Birkhoff's Representation Theorem For Finite Distributive Lattices

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Abstract

This theory proves a theorem of Birkhoff that asserts that any finite distributive lattice is isomorphic to the set of *down-sets* of that lattice's join-irreducible elements. The isomorphism preserves order, meets and joins as well as complementation in the case the lattice is a Boolean algebra. A consequence of this representation theorem is that every finite Boolean algebra is isomorphic to a powerset algebra.

Contents

theory *Birkhoff-Finite-Distributive-Lattices*
  imports
    HOL−Library.Finite-Lattice
    HOL.Transcendental
begin

unbundle *lattice-syntax*

The proof of Birkhoff's representation theorem for finite distributive lattices [1] presented here follows Davey and Priestley [2].

1 Atoms, Join Primes and Join Irreducibles

Atomic elements are defined as follows.

definition (**in** *bounded-lattice-bot*) *atomic* :: $'a \Rightarrow bool$ **where**
  $atomic\ x \equiv x \neq \bot \wedge (\forall\ y.\ y \leq x \longrightarrow y = \bot \vee y = x)$

Two related concepts are *join-prime* elements and *join-irreducible* elements.

definition (**in** *bounded-lattice-bot*) *join-prime* :: $'a \Rightarrow bool$ **where**
  $join\text{-}prime\ x \equiv x \neq \bot \wedge (\forall\ y\ z\ .\ x \leq y \sqcup z \longrightarrow x \leq y \vee x \leq z)$

definition (**in** *bounded-lattice-bot*) *join-irreducible* :: $'a \Rightarrow bool$ **where**
  $join\text{-}irreducible\ x \equiv x \neq \bot \wedge (\forall\ y\ z\ .\ y < x \longrightarrow z < x \longrightarrow y \sqcup z < x)$

lemma (**in** *bounded-lattice-bot*) *join-irreducible-def'*:
  $join\text{-}irreducible\ x = (x \neq \bot \wedge (\forall\ y\ z\ .\ x = y \sqcup z \longrightarrow x = y \vee x = z))$
  ⟨*proof*⟩

Every join-prime is also join-irreducible.

lemma (**in** *bounded-lattice-bot*) *join-prime-implies-join-irreducible*:
  assumes *join-prime x*
  shows *join-irreducible x*
  ⟨*proof*⟩

In the special case when the underlying lattice is distributive, the join-prime elements and join-irreducible elements coincide.

class *bounded-distrib-lattice-bot* = *bounded-lattice-bot* +
  assumes *sup-inf-distrib1*: $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$
begin

subclass *distrib-lattice*
  ⟨*proof*⟩

end

context *complete-distrib-lattice*
begin

subclass *bounded-distrib-lattice-bot*
  ⟨*proof*⟩

end

lemma (**in** *bounded-distrib-lattice-bot*) *join-irreducible-is-join-prime*:
  join-irreducible x = *join-prime* x
⟨*proof*⟩

Every atomic element is join-irreducible.

lemma (**in** *bounded-lattice-bot*) *atomic-implies-join-prime*:
  assumes *atomic* x
  shows *join-irreducible* x
  ⟨*proof*⟩

In the case of Boolean algebras, atomic elements and join-prime elements are one-in-the-same.

lemma (**in** *boolean-algebra*) *join-prime-is-atomic*:
  atomic x = *join-prime* x
⟨*proof*⟩

All atomic elements are disjoint.

lemma (**in** *bounded-lattice-bot*) *atomic-disjoint*:
  assumes *atomic* α
    and *atomic* β
   shows $(\alpha = \beta) \longleftrightarrow (\alpha \sqcap \beta \neq \bot)$
⟨*proof*⟩

definition (**in** *bounded-lattice-bot*) *atomic-elements* (‹$\mathcal{A}$›) **where**
  $\mathcal{A} \equiv \{a\ .\ \textit{atomic}\ a\}$

definition (**in** *bounded-lattice-bot*) *join-irreducible-elements* (‹$\mathcal{J}$›) **where**
  $\mathcal{J} \equiv \{a\ .\ \textit{join-irreducible}\ a\}$

2 Birkhoff's Representation Theorem For Finite Distributive Lattices

Birkhoff's representation theorem for finite distributive lattices follows from the fact that every non-⊥ element can be represented by the join-irreducible elements beneath it.

In this section we merely demonstrate the representation aspect of Birkhoff's theorem. In §3 we show this representation is a lattice homomorphism.

The fist step to representing elements is to show that there *exist* join-irreducible elements beneath them. This is done by showing if there is no join-irreducible element, we can make a descending chain with more elements than the finite Boolean algebra under consideration.

fun (**in** *order*) *descending-chain-list* :: *'a list* $\Rightarrow$ *bool* **where**
  descending-chain-list [] = *True*
| *descending-chain-list* [x] = *True*
| *descending-chain-list* (x # x' # *xs*)
    = ($x < x'$ $\wedge$ *descending-chain-list* (x' # *xs*))

lemma (**in** *order*) *descending-chain-list-tail*:
  assumes *descending-chain-list* (s # S)
  shows *descending-chain-list* S
  ⟨*proof*⟩

lemma (**in** *order*) *descending-chain-list-drop-penultimate*:
  assumes *descending-chain-list* (s # s' # S)
  shows *descending-chain-list* (s # S)
  ⟨*proof*⟩

lemma (**in** *order*) *descending-chain-list-less-than-others*:
  assumes *descending-chain-list* (s # S)
  shows $\forall s' \in \textit{set}\ S.\ s < s'$
  ⟨*proof*⟩

lemma (**in** *order*) *descending-chain-list-distinct*:
  assumes *descending-chain-list* S
  shows *distinct* S
  ⟨*proof*⟩

lemma (**in** *finite-distrib-lattice*) *join-irreducible-lower-bound-exists*:
  assumes $\neg\ (x \leq y)$
  shows $\exists\ z \in \mathcal{J}.\ z \leq x \wedge \neg\ (z \leq y)$
⟨*proof*⟩

definition (**in** *bounded-lattice-bot*)
  join-irreducibles-embedding :: *'a* $\Rightarrow$ *'a set* (‹⦃ - ⦄› [*50*]) **where**
  $⦃\ x\ ⦄ \equiv \{a \in \mathcal{J}.\ a \leq x\}$

We can now show every element is exactly the suprema of the join-irreducible elements beneath them in any distributive lattice.

theorem (**in** *finite-distrib-lattice*) *sup-join-prime-embedding-ident*:
  $x = \bigsqcup\ ⦃\ x\ ⦄$
⟨*proof*⟩

Just as $x = \bigsqcup\ ⦃\ x\ ⦄$, the reverse is also true; $\lambda\ x.\ ⦃\ x\ ⦄$ and $\lambda\ S.\ \bigsqcup\ S$ are inverses where $S \in \mathcal{O}\mathcal{J}$, the set of downsets in *Pow* $\mathcal{J}$.

definition (**in** *bounded-lattice-bot*) *down-irreducibles* (‹$\mathcal{O}\mathcal{J}$›) **where**
  $\mathcal{O}\mathcal{J} \equiv \{\ S \in \textit{Pow}\ \mathcal{J}\ .\ (\exists\ x\ .\ S = ⦃\ x\ ⦄)\ \}$

lemma (**in** *finite-distrib-lattice*) *join-irreducible-embedding-sup-ident*:
assumes $S \in \mathcal{O}\mathcal{J}$
shows $S = \{\!|\ \bigsqcup\ S\ |\!\}$
$\langle proof \rangle$

Given that $\lambda\ x.\ \{\!|\ x\ |\!\}$ has a left and right inverse, we can show it is a *bijection*.

The bijection below is recognizable as a form of *Birkhoff's Representation Theorem* for finite distributive lattices.

theorem (**in** *finite-distrib-lattice*) *birkhoffs-theorem*:
bij-betw $(\lambda\ x.\ \{\!|\ x\ |\!\})$ *UNIV* $\mathcal{O}\mathcal{J}$
$\langle proof \rangle$

3 Finite Ditributive Lattice Isomorphism

The form of Birkhoff's theorem presented in §2 simply gave a bijection between a finite distributive lattice and the downsets of its join-irreducible elements. This relationship can be extended to a full-blown *lattice homomorphism*. In particular we have the following properties:

- $\bot$ and $\top$ are preserved; specifically $\{\!|\ \bot\ |\!\} = \{\}$ and $\{\!|\ \top\ |\!\} = \mathcal{J}$.
- Order is preserved: $x \leq y = (\{\!|\ x\ |\!\} \subseteq \{\!|\ y\ |\!\})$.
- $\lambda\ x\ .\ \{\!|\ x\ |\!\}$ is a lower complete semi-lattice homomorphism, mapping $\{\!|\ \bigsqcup\ X\ |\!\} = (\bigcup\ x \in X\ .\ \{\!|\ x\ |\!\})$.
- In addition to preserving arbitrary joins, $\lambda\ x\ .\ \{\!|\ x\ |\!\}$ is a lattice homomorphism, since it also preserves finitary meets with $\{\!|\ x \sqcap y\ |\!\} = \{\!|\ x\ |\!\} \cap \{\!|\ y\ |\!\}$. Arbitrary meets are also preserved, but relative to a top element $\mathcal{J}$, or in other words $\{\!|\ \bigsqcap\ X\ |\!\} = \mathcal{J} \cap (\bigcap\ x \in X.\ \{\!|\ x\ |\!\})$.
- In the case of a Boolean algebra, complementation corresponds to relative set complementation via $\{\!|\ -\ x\ |\!\} = \mathcal{J} - \{\!|\ x\ |\!\}$.

lemma (**in** *finite-distrib-lattice*) *join-irreducibles-bot*:
$\{\!|\ \bot\ |\!\} = \{\}$
$\langle proof \rangle$

lemma (**in** *finite-distrib-lattice*) *join-irreducibles-top*:
$\{\!|\ \top\ |\!\} = \mathcal{J}$
$\langle proof \rangle$

lemma (**in** *finite-distrib-lattice*) *join-irreducibles-order-isomorphism*:
$x \leq y = (\{\!|\ x\ |\!\} \subseteq \{\!|\ y\ |\!\})$
$\langle proof \rangle$

lemma (**in** *finite-distrib-lattice*) *join-irreducibles-join-homomorphism*:
$\{\!\mid x \sqcup y \mid\!\} = \{\!\mid x \mid\!\} \cup \{\!\mid y \mid\!\}$
⟨*proof*⟩

lemma (**in** *finite-distrib-lattice*) *join-irreducibles-sup-homomorphism*:
$\{\!\mid \bigsqcup X \mid\!\} = (\bigcup x \in X \,.\, \{\!\mid x \mid\!\})$
⟨*proof*⟩

lemma (**in** *finite-distrib-lattice*) *join-irreducibles-meet-homomorphism*:
$\{\!\mid x \sqcap y \mid\!\} = \{\!\mid x \mid\!\} \cap \{\!\mid y \mid\!\}$
⟨*proof*⟩

Arbitrary meets are also preserved, but relative to a top element $\mathcal{J}$.

lemma (**in** *finite-distrib-lattice*) *join-irreducibles-inf-homomorphism*:
$\{\!\mid \bigsqcap X \mid\!\} = \mathcal{J} \cap (\bigcap x \in X.\ \{\!\mid x \mid\!\})$
⟨*proof*⟩

Finally, we show that complementation is preserved.

To begin, we define the class of finite Boolean algebras. This class is simply an extension of *boolean-algebra*, extended with *finite UNIV* as per the axiom class *finite*. We also also extend the language of the class with *infima* and *suprema* (i.e. $\bigsqcap A$ and $\bigsqcup A$ respectively).

class *finite-boolean-algebra* = *boolean-algebra* + *finite* + *Inf* + *Sup* +
assumes *Inf-def*: $\bigsqcap A = \mathit{Finite\text{-}Set.fold}\ (\sqcap)\ \top\ A$
assumes *Sup-def*: $\bigsqcup A = \mathit{Finite\text{-}Set.fold}\ (\sqcup)\ \bot\ A$
begin

Finite Boolean algebras are trivially a subclass of finite distributive lattices, which are necessarily *complete*.

subclass *finite-distrib-lattice-complete*
⟨*proof*⟩

subclass *bounded-distrib-lattice-bot*
⟨*proof*⟩
end

lemma (**in** *finite-boolean-algebra*) *join-irreducibles-complement-homomorphism*:
$\{\!\mid - x \mid\!\} = \mathcal{J} - \{\!\mid x \mid\!\}$
⟨*proof*⟩

4 Cardinality

Another consequence of Birkhoff's theorem from §2 is that every finite Boolean algebra has a cardinality which is a power of two. This gives a bound on the number of atoms/join-prime/irreducible elements, which must be logarithmic in the size of the finite Boolean algebra they belong to.

We first show that $\mathcal{O}\mathcal{J}$, the downsets of the join-irreducible elements $\mathcal{J}$, are the same as the powerset of $\mathcal{J}$ in any finite Boolean algebra.

```
lemma (in finite-boolean-algebra) 𝒪𝒥-is-Pow-𝒥:
  𝒪𝒥 = Pow 𝒥
⟨proof⟩


lemma (in finite-boolean-algebra) UNIV-card:
  card (UNIV::'a set) = card (Pow 𝒥)
  ⟨proof⟩

lemma finite-Pow-card:
  assumes finite X
  shows card (Pow X) = 2 powr (card X)
  ⟨proof⟩

lemma (in finite-boolean-algebra) UNIV-card-powr-2:
  card (UNIV::'a set) = 2 powr (card 𝒥)
  ⟨proof⟩

lemma (in finite-boolean-algebra) join-irreducibles-card-log-2:
  card 𝒥 = log 2 (card (UNIV :: 'a set))
⟨proof⟩

end
```

References

[1] G. Birkhoff. Rings of sets. *Duke Mathematical Journal*, 3(3):443–454, Sept. 1937.

[2] B. A. Davey and H. A. Priestley. Chapter 5. Representation: The finite case. In *Introduction to Lattices and Order*, pages 112–124. Cambridge University Press, Cambridge, UK ; New York, NY, 2nd ed edition, 2002.