

The Calculus of Communicating Systems

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Abstract

We formalise a large portion of CCS as described in Milner’s book ‘Communication and Concurrency’ using the nominal datatype package in Isabelle. Our results include many of the standard theorems of bisimulation equivalence and congruence, for both weak and strong versions. One main goal of this formalisation is to keep the machine-checked proofs as close to their pen-and-paper counterpart as possible.

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1 Overview

These theories formalise the following results from Milner’s book Communication and Concurrency.

- strong bisimilarity is a congruence
- strong bisimilarity respects the laws of structural congruence
- weak bisimilarity is preserved by all operators except sum
- weak congruence is a congruence
- all strongly bisimilar agents are also weakly congruent which in turn are weakly bisimilar. As a corollary, weak bisimilarity and weak congruence respect the laws of structural congruence.

The file naming convention is hopefully self explanatory, where the prefixes *Strong* and *Weak* denote that the file covers theories required to formalise properties of strong and weak bisimilarity respectively; if the file name contains *Sim* the theories cover simulation, file names containing *Bisim*

cover bisimulation, and file names containing *Cong* cover weak congruence; files with the suffix *Pres* deal with theories that reason about preservation properties of operators such as a certain simulation or bisimulation being preserved by a certain operator; files with the suffix *SC* reason about structural congruence.

For a complete exposition of all theories, please consult Bengtson's Ph. D. thesis [1].

2 Formalisation

```

theory Agent
  imports HOL-Nominal.Nominal
begin

atom-decl name

nominal-datatype act = actAction name    (⟨!-⟩ 100)
  | actCoAction name    (⟨⟨-⟩ 100)
  | actTau              (⟨τ⟩ 100)

nominal-datatype ccs = CCSNil          (⟨0⟩ 115)
  | Action act ccs    (⟨-.-> [120, 110] 110)
  | Sum ccs ccs      (infixl ⟨⊕⟩ 90)
  | Par ccs ccs      (infixl ⟨||⟩ 85)
  | Res «name» ccs (⟨!ν-⟩-> [105, 100] 100)
  | Bang ccs        (⟨!-> [95])

nominal-primrec coAction :: act ⇒ act
where
  coAction (|a|) = (⟨a⟩)
  | coAction (⟨a⟩) = (|a|)
  | coAction (τ) = τ
by(rule TrueI)+

lemma coActionEqvt[eqvt]:
  fixes p :: name prm
  and a :: act

  shows (p · coAction a) = coAction(p · a)
by(nominal-induct a rule: act.strong-induct) (auto simp add: eqvts)

lemma coActionSimps[simp]:
  fixes a :: act

  shows coAction(coAction a) = a
  and (coAction a = τ) = (a = τ)
by auto (nominal-induct rule: act.strong-induct, auto)+

```

lemma *coActSimp*[*simp*]: **shows** $\text{coAction } \alpha \neq \tau = (\alpha \neq \tau)$ **and** $(\text{coAction } \alpha = \tau) = (\alpha = \tau)$
by(*nominal-induct* α *rule: act.strong-induct*) *auto*

lemma *coActFresh*[*simp*]:
fixes $x :: \text{name}$
and $a :: \text{act}$

shows $x \# \text{coAction } a = x \# a$
by(*nominal-induct* a *rule: act.strong-induct*) (*auto*)

lemma *alphaRes*:
fixes $y :: \text{name}$
and $P :: \text{ccs}$
and $x :: \text{name}$

assumes $y \# P$

shows $(\nu x)P = (\nu y)((x, y) \cdot P)$

using *assms*

by(*auto simp add: ccs.inject alpha fresh-left calc-atm pt-swap-bij[OF pt-name-inst, OF at-name-inst]pt3[OF pt-name-inst, OF at-ds1[OF at-name-inst]]*)

inductive semantics :: $\text{ccs} \Rightarrow \text{act} \Rightarrow \text{ccs} \Rightarrow \text{bool}$ ($\langle \cdot \mapsto \cdot \prec \cdot \rangle$ [80, 80, 80] 80)
where

Action: $\alpha.(P) \mapsto \alpha \prec P$
| *Sum1*: $P \mapsto \alpha \prec P' \Longrightarrow P \oplus Q \mapsto \alpha \prec P'$
| *Sum2*: $Q \mapsto \alpha \prec Q' \Longrightarrow P \oplus Q \mapsto \alpha \prec Q'$
| *Par1*: $P \mapsto \alpha \prec P' \Longrightarrow P \parallel Q \mapsto \alpha \prec P' \parallel Q$
| *Par2*: $Q \mapsto \alpha \prec Q' \Longrightarrow P \parallel Q \mapsto \alpha \prec P \parallel Q'$
| *Comm*: $\llbracket P \mapsto a \prec P'; Q \mapsto (\text{coAction } a) \prec Q'; a \neq \tau \rrbracket \Longrightarrow P \parallel Q \mapsto \tau \prec P' \parallel Q'$
| *Res*: $\llbracket P \mapsto \alpha \prec P'; x \# \alpha \rrbracket \Longrightarrow (\nu x)P \mapsto \alpha \prec (\nu x)P'$
| *Bang*: $P \parallel !P \mapsto \alpha \prec P' \Longrightarrow !P \mapsto \alpha \prec P'$

equivariance semantics

nominal-inductive semantics

by(*auto simp add: abs-fresh*)

lemma *semanticsInduct*:

$\llbracket R \mapsto \beta \prec R'; \bigwedge \alpha P C. \text{Prop } C (\alpha.(P)) \alpha P;$
 $\bigwedge P \alpha P' Q C. \llbracket P \mapsto \alpha \prec P'; \bigwedge C. \text{Prop } C P \alpha P' \rrbracket \Longrightarrow \text{Prop } C (\text{ccs.Sum } P Q) \alpha P';$
 $\bigwedge Q \alpha Q' P C. \llbracket Q \mapsto \alpha \prec Q'; \bigwedge C. \text{Prop } C Q \alpha Q' \rrbracket \Longrightarrow \text{Prop } C (\text{ccs.Sum } P Q) \alpha Q';$
 $\bigwedge P \alpha P' Q C. \llbracket P \mapsto \alpha \prec P'; \bigwedge C. \text{Prop } C P \alpha P' \rrbracket \Longrightarrow \text{Prop } C (P \parallel Q) \alpha (P' \parallel Q);$

$\wedge Q \alpha Q' P \mathcal{C}. \llbracket Q \mapsto \alpha \prec Q'; \wedge \mathcal{C}. \text{Prop } \mathcal{C} Q \alpha Q' \rrbracket \implies \text{Prop } \mathcal{C} (P \parallel Q) \alpha (P \parallel Q')$;
 $\wedge P a P' Q Q' \mathcal{C}.$
 $\llbracket P \mapsto a \prec P'; \wedge \mathcal{C}. \text{Prop } \mathcal{C} P a P'; Q \mapsto (\text{coAction } a) \prec Q';$
 $\wedge \mathcal{C}. \text{Prop } \mathcal{C} Q (\text{coAction } a) Q'; a \neq \tau \rrbracket$
 $\implies \text{Prop } \mathcal{C} (P \parallel Q) (\tau) (P' \parallel Q')$;
 $\wedge P \alpha P' x \mathcal{C}.$
 $\llbracket x \# \mathcal{C}; P \mapsto \alpha \prec P'; \wedge \mathcal{C}. \text{Prop } \mathcal{C} P \alpha P'; x \# \alpha \rrbracket \implies \text{Prop } \mathcal{C} (\nu x) P \alpha (\nu x) P'$;
 $\wedge P \alpha P' \mathcal{C}. \llbracket P \parallel !P \mapsto \alpha \prec P'; \wedge \mathcal{C}. \text{Prop } \mathcal{C} (P \parallel !P) \alpha P' \rrbracket \implies \text{Prop } \mathcal{C} !P \alpha P'$

$\implies \text{Prop } (\mathcal{C}::'a::\text{fs-name}) R \beta R'$
by(*erule-tac z=C in semantics.strong-induct*) *auto*

lemma *NilTrans*[*dest*]:
shows $\mathbf{0} \mapsto \alpha \prec P' \implies \text{False}$
and $(\langle b \rangle).P \mapsto \langle c \rangle \prec P' \implies \text{False}$
and $(\langle b \rangle).P \mapsto \tau \prec P' \implies \text{False}$
and $(\langle b \rangle).P \mapsto \langle c \rangle \prec P' \implies \text{False}$
and $(\langle b \rangle).P \mapsto \tau \prec P' \implies \text{False}$
apply(*ind-cases* $\mathbf{0} \mapsto \alpha \prec P'$)
apply(*ind-cases* $(\langle b \rangle).P \mapsto \langle c \rangle \prec P'$, *auto simp add: ccs.inject*)
apply(*ind-cases* $(\langle b \rangle).P \mapsto \tau \prec P'$, *auto simp add: ccs.inject*)
apply(*ind-cases* $(\langle b \rangle).P \mapsto \langle c \rangle \prec P'$, *auto simp add: ccs.inject*)
apply(*ind-cases* $(\langle b \rangle).P \mapsto \tau \prec P'$, *auto simp add: ccs.inject*)
done

lemma *freshDerivative*:
fixes $P :: \text{ccs}$
and $a :: \text{act}$
and $P' :: \text{ccs}$
and $x :: \text{name}$

assumes $P \mapsto \alpha \prec P'$
and $x \# P$

shows $x \# \alpha$ **and** $x \# P'$
using *assms*
by(*nominal-induct rule: semantics.strong-induct*)
(auto simp add: ccs.fresh abs-fresh)

lemma *actCases*[*consumes 1, case-names cAct*]:
fixes $\alpha :: \text{act}$
and $P :: \text{ccs}$
and $\beta :: \text{act}$
and $P' :: \text{ccs}$

assumes $\alpha.(P) \mapsto \beta \prec P'$
and $\text{Prop } \alpha P$

shows $\text{Prop } \beta P'$
using *assms*
by – (*ind-cases* $\alpha.(P) \mapsto \beta \prec P'$, *auto simp add: ccs.inject*)

lemma *sumCases*[*consumes 1*, *case-names cSum1 cSum2*]:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$
and $\alpha :: \text{act}$
and $R :: \text{ccs}$

assumes $P \oplus Q \mapsto \alpha \prec R$
and $\bigwedge P'. P \mapsto \alpha \prec P' \implies \text{Prop } P'$
and $\bigwedge Q'. Q \mapsto \alpha \prec Q' \implies \text{Prop } Q'$

shows $\text{Prop } R$
using *assms*
by – (*ind-cases* $P \oplus Q \mapsto \alpha \prec R$, *auto simp add: ccs.inject*)

lemma *parCases*[*consumes 1*, *case-names cPar1 cPar2 cComm*]:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$
and $a :: \text{act}$
and $R :: \text{ccs}$

assumes $P \parallel Q \mapsto \alpha \prec R$
and $\bigwedge P'. P \mapsto \alpha \prec P' \implies \text{Prop } \alpha (P' \parallel Q)$
and $\bigwedge Q'. Q \mapsto \alpha \prec Q' \implies \text{Prop } \alpha (P \parallel Q')$
and $\bigwedge P' Q' a. \llbracket P \mapsto a \prec P'; Q \mapsto (\text{coAction } a) \prec Q'; a \neq \tau; \alpha = \tau \rrbracket \implies \text{Prop } (\tau) (P' \parallel Q')$

shows $\text{Prop } \alpha R$
using *assms*
by – (*ind-cases* $P \parallel Q \mapsto \alpha \prec R$, *auto simp add: ccs.inject*)

lemma *resCases*[*consumes 1*, *case-names cRes*]:

fixes $x :: \text{name}$
and $P :: \text{ccs}$
and $\alpha :: \text{act}$
and $P' :: \text{ccs}$

assumes $(\nu x)P \mapsto \alpha \prec P'$
and $\bigwedge P'. \llbracket P \mapsto \alpha \prec P'; x \# \alpha \rrbracket \implies \text{Prop } ((\nu x)P')$

shows $\text{Prop } P'$
proof –
from $\langle (\nu x)P \mapsto \alpha \prec P' \rangle$ **have** $x \# \alpha$ **and** $x \# P'$
by(*auto intro: freshDerivative simp add: abs-fresh*)
with *assms* **show** *?thesis*

```

    by(cases rule: semantics.strong-cases[of - - - x])
      (auto simp add: abs-fresh ccs.inject alpha)
  qed

inductive bangPred :: ccs ⇒ ccs ⇒ bool
where
  aux1: bangPred P (!P)
| aux2: bangPred P (P || !P)

lemma bangInduct[consumes 1, case-names cPar1 cPar2 cComm cBang]:
  fixes P :: ccs
  and α :: act
  and P' :: ccs
  and C :: 'a::fs-name

  assumes !P ↦α < P'
  and rPar1: ∧α P' C. [!P ↦α < P'] ⇒ Prop C (P || !P) α (P' || !P)
  and rPar2: ∧α P' C. [!P ↦α < P'; ∧C. Prop C (!P) α P'] ⇒ Prop C (P
|| !P) α (P || P')
  and rComm: ∧a P' P'' C. [!P ↦a < P'; !P ↦(coAction a) < P''; ∧C.
Prop C (!P) (coAction a) P''; a ≠ τ] ⇒ Prop C (P || !P) (τ) (P' || P'')
  and rBang: ∧α P' C. [P || !P ↦α < P'; ∧C. Prop C (P || !P) α P'] ⇒
Prop C (!P) α P'

  shows Prop C (!P) α P'
proof -
  {
    fix X α P'
    assume X ↦α < P' and bangPred P X
    hence Prop C X α P'
    proof(nominal-induct avoiding: C rule: semantics.strong-induct)
      case(Action α Pa)
      thus ?case
      by - (ind-cases bangPred P (α.(Pa)))
    next
      case(Sum1 Pa α P' Q)
      thus ?case
      by - (ind-cases bangPred P (Pa ⊕ Q))
    next
      case(Sum2 Q α Q' Pa)
      thus ?case
      by - (ind-cases bangPred P (Pa ⊕ Q))
    next
      case(Par1 Pa α P' Q)
      thus ?case
      apply -
      by(ind-cases bangPred P (Pa || Q), auto intro: rPar1 simp add: ccs.inject)
    next
  }

```

```

    case(Par2 Q α P' Pa)
  thus ?case
    apply -
      by(ind-cases bangPred P (Pa || Q), auto intro: rPar2 aux1 simp add:
ccs.inject)
  next
    case(Comm Pa a P' Q Q' C)
  thus ?case
    apply -
      by(ind-cases bangPred P (Pa || Q), auto intro: rComm aux1 simp add:
ccs.inject)
  next
    case(Res Pa α P' x)
  thus ?case
    by - (ind-cases bangPred P ((νx)Pa))
  next
    case(Bang Pa α P')
  thus ?case
    apply -
      by(ind-cases bangPred P (!Pa), auto intro: rBang aux2 simp add: ccs.inject)
  qed
}
with ⟨!P ⟶α < P'⟩ show ?thesis by(force intro: bangPred.aux1)
qed

```

inductive-set $bangRel :: (ccs \times ccs) \text{ set} \Rightarrow (ccs \times ccs) \text{ set}$

for $Rel :: (ccs \times ccs) \text{ set}$

where

$BRBang: (P, Q) \in Rel \Rightarrow (!P, !Q) \in bangRel Rel$

$| BRPar: (R, T) \in Rel \Rightarrow (P, Q) \in (bangRel Rel) \Rightarrow (R || P, T || Q) \in (bangRel Rel)$

lemma $BRBangCases[consumes 1, case-names BRBang]:$

fixes $P :: ccs$

and $Q :: ccs$

and $Rel :: (ccs \times ccs) \text{ set}$

and $F :: ccs \Rightarrow bool$

assumes $(P, !Q) \in bangRel Rel$

and $\bigwedge P. (P, Q) \in Rel \Rightarrow F (!P)$

shows $F P$

using $assms$

by - $(ind-cases (P, !Q) \in bangRel Rel, auto simp add: ccs.inject)$

lemma $BRParCases[consumes 1, case-names BRPar]:$

fixes $P :: ccs$

and $Q :: ccs$

and $Rel :: (ccs \times ccs) \text{ set}$

```

and  $F :: ccs \Rightarrow bool$ 

assumes  $(P, Q \parallel !Q) \in \text{bangRel } Rel$ 
and  $\bigwedge P R. \llbracket (P, Q) \in Rel; (R, !Q) \in \text{bangRel } Rel \rrbracket \Longrightarrow F (P \parallel R)$ 

shows  $F P$ 
using assms
by  $- (\text{ind-cases } (P, Q \parallel !Q) \in \text{bangRel } Rel, \text{auto simp add: ccs.inject})$ 

lemma bangRelSubset:
fixes  $Rel :: (ccs \times ccs) \text{ set}$ 
and  $Rel' :: (ccs \times ccs) \text{ set}$ 

assumes  $(P, Q) \in \text{bangRel } Rel$ 
and  $\bigwedge P Q. (P, Q) \in Rel \Longrightarrow (P, Q) \in Rel'$ 

shows  $(P, Q) \in \text{bangRel } Rel'$ 
using assms
by(induct rule: bangRel.induct) (auto intro: BRBang BRPar)

end

theory Tau-Chain
imports Agent
begin

definition tauChain ::  $ccs \Rightarrow ccs \Rightarrow bool$  ( $\langle - \Longrightarrow_{\tau} - \rangle [80, 80] 80$ )
where  $P \Longrightarrow_{\tau} P' \equiv (P, P') \in \{(P, P') \mid P P'. P \mapsto_{\tau} \prec P'\}^*$ 

lemma tauChainInduct[consumes 1, case-names Base Step]:
assumes  $P \Longrightarrow_{\tau} P'$ 
and  $\text{Prop } P$ 
and  $\bigwedge P' P''. \llbracket P \Longrightarrow_{\tau} P'; P' \mapsto_{\tau} \prec P''; \text{Prop } P'' \rrbracket \Longrightarrow \text{Prop } P''$ 

shows  $\text{Prop } P'$ 
using assms
by(auto simp add: tauChain-def elim: rtrancl-induct)

lemma tauChainRefl[simp]:
fixes  $P :: ccs$ 

shows  $P \Longrightarrow_{\tau} P$ 
by(auto simp add: tauChain-def)

lemma tauChainCons[dest]:
fixes  $P :: ccs$ 
and  $P' :: ccs$ 
and  $P'' :: ccs$ 

```

```

assumes  $P \Rightarrow_{\tau} P'$ 
and  $P' \mapsto_{\tau} \prec P''$ 

shows  $P \Rightarrow_{\tau} P''$ 
using assms
by(auto simp add: tauChain-def) (blast dest: rtrancl-trans)

lemma tauChainCons2[dest]:
  fixes  $P :: ccs$ 
  and  $P' :: ccs$ 
  and  $P'' :: ccs$ 

  assumes  $P' \mapsto_{\tau} \prec P''$ 
  and  $P \Rightarrow_{\tau} P'$ 

  shows  $P \Rightarrow_{\tau} P''$ 
using assms
by(auto simp add: tauChain-def) (blast dest: rtrancl-trans)

lemma tauChainAppend[dest]:
  fixes  $P :: ccs$ 
  and  $P' :: ccs$ 
  and  $P'' :: ccs$ 

  assumes  $P \Rightarrow_{\tau} P'$ 
  and  $P' \Rightarrow_{\tau} P''$ 

  shows  $P \Rightarrow_{\tau} P''$ 
using  $\langle P' \Rightarrow_{\tau} P'' \rangle \langle P \Rightarrow_{\tau} P' \rangle$ 
by(induct rule: tauChainInduct) auto

lemma tauChainSum1:
  fixes  $P :: ccs$ 
  and  $P' :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $P \Rightarrow_{\tau} P'$ 
  and  $P \neq P'$ 

  shows  $P \oplus Q \Rightarrow_{\tau} P'$ 
using assms
proof(induct rule: tauChainInduct)
  case Base
  thus ?case by simp
next
  case(Step  $P' P''$ )
  thus ?case
  by(case-tac P=P') (auto intro: Sum1 simp add: tauChain-def)

```

qed

lemma *tauChainSum2*:

fixes $P :: ccs$

and $P' :: ccs$

and $Q :: ccs$

assumes $Q \Longrightarrow_{\tau} Q'$

and $Q \neq Q'$

shows $P \oplus Q \Longrightarrow_{\tau} Q'$

using *assms*

proof(*induct rule: tauChainInduct*)

case *Base*

thus *?case* **by** *simp*

next

case(*Step Q' Q''*)

thus *?case*

by(*case-tac Q=Q'*) (*auto intro: Sum2 simp add: tauChain-def*)

qed

lemma *tauChainPar1*:

fixes $P :: ccs$

and $P' :: ccs$

and $Q :: ccs$

assumes $P \Longrightarrow_{\tau} P'$

shows $P \parallel Q \Longrightarrow_{\tau} P' \parallel Q$

using *assms*

by(*induct rule: tauChainInduct*) (*auto intro: Par1*)

lemma *tauChainPar2*:

fixes $Q :: ccs$

and $Q' :: ccs$

and $P :: ccs$

assumes $Q \Longrightarrow_{\tau} Q'$

shows $P \parallel Q \Longrightarrow_{\tau} P \parallel Q'$

using *assms*

by(*induct rule: tauChainInduct*) (*auto intro: Par2*)

lemma *tauChainRes*:

fixes $P :: ccs$

and $P' :: ccs$

and $x :: name$

assumes $P \Longrightarrow_{\tau} P'$

shows $(\nu x)P \Longrightarrow_{\tau} (\nu x)P'$
using *assms*
by(*induct rule: tauChainInduct*) (*auto dest: Res*)

lemma *tauChainRepl*:
fixes $P :: ccs$

assumes $P \parallel !P \Longrightarrow_{\tau} P'$
and $P' \neq P \parallel !P$

shows $!P \Longrightarrow_{\tau} P'$
using *assms*
apply(*induct rule: tauChainInduct*)
apply *auto*
apply(*case-tac P' ≠ P ∥ !P*)
apply *auto*
apply(*drule Bang*)
apply(*simp add: tauChain-def*)
by *auto*

end

theory *Weak-Cong-Semantics*
imports *Tau-Chain*
begin

definition *weakCongTrans* :: $ccs \Rightarrow act \Rightarrow ccs \Rightarrow bool$ ($\langle - \Longrightarrow - \prec - \rangle [80, 80, 80]$
 $80)$

where $P \Longrightarrow_{\alpha} \prec P' \equiv \exists P'' P'''. P \Longrightarrow_{\tau} P'' \wedge P'' \mapsto_{\alpha} \prec P''' \wedge P''' \Longrightarrow_{\tau} P'$

lemma *weakCongTransE*:

fixes $P :: ccs$
and $\alpha :: act$
and $P' :: ccs$

assumes $P \Longrightarrow_{\alpha} \prec P'$

obtains $P'' P'''$ **where** $P \Longrightarrow_{\tau} P''$ **and** $P'' \mapsto_{\alpha} \prec P'''$ **and** $P''' \Longrightarrow_{\tau} P'$
using *assms*
by(*auto simp add: weakCongTrans-def*)

lemma *weakCongTransI*:

fixes $P :: ccs$
and $P'' :: ccs$
and $\alpha :: act$
and $P''' :: ccs$
and $P' :: ccs$

```

assumes  $P \Longrightarrow_{\tau} P''$ 
and  $P'' \mapsto_{\alpha} \prec P'''$ 
and  $P''' \Longrightarrow_{\tau} P'$ 

shows  $P \Longrightarrow_{\alpha} \prec P'$ 
using assms
by(auto simp add: weakCongTrans-def)

lemma transitionWeakCongTransition:
  fixes  $P :: ccs$ 
  and  $\alpha :: act$ 
  and  $P' :: ccs$ 

  assumes  $P \mapsto_{\alpha} \prec P'$ 

  shows  $P \Longrightarrow_{\alpha} \prec P'$ 
using assms
by(force simp add: weakCongTrans-def)

lemma weakCongAction:
  fixes  $a :: name$ 
  and  $P :: ccs$ 

  shows  $\alpha.(P) \Longrightarrow_{\alpha} \prec P$ 
by(auto simp add: weakCongTrans-def)
  (blast intro: Action tauChainRef1)

lemma weakCongSum1:
  fixes  $P :: ccs$ 
  and  $\alpha :: act$ 
  and  $P' :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $P \Longrightarrow_{\alpha} \prec P'$ 

  shows  $P \oplus Q \Longrightarrow_{\alpha} \prec P'$ 
using assms
apply(auto simp add: weakCongTrans-def)
apply(case-tac P=P'')
apply(force simp add: tauChain-def dest: Sum1)
by(force intro: tauChainSum1)

lemma weakCongSum2:
  fixes  $Q :: ccs$ 
  and  $\alpha :: act$ 
  and  $Q' :: ccs$ 
  and  $P :: ccs$ 

  assumes  $Q \Longrightarrow_{\alpha} \prec Q'$ 

```

```

  shows  $P \oplus Q \Longrightarrow \alpha \prec Q'$ 
using assms
apply(auto simp add: weakCongTrans-def)
apply(case-tac Q=P'')
apply(force simp add: tauChain-def dest: Sum2)
by(force intro: tauChainSum2)

```

lemma *weakCongPar1*:

```

  fixes  $P :: ccs$ 
  and  $\alpha :: act$ 
  and  $P' :: ccs$ 
  and  $Q :: ccs$ 

```

```

  assumes  $P \Longrightarrow \alpha \prec P'$ 

```

```

  shows  $P \parallel Q \Longrightarrow \alpha \prec P' \parallel Q$ 
using assms
by(auto simp add: weakCongTrans-def)
(blast dest: tauChainPar1 Par1)

```

lemma *weakCongPar2*:

```

  fixes  $Q :: ccs$ 
  and  $\alpha :: act$ 
  and  $Q' :: ccs$ 
  and  $P :: ccs$ 

```

```

  assumes  $Q \Longrightarrow \alpha \prec Q'$ 

```

```

  shows  $P \parallel Q \Longrightarrow \alpha \prec P \parallel Q'$ 
using assms
by(auto simp add: weakCongTrans-def)
(blast dest: tauChainPar2 Par2)

```

lemma *weakCongSync*:

```

  fixes  $P :: ccs$ 
  and  $\alpha :: act$ 
  and  $P' :: ccs$ 
  and  $Q :: ccs$ 

```

```

  assumes  $P \Longrightarrow \alpha \prec P'$ 
  and  $Q \Longrightarrow (coAction \alpha) \prec Q'$ 
  and  $\alpha \neq \tau$ 

```

```

  shows  $P \parallel Q \Longrightarrow \tau \prec P' \parallel Q'$ 
using assms
apply(auto simp add: weakCongTrans-def)
apply(rule-tac x= P'' \parallel P''a in exI)
apply auto

```

```

apply(blast dest: tauChainPar1 tauChainPar2)
apply(rule-tac x=P''' || P'''a in exI)
apply auto
apply(rule Comm)
apply auto
apply(rule-tac P'=P' || P'''a in tauChainAppend)
by(blast dest: tauChainPar1 tauChainPar2)+

```

lemma *weakCongRes*:

```

fixes P :: ccs
and α :: act
and P' :: ccs
and x :: name

```

```

assumes P ⇒α < P'
and x ‡ α

```

```

shows (νx)P ⇒α < (νx)P'
using assms
by(auto simp add: weakCongTrans-def)
   (blast dest: tauChainRes Res)

```

lemma *weakCongRepl*:

```

fixes P :: ccs
and α :: act
and P' :: ccs

```

```

assumes P || !P ⇒α < P'

```

```

shows !P ⇒α < P'
using assms
apply(auto simp add: weakCongTrans-def)
apply(case-tac P'' = P || !P)
apply auto
apply(force intro: Bang simp add: tauChain-def)
by(force intro: tauChainRepl)

```

end

theory *Weak-Semantics*

```

imports Weak-Cong-Semantics
begin

```

```

definition weakTrans :: ccs ⇒ act ⇒ ccs ⇒ bool (α ⇒̂ - < - ⇒ [80, 80, 80] 80)
where P ⇒̂ α < P' ≡ P ⇒α < P' ∨ (α = τ ∧ P = P')

```

lemma *weakEmptyTrans*[*simp*]:

```

fixes P :: ccs

```

shows $P \Longrightarrow \hat{\tau} \prec P$
by(*auto simp add: weakTrans-def*)

lemma *weakTransCases*[*consumes 1, case-names Base Step*]:

fixes $P :: ccs$
and $\alpha :: act$
and $P' :: ccs$

assumes $P \Longrightarrow \hat{\alpha} \prec P'$
and $\llbracket \alpha = \tau; P = P' \rrbracket \Longrightarrow Prop (\tau) P$
and $P \Longrightarrow \alpha \prec P' \Longrightarrow Prop \alpha P'$

shows $Prop \alpha P'$
using *assms*
by(*auto simp add: weakTrans-def*)

lemma *weakCongTransitionWeakTransition*:

fixes $P :: ccs$
and $\alpha :: act$
and $P' :: ccs$

assumes $P \Longrightarrow \alpha \prec P'$

shows $P \Longrightarrow \hat{\alpha} \prec P'$
using *assms*
by(*auto simp add: weakTrans-def*)

lemma *transitionWeakTransition*:

fixes $P :: ccs$
and $\alpha :: act$
and $P' :: ccs$

assumes $P \mapsto \alpha \prec P'$

shows $P \Longrightarrow \hat{\alpha} \prec P'$
using *assms*
by(*auto dest: transitionWeakCongTransition weakCongTransitionWeakTransition*)

lemma *weakAction*:

fixes $a :: name$
and $P :: ccs$

shows $\alpha.(P) \Longrightarrow \hat{\alpha} \prec P$
by(*auto simp add: weakTrans-def intro: weakCongAction*)

lemma *weakSum1*:

fixes $P :: ccs$
and $\alpha :: act$
and $P' :: ccs$

```

and   Q :: ccs

assumes P  $\Longrightarrow$   $\hat{\alpha} \prec P'$ 
and     P  $\neq$  P'

shows P  $\oplus$  Q  $\Longrightarrow$   $\hat{\alpha} \prec P'$ 
using assms
by(auto simp add: weakTrans-def intro: weakCongSum1)

lemma weakSum2:
  fixes Q :: ccs
  and    $\alpha$  :: act
  and   Q' :: ccs
  and   P :: ccs

  assumes Q  $\Longrightarrow$   $\hat{\alpha} \prec Q'$ 
  and     Q  $\neq$  Q'

  shows P  $\oplus$  Q  $\Longrightarrow$   $\hat{\alpha} \prec Q'$ 
using assms
by(auto simp add: weakTrans-def intro: weakCongSum2)

lemma weakPar1:
  fixes P :: ccs
  and    $\alpha$  :: act
  and   P' :: ccs
  and   Q :: ccs

  assumes P  $\Longrightarrow$   $\hat{\alpha} \prec P'$ 

  shows P  $\parallel$  Q  $\Longrightarrow$   $\hat{\alpha} \prec P' \parallel$  Q
using assms
by(auto simp add: weakTrans-def intro: weakCongPar1)

lemma weakPar2:
  fixes Q :: ccs
  and    $\alpha$  :: act
  and   Q' :: ccs
  and   P :: ccs

  assumes Q  $\Longrightarrow$   $\hat{\alpha} \prec Q'$ 

  shows P  $\parallel$  Q  $\Longrightarrow$   $\hat{\alpha} \prec P \parallel$  Q'
using assms
by(auto simp add: weakTrans-def intro: weakCongPar2)

lemma weakSync:
  fixes P :: ccs
  and    $\alpha$  :: act

```

```

and P' :: ccs
and Q :: ccs

assumes P  $\Longrightarrow$   $\hat{\alpha} \prec P'$ 
and Q  $\Longrightarrow$   $\hat{(coAction\ \alpha)} \prec Q'$ 
and  $\alpha \neq \tau$ 

shows P || Q  $\Longrightarrow$   $\hat{\tau} \prec P' || Q'$ 
using assms
by(auto simp add: weakTrans-def intro: weakCongSync)

lemma weakRes:
  fixes P :: ccs
  and  $\alpha$  :: act
  and P' :: ccs
  and x :: name

  assumes P  $\Longrightarrow$   $\hat{\alpha} \prec P'$ 
  and  $x \# \alpha$ 

  shows  $(\nu x)P \Longrightarrow \hat{\alpha} \prec (\nu x)P'$ 
using assms
by(auto simp add: weakTrans-def intro: weakCongRes)

lemma weakRepl:
  fixes P :: ccs
  and  $\alpha$  :: act
  and P' :: ccs

  assumes P || !P  $\Longrightarrow$   $\hat{\alpha} \prec P'$ 
  and  $P' \neq P || !P$ 

  shows !P  $\Longrightarrow$   $\alpha \prec P'$ 
using assms
by(auto simp add: weakTrans-def intro: weakCongRepl)

end

theory Strong-Sim
  imports Agent
begin

definition simulation :: ccs  $\Rightarrow$  (ccs  $\times$  ccs) set  $\Rightarrow$  ccs  $\Rightarrow$  bool ( $\prec \rightsquigarrow [-] \rightarrow [80, 80, 80] 80$ )
where
  P  $\rightsquigarrow [Rel]$  Q  $\equiv \forall a Q'. Q \mapsto a \prec Q' \longrightarrow (\exists P'. P \mapsto a \prec P' \wedge (P', Q') \in Rel)$ 

lemma simI[case-names Sim]:
  fixes P :: ccs

```

```

and Rel :: (ccs × ccs) set
and Q :: ccs

assumes  $\bigwedge \alpha Q'. Q \mapsto_{\alpha} \prec Q' \implies \exists P'. P \mapsto_{\alpha} \prec P' \wedge (P', Q') \in Rel$ 

shows  $P \rightsquigarrow_{[Rel]} Q$ 
using assms
by(auto simp add: simulation-def)

lemma simE:
  fixes P :: ccs
  and Rel :: (ccs × ccs) set
  and Q :: ccs
  and  $\alpha :: act$ 
  and Q' :: ccs

  assumes  $P \rightsquigarrow_{[Rel]} Q$ 
  and  $Q \mapsto_{\alpha} \prec Q'$ 

  obtains P' where  $P \mapsto_{\alpha} \prec P'$  and  $(P', Q') \in Rel$ 
using assms
by(auto simp add: simulation-def)

lemma reflexive:
  fixes P :: ccs
  and Rel :: (ccs × ccs) set

  assumes  $Id \subseteq Rel$ 

  shows  $P \rightsquigarrow_{[Rel]} P$ 
using assms
by(auto simp add: simulation-def)

lemma transitive:
  fixes P :: ccs
  and Rel :: (ccs × ccs) set
  and Q :: ccs
  and Rel' :: (ccs × ccs) set
  and R :: ccs
  and Rel'' :: (ccs × ccs) set

  assumes  $P \rightsquigarrow_{[Rel]} Q$ 
  and  $Q \rightsquigarrow_{[Rel']} R$ 
  and  $Rel \circ Rel' \subseteq Rel''$ 

  shows  $P \rightsquigarrow_{[Rel'']} R$ 
using assms
by(force simp add: simulation-def)

```

end

theory *Weak-Sim*

imports *Weak-Semantics Strong-Sim*

begin

definition *weakSimulation* :: $ccs \Rightarrow (ccs \times ccs) \text{ set} \Rightarrow ccs \Rightarrow \text{bool}$ ($\langle - \rightsquigarrow^{\wedge} \langle - \rangle - \rangle$
[80, 80, 80] 80)

where

$P \rightsquigarrow^{\wedge} \langle \text{Rel} \rangle Q \equiv \forall a Q'. Q \mapsto a \prec Q' \longrightarrow (\exists P'. P \Longrightarrow^{\wedge} a \prec P' \wedge (P', Q') \in \text{Rel})$

lemma *weakSimI*[*case-names Sim*]:

fixes $P :: ccs$

and $\text{Rel} :: (ccs \times ccs) \text{ set}$

and $Q :: ccs$

assumes $\bigwedge \alpha Q'. Q \mapsto \alpha \prec Q' \Longrightarrow \exists P'. P \Longrightarrow^{\wedge} \alpha \prec P' \wedge (P', Q') \in \text{Rel}$

shows $P \rightsquigarrow^{\wedge} \langle \text{Rel} \rangle Q$

using *assms*

by(*auto simp add: weakSimulation-def*)

lemma *weakSimE*:

fixes $P :: ccs$

and $\text{Rel} :: (ccs \times ccs) \text{ set}$

and $Q :: ccs$

and $\alpha :: \text{act}$

and $Q' :: ccs$

assumes $P \rightsquigarrow^{\wedge} \langle \text{Rel} \rangle Q$

and $Q \mapsto \alpha \prec Q'$

obtains P' **where** $P \Longrightarrow^{\wedge} \alpha \prec P'$ **and** $(P', Q') \in \text{Rel}$

using *assms*

by(*auto simp add: weakSimulation-def*)

lemma *simTauChain*:

fixes $P :: ccs$

and $\text{Rel} :: (ccs \times ccs) \text{ set}$

and $Q :: ccs$

and $Q' :: ccs$

assumes $Q \Longrightarrow_{\tau} Q'$

and $(P, Q) \in \text{Rel}$

and $\text{Sim}: \bigwedge R S. (R, S) \in \text{Rel} \Longrightarrow R \rightsquigarrow^{\wedge} \langle \text{Rel} \rangle S$

obtains P' **where** $P \Longrightarrow_{\tau} P'$ **and** $(P', Q') \in \text{Rel}$

using $\langle Q \Longrightarrow_{\tau} Q' \rangle \langle (P, Q) \in \text{Rel} \rangle$

proof(*induct arbitrary: thesis rule: tauChainInduct*)
 case *Base*
 from $\langle P, Q \rangle \in Rel$ **show** *?case*
 by(*force intro: Base*)
next
 case(*Step* $Q'' Q'$)
 from $\langle P, Q \rangle \in Rel$ **obtain** P'' **where** $P \Rightarrow_{\tau} P''$ **and** $(P'', Q'') \in Rel$
 by(*blast intro: Step*)
 from $\langle P'', Q'' \rangle \in Rel$ **have** $P'' \rightsquigarrow^{\wedge} \langle Rel \rangle Q''$ **by**(*rule Sim*)
then obtain P' **where** $P'' \Rightarrow_{\tau} P'$ **and** $(P', Q') \in Rel$ **using** $\langle Q'' \mapsto_{\tau} Q' \rangle$
 by(*rule weakSimE*)
with $\langle P \Rightarrow_{\tau} P'' \rangle$ **show** *thesis*
 by(*force simp add: weakTrans-def weakCongTrans-def intro: Step*)
qed

lemma *simE2*:

fixes $P :: ccs$
and $Rel :: (ccs \times ccs) \text{ set}$
and $Q :: ccs$
and $\alpha :: act$
and $Q' :: ccs$

assumes $(P, Q) \in Rel$
and $Q \Rightarrow^{\wedge} \alpha \prec Q'$
and $Sim: \bigwedge R S. (R, S) \in Rel \Rightarrow R \rightsquigarrow^{\wedge} \langle Rel \rangle S$

obtains P' **where** $P \Rightarrow^{\wedge} \alpha \prec P'$ **and** $(P', Q') \in Rel$

proof –

assume *Goal*: $\bigwedge P'. \llbracket P \Rightarrow^{\wedge} \alpha \prec P'; (P', Q') \in Rel \rrbracket \Rightarrow thesis$
moreover from $\langle Q \Rightarrow^{\wedge} \alpha \prec Q' \rangle$ **have** $\exists P'. P \Rightarrow^{\wedge} \alpha \prec P' \wedge (P', Q') \in Rel$
proof(*induct rule: weakTransCases*)
 case *Base*
 from $\langle P, Q \rangle \in Rel$ **show** *?case* **by** *force*
next
 case *Step*
 from $\langle Q \Rightarrow^{\wedge} \alpha \prec Q' \rangle$ **obtain** $Q''' Q''$
where $QChain: Q \Rightarrow_{\tau} Q'''$ **and** $Q'''Trans: Q''' \mapsto_{\alpha} Q''$ **and** $Q''Chain: Q'' \Rightarrow_{\tau} Q'$
 by(*rule weakCongTransE*)
 from $QChain \langle P, Q \rangle \in Rel$ *Sim* **obtain** P''' **where** $PChain: P \Rightarrow_{\tau} P'''$
and $(P''', Q''') \in Rel$
 by(*rule simTauChain*)
 from $\langle P''', Q''' \rangle \in Rel$ **have** $P''' \rightsquigarrow^{\wedge} \langle Rel \rangle Q'''$ **by**(*rule Sim*)
then obtain P'' **where** $P'''Trans: P''' \Rightarrow^{\wedge} \alpha \prec P''$ **and** $(P'', Q'') \in Rel$
using $Q'''Trans$ **by**(*rule weakSimE*)
 from $Q''Chain \langle P'', Q'' \rangle \in Rel$ *Sim* **obtain** P' **where** $P''Chain: P'' \Rightarrow_{\tau} P'$
and $(P', Q') \in Rel$
 by(*rule simTauChain*)
 from $P'''Trans P''Chain Step$ **show** *?thesis*

```

proof(induct rule: weakTransCases)
  case Base
  from PChain  $\langle P''' \Longrightarrow_{\tau} P' \rangle$  have  $P \Longrightarrow_{\tau} P'$ 
  proof(induct rule: tauChainInduct)
    case Base
    from  $\langle P \Longrightarrow_{\tau} P' \rangle$  show ?case
    proof(induct rule: tauChainInduct)
      case Base
      show ?case by simp
    next
    case(Step  $P' P''$ )
    thus ?case by(fastforce simp add: weakTrans-def weakCongTrans-def)
    qed
  next
  case(Step  $P''' P''$ )
  thus ?case by(fastforce simp add: weakTrans-def weakCongTrans-def)
  qed
  with  $\langle P', Q' \rangle \in Rel$  show ?case by blast
next
  case Step
  thus ?case using  $\langle P', Q' \rangle \in Rel$  PChain
  by(rule-tac x=P' in exI) (force simp add: weakTrans-def weakCongTrans-def)
  qed
qed
ultimately show ?thesis
  by blast
qed

lemma reflexive:
  fixes  $P :: ccs$ 
  and  $Rel :: (ccs \times ccs) set$ 

  assumes  $Id \subseteq Rel$ 

  shows  $P \rightsquigarrow^{\langle Rel \rangle} P$ 
using assms
by(auto simp add: weakSimulation-def intro: transitionWeakCongTransition weakCongTransitionWeakTransition)

lemma transitive:
  fixes  $P :: ccs$ 
  and  $Rel :: (ccs \times ccs) set$ 
  and  $Q :: ccs$ 
  and  $Rel' :: (ccs \times ccs) set$ 
  and  $R :: ccs$ 
  and  $Rel'' :: (ccs \times ccs) set$ 

  assumes  $(P, Q) \in Rel$ 
  and  $Q \rightsquigarrow^{\langle Rel' \rangle} R$ 

```

```

and   Rel O Rel'  $\subseteq$  Rel''
and    $\bigwedge S T. (S, T) \in Rel \implies S \rightsquigarrow^{\wedge} \langle Rel \rangle T$ 

shows  $P \rightsquigarrow^{\wedge} \langle Rel'' \rangle R$ 
proof(induct rule: weakSimI)
  case(Sim  $\alpha$  R')
  thus ?case using assms
    apply(drule-tac Q=R in weakSimE, auto)
    by(drule-tac Q=Q in simE2, auto)
qed

lemma weakMonotonic:
  fixes  $P :: ccs$ 
  and    $A :: (ccs \times ccs) \text{ set}$ 
  and    $Q :: ccs$ 
  and    $B :: (ccs \times ccs) \text{ set}$ 

  assumes  $P \rightsquigarrow^{\wedge} \langle A \rangle Q$ 
  and      $A \subseteq B$ 

  shows  $P \rightsquigarrow^{\wedge} \langle B \rangle Q$ 
using assms
by(fastforce simp add: weakSimulation-def)

lemma simWeakSim:
  fixes  $P :: ccs$ 
  and    $Rel :: (ccs \times ccs) \text{ set}$ 
  and    $Q :: ccs$ 

  assumes  $P \rightsquigarrow[Rel] Q$ 

  shows  $P \rightsquigarrow^{\wedge} \langle Rel \rangle Q$ 
using assms
by(rule-tac weakSimI, auto)
  (blast dest: simE transitionWeakTransition)

end

theory Weak-Cong-Sim
  imports Weak-Cong-Semantics Weak-Sim Strong-Sim
begin

definition weakCongSimulation ::  $ccs \Rightarrow (ccs \times ccs) \text{ set} \Rightarrow ccs \Rightarrow \text{bool}$  ( $\leftarrow \rightsquigarrow^{\wedge} \langle - \rangle$ 
 $\rightarrow [80, 80, 80] 80$ )
where
   $P \rightsquigarrow^{\wedge} \langle Rel \rangle Q \equiv \forall a Q'. Q \vdash a \prec Q' \longrightarrow (\exists P'. P \Longrightarrow a \prec P' \wedge (P', Q') \in Rel)$ 

lemma weakSimI[case-names Sim]:
  fixes  $P :: ccs$ 

```

```

and Rel :: (ccs × ccs) set
and Q :: ccs

assumes  $\bigwedge \alpha Q'. Q \mapsto_{\alpha} \prec Q' \implies \exists P'. P \implies_{\alpha} \prec P' \wedge (P', Q') \in Rel$ 

shows  $P \rightsquigarrow_{\langle Rel \rangle} Q$ 
using assms
by(auto simp add: weakCongSimulation-def)

lemma weakSimE:
  fixes P :: ccs
  and Rel :: (ccs × ccs) set
  and Q :: ccs
  and  $\alpha :: act$ 
  and Q' :: ccs

  assumes  $P \rightsquigarrow_{\langle Rel \rangle} Q$ 
  and  $Q \mapsto_{\alpha} \prec Q'$ 

  obtains P' where  $P \implies_{\alpha} \prec P'$  and  $(P', Q') \in Rel$ 
using assms
by(auto simp add: weakCongSimulation-def)

lemma simWeakSim:
  fixes P :: ccs
  and Rel :: (ccs × ccs) set
  and Q :: ccs

  assumes  $P \rightsquigarrow_{[Rel]} Q$ 

  shows  $P \rightsquigarrow_{\langle Rel \rangle} Q$ 
using assms
by(rule-tac weakSimI, auto)
  (blast dest: simE transitionWeakCongTransition)

lemma weakCongSimWeakSim:
  fixes P :: ccs
  and Rel :: (ccs × ccs) set
  and Q :: ccs

  assumes  $P \rightsquigarrow_{\langle Rel \rangle} Q$ 

  shows  $P \rightsquigarrow^{\wedge}_{\langle Rel \rangle} Q$ 
using assms
by(rule-tac Weak-Sim.weakSimI, auto)
  (blast dest: weakSimE weakCongTransitionWeakTransition)

lemma test:
  assumes  $P \implies_{\tau} P'$ 

```

shows $P = P' \vee (\exists P''. P \mapsto_{\tau} \prec P'' \wedge P'' \Longrightarrow_{\tau} P')$
using *assms*
by(*induct rule: tauChainInduct*) *auto*

lemma *tauChainCasesSym*[*consumes 1, case-names cTauNil cTauStep*]:
assumes $P \Longrightarrow_{\tau} P'$
and $Prop P$
and $\bigwedge P''. \llbracket P \mapsto_{\tau} \prec P''; P'' \Longrightarrow_{\tau} P' \rrbracket \Longrightarrow Prop P'$

shows $Prop P'$
using *assms*
by(*blast dest: test*)

lemma *simE2*:
fixes $P :: ccs$
and $Rel :: (ccs \times ccs) set$
and $Q :: ccs$
and $\alpha :: act$
and $Q' :: ccs$

assumes $P \rightsquigarrow \langle Rel \rangle Q$
and $Q \Longrightarrow_{\alpha} \prec Q'$
and $Sim: \bigwedge R S. (R, S) \in Rel \Longrightarrow R \rightsquigarrow \langle Rel \rangle S$

obtains P' **where** $P \Longrightarrow_{\alpha} \prec P'$ **and** $(P', Q') \in Rel$

proof –

assume *Goal*: $\bigwedge P'. \llbracket P \Longrightarrow_{\alpha} \prec P'; (P', Q') \in Rel \rrbracket \Longrightarrow thesis$
from $\langle Q \Longrightarrow_{\alpha} \prec Q' \rangle$ **obtain** $Q''' Q''$
where $QChain: Q \Longrightarrow_{\tau} Q'''$ **and** $Q''''Trans: Q''' \mapsto_{\alpha} \prec Q''$ **and** $Q''Chain: Q'' \Longrightarrow_{\tau} Q'$
by(*rule weakCongTransE*)
from $QChain Q''''Trans$ **show** *?thesis*
proof(*induct rule: tauChainCasesSym*)
case *cTauNil*
from $\langle P \rightsquigarrow \langle Rel \rangle Q \rangle \langle Q \mapsto_{\alpha} \prec Q'' \rangle$ **obtain** P''' **where** $PTrans: P \Longrightarrow_{\alpha} \prec P'''$ **and** $(P''', Q'') \in Rel$
by(*blast dest: weakSimE*)
moreover from $Q''Chain \langle (P''', Q'') \in Rel \rangle Sim$ **obtain** P' **where** $P''Chain: P''' \Longrightarrow_{\tau} P'$ **and** $(P', Q') \in Rel$
by(*rule simTauChain*)
with $PTrans P''Chain$ **show** *?thesis*
by(*force intro: Goal simp add: weakCongTrans-def weakTrans-def*)
next
case(*cTauStep Q''''*)
from $\langle P \rightsquigarrow \langle Rel \rangle Q \rangle \langle Q \mapsto_{\tau} \prec Q'''' \rangle$ **obtain** P'''' **where** $PChain: P \Longrightarrow_{\tau} P''''$ **and** $(P''', Q''') \in Rel$
by(*drule-tac weakSimE*) *auto*
from $\langle Q'''' \Longrightarrow_{\tau} Q'' \rangle \langle (P''', Q''') \in Rel \rangle Sim$ **obtain** P''' **where** $P''''Chain: P'''' \Longrightarrow_{\tau} P'''$

```

 $P'''' \Rightarrow_{\tau} P'''$  and  $(P''', Q''') \in Rel$ 
  by(rule simTauChain)
  from  $\langle P''', Q'''\rangle \in Rel$  have  $P''' \rightsquigarrow \hat{\langle Rel \rangle} Q'''$  by(rule Sim)
  then obtain  $P''$  where  $P''''Trans: P''' \Rightarrow \hat{\alpha} \prec P''$  and  $(P'', Q'') \in Rel$ 
using  $Q''''Trans$  by(rule Weak-Sim.weakSimE)
  from  $Q''Chain \langle P'', Q''\rangle \in Rel$  Sim obtain  $P'$  where  $P''Chain: P'' \Rightarrow_{\tau} P'$ 
and  $(P', Q') \in Rel$ 
  by(rule simTauChain)
  from  $PChain P''''Chain P''''Trans P''Chain$ 
  have  $P \Rightarrow_{\alpha} \prec P'$ 
  apply(auto simp add: weakCongTrans-def weakTrans-def)
  apply(rule-tac x= $P''aa$  in exI)
  apply auto
  defer
  apply blast
  by(auto simp add: tauChain-def)

  with  $\langle P', Q'\rangle \in Rel$  show ?thesis
  by(force intro: Goal simp add: weakCongTrans-def weakTrans-def)
qed
qed

```

lemma reflexive:

```

fixes  $P :: ccs$ 
and  $Rel :: (ccs \times ccs) set$ 

```

```

assumes  $Id \subseteq Rel$ 

```

```

shows  $P \rightsquigarrow \langle Rel \rangle P$ 

```

```

using assms

```

```

by(auto simp add: weakCongSimulation-def intro: transitionWeakCongTransition)

```

lemma transitive:

```

fixes  $P :: ccs$ 
and  $Rel :: (ccs \times ccs) set$ 
and  $Q :: ccs$ 
and  $Rel' :: (ccs \times ccs) set$ 
and  $R :: ccs$ 
and  $Rel'' :: (ccs \times ccs) set$ 

```

```

assumes  $P \rightsquigarrow \langle Rel \rangle Q$ 

```

```

and  $Q \rightsquigarrow \langle Rel' \rangle R$ 

```

```

and  $Rel \circ Rel' \subseteq Rel''$ 

```

```

and  $\bigwedge S T. (S, T) \in Rel \implies S \rightsquigarrow \hat{\langle Rel \rangle} T$ 

```

```

shows  $P \rightsquigarrow \langle Rel'' \rangle R$ 

```

```

proof(induct rule: weakSimI)

```

```

case(Sim  $\alpha R'$ )

```

```

thus ?case using assms

```

```

  apply(drule-tac Q=R in weakSimE, auto)
  by(drule-tac Q=Q in simE2) auto
qed

```

lemma *weakMonotonic*:

```

  fixes P :: ccs
  and A :: (ccs × ccs) set
  and Q :: ccs
  and B :: (ccs × ccs) set

```

```

  assumes P ~<A> Q
  and A ⊆ B

```

```

  shows P ~<B> Q

```

```

using assms

```

```

by(fastforce simp add: weakCongSimulation-def)

```

end

theory *Strong-Sim-SC*

```

  imports Strong-Sim

```

begin

lemma *resNilLeft*:

```

  fixes x :: name

```

```

  shows (νx)0 ~>[Rel] 0

```

```

by(auto simp add: simulation-def)

```

lemma *resNilRight*:

```

  fixes x :: name

```

```

  shows 0 ~>[Rel] (νx)0

```

```

by(auto simp add: simulation-def elim: resCases)

```

lemma *test[simp]*:

```

  fixes x :: name

```

```

  and P :: ccs

```

```

  shows x ‡ [x].P

```

```

by(auto simp add: abs-fresh)

```

lemma *scopeExtSumLeft*:

```

  fixes x :: name

```

```

  and P :: ccs

```

```

  and Q :: ccs

```

```

  assumes x ‡ P

```

```

  and C1: ∧y R. y ‡ R ⇒ ((νy)R, R) ∈ Rel

```

and $Id \subseteq Rel$

shows $(\nu x)(P \oplus Q) \rightsquigarrow[Rel] P \oplus (\nu x)Q$
using *assms*
apply(*auto simp add: simulation-def*)
by(*elim sumCases resCases*) (*blast intro: Res Sum1 Sum2 C1 dest: freshDerivative*)+

lemma *scopeExtSumRight*:
fixes $x :: name$
and $P :: ccs$
and $Q :: ccs$

assumes $x \# P$
and $C1: \bigwedge y R. y \# R \implies (R, (\nu y)R) \in Rel$
and $Id \subseteq Rel$

shows $P \oplus (\nu x)Q \rightsquigarrow[Rel] (\nu x)(P \oplus Q)$
using *assms*
apply(*auto simp add: simulation-def*)
by(*elim sumCases resCases*) (*blast intro: Res Sum1 Sum2 C1 dest: freshDerivative*)+

lemma *scopeExtLeft*:
fixes $x :: name$
and $P :: ccs$
and $Q :: ccs$

assumes $x \# P$
and $C1: \bigwedge y R T. y \# R \implies ((\nu y)(R \parallel T), R \parallel (\nu y)T) \in Rel$

shows $(\nu x)(P \parallel Q) \rightsquigarrow[Rel] P \parallel (\nu x)Q$
using *assms*
by(*fastforce elim: parCases resCases intro: Res C1 Par1 Par2 Comm dest: freshDerivative simp add: simulation-def*)

lemma *scopeExtRight*:
fixes $x :: name$
and $P :: ccs$
and $Q :: ccs$

assumes $x \# P$
and $C1: \bigwedge y R T. y \# R \implies (R \parallel (\nu y)T, (\nu y)(R \parallel T)) \in Rel$

shows $P \parallel (\nu x)Q \rightsquigarrow[Rel] (\nu x)(P \parallel Q)$
using *assms*
by(*fastforce elim: parCases resCases intro: Res C1 Par1 Par2 Comm dest: freshDerivative simp add: simulation-def*)

```

lemma sumComm:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $Id \subseteq Rel$ 

  shows  $P \oplus Q \rightsquigarrow[Rel] Q \oplus P$ 
using assms
by(force simp add: simulation-def elim: sumCases intro: Sum1 Sum2)

lemma sumAssocLeft:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

  assumes  $Id \subseteq Rel$ 

  shows  $(P \oplus Q) \oplus R \rightsquigarrow[Rel] P \oplus (Q \oplus R)$ 
using assms
by(force simp add: simulation-def elim: sumCases intro: Sum1 Sum2)

lemma sumAssocRight:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

  assumes  $Id \subseteq Rel$ 

  shows  $P \oplus (Q \oplus R) \rightsquigarrow[Rel] (P \oplus Q) \oplus R$ 
using assms
by(intro simI, elim sumCases) (blast intro: Sum1 Sum2)+

lemma sumIdLeft:
  fixes  $P :: ccs$ 
  and  $Rel :: (ccs \times ccs) \text{ set}$ 

  assumes  $Id \subseteq Rel$ 

  shows  $P \oplus \mathbf{0} \rightsquigarrow[Rel] P$ 
using assms
by(auto simp add: simulation-def intro: Sum1)

lemma sumIdRight:
  fixes  $P :: ccs$ 
  and  $Rel :: (ccs \times ccs) \text{ set}$ 

  assumes  $Id \subseteq Rel$ 

  shows  $P \rightsquigarrow[Rel] P \oplus \mathbf{0}$ 

```

```

using assms
by(fastforce simp add: simulation-def elim: sumCases)

lemma parComm:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $C1: \bigwedge R T. (R \parallel T, T \parallel R) \in Rel$ 

  shows  $P \parallel Q \rightsquigarrow[Rel] Q \parallel P$ 
by(fastforce simp add: simulation-def elim: parCases intro: Par1 Par2 Comm C1)

lemma parAssocLeft:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

  assumes  $C1: \bigwedge S T U. ((S \parallel T) \parallel U, S \parallel (T \parallel U)) \in Rel$ 

  shows  $(P \parallel Q) \parallel R \rightsquigarrow[Rel] P \parallel (Q \parallel R)$ 
by(fastforce simp add: simulation-def elim: parCases intro: Par1 Par2 Comm C1)

lemma parAssocRight:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

  assumes  $C1: \bigwedge S T U. (S \parallel (T \parallel U), (S \parallel T) \parallel U) \in Rel$ 

  shows  $P \parallel (Q \parallel R) \rightsquigarrow[Rel] (P \parallel Q) \parallel R$ 
by(fastforce simp add: simulation-def elim: parCases intro: Par1 Par2 Comm C1)

lemma parIdLeft:
  fixes  $P :: ccs$ 
  and  $Rel :: (ccs \times ccs) set$ 

  assumes  $\bigwedge Q. (Q \parallel \mathbf{0}, Q) \in Rel$ 

  shows  $P \parallel \mathbf{0} \rightsquigarrow[Rel] P$ 
using assms
by(auto simp add: simulation-def intro: Par1)

lemma parIdRight:
  fixes  $P :: ccs$ 
  and  $Rel :: (ccs \times ccs) set$ 

  assumes  $\bigwedge Q. (Q, Q \parallel \mathbf{0}) \in Rel$ 

  shows  $P \rightsquigarrow[Rel] P \parallel \mathbf{0}$ 

```

```

using assms
by(fastforce simp add: simulation-def elim: parCases)

declare fresh-atm[simp]

lemma resActLeft:
  fixes x :: name
  and α :: act
  and P :: ccs

  assumes x # α
  and Id ⊆ Rel

  shows (νx)(α.P) ~>[Rel] (α.(νx)P)
using assms
by(fastforce simp add: simulation-def elim: actCases intro: Res Action)

lemma resActRight:
  fixes x :: name
  and α :: act
  and P :: ccs

  assumes x # α
  and Id ⊆ Rel

  shows α.(νx)P ~>[Rel] (νx)(α.P)
using assms
by(fastforce simp add: simulation-def elim: resCases actCases intro: Action)

lemma resComm:
  fixes x :: name
  and y :: name
  and P :: ccs

  assumes  $\bigwedge Q. ((\nu x)((\nu y) Q), (\nu y)((\nu x) Q)) \in Rel$ 

  shows (νx)((νy)P) ~>[Rel] (νy)(νx)P
using assms
by(fastforce simp add: simulation-def elim: resCases intro: Res)

inductive-cases bangCases[simplified ccs.distinct act.distinct]:  $!P \mapsto \alpha \prec P'$ 

lemma bangUnfoldLeft:
  fixes P :: ccs

  assumes Id ⊆ Rel

  shows P || !P ~>[Rel] !P
using assms

```

by(*fastforce simp add: simulation-def ccs.inject elim: bangCases*)

lemma *bangUnfoldRight*:

fixes $P :: ccs$

assumes $Id \subseteq Rel$

shows $!P \rightsquigarrow[Rel] P \parallel !P$

using *assms*

by(*fastforce simp add: simulation-def ccs.inject intro: Bang*)

end

theory *Strong-Bisim*

imports *Strong-Sim*

begin

lemma *monotonic*:

fixes $P :: ccs$

and $A :: (ccs \times ccs) \text{ set}$

and $Q :: ccs$

and $B :: (ccs \times ccs) \text{ set}$

assumes $P \rightsquigarrow[A] Q$

and $A \subseteq B$

shows $P \rightsquigarrow[B] Q$

using *assms*

by(*fastforce simp add: simulation-def*)

lemma *monoCoinduct*: $\bigwedge x y xa xb P Q.$

$x \leq y \implies$

$(Q \rightsquigarrow[\{(xb, xa). x xb xa\}] P) \longrightarrow$

$(Q \rightsquigarrow[\{(xb, xa). y xb xa\}] P)$

apply *auto*

apply(*rule monotonic*)

by(*auto dest: le-funE*)

coinductive-set *bisim* :: $(ccs \times ccs) \text{ set}$

where

$\llbracket P \rightsquigarrow[bisim] Q; (Q, P) \in bisim \rrbracket \implies (P, Q) \in bisim$

monos *monoCoinduct*

abbreviation

bisimJudge ($\leftarrow \sim \rightarrow [70, 70] 65$) **where** $P \sim Q \equiv (P, Q) \in bisim$

lemma *bisimCoinductAux*[*consumes 1*]:

fixes $P :: ccs$

and $Q :: ccs$

and $X :: (ccs \times ccs) \text{ set}$
assumes $(P, Q) \in X$
and $\bigwedge P Q. (P, Q) \in X \implies P \rightsquigarrow[(X \cup \text{bisim})] Q \wedge (Q, P) \in X$
shows $P \sim Q$
proof –
have $X \cup \text{bisim} = \{(P, Q). (P, Q) \in X \vee (P, Q) \in \text{bisim}\}$ **by auto**
with *assms* **show** *?thesis*
by *coinduct simp*
qed

lemma *bisimCoinduct*[*consumes 1, case-names cSim cSym*]:
fixes $P :: ccs$
and $Q :: ccs$
and $X :: (ccs \times ccs) \text{ set}$

assumes $(P, Q) \in X$
and $\bigwedge R S. (R, S) \in X \implies R \rightsquigarrow[(X \cup \text{bisim})] S$
and $\bigwedge R S. (R, S) \in X \implies (S, R) \in X$

shows $P \sim Q$
proof –
have $X \cup \text{bisim} = \{(P, Q). (P, Q) \in X \vee (P, Q) \in \text{bisim}\}$ **by auto**
with *assms* **show** *?thesis*
by *coinduct simp*
qed

lemma *bisimWeakCoinductAux*[*consumes 1*]:
fixes $P :: ccs$
and $Q :: ccs$
and $X :: (ccs \times ccs) \text{ set}$

assumes $(P, Q) \in X$
and $\bigwedge R S. (R, S) \in X \implies R \rightsquigarrow[X] S \wedge (S, R) \in X$

shows $P \sim Q$
using *assms*
by(*coinduct rule: bisimCoinductAux*) (*blast intro: monotonic*)

lemma *bisimWeakCoinduct*[*consumes 1, case-names cSim cSym*]:
fixes $P :: ccs$
and $Q :: ccs$
and $X :: (ccs \times ccs) \text{ set}$

assumes $(P, Q) \in X$
and $\bigwedge P Q. (P, Q) \in X \implies P \rightsquigarrow[X] Q$
and $\bigwedge P Q. (P, Q) \in X \implies (Q, P) \in X$

shows $P \sim Q$
proof –
have $X \cup bisim = \{(P, Q). (P, Q) \in X \vee (P, Q) \in bisim\}$ **by** *auto*
with *assms* **show** *?thesis*
by(*coinduct rule: bisimCoinduct*) (*blast intro: monotonic*)+
qed

lemma *bisimE*:
fixes $P :: ccs$
and $Q :: ccs$

assumes $P \sim Q$

shows $P \rightsquigarrow[bisim] Q$
and $Q \sim P$
using *assms*
by(*auto simp add: intro: bisim.cases*)

lemma *bisimI*:
fixes $P :: ccs$
and $Q :: ccs$

assumes $P \rightsquigarrow[bisim] Q$
and $Q \sim P$

shows $P \sim Q$
using *assms*
by(*auto intro: bisim.intros*)

lemma *reflexive*:
fixes $P :: ccs$

shows $P \sim P$
proof –
have $(P, P) \in Id$ **by** *blast*
thus *?thesis*
by(*coinduct rule: bisimCoinduct*) (*auto intro: reflexive*)
qed

lemma *symmetric*:
fixes $P :: ccs$
and $Q :: ccs$

assumes $P \sim Q$

shows $Q \sim P$
using *assms*
by(*rule bisimE*)

```

lemma transitive:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

  assumes  $P \sim Q$ 
  and  $Q \sim R$ 

  shows  $P \sim R$ 
proof –
  from assms have  $(P, R) \in \text{bisim } O \text{ bisim}$  by auto
  thus ?thesis
  by(coinduct rule: bisimCoinduct) (auto intro: transitive dest: bisimE)
qed

```

```

lemma bisimTransCoinduct[consumes 1, case-names cSim cSym]:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $(P, Q) \in X$ 
  and  $rSim: \bigwedge R S. (R, S) \in X \implies R \rightsquigarrow[(\text{bisim } O X O \text{ bisim})] S$ 
  and  $rSym: \bigwedge R S. (R, S) \in X \implies (S, R) \in X$ 

  shows  $P \sim Q$ 
proof –
  from  $\langle (P, Q) \in X \rangle$  have  $(P, Q) \in \text{bisim } O X O \text{ bisim}$ 
  by(auto intro: reflexive)
  thus ?thesis
proof(coinduct rule: bisimWeakCoinduct)
  case(cSim P Q)
  from  $\langle (P, Q) \in \text{bisim } O X O \text{ bisim} \rangle$ 
  obtain  $R S$  where  $P \sim R$  and  $(R, S) \in X$  and  $S \sim Q$ 
  by auto
  from  $\langle P \sim R \rangle$  have  $P \rightsquigarrow[\text{bisim}] R$  by(rule bisimE)
  moreover from  $\langle (R, S) \in X \rangle$  have  $R \rightsquigarrow[(\text{bisim } O X O \text{ bisim})] S$ 
  by(rule rSim)
  moreover have  $\text{bisim } O (\text{bisim } O X O \text{ bisim}) \subseteq \text{bisim } O X O \text{ bisim}$ 
  by(auto intro: transitive)
  ultimately have  $P \rightsquigarrow[(\text{bisim } O X O \text{ bisim})] S$ 
  by(rule Strong-Sim.transitive)
  moreover from  $\langle S \sim Q \rangle$  have  $S \rightsquigarrow[\text{bisim}] Q$  by(rule bisimE)
  moreover have  $(\text{bisim } O X O \text{ bisim}) O \text{ bisim} \subseteq \text{bisim } O X O \text{ bisim}$ 
  by(auto intro: transitive)
  ultimately show ?case by(rule Strong-Sim.transitive)
next
  case(cSym P Q)
  thus ?case by(auto dest: symmetric rSym)
qed
qed

```

end

theory *Strong-Sim-Pres*
 imports *Strong-Sim*
begin

lemma *actPres*:

fixes $P :: ccs$
 and $Q :: ccs$
 and $Rel :: (ccs \times ccs) \text{ set}$
 and $a :: name$
 and $Rel' :: (ccs \times ccs) \text{ set}$

assumes $(P, Q) \in Rel$

shows $\alpha.(P) \rightsquigarrow_{[Rel]} \alpha.(Q)$

using *assms*

by(*fastforce simp add: simulation-def elim: actCases intro: Action*)

lemma *sumPres*:

fixes $P :: ccs$
 and $Q :: ccs$
 and $Rel :: (ccs \times ccs) \text{ set}$

assumes $P \rightsquigarrow_{[Rel]} Q$

and $Rel \subseteq Rel'$

and $Id \subseteq Rel'$

shows $P \oplus R \rightsquigarrow_{[Rel']} Q \oplus R$

using *assms*

by(*force simp add: simulation-def elim: sumCases intro: Sum1 Sum2*)

lemma *parPresAux*:

fixes $P :: ccs$
 and $Q :: ccs$
 and $Rel :: (ccs \times ccs) \text{ set}$

assumes $P \rightsquigarrow_{[Rel]} Q$

and $(P, Q) \in Rel$

and $R \rightsquigarrow_{[Rel']} T$

and $(R, T) \in Rel'$

and $C1: \bigwedge P' Q' R' T'. \llbracket (P', Q') \in Rel; (R', T') \in Rel' \rrbracket \implies (P' \parallel R', Q' \parallel T') \in Rel''$

shows $P \parallel R \rightsquigarrow_{[Rel'']} Q \parallel T$

proof(*induct rule: simI*)

case(*Sim a QT*)

from $\langle Q \parallel T \mapsto a \prec QT \rangle$

show *?case*
proof(*induct rule: parCases*)
 case(*cPar1 Q'*)
 from $\langle P \rightsquigarrow[Rel] Q \rangle \langle Q \mapsto a \prec Q' \rangle$ **obtain** P' **where** $P \mapsto a \prec P'$ **and** $(P', Q') \in Rel$
 by(*rule simE*)
 from $\langle P \mapsto a \prec P' \rangle$ **have** $P \parallel R \mapsto a \prec P' \parallel R$ **by**(*rule Par1*)
 moreover from $\langle (P', Q') \in Rel \rangle \langle (R, T) \in Rel' \rangle$ **have** $(P' \parallel R, Q' \parallel T) \in Rel''$ **by**(*rule C1*)
 ultimately show *?case by blast*
next
 case(*cPar2 T'*)
 from $\langle R \rightsquigarrow[Rel'] T \rangle \langle T \mapsto a \prec T' \rangle$ **obtain** R' **where** $R \mapsto a \prec R'$ **and** $(R', T') \in Rel'$
 by(*rule simE*)
 from $\langle R \mapsto a \prec R' \rangle$ **have** $P \parallel R \mapsto a \prec P \parallel R'$ **by**(*rule Par2*)
 moreover from $\langle (P, Q) \in Rel \rangle \langle (R', T') \in Rel' \rangle$ **have** $(P \parallel R', Q \parallel T') \in Rel''$ **by**(*rule C1*)
 ultimately show *?case by blast*
next
 case(*cComm Q' T' a*)
 from $\langle P \rightsquigarrow[Rel] Q \rangle \langle Q \mapsto a \prec Q' \rangle$ **obtain** P' **where** $P \mapsto a \prec P'$ **and** $(P', Q') \in Rel$
 by(*rule simE*)
 from $\langle R \rightsquigarrow[Rel'] T \rangle \langle T \mapsto (coAction\ a) \prec T' \rangle$ **obtain** R' **where** $R \mapsto (coAction\ a) \prec R'$ **and** $(R', T') \in Rel'$
 by(*rule simE*)
 from $\langle P \mapsto a \prec P' \rangle \langle R \mapsto (coAction\ a) \prec R' \rangle \langle a \neq \tau \rangle$ **have** $P \parallel R \mapsto \tau \prec P' \parallel R'$ **by**(*rule Comm*)
 moreover from $\langle (P', Q') \in Rel \rangle \langle (R', T') \in Rel' \rangle$ **have** $(P' \parallel R', Q' \parallel T') \in Rel''$ **by**(*rule C1*)
 ultimately show *?case by blast*
qed
qed

lemma *parPres:*

fixes $P :: ccs$
and $Q :: ccs$
and $Rel :: (ccs \times ccs)\ set$

assumes $P \rightsquigarrow[Rel] Q$
and $(P, Q) \in Rel$
and $C1: \bigwedge S\ T\ U. (S, T) \in Rel \implies (S \parallel U, T \parallel U) \in Rel'$

shows $P \parallel R \rightsquigarrow[Rel'] Q \parallel R$

using *assms*

by(*rule-tac parPresAux[where Rel''=Rel' and Rel'=Id]*) (*auto intro: reflexive*)

lemma *resPres:*

```

fixes  $P :: ccs$ 
and  $Rel :: (ccs \times ccs) \text{ set}$ 
and  $Q :: ccs$ 
and  $x :: name$ 

assumes  $P \rightsquigarrow_{[Rel]} Q$ 
and  $\bigwedge R S y. (R, S) \in Rel \implies ((\nu y)R, (\nu y)S) \in Rel'$ 

shows  $(\nu x)P \rightsquigarrow_{[Rel']} (\nu x)Q$ 
using assms
by(fastforce simp add: simulation-def elim: resCases intro: Res)

lemma bangPres:
fixes  $P :: ccs$ 
and  $Rel :: (ccs \times ccs) \text{ set}$ 
and  $Q :: ccs$ 

assumes  $(P, Q) \in Rel$ 
and  $C1: \bigwedge R S. (R, S) \in Rel \implies R \rightsquigarrow_{[Rel]} S$ 

shows  $!P \rightsquigarrow_{[bangRel Rel]} !Q$ 
proof(induct rule: simI)
case(Sim  $\alpha$   $Q'$ )
{
  fix  $Pa \alpha Q'$ 
  assume  $!Q \mapsto_{\alpha} \prec Q'$  and  $(Pa, !Q) \in bangRel Rel$ 
  hence  $\exists P'. Pa \mapsto_{\alpha} \prec P' \wedge (P', Q') \in bangRel Rel$ 
  proof(nominal-induct arbitrary: Pa rule: bangInduct)
  case(cPar1  $\alpha$   $Q'$ )
  from  $\langle (Pa, Q \parallel !Q) \in bangRel Rel \rangle$ 
  show ?case
  proof(induct rule: BRParCases)
  case(BRPar  $P R$ )
  from  $\langle (P, Q) \in Rel \rangle$  have  $P \rightsquigarrow_{[Rel]} Q$  by(rule C1)
  with  $\langle Q \mapsto_{\alpha} \prec Q' \rangle$  obtain  $P'$  where  $P \mapsto_{\alpha} \prec P'$  and  $(P', Q') \in Rel$ 
  by(blast dest: simE)
  from  $\langle P \mapsto_{\alpha} \prec P' \rangle$  have  $P \parallel R \mapsto_{\alpha} \prec P' \parallel R$  by(rule Par1)
  moreover from  $\langle (P', Q') \in Rel \rangle \langle (R, !Q) \in bangRel Rel \rangle$  have  $(P' \parallel R,$ 
 $Q' \parallel !Q) \in bangRel Rel$ 
  by(rule bangRel.BRPar)
  ultimately show ?case by blast
  qed
next
  case(cPar2  $\alpha$   $Q'$ )
  from  $\langle (Pa, Q \parallel !Q) \in bangRel Rel \rangle$ 
  show ?case
  proof(induct rule: BRParCases)
  case(BRPar  $P R$ )
  from  $\langle (R, !Q) \in bangRel Rel \rangle$  obtain  $R'$  where  $R \mapsto_{\alpha} \prec R'$  and  $(R',$ 

```

```

Q') ∈ bangRel Rel using cPar2
  by blast
  from  $\langle R \mapsto \alpha \prec R' \rangle$  have  $P \parallel R \mapsto \alpha \prec P \parallel R'$  by(rule Par2)
  moreover from  $\langle (P, Q) \in \text{Rel} \rangle \langle (R', Q') \in \text{bangRel Rel} \rangle$  have  $(P \parallel R', Q$ 
 $\parallel Q') \in \text{bangRel Rel}$  by(rule bangRel.BRPar)
  ultimately show ?case by blast
qed
next
case(cComm a Q' Q'' Pa)
from  $\langle (Pa, Q \parallel !Q) \in \text{bangRel Rel} \rangle$ 
show ?case
proof(induct rule: BRParCases)
  case(BRPar P R)
  from  $\langle (P, Q) \in \text{Rel} \rangle$  have  $P \rightsquigarrow[\text{Rel}] Q$  by(rule C1)
  with  $\langle Q \mapsto a \prec Q' \rangle$  obtain  $P'$  where  $P \mapsto a \prec P'$  and  $(P', Q') \in \text{Rel}$ 
  by(blast dest: simE)
  from  $\langle (R, !Q) \in \text{bangRel Rel} \rangle$  obtain  $R'$  where  $R \mapsto (\text{coAction } a) \prec R'$ 
and  $(R', Q'') \in \text{bangRel Rel}$  using cComm
  by blast
  from  $\langle P \mapsto a \prec P' \rangle \langle R \mapsto (\text{coAction } a) \prec R' \rangle \langle a \neq \tau \rangle$  have  $P \parallel R \mapsto \tau$ 
 $\prec P' \parallel R'$  by(rule Comm)
  moreover from  $\langle (P', Q') \in \text{Rel} \rangle \langle (R', Q'') \in \text{bangRel Rel} \rangle$  have  $(P' \parallel R',$ 
 $Q' \parallel Q'') \in \text{bangRel Rel}$  by(rule bangRel.BRPar)
  ultimately show ?case by blast
qed
next
case(cBang α Q' Pa)
from  $\langle (Pa, !Q) \in \text{bangRel Rel} \rangle$ 
show ?case
proof(induct rule: BRBangCases)
  case(BRBang P)
  from  $\langle (P, Q) \in \text{Rel} \rangle$  have  $(!P, !Q) \in \text{bangRel Rel}$  by(rule bangRel.BRBang)
  with  $\langle (P, Q) \in \text{Rel} \rangle$  have  $(P \parallel !P, Q \parallel !Q) \in \text{bangRel Rel}$  by(rule
bangRel.BRPar)
  then obtain  $P'$  where  $P \parallel !P \mapsto \alpha \prec P'$  and  $(P', Q') \in \text{bangRel Rel}$ 
using cBang
  by blast
  from  $\langle P \parallel !P \mapsto \alpha \prec P' \rangle$  have  $!P \mapsto \alpha \prec P'$  by(rule Bang)
  thus ?case using  $\langle (P', Q') \in \text{bangRel Rel} \rangle$  by blast
qed
qed
}

moreover from  $\langle (P, Q) \in \text{Rel} \rangle$  have  $(!P, !Q) \in \text{bangRel Rel}$  by(rule BRBang)

ultimately show ?case using  $\langle !Q \mapsto \alpha \prec Q' \rangle$  by blast
qed
end

```

```

theory Strong-Bisim-Pres
  imports Strong-Bisim Strong-Sim-Pres
begin

lemma actPres:
  fixes  $P :: \text{ccs}$ 
  and  $Q :: \text{ccs}$ 
  and  $\alpha :: \text{act}$ 

  assumes  $P \sim Q$ 

  shows  $\alpha.(P) \sim \alpha.(Q)$ 
proof –
  let  $?X = \{(\alpha.(P), \alpha.(Q)) \mid P \ Q. P \sim Q\}$ 
  from assms have  $(\alpha.(P), \alpha.(Q)) \in ?X$  by auto
  thus ?thesis
  by(coinduct rule: bisimCoinduct) (auto dest: bisimE intro: actPres)
qed

lemma sumPres:
  fixes  $P :: \text{ccs}$ 
  and  $Q :: \text{ccs}$ 
  and  $R :: \text{ccs}$ 

  assumes  $P \sim Q$ 

  shows  $P \oplus R \sim Q \oplus R$ 
proof –
  let  $?X = \{(P \oplus R, Q \oplus R) \mid P \ Q \ R. P \sim Q\}$ 
  from assms have  $(P \oplus R, Q \oplus R) \in ?X$  by auto
  thus ?thesis
  by(coinduct rule: bisimCoinduct) (auto intro: sumPres reflexive dest: bisimE)
qed

lemma parPres:
  fixes  $P :: \text{ccs}$ 
  and  $Q :: \text{ccs}$ 
  and  $R :: \text{ccs}$ 

  assumes  $P \sim Q$ 

  shows  $P \parallel R \sim Q \parallel R$ 
proof –
  let  $?X = \{(P \parallel R, Q \parallel R) \mid P \ Q \ R. P \sim Q\}$ 
  from assms have  $(P \parallel R, Q \parallel R) \in ?X$  by blast
  thus ?thesis
  by(coinduct rule: bisimCoinduct, auto) (blast intro: parPres dest: bisimE)+
qed

```

```

lemma resPres:
  fixes P :: ccs
  and Q :: ccs
  and x :: name

  assumes P ~ Q

  shows  $(\nu x)P \sim (\nu x)Q$ 
proof -
  let ?X =  $\{((\nu x)P, (\nu x)Q) \mid x P Q. P \sim Q\}$ 
  from assms have  $((\nu x)P, (\nu x)Q) \in ?X$  by auto
  thus ?thesis
    by(coinduct rule: bisimCoinduct) (auto intro: resPres dest: bisimE)
qed

lemma bangPres:
  fixes P :: ccs
  and Q :: ccs

  assumes P ~ Q

  shows  $!P \sim !Q$ 
proof -
  from assms have  $(!P, !Q) \in \text{bangRel bisim}$ 
  by(auto intro: BRBang)
  thus ?thesis
  proof(coinduct rule: bisimWeakCoinduct)
    case(cSim P Q)
    from  $\langle P, Q \rangle \in \text{bangRel bisim}$  show ?case
    proof(induct)
      case(BRBang P Q)
      note  $\langle P \sim Q \rangle \text{ bisimE}(1)$ 
      thus  $!P \rightsquigarrow[\text{bangRel bisim}] !Q$  by(rule bangPres)
    next
      case(BRPar R T P Q)
      from  $\langle R \sim T \rangle$  have  $R \rightsquigarrow[\text{bisim}] T$  by(rule bisimE)
      moreover note  $\langle R \sim T \rangle \langle P \rightsquigarrow[\text{bangRel bisim}] Q \rangle \langle P, Q \rangle \in \text{bangRel bisim}$ 
      ultimately show ?case by(rule Strong-Sim-Pres.parPresAux)
    qed
  next
    case(cSym P Q)
    thus ?case
    by induct (auto dest: bisimE intro: BRPar BRBang)
  qed
qed

end

```

```

theory Struct-Cong
  imports Agent
begin

inductive structCong :: ccs  $\Rightarrow$  ccs  $\Rightarrow$  bool ( $\lambda \cdot \equiv_s \cdot$ )
  where
    Refl:  $P \equiv_s P$ 
  | Sym:  $P \equiv_s Q \Longrightarrow Q \equiv_s P$ 
  | Trans:  $\llbracket P \equiv_s Q; Q \equiv_s R \rrbracket \Longrightarrow P \equiv_s R$ 

  | ParComm:  $P \parallel Q \equiv_s Q \parallel P$ 
  | ParAssoc:  $(P \parallel Q) \parallel R \equiv_s P \parallel (Q \parallel R)$ 
  | ParId:  $P \parallel \mathbf{0} \equiv_s P$ 

  | SumComm:  $P \oplus Q \equiv_s Q \oplus P$ 
  | SumAssoc:  $(P \oplus Q) \oplus R \equiv_s P \oplus (Q \oplus R)$ 
  | SumId:  $P \oplus \mathbf{0} \equiv_s P$ 

  | ResNil:  $(\nu x)\mathbf{0} \equiv_s \mathbf{0}$ 
  | ScopeExtPar:  $x \# P \Longrightarrow (\nu x)(P \parallel Q) \equiv_s P \parallel (\nu x)Q$ 
  | ScopeExtSum:  $x \# P \Longrightarrow (\nu x)(P \oplus Q) \equiv_s P \oplus (\nu x)Q$ 
  | ScopeAct:  $x \# \alpha \Longrightarrow (\nu x)(\alpha.(P)) \equiv_s \alpha.((\nu x)P)$ 
  | ScopeCommAux:  $x \neq y \Longrightarrow (\nu x)((\nu y)P) \equiv_s (\nu y)((\nu x)P)$ 

  | BangUnfold:  $!P \equiv_s P \parallel !P$ 
equivariance structCong
nominal-inductive structCong
by(auto simp add: abs-fresh)

lemma ScopeComm:
  fixes  $x :: \textit{name}$ 
  and  $y :: \textit{name}$ 
  and  $P :: \textit{ccs}$ 

  shows  $(\nu x)((\nu y)P) \equiv_s (\nu y)((\nu x)P)$ 
by(cases x=y) (auto intro: Refl ScopeCommAux)

end

theory Strong-Bisim-SC
  imports Strong-Sim-SC Strong-Bisim-Pres Struct-Cong
begin

lemma resNil:
  fixes  $x :: \textit{name}$ 

  shows  $(\nu x)\mathbf{0} \sim \mathbf{0}$ 
proof –

```

have $(\nu x)0, 0 \in \{(\nu x)0, 0, 0, (\nu x)0\}$ **by** *simp*
thus *?thesis*
by(*coinduct rule: bisimCoinduct*)
(*auto intro: resNilLeft resNilRight*)
qed

lemma *scopeExt*:

fixes $x :: \text{name}$
and $P :: \text{ccs}$
and $Q :: \text{ccs}$

assumes $x \# P$

shows $(\nu x)(P \parallel Q) \sim P \parallel (\nu x)Q$

proof –

let $?X = \{(\nu x)(P \parallel Q), P \parallel (\nu x)Q \mid x P Q. x \# P\} \cup \{(P \parallel (\nu x)Q), (\nu x)(P \parallel Q) \mid x P Q. x \# P\}$
from *assms* **have** $(\nu x)(P \parallel Q), P \parallel (\nu x)Q \in ?X$ **by** *auto*
thus *?thesis*
by(*coinduct rule: bisimCoinduct*) (*force intro: scopeExtLeft scopeExtRight*)+
qed

lemma *sumComm*:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$

shows $P \oplus Q \sim Q \oplus P$

proof –

have $(P \oplus Q, Q \oplus P) \in \{(P \oplus Q, Q \oplus P), (Q \oplus P, P \oplus Q)\}$ **by** *simp*
thus *?thesis*
by(*coinduct rule: bisimCoinduct*) (*auto intro: sumComm reflexive*)
qed

lemma *sumAssoc*:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$
and $R :: \text{ccs}$

shows $(P \oplus Q) \oplus R \sim P \oplus (Q \oplus R)$

proof –

have $((P \oplus Q) \oplus R, P \oplus (Q \oplus R)) \in \{((P \oplus Q) \oplus R, P \oplus (Q \oplus R)), (P \oplus (Q \oplus R), (P \oplus Q) \oplus R)\}$ **by** *simp*
thus *?thesis*
by(*coinduct rule: bisimCoinduct*) (*auto intro: sumAssocLeft sumAssocRight reflexive*)
qed

lemma *sumId*:

fixes $P :: \text{ccs}$

shows $P \oplus \mathbf{0} \sim P$
proof –
have $(P \oplus \mathbf{0}, P) \in \{(P \oplus \mathbf{0}, P), (P, P \oplus \mathbf{0})\}$ **by** *simp*
thus *?thesis* **by**(*coinduct rule: bisimCoinduct*) (*auto intro: sumIdLeft sumIdRight reflexive*)
qed

lemma *parComm*:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$

shows $P \parallel Q \sim Q \parallel P$

proof –
have $(P \parallel Q, Q \parallel P) \in \{(P \parallel Q, Q \parallel P) \mid P \text{ Q. True}\} \cup \{(Q \parallel P, P \parallel Q) \mid P \text{ Q. True}\}$ **by** *auto*
thus *?thesis*
by(*coinduct rule: bisimCoinduct*) (*auto intro: parComm*)
qed

lemma *parAssoc*:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$
and $R :: \text{ccs}$

shows $(P \parallel Q) \parallel R \sim P \parallel (Q \parallel R)$

proof –
have $((P \parallel Q) \parallel R, P \parallel (Q \parallel R)) \in \{((P \parallel Q) \parallel R, P \parallel (Q \parallel R)) \mid P \text{ Q R. True}\} \cup \{(P \parallel (Q \parallel R), (P \parallel Q) \parallel R) \mid P \text{ Q R. True}\}$ **by** *auto*
thus *?thesis*
by(*coinduct rule: bisimCoinduct*) (*force intro: parAssocLeft parAssocRight*)
qed

lemma *parId*:

fixes $P :: \text{ccs}$

shows $P \parallel \mathbf{0} \sim P$

proof –
have $(P \parallel \mathbf{0}, P) \in \{(P \parallel \mathbf{0}, P) \mid P \text{ True}\} \cup \{(P, P \parallel \mathbf{0}) \mid P \text{ True}\}$ **by** *simp*
thus *?thesis* **by**(*coinduct rule: bisimCoinduct*) (*auto intro: parIdLeft parIdRight*)
qed

lemma *scopeFresh*:

fixes $x :: \text{name}$
and $P :: \text{ccs}$

assumes $x \# P$

shows $(\nu x)P \sim P$
proof –
have $(\nu x)P \sim (\nu x)P \parallel \mathbf{0}$ **by**(rule *parId*[*THEN symmetric*])
moreover have $(\nu x)P \parallel \mathbf{0} \sim \mathbf{0} \parallel (\nu x)P$ **by**(rule *parComm*)
moreover have $\mathbf{0} \parallel (\nu x)P \sim (\nu x)(\mathbf{0} \parallel P)$ **by**(rule *scopeExt*[*THEN symmetric*])
auto
moreover have $(\nu x)(\mathbf{0} \parallel P) \sim (\nu x)(P \parallel \mathbf{0})$ **by**(rule *resPres*[*OF parComm*])
moreover from $\langle x \# P \rangle$ **have** $(\nu x)(P \parallel \mathbf{0}) \sim P \parallel (\nu x)\mathbf{0}$ **by**(rule *scopeExt*)
moreover have $P \parallel (\nu x)\mathbf{0} \sim (\nu x)\mathbf{0} \parallel P$ **by**(rule *parComm*)
moreover have $(\nu x)\mathbf{0} \parallel P \sim \mathbf{0} \parallel P$ **by**(rule *parPres*[*OF resNil*])
moreover have $\mathbf{0} \parallel P \sim P \parallel \mathbf{0}$ **by**(rule *parComm*)
moreover have $P \parallel \mathbf{0} \sim P$ **by**(rule *parId*)
ultimately show *?thesis* **by**(*metis transitive*)
qed

lemma *scopeExtSum*:

fixes $x :: \text{name}$
and $P :: \text{ccs}$
and $Q :: \text{ccs}$

assumes $x \# P$

shows $(\nu x)(P \oplus Q) \sim P \oplus (\nu x)Q$

proof –

have $((\nu x)(P \oplus Q), P \oplus (\nu x)Q) \in \{((\nu x)(P \oplus Q), P \oplus (\nu x)Q), (P \oplus (\nu x)Q, (\nu x)(P \oplus Q))\}$

by *simp*

thus *?thesis* **using** $\langle x \# P \rangle$

by(*coinduct rule: bisimCoinduct*)

(*auto intro: scopeExtSumLeft scopeExtSumRight reflexive scopeFresh scopeFresh*[*THEN symmetric*])

qed

lemma *resAct*:

fixes $x :: \text{name}$
and $\alpha :: \text{act}$
and $P :: \text{ccs}$

assumes $x \# \alpha$

shows $(\nu x)(\alpha.(P)) \sim \alpha.((\nu x)P)$

proof –

have $((\nu x)(\alpha.(P)), \alpha.((\nu x)P)) \in \{((\nu x)(\alpha.(P)), \alpha.((\nu x)P)), (\alpha.((\nu x)P), (\nu x)(\alpha.(P)))\}$

by *simp*

thus *?thesis* **using** $\langle x \# \alpha \rangle$

by(*coinduct rule: bisimCoinduct*) (*auto intro: resActLeft resActRight reflexive*)

qed

lemma *resComm*:

```

fixes  $x :: name$ 
and  $y :: name$ 
and  $P :: ccs$ 

shows  $(\nu x)(\nu y)P \sim (\nu y)(\nu x)P$ 
proof –
  have  $(\nu x)(\nu y)P, (\nu y)(\nu x)P \in \{(\nu x)(\nu y)P, (\nu y)(\nu x)P\} \mid x y P.$ 
  True} by auto
  thus ?thesis
  by(coinduct rule: bisimCoinduct) (auto intro: resComm)
qed

lemma bangUnfold:
  fixes  $P$ 

  shows  $!P \sim P \parallel !P$ 
proof –
  have  $(!P, P \parallel !P) \in \{(!P, P \parallel !P), (P \parallel !P, !P)\}$  by auto
  thus ?thesis
  by(coinduct rule: bisimCoinduct) (auto intro: bangUnfoldLeft bangUnfoldRight reflexive)
qed

lemma bisimStructCong:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $P \equiv_s Q$ 

  shows  $P \sim Q$ 
using assms
apply(nominal-induct rule: Struct-Cong.strong-induct)
by(auto intro: reflexive symmetric transitive parComm parAssoc parId sumComm sumAssoc sumId resNil scopeExt scopeExtSum resAct resComm bangUnfold)

end

theory Weak-Bisim
  imports Weak-Sim Strong-Bisim-SC Struct-Cong
begin

lemma weakMonoCoinduct:  $\bigwedge x y xa xb P Q.$ 

$$x \leq y \implies$$


$$(Q \rightsquigarrow \langle \{(xb, xa). x xb xa \} \rangle P) \longrightarrow$$


$$(Q \rightsquigarrow \langle \{(xb, xa). y xb xa \} \rangle P)$$

apply auto
apply(rule weakMonotonic)
by(auto dest: le-funE)

```

coinductive-set *weakBisimulation* :: (ccs × ccs) set
where
 $\llbracket P \rightsquigarrow^{\wedge} \langle \text{weakBisimulation} \rangle Q; (Q, P) \in \text{weakBisimulation} \rrbracket \implies (P, Q) \in \text{weakBisimulation}$
monos *weakMonoCoinduct*

abbreviation

weakBisimJudge ($\leftarrow \approx \rightarrow$ [70, 70] 65) **where** $P \approx Q \equiv (P, Q) \in \text{weakBisimulation}$

lemma *weakBisimulationCoinductAux*[consumes 1]:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$
and $X :: (\text{ccs} \times \text{ccs}) \text{ set}$

assumes $(P, Q) \in X$
and $\bigwedge P Q. (P, Q) \in X \implies P \rightsquigarrow^{\wedge} \langle (X \cup \text{weakBisimulation}) \rangle Q \wedge (Q, P) \in X$

shows $P \approx Q$

proof –

have $X \cup \text{weakBisimulation} = \{(P, Q). (P, Q) \in X \vee (P, Q) \in \text{weakBisimulation}\}$ **by** *auto*

with *assms* **show** *?thesis*

by *coinduct simp*

qed

lemma *weakBisimulationCoinduct*[consumes 1, case-names *cSim cSym*]:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$
and $X :: (\text{ccs} \times \text{ccs}) \text{ set}$

assumes $(P, Q) \in X$

and $\bigwedge R S. (R, S) \in X \implies R \rightsquigarrow^{\wedge} \langle (X \cup \text{weakBisimulation}) \rangle S$

and $\bigwedge R S. (R, S) \in X \implies (S, R) \in X$

shows $P \approx Q$

proof –

have $X \cup \text{weakBisimulation} = \{(P, Q). (P, Q) \in X \vee (P, Q) \in \text{weakBisimulation}\}$ **by** *auto*

with *assms* **show** *?thesis*

by *coinduct simp*

qed

lemma *weakBisimWeakCoinductAux*[consumes 1]:

fixes $P :: \text{ccs}$
and $Q :: \text{ccs}$
and $X :: (\text{ccs} \times \text{ccs}) \text{ set}$

assumes $(P, Q) \in X$

```

and    $\bigwedge P Q. (P, Q) \in X \implies P \rightsquigarrow^{\langle X \rangle} Q \wedge (Q, P) \in X$ 

shows  $P \approx Q$ 
using assms
by(coinduct rule: weakBisimulationCoinductAux) (blast intro: weakMonotonic)

lemma weakBisimWeakCoinduct[consumes 1, case-names cSim cSym]:
  fixes  $P :: ccs$ 
  and    $Q :: ccs$ 
  and    $X :: (ccs \times ccs) \text{ set}$ 

  assumes  $(P, Q) \in X$ 
  and      $\bigwedge P Q. (P, Q) \in X \implies P \rightsquigarrow^{\langle X \rangle} Q$ 
  and      $\bigwedge P Q. (P, Q) \in X \implies (Q, P) \in X$ 

  shows  $P \approx Q$ 
proof -
  have  $X \cup \text{weakBisim} = \{(P, Q). (P, Q) \in X \vee (P, Q) \in \text{weakBisim}\}$  by auto
  with assms show ?thesis
  by(coinduct rule: weakBisimulationCoinduct) (blast intro: weakMonotonic)+
qed

lemma weakBisimulationE:
  fixes  $P :: ccs$ 
  and    $Q :: ccs$ 

  assumes  $P \approx Q$ 

  shows  $P \rightsquigarrow^{\langle \text{weakBisimulation} \rangle} Q$ 
  and    $Q \approx P$ 
using assms
by(auto simp add: intro: weakBisimulation.cases)

lemma weakBisimulationI:
  fixes  $P :: ccs$ 
  and    $Q :: ccs$ 

  assumes  $P \rightsquigarrow^{\langle \text{weakBisimulation} \rangle} Q$ 
  and      $Q \approx P$ 

  shows  $P \approx Q$ 
using assms
by(auto intro: weakBisimulation.intros)

lemma reflexive:
  fixes  $P :: ccs$ 

  shows  $P \approx P$ 
proof -

```

```

have  $(P, P) \in Id$  by blast
thus ?thesis
by(coinduct rule: weakBisimulationCoinduct) (auto intro: Weak-Sim.reflexive)
qed

```

```

lemma symmetric:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $P \approx Q$ 

  shows  $Q \approx P$ 
using assms
by(rule weakBisimulationE)

```

```

lemma transitive:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

```

```

  assumes  $P \approx Q$ 
  and  $Q \approx R$ 

```

```

  shows  $P \approx R$ 

```

```

proof –

```

```

  from assms have  $(P, R) \in \text{weakBisimulation } O \text{ weakBisimulation}$  by auto
  thus ?thesis

```

```

proof(coinduct rule: weakBisimulationCoinduct)

```

```

  case(cSim P R)

```

```

  from  $\langle (P, R) \in \text{weakBisimulation } O \text{ weakBisimulation} \rangle$ 

```

```

  obtain  $Q$  where  $P \approx Q$  and  $Q \approx R$  by auto

```

```

  note  $\langle P \approx Q \rangle$ 

```

```

  moreover from  $\langle Q \approx R \rangle$  have  $Q \rightsquigarrow^{\wedge} \langle \text{weakBisimulation} \rangle R$  by(rule weak-
BisimulationE)

```

```

  moreover have  $\text{weakBisimulation } O \text{ weakBisimulation} \subseteq (\text{weakBisimulation } O$ 
weakBisimulation)  $\cup$  weakBisimulation

```

```

  by auto

```

```

  moreover note weakBisimulationE(1)

```

```

  ultimately show ?case by(rule Weak-Sim.transitive)

```

```

next

```

```

  case(cSym P R)

```

```

  thus ?case by(blast dest: symmetric)

```

```

qed

```

```

qed

```

```

lemma bisimWeakBisimulation:

```

```

  fixes  $P :: ccs$ 

```

```

  and  $Q :: ccs$ 

```

```

assumes  $P \sim Q$ 

shows  $P \approx Q$ 
using assms
by(coinduct rule: weakBisimWeakCoinduct[where  $X = \text{bisim}$ ])
  (auto dest: bisimE simWeakSim)

lemma structCongWeakBisimulation:
  fixes  $P :: \text{ccs}$ 
  and  $Q :: \text{ccs}$ 

  assumes  $P \equiv_s Q$ 

  shows  $P \approx Q$ 
using assms

by(auto intro: bisimWeakBisimulation bisimStructCong)

lemma strongAppend:
  fixes  $P :: \text{ccs}$ 
  and  $Q :: \text{ccs}$ 
  and  $R :: \text{ccs}$ 
  and  $Rel :: (\text{ccs} \times \text{ccs}) \text{ set}$ 
  and  $Rel' :: (\text{ccs} \times \text{ccs}) \text{ set}$ 
  and  $Rel'' :: (\text{ccs} \times \text{ccs}) \text{ set}$ 

  assumes  $PSimQ: P \rightsquigarrow^{\langle Rel \rangle} Q$ 
  and  $QSimR: Q \rightsquigarrow^{[Rel']} R$ 
  and  $Trans: Rel \circ Rel' \subseteq Rel''$ 

  shows  $P \rightsquigarrow^{\langle Rel'' \rangle} R$ 
using assms
by(simp add: weakSimulation-def simulation-def) blast

lemma weakBisimWeakUpto[case-names cSim cSym, consumes 1]:
  assumes  $p: (P, Q) \in X$ 
  and  $rSim: \bigwedge P Q. (P, Q) \in X \implies P \rightsquigarrow^{\langle \text{weakBisimulation} \circ X \circ \text{bisim} \rangle} Q$ 
  and  $rSym: \bigwedge P Q. (P, Q) \in X \implies (Q, P) \in X$ 

  shows  $P \approx Q$ 
proof –
  let  $?X = \text{weakBisimulation} \circ X \circ \text{weakBisimulation}$ 
  let  $?Y = \text{weakBisimulation} \circ X \circ \text{bisim}$ 
  from  $\langle (P, Q) \in X \rangle$  have  $(P, Q) \in ?X$  by(blast intro: Strong-Bisim.reflexive reflexive)
  thus  $?thesis$ 
proof(coinduct rule: weakBisimWeakCoinduct)
  case(cSim  $P Q$ )
  {

```

```

    fix P P' Q' Q
    assume P ≈ P' and (P', Q') ∈ X and Q' ≈ Q
    from ⟨(P', Q') ∈ X⟩ have (P', Q') ∈ ?Y by(blast intro: reflexive Strong-Bisim.reflexive)
      moreover from ⟨Q' ≈ Q⟩ have Q' ≈̂ <weakBisimulation> Q by(rule
weakBisimulationE)
      moreover have ?Y O weakBisimulation ⊆ ?X by(blast dest: bisimWeak-
Bisimulation transitive)
    moreover {
      fix P Q
      assume (P, Q) ∈ ?Y
      then obtain P' Q' where P ≈ P' and (P', Q') ∈ X and Q' ≈ Q by auto
      from ⟨(P', Q') ∈ X⟩ have P' ≈̂ <?Y> Q' by(rule rSim)
      moreover from ⟨Q' ≈ Q⟩ have Q' ≈̂[bisim] Q by(rule bisimE)
      moreover have ?Y O bisim ⊆ ?Y by(auto dest: Strong-Bisim.transitive)
      ultimately have P' ≈̂ <?Y> Q by(rule strongAppend)
      moreover note ⟨P ≈ P'⟩
      moreover have weakBisimulation O ?Y ⊆ ?Y by(blast dest: transitive)
      ultimately have P ≈̂ <?Y> Q using weakBisimulationE(1)
        by(rule-tac Weak-Sim.transitive)
    }
    ultimately have P' ≈̂ <?X> Q by(rule Weak-Sim.transitive)
    moreover note ⟨P ≈ P'⟩
    moreover have weakBisimulation O ?X ⊆ ?X by(blast dest: transitive)
    ultimately have P ≈̂ <?X> Q using weakBisimulationE(1)
      by(rule-tac Weak-Sim.transitive)
  }
  with ⟨(P, Q) ∈ ?X⟩ show ?case by auto
next
case(cSym P Q)
thus ?case
  apply auto
  by(blast dest: bisimE rSym weakBisimulationE)
qed
qed

```

lemma *weakBisimUpto*[*case-names cSim cSym, consumes 1*]:

```

  assumes p: (P, Q) ∈ X
  and rSim: ⋀R S. (R, S) ∈ X ⇒ R ≈̂ <(weakBisimulation O (X ∪ weakBisim-
ulation) O bisim)> S
  and rSym: ⋀R S. (R, S) ∈ X ⇒ (S, R) ∈ X

```

shows $P \approx Q$

proof –

```

  from p have (P, Q) ∈ X ∪ weakBisimulation by simp
  thus ?thesis
    apply(coinduct rule: weakBisimWeakUpto)
    apply(auto dest: rSim rSym weakBisimulationE)
    apply(rule weakMonotonic)
    apply(blast dest: weakBisimulationE)

```

```

    apply(auto simp add: relcomp-unfold)
    by(metis reflexive Strong-Bisim.reflexive transitive)
qed

end

theory Weak-Cong
  imports Weak-Cong-Sim Weak-Bisim Strong-Bisim-SC
begin

definition weakCongruence :: ccs ⇒ ccs ⇒ bool (‹- ≅ -› [70, 70] 65)
where
  P ≅ Q ≡ P ↗<weakBisimulation> Q ∧ Q ↗<weakBisimulation> P

lemma weakCongruenceE:
  fixes P :: ccs
  and Q :: ccs

  assumes P ≅ Q

  shows P ↗<weakBisimulation> Q
  and Q ↗<weakBisimulation> P
using assms
by(auto simp add: weakCongruence-def)

lemma weakCongruenceI:
  fixes P :: ccs
  and Q :: ccs

  assumes P ↗<weakBisimulation> Q
  and Q ↗<weakBisimulation> P

  shows P ≅ Q
using assms
by(auto simp add: weakCongruence-def)

lemma weakCongISym[consumes 1, case-names cSym cSim]:
  fixes P :: ccs
  and Q :: ccs

  assumes Prop P Q
  and ∧P Q. Prop P Q ⇒ Prop Q P
  and ∧P Q. Prop P Q ⇒ (F P) ↗<weakBisimulation> (F Q)

  shows F P ≅ F Q
using assms
by(auto simp add: weakCongruence-def)

lemma weakCongISym2[consumes 1, case-names cSim]:

```

```

fixes P :: ccs
and Q :: ccs

assumes P ≅ Q
and  $\bigwedge P Q. P \cong Q \implies (F P) \rightsquigarrow\langle \text{weakBisimulation} \rangle (F Q)$ 

shows F P ≅ F Q
using assms
by(auto simp add: weakCongruence-def)

lemma reflexive:
  fixes P :: ccs

  shows P ≅ P
by(auto intro: weakCongruenceI Weak-Bisim.reflexive Weak-Cong-Sim.reflexive)

lemma symmetric:
  fixes P :: ccs
  and Q :: ccs

  assumes P ≅ Q

  shows Q ≅ P
using assms
by(auto simp add: weakCongruence-def)

lemma transitive:
  fixes P :: ccs
  and Q :: ccs
  and R :: ccs

  assumes P ≅ Q
  and Q ≅ R

  shows P ≅ R
proof –
  let ?Prop =  $\lambda P R. \exists Q. P \cong Q \wedge Q \cong R$ 
  from assms have ?Prop P R by auto
  thus ?thesis
  proof(induct rule: weakCongISym)
    case(cSym P R)
    thus ?case by(auto dest: symmetric)
  next
    case(cSim P R)
    from  $\langle ?Prop P R \rangle$  obtain Q where P ≅ Q and Q ≅ R
    by auto
    from  $\langle P \cong Q \rangle$  have P  $\rightsquigarrow\langle \text{weakBisimulation} \rangle$  Q by(rule weakCongruenceE)
    moreover from  $\langle Q \cong R \rangle$  have Q  $\rightsquigarrow\langle \text{weakBisimulation} \rangle$  R by(rule weakCongruenceE)

```

```

moreover from Weak-Bisim.transitive have weakBisimulation O weakBisimulation
 ⊆ weakBisimulation
  by auto
  ultimately show ?case using weakBisimulationE(1)
  by(rule Weak-Cong-Sim.transitive)
qed
qed

```

lemma *bisimWeakCongruence*:

```

fixes P :: ccs
and Q :: ccs

assumes P ~ Q

shows P ≅ Q
using assms
proof(induct rule: weakCongISym)
  case(cSym P Q)
  thus ?case by(rule bisimE)
next
  case(cSim P Q)
  from ⟨P ~ Q⟩ have P ↘[bisim] Q by(rule bisimE)
  hence P ↘[weakBisimulation] Q using bisimWeakBisimulation
  by(rule-tac monotonic) auto
  thus ?case by(rule simWeakSim)
qed

```

lemma *structCongWeakCongruence*:

```

fixes P :: ccs
and Q :: ccs

assumes P ≡s Q

shows P ≅ Q
using assms
by(auto intro: bisimWeakCongruence bisimStructCong)

```

lemma *weakCongruenceWeakBisimulation*:

```

fixes P :: ccs
and Q :: ccs

assumes P ≅ Q

shows P ≈ Q
proof –
  let ?X = {(P, Q) | P Q. P ≅ Q}
  from assms have (P, Q) ∈ ?X by auto
  thus ?thesis
  proof(coinduct rule: weakBisimulationCoinduct)

```

```

    case(cSim P Q)
    from ⟨(P, Q) ∈ ?X⟩ have P ≅ Q by auto
    hence P ~<weakBisimulation> Q by(rule Weak-Cong.weakCongruenceE)
    hence P ~<(?X ∪ weakBisimulation)> Q by(force intro: Weak-Cong-Sim.weakMonotonic)
    thus ?case by(rule weakCongSimWeakSim)
  next
    case(cSym P Q)
    from ⟨(P, Q) ∈ ?X⟩ show ?case by(blast dest: symmetric)
  qed
qed

```

end

```

theory Weak-Sim-Pres
  imports Weak-Sim
begin

```

lemma actPres:

```

  fixes P :: ccs
  and Q :: ccs
  and Rel :: (ccs × ccs) set
  and a :: name
  and Rel' :: (ccs × ccs) set

```

assumes (P, Q) ∈ Rel

shows $\alpha.(P) \rightsquigarrow^{\hat{\langle Rel \rangle}} \alpha.(Q)$

using *assms*

by(*fastforce simp add: weakSimulation-def elim: actCases intro: weakAction*)

lemma sumPres:

```

  fixes P :: ccs
  and Q :: ccs
  and Rel :: (ccs × ccs) set

```

assumes $P \rightsquigarrow^{\hat{\langle Rel \rangle}} Q$

and $Rel \subseteq Rel'$

and $Id \subseteq Rel'$

and $C1: \bigwedge S T U. (S, T) \in Rel \implies (S \oplus U, T) \in Rel'$

shows $P \oplus R \rightsquigarrow^{\hat{\langle Rel' \rangle}} Q \oplus R$

proof(*induct rule: weakSimI*)

case(*Sim* αQR)

from $\langle Q \oplus R \mapsto \alpha \prec QR \rangle$ show ?case

proof(*induct rule: sumCases*)

case(*cSum1* Q')

from $\langle P \rightsquigarrow^{\hat{\langle Rel \rangle}} Q \rangle \langle Q \mapsto \alpha \prec Q' \rangle$

obtain P' where $P \implies^{\hat{\alpha}} \prec P'$ and $(P', Q') \in Rel$

```

    by(blast dest: weakSimE)
  thus ?case
  proof(induct rule: weakTransCases)
    case Base
    have  $P \oplus R \Longrightarrow \hat{\tau} \prec P \oplus R$  by simp
    moreover from  $\langle (P, Q') \in Rel \rangle$  have  $(P \oplus R, Q') \in Rel'$  by(rule C1)
    ultimately show ?case by blast
  next
    case Step
    from  $\langle P \Longrightarrow \alpha \prec P' \rangle$  have  $P \oplus R \Longrightarrow \alpha \prec P'$  by(rule weakCongSum1)
    hence  $P \oplus R \Longrightarrow \hat{\alpha} \prec P'$  by(simp add: weakTrans-def)
    thus ?case using  $\langle (P', Q') \in Rel \rangle \langle Rel \subseteq Rel' \rangle$  by blast
  qed
  next
    case(cSum2 R')
    from  $\langle R \mapsto \alpha \prec R' \rangle$  have  $R \Longrightarrow \alpha \prec R'$  by(rule transitionWeakCongTransition)
    hence  $P \oplus R \Longrightarrow \alpha \prec R'$  by(rule weakCongSum2)
    hence  $P \oplus R \Longrightarrow \hat{\alpha} \prec R'$  by(simp add: weakTrans-def)
    thus ?case using  $\langle Id \subseteq Rel' \rangle$  by blast
  qed
  qed

```

lemma *parPresAux*:

```

  fixes P    :: ccs
  and   Q    :: ccs
  and   R    :: ccs
  and   T    :: ccs
  and   Rel  :: (ccs × ccs) set
  and   Rel' :: (ccs × ccs) set
  and   Rel'' :: (ccs × ccs) set

  assumes  $P \rightsquigarrow \hat{\langle Rel \rangle} Q$ 
  and      $(P, Q) \in Rel$ 
  and      $R \rightsquigarrow \hat{\langle Rel' \rangle} T$ 
  and      $(R, T) \in Rel'$ 
  and     C1:  $\bigwedge P' Q' R' T'. \llbracket (P', Q') \in Rel; (R', T') \in Rel' \rrbracket \Longrightarrow (P' \parallel R', Q' \parallel T') \in Rel''$ 

```

```

  shows  $P \parallel R \rightsquigarrow \hat{\langle Rel'' \rangle} Q \parallel T$ 
  proof(induct rule: weakSimI)
    case(Sim  $\alpha$  QT)
    from  $\langle Q \parallel T \mapsto \alpha \prec QT \rangle$ 
    show ?case
    proof(induct rule: parCases)
      case(cPar1 Q')
      from  $\langle P \rightsquigarrow \hat{\langle Rel \rangle} Q \rangle \langle Q \mapsto \alpha \prec Q' \rangle$  obtain P' where  $P \Longrightarrow \hat{\alpha} \prec P'$  and
       $(P', Q') \in Rel$ 
      by(rule weakSimE)
      from  $\langle P \Longrightarrow \hat{\alpha} \prec P' \rangle$  have  $P \parallel R \Longrightarrow \hat{\alpha} \prec P' \parallel R$  by(rule weakPar1)
    
```

```

moreover from  $\langle (P', Q') \in Rel \rangle \langle (R, T) \in Rel' \rangle$  have  $(P' \parallel R, Q' \parallel T) \in Rel''$  by(rule C1)
  ultimately show ?case by blast
next
  case(cPar2 T')
  from  $\langle R \rightsquigarrow^{\hat{\phantom{\alpha}}} \langle Rel' \rangle T \rangle \langle T \mapsto \alpha \prec T' \rangle$  obtain R' where  $R \Longrightarrow^{\hat{\phantom{\alpha}}} \alpha \prec R'$  and  $(R', T') \in Rel'$ 
    by(rule weakSimE)
  from  $\langle R \Longrightarrow^{\hat{\phantom{\alpha}}} \alpha \prec R' \rangle$  have  $P \parallel R \Longrightarrow^{\hat{\phantom{\alpha}}} \alpha \prec P \parallel R'$  by(rule weakPar2)
  moreover from  $\langle (P, Q) \in Rel \rangle \langle (R', T') \in Rel' \rangle$  have  $(P \parallel R', Q \parallel T') \in Rel''$  by(rule C1)
  ultimately show ?case by blast
next
  case(cComm Q' T'  $\alpha$ )
  from  $\langle P \rightsquigarrow^{\hat{\phantom{\alpha}}} \langle Rel \rangle Q \rangle \langle Q \mapsto \alpha \prec Q' \rangle$  obtain P' where  $P \Longrightarrow^{\hat{\phantom{\alpha}}} \alpha \prec P'$  and  $(P', Q') \in Rel$ 
    by(rule weakSimE)
  from  $\langle R \rightsquigarrow^{\hat{\phantom{\alpha}}} \langle Rel' \rangle T \rangle \langle T \mapsto (coAction \alpha) \prec T' \rangle$  obtain R' where  $R \Longrightarrow^{\hat{\phantom{\alpha}}} (coAction \alpha) \prec R'$  and  $(R', T') \in Rel'$ 
    by(rule weakSimE)
  from  $\langle P \Longrightarrow^{\hat{\phantom{\alpha}}} \alpha \prec P' \rangle \langle R \Longrightarrow^{\hat{\phantom{\alpha}}} (coAction \alpha) \prec R' \rangle \langle \alpha \neq \tau \rangle$  have  $P \parallel R \Longrightarrow \tau \prec P' \parallel R'$ 
    by(auto intro: weakCongSync simp add: weakTrans-def)
  hence  $P \parallel R \Longrightarrow \tau \prec P' \parallel R'$  by(simp add: weakTrans-def)
  moreover from  $\langle (P', Q') \in Rel \rangle \langle (R', T') \in Rel' \rangle$  have  $(P' \parallel R', Q' \parallel T') \in Rel''$  by(rule C1)
  ultimately show ?case by blast
qed
qed

```

lemma parPres:

```

fixes P :: ccs
and Q :: ccs
and R :: ccs
and Rel :: (ccs  $\times$  ccs) set
and Rel' :: (ccs  $\times$  ccs) set
assumes  $P \rightsquigarrow^{\hat{\phantom{\alpha}}} \langle Rel \rangle Q$ 
and  $(P, Q) \in Rel$ 
and C1:  $\bigwedge S T U. (S, T) \in Rel \Longrightarrow (S \parallel U, T \parallel U) \in Rel'$ 

```

shows $P \parallel R \rightsquigarrow^{\hat{}} \langle Rel' \rangle Q \parallel R$

using assms

by(rule-tac parPresAux[**where** Rel'=Id **and** Rel''=Rel']) (auto intro: reflexive)

lemma resPres:

```

fixes P :: ccs
and Rel :: (ccs  $\times$  ccs) set
and Q :: ccs
and x :: name

```

```

assumes  $P \rightsquigarrow^{\wedge} \langle Rel \rangle Q$ 
and  $\bigwedge R S y. (R, S) \in Rel \implies ((\nu y)R, (\nu y)S) \in Rel'$ 

shows  $(\nu x)P \rightsquigarrow^{\wedge} \langle Rel' \rangle (\nu x)Q$ 
using assms
by(fastforce simp add: weakSimulation-def elim: resCases intro: weakRes)

lemma bangPres:
  fixes  $P :: ccs$ 
  and  $Rel :: (ccs \times ccs) set$ 
  and  $Q :: ccs$ 

  assumes  $(P, Q) \in Rel$ 
  and  $C1: \bigwedge R S. (R, S) \in Rel \implies R \rightsquigarrow^{\wedge} \langle Rel \rangle S$ 
  and  $Par: \bigwedge R S T U. \llbracket (R, S) \in Rel; (T, U) \in Rel \rrbracket \implies (R \parallel T, S \parallel U) \in Rel'$ 
  and  $C2: bangRel Rel \subseteq Rel'$ 
  and  $C3: \bigwedge R S. (R \parallel !R, S) \in Rel' \implies (!R, S) \in Rel'$ 

  shows  $!P \rightsquigarrow^{\wedge} \langle Rel' \rangle !Q$ 
proof(induct rule: weakSimI)
  case(Sim  $\alpha Q'$ )
  {
    fix  $Pa \alpha Q'$ 
    assume  $!Q \mapsto_{\alpha} \prec Q'$  and  $(Pa, !Q) \in bangRel Rel$ 
    hence  $\exists P'. Pa \implies^{\wedge} \alpha \prec P' \wedge (P', Q') \in Rel'$ 
    proof(nominal-induct arbitrary: Pa rule: bangInduct)
      case(cPar1  $\alpha Q'$ )
      from  $\langle (Pa, Q \parallel !Q) \in bangRel Rel \rangle$ 
      show ?case
      proof(induct rule: BRParCases)
        case(BRPar  $P R$ )
        from  $\langle (P, Q) \in Rel \rangle$  have  $P \rightsquigarrow^{\wedge} \langle Rel \rangle Q$  by(rule C1)
        with  $\langle Q \mapsto_{\alpha} \prec Q' \rangle$  obtain  $P'$  where  $P \implies^{\wedge} \alpha \prec P'$  and  $(P', Q') \in Rel$ 
          by(blast dest: weakSimE)
        from  $\langle P \implies^{\wedge} \alpha \prec P' \rangle$  have  $P \parallel R \implies^{\wedge} \alpha \prec P' \parallel R$  by(rule weakPar1)
        moreover from  $\langle (P', Q') \in Rel \rangle \langle (R, !Q) \in bangRel Rel \rangle C2$  have  $(P' \parallel R, Q' \parallel !Q) \in Rel'$ 
          by(blast intro: Par)
        ultimately show ?case by blast
      qed
    next
      case(cPar2  $\alpha Q'$ )
      from  $\langle (Pa, Q \parallel !Q) \in bangRel Rel \rangle$ 
      show ?case
      proof(induct rule: BRParCases)
        case(BRPar  $P R$ )
        from  $\langle (R, !Q) \in bangRel Rel \rangle$  obtain  $R'$  where  $R \implies^{\wedge} \alpha \prec R'$  and  $(R',$ 

```

$Q') \in Rel'$ **using** *cPar2*
by *blast*
from $\langle R \Longrightarrow \hat{\alpha} \prec R' \rangle$ **have** $P \parallel R \Longrightarrow \hat{\alpha} \prec P \parallel R'$ **by**(*rule weakPar2*)
moreover from $\langle (P, Q) \in Rel \rangle \langle (R', Q') \in Rel' \rangle$ **have** $(P \parallel R', Q \parallel Q')$
 $\in Rel'$ **by**(*rule Par*)
ultimately show *?case* **by** *blast*
qed
next
case(*cComm a Q' Q'' Pa*)
from $\langle (Pa, Q \parallel !Q) \in bangRel Rel \rangle$
show *?case*
proof(*induct rule: BRParCases*)
case(*BRPar P R*)
from $\langle (P, Q) \in Rel \rangle$ **have** $P \rightsquigarrow \langle Rel \rangle Q$ **by**(*rule C1*)
with $\langle Q \longmapsto a \prec Q' \rangle$ **obtain** P' **where** $P \Longrightarrow \hat{a} \prec P'$ **and** $(P', Q') \in Rel$
by(*blast dest: weakSimE*)
from $\langle (R, !Q) \in bangRel Rel \rangle$ **obtain** R' **where** $R \Longrightarrow \hat{(coAction a)} \prec R'$
and $(R', Q'') \in Rel'$ **using** *cComm*
by *blast*
from $\langle P \Longrightarrow \hat{a} \prec P' \rangle \langle R \Longrightarrow \hat{(coAction a)} \prec R' \rangle \langle a \neq \tau \rangle$ **have** $P \parallel R$
 $\Longrightarrow \hat{\tau} \prec P' \parallel R'$
by(*auto intro: weakCongSync simp add: weakTrans-def*)
moreover from $\langle (P', Q') \in Rel \rangle \langle (R', Q'') \in Rel' \rangle$ **have** $(P' \parallel R', Q' \parallel Q'')$
 $Q'') \in Rel'$ **by**(*rule Par*)
ultimately show *?case* **by** *blast*
qed
next
case(*cBang α Q' Pa*)
from $\langle (Pa, !Q) \in bangRel Rel \rangle$
show *?case*
proof(*induct rule: BRBangCases*)
case(*BRBang P*)
from $\langle (P, Q) \in Rel \rangle$ **have** $(!P, !Q) \in bangRel Rel$ **by**(*rule bangRel.BRBang*)
with $\langle (P, Q) \in Rel \rangle$ **have** $(P \parallel !P, Q \parallel !Q) \in bangRel Rel$ **by**(*rule bangRel.BRPar*)
then obtain P' **where** $P \parallel !P \Longrightarrow \hat{\alpha} \prec P'$ **and** $(P', Q') \in Rel'$ **using**
cBang
by *blast*
from $\langle P \parallel !P \Longrightarrow \hat{\alpha} \prec P' \rangle$
show *?case*
proof(*induct rule: weakTransCases*)
case *Base*
have $!P \Longrightarrow \hat{\tau} \prec !P$ **by** *simp*
moreover from $\langle (P', Q') \in Rel' \rangle \langle P \parallel !P = P' \rangle$ **have** $(!P, Q') \in Rel'$
by(*blast intro: C3*)
ultimately show *?case* **by** *blast*
next
case *Step*
from $\langle P \parallel !P \Longrightarrow \alpha \prec P' \rangle$ **have** $!P \Longrightarrow \alpha \prec P'$ **by**(*rule weakCongRepl*)

```

    hence !P ==> ^ alpha < P' by(simp add: weakTrans-def)
    with <(P', Q') ∈ Rel'> show ?case by blast
  qed
  qed
  qed
}

moreover from <(P, Q) ∈ Rel> have (!P, !Q) ∈ bangRel Rel by(rule BRBang)

ultimately show ?case using <!Q ⟶ alpha < Q'> by blast
qed

end

theory Weak-Bisim-Pres
  imports Weak-Bisim Weak-Sim-Pres Strong-Bisim-SC
begin

lemma actPres:
  fixes P :: ccs
  and Q :: ccs
  and alpha :: act

  assumes P ≈ Q

  shows alpha.(P) ≈ alpha.(Q)
proof -
  let ?X = {(alpha.(P), alpha.(Q)) | P Q. P ≈ Q}
  from assms have (alpha.(P), alpha.(Q)) ∈ ?X by auto
  thus ?thesis
    by(coinduct rule: weakBisimulationCoinduct) (auto dest: weakBisimulationE
intro: actPres)
qed

lemma parPres:
  fixes P :: ccs
  and Q :: ccs
  and R :: ccs

  assumes P ≈ Q

  shows P || R ≈ Q || R
proof -
  let ?X = {(P || R, Q || R) | P Q R. P ≈ Q}
  from assms have (P || R, Q || R) ∈ ?X by blast
  thus ?thesis
    by(coinduct rule: weakBisimulationCoinduct, auto)
    (blast intro: parPres dest: weakBisimulationE)+
qed

```

```

lemma resPres:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $x :: name$ 

  assumes  $P \approx Q$ 

  shows  $(\nu x)P \approx (\nu x)Q$ 
proof -
  let  $?X = \{(\nu x)P, (\nu x)Q \mid x P Q. P \approx Q\}$ 
  from assms have  $(\nu x)P, (\nu x)Q \in ?X$  by auto
  thus ?thesis
    by(coinduct rule: weakBisimulationCoinduct) (auto intro: resPres dest: weak-
BisimulationE)
qed

lemma bangPres:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 

  assumes  $P \approx Q$ 

  shows  $!P \approx !Q$ 
proof -
  let  $?X = \text{bangRel weakBisimulation}$ 
  let  $?Y = \text{weakBisimulation } O \ ?X \ O \ \text{bisim}$ 
  {
    fix  $R \ T \ P \ Q$ 
    assume  $R \approx T$  and  $(P, Q) \in ?Y$ 
    from  $\langle (P, Q) \in ?Y \rangle$  obtain  $P' \ Q'$  where  $P \approx P'$  and  $(P', Q') \in ?X$  and  $Q' \sim Q$ 
    by auto
    from  $\langle P \approx P' \rangle$  have  $R \parallel P \approx R \parallel P'$ 
    by(metis parPres bisimWeakBisimulation transitive parComm)
    moreover from  $\langle R \approx T \rangle \langle (P', Q') \in ?X \rangle$  have  $(R \parallel P', T \parallel Q') \in ?X$  by(auto
dest: BRPar)
    moreover from  $\langle Q' \sim Q \rangle$  have  $T \parallel Q' \sim T \parallel Q$  by(metis Strong-Bisim-Pres.parPres
Strong-Bisim.transitive parComm)
    ultimately have  $(R \parallel P, T \parallel Q) \in ?Y$  by auto
  } note BRParAux = this

  from assms have  $(!P, !Q) \in ?X$  by(auto intro: BRBang)
  thus ?thesis
proof(coinduct rule: weakBisimWeakUpto)
  case(cSim P Q)
  from  $\langle (P, Q) \in \text{bangRel weakBisimulation} \rangle$  show ?case
  proof(induct)
    case(BRBang P Q)

```

```

note  $\langle P \approx Q \rangle$  weakBisimulationE(1) BRParAux
moreover have  $?X \subseteq ?Y$  by(auto intro: Strong-Bisim.reflexive reflexive)
moreover {
  fix  $P Q$ 
  assume  $(P \parallel !P, Q) \in ?Y$ 
  hence  $(!P, Q) \in ?Y$  using bangUnfold
  by(blast dest: Strong-Bisim.transitive transitive bisimWeakBisimulation)
}
ultimately show  $?case$  by(rule bangPres)
next
case(BRPar R T P Q)
from  $\langle R \approx T \rangle$  have  $R \rightsquigarrow^{\langle weakBisimulation \rangle} T$  by(rule weakBisimulationE)
moreover note  $\langle R \approx T \rangle \langle P \rightsquigarrow^{\langle ?Y \rangle} Q \rangle$ 
moreover from  $\langle (P, Q) \in ?X \rangle$  have  $(P, Q) \in ?Y$  by(blast intro: Strong-Bisim.reflexive reflexive)
ultimately show  $?case$  using BRParAux by(rule Weak-Sim-Pres.parPresAux)
qed
next
case(cSym P Q)
thus  $?case$ 
by induct (auto dest: weakBisimulationE intro: BRPar BRBang)
qed
qed
end

```

```

theory Weak-Cong-Sim-Pres
imports Weak-Cong-Sim
begin

```

```

lemma actPres:

```

```

fixes  $P :: ccs$ 
and  $Q :: ccs$ 
and  $Rel :: (ccs \times ccs) set$ 
and  $a :: name$ 
and  $Rel' :: (ccs \times ccs) set$ 

```

```

assumes  $(P, Q) \in Rel$ 

```

```

shows  $\alpha.(P) \rightsquigarrow^{\langle Rel \rangle} \alpha.(Q)$ 

```

```

using assms

```

```

by(fastforce simp add: weakCongSimulation-def elim: actCases intro: weakCongAction)

```

```

lemma sumPres:

```

```

fixes  $P :: ccs$ 
and  $Q :: ccs$ 
and  $Rel :: (ccs \times ccs) set$ 

```

```

assumes  $P \rightsquigarrow \langle Rel \rangle Q$ 
and  $Rel \subseteq Rel'$ 
and  $Id \subseteq Rel'$ 

shows  $P \oplus R \rightsquigarrow \langle Rel' \rangle Q \oplus R$ 
using assms
by(force simp add: weakCongSimulation-def elim: sumCases intro: weakCongSum1
weakCongSum2 transitionWeakCongTransition)

lemma parPres:
fixes  $P :: ccs$ 
and  $Q :: ccs$ 
and  $Rel :: (ccs \times ccs) \text{ set}$ 

assumes  $P \rightsquigarrow \langle Rel \rangle Q$ 
and  $(P, Q) \in Rel$ 
and  $C1: \bigwedge S T U. (S, T) \in Rel \implies (S \parallel U, T \parallel U) \in Rel'$ 

shows  $P \parallel R \rightsquigarrow \langle Rel' \rangle Q \parallel R$ 
proof(induct rule: weakSimI)
case(Sim  $\alpha QR$ )
from  $\langle Q \parallel R \mapsto \alpha \prec QR \rangle$ 
show ?case
proof(induct rule: parCases)
case(cPar1  $Q'$ )
from  $\langle P \rightsquigarrow \langle Rel \rangle Q \rangle \langle Q \mapsto \alpha \prec Q' \rangle$  obtain  $P'$  where  $P \implies \alpha \prec P'$  and
 $(P', Q') \in Rel$ 
by(rule weakSimE)
from  $\langle P \implies \alpha \prec P' \rangle$  have  $P \parallel R \implies \alpha \prec P' \parallel R$  by(rule weakCongPar1)
moreover from  $\langle (P', Q') \in Rel \rangle$  have  $(P' \parallel R, Q' \parallel R) \in Rel'$  by(rule C1)
ultimately show ?case by blast
next
case(cPar2  $R'$ )
from  $\langle R \mapsto \alpha \prec R' \rangle$  have  $R \implies \alpha \prec R'$  by(rule transitionWeakCongTransition)
hence  $P \parallel R \implies \alpha \prec P \parallel R'$  by(rule weakCongPar2)
moreover from  $\langle (P, Q) \in Rel \rangle$  have  $(P \parallel R', Q \parallel R') \in Rel'$  by(rule C1)
ultimately show ?case by blast
next
case(cComm  $Q' R' \alpha$ )
from  $\langle P \rightsquigarrow \langle Rel \rangle Q \rangle \langle Q \mapsto \alpha \prec Q' \rangle$  obtain  $P'$  where  $P \implies \alpha \prec P'$  and
 $(P', Q') \in Rel$ 
by(rule weakSimE)
from  $\langle R \mapsto (coAction \alpha) \prec R' \rangle$  have  $R \implies (coAction \alpha) \prec R'$ 
by(rule transitionWeakCongTransition)
with  $\langle P \implies \alpha \prec P' \rangle$  have  $P \parallel R \implies \tau \prec P' \parallel R'$  using  $\langle \alpha \neq \tau \rangle$ 
by(rule weakCongSync)
moreover from  $\langle (P', Q') \in Rel \rangle$  have  $(P' \parallel R', Q' \parallel R') \in Rel'$  by(rule C1)
ultimately show ?case by blast
qed

```

qed

lemma *resPres*:

fixes $P :: ccs$
and $Rel :: (ccs \times ccs) \text{ set}$
and $Q :: ccs$
and $x :: name$

assumes $P \rightsquigarrow \langle Rel \rangle Q$
and $\bigwedge R S y. (R, S) \in Rel \implies ((\nu y)R, (\nu y)S) \in Rel'$

shows $(\nu x)P \rightsquigarrow \langle Rel' \rangle (\nu x)Q$

using *assms*

by(*fastforce simp add: weakCongSimulation-def elim: resCases intro: weakCongRes*)

lemma *bangPres*:

fixes $P :: ccs$
and $Q :: ccs$
and $Rel :: (ccs \times ccs) \text{ set}$
and $Rel' :: (ccs \times ccs) \text{ set}$

assumes $(P, Q) \in Rel$
and $C1: \bigwedge R S. (R, S) \in Rel \implies R \rightsquigarrow \langle Rel' \rangle S$
and $C2: Rel \subseteq Rel'$

shows $!P \rightsquigarrow \langle \text{bangRel } Rel' \rangle !Q$

proof(*induct rule: weakSimI*)

case(*Sim* α Q')

{

fix $Pa \alpha Q'$

assume $!Q \mapsto \alpha \prec Q'$ **and** $(Pa, !Q) \in \text{bangRel } Rel$

hence $\exists P'. Pa \implies \alpha \prec P' \wedge (P', Q') \in \text{bangRel } Rel'$

proof(*nominal-induct arbitrary: Pa rule: bangInduct*)

case(*cPar1* α Q')

from $\langle (Pa, Q) \parallel !Q \rangle \in \text{bangRel } Rel$

show *?case*

proof(*induct rule: BRParCases*)

case(*BRPar* P R)

from $\langle (P, Q) \in Rel \rangle$ **have** $P \rightsquigarrow \langle Rel' \rangle Q$ **by**(*rule C1*)

with $\langle Q \mapsto \alpha \prec Q' \rangle$ **obtain** P' **where** $P \implies \alpha \prec P'$ **and** $(P', Q') \in Rel'$

by(*blast dest: weakSimE*)

from $\langle P \implies \alpha \prec P' \rangle$ **have** $P \parallel R \implies \alpha \prec P' \parallel R$ **by**(*rule weakCongPar1*)

moreover from $\langle (R, !Q) \in \text{bangRel } Rel \rangle$ $C2$ **have** $(R, !Q) \in \text{bangRel } Rel'$

by *induct (auto intro: bangRel.BRPar bangRel.BRBang)*

with $\langle (P', Q') \in Rel' \rangle$ **have** $(P' \parallel R, Q' \parallel !Q) \in \text{bangRel } Rel'$

by(*rule bangRel.BRPar*)

ultimately show *?case* **by** *blast*

qed

```

next
  case(cPar2  $\alpha$   $Q'$ )
  from  $\langle (Pa, Q \parallel !Q) \in \text{bangRel Rel} \rangle$ 
  show ?case
  proof(induct rule: BRParCases)
    case(BRPar  $P$   $R$ )
    from  $\langle (R, !Q) \in \text{bangRel Rel} \rangle$  obtain  $R'$  where  $R \Longrightarrow \alpha \prec R'$  and  $(R', Q') \in \text{bangRel Rel}'$  using cPar2
    by blast
    from  $\langle R \Longrightarrow \alpha \prec R' \rangle$  have  $P \parallel R \Longrightarrow \alpha \prec P \parallel R'$  by(rule weakCongPar2)
    moreover from  $\langle (P, Q) \in \text{Rel} \rangle \langle (R', Q') \in \text{bangRel Rel}' \rangle$  C2 have  $(P \parallel R', Q \parallel Q') \in \text{bangRel Rel}'$ 
    by(blast intro: bangRel.BRPar)
    ultimately show ?case by blast
  qed
next
  case(cComm  $a$   $Q'$   $Q''$   $Pa$ )
  from  $\langle (Pa, Q \parallel !Q) \in \text{bangRel Rel} \rangle$ 
  show ?case
  proof(induct rule: BRParCases)
    case(BRPar  $P$   $R$ )
    from  $\langle (P, Q) \in \text{Rel} \rangle$  have  $P \rightsquigarrow \langle \text{Rel}' \rangle Q$  by(rule C1)
    with  $\langle Q \longmapsto a \prec Q' \rangle$  obtain  $P'$  where  $P \Longrightarrow a \prec P'$  and  $(P', Q') \in \text{Rel}'$ 
    by(blast dest: weakSimE)
    from  $\langle (R, !Q) \in \text{bangRel Rel} \rangle$  obtain  $R'$  where  $R \Longrightarrow (\text{coAction } a) \prec R'$ 
    and  $(R', Q'') \in \text{bangRel Rel}'$  using cComm
    by blast
    from  $\langle P \Longrightarrow a \prec P' \rangle \langle R \Longrightarrow (\text{coAction } a) \prec R' \rangle \langle a \neq \tau \rangle$  have  $P \parallel R \Longrightarrow \tau \prec P' \parallel R'$  by(rule weakCongSync)
    moreover from  $\langle (P', Q') \in \text{Rel}' \rangle \langle (R', Q'') \in \text{bangRel Rel}' \rangle$  have  $(P' \parallel R', Q' \parallel Q'') \in \text{bangRel Rel}'$ 
    by(rule bangRel.BRPar)
    ultimately show ?case by blast
  qed
next
  case(cBang  $\alpha$   $Q'$   $Pa$ )
  from  $\langle (Pa, !Q) \in \text{bangRel Rel} \rangle$ 
  show ?case
  proof(induct rule: BRBangCases)
    case(BRBang  $P$ )
    from  $\langle (P, Q) \in \text{Rel} \rangle$  have  $(!P, !Q) \in \text{bangRel Rel}$  by(rule bangRel.BRBang)
    with  $\langle (P, Q) \in \text{Rel} \rangle$  have  $(P \parallel !P, Q \parallel !Q) \in \text{bangRel Rel}$  by(rule bangRel.BRPar)
    then obtain  $P'$  where  $P \parallel !P \Longrightarrow \alpha \prec P'$  and  $(P', Q') \in \text{bangRel Rel}'$ 
    using cBang
    by blast
    from  $\langle P \parallel !P \Longrightarrow \alpha \prec P' \rangle$  have  $!P \Longrightarrow \alpha \prec P'$  by(rule weakCongRepl)
    thus ?case using  $\langle (P', Q') \in \text{bangRel Rel}' \rangle$  by blast
  qed

```

```

    qed
  }

  moreover from  $\langle P, Q \rangle \in Rel$  have  $\langle !P, !Q \rangle \in \text{bangRel } Rel$  by(rule BRBang)

  ultimately show  $?case$  using  $\langle !Q \mapsto \alpha \prec Q' \rangle$  by blast
qed

end

theory Weak-Cong-Pres
  imports Weak-Cong Weak-Bisim-Pres Weak-Cong-Sim-Pres
begin

lemma actPres:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $\alpha :: act$ 

  assumes  $P \cong Q$ 

  shows  $\alpha.(P) \cong \alpha.(Q)$ 
using assms
proof(induct rule: weakCongISym2)
  case(cSim P Q)
  from  $\langle P \cong Q \rangle$  have  $P \approx Q$  by(rule weakCongruenceWeakBisimulation)
  thus  $?case$  by(rule actPres)
qed

lemma sumPres:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

  assumes  $P \cong Q$ 

  shows  $P \oplus R \cong Q \oplus R$ 
using assms
proof(induct rule: weakCongISym2)
  case(cSim P Q)
  from  $\langle P \cong Q \rangle$  have  $P \rightsquigarrow \langle \text{weakBisimulation} \rangle Q$  by(rule weakCongruenceE)
  thus  $?case$  using Weak-Bisim.reflexive
    by(rule-tac sumPres) auto
qed

lemma parPres:
  fixes  $P :: ccs$ 
  and  $Q :: ccs$ 
  and  $R :: ccs$ 

```

```

assumes  $P \cong Q$ 

shows  $P \parallel R \cong Q \parallel R$ 
using assms
proof(induct rule: weakCongISym2)
  case(cSim P Q)
    from  $\langle P \cong Q \rangle$  have  $P \rightsquigarrow \langle \text{weakBisimulation} \rangle Q$  by(rule weakCongruenceE)
    moreover from  $\langle P \cong Q \rangle$  have  $P \approx Q$  by(rule weakCongruenceWeakBisimulation)
    ultimately show ?case using Weak-Bisim-Pres.parPres
      by(rule parPres)
qed

lemma resPres:
  fixes  $P :: \text{ccs}$ 
  and  $Q :: \text{ccs}$ 
  and  $x :: \text{name}$ 

  assumes  $P \cong Q$ 

  shows  $(\nu x)P \cong (\nu x)Q$ 
using assms
proof(induct rule: weakCongISym2)
  case(cSim P Q)
    from  $\langle P \cong Q \rangle$  have  $P \rightsquigarrow \langle \text{weakBisimulation} \rangle Q$  by(rule weakCongruenceE)
    thus ?case using Weak-Bisim-Pres.resPres
      by(rule resPres)
qed

lemma weakBisimBangRel: bangRel weakBisimulation  $\subseteq$  weakBisimulation
proof auto
  fix  $P Q$ 
  assume  $(P, Q) \in \text{bangRel weakBisimulation}$ 
  thus  $P \approx Q$ 
  proof(induct rule: bangRel.induct)
    case(BRBang P Q)
      from  $\langle P \approx Q \rangle$  show  $!P \approx !Q$  by(rule Weak-Bisim-Pres.bangPres)
    next
      case(BRPar R T P Q)
        from  $\langle R \approx T \rangle$  have  $R \parallel P \approx T \parallel P$  by(rule Weak-Bisim-Pres.parPres)
        moreover from  $\langle P \approx Q \rangle$  have  $P \parallel T \approx Q \parallel T$  by(rule Weak-Bisim-Pres.parPres)
        hence  $T \parallel P \approx T \parallel Q$  by(metis bisimWeakBisimulation Weak-Bisim.transitive parComm)
        ultimately show  $R \parallel P \approx T \parallel Q$  by(rule Weak-Bisim.transitive)
      qed
    qed
qed

lemma bangPres:

```

```

fixes  $P :: ccs$ 
and    $Q :: ccs$ 

assumes  $P \cong Q$ 

shows  $!P \cong !Q$ 
using assms
proof(induct rule: weakCongISym2)
  case(cSim P Q)
  let  $?X = \{(P, Q) \mid P \ Q. P \cong Q\}$ 
  from  $\langle P \cong Q \rangle$  have  $(P, Q) \in ?X$  by auto
  moreover have  $\bigwedge P \ Q. (P, Q) \in ?X \implies P \rightsquigarrow \langle \text{weakBisimulation} \rangle Q$  by(auto
dest: weakCongruenceE)
  moreover have  $?X \subseteq \text{weakBisimulation}$  by(auto intro: weakCongruenceWeak-
Bisimulation)
  ultimately have  $!P \rightsquigarrow \langle \text{bangRel weakBisimulation} \rangle !Q$  by(rule bangPres)
  thus  $?case$  using weakBisimBangRel by(rule Weak-Cong-Sim.weakMonotonic)
qed

end

```

References

- [1] J. Bengtson. *Formalising process calculi*, volume 94. Uppsala Dissertations from the Faculty of Science and Technology, 2010.