

# The Cardinality of the Continuum

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## Abstract

This entry presents a short derivation of the cardinality of  $\mathbb{R}$ , namely that  $|\mathbb{R}| = |2^{\mathbb{N}}| = 2^{\aleph_0}$ . This is done by showing the injection  $\mathbb{R} \rightarrow 2^{\mathbb{Q}}$ ,  $x \mapsto (-\infty, x) \cap \mathbb{Q}$  (i.e. Dedekind cuts) for one direction and the injection  $2^{\mathbb{N}} \rightarrow \mathbb{Q}$ ,  $X \mapsto \sum_{n \in X} 3^{-n}$ , i.e. ternary fractions, for the other direction.

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# 1 Auxiliary material

```
theory Cardinality_Continuum_Library
  imports "HOL-Library.Equipollence" "HOL-Cardinals.Cardinals"
begin
```

## 1.1 Miscellaneous facts about cardinalities

```
lemma eqpoll_Pow [intro]:
  assumes "A ≈ B"
  shows "Pow A ≈ Pow B"
```

```
proof -
  from assms obtain f where "bij_betw f A B"
    unfolding eqpoll_def by blast
  hence "bij_betw ((') f) (Pow A) (Pow B)"
    by (rule bij_betw_Pow)
  thus ?thesis
    unfolding eqpoll_def by blast
qed
```

```
lemma lepoll_UNIV_nat_iff: "A ≲ (UNIV :: nat set) ⟷ countable A"
  unfolding countable_def lepoll_def by simp
```

```
lemma countable_eqpoll:
  assumes "countable A" "A ≈ B"
  shows "countable B"
  using assms countable_iff_bij unfolding eqpoll_def by blast
```

```
lemma countable_eqpoll_cong: "A ≈ B ⟹ countable A ⟷ countable B"
  using countable_eqpoll[of A B] countable_eqpoll[of B A]
  by (auto simp: eqpoll_sym)
```

```
lemma eqpoll_UNIV_nat_iff: "A ≈ (UNIV :: nat set) ⟷ countable A ∧
infinite A"
```

```
proof
  assume *: "A ≈ (UNIV :: nat set)"
  show "countable A ∧ infinite A"
    using eqpoll_finite_iff[OF *] countable_eqpoll_cong[OF *] by simp
next
  assume *: "countable A ∧ infinite A"
  thus "A ≈ (UNIV :: nat set)"
    by (meson countableE_infinite eqpoll_def)
qed
```

```
lemma ordLeq_finite_infinite:
  "finite A ⟹ infinite B ⟹ (card_of A, card_of B) ∈ ordLeq"
  by (meson card_of_Well_order card_of_ordLeq_finite ordLeq_total)
```

```

lemma eqpoll_imp_card_of_ordIso: "A ≈ B ⇒ |A| =o |B|"
  by (simp add: eqpoll_iff_card_of_ordIso)

lemma card_of_Func: "|Func A B| =o |B| ^c |A|"
  by (simp add: cexp_def)

lemma card_of_leq_natLeq_iff_countable:
  "|X| ≤o natLeq ↔ countable X"
proof -
  have "countable X ↔ |X| ≤o |UNIV :: nat set|"
    unfolding countable_def by (meson card_of_ordLeq top_greatest)
  with card_of_nat show ?thesis
    using ordIso_symmetric ordLeq_ordIso_trans by blast
qed

lemma card_of_Sigma_cong:
  assumes "∧x. x ∈ A ⇒ |B x| =o |B' x|"
  shows "|SIGMA x:A. B x| =o |SIGMA x:A. B' x|"
proof -
  have "∃f. bij_betw f (B x) (B' x)" if "x ∈ A" for x
    using assms that card_of_ordIso by blast
  then obtain f where f: "∧x. x ∈ A ⇒ bij_betw (f x) (B x) (B' x)"
    by metis
  have "bij_betw (λ(x,y). (x, f x y)) (SIGMA x:A. B x) (SIGMA x:A. B'
x)"
    using f by (fastforce simp: bij_betw_def inj_on_def image_def)
  thus ?thesis
    by (rule card_of_ordIsoI)
qed

lemma Cfinite_cases:
  assumes "Cfinite c"
  obtains n :: nat where "(c, natLeq_on n) ∈ ordIso"
proof -
  from assms have "card_of (Field c) =o natLeq_on (card (Field c))"
    by (simp add: cfinite_def finite_imp_card_of_natLeq_on)
  with that[of "card (Field c)"] show ?thesis
    using assms card_of_unique ordIso_transitive by blast
qed

lemma empty_nat_ordIso_czero: "{ } :: (nat × nat) set) =o czero"
proof -
  have "{ } :: (nat × nat) set) =o { } :: nat set|"
    using finite_imp_card_of_natLeq_on[of "{ } :: nat set"] by (simp add:
ordIso_symmetric)
  moreover have "{ } :: nat set| =o czero"
    by (simp add: card_of_ordIso_czero_iff_empty)
  ultimately show "{ } :: (nat × nat) set) =o czero"
    using ordIso_symmetric ordIso_transitive by blast

```

qed

```
lemma card_order_on_empty: "card_order_on {} {}"  
  unfolding card_order_on_def well_order_on_def linear_order_on_def partial_order_on_def  
    preorder_on_def antisym_def trans_def refl_on_def total_on_def  
  ordLeq_def embed_def  
  by (auto intro!: ordLeq_refl)
```

```
lemma natLeq_on_plus_ordIso: "natLeq_on (m + n) =o natLeq_on m +c natLeq_on  
n"
```

```
proof -  
  have "{0.. $m+n$ } = {0.. $m$ }  $\cup$  { $m$ .. $m+n$ }"  
    by auto  
  also have "card_of ({0.. $m$ }  $\cup$  { $m$ .. $m+n$ }) =o card_of ({0.. $m$ } <+> { $m$ .. $m+n$ })"  
    by (rule card_of_Un_Plus_ordIso) auto  
  also have "card_of ({0.. $m$ } <+> { $m$ .. $m+n$ }) =o card_of {0.. $m$ } +c card_of  
{ $m$ .. $m+n$ }"  
    by (rule Plus_csum)  
  also have "card_of {0.. $m$ } +c card_of { $m$ .. $m+n$ } =o natLeq_on m +c natLeq_on  
n"  
    using finite_imp_card_of_natLeq_on[of "{ $m$ .. $m+n$ "}]  
    by (intro csum_cong card_of_less) auto  
  finally have "|{0.. $m+n$ }| =o natLeq_on m +c natLeq_on n" .  
  moreover have "card_of {0.. $m+n$ } =o natLeq_on (m + n)"  
    by (rule card_of_less)  
  ultimately show ?thesis  
    using ordIso_symmetric ordIso_transitive by blast
```

qed

```
lemma natLeq_on_1_ord_iso: "natLeq_on 1 =o BNF_Cardinal_Arithmetic.cone"
```

```
proof -  
  have "|{0.. $1::nat$ }| =o natLeq_on 1"  
    by (rule card_of_less)  
  hence "|{0:: $nat$ }| =o natLeq_on 1"  
    by simp  
  moreover have "|{0:: $nat$ }| =o BNF_Cardinal_Arithmetic.cone"  
    by (rule single_cone)  
  ultimately show ?thesis  
    using ordIso_symmetric ordIso_transitive by blast
```

qed

```
lemma cexp_infinite_finite_ordLeq:
```

```
  assumes "Cinfinite c" "Cfinite c'"
```

```
  shows "c  $\hat{c}$  c'  $\leq$ o c"
```

```
proof -
```

```
  have c: "Card_order c"
```

```
    using assms by auto
```

```
  from assms obtain n where n: "c' =o natLeq_on n"
```

```
    using Cfinite_cases by auto
```

```

have "c ^ c' =o c ^ natLeq_on n"
  using assms(2) by (intro cexp_cong2 c n) auto
also have "c ^ natLeq_on n ≤o c"
proof (induction n)
  case 0
  have "c ^ natLeq_on 0 =o c ^ czero"
    by (intro cexp_cong2) (use assms in <auto simp: empty_nat_ordIso_czero
card_order_on_empty>)
  also have "c ^ czero =o BNF_Cardinal_Arithmetic.cone"
    by (rule cexp_czero)
  also have "BNF_Cardinal_Arithmetic.cone ≤o c"
    using assms by (simp add: Cfinite_cone Cfinite_ordLess_Cinfinite
ordLess_imp_ordLeq)
  finally show ?case .
next
  case (Suc n)
  have "c ^ natLeq_on (Suc n) =o c ^ (natLeq_on n +c natLeq_on 1)"
    using assms natLeq_on_plus_ordIso[of n 1]
    by (intro cexp_cong2) (auto simp: natLeq_on_Card_order intro: ordIso_symmetric)
  also have "c ^ (natLeq_on n +c natLeq_on 1) =o c ^ natLeq_on n *c
c ^ natLeq_on 1"
    by (rule cexp_csum)
  also have "c ^ natLeq_on n *c c ^ natLeq_on 1 ≤o c *c c"
proof (rule cprod_mono)
  show "c ^ natLeq_on n ≤o c"
    by (rule Suc.IH)
  have "c ^ natLeq_on 1 =o c ^ BNF_Cardinal_Arithmetic.cone"
    by (intro cexp_cong2 c natLeq_on_1_ord_iso natLeq_on_Card_order)
  also have "c ^ BNF_Cardinal_Arithmetic.cone =o c"
    by (intro cexp_cone c)
  finally show "c ^ natLeq_on 1 ≤o c"
    by (rule ordIso_imp_ordLeq)
qed
  also have "c *c c =o c"
    using assms(1) by (rule cprod_infinite)
  finally show "c ^ natLeq_on (Suc n) ≤o c" .
qed
finally show ?thesis .
qed

lemma cexp_infinite_finite_ordIso:
  assumes "Cinfinite c" "Cfinite c'" "BNF_Cardinal_Arithmetic.cone ≤o
c'"
  shows "c ^ c' =o c"
proof -
  have c: "Card_order c"
    using assms by auto
  have "c =o c ^ BNF_Cardinal_Arithmetic.cone"
    by (rule ordIso_symmetric, rule cexp_cone) fact

```

```

    also have "c  $\hat{c}$  BNF_Cardinal_Arithmetic.cone  $\leq_o$  c  $\hat{c}$  c'"
      by (intro cexp_mono2 c assms Card_order_cone) (use cone_not_czero
in auto)
    finally have "c  $\leq_o$  c  $\hat{c}$  c'" .
    moreover have "c  $\hat{c}$  c'  $\leq_o$  c"
      by (rule cexp_infinite_finite_ordLeq) fact+
    ultimately show ?thesis
      by (simp add: ordIso_iff_ordLeq)
qed

lemma Cfinite_ordLeq_Cinfinite:
  assumes "Cfinite c" "Cinfinite c'"
  shows "c  $\leq_o$  c'"
  using assms Cfinite_ordLess_Cinfinite ordLess_imp_ordLeq by blast

lemma cfinite_card_of_iff [simp]: "BNF_Cardinal_Arithmetic.cfinite (card_of
X)  $\longleftrightarrow$  finite X"
  by (simp add: cfinite_def)

lemma cinfinite_card_of_iff [simp]: "BNF_Cardinal_Arithmetic.cinfinite
(card_of X)  $\longleftrightarrow$  infinite X"
  by (simp add: cinfinite_def)

lemma Func_conv_PiE: "Func A B = PiE A ( $\lambda$ _. B)"
  by (auto simp: Func_def PiE_def extensional_def)

lemma finite_Func [intro]:
  assumes "finite A" "finite B"
  shows "finite (Func A B)"
  using assms unfolding Func_conv_PiE by (intro finite_PiE)

lemma ordLeq_antisym: "(c, c')  $\in$  ordLeq  $\implies$  (c', c)  $\in$  ordLeq  $\implies$  (c,
c')  $\in$  ordIso"
  using ordIso_iff_ordLeq by auto

lemma cmax_cong:
  assumes "(c1, c1')  $\in$  ordIso" "(c2, c2')  $\in$  ordIso" "Card_order c1" "Card_order
c2"
  shows "cmax c1 c2 =o cmax c1' c2'"
proof -
  have [intro]: "Card_order c1'" "Card_order c2'"
    using assms Card_order_ordIso2 by auto
  have "c1  $\leq_o$  c2  $\vee$  c2  $\leq_o$  c1"
    by (intro ordLeq_total) (use assms in auto)
  thus ?thesis
proof
  assume le: "c1  $\leq_o$  c2"
  with assms have le': "c1'  $\leq_o$  c2'"
    by (meson ordIso_iff_ordLeq ordLeq_transitive)

```

```

    have "cmax c1 c2 =o c2"
      by (rule cmax2) (use le assms in auto)
    moreover have "cmax c1' c2' =o c2'"
      by (rule cmax2) (use le' assms in auto)
    ultimately show ?thesis
      using assms ordIso_symmetric ordIso_transitive by metis
next
  assume le: "c2 ≤o c1"
  with assms have le': "c2' ≤o c1'"
    by (meson ordIso_iff_ordLeq ordLeq_transitive)
  have "cmax c1 c2 =o c1"
    by (rule cmax1) (use le assms in auto)
  moreover have "cmax c1' c2' =o c1'"
    by (rule cmax1) (use le' assms in auto)
  ultimately show ?thesis
    using assms ordIso_symmetric ordIso_transitive by metis
qed
qed

```

## 1.2 The set of finite subsets

We define an operator  $\text{FinPow}(X)$  that, given a set  $X$ , returns the set of all finite subsets of that set. For finite  $X$ , this is boring since it is obviously just the power set. For infinite  $X$ , it is however a useful concept to have.

We will show that if  $X$  is infinite then the cardinality of  $\text{FinPow}(X)$  is exactly the same as that of  $X$ .

**definition**  $\text{FinPow} :: "'a \text{ set} \Rightarrow 'a \text{ set set}"$  where  
 $\text{"FinPow } X = \{Y. Y \subseteq X \wedge \text{finite } Y\}$ "

**lemma**  $\text{finite\_FinPow}$  [intro]:  $\text{"finite } A \implies \text{finite } (\text{FinPow } A)"$   
 by (auto simp:  $\text{FinPow\_def}$ )

**lemma**  $\text{in\_FinPow\_iff}$ :  $\text{"} Y \in \text{FinPow } X \iff Y \subseteq X \wedge \text{finite } Y"$   
 by (auto simp:  $\text{FinPow\_def}$ )

**lemma**  $\text{FinPow\_subsetq\_Pow}$ :  $\text{"FinPow } X \subseteq \text{Pow } X"$   
 unfolding  $\text{FinPow\_def}$  by blast

**lemma**  $\text{FinPow\_eq\_Pow}$ :  $\text{"finite } X \implies \text{FinPow } X = \text{Pow } X"$   
 unfolding  $\text{FinPow\_def}$  using  $\text{finite\_subset}$  by blast

**theorem**  $\text{card\_of\_FinPow\_infinite}$ :

assumes  $\text{"infinite } A"$   
 shows  $\text{"|FinPow } A| =o |A|"$

proof -

have  $\text{"set ' lists } A = \text{FinPow } A"$

using  $\text{finite\_list}$  [where  $?a = 'a]$  by (force simp:  $\text{FinPow\_def}$ )

hence  $\text{"|FinPow } A| \leqo |lists } A|"$

```

    by (metis card_of_image)
  also have "|lists A| =o |A|"
    using assms by (rule card_of_lists_infinite)
  finally have "|FinPow A| ≤o |A|" .
  moreover have "inj_on (λx. {x}) A" "(λx. {x}) ' A ⊆ FinPow A"
    by (auto simp: inj_on_def FinPow_def)
  hence "|A| ≤o |FinPow A|"
    using card_of_ordLeq by blast
  ultimately show ?thesis
    by (simp add: ordIso_iff_ordLeq)
qed

```

### 1.3 The set of functions with finite support

Next, we define an operator  $\text{Func\_finsupp}_z(A, B)$  that, given sets  $A$  and  $B$  and an element  $z \in B$ , returns the set of functions  $f : A \rightarrow B$  that have *finite support*, i.e. that map all but a finite subset of  $A$  to  $z$ .

**definition**  $\text{Func\_finsupp} :: "'b \Rightarrow 'a \text{ set} \Rightarrow 'b \text{ set} \Rightarrow ('a \Rightarrow 'b) \text{ set}"$  where  
 $\text{"Func\_finsupp } z \ A \ B = \{f \in A \rightarrow B. (\forall x. x \notin A \longrightarrow f \ x = z) \wedge \text{finite } \{x. f \ x \neq z\}\}$ "

**lemma**  $\text{bij\_betw\_Func\_finsup\_Func\_finite}$ :  
 assumes "finite A"  
 shows "bij\_betw (λf. restrict f A) (Func\_finsupp z A B) (Func A B)"  
 by (rule bij\_betwI[of \_ \_ \_ "λf x. if x ∈ A then f x else z"])  
 (use assms in <auto simp: Func\_finsupp\_def Func\_def>)

**lemma**  $\text{eqpoll\_Func\_finsup\_Func\_finite}$ : "finite A  $\implies$  Func\_finsupp z A B  $\approx$  Func A B"  
 by (meson bij\_betw\_Func\_finsup\_Func\_finite eqpoll\_def)

**lemma**  $\text{card\_of\_Func\_finsup\_finite}$ : "finite A  $\implies$  |Func\_finsupp z A B| =o |B| ^c |A|"  
 using eqpoll\_Func\_finsup\_Func\_finite  
 by (metis Field\_card\_of cexp\_def eqpoll\_imp\_card\_of\_ordIso)

The cases where  $A$  and  $B$  are both finite or  $B = \{0\}$  or  $A = \emptyset$  are of course trivial.

Perhaps not completely obviously, it turns out that in all other cases, the cardinality of  $\text{Func\_finsupp}_z(A, B)$  is exactly  $\max(|A|, |B|)$ .

**theorem**  $\text{card\_of\_Func\_finsupp\_infinite}$ :  
 assumes "z ∈ B" and "B - {z} ≠ {}" and "A ≠ {}"  
 assumes "infinite A  $\vee$  infinite B"  
 shows "|Func\_finsupp z A B| =o cmax |A| |B|"

**proof** -

```

  have inf_cmax: "Cinfinite (cmax |A| |B| )"
    using assms by (simp add: Card_order_cmax cinfinite_def finite_cmax)

```

```

have "bij_betw ( $\lambda f. (\{x. f x \neq z\}, \text{restrict } f \{x. f x \neq z\})$ )
      (Func_finsupp z A B) (SIGMA X:FinPow A. Func X (B - {z}))"
  by (rule bij_betwI[of _ _ _ " $\lambda(X,f) x. \text{if } x \in -X \cup -A \text{ then } z \text{ else } f x$ "]])
      (fastforce simp: Func_def Func_finsupp_def FinPow_def fun_eq_iff
<z  $\in$  B> split: if_splits)+
  hence "|Func_finsupp z A B| =o |SIGMA X:FinPow A. Func X (B - {z})|"
    by (rule card_of_ordIsoI)

also have "|SIGMA X:FinPow A. Func X (B - {z})| =o cmax |A| |B|"
proof (rule ordLeq_antisym)
  show "|SIGMA X:FinPow A. Func X (B - {z})|  $\leq$ o cmax |A| |B|"
  proof (intro card_of_Sigma_ordLeq_infinite_Field ballI)
    show "infinite (Field (cmax |A| |B| ))"
      using assms by (simp add: finite_cmax)
  next
    show "Card_order (cmax |A| |B| )"
      by (intro Card_order_cmax) auto
  next
    show "|FinPow A|  $\leq$ o cmax |A| |B|"
    proof (cases "finite A")
      assume "infinite A"
      hence "|FinPow A| =o |A|"
        by (rule card_of_FinPow_infinite)
      also have "|A|  $\leq$ o cmax |A| |B|"
        by (simp add: ordLeq_cmax)
      finally show ?thesis .
    next
      assume A: "finite A"
      have "finite (FinPow A)"
        using A by auto
      thus "|FinPow A|  $\leq$ o cmax |A| |B|"
        using A by (intro Cfinite_ordLeq_Cinfinite inf_cmax) auto
    qed
  next
    show "|Func X (B - {z})|  $\leq$ o cmax |A| |B|" if "X  $\in$  FinPow A" for
X
  proof (cases "finite B")
    case True
      have "finite X"
        using that by (auto simp: FinPow_def)
      hence "finite (Func X (B - {z}))"
        using True by blast
      with inf_cmax show ?thesis
        by (intro Cfinite_ordLeq_Cinfinite) auto
    next
      case False
      have "|Func X (B - {z})| =o |B - {z}|  $\wedge$  c |X|"
        by (rule card_of_Func)

```

```

    also have " $|B - \{z\}| \wedge^c |X| \leq_o |B - \{z\}|$ "
      by (rule cexp_infinite_finite_ordLeq) (use False that in <auto
simp: FinPow_def>)
    also have " $|B - \{z\}| =_o |B|$ "
      using False by (simp add: infinite_card_of_diff_singl)
    also have " $|B| \leq_o \text{cmax } |A| |B|$ "
      by (simp add: ordLeq_cmax)
    finally show ?thesis .
  qed
next
have " $\text{cmax } |A| |B| =_o |A| *c |B - \{z\}|$ "
proof (cases " $|A| \leq_o |B|$ ")
  case False
  have " $\neg |B - \{z\}| =_o \text{czero}$ "
    using <B - {z} ≠ {}> by (subst card_of_ordIso_czero_iff_empty)
  auto
  from False and assms have "infinite A"
    using ordLeq_finite_infinite by blast
  from False have " $|B| \leq_o |A|$ "
    by (simp add: ordLess_imp_ordLeq)
  have " $|B - \{z\}| \leq_o |B|$ "
    by (rule card_of_mono1) auto
  also note <B| ≤o |A|>
  finally have " $|A| *c |B - \{z\}| =_o |A|$ "
    using <infinite A> <¬|B - {z}| =o czero> by (intro cprod_infinite1')
  auto
  moreover have " $\text{cmax } |A| |B| =_o |A|$ "
    using <B| ≤o |A|> by (simp add: cmax1)
  ultimately show ?thesis
    using ordIso_symmetric ordIso_transitive by blast
next
case True
from True and assms have "infinite B"
  using card_of_ordLeq_finite by blast
have " $|A| *c |B - \{z\}| =_o |A| *c |B|$ "
  using <infinite B> by (intro cprod_cong2) (simp add: infinite_card_of_diff_singl)
also have " $|A| *c |B| =_o |B|$ "
  using True <infinite B> assms(3)
  by (simp add: card_of_ordIso_czero_iff_empty cprod_infinite2')
also have " $|B| =_o \text{cmax } |A| |B|$ "
  using True by (meson card_of_Card_order cmax2 ordIso_symmetric)
finally show ?thesis
  using ordIso_symmetric ordIso_transitive by blast
qed
also have " $|A| *c |B - \{z\}| =_o |A \times (B - \{z\})|$ "
  by (metis Field_card_of card_of_refl cprod_def)
also have " $|A \times (B - \{z\})| \leq_o |\text{SIGMA } X: (\lambda x. \{x\}) 'A. B - \{z\}|$ "
  by (intro card_of_Sigma_mono[of " $\lambda x. \{x\}$ "]) auto

```

```

    also have "|SIGMA X:(λx. {x})'A. B - {z}| =o |SIGMA X:(λx. {x})'A.
Func X (B - {z})|"
    proof (rule card_of_Sigma_cong; safe)
      fix x assume x: "x ∈ A"
      have "|Func {x} (B - {z})| =o |B - {z}| ^c |{x}|"
        by (simp add: card_of_Func)
      also have "|B - {z}| ^c |{x}| =o |B - {z}| ^c BNF_Cardinal_Arithmetic.cone"
        by (intro cexp_cong2) (auto simp: single_cone)
      also have "|B - {z}| ^c Wellorder_Constructions.cone =o |B - {z}|"
        using card_of_Card_order cexp_cone by blast
      finally show "|B - {z}| =o |Func {x} (B - {z})|"
        using ordIso_symmetric by blast
    qed
    also have "|SIGMA X:(λx. {x})'A. Func X (B - {z})| ≤o |SIGMA X:FinPow
A. Func X (B - {z})|"
      by (rule card_of_Sigma_mono) (auto simp: FinPow_def)
    finally show "cmax |A| |B| ≤o |SIGMA X:FinPow A. Func X (B - {z})|"
      .
  qed
  finally show ?thesis .
qed
end

```

## 2 The Cardinality of the Continuum

```

theory Cardinality_Continuum
  imports Complex_Main Cardinality_Continuum_Library
begin

```

### 2.1 $|\mathbb{R}| \leq |2^{\mathbb{Q}}|$ via Dedekind cuts

```

lemma le_cSup_iff:
  fixes A :: "'a :: conditionally_complete_linorder set"
  assumes "A ≠ {}" "bdd_above A"
  shows "Sup A ≥ c ↔ (∀x<c. ∃y∈A. y > x)"
  using assms by (meson less_cSup_iff not_le_imp_less order_less_irrefl
order_less_le_trans)

```

We show that the function mapping a real number to all the rational numbers below it is an injective map from the reals to  $2^{\mathbb{Q}}$ . This is the same idea that is used in the Dedekind cut definition of the reals.

```

lemma inj_Dedekind_cut:
  fixes f :: "real ⇒ rat set"
  defines "f ≡ (λx::real. {r::rat. of_rat r < x})"
  shows "inj f"
proof
  fix x y :: real

```

```

assume "f x = f y"

have *: "Sup (real_of_rat ' {r. real_of_rat r < z}) = z" for z :: real
proof -
  have "real_of_rat ' {r. real_of_rat r < z} = {r∈ℚ. r < z}"
    by (auto elim!: Rats_cases)
  also have "Sup ... = z"
  proof (rule antisym)
    have "{r ∈ ℚ. r < z} ≠ {}"
      using Rats_no_bot_less less_eq_real_def by blast
    hence "Sup {r∈ℚ. r < z} ≤ Sup {...z}"
      by (rule cSup_subset_mono) auto
    also have "... = z"
      by simp
    finally show "Sup {r∈ℚ. r < z} ≤ z" .
  next
    show "Sup {r∈ℚ. r < z} ≥ z"
    proof (subst le_cSup_iff)
      show "{r∈ℚ. r < z} ≠ {}"
        using Rats_no_bot_less less_eq_real_def by blast
      show "∀y<z. ∃r∈{r∈ℚ. r < z}. y < r"
        using Rats_dense_in_real by fastforce
      show "bdd_above {r ∈ ℚ. r < z}"
        by (rule bdd_aboveI[of _ z]) auto
    qed
  qed
  finally show ?thesis .
qed

from <f x = f y> have "Sup (real_of_rat ' f x) = Sup (real_of_rat '
f y)"
  by simp
thus "x = y"
  by (simp only: * f_def)
qed

```

## 2.2 $2^{\mathbb{N}} \leq |\mathbb{R}|$ via ternary fractions

For the other direction, we construct an injective function that maps a set of natural numbers  $A$  to a real number by constructing a ternary decimal number of the form  $d_0.d_1d_2d_3\dots$  where  $d_m$  is 1 if  $m \in A$  and 0 otherwise.

We will first show a few more general results about such  $n$ -ary fraction expansions.

```

lemma geometric_sums':
  fixes c :: "'a :: real_normed_field"
  assumes "norm c < 1"
  shows "(λn. c ^ (n + m)) sums (c ^ m / (1 - c))"
proof -

```

```

have "(λn. c ^ m * c ^ n) sums (c ^ m * (1 / (1 - c)))"
  by (intro sums_mult geometric_sums assms)
thus ?thesis
  by (simp add: power_add field_simps)
qed

```

```

lemma summable_nary_fraction:
  fixes d :: real and f :: "nat ⇒ real"
  assumes "∧n. norm (f n) ≤ c" "d > 1"
  shows "summable (λn. f n / d ^ n)"
proof (rule summable_comparison_test)
  show "∃N. ∀n≥N. norm (f n / d ^ n :: real) ≤ c * (1 / d) ^ n"
    using assms by (intro exI[of _ 0]) (auto simp: field_simps)
  show "summable (λn. c * (1 / d) ^ n :: real)"
    using assms by (intro summable_mult summable_geometric) auto
qed

```

Consider two  $n$ -ary fraction expansions  $u = u_1.u_2u_3\dots$  and  $v = v_1.v_2v_3\dots$  with  $n \geq 2$ . Suppose that all the  $u_i$  and  $v_i$  are between 0 and  $n - 2$  (i.e. the highest digit does not occur). Then  $u$  and  $v$  are equal if and only if all  $u_i = v_i$  for all  $i$ .

Note that without the additional restriction the result does not hold, as e.g. the decimal numbers 0.2 and  $0.1\bar{9}$  are equal.

The reasoning boils down to showing that if  $m$  is the smallest index where the two sequences differ, then  $|u - v| \geq \frac{1}{d-1} > 0$ .

```

lemma nary_fraction_unique:
  fixes u v :: "nat ⇒ nat"
  assumes f_eq: "(∑n. real (u n) / real d ^ n) = (∑n. real (v n) /
real d ^ n)"
  assumes uv: "∧n. u n ≤ d - 2" "∧n. v n ≤ d - 2" and d: "d ≥ 2"
  shows "u = v"
proof -
  define f :: "(nat ⇒ nat) ⇒ real" where
    "f = (λu. ∑n. real (u n) / real d ^ n)"

  have "u m = v m" for m
  proof (induction m rule: less_induct)
    case (less m)
    show "u m = v m"
    proof (rule ccontr)
      assume "u m ≠ v m"

      show False
        using <u m ≠ v m> uv less.IH f_eq
    proof (induction "u m" "v m" arbitrary: u v rule: linorder_wlog)
      case (sym u v)
      from sym(1)[of v u] sym(2-) show ?case
        by (simp add: eq_commute)
    end
  end
end

```

```

next
  case (le u v)
  have uv': "real (u n) ≤ real d - 2" "real (v n) ≤ real d - 2"
for n
  by (metis d of_nat_diff of_nat_le_iff of_nat_numeral le(3,4))+
  have "f u - f v - (real (u m) - real (v m)) / real d ^ m ≤
        (real d - 2) * ((1 / real d) ^ m / (real d - 1))"
  proof (rule sums_le)
    have "(λn. (real (u n) - real (v n)) / real d ^ n) sums (f u
- f v)"
      unfolding diff_divide_distrib f_def using le d uv'
      by (intro sums_diff summable_sums summable_nary_fraction[where
c = "real d - 2"]) auto
      hence "(λn. (real (u (n + m)) - real (v (n + m))) / real d ^
(n + m)) sums
              (f u - f v - (∑ n<m. (real (u n) - real (v n)) / real
d ^ n))"
        by (rule sums_split_initial_segment)
      also have "(∑ n<m. (real (u n) - real (v n)) / real d ^ n) =
0"
        by (intro sum.neutral) (use le in auto)
      finally have "(λn. (real (u (n + m)) - real (v (n + m))) / real
d ^ (n + m)) sums (f u - f v)"
        by simp
      thus "(λn. (real (u (Suc n + m)) - real (v (Suc n + m))) / real
d ^ (Suc n + m)) sums
              (f u - f v - (real (u m) - real (v m)) / real d ^ m)"
        by (subst sums_Suc_iff) auto
    next
      have "(λn. (real d - 2) * ((1 / real d) ^ (n + Suc m))) sums
              ((real d - 2) * ((1 / real d) ^ Suc m / (1 - 1 / real
d)))"
        using d by (intro sums_mult geometric_sums') auto
      thus "(λn. (real d - 2) * ((1 / real d) ^ (n + Suc m))) sums
              ((real d - 2) * ((1 / real d) ^ m / (real d - 1)) ::
real)"
        using d by (simp add: sums_iff field_simps)
    next
      fix n :: nat
      have "(real (u (Suc n + m)) - real (v (Suc n + m))) / real d
^ (Suc n + m) ≤
              ((real d - 2) - 0) / real d ^ (Suc n + m)"
        using uv' by (intro divide_right_mono diff_mono) auto
      thus "(real (u (Suc n + m)) - real (v (Suc n + m))) / real d
^ (Suc n + m) ≤
              (real d - 2) * (1 / real d) ^ (n + Suc m)"
        by (simp add: field_simps)
    qed
  hence "f u - f v ≤

```

```

      (real d - 2) / (real d - 1) / real d ^ m + (real (u m)
- real (v m)) / real d ^ m"
      by (simp add: field_simps)
      also have "... = ((real d - 2) / (real d - 1) + real (u m) - real
(v m)) / real d ^ m"
      by (simp add: add_divide_distrib diff_divide_distrib)
      also have "... = ((real d - 2) / (real d - 1) + real_of_int (int
(u m) - int (v m))) / real d ^ m"
      using <u m ≤ v m> by simp
      also have "... ≤ ((real d - 2) / (real d - 1) + -1) / real d ^
m"
      using le_d by (intro divide_right_mono add_mono) auto
      also have "(real d - 2) / (real d - 1) + -1 = -1 / (real d - 1)"
      using d by (simp add: field_simps)
      also have "... < 0"
      using d by (simp add: field_simps)
      finally have "f u - f v < 0"
      using d by (simp add: field_simps)
      with le show False
      by (simp add: f_def)
    qed
  qed
  qed
  thus ?thesis
  by blast
qed

```

It now follows straightforwardly that mapping sets of natural numbers to ternary fraction expansions is indeed injective. For binary fractions, this would not work due to the aforementioned issue.

```

lemma inj_nat_set_to_ternary:
  fixes f :: "nat set ⇒ real"
  defines "f ≡ (λA. ∑n. (if n ∈ A then 1 else 0) / 3 ^ n)"
  shows "inj f"
proof
  fix A B :: "nat set"
  assume "f A = f B"
  have "(λn. if n ∈ A then 1 else 0 :: nat) = (λn. if n ∈ B then 1 else
0 :: nat)"
  proof (rule nary_fraction_unique)
    have *: "(∑n. (if n ∈ A then 1 else 0) / 3 ^ n) =
(∑n. real (if n ∈ A then 1 else 0) / real 3 ^ n)"
    for A by (intro suminf_cong) auto
    show "(∑n. real (if n ∈ A then 1 else 0) / real 3 ^ n) =
(∑n. real (if n ∈ B then 1 else 0) / real 3 ^ n)"
    using <f A = f B> by (simp add: f_def *)
  qed auto
  thus "A = B"
  by (metis equalityI subsetI zero_neq_one)

```

qed

### 2.3 Equipollence proof

```

theorem eqpoll_UNIV_real: "(UNIV :: real set) ≈ (UNIV :: nat set set)"
proof (rule lepoll_antisym)
  show "(UNIV :: nat set set) ≲ (UNIV :: real set)"
    unfolding lepoll_def using inj_nat_set_to_ternary by blast
next
  have "(UNIV :: real set) ≲ (UNIV :: rat set set)"
    unfolding lepoll_def using inj_Dedekind_cut by blast
  also have "... = Pow (UNIV :: rat set)"
    by simp
  also have "... ≈ Pow (UNIV :: nat set)"
    by (rule eqpoll_Pow) (auto simp: infinite_UNIV_char_0 eqpoll_UNIV_nat_iff)
  also have "... = (UNIV :: nat set set)"
    by simp
  finally show "(UNIV :: real set) ≲ (UNIV :: nat set set)" .
qed

```

We can also write the language in the language of cardinal numbers as  $|\mathbb{R}| = 2^{\aleph_0}$  using Isabelle's cardinal number library:

```

corollary card_of_UNIV_real: "|UNIV :: real set| =o ctwo ^c natLeq"
proof -
  have "|UNIV :: real set| =o |UNIV :: nat set set|"
    using eqpoll_UNIV_real by (simp add: eqpoll_iff_card_of_ordIso)
  also have "|UNIV :: nat set set| =o cpow |UNIV :: nat set|"
    by (simp add: cpow_def)
  also have "cpow |UNIV :: nat set| =o ctwo ^c |UNIV :: nat set|"
    by (rule cpow_cexp_ctwo)
  also have "ctwo ^c |UNIV :: nat set| =o ctwo ^c natLeq"
    by (intro cexp_cong2) (simp_all add: card_of_nat Card_order_ctwo)
  finally show ?thesis .
qed

```

### 2.4 Corollaries for real intervals

It is easy to show that any real interval (whether open, closed, or infinite) is equipollent to the full set of real numbers.

```

lemma eqpoll_Ioo_real:
  fixes a b :: real
  assumes "a < b"
  shows "{a<..} ≈ (UNIV :: real set)"
proof -
  have Ioo: "{a<..} ≈ {0:..<1}" if "a < b" for a b :: real
  proof -
    have "bij_betw (λx. x * (b - a) + a) {0<..<1} {a<..}"
      proof (rule bij_betwI[of _ _ _ "λy. (y - a) / (b - a)"], goal_cases)

```

```

case 1
show ?case
proof
  fix x :: real assume x: "x ∈ {0<..1}"
  have "x * (b - a) + a > 0 + a"
    using x <a < b> by (intro add_strict_right_mono mult_pos_pos)
auto
  moreover have "x * (b - a) + a < 1 * (b - a) + a"
    using x <a < b> by (intro add_strict_right_mono mult_strict_right_mono)
auto
  ultimately show "x * (b - a) + a ∈ {a<..b}"
    by simp
qed
qed (use <a < b> in <auto simp: field_simps>)
thus ?thesis
  using eqpoll_def eqpoll_sym by blast
qed

have "{a<..b} ≈ {-pi/2<..pi/2}"
  using eqpoll_trans[OF Ioo[of a b] eqpoll_sym[OF Ioo[of "-pi/2" "pi/2"]]]
assms
  by simp
also have "bij_betw tan {-pi/2<..pi/2} (UNIV :: real set)"
  by (rule bij_betwI[of _ _ _ arctan])
  (use arctan_lbound arctan_ubound in <auto simp: arctan_tan tan_arctan>)
hence "{-pi/2<..pi/2} ≈ (UNIV :: real set)"
  using eqpoll_def by blast
finally show ?thesis .
qed

lemma eqpoll_real:
  assumes "{a::real<..b} ⊆ X" "a < b"
  shows "X ≈ (UNIV :: real set)"
  using eqpoll_Ioo_real[OF assms(2)] assms(1)
  by (meson eqpoll_sym lepoll_antisym lepoll_trans1 subset_UNIV subset_imp_lepoll)

lemma eqpoll_Icc_real: "(a::real) < b ⇒ {a..b} ≈ (UNIV :: real set)"
  and eqpoll_Ioc_real: "(a::real) < b ⇒ {a<..b} ≈ (UNIV :: real set)"
  and eqpoll_Ico_real: "(a::real) < b ⇒ {a..b} ≈ (UNIV :: real set)"
  by (rule eqpoll_real[of a b]; force)+

lemma eqpoll_Ici_real: "{a::real..}" ≈ (UNIV :: real set)"
  and eqpoll_Ioi_real: "{a::real<..}" ≈ (UNIV :: real set)"
  by (rule eqpoll_real[of a "a + 1"]; force)+

lemma eqpoll_Iic_real: "{..a}::real} ≈ (UNIV :: real set)"
  and eqpoll_Iio_real: "{..a}::real} ≈ (UNIV :: real set)"
  by (rule eqpoll_real[of "a - 1" a]; force)+

```

```

lemmas eqpoll_real_ivl =
  eqpoll_Ioo_real eqpoll_Ioc_real eqpoll_Ico_real eqpoll_Icc_real
  eqpoll_Iio_real eqpoll_Iic_real eqpoll_Ici_real eqpoll_Ioi_real

lemmas card_of_ivl_real =
  eqpoll_real_ivl[THEN eqpoll_imp_card_of_ordIso, THEN ordIso_transitive[OF
_ card_of_UNIV_real]]

```

## 2.5 Corollaries for vector spaces

We will now also show some results about the cardinality of vector spaces. To do this, we use the obvious isomorphism between a vector space  $V$  with a basis  $B$  and the set of finite-support functions  $B \rightarrow V$ .

```

lemma (in vector_space) card_of_span:
  assumes "independent B"
  shows "|span B| =o |Func_finsupp 0 B (UNIV :: 'a set)|"
proof -
  define f :: "('b  $\Rightarrow$  'a)  $\Rightarrow$  'b" where "f = ( $\lambda$ g.  $\sum$  b | g b  $\neq$  0. scale
(g b) b)"
  define g :: "'b  $\Rightarrow$  'b  $\Rightarrow$  'a" where "g = representation B"
  have "bij_betw g (span B) (Func_finsupp 0 B UNIV)"
  proof (rule bij_betwI[of _ _ _ f], goal_cases)
    case 1
    thus ?case
      by (auto simp: g_def Func_finsupp_def finite_representation intro:
representation_ne_zero)
    next
    case 2
    thus ?case
      by (auto simp: f_def Func_finsupp_def intro!: span_sum span_scale
intro: span_base)
    next
    case (3 x)
    show "f (g x) = x" unfolding g_def f_def
      by (intro sum_nonzero_representation_eq) (use 3 assms in auto)
    next
    case (4 v)
    show "g (f v) = v" unfolding g_def using 4
      by (intro representation_eqI)
      (auto simp: assms f_def Func_finsupp_def intro: span_base
intro!: sum.cong span_sum span_scale split: if_splits)
  qed
  thus "|span B| =o |Func_finsupp 0 B (UNIV :: 'a set)|"
  by (simp add: card_of_ordIsoI)
qed

```

We can now easily show the following: Let  $K$  be an infinite field and  $V$  a non-trivial finite-dimensional  $K$ -vector space. Then  $|V| = |K|$ .

```

lemma (in vector_space) card_of_span_finite_dim_infinite_field:
  assumes "independent B" and "finite B" and "B ≠ {}" and "infinite
  (UNIV :: 'a set)"
  shows "|span B| =o |UNIV :: 'a set|"
proof -
  have "|span B| =o |Func_finsupp 0 B (UNIV :: 'a set)|"
    by (rule card_of_span) fact
  also have "|Func_finsupp 0 B (UNIV :: 'a set)| =o cmax |B| |UNIV ::
  'a set|"
  proof (rule card_of_Func_finsupp_infinite)
    show "UNIV - {0 :: 'a} ≠ {}"
      using assms by (metis finite.emptyI infinite_remove)
    qed (use assms in auto)
  also have "cmax |B| |UNIV :: 'a set| =o |UNIV :: 'a set|"
    using assms by (intro cmax2 ordLeq3_finite_infinite) auto
  finally show ?thesis .
qed

```

Similarly, we can show the following: Let  $V$  be an infinite-dimensional vector space  $V$  over some (not necessarily infinite) field  $K$ . Then  $|V| = \max(\dim_K(V), |K|)$ .

```

lemma (in vector_space) card_of_span_infinite_dim_infinite_field:
  assumes "independent B" "infinite B"
  shows "|span B| =o cmax |B| |UNIV :: 'a set|"
proof -
  have "|span B| =o |Func_finsupp 0 B (UNIV :: 'a set)|"
    by (rule card_of_span) fact
  also have "|Func_finsupp 0 B (UNIV :: 'a set)| =o cmax |B| |UNIV ::
  'a set|"
  proof (rule card_of_Func_finsupp_infinite)
    have "(1 :: 'a) ∈ UNIV" "(1 :: 'a) ≠ 0"
      by auto
    thus "UNIV - {0 :: 'a} ≠ {}"
      by blast
    qed (use assms in auto)
  finally show "|span B| =o cmax |B| |UNIV :: 'a set|" .
qed

```

end

```

theory Cardinality_Euclidean_Space
  imports "HOL-Analysis.Analysis" Cardinality_Continuum
begin

```

With these results, it is now easy to see that any Euclidean space (i.e. finite-dimensional real vector space) has the same cardinality as  $\mathbb{R}$ :

```

corollary card_of_UNIV_euclidean_space:
  "|UNIV :: 'a :: euclidean_space set| =o ctwo ^c natLeq"
proof -
  have "|span Basis :: 'a set| =o |UNIV :: real set|"

```

```

    by (rule card_of_span_finite_dim_infinite_field)
      (simp_all add: independent_Basis infinite_UNIV_char_0)
  also have "|UNIV :: real set| =o ctwo ^c natLeq"
    by (rule card_of_UNIV_real)
  finally show ?thesis
    by simp
qed

```

In particular, this applies to  $\mathbb{C}$  and  $\mathbb{R}^n$ :

```

corollary card_of_complex: "|UNIV :: complex set| =o ctwo ^c natLeq"
  by (rule card_of_UNIV_euclidean_space)

```

```

corollary card_of_real_vec: "|UNIV :: (real ^ 'n :: finite) set| =o ctwo
^c natLeq"
  by (rule card_of_UNIV_euclidean_space)

```

end