

The Cartan Fixed Point Theorems

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Abstract

The Cartan fixed point theorems concern the group of holomorphic automorphisms on a connected open set of \mathbb{C}^n . Ciolli et al. [1] have formalised the one-dimensional case of these theorems in HOL Light. This entry contains their proofs, ported to Isabelle/HOL. Thus it addresses the authors remark that “it would be important to write a formal proof in a language that can be read by both humans and machines.”

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theory <i>Cartan</i>	
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0.1 First Cartan Theorem

Ported from HOL Light. See Gianni Ciolli, Graziano Gentili, Marco Maggesi. A Certified Proof of the Cartan Fixed Point Theorems. *J Automated Reasoning* (2011) 47:319–336 DOI 10.1007/s10817-010-9198-6

lemma *deriv-left-inverse*:

assumes *f holomorphic-on S and g holomorphic-on T*
and *open S and open T*
and *f ' S ⊆ T*
and [*simp*]: $\bigwedge z. z \in S \implies g (f z) = z$
and *w ∈ S*
shows $deriv f w * deriv g (f w) = 1$

<proof>

lemma *Cauchy-higher-deriv-bound*:

assumes *hol f: f holomorphic-on (ball z r)*
and *cont f: continuous-on (cball z r) f*
and $0 < r$ **and** $0 < n$
and *fin : $\bigwedge w. w \in ball z r \implies f w \in ball y B0$*
shows $norm ((deriv \hat{\hat{}} n) f z) \leq (fact n) * B0 / r \hat{\hat{}} n$

<proof>

lemma *higher-deriv-comp-lemma*:

assumes *s: open s and hol f: f holomorphic-on s*
and *z ∈ s*
and *t: open t and hol g: g holomorphic-on t*
and *fst: f ' s ⊆ t*
and *n: i ≤ n*
and *dfz: deriv f z = 1 and zero: $\bigwedge i. [1 < i; i \leq n] \implies (deriv \hat{\hat{}} i) f z = 0$*
shows $(deriv \hat{\hat{}} i) (g o f) z = (deriv \hat{\hat{}} i) g (f z)$

<proof>

lemma *higher-deriv-comp-iter-lemma:*

assumes *s*: open *s* **and** *holf*: *f* holomorphic-on *s*
and *fss*: $f' s \subseteq s$
and $z \in s$ **and** [*simp*]: $f z = z$
and *n*: $i \leq n$
and *dfz*: $\text{deriv } f z = 1$ **and** *zero*: $\bigwedge i. \llbracket 1 < i; i \leq n \rrbracket \implies (\text{deriv } \tilde{i}) f z = 0$
shows $(\text{deriv } \tilde{i}) (f \tilde{m}) z = (\text{deriv } \tilde{i}) f z$

<proof>

lemma *higher-deriv-iter-top-lemma:*

assumes *s*: open *s* **and** *holf*: *f* holomorphic-on *s*
and *fss*: $f' s \subseteq s$
and $z \in s$ **and** [*simp*]: $f z = z$
and *dfz* [*simp*]: $\text{deriv } f z = 1$
and *n*: $1 < n \bigwedge i. \llbracket 1 < i; i < n \rrbracket \implies (\text{deriv } \tilde{i}) f z = 0$
shows $(\text{deriv } \tilde{n}) (f \tilde{m}) z = m * (\text{deriv } \tilde{n}) f z$

<proof>

Should be proved for n-dimensional vectors of complex numbers

theorem *first-Cartan-dim-1:*

assumes *holf*: *f* holomorphic-on *s*
and open *s* **and** connected *s* **and** bounded *s*
and *fss*: $f' s \subseteq s$
and $z \in s$ **and** [*simp*]: $f z = z$
and *dfz* [*simp*]: $\text{deriv } f z = 1$
and $w \in s$
shows $f w = w$

<proof>

Second Cartan Theorem.

lemma *Cartan-is-linear:*

assumes *holf*: *f* holomorphic-on *s*
and open *s* **and** connected *s*
and $0 \in s$
and *ins*: $\bigwedge u z. \llbracket \text{norm } u = 1; z \in s \rrbracket \implies u * z \in s$
and *feq*: $\bigwedge u z. \llbracket \text{norm } u = 1; z \in s \rrbracket \implies f (u * z) = u * f z$
shows $\exists c. \forall z \in s. f z = c * z$

<proof>

Should be proved for n-dimensional vectors of complex numbers

theorem *second-Cartan-dim-1:*

assumes *holf*: *f* holomorphic-on ball 0 *r*
and *holg*: *g* holomorphic-on ball 0 *r*
and [*simp*]: $f 0 = 0$ **and** [*simp*]: $g 0 = 0$
and *ballf*: $\bigwedge z. z \in \text{ball } 0 r \implies f z \in \text{ball } 0 r$
and *ballg*: $\bigwedge z. z \in \text{ball } 0 r \implies g z \in \text{ball } 0 r$
and *fg*: $\bigwedge z. z \in \text{ball } 0 r \implies f (g z) = z$

and $gf: \bigwedge z. z \in \text{ball } 0 \ r \implies g (f z) = z$
and $0 < r$
shows $\exists t. \forall z \in \text{ball } 0 \ r. g z = \exp(i * \text{of-real } t) * z$
<proof>
end

Bibliography

- [1] G. Cioli, G. Gentili, and M. Maggesi. A certified proof of the Cartan fixed point theorems. *J. Autom. Reason.*, 47(3):319–336, Oct. 2011.