Chvátal's Theorem on the Ramsey Number of a Triangle versus a Tree

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November 11, 2026

Abstract

We formalize Chvátal's theorem [1]: for every tree T with n vertices, the graph Ramsey number $R(K_3, T)$ equals $2n-1$. The upper bound is proved by induction on n, removing a leaf; the lower bound is the complete bipartite graph with parts of sizes $n-1$ and $n-1$. As a corollary, every connected graph with n vertices and at most $n-1$ edges, being a tree, has the same Ramsey number. Graphs, cliques and Ramsey numbers are those of the entry *Ramsey_Bounds*, and trees those of the entry *Tree_Enumeration*.

Acknowledgement. The formalization and the text of this document were prepared with the assistance of generative AI systems, and checked by the author.

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theory *Chvatal-Tree-Ramsey*
imports *Ramsey-Bounds.Ramsey-Bounds Tree-Enumeration.Tree-Graph*
begin

1 Chvátal's theorem: the Ramsey number of a triangle versus a tree

The host graphs and Ramsey numbers follow the AFP entry *Ramsey-Bounds*: a graph on a finite set W is a set of edges, cliques and independent sets are those of *HOL−Library.Ramsey*, and the definition below has the shape of *is-clique-RN*. Trees are those of the AFP entry *Tree-Enumeration* (locale *tree*: a finite connected undirected graph without cycles). A copy of a tree (V_T, E_T) in the complement of a graph is an injective map of V_T into W sending every edge of the tree to a non-edge of the graph.

definition *compl-copy* :: *'a set ⇒ 'a set set ⇒ 'b set ⇒ 'b set set ⇒ bool* **where**
compl-copy VT ET W E ⟷
(∃f. inj-on f VT ∧ f ‘ VT ⊆ W ∧ (∀u v. {u, v} ∈ ET ⟶ {f u, f v} ∉ E))

definition *is-triangle-tree-RN* **where**
is-triangle-tree-RN ≡ λU::'b itself. λVT ET r.
(∀W::'b set. ∀E. finite W ⟶ card W ≥ r ⟶
(∃K⊆W. card K = 3 ∧ clique K E) ∨ compl-copy VT ET W E)

The Ramsey number $R(K_3, T)$, with vertex sets of natural numbers as in *RN*.

definition *triangle-tree-RN* :: *'a set ⇒ 'a set set ⇒ nat* **where**
triangle-tree-RN VT ET = (LEAST r. is-triangle-tree-RN TYPE(nat) VT ET r)

lemma *is-triangle-tree-RN-mono*:
assumes *is-triangle-tree-RN U VT ET r* **and** *r ≤ s*
shows *is-triangle-tree-RN U VT ET s*
⟨*proof*⟩

2 The upper bound

Induction on the tree, adding a leaf l at v. Let R be the part of W not used by the copy of the smaller tree. If some vertex of R is not adjacent to the image of v, it receives l. Otherwise R lies in the neighbourhood of that image, so it is independent because there is no triangle, and it is large enough to hold the whole tree.

lemma *chvatal-upper-aux*:
fixes *W :: 'b set* **and** *H :: 'b set set*
assumes *tr*: *tree VT ET* **and** *finW*: *finite W*
and *noK*: *¬ (∃K⊆W. card K = 3 ∧ clique K H)* **and** *big*: *2 * card VT − 1 ≤ card W*
shows *compl-copy VT ET W H*
⟨*proof*⟩

theorem *chvatal-upper*:
assumes *tree VT ET*
shows *is-triangle-tree-RN U VT ET (2 * card VT − 1)*
⟨*proof*⟩

3 The lower bound

lemma *tree-constant-on-edges*:
assumes *tr*: *tree V E* **and** *h*: *⋀a b. {a, b} ∈ E ⟹ Q a = Q b*
and *x*: *x ∈ V* **and** *y*: *y ∈ V*
shows *Q x = Q y*
⟨*proof*⟩

The complete bipartite graph with two parts of size $n - 1$ has no triangle, and a connected graph in its complement stays inside one part.

theorem *chvatal-lower*:
fixes *U :: 'b itself*
assumes *tr*: *tree VT ET* **and** *inf*: *infinite (UNIV :: 'b set)*
shows *¬ is-triangle-tree-RN U VT ET (2 * card VT − 2)*
⟨*proof*⟩

4 Chvátal's theorem

theorem *chvatal*:
assumes *tr*: *tree VT ET*
shows *triangle-tree-RN VT ET = 2 ∗ card VT − 1*
⟨*proof*⟩

A connected graph with n vertices and at most $n - 1$ edges is a tree.

corollary *connected-few-edges-triangle-good*:
assumes *g*: *fin-connected-sgraph V E* **and** *c*: *card E + 1 ≤ card V*
shows *triangle-tree-RN V E = 2 ∗ card V − 1*
⟨*proof*⟩

5 Agreement with *RN* of *Ramsey-Bounds*

The tree K_2 is also a complete graph, so there the two notions of Ramsey number can be compared directly: a copy of K_2 in the complement is an independent pair.

lemma *compl-copy-edge-iff*:
fixes *W* :: *'b set* **and** *H* :: *'b set set*
assumes *uv*: *u ≠ v*
shows *compl-copy {u, v} {{u, v}} W H ⟷ (∃ K⊆W. card K = 2 ∧ indep K H)*
⟨*proof*⟩

lemma *ex-clique-indep-iff*:
(∃ K⊆W. clique-indep m n K E) ⟷
(∃ K⊆W. card K = m ∧ clique K E) ∨ (∃ K⊆W. card K = n ∧ indep K E)
⟨*proof*⟩

lemma *is-triangle-tree-RN-edge-iff*:
assumes *u ≠ v*
shows *is-triangle-tree-RN U {u, v} {{u, v}} r ⟷ is-clique-RN U 3 2 r*
⟨*proof*⟩

The rhs is (of course) equal to 3

corollary *triangle-tree-RN-edge*:
assumes *u ≠ v*
shows *triangle-tree-RN {u, v} {{u, v}} = RN 3 2*
⟨*proof*⟩

end

References

[1] V. Chvátal. Tree-complete graph Ramsey numbers. *Journal of Graph Theory*, 1(1):93, 1977.