\ w \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-spref-all*: $w \neq \varepsilon \Longrightarrow \alpha <_p f\ w \cdot \alpha$
⟨*proof*⟩

lemma *pref-mono-lcp*: **assumes** $w \leq_p w'$ **shows** $f\ w \cdot \alpha \leq_p f\ w' \cdot \alpha$
⟨*proof*⟩

lemma *long-bin-lcp*: **assumes** $|f\ w| \leq |\alpha|$
shows *set* $w \subseteq \{hd\ w\}$
⟨*proof*⟩

thm *sing-to-nemp*
nonerasing

lemma *bin-mismatch-code-morph*: $c_0 = \mathfrak{c}\ 0\ c_1 = \mathfrak{c}\ 1$
⟨*proof*⟩

lemma *bin-lcp-mismatch-pref-all*: $\alpha \cdot [\mathfrak{c}\ a] \leq_p f\ [a] \cdot f\ w \cdot \alpha$
⟨*proof*⟩

lemma *bin-fst-mismatch-all*: $\alpha \cdot [c_0] \leq_p f\ \mathfrak{a} \cdot f\ w \cdot \alpha$
⟨*proof*⟩

lemma *bin-snd-mismatch-all*: $\alpha \cdot [c_1] \leq_p f\ \mathfrak{b} \cdot f\ w \cdot \alpha$
⟨*proof*⟩

lemma *bin-long-mismatch*: **assumes** $|\alpha| < |f\ w|$ **shows** $\alpha \cdot [\mathfrak{c}\ (hd\ w)] \leq_p f\ w$
⟨*proof*⟩

lemma *sing-pow-mismatch*: **assumes** $f\ [a] = [b]^{@} Suc\ n$ **shows** $\mathfrak{c}\ a = b$
⟨*proof*⟩

lemma *sing-pow-mismatch-suf*: $f\ [a] = [b]^{@} Suc\ n \Longrightarrow \mathfrak{d}\ a = b$
⟨*proof*⟩

lemma *bin-lcp-swap-hd*: $f\ [a] \cdot f\ w \cdot \alpha\ \wedge_p f\ [1-a] \cdot f\ w' \cdot \alpha = \alpha$
⟨*proof*⟩

lemma *bin-lcp-neq-hd*: $a \neq b \Longrightarrow f\ [a] \cdot f\ w \cdot \alpha\ \wedge_p f\ [b] \cdot f\ w' \cdot \alpha = \alpha$
⟨*proof*⟩

lemma *bin-mismatch-swap-not-comp*: $\neg\ f\ [a] \cdot f\ w \cdot \alpha\ \bowtie f\ [1{-}a] \cdot f\ w' \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-root*: $\alpha <p\ f\ [a] \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-pref*: **assumes** *set* $w = binUNIV$
shows $\alpha \leq p\ (f\ w)$
⟨*proof*⟩

lemma *bin-lcp-pref''*: $[a] \leq f\ w \Longrightarrow [1{-}a] \leq f\ w\ \Longrightarrow \alpha \leq p\ (f\ w)$
⟨*proof*⟩
lemma *bin-lcp-pref'*: $\mathfrak{a} \leq f\ w \Longrightarrow \mathfrak{b} \leq f\ w\ \Longrightarrow \alpha \leq p\ (f\ w)$
⟨*proof*⟩

lemma *bin-lcp-mismatch-pref-all-set*: **assumes** $1{-}a \in set\ w$
shows $\alpha \cdot [\mathfrak{c}\ a] \leq p\ f\ [a] \cdot f\ w$
⟨*proof*⟩

lemma *bin-lcp-comp-hd*: $\alpha \bowtie f\ (\mathfrak{a} \cdot w0) \wedge_p\ f\ (\mathfrak{b} \cdot w1)$
⟨*proof*⟩

lemma *sing-mismatch*: **assumes** *set* $(f\ \mathfrak{a}) \subseteq \{a\}$ **shows** $c_0 = a$
⟨*proof*⟩

lemma *sing-mismatch'*: **assumes** *set* $(f\ \mathfrak{b}) \subseteq \{a\}$ **shows** $c_1 = a$
⟨*proof*⟩

lemma *bin-lcp-comp-all*: $\alpha \bowtie (f\ w)$
⟨*proof*⟩

lemma *not-comp-bin-swap*: $\neg\ f\ [a] \cdot \alpha \bowtie f\ [1{-}a] \cdot \alpha$
⟨*proof*⟩

lemma *mismatch-pref*:
assumes $\alpha \leq p\ f\ ([a] \cdot w0)$ **and** $\alpha \leq p\ f\ ([1{-}a] \cdot w1)$
shows $\alpha = f\ ([a] \cdot w0) \wedge_p f\ ([1{-}a] \cdot w1)$
⟨*proof*⟩

lemma *bin-set-UNIV-length*: **assumes** *set* $w = UNIV$ **shows** $|f\ [a]| + |f\ [1{-}a]| \leq |f\ w|$
⟨*proof*⟩

lemma *set-UNIV-bin-lcp-pref*: **assumes** *set* $w = UNIV$ **shows** $\alpha \cdot [\mathfrak{c}\ (hd\ w)] \leq p\ f\ w$
⟨*proof*⟩

lemmas *not-comp-bin-lcp-pref* = *bin-not-comp-set-UNIV*[*THEN set-UNIV-bin-lcp-pref*]

lemma *marked-lcp-conv*: *marked-morphism* $f \longleftrightarrow \alpha = \varepsilon$
⟨*proof*⟩

lemma *im-comm-lcp*: $f\ w \cdot \alpha = \alpha \cdot f\ w \Longrightarrow (\forall\ a.\ a \in set\ w \longrightarrow f\ [a] \cdot \alpha = \alpha \cdot f\ [a])$
⟨*proof*⟩

lemma *im-comm-lcp-nemp*: **assumes** $f\ w \cdot \alpha = \alpha \cdot f\ w$ **and** $w \neq \varepsilon$ **and** $\alpha \neq \varepsilon$
obtains k **where** $w = [hd\ w]^{@}k$ **and** $0 < k$
⟨*proof*⟩

lemma *bin-lcp-ims-im-lcp*: $f\ w \cdot \alpha \wedge_p f\ w' \cdot \alpha = f\ (w \wedge_p w') \cdot \alpha$
⟨*proof*⟩

lemma *per-comp*:
assumes $r <p\ f\ w \cdot r$
shows $r \bowtie f\ w \cdot \alpha$
⟨*proof*⟩

end

7.5.2 More translations

lemma *bin-code-morph-iff'*: *binary-code-morphism* $f \longleftrightarrow$ *morphism* $f \wedge f\ [a] \cdot f\ [1-a] \neq f\ [1-a] \cdot f\ [a]$
⟨*proof*⟩

lemma *bin-code-morph-iff*: *binary-code-morphism* (*bin-morph-of* $x\ y$) $\longleftrightarrow x \cdot y \neq y \cdot x$
⟨*proof*⟩

lemma *bin-noner-morph-iff*: *nonerasing-morphism* (*bin-morph-of* $x\ y$) $\longleftrightarrow x \neq \varepsilon \wedge y \neq \varepsilon$
⟨*proof*⟩

lemma *morph-bin-morph-of*: *morphism* $f \longleftrightarrow$ *bin-morph-of* $(f\ \mathfrak{a})\ (f\ \mathfrak{b}) = f$
⟨*proof*⟩

lemma *two-bin-code-morphs-nonerasing-morphs*: *binary-code-morphism g* $\Longrightarrow$ *binary-code-morphism h* $\Longrightarrow$ *two-nonerasing-morphisms g h*
⟨*proof*⟩

7.6 Marked binary morphism

lemma *marked-binary-morphI*: **assumes** *morphism f* **and** *f* [*a* :: *binA*] $\neq \varepsilon$ **and** *f* [*1−a*] $\neq \varepsilon$ **and** *hd* (*f* [*a*]) $\neq$ *hd* (*f* [*1−a*])
shows *marked-morphism f*
⟨*proof*⟩

locale *marked-binary-morphism* = *marked-morphism f* :: *binA list* $\Rightarrow$ *'a list* **for** *f*

begin

lemma *bin-marked*: *hd* (*f* $\mathfrak{a}$) $\neq$ *hd* (*f* $\mathfrak{b}$)
⟨*proof*⟩

lemma *bin-marked-sing*: *hd* (*f* [*a*]) $\neq$ *hd* (*f* [*1−a*])
⟨*proof*⟩

sublocale *binary-code-morphism*
⟨*proof*⟩

lemma *marked-lcp-emp*: $\alpha = \varepsilon$
⟨*proof*⟩

lemma *bin-marked'*: (*f* $\mathfrak{a}$)!*0* $\neq$ (*f* $\mathfrak{b}$)!*0*
⟨*proof*⟩

lemma *marked-bin-morph-prefix-code*: *r* $\bowtie$ *s* $\vee$ *f* (*r* · *z1*) $\wedge_p$ *f* (*s* · *z2*) = *f* (*r* $\wedge_p$ *s*)
⟨*proof*⟩

end

lemma *bin-marked-preimg-hd*:
assumes *marked-binary-morphism* (*f* :: *binA list* $\Rightarrow$ *binA list*)

obtains c **where** $hd\ (f\ [c]) = a$
⟨*proof*⟩

7.7 Marked version

context *binary-code-morphism*

begin

definition *marked-version* (f_m) **where** $f_m = (\lambda\ w.\ \alpha^{-1>}(f\ w \cdot \alpha))$

lemma *marked-version-conjugates*: $\alpha \cdot f_m\ w = f\ w \cdot \alpha$
⟨*proof*⟩

lemma *marked-eq-conv*: $f\ w = f\ w' \longleftrightarrow f_m\ w = f_m\ w'$
⟨*proof*⟩

lemma *marked-marked*: **assumes** *marked-morphism* f **shows** $f_m = f$
⟨*proof*⟩

lemma *marked-version-all-nemp*: $w \neq \varepsilon \Longrightarrow f_m\ w \neq \varepsilon$
⟨*proof*⟩

lemma *marked-version-binary-code-morph*: *binary-code-morphism* f_m
⟨*proof*⟩

interpretation *mv-bcm*: *binary-code-morphism* f_m
⟨*proof*⟩

lemma *marked-lcs*: *bin-lcs* $(f_m\ \mathfrak{a})\ (f_m\ \mathfrak{b}) = \beta \cdot \alpha$
⟨*proof*⟩

lemma *bin-lcp-shift*: **assumes** $|\alpha| < |f\ w|$ **shows** $(f\ w)!|\alpha| = hd\ (f_m\ w)$
⟨*proof*⟩

lemma *mismatch-fst*: $hd\ (f_m\ \mathfrak{a}) = c_0$
⟨*proof*⟩

lemma *mismatch-snd*: $hd\ (f_m\ \mathfrak{b}) = c_1$
⟨*proof*⟩

lemma *marked-hd-neq*: $hd\ (f_m\ [a]) \neq hd\ (f_m\ [1-a])$ (**is** *?P* $(a :: binA)$)
⟨*proof*⟩

lemma *marked-version-marked-morph*: *marked-morphism* f_m
⟨*proof*⟩

interpretation *mv-mbm*: *marked-binary-morphism* f_m
⟨*proof*⟩

lemma *bin-code-pref-morph*: $f\ u \cdot \alpha \leq p\ f\ w \cdot \alpha \Longrightarrow u \leq p\ w$
⟨*proof*⟩

lemma *mismatch-pref0*: $[c_0] \leq p\ f_m\ \mathfrak{a}$
⟨*proof*⟩

lemma *mismatch-pref1*: $[c_1] \leq p\ f_m\ \mathfrak{b}$
⟨*proof*⟩

lemma *marked-version-len*: $|f_m\ w| = |f\ w|$
⟨*proof*⟩

lemma *bin-code-lcp*: $(f\ r \cdot \alpha) \wedge_p (f\ s \cdot \alpha) = f\ (r \wedge_p s) \cdot \alpha$
⟨*proof*⟩

lemma *not-comp-lcp*: **assumes** $\neg\ r \bowtie s$
shows $f\ (r \wedge_p s) \cdot \alpha = f\ r \cdot f\ (r \cdot s) \wedge_p f\ s \cdot f\ (r \cdot s)$
⟨*proof*⟩

lemma *bin-morph-pref-conv*: $f\ u \cdot \alpha \leq p\ f\ v \cdot \alpha \longleftrightarrow u \leq p\ v$
⟨*proof*⟩

lemma *bin-morph-compare-conv*: $f\ u \cdot \alpha \bowtie f\ v \cdot \alpha \longleftrightarrow u \bowtie v$
⟨*proof*⟩

lemma *code-lcp'*: $\neg\ r \bowtie s \Longrightarrow \alpha \leq p\ f\ z \Longrightarrow \alpha \leq p\ f\ z' \Longrightarrow f\ (r \cdot z) \wedge_p f\ (s \cdot z')$
$= f\ (r \wedge_p s) \cdot \alpha$
⟨*proof*⟩

lemma *non-comm-im-lcp*: **assumes** $u \cdot v \neq v \cdot u$
shows $f\ (u \cdot v) \wedge_p f\ (v \cdot u) = f\ (u \cdot v \wedge_p v \cdot u) \cdot \alpha$
⟨*proof*⟩

end

— Obtaining one morphism marked from two general morphisms by shift (conjugation)

locale *binary-code-morphism-shift* = *binary-code-morphism* +
fixes α'
assumes *shift-pref*: $\alpha' \leq p\ \alpha$

begin

definition *shifted-f* **where** *shifted-f* = $(\lambda\ w.\ \alpha'^{-1>}(f\ w \cdot \alpha'))$

lemma *shift-pref-all*: $\alpha' \leq p\ f\ w \cdot \alpha'$
⟨*proof*⟩

sublocale *shifted*: *binary-code-morphism shifted-f*
⟨*proof*⟩

lemma *shifted-lcp*: $\alpha' \cdot$ *shifted.bin-code-lcp* $= \alpha$
⟨*proof*⟩

lemma $\alpha' = \alpha \Longrightarrow$ *shifted-f* $= f_m$
⟨*proof*⟩

end

7.8 Two binary code morphisms

locale *two-binary-code-morphisms* =
g: *binary-code-morphism g* +
h: *binary-code-morphism h*
for *g h* :: *binA list* ⇒ *'a list*

begin

notation *h.bin-code-lcp* (α_h)
notation *g.bin-code-lcp* (α_g)
notation *g.marked-version* (g_m)
notation *h.marked-version* (h_m)

sublocale *gm*: *marked-binary-morphism* g_m
⟨*proof*⟩

sublocale *hm*: *marked-binary-morphism* h_m
⟨*proof*⟩

sublocale *two-binary-morphisms g h*⟨*proof*⟩

sublocale *marked*: *two-marked-morphisms* g_m h_m⟨*proof*⟩

sublocale *code*: *two-code-morphisms g h*
⟨*proof*⟩

lemma *marked-two-binary-code-morphisms*: *two-binary-code-morphisms* g_m h_m
⟨*proof*⟩

lemma *revs-two-binary-code-morphisms*: *two-binary-code-morphisms* (*rev-map g*)
(*rev-map h*)
⟨*proof*⟩

lemma *swap-two-binary-code-morphisms*: *two-binary-code-morphisms h g*

⟨*proof*⟩

Each successful overflow has a unique minimal successful continuation

lemma *min-completionE*:
assumes $z \cdot g_m\ r = z' \cdot h_m\ s$
obtains $p\ q$ **where** $z \cdot g_m\ p = z' \cdot h_m\ q$ **and**
$\bigwedge r\ s.\ z \cdot g_m\ r = z' \cdot h_m\ s \implies p \leq p\ r \wedge q \leq p\ s$
⟨*proof*⟩

lemma *two-equals*:
assumes $g\ r = h\ r$ **and** $g\ s = h\ s$ **and** $\neg\ r \bowtie s$
shows $g\ (r \wedge_p s) \cdot \alpha_g = h\ (r \wedge_p s) \cdot \alpha_h$
⟨*proof*⟩

lemma *solution-sing-len-diff*: **assumes** $g \neq h$ **and** $g\ s = h\ s$ **and** $set\ s = binUNIV$
shows $|g\ [c]| \neq |h\ [c]|$
⟨*proof*⟩

lemma *alphas-pref*: **assumes** $|\alpha_h| \leq |\alpha_g|$ **and** $g\ r =_m h\ s$ **shows** $\alpha_h \leq p\ \alpha_g$
⟨*proof*⟩

end

locale *binary-codes-coincidence* = *two-binary-code-morphisms* +
assumes *alphas-len*: $|\alpha_h| \leq |\alpha_g|$ **and**
coin-ex: $\exists\ r\ s.\ g\ r =_m h\ s$
begin

lemma *alphas-pref*: $\alpha_h \leq p\ \alpha_g$
⟨*proof*⟩

definition α **where** $\alpha \equiv {\alpha_h}^{-1>}\alpha_g$
definition *critical-overflow* (c) **where** $critical\text{-}overflow \equiv {\alpha_g}^{<-1}\alpha_h$

lemma *lcp-diff*: $\alpha_h \cdot \alpha = \alpha_g$
⟨*proof*⟩

lemma *solution-marked-version-conv*: $g\ r = h\ s \longleftrightarrow \alpha \cdot\ g_m\ r = h_m\ s \cdot \alpha$
⟨*proof*⟩

end

locale *binary-code-coincidence-sym* = *two-binary-code-morphisms*
+ **assumes**
coin-ex: $\exists\ r\ s.\ g\ r =_m h\ s$
begin

lemma *coinE*: **obtains** $u\ v$ **where** $g\ u =_m h\ v$ **and** $h\ v =_m g\ u$
⟨*proof*⟩

definition α' **where** $\alpha' = (\mathit{if}\ |\alpha_g| \leq |\alpha_h|\ \mathit{then}\ \alpha_g\ \mathit{else}\ \alpha_h)$
definition g' **where** $g' = (\mathit{if}\ |\alpha_g| \leq |\alpha_h|\ \mathit{then}\ (\lambda\ w.\ \alpha'^{-1>}(g\ w \cdot \alpha'))\ \mathit{else}\ (\lambda\ w.\ \alpha'^{-1>}(h\ w \cdot \alpha')))$
definition h' **where** $h' = (\mathit{if}\ |\alpha_g| \leq |\alpha_h|\ \mathit{then}\ (\lambda\ w.\ \alpha'^{-1>}(h\ w \cdot \alpha'))\ \mathit{else}\ (\lambda\ w.\ \alpha'^{-1>}(g\ w \cdot \alpha')))$

lemma *shift-pref-fst*: $\alpha' \leq p\ \alpha_g$
⟨*proof*⟩

interpretation *gshift*: *binary-code-morphism-shift* $g\ \alpha'$
⟨*proof*⟩

interpretation *swap*: *two-binary-code-morphisms* $h\ g$
⟨*proof*⟩

lemma *shift-pref-snd*: $\alpha' \leq p\ \alpha_h$
⟨*proof*⟩

interpretation *hshift*: *binary-code-morphism-shift* $h\ \alpha'$
⟨*proof*⟩

lemma *shifted-eq-conv*:$g\ r = h\ s \longleftrightarrow g'\ r = h'\ s$
⟨*proof*⟩

lemma *shifted-eq-conv*:$g\ r = h\ r \longleftrightarrow g'\ r = h'\ r$
⟨*proof*⟩

lemma *shifted-eq-conv'*:$g = h \longleftrightarrow g' = h'$
⟨*proof*⟩

interpretation *shifted-g*: *binary-code-morphism* $(\lambda\ w.\ \alpha'^{-1>}(g\ w \cdot \alpha'))$
⟨*proof*⟩

interpretation *shifted-h*: *binary-code-morphism* $(\lambda\ w.\ \alpha'^{-1>}(h\ w \cdot \alpha'))$
⟨*proof*⟩

lemma *shifted-min-sol-conv*: $r \in g =_M h \longleftrightarrow r \in g' =_M h'$
⟨*proof*⟩

lemma *shifted-not-triv*: $g = h \longleftrightarrow g' = h'$
⟨*proof*⟩

sublocale *shifted*: *two-binary-code-morphisms* $g'\ h'$
⟨*proof*⟩

lemma *shifted-fst-lcp-emp*: *shifted.g.bin-code-lcp* $= \varepsilon$
⟨*proof*⟩

lemma *shifted-alphas*: **assumes** *le*: $|\alpha_g| \leq |\alpha_h|$
shows $\alpha' \cdot$ *shifted.g.bin-code-lcp* $= \alpha_g$ **and** $\alpha' \cdot$ *shifted.h.bin-code-lcp* $= \alpha_h$
⟨*proof*⟩

interpretation *swapped*: *binary-code-coincidence-sym h g*
⟨*proof*⟩

lemma *eq-len-eq-conv*: $\alpha_g = \alpha_h \longleftrightarrow |\alpha_g| = |\alpha_h|$
⟨*proof*⟩

lemma *shift-swapped*: *swapped*.$\alpha' = \alpha'$
⟨*proof*⟩

lemma *morphs-swapped*: **assumes** $|\alpha_g| \neq |\alpha_h|$ **shows** *swapped.g'* $= g'$ *swapped.h'* $= h'$
⟨*proof*⟩

lemma *morphs-swapped'*: **assumes** $|\alpha_g| = |\alpha_h|$ **shows** *swapped.g'* $= h'$ *swapped.h'* $= g'$
⟨*proof*⟩

lemma *shifted-lcp-len-eq*: |*shifted.g.bin-code-lcp*| = |*shifted.h.bin-code-lcp*| $\longleftrightarrow |\alpha_g| = |\alpha_h|$ **and**
shifted-lcp-len-le: |*shifted.g.bin-code-lcp*| $\leq$ |*shifted.h.bin-code-lcp*|
⟨*proof*⟩

end

locale *two-marked-binary-morphisms* = *two-marked-morphisms g h*
for *g h* :: *binA list* ⇒ *'a list*
begin

sublocale *two-binary-code-morphisms g h* ⟨*proof*⟩

lemma *not-comm-im*: **assumes** $g \neq h$ **and** $g\ s = h\ s$ **and** $s \neq \varepsilon$
and *hd s* = *a* **and** *set s* = *binUNIV*
shows $g[a] \cdot h\ [a] \neq h[a] \cdot g[a]$
⟨*proof*⟩

lemma *sol-set-not-com-hd*:
assumes

morphs-neq: $g \neq h$ **and**
sol: $g\ s = h\ s$ **and**
sol-set: $set\ s = binUNIV$
shows $g\ ([hd\ s]) \cdot\ h\ ([hd\ s]) \neq h\ ([hd\ s]) \cdot\ g\ ([hd\ s])$
⟨*proof*⟩

sublocale *g*: *marked-binary-morphism g*
⟨*proof*⟩

sublocale *h*: *marked-binary-morphism h*
⟨*proof*⟩

sublocale *revs*: *two-binary-code-morphisms rev-map g rev-map h*
⟨*proof*⟩

end

7.9 Two marked binary morphisms with blocks

locale *two-binary-marked-blocks* = *two-marked-binary-morphisms* +
assumes *both-blocks*: $\bigwedge a.\ blockP\ a$

begin

sublocale *sucs*: *two-marked-binary-morphisms suc-fst suc-snd*
⟨*proof*⟩

sublocale *sucs-enc*: *two-marked-binary-morphisms suc-fst′ suc-snd′*
⟨*proof*⟩

lemma *bin-blocks-swap*: *two-binary-marked-blocks h g*
⟨*proof*⟩

lemma *blocks-all-letters-fst*: $[b] \leq_f suc\text{-}fst\ ([a] \cdot [1-a])$
⟨*proof*⟩

lemma *blocks-all-letters-snd*: $[b] \leq_f suc\text{-}snd\ ([a] \cdot [1-a])$
⟨*proof*⟩

lemma *lcs-suf-blocks-fst*: $g.bin\text{-}code\text{-}lcs \leq_s g\ (suc\text{-}fst\ ([a] \cdot [1-a]))$
⟨*proof*⟩

lemma *lcs-suf-blocks-snd*: $h.bin\text{-}code\text{-}lcs \leq_s h\ (suc\text{-}snd\ ([a] \cdot [1-a]))$
⟨*proof*⟩

lemma *lcs-fst-suf-snd*: $g.bin\text{-}code\text{-}lcs \leq_s h.bin\text{-}code\text{-}lcs \cdot h\ \ sucs.h.bin\text{-}code\text{-}lcs$
⟨*proof*⟩

lemma *suf-comp-lcs*: $g.bin\text{-}code\text{-}lcs \bowtie_s h.bin\text{-}code\text{-}lcs$

⟨*proof*⟩

end

7.10 Binary primitivity preserving morphism given by a pair of words

definition *bin-prim* :: *'a list* ⇒ *'a list* ⇒ *bool*
  where *bin-prim x y* ⟷ *primitivity-preserving-morphism* (*bin-morph-of x y*)

lemma *bin-prim-code*:
  assumes *bin-prim x y*
  shows $x \cdot y \neq y \cdot x$
⟨*proof*⟩

7.10.1 Translating to to list concatenation

lemma *bin-concat-prim-pres-noner1*:
  assumes $x \neq y$
   and *prim-pres*: $\bigwedge$ *ws*. *ws* ∈ *lists* {*x*,*y*} ⟹ *2* ≤ |*ws*| ⟹ *primitive ws* ⟹ *primitive* (*concat ws*)
  shows $x \neq \varepsilon$
⟨*proof*⟩

lemma *bin-concat-prim-pres-noner*:
  assumes $x \neq y$
   and *prim-pres*: $\bigwedge$ *ws*. *ws* ∈ *lists* {*x*,*y*} ⟹ *2* ≤ |*ws*| ⟹ *primitive ws* ⟹ *primitive* (*concat ws*)
  shows *nonerasing-morphism* (*bin-morph-of x y*)
⟨*proof*⟩

lemma *bin-prim-concat-prim-pres-conv*:
  assumes $x \neq y$
   shows *bin-prim x y* ⟷ (∀ *ws* ∈ *lists* {*x*,*y*}. *2* ≤ |*ws*| ⟶ *primitive ws* ⟶ *primitive* (*concat ws*))
  (**is** - ⟷ *?condition*)
⟨*proof*⟩

lemma *bin-prim-concat-prim-pres*:
  assumes *bin-prim x y*
  shows *ws* ∈ *lists* {*x*, *y*} ⟹ *2* ≤ |*ws*| ⟹ *primitive ws* ⟹ *primitive* (*concat ws*)
  ⟨*proof*⟩

lemma *bin-prim-altdef1*:
  bin-prim x y ⟷
    ($x \neq y$) ∧ (∀ *ws* ∈ *lists* {*x*,*y*}. *2* ≤ |*ws*| ⟶ *primitive ws* ⟶ *primitive* (*concat ws*))
  ⟨*proof*⟩

lemma *bin-prim-altdef2*:
bin-prim x y ⟷
(*x* · *y* ≠ *y* · *x*) ∧ (∀ *ws* ∈ *lists* {*x*,*y*}. *2* ≤ |*ws*| ⟶ *primitive ws* ⟶ *primitive* (*concat ws*))
⟨*proof*⟩

7.10.2 Basic properties of *bin-prim*

lemma *bin-prim-irrefl*: ¬ *bin-prim x x*
⟨*proof*⟩

lemma *bin-prim-symm* [*sym*]: *bin-prim x y* ⟹ *bin-prim y x*
⟨*proof*⟩

lemma *bin-prim-commutes*: *bin-prim x y* ⟷ *bin-prim y x*
⟨*proof*⟩

end

theory *Equations-Basic*
imports
Periodicity-Lemma
Lyndon-Schutzenberger
Submonoids
Binary-Code-Morphisms
begin

Chapter 8

Equations on words - basics

Contains various nontrivial auxiliary or rudimentary facts related to equations. Often moderately advanced or even fairly advanced. May change significantly in the future.

8.1 Miscellanea

8.1.1 Mismatch additions

lemma *mismatch-pref-comm-len*: **assumes** $w1 \in \langle\{u,v\}\rangle$ **and** $w2 \in \langle\{u,v\}\rangle$ **and** $p \leq_p w1$
$u \cdot p \leq_p v \cdot w2$ **and** $|v| \leq |p|$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *mismatch-pref-comm*: **assumes** $w1 \in \langle\{u,v\}\rangle$ **and** $w2 \in \langle\{u,v\}\rangle$ **and**
$u \cdot w1 \cdot v \leq_p v \cdot w2 \cdot u$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *mismatch-eq-comm*: **assumes** $w1 \in \langle\{u,v\}\rangle$ **and** $w2 \in \langle\{u,v\}\rangle$ **and**
$u \cdot w1 = v \cdot w2$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemmas *mismatch-suf-comm* = *mismatch-pref-comm*[*reversed*] **and**
mismatch-suf-comm-len = *mismatch-pref-comm-len*[*reversed*, *unfolded rassoc*]

8.1.2 Conjugate words with conjugate periods

lemma *conj-pers-conj-comm-aux*:
assumes $(u \cdot v)^{@}k \cdot u = r \cdot s$ **and** $(v \cdot u)^{@}l \cdot v = (s \cdot r)^{@}m$ **and** $0 < k$ $0 < l$ **and** $2 \leq m$
shows $u \cdot v = v \cdot u$

⟨*proof*⟩

lemma *conj-pers-conj-comm*: **assumes** $\varrho\ (v \cdot (u \cdot v)^{@} k) \sim \varrho\ ((u \cdot v)^{@} m \cdot u)$ **and** $0 < k$ **and** $0 < m$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

hide-fact *conj-pers-conj-comm-aux*

8.1.3 Covering uvvu

lemma *uv-fac-uvv*: **assumes** $p \cdot u \cdot v \leq p\ u \cdot v \cdot v$ **and** $p \neq \varepsilon$ **and** $p \leq s\ w$ **and** $w \in \langle\{u,v\}\rangle$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemmas *uv-fac-uvv-suf* = *uv-fac-uvv*[*reversed*, *unfolded rassoc*]

lemma $u \leq p\ v \Longrightarrow u' \leq p\ v' \Longrightarrow u \wedge_p u' \neq u \Longrightarrow u \wedge_p u' \neq u' \Longrightarrow u \wedge_p u' = v \wedge_p v'$
⟨*proof*⟩

lemma *comm-puv-pvs-eq-uq*: **assumes** $p \cdot u \cdot v = u \cdot v \cdot p$ **and** $p \cdot v \cdot s = u \cdot q$ **and**
$p \leq p\ u\ q \leq p\ w$ **and** $s \leq p\ \ w'$ **and**
$w \in \langle\{u,v\}\rangle$ **and** $w' \in \langle\{u,v\}\rangle$ **and** $|u| \leq |s|$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma assumes $u \cdot v \cdot v \cdot u = p \cdot u \cdot v \cdot u \cdot q$ **and** $p \neq \varepsilon$ **and** $q \neq \varepsilon$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *uvu-pref-uvv*: **assumes** $p \cdot u \cdot v \cdot v \cdot s = u \cdot v \cdot u \cdot q$ **and**
$p \leq p\ u$ **and** $q \leq p\ w$ **and** $s \leq p\ \ w'$ **and**
$w \in \langle\{u,v\}\rangle$ **and** $w' \in \langle\{u,v\}\rangle$ **and** $|u| \leq |s|$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *uvu-pref-uvvu*: **assumes** $p \cdot u \cdot v \cdot v \cdot u = u \cdot v \cdot u \cdot q$ **and**
$p \leq p\ u$ **and** $q \leq p\ w$ **and** $\ \ w \in \langle\{u,v\}\rangle$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemma *uvu-pref-uvvu-interp*: **assumes** *interp*: $p\ u \cdot v \cdot v \cdot u\ s \sim_{\mathcal{I}} ws$ **and**
$[u,\ v,\ u] \leq p\ \ ws$ **and** $ws \in lists\ \{u,v\}$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

lemmas *uvu-suf-uvvu* = *uvu-pref-uvvu*[*reversed*, *unfolded rassoc*] **and**
uvu-suf-uvv = *uvu-pref-uvv*[*reversed*, *unfolded rassoc*]

lemma *uvu-suf-uvvu-interp*: $p\ u \cdot v \cdot v \cdot u\ s \sim_{\mathcal{I}} ws \Longrightarrow [u,\ v,\ u] \leq s\ \ ws$
$\Longrightarrow ws \in lists\ \{u,v\} \Longrightarrow u \cdot v = v \cdot u$
⟨*proof*⟩

8.1.4 Conjugate words

lemma *conjug-pref-suf-mismatch*: **assumes** $w1 \in \langle\{r \cdot s, s \cdot r\}\rangle$ **and** $w2 \in \langle\{r \cdot s, s \cdot r\}\rangle$
and $r \cdot w1 = w2 \cdot s$
shows $r = s \vee r = \varepsilon \vee s = \varepsilon$
⟨*proof*⟩

lemma *conjug-conjug-primroots*: **assumes** $u \neq \varepsilon$ **and** $r \neq \varepsilon$ **and** $\varrho\ (u \cdot v) = r \cdot s$ **and** $\varrho\ (v \cdot u) = s \cdot r$
obtains $k\ m$ **where** $(r \cdot s)^{@}k \cdot r = u$ **and** $(s \cdot r)^{@}m \cdot s = v$
⟨*proof*⟩

8.1.5 Predicate "commutes"

definition *commutes* :: *'a list set* ⇒ *bool*
where $commutes\ A = (\forall x\ y.\ x \in A \longrightarrow y \in A \longrightarrow x \cdot y = y \cdot x)$

lemma *commutesE*: $commutes\ A \Longrightarrow x \in A \Longrightarrow y \in A \Longrightarrow x \cdot y = y \cdot x$
⟨*proof*⟩

lemma *commutes-root*: **assumes** *commutes A*
obtains r **where** $\bigwedge x.\ x \in A \Longrightarrow x \in \langle\{r\}\rangle$
⟨*proof*⟩

lemma *commutes-primroot*: **assumes** *commutes A*
obtains r **where** $\bigwedge x.\ x \in A \Longrightarrow x \in \langle\{r\}\rangle$ **and** *primitive r*
⟨*proof*⟩

lemma *commutesI* [*intro*]: $(\bigwedge x\ y.\ x \in A \Longrightarrow y \in A \Longrightarrow x \cdot y = y \cdot x) \Longrightarrow commutes\ A$
⟨*proof*⟩

lemma *commutesI'*: **assumes** $x \neq \varepsilon$ **and** $\bigwedge y.\ y \in A \Longrightarrow x \cdot y = y \cdot x$
shows *commutes A*
⟨*proof*⟩

lemma *commutesI-root*[*intro*]: $(\bigwedge x.\ x \in A \Longrightarrow x \in \langle\{t\}\rangle) \Longrightarrow$ *commutes* A
⟨*proof*⟩

lemma *commutesI-ex-root*: $\exists\ t.\ \forall\ x \in A.\ x \in \langle\{t\}\rangle \Longrightarrow$ *commutes* A
⟨*proof*⟩

lemma *commutes-sub*: *commutes* $A \Longrightarrow B \subseteq A \Longrightarrow$ *commutes* B
⟨*proof*⟩

lemma *commutes-insert*: *commutes* $A \Longrightarrow x \in A \Longrightarrow x \neq \varepsilon \Longrightarrow\ x{\cdot}y = y{\cdot}x \Longrightarrow$ *commutes* (*insert* y A)
⟨*proof*⟩

lemma *commutes-emp* [*simp*]: *commutes* $\{\varepsilon,\ w\}$
⟨*proof*⟩

lemma *commutes-emp'*[*simp*]: *commutes* $\{w,\ \varepsilon\}$
⟨*proof*⟩

lemma *commutes-cancel*: **assumes** $y \in A$ **and** $x \cdot y \in A$ **and** *commutes* A
shows *commutes* (*insert* x A)
⟨*proof*⟩

lemma *commutes-cancel'*: **assumes** $x \in A$ **and** $x \cdot y \in A$ **and** *commutes* A
shows *commutes* (*insert* y A)
⟨*proof*⟩

8.1.6 Strong elementary lemmas

Discovered by smt

lemma *xyx-per-comm*: **assumes** $x{\cdot}y{\cdot}x \leq p\ q{\cdot}x{\cdot}y{\cdot}x$
and $q \neq \varepsilon$ **and** $q \leq p\ y \cdot q$
shows $x{\cdot}y = y{\cdot}x$
⟨*proof*⟩

lemma *two-elem-root-suf-comm*: **assumes** $u \leq p\ v \cdot u$ **and** $v \leq s\ p \cdot u$ **and** $p \in \langle\{u,v\}\rangle$
shows $u \cdot v = v \cdot u$
⟨*proof*⟩

8.1.7 Binary words without a letter square

lemma *no-repetition-list*:
assumes *set ws* $\subseteq \{a,b\}$
and *not-per*: $\neg\ ws \leq p\ [a,b] \cdot ws\ \neg\ ws \leq p\ [b,a] \cdot ws$
and *not-square*: $\neg\ [a,a] \leq f\ ws$ **and** $\neg\ [b,b] \leq f\ ws$
shows *False*
⟨*proof*⟩

lemma *no-repetition-list-bin*:
fixes *ws* :: *binA list*
assumes *not-square*: $\bigwedge$ *c*. ¬ [*c*,*c*] ≤*f* *ws*
shows *ws* ≤*p* [*hd ws*, *1*−(*hd ws*)] · *ws*
⟨*proof*⟩

lemma *per-root-hd-last-root*: **assumes** *ws* ≤*p* [*a*,*b*] · *ws* **and** *hd ws* ≠ *last ws*
shows *ws* ∈ ⟨{[*a*,*b*]}⟩
⟨*proof*⟩

lemma *no-cyclic-repetition-list*:
assumes *set ws* ⊆ {*a*,*b*} *ws* ∉ ⟨{[*a*,*b*]}⟩ *ws* ∉ ⟨{[*b*,*a*]}⟩ *hd ws* ≠ *last ws*
¬ [*a*,*a*] ≤*f* *ws* ¬ [*b*,*b*] ≤*f* *ws*
shows *False*
⟨*proof*⟩

8.1.8 Three covers

lemma *three-covers-example*:
assumes
v: $v = \mathfrak{a}$ **and**
t: $t = (\mathfrak{b} \cdot \mathfrak{a}^{@}(j{+}1))^{@}(m + l + 1) \cdot \mathfrak{b} \cdot \mathfrak{a}$ **and**
r: $r = \mathfrak{a} \cdot \mathfrak{b} \cdot (\mathfrak{a}^{@}(j{+}1) \cdot \mathfrak{b})^{@}(m + l + 1)$ **and**
t': $t' = (\mathfrak{b} \cdot \mathfrak{a}^{@}(j + 1))^{@}m \cdot \mathfrak{b} \cdot \mathfrak{a}$ **and**
r': $r' = \mathfrak{a} \cdot \mathfrak{b} \cdot (\mathfrak{a}^{@}(j + 1) \cdot \mathfrak{b})^{@}l$ **and**
w: $w = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{a}^{@}(j + 1))^{@}(m + l + 1) \cdot \mathfrak{b} \cdot \mathfrak{a}$
shows $w = v \cdot t$ **and** $w = r \cdot v$ **and** $w = r' \cdot v^{@}(j + 1) \cdot t'$ **and** $t' <_p t$ **and** $r' <_s r$
⟨*proof*⟩

lemma *three-covers-pers*: — alias Old Good Lemma
assumes $w = v \cdot t$ **and** $w = r' \cdot v^{@}j \cdot t'$ **and** $w = r \cdot v$ **and** $0 < j$ **and**
$r' <_s r$ **and** $t' <_p t$
shows *period w* (|*t*| − |*t'*|) **and** *period w* (|*r*| − |*r'*|) **and**
(|*t*| − |*t'*|) + (|*r*| − |*r'*|) = |*w*| + *j*∗|*v*| − *2*∗|*v*|
⟨*proof*⟩

lemma *three-covers-per0*: **assumes** $w = v \cdot t$ **and** $w = r' \cdot v^{@}\ j \cdot t'$ **and** $w = r \cdot v$ **and** $0 < j$
$r' <_s r$ **and** $t' <_p t$ **and** |*t'*| ≤ |*r'*|
and *primitive v*
shows *period w* (*gcd* (|*t*| − |*t'*|) (|*r*| − |*r'*|))
⟨*proof*⟩

lemma *three-covers-per*: **assumes** $w = v \cdot t$ **and** $w = r' \cdot v^{@}j \cdot t'$ **and** $w = r \cdot v$
$r' <_s r$ **and** $t' <_p t$ **and** $0 < j$
shows *period w* (*gcd* (|*t*| − |*t'*|) (|*r*| − |*r'*|))
⟨*proof*⟩

thm *per-root-modE′*

lemma assumes $w \leq p\ r \cdot w\ r \neq \varepsilon$
obtains $p\ q\ i$ **where** $w = (p \cdot q)^{@} i \cdot p\ p \cdot q = r$
⟨*proof*⟩

lemma *three-coversE*: **assumes** $w = v \cdot t$ **and** $w = r' \cdot v \cdot t'$ **and** $w = r \cdot v$ **and** $r' <s\ r$ **and** $t' <p\ t$
obtains $p\ q\ i\ k\ m$ **where** $t = (q \cdot p)^{@}(m{+}k)$ **and** $r = (p \cdot q)^{@}(m{+}k)$ **and** $t' = (q \cdot p)^{@} k$ **and** $r' = (p \cdot q)^{@} m$ **and** $v = (p \cdot q)^{@} i \cdot p$ **and** $w = (p \cdot q)^{@}(m + i + k) \cdot p$ **and** *primitive* $(p \cdot q)$ **and** $q \neq \varepsilon$ **and** $0 < m$ **and** $0 < k$
⟨*proof*⟩

lemma *three-covers-pref-suf-pow*: **assumes** $x \cdot y \leq p\ w$ **and** $y \cdot x \leq s\ w$ **and** $w \leq f\ y^{@} k$ **and** $|y| \leq |x|$
shows $x \cdot y = y \cdot x$
⟨*proof*⟩

8.1.9 Binary Equality Words

definition *binary-equality-generator* :: *binA list* ⇒ *bool* **where**
binary-equality-generator w ≡ (*set w* = *UNIV*) ∧ (∃ (*g* :: *binA list* ⇒ *nat list*) *h*. *binary-code-morphism g* ∧ *binary-code-morphism h* ∧ *g* ≠ *h* ∧ *w* ∈ *g* =$_M$ *h*)

definition *canonical-binary-equality-generator* :: *binA list* ⇒ *bool* **where**
canonical-binary-equality-generator w = (∃ (*g* :: *binA list* ⇒ *nat list*) *h*. *binary-code-morphism g* ∧ *binary-code-morphism h* ∧ *w* ∈ *g* =$_M$ *h* ∧ |*h* 𝔞| < |*g* 𝔞| ∧ |*g* 𝔟| < |*h* 𝔟| ∧ |*g* 𝔞| ≤ |*h* 𝔟|)

lemma *begE*: **assumes** *binary-equality-generator w*
obtains *g h* **where** *binary-code-morphism* (*g* :: *binA list* ⇒ *nat list*) **and** *binary-code-morphism h* **and** *g* ≠ *h* **and** *w* ∈ *g* =$_M$ *h* **and** *set w* = *UNIV*
⟨*proof*⟩

lemma *cbegE*: **assumes** *canonical-binary-equality-generator w*
obtains *g h* **where** *binary-code-morphism* (*g* :: *binA list* ⇒ *nat list*) **and** *binary-code-morphism h* **and** *w* ∈ *g* =$_M$ *h* **and** |*h* 𝔞| < |*g* 𝔞| **and** |*g* 𝔟| < |*h* 𝔟| **and** |*g* 𝔞| ≤ |*h* 𝔟|

⟨*proof*⟩

lemma *cbeg-is-beg*: **assumes** *canonical-binary-equality-generator w* **shows** *binary-equality-generator w*
⟨*proof*⟩

lemma *beg-productE*: **assumes** *binary-equality-generator* $(u{\cdot}v)$ **and** $u \neq \varepsilon$ **and** $v \neq \varepsilon$
obtains *g h* **where** *binary-code-morphism* $(g :: binA\ list \Rightarrow nat\ list)$ **and** *binary-code-morphism h* **and** $g \neq h$ **and** $(u{\cdot}v) \in g =_M h$ **and** $|g\ u| < |h\ u|$ **and** $|h\ v| < |g\ v|$
⟨*proof*⟩

lemma *bew-baiba-eq'*: **assumes** $|y| < |v|$ **and** $x \leq s\ y$ **and** $u \leq s\ v$ **and**
$y \cdot x\ ^{@}\ k \cdot y = v \cdot u\ ^{@}\ k \cdot v$
shows *commutes* $\{x,y,u,v\}$
⟨*proof*⟩

lemma *bew-baiba-eq*: **assumes** $y \cdot x \neq v \cdot u$ **and**
$y \cdot x\ ^{@}\ (k{+}1) \cdot y \cdot x = v \cdot u\ ^{@}\ (k{+}1) \cdot v \cdot u$
shows *commutes* $\{x,y,u,v\}$
⟨*proof*⟩

lemmas *less-mult-le* [*intro*] = *mult-le-mono1*[*OF Suc-leI*, *unfolded mult-Suc*]
lemma *less-mult-le'* [*intro*]: $m < (k{::}nat) \Longrightarrow m{*}r + r \leq k{*}r$
⟨*proof*⟩

lemma *bew-baibaib-eq-aux*: **assumes** $|x| < |u|$ **and** $1 < i$ **and**
eq: $x \cdot y^{@}i{\cdot}\ x \cdot y^{@}i \cdot x = u \cdot v^{@}i{\cdot}\ u \cdot v^{@}i \cdot u$
shows *commutes* $\{x,y,u,v\}$
⟨*proof*⟩

lemma *bew-baibaib-eq*: **assumes** $1 < i$ **and** $x \neq u$ **and**
eq: $x \cdot y^{@}i{\cdot}\ x \cdot y^{@}i \cdot x = u \cdot v^{@}i{\cdot}\ u \cdot v^{@}i \cdot u$
shows *commutes* $\{x,y,u,v\}$
⟨*proof*⟩

theorem *not-beg-abiab*: ¬ *binary-equality-generator* $(\mathfrak{a} \cdot \mathfrak{b}^{@}(i{+}1) \cdot \mathfrak{a} \cdot \mathfrak{b})$
⟨*proof*⟩

theorem *not-beg-baibaib*: **assumes** $1 < i$
shows ¬ *binary-equality-generator* $(\mathfrak{b} \cdot \mathfrak{a}^{@}i \cdot \mathfrak{b} \cdot \mathfrak{a}^{@}i \cdot \mathfrak{b})$
⟨*proof*⟩

end

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