Cauchy's Functional Equation

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Abstract

Cauchy's functional equation is the equation

$$h(x + y) = h(x) + h(y).$$

This entry formalizes properties of the solutions $h : \mathbb{R} \to \mathbb{R}$ of Cauchy's functional equation, including:

- A function h satisfying Cauchy's functional equation has the form $h(x) = cx$ for some c if one of the following conditions holds:
  - h is continuous at some point.
  - h is continuous.
  - h is bounded on some interval.
  - h is (anti)monotone on some interval.
  - h is measurable.
- There are non-linear solutions to Cauchy's functional equation.
- The set of non-linear solutions has cardinality $|\mathbb{R}^{\mathbb{R}}|$.
- The graph of a non-linear solution $\{(x, h(x)) \mid x \in \mathbb{R}\}$ is dense in $\mathbb{R} \times \mathbb{R}$.

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theory *Cauchy-Functional-Equation*
imports
HOL−Probability.Probability
Riesz-Representation.Regular-Measure
begin

1 Cauchy's Functional Equation

1.1 Case: h is Continuous

The proof of the following lemma is based on https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation, accessed 17 January 2026.

lemma *Cauchy-equation-Q-linear*:
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *q*:$q \in \mathbb{Q}$
shows $h\ (q * v) = q * h\ v$
⟨*proof*⟩

lemma *Cauchy-equation-isCont-continuous*:
fixes $h :: 'a :: real\text{-}normed\text{-}vector \Rightarrow 'b :: real\text{-}normed\text{-}vector$
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *h-Cont*:*isCont h x*
shows *continuous-on UNIV h*
⟨*proof*⟩

lemma *Cauchy-equation-continuous'*:
fixes $h :: - :: field\text{-}char\text{-}0 \Rightarrow real$
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *h-Cont*:*continuous-on UNIV h*
shows $h = (\lambda x.\ x * h\ 1)$
⟨*proof*⟩

corollary *Cauchy-equation-continuous*:
fixes $h :: - :: field\text{-}char\text{-}0 \Rightarrow real$
assumes $\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *continuous-on UNIV h*
shows $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

corollary
fixes $h :: - :: field\text{-}char\text{-}0 \Rightarrow real$
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$

and *isCont h a*
shows *Cauchy-equation-continuous-at'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-continuous-at*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

1.2 Case: h is Monotone

lemma *Cauchy-bounded-zero-imp-continuous-at-zero*:
fixes $h :: 'a :: \{$*linordered-field*, *linorder-topology*$\} \Rightarrow$ *real*
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *ex*:$\exists \varepsilon{>}0.\ \exists M.\ \forall x.\ |x| < \varepsilon \longrightarrow |h\ x| \leq M$
shows *isCont h 0*
⟨*proof*⟩

corollary
fixes $h :: - ::$ *field-char-0* $\Rightarrow$ *real*
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *ab*:$a < b$
and *h-mono*:*mono-on* $\{a..b\}$ *h*
shows *Cauchy-equation-mono-on'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-mono-on*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

corollary
fixes $h :: - ::$ *field-char-0* $\Rightarrow$ *real*
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *h-mono*:*mono h*
shows *Cauchy-equation-mono'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-mono*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

corollary
fixes $h :: - ::$ *field-char-0* $\Rightarrow$ *real*
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *ab*:$a < b$
and *h-mono*:*antimono-on* $\{a..b\}$ *h*
shows *Cauchy-equation-antimono-on'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-antimono-on*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

1.3 Case: h is Bounded

corollary
fixes $h :: - ::$ *field-char-0* $\Rightarrow$ *real*
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *ab*:$a < b$
and *h-bdd-above*:$\exists\, M.\ \forall\, x{\in}\{a..b\}.\ h\ x \leq M$
shows *Cauchy-equation-bounded-above-on'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-bounded-above-on*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

corollary
fixes $h :: - :: \textit{field-char-0} \Rightarrow \textit{real}$
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *ab*:$a < b$
and *h-bdd-below*:$\exists\, M.\ \forall\, x{\in}\{a..b\}.\ M \leq h\ x$
shows *Cauchy-equation-bounded-below-on'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-bounded-below-on*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

corollary
fixes $h :: - :: \textit{field-char-0} \Rightarrow \textit{real}$
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *ab*:$a < b$
and *h-bdd*:$\exists\, M.\ \forall\, x{\in}\{a..b\}.\ |h\ x| \leq M$
shows *Cauchy-equation-bounded-on'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-bounded-on*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

1.4 Case: h is Measurable

1.4.1 Lemmas for Regularity of Measures (to be moved)

lemma *lborel-inner-regular*:
assumes *A*[*measurable*]:$A \in \textit{sets borel}$ **and** *A-bdd*:*bounded A*
shows *measure lborel A* = $(\bigsqcup K{\in}\{K.\ \textit{compact}\ K \wedge K \subseteq A\}.\ \textit{measure lborel}\ K)$
⟨*proof*⟩

corollary *lborel-inner-regular-eps*:
assumes *A*[*measurable*]:$A \in \textit{sets borel}$ **and** *A-bdd*:*bounded A*
and *e*:$0 < e$
obtains *K* **where** $K \subseteq A$ *compact K measure lborel A* $<$ *measure lborel K* $+ e$
⟨*proof*⟩

lemma *lborel-outer-regular*:
assumes *A*[*measurable*]:$A \in \textit{sets borel}$ **and** *A-bdd*:*bounded A*
shows *measure lborel A* = $(\bigsqcap\ (\textit{measure lborel}\ `\ \{C.\ \textit{open}\ C \wedge A \subseteq C \wedge \textit{bounded}\ C\}))$
⟨*proof*⟩

corollary *lborel-outer-regular-eps*:
assumes *A*[*measurable*]:$A \in \textit{sets borel}$ **and** *A-bdd*:*bounded A*
and *e*:$0 < e$
obtains *U* **where** $A \subseteq U$ *open U bounded U measure lborel U* $<$ *measure lborel A* $+ e$
⟨*proof*⟩

1.4.2 Steinhaus' Theorem

The proof of the following lemma is based on https://matthewhr.wordpress.com/2012/11/14/steinhauss-theorem/.

lemma *weak-Steinhaus*:
fixes $A :: 'a :: euclidean\text{-}space\ set$
assumes [*measurable*]:$A \in sets\ borel$
and *meas*:$0 < emeasure\ lborel\ A$
shows $0 \in interior\ \{a - a' | a\ a'.\ a \in A \wedge a' \in A\}$
⟨*proof*⟩

The proof of the following lemma is based on https://math.stackexchange.com/questions/45861/show-that-fxy-fxfy-implies-f-continuous-leftrightarrow-f-measur.

lemma *Cauchy-equation-continuous-iff-measurable*:
fixes $h :: real \Rightarrow real$
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
shows $continuous\text{-}on\ UNIV\ h \longleftrightarrow h \in borel\text{-}measurable\ borel$
⟨*proof*⟩

1.4.3 Conclusion

corollary
fixes $h :: real \Rightarrow real$
assumes *h-meas*[*measurable*]:$h \in borel\text{-}measurable\ borel$
and *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
shows *Cauchy-equation-measurable'*: $h = (\lambda x.\ x * h\ 1)$
and *Cauchy-equation-measurable*: $\exists\, c.\ h = (\lambda x.\ x * c)$
⟨*proof*⟩

1.5 Corollaries

corollary *multiplicative-measurable-powr*:
fixes $A :: real \Rightarrow real$
assumes A:$A \in borel\text{-}measurable\ (restrict\text{-}space\ borel\ \{0<..\})$
and *A-pos*:$\bigwedge x.\ 0 < x \Longrightarrow 0 < A\ x$
and *eq*:$\bigwedge x\ y.\ 0 < x \Longrightarrow 0 < y \Longrightarrow A\ (x * y) = A\ x * A\ y$
obtains γ **where** $\bigwedge x.\ 0 < x \Longrightarrow A\ x = x\ powr\ \gamma$
⟨*proof*⟩

lemma *cocycle-mono-solution*:
fixes $H :: real \Rightarrow real$ **and** $\gamma :: real$
assumes *mono*:$mono\ H$
and *eq*:$\bigwedge u\ v.\ H\ (u + v) = H\ u * exp\ (\gamma * v) + H\ v$
obtains c **where** $0 \leq c$
and $\bigwedge u.\ H\ u = (if\ \gamma = 0\ then\ c * u\ else\ c * (exp\ (\gamma * u) - 1)\ /\ \gamma)$
⟨*proof*⟩

2 Non-Linear Solutions

Cf. Hamel [1] and https://en.wikipedia.org/wiki/Cauchy%27s_functional_equation.

2.1 Existence

lemma *infinite-Hamel-basis*:
fixes $B :: real\ set$
assumes $module.span\ (\lambda q\ x.\ of\text{-}rat\ q * x)\ B = UNIV$
shows $infinite\ B$
⟨*proof*⟩

lemma *rat-Hamel-basisE*:
obtains $B :: real\ set$
where $module.independent\ (\lambda q\ x.\ of\text{-}rat\ q * x)\ B$
and $module.span\ (\lambda q\ x.\ of\text{-}rat\ q * x)\ B = UNIV$
and $0 \notin B$
⟨*proof*⟩

lemma *lists-lepoll-infinite*:
assumes $inf{:}infinite\ A$
shows $lists\ A \lesssim A$
⟨*proof*⟩

lemma *UNIV-real-lepoll-basis-Diff*:
fixes $u\ v :: real$
assumes $sp{:}module.span\ (\lambda q\ x.\ of\text{-}rat\ q * x)\ B = UNIV$
shows $(UNIV :: real\ set) \lesssim B - \{u, v\}$
⟨*proof*⟩

corollary *card-Hamel-basis*:
fixes $B :: real\ set$
assumes sp: $module.span\ (\lambda q\ x.\ of\text{-}rat\ q * x)\ B = UNIV$
shows $B \approx (UNIV :: real\ set)$
⟨*proof*⟩

lemma *UNIV-fun-lepoll-nonlinear*:
$(UNIV :: (real \Rightarrow real)\ set)$
$\lesssim \{h :: real \Rightarrow real.\ (\forall x\ y.\ h\ (x + y) = h\ x + h\ y) \wedge (\forall c.\ h \neq (\lambda x.\ x * c))\}$
⟨*proof*⟩

lemma *Cauchy-equation-nonlinear-solution-ex*:
$\exists h :: real \Rightarrow real.\ (\forall x\ y.\ h\ (x + y) = h\ x + h\ y) \wedge (\forall c.\ h \neq (\lambda x.\ x * c))$
⟨*proof*⟩

lemma *Cauchy-equation-nonlinear-solution*:
obtains $h :: real \Rightarrow real$
where $\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$

and $\bigwedge c.\ h \neq (\lambda x.\ x * c)$
⟨*proof*⟩

2.2 Cardinality

lemma *card-Cauchy-equation-nonlinear-solutions*:
$\{h :: real \Rightarrow real.\ (\forall\, x\ y.\ h\ (x + y) = h\ x + h\ y) \wedge (\forall\, c.\ h \neq (\lambda x.\ x * c))\}$
$\approx (UNIV :: (real \Rightarrow real)\ set)$
⟨*proof*⟩

2.3 Density of the Graph of a Non-Linear Solution

lemma *Cauchy-equation-nonlinear-graph-indep*:
fixes $h :: real \Rightarrow real$
assumes $\bigwedge c.\ h \neq (\lambda x.\ x * c)$
obtains $x_1\ x_2$ **where** $x_1 * h\ x_2 - x_2 * h\ x_1 \neq 0$
⟨*proof*⟩

theorem *Cauchy-equation-nonlinear-graph-dense*:
fixes $h :: real \Rightarrow real$
assumes *h-eq*:$\bigwedge x\ y.\ h\ (x + y) = h\ x + h\ y$
and *nonlin*:$\bigwedge c.\ h \neq (\lambda x.\ x * c)$
shows *closure* (*range* $(\lambda x.\ (x,\ h\ x))) = UNIV$
⟨*proof*⟩

end

References

[1] G. Hamel. Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung $f(x + y) = f(x) + f(y)$. *Mathematische Annalen*, 60:459–462, 1905.