Category Theory

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Abstract

This article presents a development of Category Theory in Isabelle. A Category is defined using records and locales in Isabelle/HOL. Functors and Natural Transformations are also defined. The main result that has been formalized is that the Yoneda functor is a full and faithful embedding. We also formalize the completeness of many sorted monadic equational logic. Extensive use is made of the HOLZF theory in both cases. For an informal description see [1].

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1 Category

theory *Category*
imports *HOL−Library.FuncSet*
begin

record $('o,'m)$ *Category* =
  Obj :: $'o$ *set* (‹*obj*₁› *70*)
  Mor :: $'m$ *set* (‹*mor*₁› *70*)
  Dom :: $'m \Rightarrow 'o$ (‹*dom*₁ -› [*80*] *70*)
  Cod :: $'m \Rightarrow 'o$ (‹*cod*₁ -› [*80*] *70*)

Id :: $'o \Rightarrow 'm$ (‹*id*₁ -› [*80*] *75*)
Comp :: $'m \Rightarrow 'm \Rightarrow 'm$ (**infixl** ‹;;₁› *70*)

definition
MapsTo :: $('o,'m,'a)$ *Category-scheme* $\Rightarrow 'm \Rightarrow 'o \Rightarrow 'o \Rightarrow$ *bool* (‹- *maps*₁ - *to* -› [*60*, *60*, *60*] *65*) **where**
MapsTo CC f X Y $\equiv f \in$ *Mor CC* $\wedge$ *Dom CC f* $= X \wedge$ *Cod CC f* $= Y$

definition
CompDefined :: $('o,'m,'a)$ *Category-scheme* $\Rightarrow 'm \Rightarrow 'm \Rightarrow$ *bool* (**infixl** ‹≈>₁› *65*) **where**
CompDefined CC f g $\equiv f \in$ *Mor CC* $\wedge$ $g \in$ *Mor CC* $\wedge$ *Cod CC f* $=$ *Dom CC g*

locale *ExtCategory* =
fixes *C* :: $('o,'m,'a)$ *Category-scheme* (**structure**)
assumes *CdomExt*: (*Dom C*) $\in$ *extensional* (*Mor C*)
and *CcodExt*: (*Cod C*) $\in$ *extensional* (*Mor C*)
and *CidExt*: (*Id C*) $\in$ *extensional* (*Obj C*)
and *CcompExt*: (*case-prod* (*Comp C*)) $\in$ *extensional* ({(*f*,*g*) | *f g* . *f* ≈> *g*})

locale *Category* = *ExtCategory* +
assumes *Cdom* : $f \in$ *mor* $\Longrightarrow$ *dom f* $\in$ *obj*
and *Ccod* : $f \in$ *mor* $\Longrightarrow$ *cod f* $\in$ *obj*
and *Cidm* [*dest*]: $X \in$ *obj* $\Longrightarrow$ (*id X*) *maps X to X*
and *Cidl* : $f \in$ *mor* $\Longrightarrow$ *id* (*dom f*) ;; $f = f$
and *Cidr* : $f \in$ *mor* $\Longrightarrow f$;; *id* (*cod f*) $= f$
and *Cassoc* : $[\![f \approx> g$; $g \approx> h]\!] \Longrightarrow (f$;; $g)$;; $h = f$;; $(g$;; $h)$
and *Ccompt* : $[\![f$ *maps X to Y* ; g *maps Y to Z*$]\!] \Longrightarrow (f$;; $g)$ *maps X to Z*

definition
MakeCat :: $('o,'m,'a)$ *Category-scheme* $\Rightarrow$ $('o,'m,'a)$ *Category-scheme* **where**
MakeCat C $\equiv$ (|
Obj $=$ *Obj C* ,
Mor $=$ *Mor C* ,
Dom $=$ *restrict* (*Dom C*) (*Mor C*) ,
Cod $=$ *restrict* (*Cod C*) (*Mor C*) ,
Id $=$ *restrict* (*Id C*) (*Obj C*) ,
Comp $= \lambda f\ g$. (*restrict* (*case-prod* (*Comp C*)) ({(*f*,*g*) | *f g* . *f* ≈>$_C$ *g*})) (*f*,*g*),
... $=$ *Category.more C*
|)

lemma *MakeCatMapsTo*: *f maps*$_C$ *X to Y* $\Longrightarrow$ *f maps*$_{MakeCat\ C}$ *X to Y*
⟨*proof*⟩

lemma *MakeCatComp*: *f* ≈>$_C$ *g* $\Longrightarrow$ *f* ;;$_{MakeCat\ C}$ *g* $=$ *f* ;;$_C$ *g*
⟨*proof*⟩

lemma *MakeCatId*: *X* $\in$ *obj*$_C$ $\Longrightarrow$ *id*$_C$ *X* $=$ *id*$_{MakeCat\ C}$ *X*

⟨*proof*⟩

lemma *MakeCatObj*: $obj_{MakeCat\ C} = obj_C$
⟨*proof*⟩

lemma *MakeCatMor*: $mor_{MakeCat\ C} = mor_C$
⟨*proof*⟩

lemma *MakeCatDom*: $f \in mor_C \Longrightarrow dom_C\ f = dom_{MakeCat\ C}\ f$
⟨*proof*⟩

lemma *MakeCatCod*: $f \in mor_C \Longrightarrow cod_C\ f = cod_{MakeCat\ C}\ f$
⟨*proof*⟩

lemma *MakeCatCompDef*: $f \approx>_{MakeCat\ C} g = f \approx>_C g$
⟨*proof*⟩

lemma *MakeCatComp2*: $f \approx>_{MakeCat\ C} g \Longrightarrow f\ ;;_{MakeCat\ C}\ g = f\ ;;_C\ g$
⟨*proof*⟩

lemma *ExtCategoryMakeCat*: *ExtCategory* (*MakeCat C*)
⟨*proof*⟩

lemma *MakeCat*: *Category-axioms C* $\Longrightarrow$ *Category* (*MakeCat C*)
⟨*proof*⟩

lemma *MapsToE*[*elim*]: $[\![f\ maps_C\ X\ to\ Y\ ;\ [\![f \in mor_C\ ;\ dom_C\ f = X\ ;\ cod_C\ f = Y]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *MapsToI*[*intro*]: $[\![f \in mor_C\ ;\ dom_C\ f = X\ ;\ cod_C\ f = Y]\!] \Longrightarrow f\ maps_C\ X\ to\ Y$
⟨*proof*⟩

lemma *CompDefinedE*[*elim*]: $[\![f \approx>_C g\ ;\ [\![f \in mor_C\ ;\ g \in mor_C\ ;\ cod_C\ f = dom_C\ g]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *CompDefinedI*[*intro*]: $[\![f \in mor_C\ ;\ g \in mor_C\ ;\ cod_C\ f = dom_C\ g]\!] \Longrightarrow f \approx>_C g$
⟨*proof*⟩

lemma (**in** *Category*) *MapsToCompI*: **assumes** $f \approx> g$ **shows** $(f\ ;;\ g)\ maps\ (dom\ f)\ to\ (cod\ g)$
⟨*proof*⟩

lemma *MapsToCompDef*:

assumes f *maps*$_C$ X *to* Y **and** g *maps*$_C$ Y *to* Z
shows $f \approx>_C g$
⟨*proof*⟩

lemma (**in** *Category*) *MapsToMorDomCod*:
assumes $f \approx> g$
shows $f \mathbin{;;} g \in mor$ **and** $dom\ (f \mathbin{;;} g) = dom\ f$ **and** $cod\ (f \mathbin{;;} g) = cod\ g$
⟨*proof*⟩

lemma (**in** *Category*) *MapsToObj*:
assumes f *maps* X *to* Y
shows $X \in obj$ **and** $Y \in obj$
⟨*proof*⟩

lemma (**in** *Category*) *IdInj*:
assumes $X \in obj$ **and** $Y \in obj$ **and** $id\ X = id\ Y$
shows $X = Y$
⟨*proof*⟩

lemma (**in** *Category*) *CompDefComp*:
assumes $f \approx> g$ **and** $g \approx> h$
shows $f \approx> (g \mathbin{;;} h)$ **and** $(f \mathbin{;;} g) \approx> h$
⟨*proof*⟩

lemma (**in** *Category*) *CatIdInMor*: $X \in obj \Longrightarrow id\ X \in mor$
⟨*proof*⟩

lemma (**in** *Category*) *MapsToId*: **assumes** $X \in obj$ **shows** $id\ X \approx> id\ X$
⟨*proof*⟩

lemmas (**in** *Category*) *Simps* = *Cdom Ccod Cidm Cidl Cidr MapsToCompI IdInj MapsToId*

lemma (**in** *Category*) *LeftRightInvUniq*:
assumes *0*: $h \approx> f$ **and** *z*: $f \approx> g$
assumes *1*: $f \mathbin{;;} g = id\ (dom\ f)$
and *2*: $h \mathbin{;;} f = id\ (cod\ f)$
shows $h = g$
⟨*proof*⟩

lemma (**in** *Category*) *CatIdDomCod*:
assumes $X \in obj$
shows $dom\ (id\ X) = X$ **and** $cod\ (id\ X) = X$
⟨*proof*⟩

lemma (**in** *Category*) *CatIdCompId*:
assumes $X \in obj$
shows $id\ X \mathbin{;;} id\ X = id\ X$
⟨*proof*⟩

lemma (**in** *Category*) *CatIdUniqR*:
assumes *iota*: ι *maps* X *to* X
and *rid*: $\forall\ f\ .\ f \approx> \iota \longrightarrow f\ ;;\ \iota = f$
shows $id\ X = \iota$
⟨*proof*⟩

definition
$inverse\text{-}rel :: ('o,'m,'a)\ Category\text{-}scheme \Rightarrow 'm \Rightarrow 'm \Rightarrow bool\ (\langle cinv_1\ \text{-}\ \text{-}\rangle\ 60)$
where
$inverse\text{-}rel\ C\ f\ g \equiv (f \approx>_C g) \wedge (f\ ;;_C\ g) = (id_C\ (dom_C\ f)) \wedge (g\ ;;_C\ f) = (id_C\ (cod_C\ f))$

definition
$isomorphism :: ('o,'m,'a)\ Category\text{-}scheme \Rightarrow 'm \Rightarrow bool\ (\langle ciso_1\ \text{-}\rangle\ [70])$ **where**
$isomorphism\ C\ f \equiv \exists\ g\ .\ inverse\text{-}rel\ C\ f\ g$

lemma (**in** *Category*) *Inverse-relI*: $[\![f \approx> g\ ;\ f\ ;;\ g = id\ (dom\ f)\ ;\ g\ ;;\ f = id\ (cod\ f)]\!] \Longrightarrow (cinv\ f\ g)$
⟨*proof*⟩

lemma (**in** *Category*) *Inverse-relE*[*elim*]: $[\![cinv\ f\ g\ ;\ [\![f \approx> g\ ;\ f\ ;;\ g = id\ (dom\ f)\ ;\ g\ ;;\ f = id\ (cod\ f)]\!] \Longrightarrow P]\!] \Longrightarrow P$
⟨*proof*⟩

lemma (**in** *Category*) *Inverse-relSym*:
assumes *cinv f g*
shows *cinv g f*
⟨*proof*⟩

lemma (**in** *Category*) *InverseUnique*:
assumes *1*: *cinv f g*
and *2*: *cinv f h*
shows $g = h$
⟨*proof*⟩

lemma (**in** *Category*) *InvId*: **assumes** $X \in obj$ **shows** $(cinv\ (id\ X)\ (id\ X))$
⟨*proof*⟩

definition
$inverse :: ('o,'m,'a)\ Category\text{-}scheme \Rightarrow 'm \Rightarrow 'm\ (\langle Cinv_1\ \text{-}\rangle\ [70])$ **where**
$inverse\ C\ f \equiv THE\ g\ .\ inverse\text{-}rel\ C\ f\ g$

lemma (**in** *Category*) *inv2Inv*:
assumes *cinv f g*
shows *ciso f* **and** $Cinv\ f = g$
⟨*proof*⟩

lemma (**in** *Category*) *iso2Inv*:
assumes *ciso f*
shows *cinv f* (*Cinv f*)
⟨*proof*⟩

lemma (**in** *Category*) *InvInv*:
assumes *ciso f*
shows *ciso* (*Cinv f*) **and** (*Cinv* (*Cinv f*)) = *f*
⟨*proof*⟩

lemma (**in** *Category*) *InvIsMor*: (*cinv f g*) ⟹ (*f* ∈ *mor* ∧ *g* ∈ *mor*)
⟨*proof*⟩

lemma (**in** *Category*) *IsoIsMor*: *ciso f* ⟹ *f* ∈ *mor*
⟨*proof*⟩

lemma (**in** *Category*) *InvDomCod*:
assumes *ciso f*
shows *dom* (*Cinv f*) = *cod f* **and** *cod* (*Cinv f*) = *dom f* **and** *Cinv f* ∈ *mor*
⟨*proof*⟩

lemma (**in** *Category*) *IsoCompInv*: *ciso f* ⟹ *f* ≈> *Cinv f*
⟨*proof*⟩

lemma (**in** *Category*) *InvCompIso*: *ciso f* ⟹ *Cinv f* ≈> *f*
⟨*proof*⟩

lemma (**in** *Category*) *IsoInvId1* : *ciso f* ⟹ (*Cinv f*) ;; *f* = (*id* (*cod f*))
⟨*proof*⟩

lemma (**in** *Category*) *IsoInvId2* : *ciso f* ⟹ *f* ;; (*Cinv f*) = (*id* (*dom f*))
⟨*proof*⟩

lemma (**in** *Category*) *IsoCompDef*:
assumes *1*: *f* ≈> *g* **and** *2*: *ciso f* **and** *3*: *ciso g*
shows (*Cinv g*) ≈> (*Cinv f*)
⟨*proof*⟩

lemma (**in** *Category*) *IsoCompose*:
assumes *1*: *f* ≈> *g* **and** *2*: *ciso f* **and** *3*: *ciso g*
shows *ciso* (*f* ;; *g*) **and** *Cinv* (*f* ;; *g*) = (*Cinv g*) ;; (*Cinv f*)
⟨*proof*⟩

definition *ObjIso C A B* ≡ ∃ *k* . (*k maps*$_C$ *A to B*) ∧ *ciso*$_C$ *k*

definition
UnitCategory :: (*unit*, *unit*) *Category* **where**
UnitCategory = *MakeCat* (‖

$Obj = \{()\}$,
$Mor = \{()\}$,
$Dom = (\lambda f.())$,
$Cod = (\lambda f.())$,
$Id = (\lambda f.())$,
$Comp = (\lambda f\ g.\ ())$
(|

lemma [*simp*]: $Category(UnitCategory)$
⟨*proof*⟩

definition
$OppositeCategory :: ('o,'m,'a)\ Category\text{-}scheme \Rightarrow ('o,'m,'a)\ Category\text{-}scheme$
($\langle Op\ \text{-}\rangle\ [65]\ 65$) **where**
$OppositeCategory\ C \equiv (|$
$Obj = Obj\ C$,
$Mor = Mor\ C$,
$Dom = Cod\ C$,
$Cod = Dom\ C$,
$Id\ = Id\ C$,
$Comp = (\lambda f\ g.\ g\ ;;_C\ f)$,
$\ldots = Category.more\ C$
|)

lemma *OpCatOpCat*: $Op\ (Op\ C) = C$
⟨*proof*⟩

lemma *OpCatCatAx*: $Category\text{-}axioms\ C \Longrightarrow Category\text{-}axioms\ (Op\ C)$
⟨*proof*⟩

lemma *OpCatCatExt*: $ExtCategory\ C \Longrightarrow ExtCategory\ (Op\ C)$
⟨*proof*⟩

lemma *OpCatCat*: $Category\ C \Longrightarrow Category\ (Op\ C)$
⟨*proof*⟩

lemma *MapsToOp*: $f\ maps_C\ X\ to\ Y \Longrightarrow f\ maps_{Op\ C}\ Y\ to\ X$
⟨*proof*⟩

lemma *MapsToOpOp*: $f\ maps_{Op\ C}\ X\ to\ Y \Longrightarrow f\ maps_C\ Y\ to\ X$
⟨*proof*⟩

lemma *CompDefOp*: $f \approx>_C g \Longrightarrow g \approx>_{Op\ C} f$
⟨*proof*⟩

end

2 Universe

theory *Universe*
imports *HOL−ZF.MainZF*
begin

locale *Universe* =
  fixes *U* :: *ZF* (**structure**)
  assumes *Uempty* : *Elem Empty U*
  and *Usubset* : *Elem u U* ⟹ *subset u U*
  and *Usingle* : *Elem u U* ⟹ *Elem* (*Singleton u*) *U*
  and *Upow* : *Elem u U* ⟹ *Elem* (*Power u*) *U*
  and *Uim* : ⟦*Elem I U* ; *Elem u* (*Fun I U*) ⟧ ⟹ *Elem* (*Sum* (*Range u*)) *U*
  and *Unat* : *Elem Nat U*

lemma *ElemLambdaFun* : (⋀ *x* .*Elem x u* ⟹ *Elem* (*f x*) *U*) ⟹ *Elem* (*Lambda u f*) (*Fun u U*)
⟨*proof*⟩

lemma *RangeRepl*: *Range* (*Lambda A f*) = *Repl A f*
⟨*proof*⟩

lemma (**in** *Universe*) *Utrans*: ⟦*Elem a U* ; *Elem b a*⟧ ⟹ *Elem b U*
⟨*proof*⟩

lemma *ReplId*: *Repl A id* = *A*
⟨*proof*⟩

lemma (**in** *Universe*) *UniverseSum* : *Elem u U* ⟹ *Elem* (*Sum u*) *U*
⟨*proof*⟩

lemma (**in** *Universe*) *UniverseUnion*:
  assumes *Elem u U* **and** *Elem v U*
  shows *Elem* (*union u v*) *U*
⟨*proof*⟩

lemma *UPairSingleton*: *Upair u v* = *union* (*Singleton u*) (*Singleton v*)
⟨*proof*⟩

lemma (**in** *Universe*) *UniverseUPair*: ⟦*Elem u U* ; *Elem v U*⟧ ⟹ *Elem* (*Upair u v*) *U*
⟨*proof*⟩

lemma (**in** *Universe*) *UniversePair*: ⟦*Elem u U* ; *Elem v U*⟧ ⟹ *Elem* (*Opair u v*) *U*
⟨*proof*⟩

lemma (**in** *Universe*) ⟦*Elem u U* ; *Elem v U*⟧ ⟹ *Elem* (*Sum* (*Repl u* (%*x* .

Singleton (*Opair x v*)))) *U*
⟨*proof*⟩

lemma *SumRepl*: *Sum* (*Repl A* (*Singleton o f*)) = *Repl A f*
⟨*proof*⟩

lemma (**in** *Universe*) *UniverseProd*:
assumes *Elem u U* **and** *Elem v U*
shows *Elem* (*CartProd u v*) *U*
⟨*proof*⟩

lemma (**in** *Universe*) *UniverseSubset*: ⟦*subset u v* ; *Elem v U*⟧ ⟹ *Elem u U*
⟨*proof*⟩

definition
Product :: *ZF* ⇒ *ZF* **where**
Product U = *Sep* (*Fun U* (*Sum U*)) (%*f* . (∀ *u* . *Elem u U* ⟶ *Elem* (*app f u*) *u*))

lemma *SepSubset*: *subset* (*Sep A p*) *A*
⟨*proof*⟩

lemma *SubsetSmall*:
assumes *subset A′ A* **and** *subset A B* **shows** *subset A′ B*
⟨*proof*⟩

lemma *SubsetTrans*:
assumes (*subset a b*) **and** (*subset b c*)
shows (*subset a c*)
⟨*proof*⟩

lemma *SubsetSepTrans*: *subset A B* ⟹ *subset* (*Sep A p*) *B*
⟨*proof*⟩

lemma *ProductSubset*: *subset* (*Product u*) (*Power* (*CartProd u* (*Sum u*)))
⟨*proof*⟩

lemma (**in** *Universe*) *UniverseProduct*: *Elem u U* ⟹ *Elem* (*Product u*) *U*
⟨*proof*⟩

lemma *ZFImageRangeExplode*: *x* ∈ *range explode* ⟹ *f* ‘ *x* ∈ *range explode*
⟨*proof*⟩

definition *subsetFn X Y* ≡ λ *x* . (*if x* ∈ *Y then x else SOME y* . *y* ∈ *Y*)

lemma *subsetFn*: ⟦*Y* ≠ {} ; *Y* ⊆ *X* ⟧ ⟹ (*subsetFn X Y*) ‘ *X* = *Y*
⟨*proof*⟩

lemma *ZFSubsetRangeExplode*: $[\![X \in range\ explode\ ;\ Y \subseteq X]\!] \Longrightarrow Y \in range\ explode$
⟨*proof*⟩

lemma *ZFUnionRangeExplode*:
assumes $\bigwedge x\ .\ x \in A \Longrightarrow f\ x \in range\ explode$ **and** $A \in range\ explode$
shows $(\bigcup\ x \in A\ .\ f\ x) \in range\ explode$
⟨*proof*⟩

lemma *ZFUnionNatInRangeExplode*: $(\bigwedge\ (n :: nat)\ .\ f\ n \in range\ explode) \Longrightarrow (\bigcup\ n\ .\ f\ n) \in range\ explode$
⟨*proof*⟩

lemma *ZFProdFnInRangeExplode*: $[\![A \in range\ explode\ ;\ B \in range\ explode]\!] \Longrightarrow f\ `\ (A \times B) \in range\ explode$
⟨*proof*⟩

lemma *ZFUnionInRangeExplode*: $[\![A \in range\ explode\ ;\ B \in range\ explode]\!] \Longrightarrow A \cup B \in range\ explode$
⟨*proof*⟩

lemma *SingletonInRangeExplode*: $\{x\} \in range\ explode$
⟨*proof*⟩

definition *ZFTriple* :: [*ZF*,*ZF*,*ZF*] ⇒ *ZF* **where**
ZFTriple a b c = *Opair* (*Opair a b*) *c*
definition *ZFTFst* = *Fst o Fst*
definition *ZFTSnd* = *Snd o Fst*
definition *ZFTThd* = *Snd*

lemma *ZFTFst*: *ZFTFst* (*ZFTriple a b c*) = *a*
⟨*proof*⟩
lemma *ZFTSnd*: *ZFTSnd* (*ZFTriple a b c*) = *b*
⟨*proof*⟩
lemma *ZFTThd*: *ZFTThd* (*ZFTriple a b c*) = *c*
⟨*proof*⟩

lemma *ZFTriple*: *ZFTriple a b c* = *ZFTriple a′ b′ c′* ⟹ (*a* = *a′* ∧ *b* = *b′* ∧ *c* = *c′*)
⟨*proof*⟩

lemma *ZFSucZero*: *Nat2nat* (*SucNat Empty*) = *1*
⟨*proof*⟩

lemma *ZFZero*: *Nat2nat Empty* = *0*
⟨*proof*⟩

lemma *ZFSucNeq0*: *Elem x Nat* ⟹ *Nat2nat* (*SucNat x*) ≠ *0*

⟨*proof*⟩

end

3 Monadic Equational Theory

theory *MonadicEquationalTheory*
imports *Category Universe*
begin

record ('t,'f) *Signature* =
  BaseTypes :: 't *set* (‹*Ty*1›)
  BaseFunctions :: 'f *set* (‹*Fn*1›)
  SigDom :: 'f ⇒ 't (‹*sDom*1›)
  SigCod :: 'f ⇒ 't (‹*sCod*1›)

locale *Signature* =
  fixes *S* :: ('t,'f) *Signature* (**structure**)
  assumes *Domt*: $f \in Fn \Longrightarrow sDom\ f \in Ty$
  and *Codt*: $f \in Fn \Longrightarrow sCod\ f \in Ty$

definition *funsignature-abbrev* (‹- ∈ *Sig* - : - → -›) **where**
  $f \in Sig\ S : A \to B \equiv f \in (BaseFunctions\ S) \wedge A \in (BaseTypes\ S) \wedge B \in (BaseTypes\ S) \wedge$
  $(SigDom\ S\ f) = A \wedge (SigCod\ S\ f) = B \wedge Signature\ S$

lemma *funsignature-abbrevE*[*elim*]:
$[\![f \in Sig\ S : A \to B ; [\![f \in (BaseFunctions\ S) ; A \in (BaseTypes\ S) ; B \in (BaseTypes\ S) ;$
$(SigDom\ S\ f) = A ; (SigCod\ S\ f) = B ; Signature\ S]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

datatype ('t,'f) *Expression* = *ExprVar* (‹*Vx*›) | *ExprApp* 'f ('t,'f) *Expression* (‹- *E*@ -›)
datatype ('t,'f) *Language* = *Type* 't (‹⊢ - *Type*›) | *Term* 't ('t,'f) *Expression* 't (‹*Vx* : - ⊢ - : -›) |
  Equation 't ('t,'f) *Expression* ('t,'f) *Expression* 't (‹*Vx* : - ⊢ -≡- : -›)

inductive
  WellDefined :: ('t,'f) *Signature* ⇒ ('t,'f) *Language* ⇒ *bool* (‹*Sig* - ▷ -›) **where**
    WellDefinedTy: $A \in BaseTypes\ S \Longrightarrow Sig\ S \triangleright \vdash A\ Type$
  | *WellDefinedVar*: $Sig\ S \triangleright \vdash A\ Type \Longrightarrow Sig\ S \triangleright (Vx : A\ \vdash Vx : A)$
  | *WellDefinedFn*: $[\![Sig\ S \triangleright (Vx : A\ \vdash e : B)\ ; f \in Sig\ S : B \to C]\!] \Longrightarrow Sig\ S \triangleright (Vx : A\ \vdash (f\ E@\ e) : C)$
  | *WellDefinedEq*: $[\![Sig\ S \triangleright (Vx : A\ \vdash e1 : B)\ ; Sig\ S \triangleright (Vx : A\ \vdash e2 : B)]\!] \Longrightarrow Sig\ S \triangleright (Vx : A \vdash e1 \equiv e2 : B)$

lemmas *WellDefined.intros* [*intro*]
inductive-cases *WellDefinedTyE* [*elim*!]: $Sig\ S \triangleright \vdash A\ Type$
inductive-cases *WellDefinedVarE* [*elim*!]: $Sig\ S \triangleright (Vx : A\ \vdash Vx : A)$
inductive-cases *WellDefinedFnE* [*elim*!]: $Sig\ S \triangleright (Vx : A\ \vdash (f\ E@\ e) : C)$
inductive-cases *WellDefinedEqE* [*elim*!]: $Sig\ S \triangleright (Vx : A \vdash e1 \equiv e2 : B)$

lemma *SigId*: $Sig\ S \triangleright (Vx : A\ \vdash Vx : B) \Longrightarrow A = B$
⟨*proof*⟩

lemma *SigTyId*: $Sig\ S \triangleright (Vx : A\ \vdash Vx : A) \Longrightarrow A \in BaseTypes\ S$
⟨*proof*⟩

lemma (**in** *Signature*) *SigTy*: $\bigwedge B\ .\ Sig\ S \triangleright (Vx : A\ \vdash e : B) \Longrightarrow (A \in BaseTypes\ S \wedge B \in BaseTypes\ S)$
⟨*proof*⟩

datatype $('o,'m)\ IType = IObj\ 'o \mid IMor\ 'm \mid IBool\ bool$

record $('t,'f,'o,'m)\ Interpretation =$
  $ISignature :: ('t,'f)\ Signature\ (\langle iS_1\rangle)$
  $ICategory\ :: ('o,'m)\ Category\ \ (\langle iC_1\rangle)$
  $ITypes\ \ \ \ :: 't \Rightarrow 'o\ (\langle Ty[\![\text{-}]\!]_1\rangle)$
  $IFunctions :: 'f \Rightarrow 'm\ (\langle Fn[\![\text{-}]\!]_1\rangle)$

locale *Interpretation* =
  fixes $I :: ('t,'f,'o,'m)\ Interpretation$ (**structure**)
  assumes *ICat*: $Category\ \ iC$
  and *ISig*: $Signature\ iS$
  and $It\ : A \in BaseTypes\ iS \Longrightarrow Ty[\![A]\!] \in Obj\ iC$
  and $If\ : (f \in Sig\ iS : A \to B) \Longrightarrow Fn[\![f]\!]\ maps_{iC}\ Ty[\![A]\!]\ to\ Ty[\![B]\!]$

inductive *Interp* $(\langle L[\![\text{-}]\!]_1 \to \text{-}\rangle)$ **where**
  $InterpTy$: $Sig\ iS_I \triangleright \vdash A\ Type \Longrightarrow$
    $L[\![\vdash A\ Type]\!]_I \to (IObj\ Ty[\![A]\!]_I)$
| $InterpVar$: $L[\![\vdash A\ Type]\!]_I \to (IObj\ c) \Longrightarrow$
    $L[\![Vx : A\ \vdash Vx : A]\!]_I \to (IMor\ (Id\ iC_I\ c))$
| $InterpFn$: $[\![Sig\ iS_I \triangleright Vx : A\ \vdash e : B\ ;$
    $f \in Sig\ iS_I : B \to C\ ;$
    $L[\![Vx : A\ \vdash e : B]\!]_I \to (IMor\ g)]\!] \Longrightarrow$
    $L[\![Vx : A\ \vdash (f\ E@\ e) : C]\!]_I \to (IMor\ (g\ ;;_{ICategory\ I}\ Fn[\![f]\!]_I))$
| $InterpEq$: $[\![L[\![Vx : A\ \vdash e1 : B]\!]_I \to (IMor\ g1)\ ;$
    $L[\![Vx : A\ \vdash e2 : B]\!]_I \to (IMor\ g2)]\!] \Longrightarrow$
    $L[\![Vx : A\ \vdash e1 \equiv e2 : B]\!]_I \to (IBool\ (g1 = g2))$

lemmas *Interp.intros* [*intro*]
inductive-cases *InterpTyE* [*elim*!]: $L[\![\vdash A\ Type]\!]_I \to i$
inductive-cases *InterpVarE* [*elim*!]: $L[\![Vx : A\ \vdash Vx : A]\!]_I \to i$
inductive-cases *InterpFnE* [*elim*!]: $L[\![Vx : A\ \vdash (f\ E@\ e) : C]\!]_I \to i$

inductive-cases *InterpEqE* [*elim*!]: $L[\![Vx : A \vdash e1 \equiv e2 : B]\!]_I \to i$

lemma (**in** *Interpretation*) *InterpEqEq*[*intro*]:
$[\![L[\![Vx : A \vdash e1 : B]\!] \to (IMor\ g)\ ;\ L[\![Vx : A \vdash e2 : B]\!] \to (IMor\ g)]\!] \Longrightarrow L[\![Vx : A \vdash e1 \equiv e2 : B]\!] \to (IBool\ True)$
⟨*proof*⟩

lemma (**in** *Interpretation*) *InterpExprWellDefined*:
$L[\![Vx : A \vdash e : B]\!] \to i \Longrightarrow Sig\ iS \triangleright Vx : A \vdash e : B$
⟨*proof*⟩

lemma (**in** *Interpretation*) *WellDefined*: $L[\![\varphi]\!] \to i \Longrightarrow Sig\ iS \triangleright \varphi$
⟨*proof*⟩

lemma (**in** *Interpretation*) *Bool*: $L[\![\varphi]\!] \to (IBool\ i) \Longrightarrow \exists\ A\ B\ e\ d\ .\ \varphi = (Vx : A \vdash e \equiv d : B)$
⟨*proof*⟩

lemma (**in** *Interpretation*) *FunctionalExpr*:
$\bigwedge i\ j\ A\ B.[\![L[\![Vx : A \vdash e : B]\!] \to i;\ L[\![Vx : A \vdash e : B]\!] \to j]\!] \Longrightarrow i = j$
⟨*proof*⟩

lemma (**in** *Interpretation*) *Functional*: $[\![L[\![\varphi]\!] \to i1\ ;\ L[\![\varphi]\!] \to i2]\!] \Longrightarrow i1 = i2$
⟨*proof*⟩

lemma (**in** *Interpretation*) *MorphismsPreserved*:
$\bigwedge\ B\ i\ .\ L[\![Vx : A \vdash e : B]\!] \to i \Longrightarrow \exists\ g\ .\ i = (IMor\ g) \wedge (g\ maps_{iC}\ Ty[\![A]\!]\ to\ Ty[\![B]\!])$
⟨*proof*⟩

lemma (**in** *Interpretation*) *Expr2Mor*: $L[\![Vx : A \vdash e : B]\!] \to (IMor\ g) \Longrightarrow (g\ maps_{iC}\ Ty[\![A]\!]\ to\ Ty[\![B]\!])$
⟨*proof*⟩

lemma (**in** *Interpretation*) *WellDefinedExprInterp*: $\bigwedge\ B\ .\ (Sig\ iS \triangleright Vx : A \vdash e : B) \Longrightarrow (\exists\ i\ .\ L[\![Vx : A \vdash e : B]\!] \to i)$
⟨*proof*⟩

lemma (**in** *Interpretation*) *Sig2Mor*: **assumes** $(Sig\ iS \triangleright Vx : A \vdash e : B)$ **shows** $\exists\ g\ .\ L[\![Vx : A \vdash e : B]\!] \to (IMor\ g)$
⟨*proof*⟩

record $('t,'f)$ *Axioms* =
aAxioms :: $('t,'f)$ *Language set*
aSignature :: $('t,'f)$ *Signature* (‹*aS*1›)

locale *Axioms* =
fixes *Ax* :: $('t,'f)$ *Axioms* (**structure**)
assumes *AxT*: $(aAxioms\ Ax) \subseteq \{(Vx : A \vdash e1 \equiv e2 : B) \mid A\ B\ e1\ e2\ .\ Sig$

$(aSignature\ Ax) \triangleright (Vx : A \vdash e1 \equiv e2 : B)\}$
assumes *AxSig*: $Signature\ (aSignature\ Ax)$

primrec $Subst :: ('t,'f)\ Expression \Rightarrow ('t,'f)\ Expression \Rightarrow ('t,'f)\ Expression$ ($\langle sub$ - in -$\rangle$ $[81,81]\ 81$) **where**
$(sub\ e\ in\ Vx) = e \mid sub\ e\ in\ (f\ E@\ d) = (f\ E@\ (sub\ e\ in\ d))$

lemma *SubstXinE*: $(sub\ Vx\ in\ e) = e$
⟨*proof*⟩

lemma *SubstAssoc*: $sub\ a\ in\ (sub\ b\ in\ c) = sub\ (sub\ a\ in\ b)\ in\ c$
⟨*proof*⟩

lemma *SubstWellDefined*: $\bigwedge C\ .\ [\![Sig\ S \triangleright (Vx : A \vdash e : B); Sig\ S \triangleright (Vx : B \vdash d : C)]\!]$
$\Longrightarrow Sig\ S \triangleright (Vx : A \vdash (sub\ e\ in\ d) : C)$
⟨*proof*⟩

inductive-set (**in** *Axioms*) *Theory* **where**
Ax: $A \in (aAxioms\ Ax) \Longrightarrow A \in Theory$
| *Refl*: $Sig\ (aSignature\ Ax) \triangleright (Vx : A \vdash e : B) \Longrightarrow (Vx : A \vdash e \equiv e : B) \in Theory$
| *Symm*: $(Vx : A \vdash e1 \equiv e2 : B) \in Theory \Longrightarrow (Vx : A \vdash e2 \equiv e1 : B) \in Theory$
| *Trans*: $[\![(Vx : A \vdash e1 \equiv e2 : B) \in Theory\ ;\ (Vx : A \vdash e2 \equiv e3 : B) \in Theory]\!]$
$\Longrightarrow$
$(Vx : A \vdash e1 \equiv e3 : B) \in Theory$
| *Congr*: $[\![(Vx : A \vdash e1 \equiv e2 : B) \in Theory\ ;\ f \in Sig\ (aSignature\ Ax) : B \to C]\!]$
$\Longrightarrow$
$(Vx : A \vdash (f\ E@\ e1) \equiv (f\ E@\ e2) : C) \in Theory$
| *Subst*: $[\![Sig\ (aSignature\ Ax) \triangleright (Vx : A \vdash e1 : B)\ ;\ (Vx : B \vdash e2 \equiv e3 : C) \in Theory]\!] \Longrightarrow$
$(Vx : A \vdash (sub\ e1\ in\ e2) \equiv (sub\ e1\ in\ e3) : C) \in Theory$

lemma (**in** *Axioms*) *Equiv2WellDefined*: $\varphi \in Theory \Longrightarrow Sig\ aS \triangleright \varphi$
⟨*proof*⟩

lemma (**in** *Axioms*) *Subst'*:
$\bigwedge C\ .\ [\![Sig\ aS \triangleright Vx : B \vdash d : C\ ;\ (Vx : A \vdash e1 \equiv e2 : B) \in Theory]\!] \Longrightarrow$
$(Vx : A \vdash (sub\ e1\ in\ d) \equiv (sub\ e2\ in\ d) : C) \in Theory$
⟨*proof*⟩

locale *Model* = *Interpretation I* + *Axioms Ax*
for $I :: ('t,'f,'o,'m)\ Interpretation$ (**structure**)
and $Ax :: ('t,'f)\ Axioms$ +
assumes *AxSound*: $\varphi \in (aAxioms\ Ax) \Longrightarrow L[\![\varphi]\!] \to (IBool\ True)$
and *Seq*[*simp*]: $(aSignature\ Ax) = iS$

lemma (**in** *Interpretation*) *Equiv*:
assumes $L[\![Vx : A \vdash e1 \equiv e2 : B]\!] \to (IBool\ True)$

shows $\exists\ g\ .\ (L[\![Vx : A \vdash e1 : B]\!] \to (IMor\ g)) \wedge (L[\![Vx : A \vdash e2 : B]\!] \to (IMor\ g))$
⟨*proof*⟩

lemma (**in** *Interpretation*) *SubstComp*: $\bigwedge\ h\ C\ .\ [\![(L[\![Vx : A \vdash e : B]\!] \to (IMor\ g)) ; (L[\![Vx : B \vdash d : C]\!] \to (IMor\ h))]\!] \Longrightarrow$
$(L[\![Vx : A \vdash (sub\ e\ in\ d) : C]\!] \to (IMor\ (g\ ;;_{iC}\ h)))$
⟨*proof*⟩

lemma (**in** *Model*) *Sound*: $\varphi \in Theory \Longrightarrow\ L[\![\varphi]\!] \to (IBool\ True)$
⟨*proof*⟩

record $('t,'f)\ TermEquivClT =$
$TDomain :: 't$
$TExprSet :: ('t,'f)\ Expression\ set$
$TCodomain :: 't$

locale $ZFAxioms = Ax : Axioms\ Ax$ **for** $Ax :: (ZF,ZF)\ Axioms$ (**structure**) +
assumes $fnzf$: $BaseFunctions\ (aSignature\ Ax) \in range\ explode$

lemma [*simp*]: $ZFAxioms\ T \Longrightarrow Axioms\ T$ ⟨*proof*⟩

primrec $Expr2ZF :: (ZF,ZF)\ Expression \Rightarrow ZF$ **where**
$Expr2ZFVx$: $Expr2ZF\ Vx = ZFTriple\ (nat2Nat\ 0)\ (nat2Nat\ 0)\ Empty$
| $Expr2ZFfe$: $Expr2ZF\ (f\ E@\ e) = ZFTriple\ (SucNat\ (ZFTFst\ (Expr2ZF\ e)))$
$(nat2Nat\ 1)$
$(Opair\ f\ (Expr2ZF\ e))$

definition $ZF2Expr :: ZF \Rightarrow (ZF,ZF)\ Expression$ **where**
$ZF2Expr = inv\ Expr2ZF$

definition $ZFDepth = Nat2nat\ o\ ZFTFst$
definition $ZFType = Nat2nat\ o\ ZFTSnd$
definition $ZFData = ZFTThd$

lemma $Expr2ZFType0$: $ZFType\ (Expr2ZF\ e) = 0 \Longrightarrow e = Vx$
⟨*proof*⟩

lemma $ZFDepthInNat$: $Elem\ (ZFTFst\ (Expr2ZF\ e))\ Nat$
⟨*proof*⟩

lemma $Expr2ZFType1$: $ZFType\ (Expr2ZF\ e) = 1 \Longrightarrow$
$\exists\ f\ e'\ .\ e = (f\ E@\ e') \wedge (Suc\ (ZFDepth\ (Expr2ZF\ e'))) = (ZFDepth\ (Expr2ZF\ e))$
⟨*proof*⟩

lemma $Expr2ZFDepth0$: $ZFDepth\ (Expr2ZF\ e) = 0 \Longrightarrow ZFType\ (Expr2ZF\ e) = 0$

⟨*proof*⟩

lemma *Expr2ZFDepthSuc*: *ZFDepth* (*Expr2ZF e*) = *Suc n* ⟹ *ZFType* (*Expr2ZF e*) = *1*
⟨*proof*⟩

lemma *Expr2Data*: *ZFData* (*Expr2ZF* (*f E@ e*)) = *Opair f* (*Expr2ZF e*)
⟨*proof*⟩

lemma *Expr2ZFinj*: *inj Expr2ZF*
⟨*proof*⟩

definition *TermEquivClGen T A e B* ≡ {*e′* . (*Vx* : *A* ⊢ *e′* ≡ *e* : *B*) ∈ *Axioms.Theory T*}
definition *TermEquivCl′ T A e B* ≡ (‖ *TDomain* = *A* , *TExprSet* = *TermEquivClGen T A e B* , *TCodomain* = *B*‖)

definition *m2ZF* :: (*ZF*,*ZF*) *TermEquivClT* ⇒ *ZF* **where**
m2ZF t ≡ *ZFTriple* (*TDomain t*) (*implode* (*Expr2ZF* ‘ (*TExprSet t*))) (*TCodomain t*)

definition *ZF2m* :: (*ZF*,*ZF*) *Axioms* ⇒ *ZF* ⇒ (*ZF*,*ZF*) *TermEquivClT* **where**
ZF2m T ≡ *inv-into* {*TermEquivCl′ T A e B* | *A e B* . *True*} *m2ZF*

lemma *TDomain*: *TDomain* (*TermEquivCl′ T A e B*) = *A* ⟨*proof*⟩
lemma *TCodomain*: *TCodomain* (*TermEquivCl′ T A e B*) = *B* ⟨*proof*⟩

primrec *WellFormedToSet* :: (*ZF*,*ZF*) *Signature* ⇒ *nat* ⇒ (*ZF*,*ZF*) *Expression set* **where**
WFS0: *WellFormedToSet S 0* = {*Vx*}
| *WFSS*: *WellFormedToSet S* (*Suc n*) = (*WellFormedToSet S n*) ∪
{ *f E@ e* | *f e* . *f* ∈ *BaseFunctions S* ∧ *e* ∈ (*WellFormedToSet S n*)}

lemma *WellFormedToSetInRangeExplode*: *ZFAxioms T* ⟹ (*Expr2ZF* ‘ (*WellFormedToSet aS* $_T$ *n*)) ∈ *range explode*
⟨*proof*⟩

lemma *WellDefinedToWellFormedSet*: ⋀ *B* . (*Sig S* ▷ (*Vx* : *A* ⊢ *e* : *B*)) ⟹ ∃ *n*. *e* ∈ *WellFormedToSet S n*
⟨*proof*⟩

lemma *TermSetInSet*: *ZFAxioms T* ⟹ *Expr2ZF* ‘ (*TermEquivClGen T A e B*) ∈ *range explode*
⟨*proof*⟩

lemma *m2ZFinj-on*: *ZFAxioms T* ⟹ *inj-on m2ZF* {*TermEquivCl′ T A e B* | *A e B* . *True*}
⟨*proof*⟩

lemma *ZF2m*: *ZFAxioms T* $\Longrightarrow$ *ZF2m T* (*m2ZF* (*TermEquivCl′ T A e B*)) = (*TermEquivCl′ T A e B*)
⟨*proof*⟩

definition *TermEquivCl* (‹[-,-,-]₁›) **where** $[A,e,B]_T \equiv$ *m2ZF* (*TermEquivCl′ T A e B*)

definition *CLDomain T* ≡ *TDomain o ZF2m T*
definition *CLCodomain T* ≡ *TCodomain o ZF2m T*

definition *CanonicalComp T f g* ≡
THE h . ∃ *e e′* . *h* = $[$*CLDomain T f*,*sub e in e′*,*CLCodomain T g*$]_T$ ∧
f = $[$*CLDomain T f*,*e*,*CLCodomain T f*$]_T$ ∧ *g* = $[$*CLDomain T g*,*e′*,*CLCodomain T g*$]_T$

lemma *CLDomain*: *ZFAxioms T* $\Longrightarrow$ *CLDomain T* $[A,e,B]_T$ = *A* ⟨*proof*⟩
lemma *CLCodomain*: *ZFAxioms T* $\Longrightarrow$ *CLCodomain T* $[A,e,B]_T$ = *B* ⟨*proof*⟩

lemma *Equiv2Cl*: **assumes** *Axioms T* **and** (*Vx* : *A* ⊢ *e* ≡ *d* : *B*) ∈ *Axioms.Theory T* **shows** $[A,e,B]_T = [A,d,B]_T$
⟨*proof*⟩

lemma *Cl2Equiv*:
assumes *axt*: *ZFAxioms T* **and** *sa*: *Sig aS* $_T$ ▷ (*Vx* : *A* ⊢ *e* : *B*) **and** *cl*: $[A,e,B]_T = [A,d,B]_T$
shows (*Vx* : *A* ⊢ *e* ≡ *d* : *B*) ∈ *Axioms.Theory T*
⟨*proof*⟩

lemma *CanonicalCompWellDefined*:
assumes *zaxt*: *ZFAxioms T* **and** *Sig aS* $_T$ ▷ (*Vx* : *A* ⊢ *d* : *B*) **and** *Sig aS* $_T$ ▷ (*Vx* : *B* ⊢ *d′* : *C*)
shows *CanonicalComp T* $[A,d,B]_T$ $[B,d',C]_T$ = $[A,$*sub d in d′*$,C]_T$
⟨*proof*⟩

definition *CanonicalCat′ T* ≡ ⦇
Obj = *BaseTypes* (*aS* $_T$),
Mor = {$[A,e,B]_T$ | *A e B* . *Sig aS* $_T$ ▷ (*Vx* : *A* ⊢ *e* : *B*)},
Dom = *CLDomain T*,
Cod = *CLCodomain T*,
Id = (λ *A* . $[A,Vx,A]_T$),
Comp = *CanonicalComp T*
⦈

definition *CanonicalCat T* ≡ *MakeCat* (*CanonicalCat′ T*)

lemma *CanonicalCat′MapsTo*:
assumes *f maps*$_{CanonicalCat'\ T}$ *X to Y* **and** *zx*: *ZFAxioms T*
shows ∃ *ef* . *f* = $[X,ef,Y]_T$ ∧ *Sig* (*aSignature T*) ▷ (*Vx* : *X* ⊢ *ef* : *Y*)
⟨*proof*⟩

lemma *CanonicalCatCat'*: *ZFAxioms T* $\Longrightarrow$ *Category-axioms* (*CanonicalCat' T*)
⟨*proof*⟩

lemma *CanonicalCatCat*: *ZFAxioms T* $\Longrightarrow$ *Category* (*CanonicalCat T*)
⟨*proof*⟩

definition *CanonicalInterpretation* **where**
CanonicalInterpretation T ≡ (|
ISignature = *aSignature T*,
ICategory = *CanonicalCat T*,
ITypes = λ *A* . *A*,
IFunctions = λ *f* . [*SigDom* (*aSignature T*) *f*, *f E@ Vx*, *SigCod* (*aSignature T*) *f*]$_T$
|)

abbreviation *CI T* ≡ *CanonicalInterpretation T*

lemma *CIObj*: *Obj* (*CanonicalCat T*) = *BaseTypes* (*aSignature T*)
⟨*proof*⟩

lemma *CIMor*: *ZFAxioms T* $\Longrightarrow$ [*A*,*e*,*B*]$_T$ ∈ *Mor* (*CanonicalCat T*) = *Sig* (*aSignature T*) ▷ (*Vx* : *A* ⊢ *e* : *B*)
⟨*proof*⟩

lemma *CIDom*: ⟦*ZFAxioms T* ; [*A*,*e*,*B*]$_T$ ∈ *Mor*(*CanonicalCat T*)⟧ $\Longrightarrow$ *Dom* (*CanonicalCat T*) [*A*,*e*,*B*]$_T$ = *A*
⟨*proof*⟩

lemma *CICod*: ⟦*ZFAxioms T* ; [*A*,*e*,*B*]$_T$ ∈ *Mor*(*CanonicalCat T*)⟧ $\Longrightarrow$ *Cod* (*CanonicalCat T*) [*A*,*e*,*B*]$_T$ = *B*
⟨*proof*⟩

lemma *CIId*: ⟦*A* ∈ *BaseTypes* (*aSignature T*)⟧ $\Longrightarrow$ *Id* (*CanonicalCat T*) *A* = [*A*,*Vx*,*A*]$_T$
⟨*proof*⟩

lemma *CIComp*:
assumes *ZFAxioms T* **and** *Sig* (*aSignature T*) ▷ (*Vx* : *A* ⊢ *e* : *B*) **and** *Sig* (*aSignature T*) ▷ (*Vx* : *B* ⊢ *d* : *C*)
shows [*A*,*e*,*B*]$_T$;;$_{CanonicalCat\ T}$ [*B*,*d*,*C*]$_T$ = [*A*,*sub e in d*,*C*]$_T$
⟨*proof*⟩

lemma [*simp*]: *ZFAxioms T* $\Longrightarrow$ *Category iC*$_{CI\ T}$ ⟨*proof*⟩
lemma [*simp*]: *ZFAxioms T* $\Longrightarrow$ *Signature iS*$_{CI\ T}$ ⟨*proof*⟩

lemma *CIInterpretation*: *ZFAxioms T* $\Longrightarrow$ *Interpretation* (*CI T*)
⟨*proof*⟩

lemma *CIInterp2Mor*: *ZFAxioms T* $\Longrightarrow$ ($\bigwedge$ *B* . *Sig iS*$_{CI\ T}$ $\triangleright$ (*Vx* : *A* $\vdash$ *e* : *B*) $\Longrightarrow$ *L*$[\![Vx : A \vdash e : B]\!]_{CI\ T} \to$ (*IMor* [*A*, *e*, *B*]$_T$))
⟨*proof*⟩

lemma *CIModel*: *ZFAxioms T* $\Longrightarrow$ *Model* (*CI T*) *T*
⟨*proof*⟩

lemma *CIComplete*: **assumes** *ZFAxioms T* **and** *L*$[\![\varphi]\!]_{CI\ T} \to$ (*IBool True*) **shows** $\varphi \in$ *Axioms.Theory T*
⟨*proof*⟩

lemma *Complete*:
assumes *ZFAxioms T*
and $\bigwedge$ (*I* :: (*ZF*,*ZF*,*ZF*,*ZF*) *Interpretation*) . *Model I T* $\Longrightarrow$ (*L*$[\![\varphi]\!]_I \to$ (*IBool True*))
shows $\varphi \in$ *Axioms.Theory T*
⟨*proof*⟩

end

4 Functor

theory *Functors*
imports *Category*
begin

record (*'o1*, *'o2*, *'m1*, *'m2*, *'a*, *'b*) *Functor* =
CatDom :: (*'o1*,*'m1*,*'a*)*Category-scheme*
CatCod :: (*'o2*,*'m2*,*'b*)*Category-scheme*
MapM :: *'m1* $\Rightarrow$ *'m2*

abbreviation
FunctorMorApp :: (*'o1*, *'o2*, *'m1*, *'m2*, *'a1*, *'a2*, *'a*) *Functor-scheme* $\Rightarrow$ *'m1* $\Rightarrow$ *'m2* (**infixr** ‹##› *70*) **where**
FunctorMorApp F m $\equiv$ (*MapM F*) *m*

definition
MapO :: (*'o1*, *'o2*, *'m1*, *'m2*, *'a1*, *'a2*, *'a*) *Functor-scheme* $\Rightarrow$ *'o1* $\Rightarrow$ *'o2* **where**
MapO F X $\equiv$ *THE Y* . *Y* $\in$ *Obj*(*CatCod F*) $\wedge$ *F* ## (*Id* (*CatDom F*) *X*) = *Id* (*CatCod F*) *Y*

abbreviation
FunctorObjApp :: (*'o1*, *'o2*, *'m1*, *'m2*, *'a1*, *'a2*, *'a*) *Functor-scheme* $\Rightarrow$ *'o1* $\Rightarrow$ *'o2* (**infixr** ‹@@› *70*) **where**
FunctorObjApp F X $\equiv$ (*MapO F X*)

locale *PreFunctor* =
fixes *F* :: (*'o1*, *'o2*, *'m1*, *'m2*, *'a1*, *'a2*, *'a*) *Functor-scheme* (**structure**)
assumes *FunctorComp*: *f* $\approx>_{CatDom\ F}$ *g* $\Longrightarrow$ *F* ## (*f* ;;$_{CatDom\ F}$ *g*) = (*F* ##

$f)\ ;;_{CatCod\ F}\ (F\ \#\#\ g)$
and *FunctorId*: $X \in obj_{CatDom\ F} \Longrightarrow \exists\ Y \in obj_{CatCod\ F}\ .\ F\ \#\#\ (id_{CatDom\ F}\ X) = id_{CatCod\ F}\ Y$
and *CatDom*[*simp*]: *Category*(*CatDom F*)
and *CatCod*[*simp*]: *Category*(*CatCod F*)

locale *FunctorM* = *PreFunctor* +
assumes *FunctorCompM*: $f\ maps_{CatDom\ F}\ X\ to\ Y \Longrightarrow (F\ \#\#\ f)\ maps_{CatCod\ F}\ (F\ @@\ X)\ to\ (F\ @@\ Y)$

locale *FunctorExt* =
fixes *F* :: ('*o1*, '*o2*, '*m1*, '*m2*, '*a1*, '*a2*, '*a*) *Functor-scheme* (**structure**)
assumes *FunctorMapExt*: (*MapM F*) ∈ *extensional* (*Mor* (*CatDom F*))

locale *Functor* = *FunctorM* + *FunctorExt*

definition
MakeFtor :: ('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*,'*r*) *Functor-scheme* ⇒ ('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*,'*r*) *Functor-scheme* **where**
MakeFtor F ≡ (|
CatDom = *CatDom F* ,
CatCod = *CatCod F* ,
MapM = *restrict* (*MapM F*) (*Mor* (*CatDom F*)) ,
... = *Functor.more F*
|)

lemma *PreFunctorFunctor*[*simp*]: *Functor F* ⟹ *PreFunctor F*
⟨*proof*⟩

lemmas *functor-simps* = *PreFunctor.FunctorComp PreFunctor.FunctorId*

definition
functor-abbrev (‹*Ftor* - : - ⟶ -› [*81*]) **where**
Ftor F : *A* ⟶ *B* ≡ (*Functor F*) ∧ (*CatDom F* = *A*) ∧ (*CatCod F* = *B*)

lemma *functor-abbrevE*[*elim*]: ⟦*Ftor F* : *A* ⟶ *B* ; ⟦(*Functor F*) ; (*CatDom F* = *A*) ; (*CatCod F* = *B*)⟧ ⟹ *R*⟧ ⟹ *R*
⟨*proof*⟩

definition
functor-comp-def (‹- ≈>;;; -› [*81*]) **where**
functor-comp-def F G ≡ (*Functor F*) ∧ (*Functor G*) ∧ (*CatDom G* = *CatCod F*)

lemma *functor-comp-def*[*elim*]: ⟦*F* ≈>;;; *G* ; ⟦*Functor F* ; *Functor G* ; *CatDom G* = *CatCod F*⟧ ⟹ *R*⟧ ⟹ *R*
⟨*proof*⟩

lemma (**in** *Functor*) *FunctorMapsTo*:

assumes $f \in mor_{CatDom\ F}$
shows $F\ \#\#\ f\ maps_{CatCod\ F}\ (F\ @@\ (dom_{CatDom\ F}\ f))\ to\ (F\ @@\ (cod_{CatDom\ F}\ f))$
⟨*proof*⟩

lemma (**in** *Functor*) *FunctorCodDom*:
assumes $f \in mor_{CatDom\ F}$
shows $dom_{CatCod\ F}(F\ \#\#\ f) = F\ @@\ (dom_{CatDom\ F}\ f)$ **and** $cod_{CatCod\ F}(F\ \#\#\ f) = F\ @@\ (cod_{CatDom\ F}\ f)$
⟨*proof*⟩

lemma (**in** *Functor*) *FunctorCompPreserved*: $f \in mor_{CatDom\ F} \Longrightarrow F\ \#\#\ f \in mor_{CatCod\ F}$
⟨*proof*⟩

lemma (**in** *Functor*) *FunctorCompDef*:
assumes $f \approx>_{CatDom\ F} g$ **shows** $(F\ \#\#\ f) \approx>_{CatCod\ F} (F\ \#\#\ g)$
⟨*proof*⟩

lemma *FunctorComp*: $[\![Ftor\ F : A \longrightarrow B\ ;\ f \approx>_A g]\!] \Longrightarrow F\ \#\#\ (f\ ;;_A\ g) = (F\ \#\#\ f)\ ;;_B\ (F\ \#\#\ g)$
⟨*proof*⟩

lemma *FunctorCompDef*: $[\![Ftor\ F : A \longrightarrow B\ ;\ f \approx>_A g]\!] \Longrightarrow (F\ \#\#\ f) \approx>_B (F\ \#\#\ g)$
⟨*proof*⟩

lemma *FunctorMapsTo*:
assumes $Ftor\ F : A \longrightarrow B$ **and** $f\ maps_A\ X\ to\ Y$
shows $(F\ \#\#\ f)\ maps_B\ (F\ @@\ X)\ to\ (F\ @@\ Y)$
⟨*proof*⟩

lemma (**in** *PreFunctor*) *FunctorId2*:
assumes $X \in obj_{CatDom\ F}$
shows $F\ @@\ X \in obj_{CatCod\ F} \wedge F\ \#\#\ (id_{CatDom\ F}\ X) = id_{CatCod\ F}\ (F\ @@\ X)$
⟨*proof*⟩

lemma *FunctorId*:
assumes $Ftor\ F : C \longrightarrow D$ **and** $X \in Obj\ C$
shows $F\ \#\#\ (Id\ C\ X) = Id\ D\ (F\ @@\ X)$
⟨*proof*⟩

lemma (**in** *Functor*) *DomFunctor*: $f \in mor_{CatDom\ F} \Longrightarrow dom_{CatCod\ F}\ (F\ \#\#\ f) = F\ @@\ (dom_{CatDom\ F}\ f)$
⟨*proof*⟩

lemma (**in** *Functor*) *CodFunctor*: $f \in mor_{CatDom\ F} \Longrightarrow cod_{CatCod\ F}\ (F\ \#\#\ f) = F\ @@\ (cod_{CatDom\ F}\ f)$

⟨*proof*⟩

lemma (**in** *Functor*) *FunctorId3Dom*:
assumes $f \in mor_{CatDom\ F}$
shows $F\ \#\#\ (id_{CatDom\ F}\ (dom_{CatDom\ F}\ f)) = id_{CatCod\ F}\ (dom_{CatCod\ F}\ (F\ \#\#\ f))$
⟨*proof*⟩

lemma (**in** *Functor*) *FunctorId3Cod*:
assumes $f \in mor_{CatDom\ F}$
shows $F\ \#\#\ (id_{CatDom\ F}\ (cod_{CatDom\ F}\ f)) = id_{CatCod\ F}\ (cod_{CatCod\ F}\ (F\ \#\#\ f))$
⟨*proof*⟩

lemma (**in** *PreFunctor*) *FmToFo*: $[\![X \in obj_{CatDom\ F}\ ;\ Y \in obj_{CatCod\ F}\ ;\ F\ \#\#\ (id_{CatDom\ F}\ X) = id_{CatCod\ F}\ Y]\!] \Longrightarrow F\ @@\ X = Y$
⟨*proof*⟩

lemma *MakeFtorPreFtor*:
assumes *PreFunctor F* **shows** *PreFunctor* (*MakeFtor F*)
⟨*proof*⟩

lemma *MakeFtorMor*: $f \in mor_{CatDom\ F} \Longrightarrow MakeFtor\ F\ \#\#\ f = F\ \#\#\ f$
⟨*proof*⟩

lemma *MakeFtorObj*:
assumes *PreFunctor F* **and** $X \in obj_{CatDom\ F}$
shows *MakeFtor F* @@ *X* = *F* @@ *X*
⟨*proof*⟩

lemma *MakeFtor*: **assumes** *FunctorM F* **shows** *Functor* (*MakeFtor F*)
⟨*proof*⟩

definition
IdentityFunctor' :: ('*o*,'*m*,'*a*) *Category-scheme* ⇒ ('*o*,'*o*,'*m*,'*m*,'*a*,'*a*) *Functor* (‹*FId''* -› [*70*]) **where**
IdentityFunctor' C ≡ (|*CatDom* = *C* , *CatCod* = *C* , *MapM* = (λ *f* . *f*) |)

definition
IdentityFunctor (‹*FId* -› [*70*]) **where**
IdentityFunctor C ≡ *MakeFtor*(*IdentityFunctor' C*)

lemma *IdFtor'PreFunctor*: *Category C* ⟹ *PreFunctor* (*FId' C*)
⟨*proof*⟩

lemma *IdFtor'Obj*:
assumes *Category C* **and** $X \in obj_{CatDom\ (FId'\ C)}$
shows (*FId' C*) @@ *X* = *X*
⟨*proof*⟩

lemma *IdFtor'FtorM*:
assumes *Category C* **shows** *FunctorM* (*FId' C*)
⟨*proof*⟩

lemma *IdFtorFtor*: *Category C* ⟹ *Functor* (*FId C*)
⟨*proof*⟩

definition
ConstFunctor' :: ('*o1*,'*m1*,'*a*) *Category-scheme* ⇒
('*o2*,'*m2*,'*b*) *Category-scheme* ⇒ '*o2* ⇒ ('*o1*,'*o2*,'*m1*,'*m2*,'*a*,'*b*)
Functor **where**
ConstFunctor' A B b ≡ (|
CatDom = *A* ,
CatCod = *B* ,
MapM = (λ *f* . (*Id B*) *b*)
|)

definition *ConstFunctor A B b* ≡ *MakeFtor*(*ConstFunctor' A B b*)

lemma *ConstFtor'* :
assumes *Category A Category B b* ∈ (*Obj B*)
shows *PreFunctor* (*ConstFunctor' A B b*)
and *FunctorM* (*ConstFunctor' A B b*)
⟨*proof*⟩

lemma *ConstFtor*:
assumes *Category A Category B b* ∈ (*Obj B*)
shows *Functor* (*ConstFunctor A B b*)
⟨*proof*⟩

definition
UnitFunctor :: ('*o*,'*m*,'*a*) *Category-scheme* ⇒ ('*o*,*unit*,'*m*,*unit*,'*a*,*unit*) *Functor*
where
UnitFunctor C ≡ *ConstFunctor C UnitCategory* ()

lemma *UnitFtor*:
assumes *Category C*
shows *Functor*(*UnitFunctor C*)
⟨*proof*⟩

definition
FunctorComp' :: ('*o1*,'*o2*,'*m1*,'*m2*,'*a1*,'*a2*) *Functor* ⇒ ('*o2*,'*o3*,'*m2*,'*m3*,'*b1*,'*b2*)
Functor
⇒ ('*o1*,'*o3*,'*m1*,'*m3*,'*a1*,'*b2*) *Functor* (**infixl** ‹;;:› *71*) **where**
FunctorComp' F G ≡ (|
CatDom = *CatDom F* ,
CatCod = *CatCod G* ,
MapM = λ *f* . (*MapM G*)((*MapM F*) *f*)

⟫

definition *FunctorComp* (**infixl** ‹;;;› *71*) **where** *FunctorComp F G* ≡ *MakeFtor* (*FunctorComp′ F G*)

lemma *FtorCompComp′*:
assumes *f* ≈>$_{CatDom\ F}$ *g*
and *F* ≈>;;; *G*
shows *G* ## (*F* ## (*f* ;;$_{CatDom\ F}$ *g*)) = (*G* ## (*F* ## *f*)) ;;$_{CatCod\ G}$ (*G* ## (*F* ## *g*))
⟨*proof*⟩

lemma *FtorCompId*:
assumes *a*: *X* ∈ (*Obj* (*CatDom F*))
and *F* ≈>;;; *G*
shows *G* ## (*F* ## (*id*$_{CatDom\ F}$ *X*)) = *id*$_{CatCod\ G}$(*G* @@ (*F* @@ *X*)) ∧ *G* @@ (*F* @@ *X*) ∈ (*Obj* (*CatCod G*))
⟨*proof*⟩

lemma *FtorCompIdDef*:
assumes *a*: *X* ∈ (*Obj* (*CatDom F*)) **and** *b*: *PreFunctor* (*F* ;;: *G*)
and *F* ≈>;;; *G*
shows (*F* ;;: *G*) @@ *X* = (*G* @@ (*F* @@ *X*))
⟨*proof*⟩

lemma *FunctorCompMapsTo*:
assumes *f* ∈ *mor*$_{CatDom}$ (*F* ;;: *G*) **and** *F* ≈>;;; *G*
shows (*G* ## (*F* ## *f*)) *maps*$_{CatCod\ G}$ (*G* @@ (*F* @@ (*dom*$_{CatDom\ F}$ *f*))) *to* (*G* @@ (*F* @@ (*cod*$_{CatDom\ F}$ *f*)))
⟨*proof*⟩

lemma *FunctorCompMapsTo2*:
assumes *f* ∈ *mor*$_{CatDom}$ (*F* ;;: *G*)
and *F* ≈>;;; *G*
and *PreFunctor* (*F* ;;: *G*)
shows ((*F* ;;: *G*) ## *f*) *maps*$_{CatCod\ (F\ ;;:\ G)}$ ((*F* ;;: *G*) @@ (*dom*$_{CatDom\ (F\ ;;:\ G)}$ *f*)) *to* ((*F* ;;: *G*) @@ (*cod*$_{CatDom\ (F\ ;;:\ G)}$ *f*))
⟨*proof*⟩

lemma *FunctorCompMapsTo3*:
assumes *f maps*$_{CatDom\ (F\ ;;:\ G)}$ *X to Y*
and *F* ≈>;;; *G*
and *PreFunctor* (*F* ;;: *G*)
shows *F* ;;: *G* ## *f maps*$_{CatCod\ (F\ ;;:\ G)}$ *F* ;;: *G* @@ *X to F* ;;: *G* @@ *Y*
⟨*proof*⟩

lemma *FtorCompPreFtor*:

assumes $F \approx>;;; G$
shows *PreFunctor* $(F \;;;: G)$
⟨*proof*⟩

lemma *FtorCompM* :
assumes $F \approx>;;; G$
shows *FunctorM* $(F \;;;: G)$
⟨*proof*⟩

lemma *FtorComp*:
assumes $F \approx>;;; G$
shows *Functor* $(F \;;;\; G)$
⟨*proof*⟩

lemma (**in** *Functor*) *FunctorPreservesIso*:
assumes $ciso_{CatDom\ F}\ k$
shows $ciso_{CatCod\ F}\ (F\ \#\#\ k)$
⟨*proof*⟩

declare *PreFunctor.CatDom*[*simp*] *PreFunctor.CatCod* [*simp*]

lemma *FunctorMFunctor*[*simp*]: *Functor* $F \Longrightarrow$ *FunctorM* F
⟨*proof*⟩

locale *Equivalence* = *Functor* +
assumes *Full*: $[\![A \in Obj\ (CatDom\ F)\ ;\ B \in Obj\ (CatDom\ F)\ ;$
$h\ maps_{CatCod\ F}\ (F\ @@\ A)\ to\ (F\ @@\ B)]\!] \Longrightarrow$
$\exists\ f\ .\ (f\ maps_{CatDom\ F}\ A\ to\ B) \wedge (F\ \#\#\ f = h)$
and *Faithful*: $[\![f\ maps_{CatDom\ F}\ A\ to\ B\ ;\ g\ maps_{CatDom\ F}\ A\ to\ B\ ;\ F\ \#\#\ f = F\ \#\#\ g]\!] \Longrightarrow f = g$
and *IsoDense*: $C \in Obj\ (CatCod\ F) \Longrightarrow \exists\ A \in Obj\ (CatDom\ F)\ .\ ObjIso\ (CatCod\ F)\ (F\ @@\ A)\ C$

end

5 Natural Transformation

theory *NatTrans*
imports *Functors*
begin

record $('o1, 'o2, 'm1, 'm2, 'a, 'b)$ *NatTrans* =
NTDom :: $('o1, 'o2, 'm1, 'm2, 'a, 'b)$ *Functor*
NTCod :: $('o1, 'o2, 'm1, 'm2, 'a, 'b)$ *Functor*
NatTransMap :: $'o1 \Rightarrow 'm2$

abbreviation
NatTransApp :: $('o1, 'o2, 'm1, 'm2, 'a, 'b)$ *NatTrans* $\Rightarrow 'o1 \Rightarrow 'm2$ (**infixr** ‹$$› *70*) **where**

NatTransApp η $X \equiv$ (*NatTransMap* η) X

definition *NTCatDom* $\eta \equiv$ *CatDom* (*NTDom* η)
definition *NTCatCod* $\eta \equiv$ *CatCod* (*NTCod* η)

locale *NatTransExt* =
fixes η :: ('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*) *NatTrans* (**structure**)
assumes *NTExt* : *NatTransMap* $\eta \in$ *extensional* (*Obj* (*NTCatDom* η))

locale *NatTransP* =
fixes η :: ('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*) *NatTrans* (**structure**)
assumes *NatTransFtor*: *Functor* (*NTDom* η)
and *NatTransFtor2*: *Functor* (*NTCod* η)
and *NatTransFtorDom*: *NTCatDom* η = *CatDom* (*NTCod* η)
and *NatTransFtorCod*: *NTCatCod* η = *CatCod* (*NTDom* η)
and *NatTransMapsTo*: $X \in obj_{NTCatDom\ \eta} \Longrightarrow$
(η \$\$ X) $maps_{NTCatCod\ \eta}$ ((*NTDom* η) @@ X) *to* ((*NTCod* η) @@ X)
and *NatTrans*: $f\ maps_{NTCatDom\ \eta}\ X\ to\ Y \Longrightarrow$
((*NTDom* η) ## f) $;;_{NTCatCod\ \eta}$ (η \$\$ Y) = (η \$\$ X) $;;_{NTCatCod\ \eta}$ ((*NTCod* η) ## f)

locale *NatTrans* = *NatTransP* + *NatTransExt*

lemma [*simp*]: *NatTrans* $\eta \Longrightarrow$ *NatTransP* η
⟨*proof*⟩

definition *MakeNT* :: ('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*) *NatTrans* ⇒ ('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*) *NatTrans* **where**
MakeNT $\eta \equiv$ (|
NTDom = *NTDom* η ,
NTCod = *NTCod* η ,
NatTransMap = *restrict* (*NatTransMap* η) (*Obj* (*NTCatDom* η))
|)

definition
nt-abbrev (‹*NT* - : - ⟹ -› [*81*]) **where**
NT f : $F \Longrightarrow G \equiv$ (*NatTrans* f) ∧ (*NTDom* f = F) ∧ (*NTCod* f = G)

lemma *nt-abbrevE*[*elim*]: ⟦*NT* f : $F \Longrightarrow G$; ⟦(*NatTrans* f) ; (*NTDom* f = F) ; (*NTCod* f = G)⟧ ⟹ R⟧ ⟹ R
⟨*proof*⟩

lemma *MakeNT*: *NatTransP* $\eta \Longrightarrow$ *NatTrans* (*MakeNT* η)
⟨*proof*⟩

lemma *MakeNT-comp*: $X \in$ *Obj* (*NTCatDom* f) ⟹ (*MakeNT* f) \$\$ X = f \$\$ X
⟨*proof*⟩

lemma *MakeNT-dom*: $NTCatDom\ f = NTCatDom\ (MakeNT\ f)$
⟨*proof*⟩

lemma *MakeNT-cod*: $NTCatCod\ f = NTCatCod\ (MakeNT\ f)$
⟨*proof*⟩

lemma *MakeNTApp*: $X \in Obj\ (NTCatDom\ (MakeNT\ f)) \Longrightarrow f\ \$\$\ X = (MakeNT\ f)\ \$\$\ X$
⟨*proof*⟩

lemma *NatTransMapsTo*:
assumes $NT\ \eta : F \Longrightarrow G$ **and** $X \in Obj\ (CatDom\ F)$
shows $\eta\ \$\$\ X\ maps_{CatCod\ G}\ (F\ @@\ X)\ to\ (G\ @@\ X)$
⟨*proof*⟩

definition
$NTCompDefined :: ('o1, 'o2, 'm1, 'm2, 'a, 'b)\ NatTrans \Rightarrow ('o1, 'o2, 'm1, 'm2, 'a, 'b)\ NatTrans \Rightarrow bool$ (**infixl** ‹≈>·› 65) **where**
$NTCompDefined\ \eta1\ \eta2 \equiv NatTrans\ \eta1 \wedge NatTrans\ \eta2 \wedge NTCatDom\ \eta2 = NTCatDom\ \eta1 \wedge$
$NTCatCod\ \eta2 = NTCatCod\ \eta1 \wedge NTCod\ \eta1 = NTDom\ \eta2$

lemma *NTCompDefinedE*[*elim*]: $[\![\eta1 \approx\!>\cdot\ \eta2\ ;\ [\![NatTrans\ \eta1\ ;\ NatTrans\ \eta2\ ;\ NTCatDom\ \eta2 = NTCatDom\ \eta1\ ;$
$NTCatCod\ \eta2 = NTCatCod\ \eta1\ ;\ NTCod\ \eta1 = NTDom\ \eta2]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *NTCompDefinedI*: $[\![NatTrans\ \eta1\ ;\ NatTrans\ \eta2\ ;\ NTCatDom\ \eta2 = NTCatDom\ \eta1\ ;$
$NTCatCod\ \eta2 = NTCatCod\ \eta1\ ;\ NTCod\ \eta1 = NTDom\ \eta2]\!] \Longrightarrow \eta1 \approx\!>\cdot\ \eta2$
⟨*proof*⟩

lemma *NatTransExt0*:
assumes $NTDom\ \eta1 = NTDom\ \eta2$ **and** $NTCod\ \eta1 = NTCod\ \eta2$
and $\bigwedge X\ .\ X \in Obj\ (NTCatDom\ \eta1) \Longrightarrow \eta1\ \$\$\ X = \eta2\ \$\$\ X$
and $NatTransMap\ \eta1 \in extensional\ (Obj\ (NTCatDom\ \eta1))$
and $NatTransMap\ \eta2 \in extensional\ (Obj\ (NTCatDom\ \eta2))$
shows $\eta1 = \eta2$
⟨*proof*⟩

lemma *NatTransExt'*:
assumes $NTDom\ \eta1' = NTDom\ \eta2'$ **and** $NTCod\ \eta1' = NTCod\ \eta2'$
and $\bigwedge X\ .\ X \in Obj\ (NTCatDom\ \eta1') \Longrightarrow \eta1'\ \$\$\ X = \eta2'\ \$\$\ X$
shows $MakeNT\ \eta1' = MakeNT\ \eta2'$
⟨*proof*⟩

lemma *NatTransExt*:
assumes *NatTrans η1* **and** *NatTrans η2* **and** *NTDom η1 = NTDom η2* **and** *NTCod η1 = NTCod η2*
and $\bigwedge X\ .\ X \in Obj\ (NTCatDom\ \eta 1) \Longrightarrow \eta 1\ \$\$\ X = \eta 2\ \$\$\ X$
shows *η1 = η2*
⟨*proof*⟩

definition
IdNatTrans' :: (*'o1*, *'o2*, *'m1*, *'m2*, *'a1*, *'a2*) *Functor* ⇒ (*'o1*, *'o2*, *'m1*, *'m2*, *'a1*, *'a2*) *NatTrans* **where**
IdNatTrans' F ≡ (|
NTDom = F ,
NTCod = F ,
NatTransMap = λ *X* . $id_{CatCod\ F}\ (F\ @@\ X)$
|)

definition *IdNatTrans F* ≡ *MakeNT*(*IdNatTrans' F*)

lemma *IdNatTrans-map*: $X \in obj_{CatDom\ F} \Longrightarrow (IdNatTrans\ F)\ \$\$\ X = id_{CatCod\ F}\ (F\ @@\ X)$
⟨*proof*⟩

lemmas *IdNatTrans-defs = IdNatTrans-def IdNatTrans'-def MakeNT-def IdNatTrans-map NTCatCod-def NTCatDom-def*

lemma *IdNatTransNatTrans'*: *Functor F* ⟹ *NatTransP*(*IdNatTrans' F*)
⟨*proof*⟩

lemma *IdNatTransNatTrans*: *Functor F* ⟹ *NatTrans* (*IdNatTrans F*)
⟨*proof*⟩

definition
NatTransComp' :: (*'o1*, *'o2*, *'m1*, *'m2*, *'a*, *'b*) *NatTrans* ⇒
(*'o1*, *'o2*, *'m1*, *'m2*, *'a*, *'b*) *NatTrans* ⇒
(*'o1*, *'o2*, *'m1*, *'m2*, *'a*, *'b*) *NatTrans* (**infixl** ‹·1› 75) **where**
NatTransComp' η1 η2 = (|
NTDom = NTDom η1 ,
NTCod = NTCod η2 ,
NatTransMap = λ *X* . (*η1 $$ X*) ;;$_{NTCatCod\ \eta 1}$ (*η2 $$ X*)
|)

definition *NatTransComp* (**infixl** ‹·› 75) **where** *η1 · η2* ≡ *MakeNT*(*η1 ·1 η2*)

lemma *NatTransComp-Comp1*: ⟦*x* ∈ *Obj* (*NTCatDom f*) ; *f* ≈>· *g*⟧ ⟹ (*f · g*) $$ *x* = (*f $$ x*) ;;$_{NTCatCod\ g}$ (*g $$ x*)
⟨*proof*⟩

lemma *NatTransComp-Comp2*: ⟦*x* ∈ *Obj* (*NTCatDom f*) ; *f* ≈>· *g*⟧ ⟹ (*f · g*)

$\$\$\ x = (f\ \$\$\ x)\ ;;_{NTCatCod\ f}\ (g\ \$\$\ x)$
⟨*proof*⟩

lemmas *NatTransComp-defs* = *NatTransComp-def NatTransComp′-def MakeNT-def*

NatTransComp-Comp1 NTCatCod-def NTCatDom-def

lemma [*simp*]: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NatTrans* $\eta 1$ ⟨*proof*⟩
lemma [*simp*]: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NatTrans* $\eta 2$ ⟨*proof*⟩
lemma *NTCatDom*: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NTCatDom* $\eta 1$ = *NTCatDom* $\eta 2$ ⟨*proof*⟩
lemma *NTCatCod*: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NTCatCod* $\eta 1$ = *NTCatCod* $\eta 2$ ⟨*proof*⟩
lemma [*simp*]: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NTCatDom* $(\eta 1\ \cdot 1\ \eta 2)$ = *NTCatDom* $\eta 1$ ⟨*proof*⟩
lemma [*simp*]: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NTCatCod* $(\eta 1\ \cdot 1\ \eta 2)$ = *NTCatCod* $\eta 1$ ⟨*proof*⟩
lemma [*simp*]: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NTCatDom* $(\eta 1\ \cdot\ \eta 2)$ = *NTCatDom* $\eta 1$ ⟨*proof*⟩
lemma [*simp*]: $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2 \Longrightarrow$ *NTCatCod* $(\eta 1\ \cdot\ \eta 2)$ = *NTCatCod* $\eta 1$ ⟨*proof*⟩
lemma [*simp*]: *NatTrans* $\eta \Longrightarrow$ *Category*(*NTCatDom* η) ⟨*proof*⟩
lemma [*simp*]: *NatTrans* $\eta \Longrightarrow$ *Category*(*NTCatCod* η) ⟨*proof*⟩
lemma *DDDC*: **assumes** *NatTrans f* **shows** *CatDom* (*NTDom f*) = *CatDom* (*NTCod f*)
⟨*proof*⟩
lemma *CCCD*: **assumes** *NatTrans f* **shows** *CatCod* (*NTCod f*) = *CatCod* (*NTDom f*)
⟨*proof*⟩

lemma *IdNatTransCompDefDom*: *NatTrans f* $\Longrightarrow$ (*IdNatTrans* (*NTDom f*)) $\approx\!\!>\!\!\cdot$ *f*
⟨*proof*⟩

lemma *IdNatTransCompDefCod*: *NatTrans f* $\Longrightarrow$ *f* $\approx\!\!>\!\!\cdot$ (*IdNatTrans* (*NTCod f*))
⟨*proof*⟩

lemma *NatTransCompDefCod*:
assumes *NatTrans* η **and** *f maps*$_{NTCatDom\ \eta}$ *X to Y*
shows $(\eta\ \$\$\ X) \approx\!\!>_{NTCatCod\ \eta}\ (NTCod\ \eta\ \#\#\ f)$
⟨*proof*⟩

lemma *NatTransCompDefDom*:
assumes *NatTrans* η **and** *f maps*$_{NTCatDom\ \eta}$ *X to Y*
shows $(NTDom\ \eta\ \#\#\ f)\ \approx\!\!>_{NTCatCod\ \eta}\ (\eta\ \$\$\ Y)$
⟨*proof*⟩

lemma *NatTransCompCompDef*:
assumes $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2$ **and** $X \in obj_{NTCatDom\ \eta 1}$
shows $(\eta 1\ \$\$\ X) \approx\!\!>_{NTCatCod\ \eta 1}\ (\eta 2\ \$\$\ X)$
⟨*proof*⟩

lemma *NatTransCompNatTrans′*:
assumes $\eta 1 \approx\!\!>\!\!\cdot\ \eta 2$

shows *NatTransP* ($\eta 1$ $\cdot 1$ $\eta 2$)
⟨*proof*⟩

lemma *NatTransCompNatTrans*: $\eta 1 \approx\!\!>\cdot \eta 2 \Longrightarrow$ *NatTrans* ($\eta 1 \cdot \eta 2$)
⟨*proof*⟩

definition
CatExp' :: ('*o1*,'*m1*,'*a*) *Category-scheme* ⇒ ('*o2*,'*m2*,'*b*) *Category-scheme* ⇒
(('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*) *Functor*,
('*o1*, '*o2*, '*m1*, '*m2*, '*a*, '*b*) *NatTrans*) *Category* **where**
CatExp' A B ≡ ⦇
Category.Obj = {*F* . *Ftor F* : *A* ⟶ *B*} ,
Category.Mor = {η . *NatTrans* η ∧ *NTCatDom* η = *A* ∧ *NTCatCod* η = *B*}
,
Category.Dom = *NTDom* ,
Category.Cod = *NTCod* ,
Category.Id = *IdNatTrans* ,
Category.Comp = λ*f g*. (*f* · *g*)
⦈

definition *CatExp A B* ≡ *MakeCat*(*CatExp' A B*)

lemma *IdNatTransMapL*:
assumes *NT*: *NatTrans f*
shows *IdNatTrans* (*NTDom f*) · *f* = *f*
⟨*proof*⟩

lemma *IdNatTransMapR*:
assumes *NT*: *NatTrans f*
shows *f* · *IdNatTrans* (*NTCod f*) = *f*
⟨*proof*⟩

lemma *NatTransCompDefined*:
assumes *f* ≈>· *g* **and** *g* ≈>· *h*
shows (*f* · *g*) ≈>· *h* **and** *f* ≈>· (*g* · *h*)
⟨*proof*⟩

lemma *NatTransCompAssoc*:
assumes *f* ≈>· *g* **and** *g* ≈>· *h*
shows (*f* · *g*) · *h* = *f* · (*g* · *h*)
⟨*proof*⟩

lemma *CatExpCatAx*:
assumes *Category A* **and** *Category B*
shows *Category-axioms* (*CatExp' A B*)
⟨*proof*⟩

lemma *CatExpCat*: ⟦*Category A* ; *Category B*⟧ ⟹ *Category* (*CatExp A B*)
⟨*proof*⟩

lemmas *CatExp-defs* = *CatExp-def CatExp'-def MakeCat-def*

lemma *CatExpDom*: $f \in Mor\ (CatExp\ A\ B) \Longrightarrow dom_{CatExp\ A\ B}\ f = NTDom\ f$
⟨*proof*⟩

lemma *CatExpCod*: $f \in Mor\ (CatExp\ A\ B) \Longrightarrow cod_{CatExp\ A\ B}\ f = NTCod\ f$
⟨*proof*⟩

lemma *CatExpId*: $X \in Obj\ (CatExp\ A\ B) \Longrightarrow Id\ (CatExp\ A\ B)\ X = IdNatTrans\ X$
⟨*proof*⟩

lemma *CatExpNatTransCompDef*: **assumes** $f \approx>_{CatExp\ A\ B}\ g$ **shows** $f \approx>_{\cdot}\ g$
⟨*proof*⟩

lemma *CatExpDist*:
assumes $X \in Obj\ A$ **and** $f \approx>_{CatExp\ A\ B}\ g$
shows $(f\ ;;_{CatExp\ A\ B}\ g)\ \$\$\ X = (f\ \$\$\ X)\ ;;_B\ (g\ \$\$\ X)$
⟨*proof*⟩

lemma *CatExpMorNT*: $f \in Mor\ (CatExp\ A\ B) \Longrightarrow NatTrans\ f$
⟨*proof*⟩

end

6 The Category of Sets

theory *SetCat*
imports *Functors Universe*
begin

notation *Elem* (**infixl** ‹|∈|› *70*)
notation *HOLZF.subset* (**infixl** ‹|⊆|› *71*)
notation *CartProd* (**infixl** ‹|×|› *75*)

definition
ZFfun :: $ZF \Rightarrow ZF \Rightarrow (ZF \Rightarrow ZF) \Rightarrow ZF$ **where**
$ZFfun\ d\ r\ f \equiv Opair\ (Opair\ d\ r)\ (Lambda\ d\ f)$

definition
ZFfunDom :: $ZF \Rightarrow ZF$ (‹|*dom*|-› [*72*] *72*) **where**
$ZFfunDom\ f \equiv Fst\ (Fst\ f)$

definition
ZFfunCod :: $ZF \Rightarrow ZF$ (‹|*cod*|-› [*72*] *72*) **where**
$ZFfunCod\ f \equiv Snd\ (Fst\ f)$

definition
ZFfunApp :: *ZF* ⇒ *ZF* ⇒ *ZF* (**infixl** ‹|@|› *73*) **where**
ZFfunApp f x ≡ *app* (*Snd f*) *x*

definition
ZFfunComp :: *ZF* ⇒ *ZF* ⇒ *ZF* (**infixl** ‹|o|› *72*) **where**
ZFfunComp f g ≡ *ZFfun* (|*dom*| *f*) (|*cod*| *g*) (λ*x*. *g* |@| (*f* |@| *x*))

definition
isZFfun :: *ZF* ⇒ *bool* **where**
isZFfun drf ≡ *let f* = *Snd drf in*
isOpair drf ∧ *isOpair* (*Fst drf*) ∧ *isFun f* ∧ (*f* |⊆| (*Domain f*) |×| (*Range f*))
∧ (*Domain f* = |*dom*| *drf*) ∧ (*Range f* |⊆| |*cod*| *drf*)

lemma *isZFfunE*[*elim*]: ⟦*isZFfun f* ;
⟦*isOpair f* ; *isOpair* (*Fst f*) ; *isFun* (*Snd f*) ;
((*Snd f*) |⊆| (*Domain* (*Snd f*)) |×| (*Range* (*Snd f*))) ;
(*Domain* (*Snd f*) = |*dom*| *f*) ∧ (*Range* (*Snd f*) |⊆| |*cod*| *f*)⟧ ⟹ *R*⟧ ⟹ *R*
⟨*proof*⟩

definition
SET′ :: (*ZF*, *ZF*) *Category* **where**
SET′ ≡ (⦇
Category.Obj = {*x* . *True*} ,
Category.Mor = {*f* . *isZFfun f*} ,
Category.Dom = *ZFfunDom* ,
Category.Cod = *ZFfunCod* ,
Category.Id = λ*x*. *ZFfun x x* (λ*x* . *x*) ,
Category.Comp = *ZFfunComp*
⦈)

definition *SET* ≡ *MakeCat SET′*

lemma *ZFfunDom*: |*dom*| (*ZFfun A B f*) = *A*
⟨*proof*⟩

lemma *ZFfunCod*: |*cod*| (*ZFfun A B f*) = *B*
⟨*proof*⟩

lemma *SETfun*:
assumes ∀ *x* . *x* |∈| *A* ⟶ (*f x*) |∈| *B*
shows *isZFfun* (*ZFfun A B f*)
⟨*proof*⟩

lemma *ZFCartProd*:
assumes *x* |∈| *A* |×| *B*
shows *Fst x* |∈| *A* ∧ *Snd x* |∈| *B* ∧ *isOpair x*
⟨*proof*⟩

lemma *ZFfunDomainOpair*:
assumes *isFun f*
and $x \mid\in\mid Domain\ f$
shows $Opair\ x\ (app\ f\ x) \mid\in\mid f$
⟨*proof*⟩

lemma *ZFFunToLambda*:
assumes *1*: *isFun f*
and *2*: $f \mid\subseteq\mid (Domain\ f) \mid\times\mid (Range\ f)$
shows $f = Lambda\ (Domain\ f)\ (\lambda x.\ app\ f\ x)$
⟨*proof*⟩

lemma *ZFfunApp*:
assumes $x \mid\in\mid A$
shows $(ZFfun\ A\ B\ f) \mid@\mid x = f\ x$
⟨*proof*⟩

lemma *ZFfun*:
assumes *isZFfun f*
shows $f = ZFfun\ (\mid dom\mid f)\ (\mid cod\mid f)\ (\lambda x.\ f \mid@\mid x)$
⟨*proof*⟩

lemma *ZFfun-ext*:
assumes $\forall\ x\ .\ x \mid\in\mid A \longrightarrow f\ x = g\ x$
shows $(ZFfun\ A\ B\ f) = (ZFfun\ A\ B\ g)$
⟨*proof*⟩

lemma *ZFfunExt*:
assumes $\mid dom\mid f = \mid dom\mid g$ **and** $\mid cod\mid f = \mid cod\mid g$ **and** *funf*: *isZFfun f* **and** *fung*: *isZFfun g*
and $\bigwedge x\ .\ x \mid\in\mid (\mid dom\mid f) \Longrightarrow f \mid@\mid x = g \mid@\mid x$
shows $f = g$
⟨*proof*⟩

lemma *ZFfunDomAppCod*:
assumes *isZFfun f*
and $x \mid\in\mid \mid dom\mid f$
shows $f \mid@\mid x \mid\in\mid \mid cod\mid f$
⟨*proof*⟩

lemma *ZFfunComp*:
assumes $\forall\ x\ .\ x \mid\in\mid A \longrightarrow f\ x \mid\in\mid B$
shows $(ZFfun\ A\ B\ f) \mid o\mid (ZFfun\ B\ C\ g) = ZFfun\ A\ C\ (g\ o\ f)$
⟨*proof*⟩

lemma *ZFfunCompApp*:
assumes *a*:*isZFfun f* **and** *b*:*isZFfun g* **and** *c*:$\mid dom\mid g = \mid cod\mid f$
shows $f \mid o\mid g = ZFfun\ (\mid dom\mid f)\ (\mid cod\mid g)\ (\lambda\ x\ .\ g \mid@\mid (f \mid@\mid x))$

⟨*proof*⟩

lemma *ZFfunCompAppZFfun*:
assumes *isZFfun f* **and** *isZFfun g* **and** $|dom|g = |cod|f$
shows *isZFfun* (*f* $|o|$ *g*)
⟨*proof*⟩

lemma *ZFfunCompAssoc*:
assumes *a*: *isZFfun f* **and** *b*:*isZFfun h* **and** *c*:$|cod|g = |dom|h$
and *d*:*isZFfun g* **and** *e*:$|cod|f = |dom|g$
shows $f\ |o|\ g\ |o|\ h = f\ |o|\ (g\ |o|\ h)$
⟨*proof*⟩

lemma *ZFfunCompAppDomCod*:
assumes *isZFfun f* **and** *isZFfun g* **and** $|dom|g = |cod|f$
shows $|dom|\ (f\ |o|\ g) = |dom|\ f \wedge |cod|\ (f\ |o|\ g) = |cod|\ g$
⟨*proof*⟩

lemma *ZFfunIdLeft*:
assumes *a*: *isZFfun f* **shows** (*ZFfun* ($|dom|f$) ($|dom|f$) ($\lambda x.\ x$)) $|o|\ f = f$
⟨*proof*⟩

lemma *ZFfunIdRight*:
assumes *a*: *isZFfun f* **shows** $f\ |o|$ (*ZFfun* ($|cod|f$) ($|cod|f$) ($\lambda x.\ x$)) $= f$
⟨*proof*⟩

lemma *SETCategory*: *Category*(*SET*)
⟨*proof*⟩

lemma *SETobj*: $X \in Obj\ (SET)$
⟨*proof*⟩

lemma *SETcod*: *isZFfun* (*ZFfun A B f*) $\Longrightarrow cod_{SET}$ *ZFfun A B f* $= B$
⟨*proof*⟩

lemma *SETmor*: (*isZFfun f*) $= (f \in mor_{SET})$
⟨*proof*⟩

lemma *SETdom*: *isZFfun* (*ZFfun A B f*) $\Longrightarrow dom_{SET}$ *ZFfun A B f* $= A$
⟨*proof*⟩

lemma *SETId*: **assumes** $x\ |\in|\ X$ **shows** (*Id SET X*) $|@|\ x = x$
⟨*proof*⟩

lemma *SETCompE*[*elim*]: $[\![f \approx>_{SET} g$; $[\![$*isZFfun f* ; *isZFfun g* ; $|cod|\ f = |dom|\ g]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

lemma *SETmapsTo*: *f maps*$_{SET}$ *X to Y* $\Longrightarrow$ *isZFfun f* $\wedge$ $|dom|\ f = X \wedge |cod|\ f$

$= Y$
⟨*proof*⟩

lemma *SETComp*: **assumes** $f \approx>_{SET} g$ **shows** $f ;;_{SET} g = f \mid o \mid g$
⟨*proof*⟩

lemma *SETCompAt*:
assumes $f \approx>_{SET} g$ **and** $x \mid \in \mid \ \mid dom \mid f$ **shows** $(f ;;_{SET} g) \mid @ \mid x = g \mid @ \mid (f \mid @ \mid x)$
⟨*proof*⟩

lemma *SETZFfun*:
assumes $f\ maps_{SET}\ X\ to\ Y$ **shows** $f = ZFfun\ X\ Y\ (\lambda x\ .\ f \mid @ \mid x)$
⟨*proof*⟩

lemma *SETfunDomAppCod*:
assumes $f\ maps_{SET}\ X\ to\ Y$ **and** $x \mid \in \mid X$
shows $f \mid @ \mid x \mid \in \mid Y$
⟨*proof*⟩

record $('o,'m)\ LSCategory = ('o,'m)\ Category +$
$mor2ZF :: 'm \Rightarrow ZF\ (‹m2z₁-›\ [70]\ 70)$

definition
$ZF2mor\ (‹z2m₁-›\ [70]\ 70)$ **where**
$ZF2mor\ C\ f \equiv THE\ m\ .\ m \in mor_C \wedge m2z_C\ m = f$

definition
$HOMCollection\ C\ X\ Y \equiv \{m2z_C\ f \mid f\ .\ f\ maps_C\ X\ to\ Y\}$

definition
$HomSet\ (‹Hom₁\ -\ -›\ [65,\ 65]\ 65)$ **where**
$HomSet\ C\ X\ Y \equiv implode\ (HOMCollection\ C\ X\ Y)$

locale $LSCategory = Category +$
assumes $mor2ZFInj$: $[\![x \in mor\ ;\ y \in mor\ ;\ m2z\ x = m2z\ y]\!] \Longrightarrow x = y$
and $HOMSetIsSet$: $[\![X \in obj\ ;\ Y \in obj]\!] \Longrightarrow HOMCollection\ C\ X\ Y \in range\ explode$
and $m2zExt$: $mor2ZF\ C \in extensional\ (Mor\ C)$

lemma [*elim*]: $[\![LSCategory\ C\ ;$
$[\![Category\ C\ ;\ [\![x \in mor_C\ ;\ y \in mor_C\ ;\ m2z_C\ x = m2z_C\ y]\!] \Longrightarrow\ x = y;$
$[\![X \in obj_C\ ;\ Y \in obj_C]\!] \Longrightarrow HOMCollection\ C\ X\ Y \in range\ explode]\!] \Longrightarrow R]\!] \Longrightarrow R$
⟨*proof*⟩

definition
$HomFtorMap :: ('o,'m,'a)\ LSCategory\text{-}scheme \Rightarrow 'o \Rightarrow 'm \Rightarrow ZF\ (‹Hom₁[-,-]›$

[65,65] 65) **where**
HomFtorMap C X g ≡ *ZFfun* (*Hom*$_C$ *X* (*dom*$_C$ *g*)) (*Hom*$_C$ *X* (*cod*$_C$ *g*)) (λ *f* . *m2z*$_C$ ((*z2m*$_C$ *f*) ;;$_C$ *g*))

definition
HomFtor′ :: ('*o*,'*m*,'*a*) *LSCategory-scheme* ⇒ '*o* ⇒
('*o*,*ZF*,'*m*,*ZF*,(|*mor2ZF* :: '*m* ⇒ *ZF*, ... :: '*a*|),*unit*) *Functor* (‹*HomP*$_1$[-,−]› [65] 65) **where**
HomFtor′ C X ≡ (|
CatDom = *C*,
CatCod = *SET* ,
MapM = λ *g* . *Hom*$_C$[*X*,*g*]
|)

definition *HomFtor* (‹*Hom*$_1$[-,−]› [65] 65) **where** *HomFtor C X* ≡ *MakeFtor* (*HomFtor′ C X*)

lemma [*simp*]: *LSCategory C* ⟹ *Category C*
⟨*proof*⟩

lemma (**in** *LSCategory*) *m2zz2m*:
assumes *f maps X to Y* **shows** (*m2z f*) |∈| (*Hom X Y*)
⟨*proof*⟩

lemma (**in** *LSCategory*) *m2zz2mInv*:
assumes *f* ∈ *mor*
shows *z2m* (*m2z f*) = *f*
⟨*proof*⟩

lemma (**in** *LSCategory*) *z2mm2z*:
assumes *X* ∈ *obj* **and** *Y* ∈ *obj* **and** *f* |∈| (*Hom X Y*)
shows *z2m f maps X to Y* ∧ *m2z* (*z2m f*) = *f*
⟨*proof*⟩

lemma *HomFtorMapLemma1*:
assumes *a*: *LSCategory C* **and** *b*: *X* ∈ *obj*$_C$ **and** *c*: *f* ∈ *mor*$_C$ **and** *d*: *x* |∈| (*Hom*$_C$ *X* (*dom*$_C$ *f*))
shows (*m2z*$_C$ ((*z2m*$_C$ *x*) ;;$_C$ *f*)) |∈| (*Hom*$_C$ *X* (*cod*$_C$ *f*))
⟨*proof*⟩

lemma *HomFtorInMor′*:
assumes *LSCategory C* **and** *X* ∈ *obj*$_C$ **and** *f* ∈ *mor*$_C$
shows *Hom*$_C$[*X*,*f*] ∈ *mor*$_{SET'}$
⟨*proof*⟩

lemma *HomFtorMor′*:
assumes *LSCategory C* **and** *X* ∈ *obj*$_C$ **and** *f* ∈ *mor*$_C$
shows *Hom*$_C$[*X*,*f*] *maps*$_{SET'}$ *Hom*$_C$ *X* (*dom*$_C$ *f*) *to Hom*$_C$ *X* (*cod*$_C$ *f*)
⟨*proof*⟩

lemma *HomFtorMapsTo*:
$[\![LSCategory\ C\ ;\ X \in obj_C\ ;\ f \in mor_C\ ]\!] \Longrightarrow Hom_C[X,f]\ maps_{SET}\ Hom_C\ X\ (dom_C\ f)\ to\ Hom_C\ X\ (cod_C\ f)$
⟨*proof*⟩

lemma *HomFtorMor*:
assumes $LSCategory\ C$ **and** $X \in obj_C$ **and** $f \in mor_C$
shows $Hom_C[X,f] \in Mor\ SET$ **and** $dom_{SET}\ (Hom_C[X,f]) = Hom_C\ X\ (dom_C\ f)$
and $cod_{SET}\ (Hom_C[X,f]) = Hom_C\ X\ (cod_C\ f)$
⟨*proof*⟩

lemma *HomFtorCompDef′*:
assumes $LSCategory\ C$ **and** $X \in obj_C$ **and** $f \approx>_C g$
shows $(Hom_C[X,f]) \approx>_{SET'} (Hom_C[X,g])$
⟨*proof*⟩

lemma *HomFtorDist′*:
assumes *a*: $LSCategory\ C$ **and** *b*: $X \in obj_C$ **and** *c*: $f \approx>_C g$
shows $(Hom_C[X,f])\ ;;_{SET'}\ (Hom_C[X,g]) = Hom_C[X,f\ ;;_C\ g]$
⟨*proof*⟩

lemma *HomFtorDist*:
assumes $LSCategory\ C$ **and** $X \in obj_C$ **and** $f \approx>_C g$
shows $(Hom_C[X,f])\ ;;_{SET}\ (Hom_C[X,g]) = Hom_C[X,f\ ;;_C\ g]$
⟨*proof*⟩

lemma *HomFtorId′*:
assumes *a*: $LSCategory\ C$ **and** *b*: $X \in obj_C$ **and** *c*: $Y \in obj_C$
shows $Hom_C[X,id_C\ Y] = id_{SET'}\ (Hom_C\ X\ Y)$
⟨*proof*⟩

lemma *HomFtorId*:
assumes $LSCategory\ C$ **and** $X \in obj_C$ **and** $Y \in obj_C$
shows $Hom_C[X,id_C\ Y] = id_{SET}\ (Hom_C\ X\ Y)$
⟨*proof*⟩

lemma *HomFtorObj′*:
assumes *a*: $LSCategory\ C$
and *b*: $PreFunctor\ (HomP_C[X,-])$ **and** *c*: $X \in obj_C$ **and** *d*: $Y \in obj_C$
shows $(HomP_C[X,-])\ @@\ Y = Hom_C\ X\ Y$
⟨*proof*⟩

lemma *HomFtorFtor′*:
assumes *a*: $LSCategory\ C$
and *b*: $X \in obj_C$
shows $FunctorM\ (HomP_C[X,-])$
⟨*proof*⟩

lemma *HomFtorFtor*:
assumes *a*: *LSCategory C*
and *b*: $X \in obj_C$
shows *Functor* ($Hom_C[X,-]$)
⟨*proof*⟩

lemma *HomFtorObj*:
assumes *LSCategory C*
and $X \in obj_C$ **and** $Y \in obj_C$
shows $(Hom_C[X,-])$ @@ $Y = Hom_C\ X\ Y$
⟨*proof*⟩

definition
HomFtorMapContra :: ('*o*,'*m*,'*a*) *LSCategory-scheme* ⇒ '*m* ⇒ '*o* ⇒ *ZF* (‹$HomC_1$[-,-]› [*65*,*65*] *65*) **where**
HomFtorMapContra C g X ≡ *ZFfun* (Hom_C (cod_C *g*) *X*) (Hom_C (dom_C *g*) *X*) (λ *f* . $m2z_C$ (*g* ;;$_C$ ($z2m_C$ *f*)))

definition
HomFtorContra' :: ('*o*,'*m*,'*a*) *LSCategory-scheme* ⇒ '*o* ⇒
('*o*,*ZF*,'*m*,*ZF*,(|*mor2ZF* :: '*m* ⇒ *ZF*, ... :: '*a*|),*unit*) *Functor* (‹$HomP_1$[−,-]› [*65*] *65*) **where**
HomFtorContra' C X ≡ (|
CatDom = (*Op C*),
CatCod = *SET* ,
MapM = λ *g* . $HomC_C$[*g*,*X*]
|)

definition *HomFtorContra* (‹Hom_1[−,-]› [*65*] *65*) **where** *HomFtorContra C X* ≡ *MakeFtor*(*HomFtorContra' C X*)

lemma *HomContraAt*: *x* |∈| (Hom_C (cod_C *f*) *X*) ⟹ ($HomC_C$[*f*,*X*]) |@| *x* = $m2z_C$ (*f* ;;$_C$ ($z2m_C$ *x*))
⟨*proof*⟩

lemma *mor2ZF-Op*: *mor2ZF* (*Op C*) = *mor2ZF C*
⟨*proof*⟩

lemma *mor-Op*: $mor_{Op\ C}$ = mor_C ⟨*proof*⟩
lemma *obj-Op*: $obj_{Op\ C}$ = obj_C ⟨*proof*⟩

lemma *ZF2mor-Op*: *ZF2mor* (*Op C*) *f* = *ZF2mor C f*
⟨*proof*⟩

lemma *mapsTo-Op*: *f* $maps_{Op\ C}$ *Y to X* = *f* $maps_C$ *X to Y*
⟨*proof*⟩

lemma *HOMCollection-Op*: *HOMCollection* (*Op C*) *X Y* = *HOMCollection C Y*

X
⟨*proof*⟩

lemma *Hom-Op*: $Hom_{Op\ C}\ X\ Y = Hom_C\ Y\ X$
⟨*proof*⟩

lemma *HomFtorContra'*: $HomP_C[-,X] = HomP_{Op\ C}[X,-]$
⟨*proof*⟩

lemma *HomFtorContra*: $Hom_C[-,X] = Hom_{Op\ C}[X,-]$
⟨*proof*⟩

lemma *HomFtorContraDom*: $CatDom\ (Hom_C[-,X]) = Op\ C$
⟨*proof*⟩

lemma *HomFtorContraCod*: $CatCod\ (Hom_C[-,X]) = SET$
⟨*proof*⟩

lemma *LSCategory-Op*: **assumes** $LSCategory\ C$ **shows** $LSCategory\ (Op\ C)$
⟨*proof*⟩

lemma *HomFtorContraFtor*:
assumes $LSCategory\ C$
and $X \in obj_C$
shows $Ftor\ (Hom_C[-,X]) : (Op\ C) \longrightarrow SET$
⟨*proof*⟩

lemma *HomFtorOpObj*:
assumes $LSCategory\ C$
and $X \in obj_C$ **and** $Y \in obj_C$
shows $(Hom_C[-,X])\ @@\ Y = Hom_C\ Y\ X$
⟨*proof*⟩

lemma *HomCHomOp*: $HomC_C[g,X] = Hom_{Op\ C}[X,g]$
⟨*proof*⟩

lemma *HomFtorContraMapsTo*:
assumes $LSCategory\ C$ **and** $X \in obj_C$ **and** $f \in mor_C$
shows $HomC_C[f,X]\ maps_{SET}\ Hom_C\ (cod_C\ f)\ X\ \ to\ Hom_C\ (dom_C\ f)\ X$
⟨*proof*⟩

lemma *HomFtorContraMor*:
assumes $LSCategory\ C$ **and** $X \in obj_C$ **and** $f \in mor_C$
shows $HomC_C[f,X] \in Mor\ SET$ **and** $dom_{SET}\ (HomC_C[f,X]) = Hom_C\ (cod_C\ f)\ X$
and $cod_{SET}\ (HomC_C[f,X]) = Hom_C\ (dom_C\ f)\ X$
⟨*proof*⟩

lemma *HomContraMor*:
  assumes *LSCategory C* **and** $f \in Mor\ C$
  shows $(Hom_C[-,X])\ \#\#\ f = HomC_C[f,X]$
⟨*proof*⟩

lemma *HomCHom*:
  assumes *LSCategory C* **and** $f \in Mor\ C$ **and** $g \in Mor\ C$
  shows $(HomC_C[g,dom_C\ f])\ ;;_{SET}\ (Hom_C[dom_C\ g,f]) = (Hom_C[cod_C\ g,f])\ ;;_{SET}$
$(HomC_C[g,cod_C\ f])$
⟨*proof*⟩

end

7 Yoneda

theory *Yoneda*
imports *NatTrans SetCat*
begin

definition $YFtorNT'\ C\ f \equiv (\!|NTDom = Hom_C[-,dom_C\ f]\ ,\ NTCod = Hom_C[-,cod_C\ f]\ ,$
$NatTransMap = \lambda\ B\ .\ Hom_C[B,f]|\!)$

definition $YFtorNT\ C\ f \equiv MakeNT\ (YFtorNT'\ C\ f)$

lemmas *YFtorNT-defs* = *YFtorNT'-def YFtorNT-def MakeNT-def*

lemma *YFtorNTCatDom*: $NTCatDom\ (YFtorNT\ C\ f) = Op\ C$
⟨*proof*⟩

lemma *YFtorNTCatCod*: $NTCatCod\ (YFtorNT\ C\ f) = SET$
⟨*proof*⟩

lemma *YFtorNTApp1*: **assumes** $X \in Obj\ (NTCatDom\ (YFtorNT\ C\ f))$ **shows**
$(YFtorNT\ C\ f)\ \$\$\ X = Hom_C[X,f]$
⟨*proof*⟩

definition
  $YFtor'\ C \equiv (\!|$
    $CatDom = C\ ,$
    $CatCod = CatExp\ (Op\ C)\ SET\ ,$
    $MapM = \lambda\ f\ .\ YFtorNT\ C\ f$
  $|\!)$

definition $YFtor\ C \equiv MakeFtor(YFtor'\ C)$

lemmas *YFtor-defs* = *YFtor′-def YFtor-def MakeFtor-def*

lemma *YFtorNTNatTrans′*:
assumes *LSCategory C* **and** *f* ∈ *Mor C*
shows *NatTransP* (*YFtorNT′ C f*)
⟨*proof*⟩

lemma *YFtorNTNatTrans*:
assumes *LSCategory C* **and** *f* ∈ *Mor C*
shows *NatTrans* (*YFtorNT C f*)
⟨*proof*⟩

lemma *YFtorNTMor*:
assumes *LSCategory C* **and** *f* ∈ *Mor C*
shows *YFtorNT C f* ∈ *Mor* (*CatExp* (*Op C*) *SET*)
⟨*proof*⟩

lemma *YFtorNtMapsTo*:
assumes *LSCategory C* **and** *f* ∈ *Mor C*
shows *YFtorNT C f maps*$_{CatExp\ (Op\ C)\ SET}$ (*Hom*$_C$[−,*dom*$_C$ *f*]) *to* (*Hom*$_C$[−,*cod*$_C$ *f*])
⟨*proof*⟩

lemma *YFtorNTCompDef*:
assumes *LSCategory C* **and** *f* ≈>$_C$ *g*
shows *YFtorNT C f* ≈>$_{CatExp\ (Op\ C)\ SET}$ *YFtorNT C g*
⟨*proof*⟩

lemma *PreSheafCat*: *LSCategory C* ⟹ *Category* (*CatExp* (*Op C*) *SET*)
⟨*proof*⟩

lemma *YFtor′Obj1*:
assumes *X* ∈ *Obj* (*CatDom* (*YFtor′ C*)) **and** *LSCategory C*
shows (*YFtor′ C*) ## (*Id* (*CatDom* (*YFtor′ C*)) *X*) = *Id* (*CatCod* (*YFtor′ C*)) (*Hom*$_C$ [−,*X*])
⟨*proof*⟩

lemma *YFtorPreFtor*:
assumes *LSCategory C*
shows *PreFunctor* (*YFtor′ C*)
⟨*proof*⟩

lemma *YFtor′Obj*:
assumes *X* ∈ *Obj* (*CatDom* (*YFtor′ C*))
and *LSCategory C*
shows (*YFtor′ C*) @@ *X* = *Hom*$_C$ [−,*X*]
⟨*proof*⟩

lemma *YFtorFtor′*:

assumes *LSCategory C*
shows *FunctorM* (*YFtor′ C*)
⟨*proof*⟩

lemma *YFtorFtor*: **assumes** *LSCategory C* **shows** *Ftor* (*YFtor C*) : *C* ⟶ (*CatExp* (*Op C*) *SET*)
⟨*proof*⟩

lemma *YFtorObj*:
assumes *LSCategory C* **and** *X* ∈ *Obj C*
shows (*YFtor C*) @@ *X* = Hom_C [−,*X*]
⟨*proof*⟩

lemma *YFtorObj2*:
assumes *LSCategory C* **and** *X* ∈ *Obj C* **and** *Y* ∈ *Obj C*
shows ((*YFtor C*) @@ *Y*) @@ *X* = Hom_C *X Y*
⟨*proof*⟩

lemma *YFtorMor*: ⟦*LSCategory C* ; *f* ∈ *Mor C*⟧ ⟹ (*YFtor C*) ## *f* = *YFtorNT C f*
⟨*proof*⟩

definition *YMap C X η* ≡ (*η* $$ *X*) |@| ($m2z_C$ (id_C *X*))
definition *YMapInv′ C X F x* ≡ ⦇
NTDom = ((*YFtor C*) @@ *X*),
NTCod = *F*,
NatTransMap = λ *B* . *ZFfun* (Hom_C *B X*) (*F* @@ *B*) (λ *f* . (*F* ## ($z2m_C$ *f*)) |@| *x*)
⦈
definition *YMapInv C X F x* ≡ *MakeNT* (*YMapInv′ C X F x*)

lemma *YMapInvApp*:
assumes *X* ∈ *Obj C* **and** *B* ∈ *Obj C* **and** *LSCategory C*
shows (*YMapInv C X F x*) $$ *B* = *ZFfun* (Hom_C *B X*) (*F* @@ *B*) (λ *f* . (*F* ## ($z2m_C$ *f*)) |@| *x*)
⟨*proof*⟩

lemma *YMapImage*:
assumes *LSCategory C* **and** *Ftor F* : (*Op C*) ⟶ *SET* **and** *X* ∈ *Obj C*
and *NT η* : (*YFtor C* @@ *X*) ⟹ *F*
shows (*YMap C X η*) |∈| (*F* @@ *X*)
⟨*proof*⟩

lemma *YMapInvNatTransP*:
assumes *LSCategory C* **and** *Ftor F* : (*Op C*) ⟶ *SET* **and** *xobj*: *X* ∈ *Obj C* **and** *xinF*: *x* |∈| (*F* @@ *X*)
shows *NatTransP* (*YMapInv′ C X F x*)

⟨*proof*⟩

lemma *YMapInvNatTrans*:
assumes *LSCategory C* **and** *Ftor F* : (*Op C*) ⟶ *SET* **and** *X* ∈ *Obj C* **and** *x* |∈| (*F* @@ *X*)
shows *NatTrans* (*YMapInv C X F x*)
⟨*proof*⟩

lemma *YMapInvImage*:
assumes *LSCategory C* **and** *Ftor F* : (*Op C*) ⟶ *SET* **and** *X* ∈ *Obj C*
and *x* |∈| (*F* @@ *X*)
shows *NT* (*YMapInv C X F x*) : (*YFtor C* @@ *X*) ⟹ *F*
⟨*proof*⟩

lemma *YMap1*:
assumes *LSCat*: *LSCategory C* **and** *Fftor*: *Ftor F* : (*Op C*) ⟶ *SET* **and** *XObj*: *X* ∈ *Obj C*
and *NT*: *NT η* : (*YFtor C* @@ *X*) ⟹ *F*
shows *YMapInv C X F* (*YMap C X η*) = *η*
⟨*proof*⟩

lemma *YMap2*:
assumes *LSCategory C* **and** *Ftor F* : (*Op C*) ⟶ *SET* **and** *X* ∈ *Obj C*
and *x* |∈| (*F* @@ *X*)
shows *YMap C X* (*YMapInv C X F x*) = *x*
⟨*proof*⟩

lemma *YFtorNT-YMapInv*:
assumes *LSCategory C* **and** *f maps*$_C$ *X to Y*
shows *YFtorNT C f* = *YMapInv C X* (*Hom*$_C$[−, *Y*]) (*m2z*$_C$ *f*)
⟨*proof*⟩

lemma *YMapYoneda*:
assumes *LSCategory C* **and** *f maps*$_C$ *X to Y*
shows *YFtor C* ## *f* = *YMapInv C X* (*YFtor C* @@ *Y*) (*m2z*$_C$ *f*)
⟨*proof*⟩

lemma *YonedaFull*:
assumes *LSCategory C* **and** *X* ∈ *Obj C* **and** *Y* ∈ *Obj C*
and *NT η* : (*YFtor C* @@ *X*) ⟹ (*YFtor C* @@ *Y*)
shows *YFtor C* ## (*z2m*$_C$ (*YMap C X η*)) = *η*
and *z2m*$_C$ (*YMap C X η*) *maps*$_C$ *X to Y*
⟨*proof*⟩

lemma *YonedaFaithful*:
assumes *LSCategory C* **and** *f maps*$_C$ *X to Y* **and** *g maps*$_C$ *X to Y*
and *YFtor C* ## *f* = *YFtor C* ## *g*
shows *f* = *g*
⟨*proof*⟩

lemma *YonedaEmbedding*:
assumes *LSCategory C* **and** $A \in Obj\ C$ **and** $B \in Obj\ C$ **and** $(YFtor\ C)\ @@\ A = (YFtor\ C)\ @@\ B$
shows $A = B$
⟨*proof*⟩

end

References

[1] A. Katovsky. Category theory in Isabelle/HOL, 2010. http://www.srcf.ucam.org/~apk32/Isabelle/Category/Cat.pdf.