

# A Fully Verified Executable LTL Model Checker

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## Abstract

We present an LTL model checker whose code has been completely verified using the Isabelle theorem prover. The checker consists of over 4000 lines of ML code. The code is produced using the Isabelle Refinement Framework, which allows us to split its correctness proof into (1) the proof of an abstract version of the checker, consisting of a few hundred lines of “formalized pseudocode”, and (2) a verified refinement step in which mathematical sets and other abstract structures are replaced by implementations of efficient structures like red-black trees and functional arrays. This leads to a checker that, while still slower than unverified checkers, can already be used as a trusted reference implementation against which advanced implementations can be tested.

An early version of this model checker is described elsewhere [1].

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# 1 Nested DFS using Standard Invariants Approach

```

theory NDFS-SI
imports
  CAVA-Automata.Automata-Impl
  CAVA-Automata.Lasso
  NDFS-SI-Statistics
  CAVA-Base.CAVA-Code-Target
begin

```

Implementation of a nested DFS algorithm for accepting cycle detection using the refinement framework. The algorithm uses the improvement of Holzmann et al. [2], i.e., it reports a cycle if the inner DFS finds a path back to the stack of the outer DFS. Moreover, an early cycle detection optimization is implemented [4], i.e., the outer DFS may already report a cycle on a back-edge involving an accepting node.

The algorithm returns a witness in case that an accepting cycle is detected. The design approach to this algorithm is to first establish invariants of a generic DFS-Algorithm, which are then used to instantiate the concrete nested DFS-algorithm for Büchi emptiness check. This formalization can be seen as a predecessor of the formalization of Gabow's algorithm [3], where this technique has been further developed.

## 1.1 Tools for DFS Algorithms

### 1.1.1 Invariants

**definition** *gen-dfs-pre*  $E U S V u0 \equiv$   
 $E''U \subseteq U$  — Upper bound is closed under transitions  
 $\wedge$  *finite*  $U$  — Upper bound is finite  
 $\wedge V \subseteq U$  — Visited set below upper bound  
 $\wedge u0 \in U$  — Start node in upper bound  
 $\wedge E''(V-S) \subseteq V$  — Visited nodes closed under reachability, or on stack  
 $\wedge u0 \notin V$  — Start node not yet visited  
 $\wedge S \subseteq V$  — Stack is visited  
 $\wedge (\forall v \in S. (v, u0) \in (E \cap S \times UNIV)^*)$  —  $u0$  reachable from stack, only over stack

**definition** *gen-dfs-var*  $U \equiv$  *finite-psupset*  $U$

**definition** *gen-dfs-fe-inv*  $E U S V0 u0 it V brk \equiv$   
 $(\neg brk \longrightarrow E''(V-S) \subseteq V)$  — Visited set closed under reachability  
 $\wedge E''\{u0\} - it \subseteq V$  — Successors of  $u0$  visited  
 $\wedge V0 \subseteq V$  — Visited set increasing  
 $\wedge V \subseteq V0 \cup (E - UNIV \times S)^*$  “  $(E''\{u0\} - it - S)$  — All visited nodes reachable

**definition** *gen-dfs-post*  $E U S V0 u0 V brk \equiv$   
 $(\neg brk \longrightarrow E^*(V-S) \subseteq V)$  — Visited set closed under reachability  
 $\wedge u0 \in V$  —  $u0$  visited  
 $\wedge V0 \subseteq V$  — Visited set increasing  
 $\wedge V \subseteq V0 \cup (E - UNIV \times S)^* \{u0\}$  — All visited nodes reachable

**definition** *gen-dfs-outer*  $E U V0 it V brk \equiv$   
 $V0 \subseteq U$  — Start nodes below upper bound  
 $\wedge E^*U \subseteq U$  — Upper bound is closed under transitions  
 $\wedge finite U$  — Upper bound is finite  
 $\wedge V \subseteq U$  — Visited set below upper bound  
 $\wedge (\neg brk \longrightarrow E^*V \subseteq V)$  — Visited set closed under reachability  
 $\wedge (V0 - it \subseteq V)$  — Start nodes already iterated over are visited

### 1.1.2 Invariant Preservation

**lemma** *gen-dfs-outer-initial*:  
**assumes** *finite*  $(E^*V0)$   
**shows** *gen-dfs-outer*  $E (E^*V0) V0 V0 \{\}$  *brk*  
**using** *assms unfolding gen-dfs-outer-def*  
**by** (*auto intro: rev-ImageI*)

**lemma** *gen-dfs-pre-initial*:  
**assumes** *gen-dfs-outer*  $E U V0 it V False$   
**assumes**  $v0 \in U - V$   
**shows** *gen-dfs-pre*  $E U \{\}$   $V v0$   
**using** *assms unfolding gen-dfs-pre-def gen-dfs-outer-def*  
**apply** *auto*  
**done**

**lemma** *fin-U-imp-wf*:  
**assumes** *finite*  $U$   
**shows** *wf*  $(gen-dfs-var U)$   
**using** *assms unfolding gen-dfs-var-def* **by** *auto*

**lemma** *gen-dfs-pre-imp-wf*:  
**assumes** *gen-dfs-pre*  $E U S V u0$   
**shows** *wf*  $(gen-dfs-var U)$   
**using** *assms unfolding gen-dfs-pre-def gen-dfs-var-def* **by** *auto*

**lemma** *gen-dfs-pre-imp-fin*:  
**assumes** *gen-dfs-pre*  $E U S V u0$   
**shows** *finite*  $(E \{u0\})$   
**apply** (*rule finite-subset[where B=U]*)  
**using** *assms unfolding gen-dfs-pre-def*  
**by** *auto*

Inserted  $u0$  on stack and to visited set

**lemma** *gen-dfs-pre-imp-fe*:  
 **assumes** *gen-dfs-pre*  $E U S V u0$   
 **shows** *gen-dfs-fe-inv*  $E U (\text{insert } u0 S) (\text{insert } u0 V) u0$   
  $(E''\{u0\}) (\text{insert } u0 V) \text{False}$   
 **using** *assms* **unfolding** *gen-dfs-pre-def gen-dfs-fe-inv-def*  
 **apply** *auto*  
 **done**

**lemma** *gen-dfs-fe-inv-pres-visited*:  
 **assumes** *gen-dfs-pre*  $E U S V u0$   
 **assumes** *gen-dfs-fe-inv*  $E U (\text{insert } u0 S) (\text{insert } u0 V) u0 \text{it } V' \text{False}$   
 **assumes**  $t \in \text{it} \quad \text{it} \subseteq E''\{u0\} \quad t \in V'$   
 **shows** *gen-dfs-fe-inv*  $E U (\text{insert } u0 S) (\text{insert } u0 V) u0 (\text{it} - \{t\}) V' \text{False}$   
 **using** *assms* **unfolding** *gen-dfs-fe-inv-def*  
 **apply** *auto*  
 **done**

**lemma** *gen-dfs-upper-aux*:  
 **assumes**  $(x,y) \in E'^*$   
 **assumes**  $(u0,x) \in E$   
 **assumes**  $u0 \in U$   
 **assumes**  $E' \subseteq E$   
 **assumes**  $E''U \subseteq U$   
 **shows**  $y \in U$   
 **using** *assms*  
 **by** *induct auto*

**lemma** *gen-dfs-fe-inv-imp-var*:  
 **assumes** *gen-dfs-pre*  $E U S V u0$   
 **assumes** *gen-dfs-fe-inv*  $E U (\text{insert } u0 S) (\text{insert } u0 V) u0 \text{it } V' \text{False}$   
 **assumes**  $t \in \text{it} \quad \text{it} \subseteq E''\{u0\} \quad t \notin V'$   
 **shows**  $(V', V) \in \text{gen-dfs-var } U$   
 **using** *assms* **unfolding** *gen-dfs-fe-inv-def gen-dfs-pre-def gen-dfs-var-def*  
 **apply** (*clarsimp simp add: finite-psupset-def*)  
 **apply** (*blast dest: gen-dfs-upper-aux*)  
 **done**

**lemma** *gen-dfs-fe-inv-imp-pre*:  
 **assumes** *gen-dfs-pre*  $E U S V u0$   
 **assumes** *gen-dfs-fe-inv*  $E U (\text{insert } u0 S) (\text{insert } u0 V) u0 \text{it } V' \text{False}$   
 **assumes**  $t \in \text{it} \quad \text{it} \subseteq E''\{u0\} \quad t \notin V'$   
 **shows** *gen-dfs-pre*  $E U (\text{insert } u0 S) V' t$   
 **using** *assms* **unfolding** *gen-dfs-fe-inv-def gen-dfs-pre-def*  
 **apply** *clarsimp*  
 **apply** (*intro conjI*)  
 **apply** (*blast dest: gen-dfs-upper-aux*)  
 **apply** *blast*

```

apply blast
apply blast
apply clarsimp
apply (rule rtrancl-into-rtrancl[where  $b=u0$ ])
apply (auto intro: rev-subsetD[OF - rtrancl-mono[where  $r=E \cap S \times UNIV$ ]])
□
apply blast
done

```

```

lemma gen-dfs-post-imp-fe-inv:
assumes gen-dfs-pre  $E U S V u0$ 
assumes gen-dfs-fe-inv  $E U (insert\ u0\ S) (insert\ u0\ V) u0\ it\ V'\ False$ 
assumes  $t \in it \quad it \subseteq E \setminus \{u0\} \quad t \notin V'$ 
assumes gen-dfs-post  $E U (insert\ u0\ S) V' t V'' cyc$ 
shows gen-dfs-fe-inv  $E U (insert\ u0\ S) (insert\ u0\ V) u0 (it - \{t\}) V'' cyc$ 
using assms unfolding gen-dfs-fe-inv-def gen-dfs-post-def gen-dfs-pre-def
apply clarsimp
apply (intro conjI)
apply blast
apply blast
apply blast
apply (erule order-trans)
apply simp
apply (rule conjI)
  apply (erule order-trans[
    where  $y=insert\ u0\ (V \cup (E - UNIV \times insert\ u0\ S))^*$ 
    “( $E \setminus \{u0\} - it - insert\ u0\ S$ )”])
  apply blast

  apply (cases cyc)
    apply simp
    apply blast

  apply simp
  apply blast
done

```

```

lemma gen-dfs-post-aux:
assumes  $1: (u0, x) \in E$ 
assumes  $2: (x, y) \in (E - UNIV \times insert\ u0\ S)^*$ 
assumes  $3: S \subseteq V \quad x \notin V$ 
shows  $(u0, y) \in (E - UNIV \times S)^*$ 
proof -
from  $1\ 3$  have  $(u0, x) \in (E - UNIV \times S)$  by blast
also have  $(x, y) \in (E - UNIV \times S)^*$ 
  apply (rule-tac rev-subsetD[OF  $2\ rtrancl-mono$ ])
  by auto
finally show ?thesis .
qed

```

```

lemma gen-dfs-fe-imp-post-brk:
  assumes gen-dfs-pre E U S V u0
  assumes gen-dfs-fe-inv E U (insert u0 S) (insert u0 V) u0 it V' True
  assumes it ⊆ E*{u0}
  shows gen-dfs-post E U S V u0 V' True
  using assms unfolding gen-dfs-pre-def gen-dfs-fe-inv-def gen-dfs-post-def
  apply clarify
  apply (intro conjI)
  apply simp
  apply simp
  apply simp
  apply clarsimp
  apply (blast intro: gen-dfs-post-aux)
  done

```

```

lemma gen-dfs-fe-inv-imp-post:
  assumes gen-dfs-pre E U S V u0
  assumes gen-dfs-fe-inv E U (insert u0 S) (insert u0 V) u0 {} V' cyc
  assumes cyc → cyc'
  shows gen-dfs-post E U S V u0 V' cyc'
  using assms unfolding gen-dfs-pre-def gen-dfs-fe-inv-def gen-dfs-post-def
  apply clarsimp
  apply (intro conjI)
  apply blast
  apply (auto intro: gen-dfs-post-aux) []
  done

```

```

lemma gen-dfs-pre-imp-post-brk:
  assumes gen-dfs-pre E U S V u0
  shows gen-dfs-post E U S V u0 (insert u0 V) True
  using assms unfolding gen-dfs-pre-def gen-dfs-post-def
  apply auto
  done

```

### 1.1.3 Consequences of Postcondition

```

lemma gen-dfs-post-imp-reachable:
  assumes gen-dfs-pre E U S V0 u0
  assumes gen-dfs-post E U S V0 u0 V brk
  shows V ⊆ V0 ∪ E*{u0}
  using assms unfolding gen-dfs-post-def gen-dfs-pre-def
  apply clarsimp
  apply (blast intro: rev-subsetD[OF - rtrancl-mono])
  done

```

```

lemma gen-dfs-post-imp-complete:
  assumes gen-dfs-pre E U {} V0 u0

```

```

assumes gen-dfs-post  $E U \{\}$   $V0 u0 V$  False
shows  $V0 \cup E^* \{\{u0\}\} \subseteq V$ 
using assms unfolding gen-dfs-post-def gen-dfs-pre-def
apply clarsimp
apply (blast dest: Image-closed-trancl)
done

```

```

lemma gen-dfs-post-imp-eq:
assumes gen-dfs-pre  $E U \{\}$   $V0 u0$ 
assumes gen-dfs-post  $E U \{\}$   $V0 u0 V$  False
shows  $V = V0 \cup E^* \{\{u0\}\}$ 
using gen-dfs-post-imp-reachable[OF assms] gen-dfs-post-imp-complete[OF assms]
by blast

```

```

lemma gen-dfs-post-imp-below-U:
assumes gen-dfs-pre  $E U S V0 u0$ 
assumes gen-dfs-post  $E U S V0 u0 V$  False
shows  $V \subseteq U$ 
using assms unfolding gen-dfs-pre-def gen-dfs-post-def
apply clarsimp
apply (blast intro: rev-subsetD[OF - rtrancl-mono] dest: Image-closed-trancl)
done

```

```

lemma gen-dfs-post-imp-outer:
assumes gen-dfs-outer  $E U V0 it Vis0$  False
assumes gen-dfs-post  $E U \{\}$   $Vis0 v0 Vis$  False
assumes  $v0 \in it \quad it \subseteq V0 \quad v0 \notin Vis0$ 
shows gen-dfs-outer  $E U V0 (it - \{v0\}) Vis$  False
proof –
  {
    assume  $v0 \in it \quad it \subseteq V0 \quad V0 \subseteq U \quad E \{\{U \subseteq U\}$ 
    hence  $E^* \{\{v0\}\} \subseteq U$ 
    by (metis (full-types) empty-subsetI insert-subset rtrancl-reachable-induct
subset-trans)
  } note  $AUX=this$ 

```

```

show ?thesis
using assms
unfolding gen-dfs-outer-def gen-dfs-post-def
using  $AUX$ 
by auto

```

**qed**

```

lemma gen-dfs-outer-already-vis:
assumes  $v0 \in it \quad it \subseteq V0 \quad v0 \in V$ 
assumes gen-dfs-outer  $E U V0 it V$  False
shows gen-dfs-outer  $E U V0 (it - \{v0\}) V$  False
using assms
unfolding gen-dfs-outer-def

```

by *auto*

## 1.2 Abstract Algorithm

### 1.2.1 Inner (red) DFS

A witness of the red algorithm is a node on the stack and a path to this node

**type-synonym**  $'v$  red-witness = ( $'v$  list  $\times$   $'v$ ) option

Prepend node to red witness

**fun** prep-wit-red ::  $'v \Rightarrow 'v$  red-witness  $\Rightarrow 'v$  red-witness **where**  
  prep-wit-red  $v$  None = None  
| prep-wit-red  $v$  (Some ( $p,u$ )) = Some ( $v\#p,u$ )

Initial witness for node  $u$  with onstack successor  $v$

**definition** red-init-witness ::  $'v \Rightarrow 'v \Rightarrow 'v$  red-witness **where**  
  red-init-witness  $u$   $v$  = Some ( $[u],v$ )

**definition** red-dfs **where**

```
red-dfs  $E$  onstack  $V$   $u$   $\equiv$ 
  RECT ( $\lambda D$  ( $V,u$ ). do {
    let  $V = \text{insert } u$   $V$ ;
    NDFS-SI-Statistics.vis-red-nres;

    — Check whether we have a successor on stack
    brk  $\leftarrow$  FOREACHC ( $E''\{u\}$ ) ( $\lambda brk.$  brk=None)
    ( $\lambda t$  -.
      if  $t \in \text{onstack}$  then
        RETURN (red-init-witness  $u$   $t$ )
      else
        RETURN None
    )
    None;

    — Recurse for successors
    case brk of
      None  $\Rightarrow$ 
        FOREACHC ( $E''\{u\}$ ) ( $\lambda(V,brk).$  brk=None)
        ( $\lambda t$  ( $V,-$ ).
          if  $t \notin V$  then do {
            ( $V,brk$ )  $\leftarrow$   $D$  ( $V,t$ );
            RETURN ( $V,$ prep-wit-red  $u$  brk)
          } else RETURN ( $V,None$ )
        ) ( $V,None$ )
      | -  $\Rightarrow$  RETURN ( $V,brk$ )
    }) ( $V,u$ )
```

**datatype** *'v blue-witness* =  
*NO-CYC* — No cycle detected  
 | *REACH 'v 'v list* — Path from current start node to node on stack, path  
 contains accepting node.  
 | *CIRC 'v list 'v list* — *CIRI pr pl*: Lasso found from current start node.

Prepend node to witness

**primrec** *prep-wit-blue* :: *'v* ⇒ *'v blue-witness* ⇒ *'v blue-witness* **where**  
*prep-wit-blue u0 NO-CYC* = *NO-CYC*  
 | *prep-wit-blue u0 (REACH u p)* = (  
 if *u0=u* then  
*CIRC [] (u0#p)*  
 else  
*REACH u (u0#p)*  
 )  
 | *prep-wit-blue u0 (CIRC pr pl)* = *CIRC (u0#pr) pl*

Initialize blue witness

**fun** *init-wit-blue* :: *'v* ⇒ *'v red-witness* ⇒ *'v blue-witness* **where**  
*init-wit-blue u0 None* = *NO-CYC*  
 | *init-wit-blue u0 (Some (p,u))* = (  
 if *u=u0* then  
*CIRC [] p*  
 else *REACH u p*)

**definition** *init-wit-blue-early* :: *'v* ⇒ *'v* ⇒ *'v blue-witness*  
**where** *init-wit-blue-early s t* ≡ if *s=t* then *CIRC [] [s]* else *REACH t [s]*

Extract result from witness

**term** *lasso-ext*

**definition** *extract-res cyc*  
 ≡ (case *cyc* of  
*CIRC pr pl* ⇒ *Some (pr,pl)*  
 | - ⇒ *None*)

## 1.2.2 Outer (Blue) DFS

**definition** *blue-dfs*  
 :: (*'a,-*) *b-graph-rec-scheme* ⇒ (*'a list* × *'a list*) *option nres*  
**where**  
*blue-dfs G* ≡ do {  
*NDFS-SI-Statistics.start-nres*;  
 (*-, -, cyc*) ← *FOREACHc (g-V0 G) (λ(-, -, cyc). cyc=NO-CYC)*  
 (λ*v0* (*blues, reds, -*). do {  
 if *v0 ∉ blues* then do {

```

(blues,reds,-,cyc) ← RECT (λD (blues,reds,onstack,s). do {
  let blues=insert s blues;
  let onstack=insert s onstack;
  let s-acc = s ∈ bg-F G;
  NDFS-SI-Statistics.vis-blue-nres;
  (blues,reds,onstack,cyc) ←
  FOREACHC ((g-E G)“{s}) (λ(-,-,cyc). cyc=NO-CYC)
  (λt (blues,reds,onstack,cyc).
    if t ∈ onstack ∧ (s-acc ∨ t ∈ bg-F G) then (
      RETURN (blues,reds,onstack, init-wit-blue-early s t)
    ) else if t∉blues then do {
      (blues,reds,onstack,cyc) ← D (blues,reds,onstack,t);
      RETURN (blues,reds,onstack,(prep-wit-blue s cyc))
    } else do {
      NDFS-SI-Statistics.match-blue-nres;
      RETURN (blues,reds,onstack,cyc)
    }
  })
  (blues,reds,onstack,NO-CYC);

(reds,cyc) ←
if cyc=NO-CYC ∧ s-acc then do {
  (reds,rcyc) ← red-dfs (g-E G) onstack reds s;
  RETURN (reds, init-wit-blue s rcyc)
} else RETURN (reds,cyc);

let onstack=onstack - {s};
RETURN (blues,reds,onstack,cyc)
}) (blues,reds,{},v0);
RETURN (blues, reds, cyc)
} else do {
  RETURN (blues, reds, NO-CYC)
}
}) ({} , {} , NO-CYC);
NDFS-SI-Statistics.stop-nres;
RETURN (extract-res cyc)
}

```

**concrete-definition** *blue-dfs-fe* uses *blue-dfs-def*

```

is blue-dfs G ≡ do {
  NDFS-SI-Statistics.start-nres;
  (-,-,cyc) ← ?FE;
  NDFS-SI-Statistics.stop-nres;
  RETURN (extract-res cyc)
}

```

**concrete-definition** *blue-dfs-body* uses *blue-dfs-fe-def*

```

is - ≡ FOREACHc (g-V0 G) (λ(-,-,cyc). cyc=NO-CYC)
(λv0 (blues,reds,-). do {

```

```

    if  $v0 \notin blues$  then do {
      ( $blues, reds, -, cyc$ )  $\leftarrow$   $REC_T$  ? $B$  ( $blues, reds, \{\}, v0$ );
      RETURN ( $blues, reds, cyc$ )
    } else do {RETURN ( $blues, reds, NO-CYC$ )}
  }) ( $\{\}, \{\}, NO-CYC$ )

```

**thm** *blue-dfs-body-def*

### 1.3 Correctness

Additional invariant to be maintained between calls of red dfs

**definition** *red-dfs-inv*  $E U reds onstack \equiv$   
 $E^*U \subseteq U$  — Upper bound is closed under transitions  
 $\wedge$  *finite*  $U$  — Upper bound is finite  
 $\wedge$   $reds \subseteq U$  — Red set below upper bound  
 $\wedge$   $E^*reds \subseteq reds$  — Red nodes closed under reachability  
 $\wedge$   $E^*reds \cap onstack = \{\}$  — No red node with edge to stack

**lemma** *red-dfs-inv-initial*:

```

assumes finite ( $E^*V0$ )
shows red-dfs-inv  $E$  ( $E^*V0$ )  $\{\}$   $\{\}$ 
using assms unfolding red-dfs-inv-def
apply (auto intro: rev-ImageI)
done

```

Correctness of the red DFS.

**theorem** *red-dfs-correct*:

```

fixes  $v0 u0 :: 'v$ 
assumes PRE:
  red-dfs-inv  $E U reds onstack$ 
   $u0 \in U$ 
   $u0 \notin reds$ 
shows red-dfs  $E onstack reds u0$ 
   $\leq SPEC$  ( $\lambda(reds', cyc). case$  cyc of
    Some ( $p, v$ )  $\Rightarrow v \in onstack \wedge p \neq [] \wedge path$   $E u0 p v$ 
  | None  $\Rightarrow$ 
    red-dfs-inv  $E U reds' onstack$ 
     $\wedge u0 \in reds'$ 
     $\wedge reds' \subseteq reds \cup E^* \{u0\}$ 
  )

```

**proof** —

```

let ?dfs-red =
   $REC_T$  ( $\lambda D (V, u). do$  {
    let  $V = insert$   $u V$ ;
    NDFS-SI-Statistics.vis-red-nres;

    — Check whether we have a successor on stack
     $brk \leftarrow FOREACH_C$  ( $E^*\{u\}$ ) ( $\lambda brk. brk = None$ )

```

```

( $\lambda t$  -. if  $t \in onstack$  then
  RETURN (red-init-witness  $u$   $t$ )
  else RETURN None)
None;

```

— Recurse for successors

```

case brk of
None  $\Rightarrow$ 
  FOREACHC (E“{ $u$ }) ( $\lambda(V, brk)$ . brk=None)
  ( $\lambda t$  (V,-).
    if  $t \notin V$  then do {
      (V, brk)  $\leftarrow$  D (V,  $t$ );
      RETURN (V, prep-wit-red  $u$  brk)
    } else RETURN (V, None))
  (V, None)
| -  $\Rightarrow$  RETURN (V, brk)
}) (V,  $u$ )

```

let  $REC_T$  ?body ?init = ?dfs-red

**define** pre where pre = ( $\lambda S$  (V,  $u0$ ). gen-dfs-pre E U S V  $u0 \wedge E^{\text{“}}V \cap onstack$   
= { })

**define** post where post = ( $\lambda S$  (V0,  $u0$ ) (V, cyc). gen-dfs-post E U S V0  $u0$  V  
(cyc  $\neq$  None)  
 $\wedge$  (case cyc of None  $\Rightarrow E^{\text{“}}V \cap onstack = \{ \}$   
| Some (p, v)  $\Rightarrow v \in onstack \wedge p \neq [] \wedge path$  E  $u0$  p v))

**define** fe-inv where fe-inv = ( $\lambda S$  V0  $u0$  it (V, cyc).  
gen-dfs-fe-inv E U S V0  $u0$  it V (cyc  $\neq$  None)  
 $\wedge$  (case cyc of None  $\Rightarrow E^{\text{“}}V \cap onstack = \{ \}$   
| Some (p, v)  $\Rightarrow v \in onstack \wedge p \neq [] \wedge path$  E  $u0$  p v))

**from** PRE **have** GENPRE: gen-dfs-pre E U { } reds  $u0$

**unfolding** red-dfs-inv-def gen-dfs-pre-def

**by** auto

**with** PRE **have** PRE': pre { } (reds,  $u0$ )

**unfolding** pre-def red-dfs-inv-def

**by** auto

**have** IMP-POST: SPEC (post { } (reds,  $u0$ ))

$\leq$  SPEC ( $\lambda$ (reds', cyc). case cyc of

Some (p, v)  $\Rightarrow v \in onstack \wedge p \neq [] \wedge path$  E  $u0$  p v

| None  $\Rightarrow$

red-dfs-inv E U reds' onstack

$\wedge u0 \in reds'$

$\wedge reds' \subseteq reds \cup E^* \text{ “ } \{ u0 \}$ )

```

apply (clarsimp split: option.split)
apply (intro impI conjI allI)
apply simp-all
proof -
  fix reds' p v
  assume post {} (reds, u0) (reds', Some (p, v))
  thus v ∈ onstack and p ≠ [] and path E u0 p v
    unfolding post-def by auto
next
  fix reds'
  assume post {} (reds, u0) (reds', None)
  hence GPOST: gen-dfs-post E U {} reds u0 reds' False
    and NS: E “reds' ∩ onstack = {}”
    unfolding post-def by auto

from GPOST show u0 ∈ reds' unfolding gen-dfs-post-def by auto

show red-dfs-inv E U reds' onstack
  unfolding red-dfs-inv-def
  apply (intro conjI)
  using GENPRE[unfolded gen-dfs-pre-def]
  apply (simp-all) [2]
  apply (rule gen-dfs-post-imp-below-U[OF GENPRE GPOST])
  using GPOST[unfolded gen-dfs-post-def] apply simp
  apply fact
  done

from GPOST show reds' ⊆ reds ∪ E* “{u0}”
  unfolding gen-dfs-post-def by auto
qed

{
  fix σ S
  assume INV0: pre S σ
  have RECT ?body σ
    ≤ SPEC (post S σ)

  apply (rule RECT-rule-arb[where
    pre=pre and
    V=gen-dfs-var U < *lex* > {} and
    arb=S
  ])

  apply refine-mono

  using INV0[unfolded pre-def] apply (auto intro: gen-dfs-pre-imp-wf) []

  apply fact

```

**apply** (*rename-tac D S u*)  
**apply** (*intro refine-vcg*)

**apply** (*rule-tac I= $\lambda$ it cyc.*  
 (*case cyc of None  $\Rightarrow$  ( $E^{\{b\}} - it$ )  $\cap$  onstack =  $\{\}$*   
 | *Some (p,v)  $\Rightarrow$  ( $v \in$ onstack  $\wedge$   $p \neq []$   $\wedge$  path  $E$   $b$   $p$   $v$ )*)  
**in** *FOREACHc-rule*)  
**apply** (*auto simp add: pre-def gen-dfs-pre-imp-fin*) []  
**apply** *auto* []  
**apply** (*auto*  
*split: option.split*  
*simp: red-init-witness-def intro: path1*) []

**apply** (*intro refine-vcg*)

**apply** (*rule-tac I= $\lambda$ fe-inv (insert b S) (insert b a) b in*  
*FOREACHc-rule*  
 )  
**apply** (*auto simp add: pre-def gen-dfs-pre-imp-fin*) []

**apply** (*auto simp add: pre-def fe-inv-def gen-dfs-pre-imp-fe*) []

**apply** (*intro refine-vcg*)

**apply** (*rule order-trans*)  
**apply** (*rprems*)  
**apply** (*clarsimp simp add: pre-def fe-inv-def*)  
**apply** (*rule gen-dfs-fe-inv-imp-pre, assumption+*) []  
**apply** (*auto simp add: pre-def fe-inv-def intro: gen-dfs-fe-inv-imp-var*) []

**apply** (*clarsimp simp add: pre-def post-def fe-inv-def*  
*split: option.split-asm prod.split-asm*  
 ) []  
**apply** (*blast intro: gen-dfs-post-imp-fe-inv*)  
**apply** (*blast intro: gen-dfs-post-imp-fe-inv path-prepend*)

**apply** (*auto simp add: pre-def post-def fe-inv-def*  
*intro: gen-dfs-fe-inv-pres-visited*) []

**apply** (*auto simp add: pre-def post-def fe-inv-def*  
*intro: gen-dfs-fe-inv-imp-post*) []

**apply** (*auto simp add: pre-def post-def fe-inv-def*  
*intro: gen-dfs-fe-imp-post-brk*) []

**apply** (*auto simp add: pre-def post-def fe-inv-def*)

```

    intro: gen-dfs-pre-imp-post-brk) []

apply (auto simp add: pre-def post-def fe-inv-def
    intro: gen-dfs-pre-imp-post-brk) []

done
} note GEN=this

note GEN[OF PRE]
also note IMP-POST
finally show ?thesis
  unfolding red-dfs-def .
qed

```

Main theorem: Correctness of the blue DFS

```

theorem blue-dfs-correct:
  fixes G :: ('v,-) b-graph-rec-scheme
  assumes b-graph G
  assumes finitely-reachable: finite ((g-E G)* “ g-V0 G)
  shows blue-dfs G ≤ SPEC (λr.
    case r of None ⇒ (∀ L. ¬b-graph.is-lasso-prpl G L)
    | Some L ⇒ b-graph.is-lasso-prpl G L)
proof –
  interpret b-graph G by fact

  let ?A = bg-F G
  let ?E = g-E G
  let ?V0 = g-V0 G

  let ?U = ?E* “ ?V0

  define add-inv where add-inv = (λblues reds onstack.
    ¬(∃ v∈(blues-onstack)∩?A. (v,v)∈?E+) — No cycles over finished, accepting
    states
    ∧ reds ⊆ blues — Red nodes are also blue
    ∧ reds ∩ onstack = {} — No red nodes on stack
    ∧ red-dfs-inv ?E ?U reds onstack)

  define cyc-post where cyc-post = (λblues reds onstack u0 cyc. (case cyc of
    NO-CYC ⇒ add-inv blues reds onstack
    | REACH u p ⇒
      path ?E u0 p u
      ∧ u ∈ onstack- {u0}
      ∧ insert u (set p) ∩ ?A ≠ {})
    | CIRC pr pl ⇒ ∃ v.
      pl≠[]
      ∧ path ?E v pl v
      ∧ path ?E u0 pr v
      ∧ set pl ∩ ?A ≠ {})

```

))

**define** *pre* **where** *pre* = ( $\lambda(blues,reds,onstack,u::'v)$ .  
*gen-dfs-pre* ?E ?U *onstack blues u*  $\wedge$  *add-inv blues reds onstack*)

**define** *post* **where** *post* = ( $\lambda(blues0,reds0::'v set,onstack0,u0)$  (*blues,reds,onstack,cyc*).

*onstack* = *onstack0*  
 $\wedge$  *gen-dfs-post* ?E ?U *onstack0 blues0 u0 blues* (*cyc*  $\neq$  *NO-CYC*)  
 $\wedge$  *cyc-post blues reds onstack u0 cyc*)

**define** *fe-inv* **where** *fe-inv* = ( $\lambda blues0 u0 onstack0 it$  (*blues,reds,onstack,cyc*).  
*onstack* = *onstack0*  
 $\wedge$  *gen-dfs-fe-inv* ?E ?U *onstack0 blues0 u0 it blues* (*cyc*  $\neq$  *NO-CYC*)  
 $\wedge$  *cyc-post blues reds onstack u0 cyc*)

**define** *outer-inv* **where** *outer-inv* = ( $\lambda it$  (*blues,reds,cyc*).

*case cyc of*  
*NO-CYC*  $\Rightarrow$   
  *add-inv blues reds* {}  
   $\wedge$  *gen-dfs-outer* ?E ?U ?V0 *it blues False*  
| *CIRC pr pl*  $\Rightarrow \exists v0 \in ?V0. \exists v.$   
  *pl*  $\neq \square$   
   $\wedge$  *path* ?E *v pl v*  
   $\wedge$  *path* ?E *v0 pr v*  
   $\wedge$  *set pl*  $\cap$  ?A  $\neq \{\}$   
| -  $\Rightarrow$  *False*)

**have** *OUTER-INITIAL*: *outer-inv V0* ({}, {}, *NO-CYC*)  
**unfolding** *outer-inv-def add-inv-def*  
**using** *finitely-reachable*  
**apply** (*auto intro: red-dfs-inv-initial gen-dfs-outer-initial*)  
**done**

{  
  **fix** *onstack blues u0 reds*  
  **assume** *pre* (*blues,reds,onstack,u0*)  
  **hence** *fe-inv* (*insert u0 blues*) *u0* (*insert u0 onstack*) (?E "{u0}")  
    (*insert u0 blues,reds,insert u0 onstack,NO-CYC*)  
  **unfolding** *fe-inv-def add-inv-def cyc-post-def*  
  **apply** *clarsimp*  
  **apply** (*intro conjI*)  
  **apply** (*simp add: pre-def gen-dfs-pre-imp-fe*)  
  **apply** (*auto simp: pre-def add-inv-def*) []  
  **apply** (*auto simp: pre-def add-inv-def*) []  
  **apply** (*auto simp: pre-def add-inv-def gen-dfs-pre-def*) []  
  **apply** (*auto simp: pre-def add-inv-def*) []  
  
  **apply** (*unfold pre-def add-inv-def red-dfs-inv-def gen-dfs-pre-def*) []

```

    apply clarsimp
    apply blast
  done
} note PRE-IMP-FE = this

have [simp]:  $\bigwedge u \text{ cyc. prep-wit-blue } u \text{ cyc} = \text{NO-CYC} \longleftrightarrow \text{cyc} = \text{NO-CYC}$ 
  by (case-tac cyc) auto

{
  fix blues0 reds0 onstack0 and u0::'v and
    blues reds onstack blues' reds' onstack'
    cyc it t
  assume PRE: pre (blues0, reds0, onstack0, u0)
  assume FEI: fe-inv (insert u0 blues0) u0 (insert u0 onstack0)
    it (blues, reds, onstack, NO-CYC)
  assume IT:  $t \in it \quad it \subseteq ?E^* \{u0\} \quad t \notin blues$ 
  assume POST: post (blues, reds, onstack, t) (blues', reds', onstack', cyc)
  note [simp del] = path-simps
  have fe-inv (insert u0 blues0) u0 (insert u0 onstack0) (it - {t})
    (blues', reds', onstack', prep-wit-blue u0 cyc)
  unfolding fe-inv-def
  using PRE FEI IT POST
  unfolding fe-inv-def post-def pre-def
  apply (clarsimp)
  apply (intro allI impI conjI)
  apply (blast intro: gen-dfs-post-imp-fe-inv)
  unfolding cyc-post-def
  apply (auto split: blue-witness.split-asm simp: path-simps)
  done
} note FE-INV-PRES = this

{
  fix blues reds onstack u0
  assume pre (blues, reds, onstack, u0)
  hence  $u0 \in ?E^* \{?V0\}$ 
  unfolding pre-def gen-dfs-pre-def by auto
} note PRE-IMP-REACH = this

{
  fix blues0 reds0 onstack0 u0 blues reds onstack
  assume A: pre (blues0, reds0, onstack0, u0)
    fe-inv (insert u0 blues0) u0 (insert u0 onstack0)
    {} (blues, reds, onstack, NO-CYC)
     $u0 \in ?A$ 
  have  $u0 \notin reds$  using A
  unfolding fe-inv-def add-inv-def pre-def cyc-post-def
  apply auto
  done
}

```

```

} note FE-IMP-RED-PRE = this

{
  fix blues0 reds0 onstack0 u0 blues reds onstack rcyc reds'
  assume PRE: pre (blues0, reds0, onstack0, u0)
  assume FEI: fe-inv (insert u0 blues0) u0 (insert u0 onstack0)
    {} (blues, reds, onstack, NO-CYC)
  assume ACC: u0 ∈ ?A
  assume SPECR: case rcyc of
    Some (p, v) ⇒ v ∈ onstack ∧ p ≠ [] ∧ path ?E u0 p v
  | None ⇒
    red-dfs-inv ?E ?U reds' onstack
    ∧ u0 ∈ reds'
    ∧ reds' ⊆ reds ∪ ?E* “ {u0}
  have post (blues0, reds0, onstack0, u0)
    (blues, reds', onstack - {u0}, init-wit-blue u0 rcyc)
  unfolding post-def add-inv-def cyc-post-def
  apply (clarsimp)
  apply (intro conjI)
  proof goal-cases
  from PRE FEI show OS0[symmetric]: onstack - {u0} = onstack0
    by (auto simp: pre-def fe-inv-def add-inv-def gen-dfs-pre-def)

  from PRE FEI have u0 ∈ onstack
    unfolding pre-def gen-dfs-pre-def fe-inv-def gen-dfs-fe-inv-def
    by auto

  from PRE FEI
  show POST: gen-dfs-post ?E (?E* “ ?V0) onstack0 blues0 u0 blues
    (init-wit-blue u0 rcyc ≠ NO-CYC)
    by (auto simp: pre-def fe-inv-def intro: gen-dfs-fe-inv-imp-post)

  case 3

  from FEI have [simp]: onstack = insert u0 onstack0
    unfolding fe-inv-def by auto
  from FEI have u0 ∈ blues unfolding fe-inv-def gen-dfs-fe-inv-def by auto

  show ?case
    apply (cases rcyc)
    apply (simp-all add: split-paired-all)
  proof -
    assume [simp]: rcyc = None
    show (∀ v ∈ (blues - (onstack0 - {u0})) ∩ ?A. (v, v) ∉ ?E+) ∧
      reds' ⊆ blues ∧
      reds' ∩ (onstack0 - {u0}) = {} ∧
      red-dfs-inv ?E (?E* “ ?V0) reds' (onstack0 - {u0})
    proof (intro conjI)
      from SPECR have RINV: red-dfs-inv ?E ?U reds' onstack

```

```

and  $u0 \in \text{reds}'$ 
and  $\text{REDS}'R: \text{reds}' \subseteq \text{reds} \cup ?E^* \langle \{u0\} \rangle$ 
by auto

from  $RINV$  show
 $RINV'$ :  $\text{red-dfs-inv } ?E ( ?E^* \langle \{V0\} \rangle \text{reds}' (\text{onstack0} - \{u0\}))$ 
unfolding  $\text{red-dfs-inv-def}$  by auto

from  $RINV'$ [unfolded red-dfs-inv-def] have
 $\text{REDS}'CL: ?E \langle \text{reds}' \subseteq \text{reds}' \rangle$ 
and  $DJ': ?E \langle \text{reds}' \cap (\text{onstack0} - \{u0\}) = \{\} \rangle$  by auto

from  $RINV$ [unfolded red-dfs-inv-def] have
 $DJ: ?E \langle \text{reds}' \cap (\text{onstack}) = \{\} \rangle$  by auto

show  $\text{reds}' \subseteq \text{blues}$ 
proof
fix  $v$  assume  $v \in \text{reds}'$ 
with  $\text{REDS}'R$  have  $v \in \text{reds} \vee (u0, v) \in ?E^*$  by blast
thus  $v \in \text{blues}$  proof
assume  $v \in \text{reds}$ 
moreover with  $FEI$  have  $\text{reds} \subseteq \text{blues}$ 
unfolding  $\text{fe-inv-def add-inv-def cyc-post-def}$  by auto
ultimately show ?thesis ..
next
from  $POST$ [unfolded gen-dfs-post-def OS0] have
 $CL: ?E \langle (\text{blues} - (\text{onstack0} - \{u0\})) \subseteq \text{blues} \rangle$  and  $u0 \in \text{blues}$ 
by auto
from  $PRE$   $FEI$  have  $\text{onstack0} \subseteq \text{blues}$ 
unfolding  $\text{pre-def fe-inv-def gen-dfs-pre-def gen-dfs-fe-inv-def}$ 
by auto

assume  $(u0, v) \in ?E^*$ 
thus  $v \in \text{blues}$ 
proof (cases rule: rtrancl-last-visit[where  $S = \text{onstack} - \{u0\}$ ])
case no-visit
thus  $v \in \text{blues}$  using  $\langle u0 \in \text{blues} \rangle CL$ 
by induct (auto elim: rtranclE)
next
case (last-visit-point  $u$ )
then obtain  $uh$  where  $(u0, uh) \in ?E^*$  and  $(uh, u) \in ?E$ 
by (metis tranclD2)
with  $\text{REDS}'CL DJ' \langle u0 \in \text{reds}' \rangle$  have  $uh \in \text{reds}'$ 
by (auto dest: Image-closed-trancl)
with  $DJ' \langle (uh, u) \in ?E \rangle \langle u \in \text{onstack} - \{u0\} \rangle$  have False
by simp blast
thus ?thesis ..
qed
qed

```

qed

**show**  $\forall v \in (\text{blues} - (\text{onstack0} - \{u0\})) \cap ?A. (v, v) \notin ?E^+$

**proof**

**fix**  $v$

**assume**  $A: v \in (\text{blues} - (\text{onstack0} - \{u0\})) \cap ?A$

**show**  $(v, v) \notin ?E^+$  **proof** (cases  $v=u0$ )

**assume**  $v \neq u0$

**with**  $A$  **have**  $v \in (\text{blues} - (\text{insert } u0 \text{ onstack})) \cap ?A$  **by** *auto*

**with** *FEI* **show** *?thesis*

**unfolding** *fe-inv-def add-inv-def cyc-post-def* **by** *auto*

**next**

**assume** [*simp*]:  $v=u0$

**show** *?thesis* **proof**

**assume**  $(v, v) \in ?E^+$

**then obtain**  $uh$  **where**  $(u0, uh) \in ?E^*$  **and**  $(uh, u0) \in ?E$

**by** (*auto dest: tranclD2*)

**with** *REDS'CL DJ*  $\langle u0 \in \text{reds}' \rangle$  **have**  $uh \in \text{reds}'$

**by** (*auto dest: Image-closed-trancl*)

**with** *DJ*  $\langle (uh, u0) \in ?E \rangle$   $\langle u0 \in \text{onstack} \rangle$  **show** *False* **by** *blast*

qed

qed

qed

**show**  $\text{reds}' \cap (\text{onstack0} - \{u0\}) = \{\}$

**proof** (*rule ccontr*)

**assume**  $\text{reds}' \cap (\text{onstack0} - \{u0\}) \neq \{\}$

**then obtain**  $v$  **where**  $v \in \text{reds}'$  **and**  $v \in \text{onstack0}$  **and**  $v \neq u0$  **by** *auto*

**from**  $\langle v \in \text{reds}' \rangle$  *REDS'R* **have**  $v \in \text{reds} \vee (u0, v) \in ?E^*$

**by** *auto*

**thus** *False* **proof**

**assume**  $v \in \text{reds}$

**with** *FEI*[*unfolded fe-inv-def add-inv-def cyc-post-def*]

$\langle v \in \text{onstack0} \rangle$

**show** *False* **by** *auto*

**next**

**assume**  $(u0, v) \in ?E^*$

**with**  $\langle v \neq u0 \rangle$  **obtain**  $uh$  **where**  $(u0, uh) \in ?E^*$  **and**  $(uh, v) \in ?E$

**by** (*auto elim: rtranclE*)

**with** *REDS'CL DJ*  $\langle u0 \in \text{reds}' \rangle$  **have**  $uh \in \text{reds}'$

**by** (*auto dest: Image-closed-trancl*)

**with** *DJ*  $\langle (uh, v) \in ?E \rangle$   $\langle v \in \text{onstack0} \rangle$  **show** *False* **by** *simp blast*

qed

qed

qed

**next**

**fix**  $u p$

**assume** [*simp*]:  $\text{rcyc} = \text{Some } (p, u)$

```

show
  (u = u0  $\longrightarrow$  p  $\neq$  []  $\wedge$  path ?E u0 p u0  $\wedge$  set p  $\cap$  ?A  $\neq$  {})  $\wedge$ 
  (u  $\neq$  u0  $\longrightarrow$ 
    path ?E u0 p u  $\wedge$  u  $\in$  onstack0  $\wedge$  (u  $\in$  ?A  $\vee$  set p  $\cap$  ?A  $\neq$  {}))
proof (intro conjI impI)
  from SPECR  $\langle$ u0 $\in$ ?A $\rangle$  show
    u $\neq$ u0  $\implies$  u $\in$ onstack0
    p $\neq$ []
    path ?E u0 p u
    u=u0  $\implies$  path ?E u0 p u0
    set p  $\cap$  F  $\neq$  {}
    u  $\in$  F  $\vee$  set p  $\cap$  F  $\neq$  {}
    by (auto simp: neq-Nil-conv path-simps)
  qed
qed
qed
note RED-IMP-POST = this

{
  fix blues0 reds0 onstack0 u0 blues reds onstack and cyc :: 'v blue-witness
  assume PRE: pre (blues0,reds0,onstack0,u0)
  and FEI: fe-inv (insert u0 blues0) u0 (insert u0 onstack0)
    {} (blues,reds,onstack,NO-CYC)
  and FC[simp]: cyc=NO-CYC
  and NCOND: u0 $\notin$ ?A

  from PRE FEI have OS0: onstack0 = onstack - {u0}
    by (auto simp: pre-def fe-inv-def add-inv-def gen-dfs-pre-def) []

  from PRE FEI have u0 $\in$ onstack
    unfolding pre-def gen-dfs-pre-def fe-inv-def gen-dfs-fe-inv-def
    by auto
  with OS0 have OS1: onstack = insert u0 onstack0 by auto

  have post (blues0,reds0,onstack0,u0) (blues,reds,onstack - {u0},NO-CYC)
    apply (clarsimp simp: post-def cyc-post-def) []
    apply (intro conjI impI)
    apply (simp add: OS0)
    using PRE FEI apply (auto
      simp: pre-def fe-inv-def intro: gen-dfs-fe-inv-imp-post) []

  using FEI[unfolded fe-inv-def cyc-post-def] unfolding add-inv-def
  apply clarsimp
  apply (intro conjI)
  using NCOND apply auto []
  apply auto []
  apply (clarsimp simp: red-dfs-inv-def, blast) []
  done
note NCOND-IMP-POST=this
}

```

```

{
  fix blues0 reds0 onstack0 u0 blues reds onstack it
    and cyc :: 'v blue-witness
  assume PRE: pre (blues0,reds0,onstack0,u0)
  and FEI: fe-inv (insert u0 blues0) u0 (insert u0 onstack0)
    it (blues,reds,onstack,cyc)
  and NC: cyc≠NO-CYC
  and IT: it⊆?E''{u0}
  from PRE FEI have OS0: onstack0 = onstack - {u0}
    by (auto simp: pre-def fe-inv-def add-inv-def gen-dfs-pre-def) []

  from PRE FEI have u0∈onstack
    unfolding pre-def gen-dfs-pre-def fe-inv-def gen-dfs-fe-inv-def
    by auto
  with OS0 have OS1: onstack = insert u0 onstack0 by auto

  have post (blues0,reds0,onstack0,u0) (blues,reds,onstack - {u0},cyc)
    apply (clarsimp simp: post-def) []
    apply (intro conjI impI)
    apply (simp add: OS0)
    using PRE FEI IT NC apply (auto
      simp: pre-def fe-inv-def intro: gen-dfs-fe-imp-post-brk) []
    using FEI[unfolded fe-inv-def] NC
    unfolding cyc-post-def
    apply (auto split: blue-witness.split simp: OS1) []
    done
} note BREAK-IMP-POST = this

```

```

{
  fix blues0 reds0 onstack0 and u0::'v and
    blues reds onstack cyc it t
  assume PRE: pre (blues0,reds0,onstack0,u0)
  assume FEI: fe-inv (insert u0 blues0) u0 (insert u0 onstack0)
    it (blues,reds,onstack,NO-CYC)
  assume IT: it⊆?E''{u0}    t∈it
  assume T-OS: t ∈ onstack
  assume U0ACC: u0∈F ∨ t∈F

  from T-OS have TIB: t ∈ blues using PRE FEI
    by (auto simp add: fe-inv-def pre-def gen-dfs-fe-inv-def gen-dfs-pre-def)

  have fe-inv (insert u0 blues0) u0 (insert u0 onstack0) (it - {t})
    (blues,reds,onstack,init-wit-blue-early u0 t)
    unfolding fe-inv-def
    apply (clarsimp simp: it-step-insert-iff[OF IT])
    apply (intro conjI)

```

```

using PRE FEI apply (simp add: fe-inv-def pre-def)

using FEI TIB apply (auto simp add: fe-inv-def gen-dfs-fe-inv-def) []

unfolding cyc-post-def init-wit-blue-early-def
using IT T-OS U0ACC apply (auto simp: path-simps) []
done
} note EARLY-DET-OPT = this

{
  fix  $\sigma$ 
  assume INV0: pre  $\sigma$ 

  have RECT (blue-dfs-body G)  $\sigma \leq$  SPEC (post  $\sigma$ )
  apply (intro refine-vcg
    RECT-rule[where pre= $\sigma$ 
    and V= $\text{gen-dfs-var } ?U <*\text{lex}*> \{\}$ ]
  )
  apply (unfold blue-dfs-body-def, refine-mono) []
  apply (blast intro!: fin-U-imp-wf finitely-reachable)
  apply (rule INV0)

  apply (simp (no-asm) only: blue-dfs-body-def)
  apply (refine-rcg refine-vcg)

  apply (rule-tac
    I= $\text{fe-inv (insert bb a) bb (insert bb ab)}$ 
    in FOREACHc-rule')

  apply (auto simp add: pre-def gen-dfs-pre-imp-fin) []

  apply (blast intro: PRE-IMP-FE)

  apply (intro refine-vcg)

  apply (blast intro: EARLY-DET-OPT)

  apply (rule order-trans)
  apply (rprems)
  apply (clarsimp simp add: pre-def fe-inv-def cyc-post-def)
  apply (rule gen-dfs-fe-inv-imp-pre, assumption+) []

```

```

apply (auto simp add: pre-def fe-inv-def intro: gen-dfs-fe-inv-imp-var) []

apply (auto intro: FE-INV-PRES) []

apply (auto simp add: pre-def post-def fe-inv-def
  intro: gen-dfs-fe-inv-pres-visited) []

apply (intro refine-vcg)

apply (rule order-trans)
apply (rule red-dfs-correct[where  $U = ?E^*$  “ $?V0$ ”])
apply (auto simp add: fe-inv-def add-inv-def cyc-post-def) []
apply (auto intro: PRE-IMP-REACH) []
apply (auto dest: FE-IMP-RED-PRE) []

apply (intro refine-vcg)
apply clarsimp
apply (rule RED-IMP-POST, assumption+) []

apply (clarsimp, blast intro: NCOND-IMP-POST) []

apply (intro refine-vcg)
apply simp

apply (clarsimp, blast intro: BREAK-IMP-POST) []
done
} note GEN=this

{
  fix v0 it blues reds
  assume v0 ∈ it  it ⊆ V0  v0 ∉ blues
    outer-inv it (blues, reds, NO-CYC)
  hence pre (blues, reds, {}, v0)
    unfolding pre-def outer-inv-def
    by (auto intro: gen-dfs-pre-initial)
} note OUTER-IMP-PRE = this

{
  fix v0 it blues0 reds0 blues reds onstack cyc
  assume v0 ∈ it  it ⊆ V0  v0 ∉ blues0
    outer-inv it (blues0, reds0, NO-CYC)
    post (blues0, reds0, {}, v0) (blues, reds, onstack, cyc)
  hence outer-inv (it - {v0}) (blues, reds, cyc)
    unfolding post-def outer-inv-def cyc-post-def
    by (fastforce split: blue-witness.split intro: gen-dfs-post-imp-outer)
} note POST-IMP-OUTER = this

{

```

```

fix v0 it blues reds
assume  $v0 \in it \quad it \subseteq V0 \quad \text{outer-inv } it \text{ (blues, reds, NO-CYC)}$ 
 $v0 \in blues$ 
hence  $\text{outer-inv } (it - \{v0\}) \text{ (blues, reds, NO-CYC)}$ 
unfolding outer-inv-def
by (auto intro: gen-dfs-outer-already-vis)
} note OUTER-ALREX = this

{
fix it blues reds cyc
assume  $\text{outer-inv } it \text{ (blues, reds, cyc)} \quad cyc \neq \text{NO-CYC}$ 
hence case extract-res cyc of
 $None \Rightarrow \forall L. \neg \text{is-lasso-prpl } L$ 
 $| \text{Some } x \Rightarrow \text{is-lasso-prpl } x$ 
unfolding outer-inv-def extract-res-def is-lasso-prpl-def
 $\text{is-lasso-prpl-pre-def}$ 
apply (cases cyc)
apply auto
done
} note IMP-POST-CYC = this

{ fix pr pl blues reds
assume ADD-INV: add-inv blues reds {}
assume GEN-INV: gen-dfs-outer E (E* “ V0) V0 {} blues False
assume LASSO: is-lasso-prpl (pr, pl)

from LASSO[unfolded is-lasso-prpl-def is-lasso-prpl-pre-def]
obtain  $v0 \text{ va where}$ 
 $v0 \in V0 \quad pl \neq [] \text{ and}$ 
 $PR: \text{path } E \text{ } v0 \text{ } pr \text{ va and } PL: \text{path } E \text{ } va \text{ } pl \text{ va and}$ 
 $F: \text{set } pl \cap F \neq \{\}$ 
by auto

from  $F$  obtain  $pl1 \text{ } vf \text{ } pl2 \text{ where } [simp]: pl = pl1 @ vf \# pl2 \text{ and } vf \in F$ 
by (fastforce simp: in-set-conv-decomp)

from  $PR \text{ } PL$  have  $\text{path } E \text{ } v0 \text{ } (pr @ pl1) \text{ } vf \quad \text{path } E \text{ } vf \text{ } (vf \# pl2 @ pl1) \text{ } vf$ 
by (auto simp: path-simps)
hence  $(v0, vf) \in E^* \text{ and } (vf, vf) \in E^+$ 
by (auto dest: path-is-rtrancl path-is-trancl)

from GEN-INV  $\langle v0 \in V0 \rangle \langle (v0, vf) \in E^* \rangle$  have  $vf \in blues$ 
unfolding gen-dfs-outer-def
apply (clarsimp)
by (metis Image-closed-trancl rev-ImageI rev-subsetD)

from ADD-INV[unfolded add-inv-def]  $\langle vf \in blues \rangle \langle vf \in F \rangle \langle (vf, vf) \in E^+ \rangle$ 
have False by auto

```

```

} note IMP-POST-NOCYC-AUX = this

{
  fix blues reds cyc
  assume outer-inv {} (blues, reds, cyc)
  hence case extract-res cyc of
    None  $\Rightarrow \forall L. \neg \text{is-lasso-prpl } L$ 
  | Some x  $\Rightarrow \text{is-lasso-prpl } x$ 
  apply (cases cyc)
  apply (simp-all add: IMP-POST-CYC)
  unfolding outer-inv-def extract-res-def
  apply (auto intro: IMP-POST-NOCYC-AUX)
  done
} note IMP-POST-NOCYC = this

```

```

show ?thesis
  unfolding blue-dfs-fe.refine blue-dfs-body.refine
  apply (refine-rcg
    FOREACHc-rule[where I=outer-inv]
    refine-vcg
  )

  apply (simp add: finitely-reachable finite-V0)

  apply (rule OUTER-INITIAL)

  apply (rule order-trans[OF GEN])
  apply (clarsimp, blast intro: OUTER-IMP-PRE)

  apply (clarsimp, blast intro: POST-IMP-OUTER)

  apply (clarsimp, blast intro: OUTER-ALREX)

  apply (clarsimp, blast intro: IMP-POST-NOCYC)

  apply (clarsimp, blast intro: IMP-POST-CYC)
  done

```

qed

## 1.4 Refinement

### 1.4.1 Setup for Custom Datatypes

This effort can be automated, but currently, such an automation is not yet implemented

**abbreviation** *red-wit-rel*  $R \equiv \langle\langle R \rangle \text{list-rel}, R \rangle \text{prod-rel} \rangle \text{option-rel}$   
**abbreviation** *i-red-wit*  $I \equiv \langle\langle I \rangle_i \text{i-list}, I \rangle_i \text{i-prod} \rangle_i \text{i-option}$

**abbreviation**  $blue-wit-rel \equiv (Id::(-\ blue-witness \times -)\ set)$   
**consts**  $i-blue-wit :: interface$

**lemmas**  $[autoref-rel-intf] = REL-INTFI[of\ blue-wit-rel\ i-blue-wit]$

**term**  $init-wit-blue-early$

**lemma**  $[autoref-itype]:$

$NO-CYC ::_i\ i-blue-wit$   
 $(=) ::_i\ i-blue-wit \rightarrow_i\ i-blue-wit \rightarrow_i\ i-bool$   
 $init-wit-blue ::_i\ I \rightarrow_i\ i-red-wit\ I \rightarrow_i\ i-blue-wit$   
 $init-wit-blue-early ::_i\ I \rightarrow_i\ I \rightarrow_i\ i-blue-wit$   
 $prep-wit-blue ::_i\ I \rightarrow_i\ i-blue-wit \rightarrow_i\ i-blue-wit$   
 $red-init-witness ::_i\ I \rightarrow_i\ I \rightarrow_i\ i-red-wit\ I$   
 $prep-wit-red ::_i\ I \rightarrow_i\ i-red-wit\ I \rightarrow_i\ i-red-wit\ I$   
 $extract-res ::_i\ i-blue-wit \rightarrow_i\ \langle\langle I \rangle_i i-list, \langle I \rangle_i i-list \rangle_i i-prod \rangle_i i-option$   
 $red-dfs ::_i\ \langle I \rangle_i i-slg \rightarrow_i\ \langle I \rangle_i i-set \rightarrow_i\ \langle I \rangle_i i-set \rightarrow_i\ I$   
 $\rightarrow_i\ \langle\langle I \rangle_i i-set, i-red-wit\ I \rangle_i i-prod \rangle_i i-nres$   
 $blue-dfs ::_i\ i-bg\ i-unit\ I$   
 $\rightarrow_i\ \langle\langle\langle I \rangle_i i-list, \langle I \rangle_i i-list \rangle_i i-prod \rangle_i i-option \rangle_i i-nres$   
**by**  $auto$

**context begin interpretation**  $autoref-syn$  .

**lemma**  $[autoref-op-pat]: NO-CYC \equiv OP\ NO-CYC ::_i\ i-blue-wit$  **by**  $simp$   
**end**

**term**  $lasso-rel-ext$

**lemma**  $autoref-wit[autoref-rules-raw]:$

$(NO-CYC, NO-CYC) \in blue-wit-rel$   
 $((=), (=)) \in blue-wit-rel \rightarrow blue-wit-rel \rightarrow bool-rel$   
 $\bigwedge R. PREFER-id\ R$   
 $\implies (init-wit-blue, init-wit-blue) \in R \rightarrow red-wit-rel\ R \rightarrow blue-wit-rel$   
 $\bigwedge R. PREFER-id\ R$   
 $\implies (init-wit-blue-early, init-wit-blue-early) \in R \rightarrow R \rightarrow blue-wit-rel$   
 $\bigwedge R. PREFER-id\ R$   
 $\implies (prep-wit-blue, prep-wit-blue) \in R \rightarrow blue-wit-rel \rightarrow blue-wit-rel$   
 $\bigwedge R. PREFER-id\ R$   
 $\implies (red-init-witness, red-init-witness) \in R \rightarrow R \rightarrow red-wit-rel\ R$   
 $\bigwedge R. PREFER-id\ R$   
 $\implies (prep-wit-red, prep-wit-red) \in R \rightarrow red-wit-rel\ R \rightarrow red-wit-rel\ R$   
 $\bigwedge R. PREFER-id\ R$   
 $\implies (extract-res, extract-res)$   
 $\in blue-wit-rel \rightarrow \langle\langle R \rangle list-rel \times_r \langle R \rangle list-rel \rangle option-rel$   
**by**  $(simp-all)$

## 1.4.2 Actual Refinement

**term** *red-dfs*

**term** *map2set-rel* (*rbt-map-rel ord*)

**term** *rbt-set-rel*

**schematic-goal** *red-dfs-refine-aux*: (*?f::?'c*, *red-dfs::('a::linorder × -) set⇒-*) ∈ *?R*

**supply** [*autoref-tyrel*] = *ty-REL*[**where** *'a='a set* **and** *R=(Id) dflt-rs-rel*]  
**unfolding** *red-dfs-def*[*abs-def*]  
**apply** (*autoref* (*trace,keep-goal*))  
**done**

**concrete-definition** *impl-red-dfs* **uses** *red-dfs-refine-aux*

**lemma** *impl-red-dfs-autoref*[*autoref-rules*]:

**fixes** *R :: ('a × 'a::linorder) set*  
**assumes** *PREFER-id R*  
**shows** (*impl-red-dfs*, *red-dfs*) ∈  
 $\langle R \rangle \text{slg-rel} \rightarrow \langle R \rangle \text{dflt-rs-rel} \rightarrow \langle R \rangle \text{dflt-rs-rel} \rightarrow R$   
 $\rightarrow \langle \langle R \rangle \text{dflt-rs-rel} \times_r \text{red-wit-rel } R \rangle \text{nres-rel}$   
**using** *assms impl-red-dfs.refine* **by** *simp*

**thm** *autoref-itype(1-10)*

**schematic-goal** *code-red-dfs-aux*:

**shows** *RETURN ?c ≤ impl-red-dfs E onstack V u*  
**unfolding** *impl-red-dfs-def*  
**by** (*refine-transfer* (*post*) *the-resI*)

**concrete-definition** *code-red-dfs* **uses** *code-red-dfs-aux*

**prepare-code-thms** *code-red-dfs-def*

**declare** *code-red-dfs.refine*[*refine-transfer*]

**export-code** *code-red-dfs* **checking** *SML*

**schematic-goal** *red-dfs-hash-refine-aux*: (*?f::?'c*, *red-dfs::('a::hashable × -) set⇒-*) ∈ *?R*

**supply** [*autoref-tyrel*] = *ty-REL*[**where** *'a='a set* **and** *R=(Id)hs.rel*]  
**unfolding** *red-dfs-def*[*abs-def*]  
**apply** (*autoref* (*trace,keep-goal*))  
**done**

**concrete-definition** *impl-red-dfs-hash* **uses** *red-dfs-hash-refine-aux*

**thm** *impl-red-dfs-hash.refine*

**lemma** *impl-red-dfs-hash-autoref*[*autoref-rules*]:

**fixes** *R :: ('a × 'a::hashable) set*  
**assumes** *PREFER-id R*  
**shows** (*impl-red-dfs-hash*, *red-dfs*) ∈

$\langle R \rangle \text{slg-rel} \rightarrow \langle R \rangle \text{hs-rel} \rightarrow \langle R \rangle \text{hs-rel} \rightarrow R$   
 $\rightarrow \langle \langle R \rangle \text{hs-rel} \times_r \text{red-wit-rel } R \rangle \text{nres-rel}$   
**using** *assms impl-red-dfs-hash.refine* **by** *simp*

**schematic-goal** *code-red-dfs-hash-aux*:  
**shows** *RETURN ?c*  $\leq$  *impl-red-dfs-hash E onstack V u*  
**unfolding** *impl-red-dfs-hash-def*  
**by** (*refine-transfer (post) the-resI*)  
**concrete-definition** *code-red-dfs-hash* **uses** *code-red-dfs-hash-aux*  
**prepare-code-thms** *code-red-dfs-hash-def*  
**declare** *code-red-dfs-hash.refine*[*refine-transfer*]

**export-code** *code-red-dfs-hash* **checking** *SML*

**schematic-goal** *red-dfs-ahs-refine-aux*: (*?f::?c, red-dfs::('a::hashable  $\times$  -) set $\Rightarrow$ -)*)  
 $\in$  *?R*  
**supply** [*autoref-tyrel*] = *ty-REL*[**where** *'a='a::hashable set* **and** *R=(Id)ahs.rel*]  
**unfolding** *red-dfs-def*[*abs-def*]  
**apply** (*autoref (trace,keep-goal)*)  
**done**  
**concrete-definition** *impl-red-dfs-ahs* **uses** *red-dfs-ahs-refine-aux*

**lemma** *impl-red-dfs-ahs-autoref*[*autoref-rules*]:  
**fixes** *R :: ('a  $\times$  'a::hashable) set*  
**assumes** *PREFER-id R*  
**shows** (*impl-red-dfs-ahs, red-dfs*)  $\in$   
 $\langle R \rangle \text{slg-rel} \rightarrow \langle R \rangle \text{ahs-rel} \rightarrow \langle R \rangle \text{ahs-rel} \rightarrow R$   
 $\rightarrow \langle \langle R \rangle \text{ahs-rel} \times_r \text{red-wit-rel } R \rangle \text{nres-rel}$   
**using** *assms impl-red-dfs-ahs.refine* **by** *simp*

**schematic-goal** *code-red-dfs-ahs-aux*:  
**shows** *RETURN ?c*  $\leq$  *impl-red-dfs-ahs E onstack V u*  
**unfolding** *impl-red-dfs-ahs-def*  
**by** (*refine-transfer the-resI*)  
**concrete-definition** *code-red-dfs-ahs* **uses** *code-red-dfs-ahs-aux*  
**prepare-code-thms** *code-red-dfs-ahs-def*  
**declare** *code-red-dfs-ahs.refine*[*refine-transfer*]

**export-code** *code-red-dfs-ahs* **checking** *SML*

**schematic-goal** *blue-dfs-refine-aux*: (*?f::?c, blue-dfs::('a::linorder b-graph-rec $\Rightarrow$ -)*)  
 $\in$  *?R*  
**supply** [*autoref-tyrel*] =  
*ty-REL*[**where** *'a='a* **and** *R=Id*]  
*ty-REL*[**where** *'a='a set* **and** *R=(Id)dflt-rs-rel*]  
**unfolding** *blue-dfs-def*[*abs-def*]  
**apply** (*autoref (trace,keep-goal)*)

```

done
concrete-definition impl-blue-dfs uses blue-dfs-refine-aux

thm impl-blue-dfs.refine

lemma impl-blue-dfs-autoref[autoref-rules]:
  fixes  $R :: ('a \times 'a::\text{linorder}) \text{ set}$ 
  assumes PREFER-id  $R$ 
  shows (impl-blue-dfs, blue-dfs)
   $\in \text{bg-impl-rel-ext unit-rel } R$ 
   $\rightarrow \langle\langle R \rangle\text{list-rel} \times_r \langle R \rangle\text{list-rel} \rangle \text{Relators.option-rel} \rangle \text{nres-rel}$ 
  using assms impl-blue-dfs.refine by simp

schematic-goal code-blue-dfs-aux:
  shows RETURN ?c  $\leq \text{impl-blue-dfs } G$ 
  unfolding impl-blue-dfs-def
  apply (refine-transfer (post) the-resI
    order-trans[OF det-RETURN code-red-dfs.refine])
  done
concrete-definition code-blue-dfs uses code-blue-dfs-aux
prepare-code-thms code-blue-dfs-def
declare code-blue-dfs.refine[refine-transfer]

export-code code-blue-dfs checking SML

schematic-goal blue-dfs-hash-refine-aux: (?f::?'c, blue-dfs::('a::hashable b-graph-rec $\Rightarrow$ -))
 $\in ?R$ 
  supply [autoref-tyrel] =
    ty-REL[where  $'a='a$  and  $R=Id$ ]
    ty-REL[where  $'a='a::\text{hashable set}$  and  $R=\langle Id \rangle \text{hs.rel}$ ]
  unfolding blue-dfs-def[abs-def]
  using [[autoref-trace-failed-id]]
  apply (autoref (trace,keep-goal))
  done
concrete-definition impl-blue-dfs-hash uses blue-dfs-hash-refine-aux

lemma impl-blue-dfs-hash-autoref[autoref-rules]:
  fixes  $R :: ('a \times 'a::\text{hashable}) \text{ set}$ 
  assumes PREFER-id  $R$ 
  shows (impl-blue-dfs-hash, blue-dfs)  $\in \text{bg-impl-rel-ext unit-rel } R$ 
   $\rightarrow \langle\langle R \rangle\text{list-rel} \times_r \langle R \rangle\text{list-rel} \rangle \text{Relators.option-rel} \rangle \text{nres-rel}$ 
  using assms impl-blue-dfs-hash.refine by simp

schematic-goal code-blue-dfs-hash-aux:
  shows RETURN ?c  $\leq \text{impl-blue-dfs-hash } G$ 
  unfolding impl-blue-dfs-hash-def
  apply (refine-transfer the-resI
    order-trans[OF det-RETURN code-red-dfs-hash.refine])
  done

```

```

concrete-definition code-blue-dfs-hash uses code-blue-dfs-hash-aux
prepare-code-thms code-blue-dfs-hash-def
declare code-blue-dfs-hash.refine[refine-transfer]

export-code code-blue-dfs-hash checking SML

schematic-goal blue-dfs-ahs-refine-aux: (?f::?c, blue-dfs::('a::hashable b-graph-rec⇒-))
∈ ?R
  supply [autoref-tyrel] =
    ty-REL[where 'a='a and R=Id]
    ty-REL[where 'a='a::hashable set and R=⟨Id⟩ahs.rel]
  unfolding blue-dfs-def[abs-def]
  apply (autoref (trace,keep-goal))
  done
concrete-definition impl-blue-dfs-ahs uses blue-dfs-ahs-refine-aux

lemma impl-blue-dfs-ahs-autoref[autoref-rules]:
  fixes R :: ('a × 'a::hashable) set
  assumes MINOR-PRIO-TAG 5
  assumes PREFER-id R
  shows (impl-blue-dfs-ahs, blue-dfs) ∈ bg-impl-rel-ext unit-rel R
  → ⟨⟨R⟩list-rel ×r ⟨R⟩list-rel⟩Relators.option-rel⟩nres-rel
  using assms impl-blue-dfs-ahs.refine by simp

thm impl-blue-dfs-ahs-def

schematic-goal code-blue-dfs-ahs-aux:
  shows RETURN ?c ≤ impl-blue-dfs-ahs G
  unfolding impl-blue-dfs-ahs-def
  apply (refine-transfer the-resI
    order-trans[OF det-RETURN code-red-dfs-ahs.refine])
  done
concrete-definition code-blue-dfs-ahs uses code-blue-dfs-ahs-aux
prepare-code-thms code-blue-dfs-ahs-def
declare code-blue-dfs-ahs.refine[refine-transfer]

export-code code-blue-dfs-ahs checking SML

Correctness theorem

theorem code-blue-dfs-correct:
  assumes G: b-graph G finite ((g-E G)* “g-V0 G)
  assumes REL: (Gi,G)∈bg-impl-rel-ext unit-rel Id
  shows RETURN (code-blue-dfs Gi) ≤ SPEC (λr.
    case r of None ⇒ ∀ prpl. ¬b-graph.is-lasso-prpl G prpl
    | Some L ⇒ b-graph.is-lasso-prpl G L)
proof –
  note code-blue-dfs.refine
  also note impl-blue-dfs.refine[param-fo, OF REL, THEN nres-relD]
  also note blue-dfs-correct[OF G]

```

**finally show** ?thesis **by** (simp cong: option.case-cong)  
**qed**

**theorem** code-blue-dfs-correct':  
**assumes**  $G$ : b-graph  $G$  finite (( $g$ -E  $G$ )<sup>\*</sup> “  $g$ -V0  $G$ )  
**assumes**  $REL$ : ( $G_i, G$ ) $\in$ bg-impl-rel-ext unit-rel Id  
**shows** case code-blue-dfs  $G_i$  of  
  None  $\Rightarrow \forall$  prpl.  $\neg$ b-graph.is-lasso-prpl  $G$  prpl  
  | Some  $L \Rightarrow$  b-graph.is-lasso-prpl  $G$   $L$   
**using** code-blue-dfs-correct[OF  $G$   $REL$ ]  
**by** simp

**theorem** code-blue-dfs-hash-correct:  
**assumes**  $G$ : b-graph  $G$  finite (( $g$ -E  $G$ )<sup>\*</sup> “  $g$ -V0  $G$ )  
**assumes**  $REL$ : ( $G_i, G$ ) $\in$ bg-impl-rel-ext unit-rel Id  
**shows** RETURN (code-blue-dfs-hash  $G_i$ )  $\leq$  SPEC ( $\lambda r$ .  
  case  $r$  of None  $\Rightarrow \forall$  prpl.  $\neg$ b-graph.is-lasso-prpl  $G$  prpl  
  | Some  $L \Rightarrow$  b-graph.is-lasso-prpl  $G$   $L$ )

**proof** –  
**note** code-blue-dfs-hash.refine  
**also note** impl-blue-dfs-hash.refine[param-fo, OF  $REL$ , THEN nres-relD]  
**also note** blue-dfs-correct[OF  $G$ ]  
**finally show** ?thesis **by** (simp cong: option.case-cong)  
**qed**

**theorem** code-blue-dfs-hash-correct':  
**assumes**  $G$ : b-graph  $G$  finite (( $g$ -E  $G$ )<sup>\*</sup> “  $g$ -V0  $G$ )  
**assumes**  $REL$ : ( $G_i, G$ ) $\in$ bg-impl-rel-ext unit-rel Id  
**shows** case code-blue-dfs-hash  $G_i$  of  
  None  $\Rightarrow \forall$  prpl.  $\neg$ b-graph.is-lasso-prpl  $G$  prpl  
  | Some  $L \Rightarrow$  b-graph.is-lasso-prpl  $G$   $L$   
**using** code-blue-dfs-hash-correct[OF  $G$   $REL$ ]  
**by** simp

**theorem** code-blue-dfs-ahs-correct:  
**assumes**  $G$ : b-graph  $G$  finite (( $g$ -E  $G$ )<sup>\*</sup> “  $g$ -V0  $G$ )  
**assumes**  $REL$ : ( $G_i, G$ ) $\in$ bg-impl-rel-ext unit-rel Id  
**shows** RETURN (code-blue-dfs-ahs  $G_i$ )  $\leq$  SPEC ( $\lambda r$ .  
  case  $r$  of None  $\Rightarrow \forall$  prpl.  $\neg$ b-graph.is-lasso-prpl  $G$  prpl  
  | Some  $L \Rightarrow$  b-graph.is-lasso-prpl  $G$   $L$ )

**proof** –  
**note** code-blue-dfs-ahs.refine  
**also note** impl-blue-dfs-ahs.refine[param-fo, OF  $REL$ , THEN nres-relD]  
**also note** blue-dfs-correct[OF  $G$ ]  
**finally show** ?thesis **by** (simp cong: option.case-cong)  
**qed**

**theorem** code-blue-dfs-ahs-correct':  
**assumes**  $G$ : b-graph  $G$  finite (( $g$ -E  $G$ )<sup>\*</sup> “  $g$ -V0  $G$ )

```

assumes REL:  $(Gi, G) \in \text{bg-impl-rel-ext unit-rel Id}$ 
shows case code-blue-dfs-ahs Gi of
  None  $\Rightarrow \forall \text{prpl}. \neg \text{b-graph.is-lasso-prpl } G \text{ prpl}$ 
  | Some L  $\Rightarrow \text{b-graph.is-lasso-prpl } G L$ 
using code-blue-dfs-ahs-correct[OF G REL]
by simp

```

Export for benchmarking

```

schematic-goal acc-of-list-impl-hash:
  notes [autoref-tyrel] =
    ty-REL[where 'a=nat set and R=<nat-rel>iam-set-rel]

  shows (?f::?'c, λl::nat list.
    let s=(set l)::r<nat-rel>iam-set-rel
    in (λx::nat. x∈s)
  )  $\in ?R$ 
  apply (autoref (keep-goal))
done

```

```

concrete-definition acc-of-list-impl-hash uses acc-of-list-impl-hash
export-code acc-of-list-impl-hash checking SML

```

```

definition code-blue-dfs-nat
   $\equiv \text{code-blue-dfs} :: - \Rightarrow (\text{nat list} \times -) \text{ option}$ 
definition code-blue-dfs-hash-nat
   $\equiv \text{code-blue-dfs-hash} :: - \Rightarrow (\text{nat list} \times -) \text{ option}$ 
definition code-blue-dfs-ahs-nat
   $\equiv \text{code-blue-dfs-ahs} :: - \Rightarrow (\text{nat list} \times -) \text{ option}$ 

```

```

definition succ-of-list-impl-int  $\equiv$ 
  succ-of-list-impl o map  $(\lambda(u,v). (\text{nat-of-integer } u, \text{nat-of-integer } v))$ 

```

```

definition acc-of-list-impl-hash-int  $\equiv$ 
  acc-of-list-impl-hash o map nat-of-integer

```

```

export-code
  code-blue-dfs-nat
  code-blue-dfs-hash-nat
  code-blue-dfs-ahs-nat
  succ-of-list-impl-int
  acc-of-list-impl-hash-int
  nat-of-integer
  integer-of-nat
  lasso-ext
in SML module-name HPY-new-hash
file  $\langle \text{nested-dfs-hash.sml} \rangle$ 

```

**end**

## 2 Abstract Model-Checker

```

theory CAVA-Abstract
imports
  CAVA-Base.CAVA-Base
  CAVA-Automata.Automata
  LTL.LTL
begin

```

This theory defines the abstract version of the cava model checker, as well as a generic implementation.

### 2.1 Specification of an LTL Model-Checker

Abstractly, an LTL model-checker consists of three components:

1. A conversion of LTL-formula to Indexed Generalized Buchi Automata (IGBA) over sets of atomic propositions.
2. An intersection construction, which takes a system and an IGBA, and creates an Indexed Generalized Buchi Graph (IGBG) and a projection function to project runs of the IGBG back to runs of the system.
3. An emptiness check for IGBGs.

Given an LTL formula, the LTL to Buchi conversion returns a Generalized Buchi Automaton that accepts the same language.

**definition** *ltl-to-gba-spec*

```

:: 'prop ltlc  $\Rightarrow$  ('q, 'prop set, -) igba-rec-scheme nres
— Conversion of LTL formula to generalized buchi automaton
where ltl-to-gba-spec  $\varphi \equiv SPEC (\lambda gba.$ 
   $igba.lang\ gba = language-ltlc\ \varphi \wedge igba\ gba \wedge finite\ ((g-E\ gba)^* \text{“} g-V0\ gba))$ 

```

**definition** *inter-spec*

```

:: ('s, 'prop set, -) sa-rec-scheme
 $\Rightarrow$  ('q, 'prop set, -) igba-rec-scheme
 $\Rightarrow$  (('prod-state, -) igb-graph-rec-scheme  $\times$  ('prod-state  $\Rightarrow$  's)) nres
— Intersection of system and IGBA
where  $\bigwedge_{sys\ ba. inter-spec\ sys\ ba \equiv do \{$ 
  ASSERT (sa sys);
  ASSERT (finite ((g-E sys)* “ g-V0 sys));
  ASSERT (igba ba);
  ASSERT (finite ((g-E ba)* “ g-V0 ba));
  SPEC ( $\lambda(G, project). igb-graph\ G \wedge finite\ ((g-E\ G)^* \text{“} g-V0\ G) \wedge (\forall r.$ 
    ( $\exists r'. igb-graph.is-acc-run\ G\ r' \wedge r = project\ o\ r')$ )
     $\longleftrightarrow (graph-defs.is-run\ sys\ r \wedge sa-L\ sys\ o\ r \in igba.lang\ ba))$ )
  }

```

**definition** *find-ce-spec*

```

:: ('q,-) igb-graph-rec-scheme ⇒ 'q word option option nres
— Check Generalized Buchi graph for emptiness, with optional counterexample
where find-ce-spec G ≡ do {
  ASSERT (igb-graph G);
  ASSERT (finite ((g-E G)* “ g-V0 G));
  SPEC (λres. case res of
    None ⇒ (∀ r. ¬igb-graph.is-acc-run G r)
  | Some None ⇒ (∃ r. igb-graph.is-acc-run G r)
  | Some (Some r) ⇒ igb-graph.is-acc-run G r
  )}

```

Using the specifications from above, we can specify the essence of the model-checking algorithm: Convert the LTL-formula to a GBA, make an intersection with the system and check the result for emptiness.

**definition** *abs-model-check*

```

:: 'ba-state itself ⇒ 'ba-more itself
⇒ 'prod-state itself ⇒ 'prod-more itself
⇒ ('s,'prop set,-) sa-rec-scheme ⇒ 'prop ltlc
⇒ 's word option option nres
where
abs-model-check - - - sys φ ≡ do {
  gba :: ('ba-state,-,'ba-more) igba-rec-scheme
  ← ltl-to-gba-spec (Not-ltlc φ);
  ASSERT (igba gba);
  ASSERT (sa sys);
  (Gprod::('prod-state,'prod-more)igb-graph-rec-scheme, map-state)
  ← inter-spec sys gba;
  ASSERT (igb-graph Gprod);
  ce ← find-ce-spec Gprod;

  case ce of
    None ⇒ RETURN None
  | Some None ⇒ RETURN (Some None)
  | Some (Some r) ⇒ RETURN (Some (Some (map-state o r)))
}

```

The main correctness theorem states that our abstract model checker really checks whether the system satisfies the formula, and a correct counterexample is returned (if any). Note that, if the model does not satisfy the formula, returning a counterexample is optional.

**theorem** *abs-model-check-correct*:

```

abs-model-check T1 T2 T3 T4 sys φ ≤ do {
  ASSERT (sa sys);
  ASSERT (finite ((g-E sys)* “ g-V0 sys));
  SPEC (λres. case res of
    None ⇒ sa.lang sys ⊆ language-ltlc φ

```

```

  | Some None  $\Rightarrow \neg sa.lang\ sys \subseteq language\ ltlc\ \varphi$ 
  | Some (Some r)  $\Rightarrow graph\ defs.is\ run\ sys\ r \wedge sa\ L\ sys \circ r \notin language\ ltlc\ \varphi$ 
}
unfolding abs-model-check-def ltl-to-gba-spec-def inter-spec-def
  find-ce-spec-def
apply (refine-rcg refine-vcg ASSERT-leI le-ASSERTI)
apply (auto simp: sa.lang-def
  sa.accept-def[THEN meta-eq-to-obj-eq, THEN ext[of sa.accept sys] ])

done

```

## 2.2 Generic Implementation

In this section, we define a generic implementation of an LTL model checker, that is parameterized with implementations of its components.

**abbreviation**  $ltl\text{-}rel \equiv Id :: ('a\ ltlc \times -)\ set$

**locale**  $impl\text{-}model\text{-}checker =$

— Assembly of a generic model-checker

**fixes**  $sa\text{-}rel :: ('sai \times ('s, 'prop\ set, 'sa\text{-}more)\ sa\text{-}rec\text{-}scheme)\ set$

**fixes**  $igba\text{-}rel :: ('igbai \times ('q, 'prop\ set, 'igba\text{-}more)\ igba\text{-}rec\text{-}scheme)\ set$

**fixes**  $igbg\text{-}rel :: ('igbgi \times ('sq, 'igbg\text{-}more)\ igb\text{-}graph\text{-}rec\text{-}scheme)\ set$

**fixes**  $ce\text{-}rel :: ('cei \times 'sq\ word)\ set$

**fixes**  $mce\text{-}rel :: ('mcei \times 's\ word)\ set$

**fixes**  $ltl\text{-}to\text{-}gba\text{-}impl :: 'cfg\text{-}l2b \Rightarrow 'prop\ ltlc \Rightarrow 'igbai$

**fixes**  $inter\text{-}impl :: 'cfg\text{-}int \Rightarrow 'sai \Rightarrow 'igbai \Rightarrow 'igbgi \times ('sq \Rightarrow 's)$

**fixes**  $find\text{-}ce\text{-}impl :: 'cfg\text{-}ce \Rightarrow 'igbgi \Rightarrow 'cei\ option\ option$

**fixes**  $map\text{-}run\text{-}impl :: ('sq \Rightarrow 's) \Rightarrow 'cei \Rightarrow 'mcei$

**assumes** [ $relator\text{-}props, simp, intro!$ ]:  $single\text{-}valued\ mce\text{-}rel$

**assumes**  $ltl\text{-}to\text{-}gba\text{-}refine:$

$\wedge\ cfg. (ltl\text{-}to\text{-}gba\text{-}impl\ cfg, ltl\text{-}to\text{-}gba\text{-}spec)$

$\in ltl\text{-}rel \rightarrow \langle igba\text{-}rel \rangle plain\text{-}nres\text{-}rel$

**assumes**  $inter\text{-}refine:$

$\wedge\ cfg. (inter\text{-}impl\ cfg, inter\text{-}spec)$

$\in sa\text{-}rel \rightarrow igba\text{-}rel \rightarrow \langle igbg\text{-}rel \times_r (Id \rightarrow Id) \rangle plain\text{-}nres\text{-}rel$

**assumes**  $find\text{-}ce\text{-}refine:$

$\wedge\ cfg. (find\text{-}ce\text{-}impl\ cfg, find\text{-}ce\text{-}spec)$

$\in igbg\text{-}rel \rightarrow \langle \langle ce\text{-}rel \rangle option\text{-}rel \rangle option\text{-}rel \rangle plain\text{-}nres\text{-}rel$

**assumes**  $map\text{-}run\text{-}refine: (map\text{-}run\text{-}impl, (o)) \in (Id \rightarrow Id) \rightarrow ce\text{-}rel \rightarrow mce\text{-}rel$

**begin**

**fun**  $cfg\text{-}l2b$  **where**  $cfg\text{-}l2b\ (c1, c2, c3) = c1$

**fun** *cfg-int* **where** *cfg-int* (*c1,c2,c3*) = *c2*  
**fun** *cfg-ce* **where** *cfg-ce* (*c1,c2,c3*) = *c3*

**definition** *impl-model-check*  
:: ('*cfg-l2b* × '*cfg-int* × '*cfg-ce*)  
⇒ '*sai* ⇒ '*prop ltlc* ⇒ '*mcei option option*

**where**  
*impl-model-check* *cfg sys φ* ≡ *let*  
*ba* = *ltl-to-gba-impl* (*cfg-l2b* *cfg*) (*Not-ltlc φ*);  
(*G, map-q*) = *inter-impl* (*cfg-int* *cfg*) *sys ba*;  
*ce* = *find-ce-impl* (*cfg-ce* *cfg*) *G*  
*in*  
*case ce of*  
*None* ⇒ *None*  
| *Some None* ⇒ *Some None*  
| *Some (Some ce)* ⇒ *Some (Some (map-run-impl map-q ce))*

**lemma** *impl-model-check-refine*:  
(*impl-model-check* *cfg, abs-model-check*  
TYPE('q) TYPE('igba-more) TYPE('sq) TYPE('igbg-more))  
∈ *sa-rel* → *ltl-rel* → ⟨⟨⟨*mce-rel*⟩*option-rel*⟩*option-rel*⟩*plain-nres-rel*  
**apply** (*intro fun-relI plain-nres-relI*)  
**unfolding** *abs-model-check-def impl-model-check-def*

**apply** (*simp only: let-to-bind-conv pull-out-let-conv*  
*pull-out-RETURN-case-option*)

**apply** (*refine-rcg*  
*ltl-to-gba-refine*[*param-fo, THEN plain-nres-relD*]  
*rel-arg-cong*[**where** *f=Not-ltlc*]  
*inter-refine*[*param-fo, THEN plain-nres-relD*]  
*find-ce-refine*[*param-fo, THEN plain-nres-relD*]  
)  
)

**apply** (*simp-all split: option.split*)  
**apply** (*auto elim: option-relE*)  
**apply** (*parametricity add: map-run-refine*)  
**apply** *simp*  
**done**

**theorem** *impl-model-check-correct*:  
**assumes** *R: (sysi,sys) ∈ sa-rel*  
**assumes** [*simp*]: *sa sys finite ((g-E sys)\* “g-V0 sys)*  
**shows** *case impl-model-check* *cfg sysi φ of*  
*None*  
⇒ *sa.lang sys* ⊆ *language-ltlc φ*  
| *Some None*  
⇒ ¬ *sa.lang sys* ⊆ *language-ltlc φ*  
| *Some (Some ri)*

```

    ⇒ (∃ r. (ri,r)∈mce-rel
      ∧ graph-defs.is-run sys r ∧ sa-L sys o r ∉ language-ltlc φ)
proof –
  note impl-model-check-refine[
    where cfg=cfg,
    param-fo,
    THEN plain-nres-relD,
    OF R IdI[of φ]]
  also note abs-model-check-correct
  finally show ?thesis
  apply (simp split: option.split)
  apply (simp add: refine-pw-simps pw-le-iff)
  apply (auto elim!: option-relE) []
  done
qed

```

```

theorem impl-model-check-correct-no-ce:
  assumes (sysi,sys)∈sa-rel
  assumes SA: sa sys finite ((g-E sys)* “ g-V0 sys)
  shows impl-model-check cfg sysi φ = None
  ↔ sa.lang sys ⊆ language-ltlc φ
  using impl-model-check-correct[where cfg=cfg, OF assms, of φ]
  by (auto
    split: option.splits
    simp: sa.lang-def[OF SA(1)] sa.accept-def[OF SA(1), abs-def])

```

end

end

### 3 Boolean Programs

```

theory BoolProgs
imports
  CAVA-Base.CAVA-Base
  Word-Lib.Generic-set-bit
begin

```

#### 3.1 Syntax and Semantics

```

datatype bexp = TT | FF | V nat | Not bexp | And bexp bexp | Or bexp bexp

```

```

type-synonym state = bitset

```

```

fun bval :: bexp ⇒ state ⇒ bool where
  bval TT s = True |
  bval FF s = False |
  bval (V n) s = bs-mem n s |

```

$\text{bval } (\text{Not } b) \ s = (\neg \text{bval } b \ s) \mid$   
 $\text{bval } (\text{And } b_1 \ b_2) \ s = (\text{bval } b_1 \ s \ \& \ \text{bval } b_2 \ s) \mid$   
 $\text{bval } (\text{Or } b_1 \ b_2) \ s = (\text{bval } b_1 \ s \mid \text{bval } b_2 \ s)$

**datatype** *instr* =  
*AssI* *nat list* *beexp list* |  
*TestI* *beexp int* |  
*ChoiceI* (*beexp \* int*) *list* |  
*GotoI* *int*

**type-synonym** *config* = *nat \* state*  
**type-synonym** *bprog* = *instr array*

Semantics Notice: To be equivalent in semantics with SPIN, there is no such thing as a finite run:

- Deadlocks (i.e. empty Choice) are self-loops
- program termination is self-loop

**fun** *exec* :: *instr*  $\Rightarrow$  *config*  $\Rightarrow$  *config list* **where**  
*exec instr (pc,s) = (case instr of*  
*AssI ns bs*  $\Rightarrow$  *let* *bvs* = *zip ns (map* ( $\lambda b. \text{bval } b \ s$ ) *bs)* *in*  
 $[(pc + 1, \text{foldl } (\lambda s \ (n,bv). \text{set-bit } s \ n \ bv) \ s \ bvs)] \mid$   
*TestI b d*  $\Rightarrow$   $[\text{if } \text{bval } b \ s \ \text{then } (pc+1, \ s) \ \text{else } (\text{nat}(\text{int}(pc+1)+d), \ s)] \mid$   
*ChoiceI bis*  $\Rightarrow$  *let* *succs* =  $[(\text{nat}(\text{int}(pc+1)+i), \ s) \cdot (b,i) <- \text{bis}, \ \text{bval } b \ s]$   
 $\text{in if } \text{succs} = [] \ \text{then } [(pc,s)] \ \text{else } \text{succs} \mid$   
*GotoI d*  $\Rightarrow$   $[(\text{nat}(\text{int}(pc+1)+d),s)]$

**function** *exec'* :: *bprog*  $\Rightarrow$  *state*  $\Rightarrow$  *nat*  $\Rightarrow$  *nat list* **where**  
*exec' ins s pc = (*  
*if* *pc* < *array-length ins* *then (*  
*case* (*array-get ins pc*) *of*  
*AssI ns bs*  $\Rightarrow$  [*pc*] |  
*TestI b d*  $\Rightarrow$  (  
 $\text{if } \text{bval } b \ s \ \text{then } \text{exec}' \ \text{ins } s \ (pc+1)$   
 $\text{else let } pc' = (\text{nat}(\text{int}(pc+1)+d)) \ \text{in if } pc' > pc \ \text{then } \text{exec}' \ \text{ins } s \ pc'$   
 $\text{else } [pc']$   
 $) \mid$   
*ChoiceI bis*  $\Rightarrow$  *let* *succs* =  $[(\text{nat}(\text{int}(pc+1)+i)) \cdot (b,i) <- \text{bis}, \ \text{bval } b \ s]$   
 $\text{in if } \text{succs} = [] \ \text{then } [pc] \ \text{else } \text{concat } (\text{map } (\lambda pc'. \ \text{if } pc' > pc \ \text{then}$   
 $\text{exec}' \ \text{ins } s \ pc' \ \text{else } [pc']) \ \text{succs}) \mid$   
*GotoI d*  $\Rightarrow$  *let* *pc'* =  $\text{nat}(\text{int}(pc+1)+d)$  *in (if* *pc'* > *pc* *then* *exec'* *ins s pc'* *else*  
 $[pc'])$   
 $) \ \text{else } [pc]$   
 $)$   
**by** *pat-completeness auto*  
**termination**  
**apply** (*relation measure* ( $\%(ins,s,pc). \ \text{array-length } ins - pc$ ))

```

apply auto
done

fun nexts1 :: bprog ⇒ config ⇒ config list where
nexts1 ins (pc,s) = (
  if pc < array-length ins then
    exec (array-get ins pc) (pc,s)
  else
    [(pc,s)]

fun nexts :: bprog ⇒ config ⇒ config list where
nexts ins (pc,s) = concat (
  map
    ( $\lambda(pc,s).$  map ( $\lambda pc.$  (pc,s)) (exec' ins s pc))
    (nexts1 ins (pc,s))
  )

declare nexts.simps [simp del]

datatype
  com = SKIP
  | Assign nat list bexp list
  | Seq com com
  | GC (bexp * com)list
  | IfTE bexp com com
  | While bexp com

locale BoolProg-Syntax begin
  notation
    Assign      ( $\langle - ::= - \rangle$  [999, 61] 61)
    and Seq    ( $\langle - ; - \rangle$  [60, 61] 60)
    and GC     ( $\langle IF - FI \rangle$ )
    and IfTE   ( $\langle (IF - / THEN - / ELSE -) \rangle$  [0, 61, 61] 61)
    and While  ( $\langle (WHILE - / DO -) \rangle$  [0, 61] 61)
end

context begin interpretation BoolProg-Syntax .
fun comp' :: com ⇒ instr list where
comp' SKIP = [] |
comp' (Assign n b) = [AssI n b] |
comp' (c1;c2) = comp' c1 @ comp' c2 |
comp' (IF gcs FI) =
  (let cgcs = map ( $\lambda(b,c).$  (b,comp' c)) gcs in
    let addbc = ( $\lambda(b,cc)$  (bis,ins).
      let cc' = cc @ (if ins = [] then [] else [GotoI (int(length ins)])) in
      let bis' = map ( $\lambda(b,i).$  (b, i + int(length cc'))) bis
      in ((b,0)#bis', cc' @ ins) in
      let (bis,ins) = foldr addbc cgcs ([],[])
      in ChoiceI bis # ins) |

```

```

comp' (IF b THEN c1 ELSE c2) =
  (let ins1 = comp' c1 in let ins2 = comp' c2 in
   let i1 = int(length ins1 + 1) in let i2 = int(length ins2)
   in TestI b i1 # ins1 @ GotoI i2 # ins2) |
comp' (WHILE b DO c) =
  (let ins = comp' c in
   let i = int(length ins + 1)
   in TestI b i # ins @ [GotoI (-(i+1))])

```

```

value comp' (IF [(V 0, [1,0] ::= [TT, FF]), (V 1, [0] ::= [TT])] FI)

```

**end**

```

definition comp :: com ⇒ bprog where
  comp = array-of-list ∘ comp'

```

```

fun opt' where
  opt' (GotoI d) ys = (let next = λi. (case i of GotoI d ⇒ d + 1 | - ⇒ 0)
    in if d < 0 ∨ nat d ≥ length ys then (GotoI d)#ys
    else let d' = d + next (ys ! nat d)
    in (GotoI d' # ys))
| opt' x ys = x#ys

```

```

definition opt :: instr list ⇒ instr list where
  opt instr = foldr opt' instr []

```

```

definition optcomp :: com ⇒ bprog where
  optcomp ≡ array-of-list ∘ opt ∘ comp'

```

### 3.2 Finiteness of reachable configurations

**inductive-set** *reachable-configs*

```

for bp :: bprog
and cs :: config — start configuration

```

**where**

```

cs ∈ reachable-configs bp cs |
c ∈ reachable-configs bp cs ⇒ x ∈ set (nexts bp c) ⇒ x ∈ reachable-configs bp
cs

```

```

lemmas reachable-configs-induct = reachable-configs.induct[split-format(complete),case-names
0 1]

```

**fun** *offsets* :: instr ⇒ int set **where**

```

offsets (AssI - -) = {0} |
offsets (TestI - i) = {0,i} |
offsets (ChoiceI bis) = set(map snd bis) ∪ {0} |

```

$offsets (GotoI i) = \{i\}$

**definition**  $offsets-is :: instr\ list \Rightarrow int\ set$  **where**  
 $offsets-is\ ins = (UN\ instr : set\ ins.\ offsets\ instr)$

**definition**  $max-next-pcs :: instr\ list \Rightarrow nat\ set$  **where**  
 $max-next-pcs\ ins = \{nat(int(length\ ins + 1) + i) \mid i.\ i : offsets-is\ ins\}$

**lemma**  $finite-max-next-pcs: finite(max-next-pcs\ bp)$

**proof** –

**{ fix instr have finite (offsets instr) by (cases instr) auto }**  
**moreover**  
**{ fix ins have max-next-pcs ins = (UN i : offsets-is ins. {nat(int(length ins + 1) + i)})**  
**by (auto simp add: max-next-pcs-def) }**  
**ultimately show ?thesis by (auto simp add: offsets-is-def)**  
**qed**

**lemma** (in  $linorder$ )  $le-Max-insertI1: \llbracket finite\ A; x \leq b \rrbracket \Longrightarrow x \leq Max (insert\ b\ A)$

**by** ( $metis\ Max-ge\ finite.insertI\ insert-iff\ order-trans$ )

**lemma** (in  $linorder$ )  $le-Max-insertI2: \llbracket finite\ A; A \neq \{\}; x \leq Max\ A \rrbracket \Longrightarrow x \leq Max (insert\ b\ A)$

**by** ( $auto\ simp\ add: max-def\ not-le\ simp\ del: Max-less-iff$ )

**lemma**  $max-next-pcs-not-empty:$

$pc < length\ bp \Longrightarrow x : set (exec (bp!pc) (pc,s)) \Longrightarrow max-next-pcs\ bp \neq \{\}$   
**apply** ( $drule\ nth-mem$ )  
**apply** ( $fastforce\ simp: max-next-pcs-def\ offsets-is-def\ split: instr.splits$ )  
**done**

**lemma**  $Max-lem2:$

**assumes**  $pc < length\ bp$   
**and**  $(pc', s') \in set (exec (bp!pc) (pc, s))$   
**shows**  $pc' \leq Max (max-next-pcs\ bp)$   
**using**  $assms$   
**proof** ( $cases\ bp\ !\ pc$ )  
**case** ( $ChoiceI\ l$ )  
**show**  $?thesis$   
**proof** ( $cases\ pc' = pc$ )  
**case**  $True$  **with**  $assms\ ChoiceI$  **show**  $?thesis$   
**by** ( $auto\ simp: Max-ge-iff\ max-next-pcs-not-empty\ finite-max-next-pcs$ )  
( $force\ simp\ add: max-next-pcs-def\ offsets-is-def\ dest: nth-mem$ )  
**next**  
**case**  $False$  **with**  $ChoiceI\ assms$  **obtain**  $b\ i$  **where**  
 $bi: bval\ b\ s \quad (b,i) \in set\ l \quad pc' = nat(int(pc+1)+i)$   
**by** ( $auto\ split: if-split-asm$ )

```

with ChoiceI assms have  $i \in \bigcup(\text{offsets } '(\text{set } bp))$  by (force dest: nth-mem)
with bi assms have  $\exists a. (a \in \text{max-next-pcs } bp \wedge pc' \leq a)$ 
  unfolding max-next-pcs-def offsets-is-def by force
thus ?thesis
by (auto simp: Max-ge-iff max-next-pcs-not-empty[OF assms] finite-max-next-pcs)

qed
qed (auto simp: Max-ge-iff max-next-pcs-not-empty finite-max-next-pcs,
  (force simp add: max-next-pcs-def offsets-is-def dest: nth-mem split: if-split-asm)+)

lemma Max-lem1:  $\llbracket pc < \text{length } bp; (pc', s') \in \text{set } (\text{exec } (bp ! pc) (pc, s)) \rrbracket$ 
   $\implies pc' \leq \text{Max } (\text{insert } x (\text{max-next-pcs } bp))$ 
apply(rule le-Max-insertI2)
apply (simp add: finite-max-next-pcs)
apply(simp add: max-next-pcs-not-empty)
apply(auto intro!: Max-lem2 simp del:exec.simps)
done

definition pc-bound bp  $\equiv \text{max}$ 
  ( $\text{Max } (\text{max-next-pcs } (\text{list-of-array } bp)) + 1$ )
  ( $\text{array-length } bp + 1$ )

declare exec'.simps[simp del]

lemma [simp]:  $\text{length } (\text{list-of-array } a) = \text{array-length } a$  by (cases a) auto

lemma aux2:
  assumes A:  $pc < \text{array-length } ins$ 
  assumes B:  $ofs \in \text{offsets-is } (\text{list-of-array } ins)$ 
  shows  $\text{nat } (1 + \text{int } pc + ofs) < \text{pc-bound } ins$ 
proof –
  have  $\text{nat } (\text{int } (1 + \text{array-length } ins) + ofs)$ 
     $\in \text{max-next-pcs } (\text{list-of-array } ins)$ 
  using B unfolding max-next-pcs-def
  by auto
with A show ?thesis
  unfolding pc-bound-def
  apply –
  apply (rule max.strict-coboundedI1)
  apply auto
  apply (drule Max-ge[OF finite-max-next-pcs])
  apply simp
done
qed

lemma array-idx-in-set:
   $\llbracket pc < \text{array-length } ins; \text{array-get } ins \ pc = x \rrbracket$ 
   $\implies x \in \text{set } (\text{list-of-array } ins)$ 
by (induct ins) auto

```

```

lemma rcs-aux:
  assumes  $pc < pc\text{-bound } bp$ 
  assumes  $pc' \in set (exec' bp s pc)$ 
  shows  $pc' < pc\text{-bound } bp$ 
  using assms
proof (induction bp s pc arbitrary: pc' rule: exec'.induct[case-names C])
  case ( $C ins s pc pc'$ )
  from  $C.premis$  show ?case
    apply (subst (asm) exec'.simps)
    apply (split if-split-asm instr.split-asm) +
    apply (simp add: pc-bound-def)

    apply (simp split: if-split-asm add: Let-def)
    apply (frule (2) C.IH(1), auto) []
    apply (auto simp: pc-bound-def) []
    apply (frule (2) C.IH(2), auto) []
    apply (rename-tac bexp int')
    apply (subgoal-tac int'  $\in$  offsets-is (list-of-array ins))
    apply (blast intro: aux2)
    apply (auto simp: offsets-is-def) []
    apply (rule-tac x=TestI bexp int' in bexI, auto simp: array-idx-in-set) []

    apply (rename-tac list)
    apply (auto split: if-split-asm) []
    apply (frule (1) C.IH(3), auto) []
    apply (force)
    apply (force)
    apply (subgoal-tac ba  $\in$  offsets-is (list-of-array ins))
    apply (blast intro: aux2)
    apply (auto simp: offsets-is-def) []
    apply (rule-tac x=ChoiceI list in bexI, auto simp: array-idx-in-set) []

    apply (rename-tac int')
    apply (simp split: if-split-asm add: Let-def)
    apply (frule (1) C.IH(4), auto) []
    apply (subgoal-tac int'  $\in$  offsets-is (list-of-array ins))
    apply (blast intro: aux2)
    apply (auto simp: offsets-is-def) []
    apply (rule-tac x=GotoI int' in bexI, auto simp: array-idx-in-set) []

  apply simp
  done
qed

primrec bexp-vars ::  $bexp \Rightarrow nat\ set$  where
  bexp-vars TT = {}
| bexp-vars FF = {}

```

|  $bexp\text{-vars } (V\ n) = \{n\}$   
|  $bexp\text{-vars } (Not\ b) = bexp\text{-vars } b$   
|  $bexp\text{-vars } (And\ b1\ b2) = bexp\text{-vars } b1 \cup bexp\text{-vars } b2$   
|  $bexp\text{-vars } (Or\ b1\ b2) = bexp\text{-vars } b1 \cup bexp\text{-vars } b2$

**primrec**  $instr\text{-vars} :: instr \Rightarrow nat\ set$  **where**  
 $instr\text{-vars } (AssI\ xs\ bs) = set\ xs \cup \bigcup (bexp\text{-vars}'set\ bs)$   
|  $instr\text{-vars } (TestI\ b\ -) = bexp\text{-vars } b$   
|  $instr\text{-vars } (ChoiceI\ cs) = \bigcup (bexp\text{-vars}'fst'set\ cs)$   
|  $instr\text{-vars } (GotoI\ -) = \{\}$

**find-consts**  $'a\ array \Rightarrow 'a\ list$

**definition**  $bprog\text{-vars} :: bprog \Rightarrow nat\ set$  **where**  
 $bprog\text{-vars } bp = \bigcup (instr\text{-vars}'set\ (list\text{-of}\ array\ bp))$

**definition**  $state\text{-bound } bp\ s0$   
 $\equiv \{s.\ bs\text{-}\alpha\ s - bprog\text{-vars } bp = bs\text{-}\alpha\ s0 - bprog\text{-vars } bp\}$   
**abbreviation**  $config\text{-bound } bp\ s0 \equiv \{0..< pc\text{-bound } bp\} \times state\text{-bound } bp\ s0$

**lemma**  $exec\text{-bound}$ :

**assumes**  $PCB$ :  $pc < array\text{-length } bp$   
**assumes**  $SB$ :  $s \in state\text{-bound } bp\ s0$   
**shows**  $set\ (exec\ (array\text{-get } bp\ pc)\ (pc,s)) \subseteq config\text{-bound } bp\ s0$   
**proof** ( $clarsimp\ simp\ del: exec.simps, intro\ conjI$ )

**obtain**  $instrs$  **where**  $BP\text{-eq}[simp]$ :  $bp = Array\ instrs$  **by** ( $cases\ bp$ )  
**from**  $PCB$  **have**  $PCB'[simp]$ :  $pc < length\ instrs$  **by**  $simp$

**fix**  $pc'\ s'$   
**assume**  $STEP$ :  $(pc',s') \in set\ (exec\ (array\text{-get } bp\ pc)\ (pc,s))$   
**hence**  $STEP'$ :  $(pc',s') \in set\ (exec\ (instrs!pc)\ (pc,s))$  **by**  $simp$

**show**  $pc' < pc\text{-bound } bp$   
**using**  $Max\text{-lem2}[OF\ PCB'\ STEP']$   
**unfolding**  $pc\text{-bound}\text{-def}$  **by**  $simp$

**show**  $s' \in state\text{-bound } bp\ s0$   
**using**  $STEP'\ SB$   
**proof** ( $cases\ instrs!pc$ )  
**case** ( $AssI\ xs\ vs$ )

**have**  $set\ xs \subseteq instr\text{-vars } (instrs!pc)$   
**by** ( $simp\ add: AssI$ )  
**also have**  $\dots \subseteq bprog\text{-vars } bp$   
**apply** ( $simp\ add: bprog\text{-vars}\text{-def}$ )  
**by** ( $metis\ PCB'\ UN\text{-upper}\ nth\text{-mem}$ )  
**finally have**  $XSB$ :  $set\ xs \subseteq bprog\text{-vars } bp$  .

```

{
  fix x s v
  assume A: x ∈ bprog-vars bp    s ∈ state-bound bp s0

  have SB-CNV: bs-α (set-bit s x v)
    = (if v then (insert x (bs-α s)) else (bs-α s - {x}))
    by (cases v) (simp-all add: set-bit-eq flip: bs-insert-def bs-delete-def)

  from A have set-bit s x v ∈ state-bound bp s0
    unfolding state-bound-def
    by (auto simp: SB-CNV)
} note aux=this

{
  fix vs
  have foldl (λs (x, y). set-bit s x y) s (zip xs vs)
    ∈ state-bound (Array instrs) s0
  using SB XSB
  apply (induct xs arbitrary: vs s)
  apply simp
  apply (case-tac vs)
  apply simp
  using aux
  apply (auto)
  done
} note aux2=this

thus ?thesis using STEP'
  by (simp add: AssI)
qed (auto split: if-split-asm)
qed

lemma in-bound-step:
  notes [simp del] = exec.simps
  assumes BOUND: c ∈ config-bound bp s0
  assumes STEP: c' ∈ set (nexts bp c)
  shows c' ∈ config-bound bp s0
  using BOUND STEP
  apply (cases c)
  apply (auto
    simp add: nexts.simps
    split: if-split-asm)
  apply (frule (2) exec-bound[THEN subsetD])
  apply clarsimp
  apply (frule (1) rcs-aux)
  apply simp

```

**apply** (*frule* (2) *exec-bound*[*THEN subsetD*])  
**apply** *clarsimp*

**apply** (*frule* (1) *rcaux*)  
**apply** *simp*  
**done**

**lemma** *reachable-configs-in-bound*:

$c \in \text{config-bound } bp \ s0 \implies \text{reachable-configs } bp \ c \subseteq \text{config-bound } bp \ s0$

**proof**

**fix**  $c'$

**assume**  $c' \in \text{reachable-configs } bp \ c \quad c \in \text{config-bound } bp \ s0$

**thus**  $c' \in \text{config-bound } bp \ s0$

**apply** *induction*

**apply** *simp*

**by** (*rule in-bound-step*)

**qed**

**lemma** *reachable-configs-out-of-bound*:  $(pc', s') \in \text{reachable-configs } bp \ (pc, s)$

$\implies \neg pc < \text{pc-bound } bp \implies (pc', s') = (pc, s)$

**proof** (*induct rule: reachable-configs-induct*)

**case** (1  $pc' \ s' \ pc'' \ s''$ )

**hence** [*simp*]:  $pc' = pc \quad s' = s$  **by** *auto*

**from** 1(4) **have**  $\neg pc < \text{array-length } bp$  **unfolding** *pc-bound-def* **by** *auto*

**with** 1(3) **show** ?*case*

**by** (*auto simp add: nexts.simps exec'.simps*)

**qed** *auto*

**lemma** *finite-bexp-vars*[*simp, intro!*]: *finite* (*bexp-vars* *be*)

**by** (*induction be*) *auto*

**lemma** *finite-instr-vars*[*simp, intro!*]: *finite* (*instr-vars* *ins*)

**by** (*cases ins*) *auto*

**lemma** *finite-bprog-vars*[*simp, intro!*]: *finite* (*bprog-vars* *bp*)

**unfolding** *bprog-vars-def* **by** *simp*

**lemma** *finite-state-bound*[*simp, intro!*]: *finite* (*state-bound* *bp* *s0*)

**unfolding** *state-bound-def*

**apply** (*rule finite-imageD*[**where**  $f = bs-\alpha$ ])

**apply** (*rule finite-subset*[**where**

$B = \{s. s - \text{bprog-vars } bp = bs-\alpha \ s0 - \text{bprog-vars } bp\}$ ])

**apply** *auto* []

**apply** (*rule finite-if-eq-beyond-finite*)

**apply** *simp*

**apply** (*rule inj-onI*)

**apply** (*fold bs-eq-def*)

**apply** (*auto simp: bs-eq-correct*)

```

done

lemma finite-config-bound[simp, intro!]: finite (config-bound bp s0)
  by blast

lemma reachable-configs-finite[simp, intro!]:
  finite (reachable-configs bp c)
proof (cases c, clarsimp)
  fix pc s
  show finite (reachable-configs bp (pc, s))
  proof (cases pc < pc-bound bp)
    case False from reachable-configs-out-of-bound[OF - False, where s=s]
    have reachable-configs bp (pc, s) ⊆ {(pc,s)} by auto
    thus ?thesis by (rule finite-subset) auto
  next
    case True
    hence (pc,s) ∈ config-bound bp s
    by (simp add: state-bound-def)
    thus ?thesis
    by (rule finite-subset[OF reachable-configs-in-bound]) simp
  qed
qed

definition bpc-is-run bpc r ≡ let (bp,c)=bpc in r 0 = c ∧ (∀i. r (Suc i) ∈ set
  (BoolProgs.nexts bp (r i)))
definition bpc-props c ≡ bs-α (snd c)
definition bpc-lang bpc ≡ {bpc-props o r | r. bpc-is-run bpc r}

fun print-config ::
  (nat ⇒ string) ⇒ (bitset ⇒ string) ⇒ config ⇒ string where
  print-config f fx (p,s) = f p @ " " @ fx s

end

```

## References

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