

Cardinality of Number Partitions

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Abstract

This entry provides a basic library for number partitions, defines the two-argument partition function through its recurrence relation and relates this partition function to the cardinality of number partitions. The main proof shows that the recursively-defined partition function with arguments n and k equals the cardinality of number partitions of n with exactly k parts. The combinatorial proof follows the proof sketch of Theorem 2.4.1 in Mazur’s textbook “Combinatorics: A Guided Tour” [2]. This entry can serve as starting point for various more intrinsic properties about number partitions, the partition function and related recurrence relations.

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1 Additions to Isabelle's Main Theories

```
theory Additions-to-Main
imports HOL-Library.Multiset
begin
```

1.1 Addition to Finite-Set Theory

```
lemma bound-domain-and-range-impl-finitely-many-functions:
  finite {f::nat⇒nat. (∀ i. f i ≤ n) ∧ (∀ i≥m. f i = 0)}
⟨proof⟩
```

1.2 Addition to Set-Interval Theory

```
lemma sum-atMost-remove-nat:
  assumes k ≤ (n :: nat)
  shows (∑ i≤n. f i) = f k + (∑ i∈{..n}-{k}. f i)
⟨proof⟩
```

1.3 Additions to Multiset Theory

```
lemma set-mset-Abs-multiset:
  assumes finite {x. f x > 0}
  shows set-mset (Abs-multiset f) = {x. f x > 0}
⟨proof⟩
```

```
lemma sum-mset-sum-count:
  sum-mset M = (∑ i∈set-mset M. count M i * i)
⟨proof⟩
```

```
lemma sum-mset-eq-sum-on-supersets:
  assumes finite A set-mset M ⊆ A
  shows (∑ i∈set-mset M. count M i * i) = (∑ i∈A. count M i * i)
⟨proof⟩
```

end

2 Number Partitions

```
theory Number-Partition
imports Additions-to-Main
begin
```

2.1 Number Partitions as $\text{nat} \Rightarrow \text{nat}$ Functions

```
definition partitions :: (nat ⇒ nat) ⇒ nat ⇒ bool (infix partitions 50)
where
```

$$p \text{ partitions } n = ((\forall i. p \ i \neq 0 \longrightarrow 1 \leq i \wedge i \leq n) \wedge (\sum i \leq n. p \ i * i) = n)$$

lemma *partitionsI*:

assumes $\bigwedge i. p\ i \neq 0 \implies 1 \leq i \wedge i \leq n$

assumes $(\sum_{i \leq n}. p\ i * i) = n$

shows *p partitions n*

<proof>

lemma *partitionsE*:

assumes *p partitions n*

obtains $\bigwedge i. p\ i \neq 0 \implies 1 \leq i \wedge i \leq n \ (\sum_{i \leq n}. p\ i * i) = n$

<proof>

lemma *partitions-zero*:

p partitions 0 $\longleftrightarrow p = (\lambda i. 0)$

<proof>

lemma *partitions-one*:

p partitions (Suc 0) $\longleftrightarrow p = (\lambda i. 0)(1 := 1)$

<proof>

2.2 Bounds and Finiteness of Number Partitions

lemma *partitions-imp-finite-elements*:

assumes *p partitions n*

shows *finite {i. 0 < p i}*

<proof>

lemma *partitions-bounds*:

assumes *p partitions n*

shows $p\ i \leq n$

<proof>

lemma *partitions-parts-bounded*:

assumes *p partitions n*

shows $\text{sum } p\ \{..n\} \leq n$

<proof>

lemma *finite-partitions*:

finite {p. p partitions n}

<proof>

lemma *finite-partitions-k-parts*:

finite {p. p partitions n \wedge sum p {..n} = k}

<proof>

lemma *partitions-remaining-Max-part*:

assumes *p partitions n*

assumes $0 < p\ k$

shows $\forall i. n - k < i \wedge i \neq k \implies p\ i = 0$

<proof>

2.3 Operations of Number Partitions

lemma *partitions-remove1-bounds*:

assumes *partitions*: p partitions n

assumes *gr0*: $0 < p\ k$

assumes *neq*: $(p(k := p\ k - 1))\ i \neq 0$

shows $1 \leq i \wedge i \leq n - k$

<proof>

lemma *partitions-remove1*:

assumes *partitions*: p partitions n

assumes *gr0*: $0 < p\ k$

shows $p(k := p\ k - 1)$ partitions $(n - k)$

<proof>

lemma *partitions-insert1*:

assumes *p*: p partitions n

assumes $k > 0$

shows $(p(k := p\ k + 1))$ partitions $(n + k)$

<proof>

lemma *count-remove1*:

assumes p partitions n

assumes $0 < p\ k$

shows $(\sum_{i \leq n - k}. (p(k := p\ k - 1))\ i) = (\sum_{i \leq n}. p\ i) - 1$

<proof>

lemma *count-insert1*:

assumes p partitions n

shows $sum\ (p(k := p\ k + 1))\ \{..n + k\} = (\sum_{i \leq n}. p\ i) + 1$

<proof>

lemma *partitions-decrease1*:

assumes p : p partitions m

assumes *sum*: $sum\ p\ \{..m\} = k$

assumes $p\ 1 = 0$

shows $(\lambda i. p\ (i + 1))$ partitions $m - k$

<proof>

lemma *partitions-increase1*:

assumes *partitions*: p partitions $m - k$

assumes *k*: $sum\ p\ \{..m - k\} = k$

shows $(\lambda i. p\ (i - 1))$ partitions m

<proof>

lemma *count-decrease1*:

assumes p : p partitions m

assumes *sum*: $sum\ p\ \{..m\} = k$

assumes $p\ 1 = 0$

shows $sum\ (\lambda i. p\ (i + 1))\ \{..m - k\} = k$

<proof>

lemma *count-increase1*:

assumes *partitions*: p partitions $m - k$

assumes k : $\text{sum } p \{..m - k\} = k$

shows $(\sum_{i \leq m}. p (i - 1)) = k$

<proof>

2.4 Number Partitions as Multisets on Natural Numbers

definition *number-partition* :: $\text{nat} \Rightarrow \text{nat multiset} \Rightarrow \text{bool}$

where

number-partition n $N = (\text{sum-mset } N = n \wedge 0 \notin \# N)$

2.4.1 Relationship to Definition on Functions

lemma *count-partitions-iff*:

count N partitions $n \iff \text{number-partition } n$ N

<proof>

lemma *partitions-iff-Abs-multiset*:

p partitions $n \iff \text{finite } \{x. 0 < p x\} \wedge \text{number-partition } n$ (*Abs-multiset* p)

<proof>

lemma *size-nat-multiset-eq*:

fixes N :: nat multiset

assumes *number-partition* n N

shows $\text{size } N = \text{sum } (\text{count } N) \{..n\}$

<proof>

end

3 Cardinality of Number Partitions

theory *Card-Number-Partitions*

imports *Number-Partition*

begin

3.1 The Partition Function

fun *Partition* :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{nat}$

where

Partition 0 0 = 1

| *Partition* 0 (Suc k) = 0

| *Partition* (Suc m) 0 = 0

| *Partition* (Suc m) (Suc k) = *Partition* m k + *Partition* ($m - k$) (Suc k)

lemma *Partition-less*:

assumes $m < k$
shows $\text{Partition } m \ k = 0$
 $\langle \text{proof} \rangle$

lemma *Partition-sum-Partition-diff*:
assumes $k \leq m$
shows $\text{Partition } m \ k = (\sum_{i \leq k}. \text{Partition } (m - k) \ i)$
 $\langle \text{proof} \rangle$

lemma *Partition-parts1*:
 $\text{Partition } (\text{Suc } m) \ (\text{Suc } 0) = 1$
 $\langle \text{proof} \rangle$

lemma *Partition-diag*:
 $\text{Partition } (\text{Suc } m) \ (\text{Suc } m) = 1$
 $\langle \text{proof} \rangle$

lemma *Partition-diag1*:
 $\text{Partition } (\text{Suc } (\text{Suc } m)) \ (\text{Suc } m) = 1$
 $\langle \text{proof} \rangle$

lemma *Partition-parts2*:
shows $\text{Partition } m \ 2 = m \ \text{div } 2$
 $\langle \text{proof} \rangle$

3.2 Cardinality of Number Partitions

lemma *set-rewrite1*:
 $\{p. p \ \text{partitions } \text{Suc } m \wedge \text{sum } p \ \{..\text{Suc } m\} = \text{Suc } k \wedge p \ 1 \neq 0\}$
 $= (\lambda p. p(1 := p \ 1 + 1)) \ \{p. p \ \text{partitions } m \wedge \text{sum } p \ \{..m\} = k\}$ (**is** $?S = ?T$)
 $\langle \text{proof} \rangle$

lemma *set-rewrite2*:
 $\{p. p \ \text{partitions } m \wedge \text{sum } p \ \{..m\} = k \wedge p \ 1 = 0\}$
 $= (\lambda p. (\lambda i. p \ (i - 1))) \ \{p. p \ \text{partitions } (m - k) \wedge \text{sum } p \ \{..m - k\} = k\}$
(**is** $?S = ?T$)
 $\langle \text{proof} \rangle$

theorem *card-partitions-k-parts*:
 $\text{card } \{p. p \ \text{partitions } n \wedge (\sum_{i \leq n}. p \ i) = k\} = \text{Partition } n \ k$
 $\langle \text{proof} \rangle$

theorem *card-partitions*:
 $\text{card } \{p. p \ \text{partitions } n\} = (\sum_{k \leq n}. \text{Partition } n \ k)$
 $\langle \text{proof} \rangle$

lemma *card-partitions-atmost-k-parts*:
 $\text{card } \{p. p \ \text{partitions } n \wedge \text{sum } p \ \{..n\} \leq k\} = \text{Partition } (n + k) \ k$

<proof>

3.3 Cardinality of Number Partitions as Multisets of Natural Numbers

lemma *bij-betw-multiset-number-partition-with-size:*

bij-betw count $\{N. \text{number-partition } n \ N \wedge \text{size } N = k\} \{p. \text{ } p \text{ partitions } n \wedge \text{sum } p \{..n\} = k\}$
<proof>

lemma *bij-betw-multiset-number-partition-with-atmost-size:*

bij-betw count $\{N. \text{number-partition } n \ N \wedge \text{size } N \leq k\} \{p. \text{ } p \text{ partitions } n \wedge \text{sum } p \{..n\} \leq k\}$
<proof>

theorem *card-number-partitions-with-atmost-k-parts:*

shows $\text{card } \{N. \text{number-partition } n \ N \wedge \text{size } N \leq x\} = \text{Partition } (n + x) \ x$
<proof>

theorem *card-partitions-with-k-parts:*

$\text{card } \{N. \text{number-partition } n \ N \wedge \text{size } N = k\} = \text{Partition } n \ k$
<proof>

3.4 Cardinality of Number Partitions with only 1-parts

lemma *number-partition1-eq-replicate-mset:*

$\{N. (\forall n. n \in \# \ N \longrightarrow n = 1) \wedge \text{number-partition } n \ N\} = \{\text{replicate-mset } n \ 1\}$
<proof>

lemma *card-number-partitions-with-only-parts-1-eq-1:*

assumes $n \leq x$
shows $\text{card } \{N. (\forall n. n \in \# \ N \longrightarrow n = 1) \wedge \text{number-partition } n \ N \wedge \text{size } N \leq x\} = 1$ (**is** $\text{card } ?N = -$)
<proof>

lemma *card-number-partitions-with-only-parts-1-eq-0:*

assumes $x < n$
shows $\text{card } \{N. (\forall n. n \in \# \ N \longrightarrow n = 1) \wedge \text{number-partition } n \ N \wedge \text{size } N \leq x\} = 0$ (**is** $\text{card } ?N = -$)
<proof>

end

References

- [1] M. Junker. Diskrete Algebraische Strukturen, 2010. German lecture notes from Mathematisches Institut Albert-Ludwigs-Universität Freiburg.

- [2] D. R. Mazur. *Combinatorics: a guided tour*. MAA textbooks. Mathematical Association of America, 2010.