Automatic Data Refinement

Peter Lammich

June 25, 2026

Abstract

We present the Autoref tool for Isabelle/HOL, which automatically refines algorithms specified over abstract concepts like maps and sets to algorithms over concrete implementations like red-black-trees, and produces a refinement theorem. It is based on ideas borrowed from relational parametricity due to Reynolds and Wadler. The tool allows for rapid prototyping of verified, executable algorithms. Moreover, it can be configured to fine-tune the result to the user's needs. Our tool is able to automatically instantiate generic algorithms, which greatly simplifies the implementation of executable data structures.

This AFP-entry provides the basic tool, which is then used by the Refinement and Collection Framework to provide automatic data refinement for the nondeterminism monad and various collection datastructures.

Contents

1 Parametricity Solver **5**
- 1.1 Relators . . . 5
  - 1.1.1 Basic Definitions . . . 5
  - 1.1.2 Basic HOL Relators . . . 6
  - 1.1.3 Automation . . . 12
  - 1.1.4 Setup . . . 17
  - 1.1.5 Invariant and Abstraction . . . 21
  - 1.1.6 Miscellanneous . . . 22
  - 1.1.7 Conversion between Predicate and Set Based Relators 22
  - 1.1.8 More Properties . . . 24
- 1.2 Basic Parametricity Reasoning . . . 25
  - 1.2.1 Auxiliary Lemmas . . . 25
  - 1.2.2 ML-Setup . . . 25
  - 1.2.3 Convenience Tools . . . 31
- 1.3 Parametricity Theorems for HOL . . . 32
  - 1.3.1 Sets . . . 32
  - 1.3.2 Standard HOL Constructs . . . 34
  - 1.3.3 Functions . . . 34
  - 1.3.4 Boolean . . . 34
  - 1.3.5 Nat . . . 35
  - 1.3.6 Int . . . 35
  - 1.3.7 Product . . . 36
  - 1.3.8 Option . . . 37
  - 1.3.9 Sum . . . 38
  - 1.3.10 List . . . 39

2 Automatic Refinement **45**
- 2.1 Automatic Refinement Tool . . . 45
  - 2.1.1 Standard setup . . . 45
  - 2.1.2 Tools . . . 46
  - 2.1.3 Advanced Debugging . . . 47
- 2.2 Standard HOL Bindings . . . 49
  - 2.2.1 Structural Expansion . . . 49

2.2.2 Booleans 50
2.2.3 Standard Type Classes 51
2.2.4 Functional Combinators 52
2.2.5 Unit 53
2.2.6 Nat 53
2.2.7 Int 54
2.2.8 Product 54
2.2.9 Option 55
2.2.10 Sum-Types 56
2.2.11 List 57
2.2.12 Examples 61
2.3 Entry Point for the Automatic Refinement Tool 62
2.3.1 Convenience 62

Chapter 1

Parametricity Solver

1.1 Relators

theory *Relators*
imports *../Lib/Refine-Lib*
begin

We define the concept of relators. The relation between a concrete type and an abstract type is expressed by a relation of type $('c \times 'a)$ *set*. For each composed type, say *'a list*, we can define a *relator*, that takes as argument a relation for the element type, and returns a relation for the list type. For most datatypes, there exists a *natural relator*. For algebraic datatypes, this is the relator that preserves the structure of the datatype, and changes the components. For example, *list-rel*$::('c \times 'a)$ *set* $\Rightarrow$ $('c$ *list* $\times 'a$ *list*$)$ *set* is the natural relator for lists.

However, relators can also be used to change the representation, and thus relate an implementation with an abstract type. For example, the relator *list-set-rel*$::('c \times 'a)$ *set* $\Rightarrow$ $('c$ *list* $\times 'a$ *set*$)$ *set* relates lists with the set of their elements.

In this theory, we define some basic notions for relators, and then define natural relators for all HOL-types, including the function type. For each relator, we also show a single-valuedness property, and initialize a solver for single-valued properties.

1.1.1 Basic Definitions

For smoother handling of relator unification, we require relator arguments to be applied by a special operator, such that we avoid higher-order unification problems. We try to set up some syntax to make this more transparent, and give relators a type-like prefix-syntax.

definition *relAPP*
:: $(('c1 \times 'a1)$ *set* $\Rightarrow$ -$) \Rightarrow ('c1 \times 'a1)$ *set* $\Rightarrow$ -

where *relAPP f x* ≡ *f x*

syntax *-rel-APP* :: *args* ⇒ *'a* ⇒ *'b* (‹⟨-⟩-› [*0*,*900*] *900*)

syntax-consts *-rel-APP* == *relAPP*

translations
⟨*x*,*xs*⟩*R* == ⟨*xs*⟩(*CONST relAPP R x*)
⟨*x*⟩*R* == *CONST relAPP R x*

ML ‹
structure Refine-Relators-Thms = struct
structure rel-comb-def-rules = Named-Thms (
val name = @{binding refine-rel-defs}
val description = Refinement Framework: ^
Relator definitions
);
end
›

setup *Refine-Relators-Thms.rel-comb-def-rules.setup*

1.1.2 Basic HOL Relators

Function

definition *fun-rel* **where**
fun-rel-def-internal: *fun-rel A B* ≡ { (*f*,*f'*). ∀(*a*,*a'*)∈*A*. (*f a*, *f' a'*)∈*B* }
abbreviation *fun-rel-syn* (**infixr** ‹→› *60*) **where** *A*→*B* ≡ ⟨*A*,*B*⟩*fun-rel*

lemma *fun-rel-def*[*refine-rel-defs*]:
A→*B* ≡ { (*f*,*f'*). ∀(*a*,*a'*)∈*A*. (*f a*, *f' a'*)∈*B* }
by (*simp add*: *relAPP-def fun-rel-def-internal*)

lemma *fun-relI*[*intro*!]: ⟦⋀*a a'*. (*a*,*a'*)∈*A* ⟹ (*f a*,*f' a'*)∈*B*⟧ ⟹ (*f*,*f'*)∈*A*→*B*
by (*auto simp*: *fun-rel-def*)

lemma *fun-relD*:
shows ((*f*,*f'*)∈(*A*→*B*)) ⟹
(⋀*x x'*. ⟦ (*x*,*x'*)∈*A* ⟧ ⟹ (*f x*, *f' x'*)∈*B*)
apply *rule*
by (*auto simp*: *fun-rel-def*)

lemma *fun-relD1*:
assumes (*f*,*f'*)∈*Ra*→*Rr*
assumes *f x* = *r*
shows ∀*x'*. (*x*,*x'*)∈*Ra* ⟶ (*r*,*f' x'*)∈*Rr*
using *assms* **by** (*auto simp*: *fun-rel-def*)

lemma *fun-relD2*:
assumes $(f,f')\in Ra\to Rr$
assumes $f'\ x' = r'$
shows $\forall x.\ (x,x')\in Ra \longrightarrow (f\ x,r')\in Rr$
using *assms* **by** (*auto simp*: *fun-rel-def*)

lemma *fun-relE1*:
assumes $(f,f')\in Id \to Rv$
assumes $t' = f'\ x$
shows $(f\ x,t')\in Rv$ **using** *assms*
by (*auto elim*: *fun-relD*)

lemma *fun-relE2*:
assumes $(f,f')\in Id \to Rv$
assumes $t = f\ x$
shows $(t,f'\ x)\in Rv$ **using** *assms*
by (*auto elim*: *fun-relD*)

Terminal Types

abbreviation *unit-rel* :: (*unit*×*unit*) *set* **where** *unit-rel* == *Id*

abbreviation *nat-rel* ≡ *Id*::(*nat*×-) *set*
abbreviation *int-rel* ≡ *Id*::(*int*×-) *set*
abbreviation *bool-rel* ≡ *Id*::(*bool*×-) *set*

Product

definition *prod-rel* **where**
prod-rel-def-internal: *prod-rel R1 R2*
≡ { $((a,b),(a',b'))$. $(a,a')\in R1 \wedge (b,b')\in R2$ }

abbreviation *prod-rel-syn* (**infixr** ‹×$_r$› *70*) **where** $a\times_r b \equiv \langle a,b\rangle$*prod-rel*

lemma *prod-rel-def*[*refine-rel-defs*]:
$(\langle R1,R2\rangle$*prod-rel*$) \equiv$ { $((a,b),(a',b'))$. $(a,a')\in R1 \wedge (b,b')\in R2$ }
by (*simp add*: *prod-rel-def-internal relAPP-def*)

lemma *prod-relI*: $[\![(a,a')\in R1;\ (b,b')\in R2]\!] \Longrightarrow ((a,b),(a',b'))\in\langle R1,R2\rangle$*prod-rel*
by (*auto simp*: *prod-rel-def*)
lemma *prod-relE*:
assumes $(p,p')\in\langle R1,R2\rangle$*prod-rel*
obtains *a b a' b'* **where** $p=(a,b)$ **and** $p'=(a',b')$
and $(a,a')\in R1$ **and** $(b,b')\in R2$
using *assms*
by (*auto simp*: *prod-rel-def*)

lemma *prod-rel-simp*[*simp*]:
$((a,b),(a',b'))\in\langle R1,R2\rangle$*prod-rel* $\longleftrightarrow (a,a')\in R1 \wedge (b,b')\in R2$
by (*auto intro*: *prod-relI elim*: *prod-relE*)

lemma *in-Domain-prod-rel-iff*[*iff*]: $(a,b)\in Domain\ (A\times_r B) \longleftrightarrow a\in Domain\ A \wedge b\in Domain\ B$
by (*auto simp*: *prod-rel-def*)

lemma *prod-rel-comp*: $(A \times_r B)\ O\ (C \times_r D) = (A\ O\ C) \times_r (B\ O\ D)$
unfolding *prod-rel-def*
by *auto*

Option

definition *option-rel* **where**
option-rel-def-internal:
option-rel R $\equiv$ { (*Some a*,*Some a'*) | *a a'*. $(a,a')\in R$ } $\cup$ {(*None*,*None*)}

lemma *option-rel-def*[*refine-rel-defs*]:
$\langle R\rangle$*option-rel* $\equiv$ { (*Some a*,*Some a'*) | *a a'*. $(a,a')\in R$ } $\cup$ {(*None*,*None*)}
by (*simp add*: *option-rel-def-internal relAPP-def*)

lemma *option-relI*:
$(None,None)\in\langle R\rangle$ *option-rel*
$[\![\ (a,a')\in R\ ]\!] \Longrightarrow (Some\ a,\ Some\ a')\in\langle R\rangle$*option-rel*
by (*auto simp*: *option-rel-def*)

lemma *option-relE*:
assumes $(x,x')\in\langle R\rangle$*option-rel*
obtains $x=None$ **and** $x'=None$
| *a a'* **where** $x=Some\ a$ **and** $x'=Some\ a'$ **and** $(a,a')\in R$
using *assms* **by** (*auto simp*: *option-rel-def*)

lemma *option-rel-simp*[*simp*]:
$(None,a)\in\langle R\rangle$*option-rel* $\longleftrightarrow a=None$
$(c,None)\in\langle R\rangle$*option-rel* $\longleftrightarrow c=None$
$(Some\ x,Some\ y)\in\langle R\rangle$*option-rel* $\longleftrightarrow (x,y)\in R$
by (*auto intro*: *option-relI elim*: *option-relE*)

Sum

definition *sum-rel* **where** *sum-rel-def-internal*:
sum-rel Rl Rr
$\equiv$ { (*Inl a*, *Inl a'*) | *a a'*. $(a,a')\in Rl$ } $\cup$
{ (*Inr a*, *Inr a'*) | *a a'*. $(a,a')\in Rr$ }

lemma *sum-rel-def*[*refine-rel-defs*]:
$\langle Rl,Rr\rangle$*sum-rel* $\equiv$
{ (*Inl a*, *Inl a'*) | *a a'*. $(a,a')\in Rl$ } $\cup$
{ (*Inr a*, *Inr a'*) | *a a'*. $(a,a')\in Rr$ }
by (*simp add*: *sum-rel-def-internal relAPP-def*)

lemma *sum-rel-simp*[*simp*]:

$\bigwedge a\ a'.\ (Inl\ a,\ Inl\ a') \in \langle Rl,Rr\rangle sum\text{-}rel \longleftrightarrow (a,a')\in Rl$
$\bigwedge a\ a'.\ (Inr\ a,\ Inr\ a') \in \langle Rl,Rr\rangle sum\text{-}rel \longleftrightarrow (a,a')\in Rr$
$\bigwedge a\ a'.\ (Inl\ a,\ Inr\ a') \notin \langle Rl,Rr\rangle sum\text{-}rel$
$\bigwedge a\ a'.\ (Inr\ a,\ Inl\ a') \notin \langle Rl,Rr\rangle sum\text{-}rel$
unfolding *sum-rel-def* **by** *auto*

lemma *sum-relI*:
$(l,l')\in Rl \Longrightarrow (Inl\ l,\ Inl\ l') \in \langle Rl,Rr\rangle sum\text{-}rel$
$(r,r')\in Rr \Longrightarrow (Inr\ r,\ Inr\ r') \in \langle Rl,Rr\rangle sum\text{-}rel$
by *simp-all*

lemma *sum-relE*:
assumes $(x,x')\in\langle Rl,Rr\rangle sum\text{-}rel$
obtains
$l\ l'$ **where** $x=Inl\ l$ **and** $x'=Inl\ l'$ **and** $(l,l')\in Rl$
$|\ r\ r'$ **where** $x=Inr\ r$ **and** $x'=Inr\ r'$ **and** $(r,r')\in Rr$
using *assms* **by** (*auto simp*: *sum-rel-def*)

Lists

definition *list-rel* **where** *list-rel-def-internal*:
$list\text{-}rel\ R \equiv \{(l,l').\ list\text{-}all2\ (\lambda x\ x'.\ (x,x')\in R)\ l\ l'\}$

lemma *list-rel-def*[*refine-rel-defs*]:
$\langle R\rangle list\text{-}rel \equiv \{(l,l').\ list\text{-}all2\ (\lambda x\ x'.\ (x,x')\in R)\ l\ l'\}$
by (*simp add*: *list-rel-def-internal relAPP-def*)

lemma *list-rel-induct*[*induct set*,*consumes 1*, *case-names Nil Cons*]:
assumes $(l,l')\in\langle R\rangle\ list\text{-}rel$
assumes $P\ []\ []$
assumes $\bigwedge x\ x'\ l\ l'.\ [\![\ (x,x')\in R;\ (l,l')\in\langle R\rangle list\text{-}rel;\ P\ l\ l'\ ]\!]$
$\Longrightarrow P\ (x\#l)\ (x'\#l')$
shows $P\ l\ l'$
using *assms* **unfolding** *list-rel-def*
apply *simp*
by (*rule list-all2-induct*)

lemma *list-rel-eq-listrel*: $list\text{-}rel = listrel$
apply (*rule ext*)
apply *safe*
proof *goal-cases*
case (*1 x a b*) **thus** *?case*
unfolding *list-rel-def-internal*
apply *simp*
apply (*induct a b rule*: *list-all2-induct*)
apply (*auto intro*: *listrel.intros*)
done
next
case *2* **thus** *?case*

```
    apply (induct)
    apply (auto simp: list-rel-def-internal)
    done
qed

lemma list-relI:
  ([],[])∈⟨R⟩list-rel
  ⟦ (x,x')∈R; (l,l')∈⟨R⟩list-rel ⟧ ⟹ (x#l,x'#l')∈⟨R⟩list-rel
  by (auto simp: list-rel-def)

lemma list-rel-simp[simp]:
  ([],l')∈⟨R⟩list-rel ⟷ l'=[]
  (l,[])∈⟨R⟩list-rel ⟷ l=[]
  ([],[])∈⟨R⟩list-rel
  (x#l,x'#l')∈⟨R⟩list-rel ⟷ (x,x')∈R ∧ (l,l')∈⟨R⟩list-rel
  by (auto simp: list-rel-def)

lemma list-relE1:
  assumes (l,[])∈⟨R⟩list-rel obtains l=[] using assms by auto

lemma list-relE2:
  assumes ([],l)∈⟨R⟩list-rel obtains l=[] using assms by auto

lemma list-relE3:
  assumes (x#xs,l')∈⟨R⟩list-rel obtains x' xs' where
  l'=x'#xs' and (x,x')∈R and (xs,xs')∈⟨R⟩list-rel
  using assms
  apply (cases l')
  apply auto
  done

lemma list-relE4:
  assumes (l,x'#xs')∈⟨R⟩list-rel obtains x xs where
  l=x#xs and (x,x')∈R and (xs,xs')∈⟨R⟩list-rel
  using assms
  apply (cases l)
  apply auto
  done

lemmas list-relE = list-relE1 list-relE2 list-relE3 list-relE4

lemma list-rel-imp-same-length:
    (l, l') ∈ ⟨R⟩list-rel ⟹ length l = length l'
  unfolding list-rel-eq-listrel relAPP-def
  by (rule listrel-eq-len)

lemma list-rel-split-right-iff:
  (x#xs,l)∈⟨R⟩list-rel ⟷ (∃ y ys. l=y#ys ∧ (x,y)∈R ∧ (xs,ys)∈⟨R⟩list-rel)
  by (cases l) auto
```

lemma *list-rel-split-left-iff*:
(*l*,*y*#*ys*)∈⟨*R*⟩*list-rel* ⟷ (∃ *x xs*. *l*=*x*#*xs* ∧ (*x*,*y*)∈*R* ∧ (*xs*,*ys*)∈⟨*R*⟩*list-rel*)
by (*cases l*) *auto*

Sets

Pointwise refinement: The abstract set is the image of the concrete set, and the concrete set only contains elements that have an abstract counterpart

definition *set-rel* **where**
set-rel-def-internal:
set-rel R ≡ {(*A*,*B*). (∀ *x*∈*A*. ∃ *y*∈*B*. (*x*,*y*)∈*R*) ∧ (∀ *y*∈*B*. ∃ *x*∈*A*. (*x*,*y*)∈*R*)}

term *set-rel*

lemma *set-rel-def*[*refine-rel-defs*]:
⟨*R*⟩*set-rel* ≡ {(*A*,*B*). (∀ *x*∈*A*. ∃ *y*∈*B*. (*x*,*y*)∈*R*) ∧ (∀ *y*∈*B*. ∃ *x*∈*A*. (*x*,*y*)∈*R*)}
by (*simp add*: *set-rel-def-internal relAPP-def*)

lemma *set-rel-alt*: ⟨*R*⟩*set-rel* = {(*A*,*B*). *A* ⊆ *R*$^{-1}$ ‘‘*B* ∧ *B* ⊆ *R*‘‘*A*}
unfolding *set-rel-def* **by** *auto*

lemma *set-relI*[*intro?*]:
assumes ⋀*x*. *x*∈*A* ⟹ ∃ *y*∈*B*. (*x*,*y*)∈*R*
assumes ⋀*y*. *y*∈*B* ⟹ ∃ *x*∈*A*. (*x*,*y*)∈*R*
shows (*A*,*B*)∈⟨*R*⟩*set-rel*
using *assms* **unfolding** *set-rel-def* **by** *blast*

Original definition of *set-rel* in refinement framework. Abandoned in favour of more symmetric definition above:

definition *old-set-rel* **where** *old-set-rel-def-internal*:
old-set-rel R ≡ {(*S*,*S*′). *S*′=*R*‘‘*S* ∧ *S*⊆*Domain R*}

lemma *old-set-rel-def*[*refine-rel-defs*]:
⟨*R*⟩*old-set-rel* ≡ {(*S*,*S*′). *S*′=*R*‘‘*S* ∧ *S*⊆*Domain R*}
by (*simp add*: *old-set-rel-def-internal relAPP-def*)

Old definition coincides with new definition for single-valued element relations. This is probably the reason why the old definition worked for most applications.

lemma *old-set-rel-sv-eq*: *single-valued R* ⟹ ⟨*R*⟩*old-set-rel* = ⟨*R*⟩*set-rel*
unfolding *set-rel-def old-set-rel-def single-valued-def*
by *blast*

lemma *set-rel-simp*[*simp*]:
({},{})∈⟨*R*⟩*set-rel*

by (*auto simp*: *set-rel-def*)

lemma *set-rel-empty-iff*[*simp*]:
($\{\}$,*y*)∈⟨*A*⟩*set-rel* ⟷ *y*=$\{\}$
(*x*,$\{\}$)∈⟨*A*⟩*set-rel* ⟷ *x*=$\{\}$
by (*auto simp*: *set-rel-def*; *fastforce*)+

lemma *set-relD1*: (*s*,*s'*)∈⟨*R*⟩*set-rel* ⟹ *x*∈*s* ⟹ ∃ *x'*∈*s'*. (*x*,*x'*)∈*R*
unfolding *set-rel-def* **by** *blast*

lemma *set-relD2*: (*s*,*s'*)∈⟨*R*⟩*set-rel* ⟹ *x'*∈*s'* ⟹ ∃ *x*∈*s*. (*x*,*x'*)∈*R*
unfolding *set-rel-def* **by** *blast*

lemma *set-relE1*[*consumes 2*]:
assumes (*s*,*s'*)∈⟨*R*⟩*set-rel* *x*∈*s*
obtains *x'* **where** *x'*∈*s'* (*x*,*x'*)∈*R*
using *set-relD1*[*OF assms*] **..**

lemma *set-relE2*[*consumes 2*]:
assumes (*s*,*s'*)∈⟨*R*⟩*set-rel* *x'*∈*s'*
obtains *x* **where** *x*∈*s* (*x*,*x'*)∈*R*
using *set-relD2*[*OF assms*] **..**

1.1.3 Automation

A solver for relator properties

lemma *relprop-triggers*:
⋀*R*. *single-valued R* ⟹ *single-valued R*
⋀*R*. *R*=*Id* ⟹ *R*=*Id*
⋀*R*. *R*=*Id* ⟹ *Id*=*R*
⋀*R*. *Range R* = *UNIV* ⟹ *Range R* = *UNIV*
⋀*R*. *Range R* = *UNIV* ⟹ *UNIV* = *Range R*
⋀*R R'*. *R*⊆*R'* ⟹ *R*⊆*R'*
by *auto*

ML ‹
structure relator-props = *Named-Thms* (
val name = @{*binding relator-props*}
val description = *Additional relator properties*
)

structure solve-relator-props = *Named-Thms* (
val name = @{*binding solve-relator-props*}
val description = *Relator properties that solve goal*
)

›
setup *relator-props.setup*

setup *solve-relator-props.setup*

declaration ‹
Tagged-Solver.declare-solver
@{*thms relprop-triggers*}
@{*binding relator-props-solver*}
Additional relator properties solver
(*fn ctxt* => (*REPEAT-ALL-NEW* (*CHANGED o* (
match-tac ctxt (*solve-relator-props.get ctxt*) *ORELSE'*
match-tac ctxt (*relator-props.get ctxt*)
))))
›

declaration ‹
Tagged-Solver.declare-solver
[]
@{*binding force-relator-props-solver*}
Additional relator properties solver (*instantiate schematics*)
(*fn ctxt* => (*REPEAT-ALL-NEW* (*CHANGED o* (
resolve-tac ctxt (*solve-relator-props.get ctxt*) *ORELSE'*
match-tac ctxt (*relator-props.get ctxt*)
))))
›

lemma
relprop-id-orient[*relator-props*]: *R*=*Id* ⟹ *Id*=*R* **and**
relprop-eq-refl[*solve-relator-props*]: *t* = *t*
by *auto*

lemma
relprop-UNIV-orient[*relator-props*]: *R*=*UNIV* ⟹ *UNIV*=*R*
by *auto*

ML-Level utilities

ML ‹
signature RELATORS = *sig*
val mk-relT: *typ* ∗ *typ* −> *typ*
val dest-relT: *typ* −> *typ* ∗ *typ*

val mk-relAPP: *term* −> *term* −> *term*
val list-relAPP: *term list* −> *term* −> *term*
val strip-relAPP: *term* −> *term list* ∗ *term*
val mk-fun-rel: *term* −> *term* −> *term*

val list-rel: *term list* −> *term* −> *term*

val rel-absT: *term* −> *typ*
val rel-concT: *term* −> *typ*

```
  val mk-prodrel: term * term -> term
  val is-prodrel: term -> bool
  val dest-prodrel: term -> term * term

  val strip-prodrel-left: term -> term list
  val list-prodrel-left: term list -> term


  val declare-natural-relator:
    (string*string) -> Context.generic -> Context.generic
  val remove-natural-relator: string -> Context.generic -> Context.generic
  val natural-relator-of: Proof.context -> string -> string option

  val mk-natural-relator: Proof.context -> term list -> string -> term option

  val setup: theory -> theory
end

structure Relators :RELATORS = struct
  val mk-relT = HOLogic.mk-prodT #> HOLogic.mk-setT

  fun dest-relT (Type (@{type-name set},[Type (@{type-name prod},[cT,aT])]))
    = (cT,aT)
    | dest-relT ty = raise TYPE (dest-relT,[ty],[])

  fun mk-relAPP x f = let
    val xT = fastype-of x
    val fT = fastype-of f
    val rT = range-type fT
  in
    Const (@{const-name relAPP},fT-->xT-->rT)$f$x
  end

  val list-relAPP = fold mk-relAPP

  fun strip-relAPP R = let
    fun aux @{mpat ⟨?R⟩ ?S} l = aux S (R::l)
      | aux R l = (l,R)
  in aux R [] end

  val rel-absT = fastype-of #> HOLogic.dest-setT #> HOLogic.dest-prodT #>
snd
  val rel-concT = fastype-of #> HOLogic.dest-setT #> HOLogic.dest-prodT #>
fst

  fun mk-fun-rel r1 r2 = let
    val (r1T,r2T) = (fastype-of r1,fastype-of r2)
    val (c1T,a1T) = dest-relT r1T
```

```
    val (c2T,a2T) = dest-relT r2T
    val (cT,aT) = (c1T --> c2T, a1T --> a2T)
    val rT = mk-relT (cT,aT)
  in
    list-relAPP [r1,r2] (Const (@{const-name fun-rel},r1T-->r2T-->rT))
  end

  val list-rel = fold-rev mk-fun-rel

  fun mk-prodrel (A,B) = @{mk-term ?A ×r ?B}
  fun is-prodrel @{mpat - ×r -} = true | is-prodrel - = false
fun dest-prodrel @{mpat ?A ×r ?B} = (A,B) | dest-prodrel t = raise TERM(dest-prodrel,[t])

  fun strip-prodrel-left @{mpat ?A ×r ?B} = strip-prodrel-left A @ [B]
    | strip-prodrel-left @{mpat (typs) unit-rel} = []
    | strip-prodrel-left R = [R]

  val list-prodrel-left = Refine-Util.list-binop-left @{term unit-rel} mk-prodrel

  structure natural-relators = Generic-Data (
    type T = string Symtab.table
    val empty = Symtab.empty
    val merge = Symtab.join (fn - => fn (-,cn) => cn)
  )

  fun declare-natural-relator tcp =
    natural-relators.map (Symtab.update tcp)

  fun remove-natural-relator tname =
    natural-relators.map (Symtab.delete-safe tname)

  fun natural-relator-of ctxt =
    Symtab.lookup (natural-relators.get (Context.Proof ctxt))

  (* [R1,...,Rn] T is mapped to ⟨R1,...,Rn⟩ Trel *)
  fun mk-natural-relator ctxt args Tname =
    case natural-relator-of ctxt Tname of
      NONE => NONE
    | SOME Cname => SOME let
        val argsT = map fastype-of args
        val (cTs, aTs) = map dest-relT argsT |> split-list
        val aT = Type (Tname,aTs)
        val cT = Type (Tname,cTs)
        val rT = mk-relT (cT,aT)
      in
        list-relAPP args (Const (Cname,argsT--->rT))
      end

  fun
```

```
    natural-relator-from-term (t as Const (name,T)) = let
      fun err msg = raise TERM (msg,[t])

      val (argTs,bodyT) = strip-type T
    val (conTs,absTs) = argTs |> map (HOLogic.dest-setT #> HOLogic.dest-prodT)
|> split-list
      val (bconT,babsT) = bodyT |> HOLogic.dest-setT |> HOLogic.dest-prodT
      val (Tcon,bconTs) = dest-Type bconT
      val (Tcon',babsTs) = dest-Type babsT

      val - = Tcon = Tcon' orelse err Type constructors do not match
      val - = conTs = bconTs orelse err Concrete types do not match
      val - = absTs = babsTs orelse err Abstract types do not match

    in
      (Tcon,name)
    end
  | natural-relator-from-term t =
      raise TERM (Expected constant,[t]) (* TODO: Localize this! *)

  local
    fun decl-natrel-aux t context = let
      fun warn msg = let
        val tP =
          Context.cases Syntax.pretty-term-global Syntax.pretty-term
            context t
        val m = Pretty.block [
          Pretty.str Ignoring invalid natural-relator declaration:,
          Pretty.brk 1,
          Pretty.str msg,
          Pretty.brk 1,
          tP
        ] |> Pretty.string-of
        val - = warning m
      in context end
    in
      try ‹declare-natural-relator (natural-relator-from-term t) context
        catch TERM (msg,-) => warn msg
          | exn => warn ›
    end
  in
    val natural-relator-attr = Scan.repeat1 Args.term >> (fn ts =>
      Thm.declaration-attribute ( fn - => fold decl-natrel-aux ts)
    )
  end


  val setup = I
    #> Attrib.setup
```

@{*binding natural-relator*} *natural-relator-attr Declare natural relator*

end
›

setup *Relators.setup*

1.1.4 Setup

Natural Relators

declare [[*natural-relator*
unit-rel int-rel nat-rel bool-rel
fun-rel prod-rel option-rel sum-rel list-rel
]]

ML-val ‹
Relators.mk-natural-relator
@{*context*}
[@{*term Ra*::('*c*×'*a*) *set*},@{*term* ⟨*Rb*⟩*option-rel*}]
@{*type-name prod*}
|> *the*
|> *Thm.cterm-of* @{*context*}
;
Relators.mk-fun-rel @{*term* ⟨*Id*⟩*option-rel*} @{*term* ⟨*Id*⟩*list-rel*}
|> *Thm.cterm-of* @{*context*}
›

Additional Properties

lemmas [*relator-props*] =
single-valued-Id
subset-refl
refl

lemma *eq-UNIV-iff*: *S*=*UNIV* ⟷ (∀ *x*. *x*∈*S*) **by** *auto*

lemma *fun-rel-sv*[*relator-props*]:
assumes *RAN*: *Range Ra* = *UNIV*
assumes *SV*: *single-valued Rv*
shows *single-valued* (*Ra* → *Rv*)
proof (*intro single-valuedI ext impI allI*)
fix *f g h x'*
assume *R1*: (*f*,*g*)∈*Ra*→*Rv*
and *R2*: (*f*,*h*)∈*Ra*→*Rv*

from *RAN* **obtain** *x* **where** *AR*: (*x*,*x'*)∈*Ra* **by** *auto*

from *fun-relD*[*OF R1 AR*] **have** $(f\ x, g\ x') \in Rv$ **.**
moreover from *fun-relD*[*OF R2 AR*] **have** $(f\ x, h\ x') \in Rv$ **.**
ultimately show $g\ x' = h\ x'$ **using** *SV* **by** (*auto dest*: *single-valuedD*)
qed

lemmas [*relator-props*] = *Range-Id*

lemma *fun-rel-id*[*relator-props*]: $[\![R1{=}Id;\ R2{=}Id]\!] \Longrightarrow R1 \to R2 = Id$
by (*auto simp*: *fun-rel-def*)

lemma *fun-rel-id-simp*[*simp*]: $Id{\to}Id = Id$ **by** *tagged-solver*

lemma *fun-rel-comp-dist*[*relator-props*]:
$(R1{\to}R2)\ O\ (R3{\to}R4) \subseteq ((R1\ O\ R3) \to (R2\ O\ R4))$
by (*auto simp*: *fun-rel-def*)

lemma *fun-rel-mono*[*relator-props*]: $[\![\ R1{\subseteq}R2;\ R3{\subseteq}R4\ ]\!] \Longrightarrow R2{\to}R3 \subseteq R1{\to}R4$
by (*force simp*: *fun-rel-def*)

lemma *prod-rel-sv*[*relator-props*]:
$[\![single\text{-}valued\ R1;\ single\text{-}valued\ R2]\!] \Longrightarrow single\text{-}valued\ (\langle R1,R2\rangle prod\text{-}rel)$
by (*auto intro*: *single-valuedI dest*: *single-valuedD simp*: *prod-rel-def*)

lemma *prod-rel-id*[*relator-props*]: $[\![R1{=}Id;\ R2{=}Id]\!] \Longrightarrow \langle R1,R2\rangle prod\text{-}rel = Id$
by (*auto simp*: *prod-rel-def*)

lemma *prod-rel-id-simp*[*simp*]: $\langle Id,Id\rangle prod\text{-}rel = Id$ **by** *tagged-solver*

lemma *prod-rel-mono*[*relator-props*]:
$[\![\ R2{\subseteq}R1;\ R3{\subseteq}R4\ ]\!] \Longrightarrow \langle R2,R3\rangle prod\text{-}rel \subseteq \langle R1,R4\rangle prod\text{-}rel$
by (*auto simp*: *prod-rel-def*)

lemma *prod-rel-range*[*relator-props*]: $[\![Range\ Ra{=}UNIV;\ Range\ Rb{=}UNIV]\!]$
$\Longrightarrow Range\ (\langle Ra,Rb\rangle prod\text{-}rel) = UNIV$
apply (*auto simp*: *prod-rel-def*)
by (*metis Range-iff UNIV-I*)+

lemma *option-rel-sv*[*relator-props*]:
$[\![single\text{-}valued\ R]\!] \Longrightarrow single\text{-}valued\ (\langle R\rangle option\text{-}rel)$
by (*auto intro*: *single-valuedI dest*: *single-valuedD simp*: *option-rel-def*)

lemma *option-rel-id*[*relator-props*]:
$R{=}Id \Longrightarrow \langle R\rangle option\text{-}rel = Id$ **by** (*auto simp*: *option-rel-def*)

lemma *option-rel-id-simp*[*simp*]: $\langle Id\rangle option\text{-}rel = Id$ **by** *tagged-solver*

lemma *option-rel-mono*[*relator-props*]: $R{\subseteq}R' \Longrightarrow \langle R\rangle option\text{-}rel \subseteq \langle R'\rangle option\text{-}rel$
by (*auto simp*: *option-rel-def*)

lemma *option-rel-range*: *Range R = UNIV ⟹ Range (⟨R⟩option-rel) = UNIV*
apply (*auto simp*: *option-rel-def Range-iff*)
by (*metis Range-iff UNIV-I option.exhaust*)

lemma *option-rel-inter*[*simp*]: *⟨R1 ∩ R2⟩option-rel = ⟨R1⟩option-rel ∩ ⟨R2⟩option-rel*
by (*auto simp*: *option-rel-def*)

lemma *option-rel-constraint*[*simp*]:
(x,x)∈⟨UNIV×C⟩option-rel ⟷ (∀ v. x=Some v ⟶ v∈C)
by (*auto simp*: *option-rel-def*)

lemma *sum-rel-sv*[*relator-props*]:
⟦single-valued Rl; single-valued Rr⟧ ⟹ single-valued (⟨Rl,Rr⟩sum-rel)
by (*auto intro*: *single-valuedI dest*: *single-valuedD simp*: *sum-rel-def*)

lemma *sum-rel-id*[*relator-props*]: *⟦Rl=Id; Rr=Id⟧ ⟹ ⟨Rl,Rr⟩sum-rel = Id*
apply (*auto elim*: *sum-relE*)
apply (*case-tac b*)
apply *simp-all*
done

lemma *sum-rel-id-simp*[*simp*]: *⟨Id,Id⟩sum-rel = Id* **by** *tagged-solver*

lemma *sum-rel-mono*[*relator-props*]:
⟦ Rl⊆Rl'; Rr⊆Rr' ⟧ ⟹ ⟨Rl,Rr⟩sum-rel ⊆ ⟨Rl',Rr'⟩sum-rel
by (*auto simp*: *sum-rel-def*)

lemma *sum-rel-range*[*relator-props*]:
⟦ Range Rl=UNIV; Range Rr=UNIV ⟧ ⟹ Range (⟨Rl,Rr⟩sum-rel) = UNIV
apply (*auto simp*: *sum-rel-def Range-iff*)
by (*metis Range-iff UNIV-I sumE*)

lemma *list-rel-sv-iff*:
single-valued (⟨R⟩list-rel) ⟷ single-valued R
apply (*intro iffI*[*rotated*] *single-valuedI allI impI*)
apply (*clarsimp simp*: *list-rel-def*)
proof −
fix *x y z*
assume *SV*: *single-valued R*
assume *list-all2* (*λx x'*. (*x*, *x'*) ∈ *R*) *x y* **and**
list-all2 (*λx x'*. (*x*, *x'*) ∈ *R*) *x z*
thus *y=z*
apply (*induct arbitrary*: *z rule*: *list-all2-induct*)
apply *simp*
apply (*case-tac z*)
apply *force*
apply (*force intro*: *single-valuedD*[*OF SV*])

```
    done
next
  fix x y z
  assume SV: single-valued (⟨R⟩list-rel)
  assume (x,y)∈R    (x,z)∈R
  hence ([x],[y])∈⟨R⟩list-rel and ([x],[z])∈⟨R⟩list-rel
    by (auto simp: list-rel-def)
  with single-valuedD[OF SV] show y=z by blast
qed

lemma list-rel-sv[relator-props]:
  single-valued R ⟹ single-valued (⟨R⟩list-rel)
  by (simp add: list-rel-sv-iff)

lemma list-rel-id[relator-props]: ⟦R=Id⟧ ⟹ ⟨R⟩list-rel = Id
  by (auto simp add: list-rel-def list-all2-eq[symmetric])

lemma list-rel-id-simp[simp]: ⟨Id⟩list-rel = Id by tagged-solver

lemma list-rel-mono[relator-props]:
  assumes A: R⊆R'
  shows ⟨R⟩list-rel ⊆ ⟨R'⟩list-rel
proof clarsimp
  fix l l'
  assume (l,l')∈⟨R⟩list-rel
  thus (l,l')∈⟨R'⟩list-rel
    apply induct
    using A
    by auto
qed

lemma list-rel-range[relator-props]:
  assumes A: Range R = UNIV
  shows Range (⟨R⟩list-rel) = UNIV
proof (clarsimp simp: eq-UNIV-iff)
  fix l
  show l∈Range (⟨R⟩list-rel)
    apply (induct l)
    using A[unfolded eq-UNIV-iff]
    by (auto simp: Range-iff intro: list-relI)
qed

lemma bijective-imp-sv:
  bijective R ⟹ single-valued R
  bijective R ⟹ single-valued (R⁻¹)
  by (simp-all add: bijective-alt)


declare bijective-Id[relator-props]
```

declare *bijective-Empty*[*relator-props*]

Pointwise refinement for set types:

lemma *set-rel-sv*[*relator-props*]:
single-valued $R \Longrightarrow$ *single-valued* $(\langle R\rangle$*set-rel*$)$
unfolding *single-valued-def set-rel-def* **by** *blast*

lemma *set-rel-id*[*relator-props*]: $R{=}Id \Longrightarrow \langle R\rangle$*set-rel* $= Id$
by (*auto simp add*: *set-rel-def*)

lemma *set-rel-id-simp*[*simp*]: $\langle Id\rangle$*set-rel* $= Id$ **by** *tagged-solver*

lemma *set-rel-csv*[*relator-props*]:
$[\![$ *single-valued* (R^{-1}) $]\!]$
$\Longrightarrow$ *single-valued* $((\langle R\rangle$*set-rel*$)^{-1})$
unfolding *single-valued-def set-rel-def converse-iff*
by *fast*

1.1.5 Invariant and Abstraction

Quite often, a relation can be described as combination of an abstraction function and an invariant, such that the invariant describes valid values on the concrete domain, and the abstraction function maps valid concrete values to its corresponding abstract value.

definition *build-rel* **where**
build-rel $\alpha\ I \equiv \{(c,a)\ .\ a{=}\alpha\ c \wedge I\ c\}$
abbreviation *br*$\equiv$*build-rel*
lemmas *br-def*[*refine-rel-defs*] $=$ *build-rel-def*

lemma *in-br-conv*: $(c,a){\in}br\ \alpha\ I \longleftrightarrow a{=}\alpha\ c \wedge I\ c$
by (*auto simp*: *br-def*)

lemma *brI*[*intro?*]: $[\![\ a{=}\alpha\ c;\ I\ c\ ]\!] \Longrightarrow (c,a){\in}br\ \alpha\ I$
by (*simp add*: *br-def*)

lemma *br-id*[*simp*]: *br id* $(\lambda$*-*. *True*$) = Id$
unfolding *build-rel-def* **by** *auto*

lemma *br-chain*:
(*build-rel* $\beta\ J$) O (*build-rel* $\alpha\ I$) $=$ *build-rel* $(\alpha{\circ}\beta)$ $(\lambda s.\ J\ s \wedge I\ (\beta\ s))$
unfolding *build-rel-def* **by** *auto*

lemma *br-sv*[*simp*, *intro*!,*relator-props*]: *single-valued* $(br\ \alpha\ I)$
unfolding *build-rel-def*
apply (*rule single-valuedI*)
apply *auto*
done

lemma *converse-br-sv-iff*[*simp*]:
single-valued (*converse* (*br* α *I*)) $\longleftrightarrow$ *inj-on* α (*Collect I*)
by (*auto intro*!: *inj-onI single-valuedI dest*: *single-valuedD inj-onD*
simp: *br-def*) []

lemmas [*relator-props*] = *single-valued-relcomp*

lemma *br-comp-alt*: *br* α *I O R* = { (*c*,*a*) . *I c* $\wedge$ (α *c*,*a*)$\in$*R* }
by (*auto simp add*: *br-def*)

lemma *br-comp-alt'*:
{(*c*,*a*) . *a*=α *c* $\wedge$ *I c*} *O R* = { (*c*,*a*) . *I c* $\wedge$ (α *c*,*a*)$\in$*R* }
by *auto*

lemma *single-valued-as-brE*:
assumes *single-valued R*
obtains α *invar* **where** *R*=*br* α *invar*
apply (*rule that*[*of* λx. *THE y*. (*x*,*y*)$\in$*R* λx. *x*$\in$*Domain R*])
using *assms* **unfolding** *br-def*
by (*auto dest*: *single-valuedD*
intro: *the-equality*[*symmetric*] *theI*)

lemma *sv-add-invar*:
single-valued R $\Longrightarrow$ *single-valued* {(*c*, *a*). (*c*, *a*) $\in$ *R* $\wedge$ *I c*}
by (*auto dest*: *single-valuedD intro*: *single-valuedI*)

lemma *br-Image-conv*[*simp*]: *br* α *I* `` *S* = {α *x* | *x*. *x*$\in$*S* $\wedge$ *I x*}
by (*auto simp*: *br-def*)

1.1.6 Miscellanneous

lemma *rel-cong*: (*f*,*g*)$\in$*Id* $\Longrightarrow$ (*x*,*y*)$\in$*Id* $\Longrightarrow$ (*f x*, *g y*)$\in$*Id* **by** *simp*
lemma *rel-fun-cong*: (*f*,*g*)$\in$*Id* $\Longrightarrow$ (*f x*, *g x*)$\in$*Id* **by** *simp*
lemma *rel-arg-cong*: (*x*,*y*)$\in$*Id* $\Longrightarrow$ (*f x*, *f y*)$\in$*Id* **by** *simp*

1.1.7 Conversion between Predicate and Set Based Relators

Autoref uses set-based relators of type ($'a \times 'b$) *set*, while the transfer and lifting package of Isabelle/HOL uses predicate based relators of type $'a \Rightarrow 'b \Rightarrow$ *bool*. This section defines some utilities to convert between the two.

definition *rel2p R x y* $\equiv$ (*x*,*y*)$\in$*R*
definition *p2rel P* $\equiv$ {(*x*,*y*). *P x y*}

lemma *rel2pD*: $[\![$*rel2p R a b*$]\!] \Longrightarrow$ (*a*,*b*)$\in$*R* **by** (*auto simp*: *rel2p-def*)
lemma *p2relD*: $[\![$(*a*,*b*) $\in$ *p2rel R*$]\!] \Longrightarrow$ *R a b* **by** (*auto simp*: *p2rel-def*)

lemma *rel2p-inv*[*simp*]:
rel2p (*p2rel P*) = *P*
p2rel (*rel2p R*) = *R*

by (*auto simp*: *rel2p-def*[*abs-def*] *p2rel-def*)

named-theorems *rel2p*
named-theorems *p2rel*

lemma *rel2p-dflt*[*rel2p*]:
rel2p Id = (=)
rel2p (*A*→*B*) = *rel-fun* (*rel2p A*) (*rel2p B*)
rel2p (*A*×$_r$*B*) = *rel-prod* (*rel2p A*) (*rel2p B*)
rel2p (⟨*A*,*B*⟩*sum-rel*) = *rel-sum* (*rel2p A*) (*rel2p B*)
rel2p (⟨*A*⟩*option-rel*) = *rel-option* (*rel2p A*)
rel2p (⟨*A*⟩*list-rel*) = *list-all2* (*rel2p A*)
by (*auto*
simp: *rel2p-def*[*abs-def*]
intro!: *ext*
simp: *fun-rel-def rel-fun-def*
simp: *sum-rel-def elim*: *rel-sum.cases*
simp: *option-rel-def elim*: *option.rel-cases*
simp: *list-rel-def*
simp: *set-rel-def rel-set-def Image-def*
)

lemma *p2rel-dflt*[*p2rel*]:
p2rel (=) = *Id*
p2rel (*rel-fun A B*) = *p2rel A* → *p2rel B*
p2rel (*rel-prod A B*) = *p2rel A* ×$_r$ *p2rel B*
p2rel (*rel-sum A B*) = ⟨*p2rel A*, *p2rel B*⟩*sum-rel*
p2rel (*rel-option A*) = ⟨*p2rel A*⟩*option-rel*
p2rel (*list-all2 A*) = ⟨*p2rel A*⟩*list-rel*
by (*auto*
simp: *p2rel-def*[*abs-def*]
simp: *fun-rel-def rel-fun-def*
simp: *sum-rel-def elim*: *rel-sum.cases*
simp: *option-rel-def elim*: *option.rel-cases*
simp: *list-rel-def*
)

lemma [*rel2p*]: *rel2p* (⟨*A*⟩*set-rel*) = *rel-set* (*rel2p A*)
unfolding *set-rel-def rel-set-def rel2p-def*[*abs-def*]
by *blast*

lemma [*p2rel*]: *left-unique A* ⟹ *p2rel* (*rel-set A*) = (⟨*p2rel A*⟩*set-rel*)
unfolding *set-rel-def rel-set-def p2rel-def*[*abs-def*]
by *blast*

lemma *rel2p-comp*: *rel2p A OO rel2p B* = *rel2p* (*A O B*)
by (*auto simp*: *rel2p-def*[*abs-def*] *intro*!: *ext*)

lemma *rel2p-inj*[*simp*]: *rel2p A* = *rel2p B* ⟷ *A*=*B*
by (*auto simp*: *rel2p-def*[*abs-def*]; *meson*)

lemma *rel2p-left-unique*: *left-unique* (*rel2p A*) = *single-valued* (A^{-1})
unfolding *left-unique-def rel2p-def single-valued-def* **by** *blast*

lemma *rel2p-right-unique*: *right-unique* (*rel2p A*) = *single-valued A*
unfolding *right-unique-def rel2p-def single-valued-def* **by** *blast*

lemma *rel2p-bi-unique*: *bi-unique* (*rel2p A*) ⟷ *single-valued A* ∧ *single-valued* (A^{-1})
unfolding *bi-unique-alt-def* **by** (*auto simp*: *rel2p-left-unique rel2p-right-unique*)

lemma *p2rel-left-unique*: *single-valued* ((*p2rel A*)$^{-1}$) = *left-unique A*
unfolding *left-unique-def p2rel-def single-valued-def* **by** *blast*

lemma *p2rel-right-unique*: *single-valued* (*p2rel A*) = *right-unique A*
unfolding *right-unique-def p2rel-def single-valued-def* **by** *blast*

1.1.8 More Properties

lemma *list-rel-compp*: ⟨*A O B*⟩*list-rel* = ⟨*A*⟩*list-rel O* ⟨*B*⟩*list-rel*
using *list.rel-compp*[*of rel2p A rel2p B*]
by (*auto simp*: *rel2p*(*2*−)[*symmetric*] *rel2p-comp*)

lemma *option-rel-compp*: ⟨*A O B*⟩*option-rel* = ⟨*A*⟩*option-rel O* ⟨*B*⟩*option-rel*
using *option.rel-compp*[*of rel2p A rel2p B*]
by (*auto simp*: *rel2p*(*2*−)[*symmetric*] *rel2p-comp*)

lemma *prod-rel-compp*: ⟨*A O B, C O D*⟩*prod-rel* = ⟨*A,C*⟩*prod-rel O* ⟨*B,D*⟩*prod-rel*
using *prod.rel-compp*[*of rel2p A rel2p B rel2p C rel2p D*]
by (*auto simp*: *rel2p*(*2*−)[*symmetric*] *rel2p-comp*)

lemma *sum-rel-compp*: ⟨*A O B, C O D*⟩*sum-rel* = ⟨*A,C*⟩*sum-rel O* ⟨*B,D*⟩*sum-rel*
using *sum.rel-compp*[*of rel2p A rel2p B rel2p C rel2p D*]
by (*auto simp*: *rel2p*(*2*−)[*symmetric*] *rel2p-comp*)

lemma *set-rel-compp*: ⟨*A O B*⟩*set-rel* = ⟨*A*⟩*set-rel O* ⟨*B*⟩*set-rel*
using *rel-set-OO*[*of rel2p A rel2p B*]
by (*auto simp*: *rel2p*(*2*−)[*symmetric*] *rel2p-comp*)

lemma *map-in-list-rel-conv*:
shows (*l*, *map* α *l*) ∈ ⟨*br* α *I*⟩*list-rel* ⟷ (∀ *x*∈*set l*. *I x*)
by (*induction l*) (*auto simp*: *in-br-conv*)

lemma *br-set-rel-alt*: (*s′*,*s*)∈⟨*br* α *I*⟩*set-rel* ⟷ (*s*=α‘*s′* ∧ (∀ *x*∈*s′*. *I x*))
by (*auto simp*: *set-rel-def br-def*)

lemma *finite-Image-sv*: *single-valued R* $\Longrightarrow$ *finite s* $\Longrightarrow$ *finite* (*R``s*)
by (*erule single-valued-as-brE*) *simp*

lemma *finite-set-rel-transfer*: $[\![(s,s')\in\langle R\rangle$*set-rel*; *single-valued R*; *finite s*$]\!] \Longrightarrow$ *finite s'*
unfolding *set-rel-alt*
by (*blast intro*: *finite-subset*[*OF - finite-Image-sv*])

lemma *finite-set-rel-transfer-back*: $[\![(s,s')\in\langle R\rangle$*set-rel*; *single-valued* (R^{-1}); *finite s'*$]\!]$ $\Longrightarrow$ *finite s*
unfolding *set-rel-alt*
by (*blast intro*: *finite-subset*[*OF - finite-Image-sv*])

end

1.2 Basic Parametricity Reasoning

theory *Param-Tool*
imports *Relators*
begin

1.2.1 Auxiliary Lemmas

lemma *tag-both*: $[\![$ (*Let x f*,*Let x' f'*)$\in R$ $]\!] \Longrightarrow$ (*f x*,*f' x'*)$\in R$ **by** *simp*
lemma *tag-rhs*: $[\![$ (*c*,*Let x f*)$\in R$ $]\!] \Longrightarrow$ (*c*,*f x*)$\in R$ **by** *simp*
lemma *tag-lhs*: $[\![$ (*Let x f*,*a*)$\in R$ $]\!] \Longrightarrow$ (*f x*,*a*)$\in R$ **by** *simp*

lemma *tagged-fun-relD-both*:
$[\![$ (*f*,*f'*)$\in A\to B$; (*x*,*x'*)$\in A$ $]\!] \Longrightarrow$ (*Let x f*,*Let x' f'*)$\in B$
and *tagged-fun-relD-rhs*: $[\![$ (*f*,*f'*)$\in A\to B$; (*x*,*x'*)$\in A$ $]\!] \Longrightarrow$ (*f x*,*Let x' f'*)$\in B$
and *tagged-fun-relD-lhs*: $[\![$ (*f*,*f'*)$\in A\to B$; (*x*,*x'*)$\in A$ $]\!] \Longrightarrow$ (*Let x f*,*f' x'*)$\in B$
and *tagged-fun-relD-none*: $[\![$ (*f*,*f'*)$\in A\to B$; (*x*,*x'*)$\in A$ $]\!] \Longrightarrow$ (*f x*,*f' x'*)$\in B$
by (*simp-all add*: *fun-relD*)

1.2.2 ML-Setup

ML ‹
signature PARAMETRICITY = sig
type param-ruleT = {
lhs: *term*,
rhs: *term*,
R: *term*,
rhs-head: *term*,
arity: *int*
}

val dest-param-term: *term -> param-ruleT*

```
  val dest-param-rule: thm -> param-ruleT
  val dest-param-goal: Proof.context -> int -> thm -> param-ruleT

  val safe-fun-relD-tac: Proof.context -> tactic'

  val adjust-arity: int -> thm -> thm
  val adjust-arity-tac: int -> Proof.context -> tactic'
  val unlambda-tac: Proof.context -> tactic'
  val prepare-tac: Proof.context -> tactic'

  val fo-rule: thm -> thm

  (*** Basic tactics ***)
  val param-rule-tac: Proof.context -> thm -> tactic'
  val param-rules-tac: Proof.context -> thm list -> tactic'
  val asm-param-tac: Proof.context -> tactic'


  (*** Nets of parametricity rules ***)
  type param-net
  val net-empty: param-net
  val net-add: thm -> param-net -> param-net
  val net-del: thm -> param-net -> param-net
  val net-add-int: Context.generic -> thm -> param-net -> param-net
  val net-del-int: Context.generic -> thm -> param-net -> param-net
  val net-tac: param-net -> Proof.context -> tactic'

  (*** Default parametricity rules ***)
  val add-dflt: thm -> Context.generic -> Context.generic
  val add-dflt-attr: attribute
  val del-dflt: thm -> Context.generic -> Context.generic
  val del-dflt-attr: attribute
  val get-dflt: Proof.context -> param-net

  (** Configuration **)
  val cfg-use-asm: bool Config.T
  val cfg-single-step: bool Config.T

  (** Setup **)
  val setup: theory -> theory
end

structure Parametricity : PARAMETRICITY = struct
  type param-ruleT = {
    lhs: term,
    rhs: term,
    R: term,
    rhs-head: term,
    arity: int
```

```
}

fun dest-param-term t =
  case
    strip-all-body t |> Logic.strip-imp-concl |> HOLogic.dest-Trueprop
  of
    @{mpat (?lhs,?rhs):?R} => let
      val (rhs-head,arity) =
        case strip-comb rhs of
          (c as Const -,l) => (c,length l)
        | (c as Free -,l) => (c,length l)
        | (c as Abs -,l) => (c,length l)
        | - => raise TERM (dest-param-term: Head,[t])
    in
      { lhs = lhs, rhs = rhs, R=R, rhs-head = rhs-head, arity = arity }
    end
  | t => raise TERM (dest-param-term: Expected (-,-):-,[t])

val dest-param-rule = dest-param-term o Thm.prop-of
fun dest-param-goal ctxt i st =
  if i > Thm.nprems-of st then
    raise THM (dest-param-goal,i,[st])
  else
    dest-param-term (Logic.concl-of-goal (Thm.prop-of st) i)


fun safe-fun-relD-tac ctxt = let
  fun t a b = fo-resolve-tac [a] ctxt THEN' resolve-tac ctxt [b]
in
  DETERM o (
    t @{thm tag-both} @{thm tagged-fun-relD-both} ORELSE'
    t @{thm tag-rhs} @{thm tagged-fun-relD-rhs} ORELSE'
    t @{thm tag-lhs} @{thm tagged-fun-relD-lhs} ORELSE'
    resolve-tac ctxt @{thms tagged-fun-relD-none}
  )
end

fun adjust-arity i thm =
  if i = 0 then thm
  else if i<0 then funpow (~i) (fn thm => thm RS @{thm fun-relI}) thm
  else funpow i (fn thm => thm RS @{thm fun-relD}) thm

fun NTIMES k tac =
  if k <= 0 then K all-tac
  else tac THEN' NTIMES (k-1) tac

fun adjust-arity-tac n ctxt i st =
  (if n = 0 then K all-tac
  else if n>0 then NTIMES n (DETERM o resolve-tac ctxt @{thms fun-relI})
```

```
  else NTIMES (~n) (safe-fun-relD-tac ctxt)) i st

fun unlambda-tac ctxt i st =
  case try (dest-param-goal ctxt i) st of
    NONE => Seq.empty
  | SOME g => let
      val n = Term.strip-abs (#rhs-head g) |> #1 |> length
    in NTIMES n (resolve-tac ctxt @{thms fun-relI}) i st end

fun prepare-tac ctxt =
  Subgoal.FOCUS (K (PRIMITIVE (Drule.eta-contraction-rule))) ctxt
  THEN' unlambda-tac ctxt


fun could-param-rl ctxt rl i st =
  if i > Thm.nprems-of st then NONE
  else (
    case (try (dest-param-goal ctxt i) st, try dest-param-term rl) of
      (SOME g, SOME r) =>
        if Term.could-unify (#rhs-head g, #rhs-head r) then
          SOME (#arity r - #arity g)
        else NONE
    | - => NONE
  )

fun param-rule-tac-aux ctxt rl i st =
  case could-param-rl ctxt (Thm.prop-of rl) i st of
    SOME adj => (adjust-arity-tac adj ctxt THEN' resolve-tac ctxt [rl]) i st
  | - => Seq.empty

fun param-rule-tac ctxt rl =
  prepare-tac ctxt THEN' param-rule-tac-aux ctxt rl

fun param-rules-tac ctxt rls =
  prepare-tac ctxt THEN' FIRST' (map (param-rule-tac-aux ctxt) rls)

fun asm-param-tac-aux ctxt i st =
  if i > Thm.nprems-of st then Seq.empty
  else let
    val prems = Logic.prems-of-goal (Thm.prop-of st) i |> tag-list 1

    fun tac (n,t) i st = case could-param-rl ctxt t i st of
      SOME adj => (adjust-arity-tac adj ctxt THEN' rprem-tac n ctxt) i st
    | NONE => Seq.empty
  in
    FIRST' (map tac prems) i st
  end

fun asm-param-tac ctxt = prepare-tac ctxt THEN' asm-param-tac-aux ctxt
```

```
type param-net = (param-ruleT * thm) Item-Net.T

local
  val param-get-key = single o #rhs-head o #1
in
  val net-empty = Item-Net.init (Thm.eq-thm o apply2 #2) param-get-key
end

fun wrap-pr-op f context thm = case try ('dest-param-rule) thm of
  NONE =>
    let
      val msg = Ignoring invalid parametricity theorem:
        ^ Thm.string-of-thm (Context.proof-of context) thm
      val - = warning msg
    in I end
| SOME p => f p

val net-add-int = wrap-pr-op Item-Net.update
val net-del-int = wrap-pr-op Item-Net.remove

val net-add = Item-Net.update o 'dest-param-rule
val net-del = Item-Net.remove o 'dest-param-rule

fun net-tac-aux net ctxt i st =
  if i > Thm.nprems-of st then
    Seq.empty
  else
    let
      val g = dest-param-goal ctxt i st
      val rls = Item-Net.retrieve net (#rhs-head g)

      fun tac (r,thm) =
        adjust-arity-tac (#arity r - #arity g) ctxt
        THEN' DETERM o resolve-tac ctxt [thm]
    in
      FIRST' (map tac rls) i st
    end

fun net-tac net ctxt = prepare-tac ctxt THEN' net-tac-aux net ctxt

structure dflt-rules = Generic-Data (
  type T = param-net
  val empty = net-empty
  val merge = Item-Net.merge
)

fun add-dflt thm context = dflt-rules.map (net-add-int context thm) context
fun del-dflt thm context = dflt-rules.map (net-del-int context thm) context
```

```
val add-dflt-attr = Thm.declaration-attribute add-dflt
val del-dflt-attr = Thm.declaration-attribute del-dflt

val get-dflt = dflt-rules.get o Context.Proof

val cfg-use-asm =
  Attrib.setup-config-bool @{binding param-use-asm} (K true)
val cfg-single-step =
  Attrib.setup-config-bool @{binding param-single-step} (K false)

local
  open Refine-Util

  val param-modifiers =
    [Args.add -- Args.colon >> K (Method.modifier add-dflt-attr here),
     Args.del -- Args.colon >> K (Method.modifier del-dflt-attr here),
     Args.$$$ only -- Args.colon >>
      K {init = Context.proof-map (dflt-rules.map (K net-empty)),
         attribute = add-dflt-attr, pos = here}]

  val param-flags =
     parse-bool-config use-asm cfg-use-asm
  || parse-bool-config single-step cfg-single-step

in

  val parametricity-method =
    parse-paren-lists param-flags |-- Method.sections param-modifiers >>
    (fn - => fn ctxt =>
      let
        val net2 = get-dflt ctxt
        val asm-tac =
          if Config.get ctxt cfg-use-asm then
            asm-param-tac ctxt
          else K no-tac

        val RPT =
          if Config.get ctxt cfg-single-step then I
          else REPEAT-ALL-NEW-FWD

      in
        SIMPLE-METHOD' (
          RPT (
            (assume-tac ctxt
              ORELSE' net-tac net2 ctxt
              ORELSE' asm-tac)
          )
        )
      end
```

```
      )
  end

  fun fo-rule thm = case Thm.concl-of thm of
      @{mpat Trueprop ((-,-)∈-→-)} => fo-rule (thm RS @{thm fun-relD})
    | - => thm

  val param-fo-attr = Scan.succeed (Thm.rule-attribute [] (K fo-rule))

  val setup = I
    #> Attrib.setup @{binding param}
        (Attrib.add-del add-dflt-attr del-dflt-attr)
        declaration of parametricity theorem
    #> Global-Theory.add-thms-dynamic (@{binding param},
        map #2 o Item-Net.content o dflt-rules.get)
    #> Method.setup @{binding parametricity} parametricity-method
        Parametricity solver
    #> Attrib.setup @{binding param-fo} param-fo-attr
        Parametricity: Rule in first−order form

end
›

setup Parametricity.setup
```

1.2.3 Convenience Tools

```
ML ‹
  (* Prefix p- or wrong type supresses generation of relAPP *)

  fun cnv-relAPP t = let
    fun consider (Var ((name,-),T)) =
      if String.isPrefix p- name then false
      else (
        case T of
          Type(@{type-name set},[Type(@{type-name prod},-)]) => true
        | - => false)
    | consider - = true

    fun strip-rcomb u : term * term list =
      let
        fun stripc (x as (f$t, ts)) =
          if consider t then stripc (f, t::ts) else x
        | stripc  x =  x
      in  stripc(u,[])  end;

    val (f,a) = strip-rcomb t
  in
    Relators.list-relAPP a f
```

```
  end

  fun to-relAPP-conv ctxt = Refine-Util.f-tac-conv ctxt
    cnv-relAPP
    (fn goal-ctxt => ALLGOALS (simp-tac
      (put-simpset HOL-basic-ss goal-ctxt addsimps @{thms relAPP-def})))


  val to-relAPP-attr = Thm.rule-attribute [] (fn context => let
    val ctxt = Context.proof-of context
  in
    Conv.fconv-rule (Conv.arg1-conv (to-relAPP-conv ctxt))
  end)
›

  attribute-setup to-relAPP = ‹Scan.succeed (to-relAPP-attr)›
    Convert relator definition to prefix−form


end
```

1.3 Parametricity Theorems for HOL

```
theory Param-HOL
imports Param-Tool
begin
```

1.3.1 Sets

```
lemma param-empty[param]:
  ({},{})∈⟨R⟩set-rel by (auto simp: set-rel-def)

lemma param-member[param]:
  ⟦single-valued R; single-valued (R⁻¹)⟧ ⟹ ((∈), (∈)) ∈ R → ⟨R⟩set-rel → bool-rel

  unfolding set-rel-def
  by (blast dest: single-valuedD)


lemma param-insert[param]:
  (insert,insert)∈R→⟨R⟩set-rel→⟨R⟩set-rel
  by (auto simp: set-rel-def)

lemma param-union[param]:
  ((∪), (∪)) ∈ ⟨R⟩set-rel → ⟨R⟩set-rel → ⟨R⟩set-rel
  by (auto simp: set-rel-def)

lemma param-inter[param]:
  assumes single-valued R    single-valued (R⁻¹)
```

shows $((\cap), (\cap)) \in \langle R\rangle$*set-rel* $\to \langle R\rangle$*set-rel* $\to \langle R\rangle$*set-rel*
using *assms*
unfolding *set-rel-def*
by (*blast dest*: *single-valuedD*)

lemma *param-diff*[*param*]:
assumes *single-valued R* *single-valued* (R^{-1})
shows $((-), (-)) \in \langle R\rangle$*set-rel* $\to \langle R\rangle$*set-rel* $\to \langle R\rangle$*set-rel*
using *assms*
unfolding *set-rel-def*
by (*blast dest*: *single-valuedD*)

lemma *param-subseteq*[*param*]:
$[\![$*single-valued R*; *single-valued* $(R^{-1})]\!] \Longrightarrow ((\subseteq), (\subseteq)) \in \langle R\rangle$*set-rel* $\to \langle R\rangle$*set-rel* $\to$ *bool-rel*
unfolding *set-rel-def*
by (*blast dest*: *single-valuedD*)

lemma *param-subset*[*param*]:
$[\![$*single-valued R*; *single-valued* $(R^{-1})]\!] \Longrightarrow ((\subset), (\subset)) \in \langle R\rangle$*set-rel* $\to \langle R\rangle$*set-rel* $\to$ *bool-rel*
unfolding *set-rel-def*
by (*blast dest*: *single-valuedD*)

lemma *param-Ball*[*param*]: (*Ball*,*Ball*)$\in\langle Ra\rangle$*set-rel*$\to(Ra\to Id)\to Id$
by (*force simp*: *set-rel-alt dest*: *fun-relD*)

lemma *param-Bex*[*param*]: (*Bex*,*Bex*)$\in\langle Ra\rangle$*set-rel*$\to(Ra\to Id)\to Id$
by (*fastforce simp*: *set-rel-def dest*: *fun-relD*)

lemma *param-set*[*param*]:
single-valued Ra $\Longrightarrow$ (*set*,*set*)$\in\langle Ra\rangle$*list-rel* $\to \langle Ra\rangle$*set-rel*
proof
fix *l l′*
assume *A*: *single-valued Ra*
assume (*l*,*l′*)$\in\langle Ra\rangle$*list-rel*
thus (*set l*, *set l′*)$\in\langle Ra\rangle$*set-rel*
apply (*induct*)
apply *simp*
apply *simp*
using *A* **apply** (*parametricity*)
done
qed

lemma *param-Collect*[*param*]:
$[\![$*Domain A* = *UNIV*; *Range A* = *UNIV*$]\!] \Longrightarrow$ (*Collect*,*Collect*)$\in(A\to$*bool-rel*$) \to \langle A\rangle$*set-rel*
unfolding *set-rel-def*

apply (*clarsimp*; *safe*)
subgoal using *fun-relD1* **by** *fastforce*
subgoal using *fun-relD2* **by** *fastforce*
done

lemma *param-finite*[*param*]: ⟦
single-valued R; *single-valued* (R^{-1})
⟧ ⟹ (*finite*,*finite*) ∈ ⟨*R*⟩*set-rel* → *bool-rel*
using *finite-set-rel-transfer finite-set-rel-transfer-back* **by** *blast*

lemma *param-card*[*param*]: ⟦*single-valued R*; *single-valued* (R^{-1})⟧
⟹ (*card*, *card*) ∈ ⟨*R*⟩*set-rel* →*nat-rel*
apply (*rule rel2pD*)
apply (*simp only*: *rel2p*)
apply (*rule card-transfer*)
by (*simp add*: *rel2p-bi-unique*)

1.3.2 Standard HOL Constructs

lemma *param-if*[*param*]:
assumes (*c*,*c'*)∈*Id*
assumes ⟦*c*;*c'*⟧ ⟹ (*t*,*t'*)∈*R*
assumes ⟦¬*c*;¬*c'*⟧ ⟹ (*e*,*e'*)∈*R*
shows (*If c t e*, *If c' t' e'*)∈*R*
using *assms* **by** *auto*

lemma *param-Let*[*param*]:
(*Let*,*Let*)∈*Ra* → (*Ra*→*Rr*) → *Rr*
by (*auto dest*: *fun-relD*)

1.3.3 Functions

lemma *param-id*[*param*]: (*id*,*id*)∈*R*→*R* **unfolding** *id-def* **by** *parametricity*

lemma *param-fun-comp*[*param*]: ((*o*), (*o*)) ∈ (*Ra*→*Rb*) → (*Rc*→*Ra*) → *Rc*→*Rb*
unfolding *comp-def*[*abs-def*] **by** *parametricity*

lemma *param-fun-upd*[*param*]:
((=), (=)) ∈ *Ra*→*Ra*→*Id*
⟹ (*fun-upd*,*fun-upd*) ∈ (*Ra*→*Rb*) → *Ra* → *Rb* → *Ra* → *Rb*
unfolding *fun-upd-def*[*abs-def*]
by (*parametricity*)

1.3.4 Boolean

lemma *rec-bool-is-case*: *old.rec-bool* = *case-bool*
by (*rule ext*)+ (*auto split*: *bool.split*)

lemma *param-bool*[*param*]:
(*True*,*True*)∈*Id*

(*False*,*False*)∈*Id*
(*conj*,*conj*)∈*Id*→*Id*→*Id*
(*disj*,*disj*)∈*Id*→*Id*→*Id*
(*Not*,*Not*)∈*Id*→*Id*
(*case-bool*,*case-bool*)∈*R*→*R*→*Id*→*R*
(*old.rec-bool*,*old.rec-bool*)∈*R*→*R*→*Id*→*R*
((⟷), (⟷))∈*Id*→*Id*→*Id*
((⟶), (⟶))∈*Id*→*Id*→*Id*
by (*auto split*: *bool.split simp*: *rec-bool-is-case*)

lemma *param-and-cong1*: ⟦ (*a*,*a'*)∈*bool-rel*; ⟦*a*; *a'*⟧ ⟹ (*b*,*b'*)∈*bool-rel* ⟧ ⟹ (*a*∧*b*,*a'*∧*b'*)∈*bool-rel*
by *blast*
lemma *param-and-cong2*: ⟦ (*a*,*a'*)∈*bool-rel*; ⟦*a*; *a'*⟧ ⟹ (*b*,*b'*)∈*bool-rel* ⟧ ⟹ (*b*∧*a*,*b'*∧*a'*)∈*bool-rel*
by *blast*

1.3.5 Nat

lemma *param-nat1*[*param*]:
(*0*, *0*::*nat*) ∈ *Id*
(*Suc*, *Suc*) ∈ *Id* → *Id*
(*1*, *1*::*nat*) ∈ *Id*
(*numeral n*::*nat*,*numeral n*::*nat*) ∈ *Id*
((<), (<) ::*nat* ⇒ -) ∈ *Id* → *Id* → *Id*
((≤), (≤) ::*nat* ⇒ -) ∈ *Id* → *Id* → *Id*
((=), (=) ::*nat* ⇒ -) ∈ *Id* → *Id* → *Id*
((+) ::*nat*⇒-,(+))∈*Id*→*Id*→*Id*
((−) ::*nat*⇒-,(−))∈*Id*→*Id*→*Id*
((∗) ::*nat*⇒-,(∗))∈*Id*→*Id*→*Id*
((*div*) ::*nat*⇒-,(*div*))∈*Id*→*Id*→*Id*
((*mod*) ::*nat*⇒-,(*mod*))∈*Id*→*Id*→*Id*
by *auto*

lemma *param-case-nat*[*param*]:
(*case-nat*,*case-nat*)∈*Ra* → (*Id* → *Ra*) → *Id* → *Ra*
apply (*intro fun-relI*)
apply (*auto split*: *nat.split dest*: *fun-relD*)
done

lemma *param-rec-nat*[*param*]:
(*rec-nat*,*rec-nat*) ∈ *R* → (*Id* → *R* → *R*) → *Id* → *R*
proof (*intro fun-relI*, *goal-cases*)
case (*1 s s' f f' n n'*) **thus** *?case*
apply (*induct n' arbitrary*: *n s s'*)
apply (*fastforce simp*: *fun-rel-def*)+
done
qed

1.3.6 Int

lemma *param-int*[*param*]:

$(0,\ 0::int) \in Id$
$(1,\ 1::int) \in Id$
$(numeral\ n::int, numeral\ n::int) \in Id$
$((<),\ (<)\ ::int \Rightarrow \text{-}) \in Id \to Id \to Id$
$((\leq),\ (\leq)\ ::int \Rightarrow \text{-}) \in Id \to Id \to Id$
$((=),\ (=)\ ::int \Rightarrow \text{-}) \in Id \to Id \to Id$
$((+)\ ::int \Rightarrow \text{-}, (+)) \in Id \to Id \to Id$
$((-)\ ::int \Rightarrow \text{-}, (-)) \in Id \to Id \to Id$
$((*)\ ::int \Rightarrow \text{-}, (*)) \in Id \to Id \to Id$
$((div)\ ::int \Rightarrow \text{-}, (div)) \in Id \to Id \to Id$
$((mod)\ ::int \Rightarrow \text{-}, (mod)) \in Id \to Id \to Id$
by *auto*

1.3.7 Product

lemma *param-unit*[*param*]: $((),()) \in$ *unit-rel* **by** *auto*

lemma *rec-prod-is-case*: *old.rec-prod* = *case-prod*
by (*rule ext*)+ (*auto split*: *bool.split*)

lemma *param-prod*[*param*]:
$(Pair, Pair) \in Ra \to Rb \to \langle Ra, Rb\rangle prod\text{-}rel$
$(case\text{-}prod, case\text{-}prod) \in (Ra \to Rb \to Rr) \to \langle Ra, Rb\rangle prod\text{-}rel \to Rr$
$(old.rec\text{-}prod, old.rec\text{-}prod) \in (Ra \to Rb \to Rr) \to \langle Ra, Rb\rangle prod\text{-}rel \to Rr$
$(fst, fst) \in \langle Ra, Rb\rangle prod\text{-}rel \to Ra$
$(snd, snd) \in \langle Ra, Rb\rangle prod\text{-}rel \to Rb$
by (*auto dest*: *fun-relD split*: *prod.split*
simp: *prod-rel-def rec-prod-is-case*)

lemma *param-case-prod′*:
$[\![\ (p,p') \in \langle Ra, Rb\rangle prod\text{-}rel;$
$\bigwedge a\ b\ a'\ b'.\ [\![\ p=(a,b);\ p'=(a',b');\ (a,a') \in Ra;\ (b,b') \in Rb\ ]\!]$
$\Longrightarrow (f\ a\ b,\ f'\ a'\ b') \in R$
$]\!] \Longrightarrow (case\text{-}prod\ f\ p,\ case\text{-}prod\ f'\ p') \in R$
by (*auto split*: *prod.split*)

lemma *param-case-prod″*:
$[\![$
$\bigwedge a\ b\ a'\ b'.\ [\![p=(a,b);\ p'=(a',b')]\!] \Longrightarrow (f\ a\ b, f'\ a'\ b') \in R$
$]\!] \Longrightarrow (case\text{-}prod\ f\ p,\ case\text{-}prod\ f'\ p') \in R$
by (*auto split*: *prod.split*)

lemma *param-map-prod*[*param*]:
$(map\text{-}prod,\ map\text{-}prod)$
$\in (Ra \to Rb) \to (Rc \to Rd) \to \langle Ra, Rc\rangle prod\text{-}rel \to \langle Rb, Rd\rangle prod\text{-}rel$
unfolding *map-prod-def*[*abs-def*]
by *parametricity*

lemma *param-apfst*[*param*]:
(*apfst*,*apfst*)∈(*Ra*→*Rb*)→⟨*Ra*,*Rc*⟩*prod-rel*→⟨*Rb*,*Rc*⟩*prod-rel*
unfolding *apfst-def*[*abs-def*] **by** *parametricity*

lemma *param-apsnd*[*param*]:
(*apsnd*,*apsnd*)∈(*Rb*→*Rc*)→⟨*Ra*,*Rb*⟩*prod-rel*→⟨*Ra*,*Rc*⟩*prod-rel*
unfolding *apsnd-def*[*abs-def*] **by** *parametricity*

lemma *param-curry*[*param*]:
(*curry*,*curry*) ∈ (⟨*Ra*,*Rb*⟩*prod-rel* → *Rc*) → *Ra* → *Rb* → *Rc*
unfolding *curry-def* **by** *parametricity*

lemma *param-uncurry*[*param*]: (*uncurry*,*uncurry*) ∈ (*A*→*B*→*C*) → *A*×$_r$*B*→*C*
unfolding *uncurry-def*[*abs-def*] **by** *parametricity*

lemma *param-prod-swap*[*param*]: (*prod.swap*, *prod.swap*)∈*A*×$_r$*B* → *B*×$_r$*A* **by** *auto*

context *partial-function-definitions* **begin**
lemma
assumes *M*: *monotone le-fun le-fun F*
and *M'*: *monotone le-fun le-fun F'*
assumes *ADM*:
admissible (λ*a*. ∀ *x xa*. (*x*, *xa*) ∈ *Rb* ⟶ (*a x*, *fixp-fun F' xa*) ∈ *Ra*)
assumes *bot*: ⋀*x xa*. (*x*, *xa*) ∈ *Rb* ⟹ (*lub* {}, *fixp-fun F' xa*) ∈ *Ra*
assumes *F*: (*F*,*F'*)∈(*Rb*→*Ra*)→*Rb*→*Ra*
assumes *A*: (*x*,*x'*)∈*Rb*
shows (*fixp-fun F x*, *fixp-fun F' x'*)∈*Ra*
using *A*
apply (*induct arbitrary*: *x x'* *rule*: *ccpo.fixp-induct*[*OF ccpo - M*])
apply (*rule ADM*)
apply(*simp add*: *fun-lub-def bot*)
apply (*subst ccpo.fixp-unfold*[*OF ccpo M'*])
apply (*parametricity add*: *F*)
done
end

1.3.8 Option

lemma *param-option*[*param*]:
(*None*,*None*)∈⟨*R*⟩*option-rel*
(*Some*,*Some*)∈*R* → ⟨*R*⟩*option-rel*
(*case-option*,*case-option*)∈*Rr*→(*R* → *Rr*)→⟨*R*⟩*option-rel* → *Rr*
(*rec-option*,*rec-option*)∈*Rr*→(*R* → *Rr*)→⟨*R*⟩*option-rel* → *Rr*
by (*auto split*: *option.split*
simp: *option-rel-def case-option-def*[*symmetric*]
dest: *fun-relD*)

lemma *param-map-option*[*param*]: (*map-option*, *map-option*) ∈ (*A* → *B*) → ⟨*A*⟩*option-rel*
→ ⟨*B*⟩*option-rel*

apply (*intro fun-relI*)
apply (*auto elim*!: *option-relE dest*: *fun-relD*)
done

lemma *param-case-option′*:
⟦ (*x*,*x′*)∈⟨*Rv*⟩*option-rel*;
⟦*x*=*None*; *x′*=*None* ⟧ ⟹ (*fn*,*fn′*)∈*R*;
⋀*v v′*. ⟦ *x*=*Some v*; *x′*=*Some v′*; (*v*,*v′*)∈*Rv* ⟧ ⟹ (*fs v*, *fs′ v′*)∈*R*
⟧ ⟹ (*case-option fn fs x*, *case-option fn′ fs′ x′*) ∈ *R*
by (*auto split*: *option.split*)

lemma *the-paramL*: ⟦*l*≠*None*; (*l*,*r*)∈⟨*R*⟩*option-rel*⟧ ⟹ (*the l*, *the r*)∈*R*
apply (*cases l*)
by (*auto elim*: *option-relE*)

lemma *the-paramR*: ⟦*r*≠*None*; (*l*,*r*)∈⟨*R*⟩*option-rel*⟧ ⟹ (*the l*, *the r*)∈*R*
apply (*cases l*)
by (*auto elim*: *option-relE*)

lemma *the-default-param*[*param*]:
(*the-default*, *the-default*) ∈ *R* → ⟨*R*⟩*option-rel* → *R*
unfolding *the-default-def*
by *parametricity*

1.3.9 Sum

lemma *rec-sum-is-case*: *old.rec-sum* = *case-sum*
by (*rule ext*)+ (*auto split*: *sum.split*)

lemma *param-sum*[*param*]:
(*Inl*,*Inl*) ∈ *Rl* → ⟨*Rl*,*Rr*⟩*sum-rel*
(*Inr*,*Inr*) ∈ *Rr* → ⟨*Rl*,*Rr*⟩*sum-rel*
(*case-sum*,*case-sum*) ∈ (*Rl* → *R*) → (*Rr* → *R*) → ⟨*Rl*,*Rr*⟩*sum-rel* → *R*
(*old.rec-sum*,*old.rec-sum*) ∈ (*Rl* → *R*) → (*Rr* → *R*) → ⟨*Rl*,*Rr*⟩*sum-rel* → *R*
by (*fastforce split*: *sum.split dest*: *fun-relD*
simp: *rec-sum-is-case*)+

lemma *param-case-sum′*:
⟦ (*s*,*s′*)∈⟨*Rl*,*Rr*⟩*sum-rel*;
⋀*l l′*. ⟦ *s*=*Inl l*; *s′*=*Inl l′*; (*l*,*l′*)∈*Rl* ⟧ ⟹ (*fl l*, *fl′ l′*)∈*R*;
⋀*r r′*. ⟦ *s*=*Inr r*; *s′*=*Inr r′*; (*r*,*r′*)∈*Rr* ⟧ ⟹ (*fr r*, *fr′ r′*)∈*R*
⟧ ⟹ (*case-sum fl fr s*, *case-sum fl′ fr′ s′*)∈*R*
by (*auto split*: *sum.split*)

primrec *is-Inl* **where** *is-Inl* (*Inl* -) = *True* | *is-Inl* (*Inr* -) = *False*
primrec *is-Inr* **where** *is-Inr* (*Inr* -) = *True* | *is-Inr* (*Inl* -) = *False*

lemma *is-Inl-param*[*param*]: (*is-Inl*,*is-Inl*) ∈ ⟨*Ra*,*Rb*⟩*sum-rel* → *bool-rel*
unfolding *is-Inl-def* **by** *parametricity*

lemma *is-Inr-param*[*param*]: (*is-Inr*,*is-Inr*) ∈ ⟨*Ra*,*Rb*⟩*sum-rel* → *bool-rel*
unfolding *is-Inr-def* **by** *parametricity*

lemma *sum-projl-param*[*param*]:
⟦*is-Inl s*; (*s'*,*s*)∈⟨*Ra*,*Rb*⟩*sum-rel*⟧
⟹ (*Sum-Type.sum.projl s'*,*Sum-Type.sum.projl s*) ∈ *Ra*
apply (*cases s*)
apply (*auto elim*: *sum-relE*)
done

lemma *sum-projr-param*[*param*]:
⟦*is-Inr s*; (*s'*,*s*)∈⟨*Ra*,*Rb*⟩*sum-rel*⟧
⟹ (*Sum-Type.sum.projr s'*,*Sum-Type.sum.projr s*) ∈ *Rb*
apply (*cases s*)
apply (*auto elim*: *sum-relE*)
done

1.3.10 List

lemma *list-rel-append1*: (*as* @ *bs*, *l*) ∈ ⟨*R*⟩*list-rel*
⟷ (∃ *cs ds*. *l* = *cs*@*ds* ∧ (*as*,*cs*)∈⟨*R*⟩*list-rel* ∧ (*bs*,*ds*)∈⟨*R*⟩*list-rel*)
apply (*simp add*: *list-rel-def list-all2-append1*)
apply *auto*
apply (*metis list-all2-lengthD*)
done

lemma *list-rel-append2*: (*l*,*as* @ *bs*) ∈ ⟨*R*⟩*list-rel*
⟷ (∃ *cs ds*. *l* = *cs*@*ds* ∧ (*cs*,*as*)∈⟨*R*⟩*list-rel* ∧ (*ds*,*bs*)∈⟨*R*⟩*list-rel*)
apply (*simp add*: *list-rel-def list-all2-append2*)
apply *auto*
apply (*metis list-all2-lengthD*)
done

lemma *param-append*[*param*]:
(*append*, *append*)∈⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel*
by (*auto simp*: *list-rel-def list-all2-appendI*)

lemma *param-list1*[*param*]:
(*Nil*,*Nil*)∈⟨*R*⟩*list-rel*
(*Cons*,*Cons*)∈*R* → ⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel*
(*case-list*,*case-list*)∈*Rr*→(*R*→⟨*R*⟩*list-rel*→*Rr*)→⟨*R*⟩*list-rel*→*Rr*
apply (*force dest*: *fun-relD split*: *list.split*)+
done

lemma *param-rec-list*[*param*]:
(*rec-list*,*rec-list*)
∈ *Ra* → (*Rb* → ⟨*Rb*⟩*list-rel* → *Ra* → *Ra*) → ⟨*Rb*⟩*list-rel* → *Ra*
proof (*intro fun-relI*, *goal-cases*)

```
  case prems: (1 a a' f f' l l')
  from prems(3) show ?case
    using prems(1,2)
    apply (induct arbitrary: a a')
    apply simp
    apply (fastforce dest: fun-relD)
    done
qed

lemma param-case-list':
  ⟦ (l,l')∈⟨Rb⟩list-rel;
     ⟦l=[]; l'=[]⟧ ⟹ (n,n')∈Ra;
     ⋀x xs x' xs'. ⟦ l=x#xs; l'=x'#xs'; (x,x')∈Rb; (xs,xs')∈⟨Rb⟩list-rel ⟧
     ⟹ (c x xs, c' x' xs')∈Ra
   ⟧ ⟹ (case-list n c l, case-list n' c' l') ∈ Ra
  by (auto split: list.split)

lemma param-map[param]:
  (map,map)∈(R1→R2) → ⟨R1⟩list-rel → ⟨R2⟩list-rel
proof (rule fun-relI)+
  fix f g and xs ys
  assume ‹(f, g) ∈ R1 → R2›
  assume ‹(xs, ys) ∈ ⟨R1⟩list-rel›
  then show ‹(map f xs, map g ys) ∈ ⟨R2⟩list-rel›
  proof induction
    case 1
    show ?case
      by simp
  next
    case (2 x y xs ys)
    with ‹(f, g) ∈ R1 → R2› show ?case
      by (simp add: fun-relD2)
  qed
qed

lemma param-fold[param]:
  (fold,fold)∈(Re→Rs→Rs) → ⟨Re⟩list-rel → Rs → Rs
  (foldl,foldl)∈(Rs→Re→Rs) → Rs → ⟨Re⟩list-rel → Rs
  (foldr,foldr)∈(Re→Rs→Rs) → ⟨Re⟩list-rel → Rs → Rs
  unfolding List.fold-def List.foldr-def List.foldl-def
  by (parametricity)+

lemma param-list-all[param]: (list-all,list-all) ∈ (A→bool-rel) → ⟨A⟩list-rel →
bool-rel
  by (fold rel2p-def) (simp add: rel2p List.list-all-transfer)

context begin
  private primrec list-all2-alt :: ('a ⇒ 'b ⇒ bool) ⇒ 'a list ⇒ 'b list ⇒ bool
where
```

list-all2-alt P [] ys ⟷ (case ys of [] ⇒ True | - ⇒ False)
| *list-all2-alt P (x#xs) ys ⟷ (case ys of [] ⇒ False | y#ys ⇒ P x y ∧ list-all2-alt P xs ys)*

private lemma *list-all2-alt*: *list-all2 P xs ys = list-all2-alt P xs ys*
by (*induction xs arbitrary*: *ys*) (*auto split*: *list.splits*)

lemma *param-list-all2*[*param*]: (*list-all2*, *list-all2*) ∈ (*A*→*B*→*bool-rel*) → ⟨*A*⟩*list-rel* → ⟨*B*⟩*list-rel* → *bool-rel*
unfolding *list-all2-alt*[*abs-def*]
unfolding *list-all2-alt-def*[*abs-def*]
by *parametricity*

end

lemma *param-hd*[*param*]: *l*≠[] ⟹ (*l'*,*l*)∈⟨*A*⟩*list-rel* ⟹ (*hd l'*, *hd l*)∈*A*
unfolding *hd-def* **by** (*auto split*: *list.splits*)

lemma *param-last*[*param*]:
assumes *y* ≠ []
assumes (*x*, *y*) ∈ ⟨*A*⟩*list-rel*
shows (*last x*, *last y*) ∈ *A*
using *assms*(*2*,*1*)
by (*induction rule*: *list-rel-induct*) *auto*

lemma *param-rotate1*[*param*]: (*rotate1*, *rotate1*) ∈ ⟨*A*⟩*list-rel* → ⟨*A*⟩*list-rel*
unfolding *rotate1-def* **by** *parametricity*

schematic-goal *param-take*[*param*]: (*take*,*take*)∈(*?R*::(-×-) *set*)
unfolding *take-def*
by (*parametricity*)

schematic-goal *param-drop*[*param*]: (*drop*,*drop*)∈(*?R*::(-×-) *set*)
unfolding *drop-def*
by (*parametricity*)

schematic-goal *param-length*[*param*]:
(*length*,*length*)∈(*?R*::(-×-) *set*)
unfolding *size-list-overloaded-def size-list-def*
by (*parametricity*)

fun *list-eq* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* ⇒ *bool* **where**
list-eq eq [] [] ⟷ True
| *list-eq eq (a#l) (a'#l')*
⟷ (*if eq a a' then list-eq eq l l' else False*)
| *list-eq - - - ⟷ False*

lemma *param-list-eq*[*param*]:
(*list-eq*,*list-eq*) ∈

```
    (R → R → Id) → ⟨R⟩list-rel → ⟨R⟩list-rel → Id
proof (intro fun-relI, goal-cases)
  case prems: (1 eq eq' l1 l1' l2 l2')
  thus ?case
    apply −
    apply (induct eq' l1' l2' arbitrary: l1 l2 rule: list-eq.induct)
    apply (simp-all only: list-eq.simps |
      elim list-relE |
      parametricity)+
    done
qed

lemma id-list-eq-aux[simp]: (list-eq (=)) = (=)
proof (intro ext)
  fix l1 l2 :: 'a list
  show list-eq (=) l1 l2 = (l1 = l2)
    apply (induct (=) :: 'a ⇒ - l1 l2 rule: list-eq.induct)
    apply simp-all
    done
qed

lemma param-list-equals[param]:
  ⟦ ((=), (=)) ∈ R→R→Id ⟧
  ⟹ ((=), (=)) ∈ ⟨R⟩list-rel → ⟨R⟩list-rel → Id
  unfolding id-list-eq-aux[symmetric]
  by (parametricity)

lemma param-tl[param]:
  (tl,tl) ∈ ⟨R⟩list-rel → ⟨R⟩list-rel
  unfolding tl-def[abs-def]
  by (parametricity)


primrec list-all-rec where
  list-all-rec P [] ⟷ True
| list-all-rec P (a#l) ⟷ P a ∧ list-all-rec P l

primrec list-ex-rec where
  list-ex-rec P [] ⟷ False
| list-ex-rec P (a#l) ⟷ P a ∨ list-ex-rec P l

lemma list-all-rec-eq: (∀ x∈set l. P x) = list-all-rec P l
  by (induct l) auto

lemma list-ex-rec-eq: (∃ x∈set l. P x) = list-ex-rec P l
  by (induct l) auto

lemma param-list-ball[param]:
  ⟦(P,P')∈(Ra→Id); (l,l')∈⟨Ra⟩ list-rel⟧
```

$\Longrightarrow (\forall\, x{\in}set\ l.\ P\ x,\ \forall\, x{\in}set\ l'.\ P'\ x) \in Id$
unfolding *list-all-rec-eq*
unfolding *list-all-rec-def*
by (*parametricity*)

lemma *param-list-bex*[*param*]:
$[\![(P,P')\in(Ra\to Id);\ (l,l')\in\langle Ra\rangle\ list\text{-}rel]\!]$
$\Longrightarrow (\exists\, x{\in}set\ l.\ P\ x,\ \exists\, x{\in}set\ l'.\ P'\ x) \in Id$
unfolding *list-ex-rec-eq*[*abs-def*]
unfolding *list-ex-rec-def*
by (*parametricity*)

lemma *param-rev*[*param*]: $(rev,rev) \in \langle R\rangle list\text{-}rel \to \langle R\rangle list\text{-}rel$
unfolding *rev-def*
by (*parametricity*)

lemma *param-foldli*[*param*]: (*foldli*, *foldli*)
$\in \langle Re\rangle list\text{-}rel \to (Rs\to Id) \to (Re\to Rs\to Rs) \to Rs \to Rs$
unfolding *foldli-def*
by *parametricity*

lemma *param-foldri*[*param*]: (*foldri*, *foldri*)
$\in \langle Re\rangle list\text{-}rel \to (Rs\to Id) \to (Re\to Rs\to Rs) \to Rs \to Rs$
unfolding *foldri-def*[*abs-def*]
by *parametricity*

lemma *param-nth*[*param*]:
assumes *I*: $i'<length\ l'$
assumes *IR*: $(i,i')\in nat\text{-}rel$
assumes *LR*: $(l,l')\in\langle R\rangle list\text{-}rel$
shows $(l!i,l'!i') \in R$
using *LR I IR*
by (*induct arbitrary*: *i i' rule*: *list-rel-induct*)
(*auto simp*: *nth.simps split*: *nat.split*)

lemma *param-replicate*[*param*]:
$(replicate,replicate) \in nat\text{-}rel \to R \to \langle R\rangle list\text{-}rel$
unfolding *replicate-def* **by** *parametricity*

lemma *param-list-update*[*param*]:
$(list\text{-}update,list\text{-}update) \in \langle Ra\rangle list\text{-}rel \to nat\text{-}rel \to Ra \to \langle Ra\rangle list\text{-}rel$
unfolding *list-update-def*[*abs-def*] **by** *parametricity*

lemma *param-zip*[*param*]:
$(zip,\ zip) \in \langle Ra\rangle list\text{-}rel \to \langle Rb\rangle list\text{-}rel \to \langle\langle Ra,Rb\rangle prod\text{-}rel\rangle list\text{-}rel$
unfolding *zip-def* **by** *parametricity*

lemma *param-upt*[*param*]:
$(upt,\ upt) \in nat\text{-}rel \to nat\text{-}rel \to \langle nat\text{-}rel\rangle list\text{-}rel$

unfolding *upt-def*[*abs-def*] **by** *parametricity*

lemma *param-concat*[*param*]: (*concat*, *concat*) ∈
⟨⟨*R*⟩*list-rel*⟩*list-rel* → ⟨*R*⟩*list-rel*
unfolding *concat-def*[*abs-def*] **by** *parametricity*

lemma *param-all-interval-nat′* [*param*]:
(*List.all-interval*, *List.all-interval*)
∈ (*nat-rel* → *bool-rel*) → *nat-rel* → *nat-rel* → *bool-rel*
unfolding *List.all-interval-iff* [*abs-def*]
apply *parametricity*
apply *simp*
done

context
begin

qualified definition *all-interval-nat* :: ‹(*nat* ⇒ *bool*) ⇒ *nat* ⇒ *nat* ⇒ *bool*›
where *all-interval-nat-eq* [*simp*]: ‹*all-interval-nat P m n* ⟷ (∀ *i*∈{*m*..<*n*}. *P* *i*)›

end

lemma *param-all-interval-nat* [*param*]:
‹(*Param-HOL.all-interval-nat*, *Param-HOL.all-interval-nat*)
∈ (*nat-rel* → *bool-rel*) → *nat-rel* → *nat-rel* → *bool-rel*›
unfolding *Param-HOL.all-interval-nat-eq* [*abs-def*]
apply *parametricity*
apply *simp*
done

lemma *param-dropWhile*[*param*]:
(*dropWhile*, *dropWhile*) ∈ (*a* → *bool-rel*) → ⟨*a*⟩*list-rel* → ⟨*a*⟩*list-rel*
unfolding *dropWhile-def* **by** *parametricity*

lemma *param-takeWhile*[*param*]:
(*takeWhile*, *takeWhile*) ∈ (*a* → *bool-rel*) → ⟨*a*⟩*list-rel* → ⟨*a*⟩*list-rel*
unfolding *takeWhile-def* **by** *parametricity*

end

Chapter 2

Automatic Refinement

2.1 Automatic Refinement Tool

theory *Autoref-Tool*
imports
Autoref-Translate
Autoref-Gen-Algo
Autoref-Relator-Interface
begin

2.1.1 Standard setup

Declaration of standard phases

```
declaration ‹fn phi => let open Autoref-Phases in
  I
  #> register-phase id-op 10 Autoref-Id-Ops.id-phase phi
  #> register-phase rel-inf 20
       Autoref-Rel-Inf.roi-phase phi
  #> register-phase fix-rel 22
       Autoref-Fix-Rel.phase phi
  #> register-phase trans 30
       Autoref-Translate.trans-phase phi
end
›
```

Main method

```
method-setup autoref = ‹let
    open Refine-Util
    val autoref-flags =
          parse-bool-config trace Autoref-Phases.cfg-trace
      || parse-bool-config debug Autoref-Phases.cfg-debug
      || parse-bool-config keep-goal Autoref-Phases.cfg-keep-goal

    val autoref-phases =
```

```
    Args.$$$ phases |-- Args.colon |-- Scan.repeat1 Args.name

  in
    parse-paren-lists autoref-flags
    |-- Scan.option (Scan.lift (autoref-phases)) >>
    ( fn phases => fn ctxt => SIMPLE-METHOD' (
      (
        case phases of
          NONE => Autoref-Phases.all-phases-tac
        | SOME names => Autoref-Phases.phases-tacN names
      ) (Autoref-Phases.init-data ctxt)
      (* TODO: If we want more fine-grained initialization here, solvers have
         to depend on phases, or on data that they initialize if necessary *)
    ))

  end

› Automatic Refinement
```

2.1.2 Tools

```
setup ‹
  let
    fun higher-order-rl-of ctxt thm = case Thm.concl-of thm of
      @{mpat Trueprop ((-,?t)∈-)} => let
        val (f,args) = strip-comb t
      in
        if length args = 0 then
          thm
        else let
          val cT = TVar(('c,0), @{sort type})
          val c = Var ((c,0),cT)
          val R = Var ((R,0), HOLogic.mk-setT (HOLogic.mk-prodT (cT, fastype-of
f)))
          val goal =
            HOLogic.mk-mem (HOLogic.mk-prod (c,f), R)
            |> HOLogic.mk-Trueprop
          val goal-ctxt = Variable.declare-term goal ctxt
          val res-thm =
            Goal.prove-internal ctxt [] (Thm.cterm-of ctxt goal)
              (fn - =>
                REPEAT (resolve-tac goal-ctxt @{thms fun-relI} 1)
                THEN (resolve-tac goal-ctxt [thm] 1)
                THEN (ALLGOALS (assume-tac goal-ctxt)))
        in
          res-thm
        end
      end
    | - => raise THM(Expected autoref rule,~1,[thm])
```

```
    val higher-order-rl-attr =
      Thm.rule-attribute [] (higher-order-rl-of o Context.proof-of)
  in
    Attrib.setup @{binding autoref-higher-order-rule}
      (Scan.succeed higher-order-rl-attr) Autoref: Convert rule to higher−order form
  end
›
```

2.1.3 Advanced Debugging

```
method-setup autoref-trans-step-keep = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
    Autoref-Translate.trans-dbg-step-tac (Autoref-Phases.init-data ctxt)
  ))
› Single translation step, leaving unsolved side−coditions

method-setup autoref-trans-step = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
    Autoref-Translate.trans-step-tac (Autoref-Phases.init-data ctxt)
  ))
› Single translation step

method-setup autoref-trans-step-only = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
    Autoref-Translate.trans-step-only-tac (Autoref-Phases.init-data ctxt)
  ))
› Single translation step, not attempting to solve side−coditions

method-setup autoref-side = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
    Autoref-Translate.side-dbg-tac (Autoref-Phases.init-data ctxt)
  ))
› Solve side condition, leave unsolved subgoals

method-setup autoref-try-solve = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
    Autoref-Fix-Rel.try-solve-tac (Autoref-Phases.init-data ctxt)
  ))
› Try to solve constraint and trace debug information

method-setup autoref-solve-step = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
    Autoref-Fix-Rel.solve-step-tac (Autoref-Phases.init-data ctxt)
  ))
› Single−step of constraint solver

method-setup autoref-id-op = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
```

```
    Autoref-Id-Ops.id-tac ctxt
  ))
›

method-setup autoref-solve-id-op = ‹
  Scan.succeed (fn ctxt => SIMPLE-METHOD' (
    Autoref-Id-Ops.id-tac (Config.put Autoref-Id-Ops.cfg-ss-id-op false ctxt)
  ))
›



ML ‹
  structure Autoref-Debug = struct
    fun print-thm-pairs ctxt = let
      val ctxt = Autoref-Phases.init-data ctxt
      val p = Autoref-Fix-Rel.thm-pairsD-get ctxt
        |> Autoref-Fix-Rel.pretty-thm-pairs ctxt
        |> Pretty.string-of
    in
      warning p
    end

    fun print-thm-pairs-matching ctxt cpat = let
      val pat = Thm.term-of cpat
      val ctxt = Autoref-Phases.init-data ctxt
      val thy = Proof-Context.theory-of ctxt

      fun matches NONE = false
        | matches (SOME (-,(f,-))) = Pattern.matches thy (pat,f)

      val p = Autoref-Fix-Rel.thm-pairsD-get ctxt
        |> filter (matches o #1)
        |> Autoref-Fix-Rel.pretty-thm-pairs ctxt
        |> Pretty.string-of
    in
      warning p
    end
  end
›
```

General casting-tag, that allows type-casting on concrete level, while being identity on abstract level.

definition [*simp*]: $CAST \equiv id$
lemma [*autoref-itype*]: $CAST ::_i I \rightarrow_i I$ **by** *simp*

Hide internal stuff

notation (*input*) *rel-ANNOT* (**infix** ‹:::$_r$› *10*)
notation (*input*) *ind-ANNOT* (**infix** ‹::#$_r$› *10*)

locale *autoref-syn* **begin**
notation (*input*) *APP* (**infixl** ‹$› *900*)
notation (*input*) *rel-ANNOT* (**infix** ‹:::› *10*)
notation (*input*) *ind-ANNOT* (**infix** ‹::#› *10*)
notation *OP* (‹*OP*›)
notation (*input*) *ABS* (**binder** ‹λ″› *10*)
end

no-notation (*input*) *APP* (**infixl** ‹$› *900*)
no-notation (*input*) *ABS* (**binder** ‹λ″› *10*)

no-notation (*input*) *rel-ANNOT* (**infix** ‹:::› *10*)
no-notation (*input*) *ind-ANNOT* (**infix** ‹::#› *10*)

hide-const (**open**) *PROTECT ANNOT OP APP ABS ID-FAIL rel-annot ind-annot*

end

2.2 Standard HOL Bindings

theory *Autoref-Bindings-HOL*
imports *Tool/Autoref-Tool*
begin

2.2.1 Structural Expansion

In some situations, autoref imitates the operations on typeclasses and the typeclass hierarchy. This may result in structural mismatches, e.g., a hash-code side-condition may look like:

is-hashcode (*prod-eq* (=) (=)) *hashcode*

This cannot be discharged by the rule

is-hashcode (=) *hashcode*

In order to handle such cases, we introduce a set of simplification lemmas that expand the structure of an operator as far as possible. These lemmas are integrated into a tagged solver, that can prove equality between operators modulo structural expansion.

definition [*simp*]: *STRUCT-EQ-tag x y* ≡ *x* = *y*
lemma *STRUCT-EQ-tagI*: *x*=*y* ⟹ *STRUCT-EQ-tag x y* **by** *simp*

ML ‹
structure Autoref-Struct-Expand = *struct*
structure autoref-struct-expand = *Named-Thms* (

```
    val name = @{binding autoref-struct-expand}
    val description = Autoref: Structural expansion lemmas
  )

  fun expand-tac ctxt = let
    val ss = put-simpset HOL-basic-ss ctxt addsimps autoref-struct-expand.get ctxt
  in
    SOLVED' (asm-simp-tac ss)
  end


  val setup = autoref-struct-expand.setup
  val decl-setup = fn phi =>
   Tagged-Solver.declare-solver @{thms STRUCT-EQ-tagI} @{binding STRUCT-EQ}

      Autoref: Equality modulo structural expansion
      (expand-tac) phi

 end
›
```

setup *Autoref-Struct-Expand.setup*
declaration *Autoref-Struct-Expand.decl-setup*

Sometimes, also relators must be expanded. Usually to check them to be the identity relator

definition [*simp*]: *REL-IS-ID R* ≡ *R=Id*
definition [*simp*]: *REL-FORCE-ID R* ≡ *R=Id*
lemma *REL-IS-ID-trigger*: *R=Id* ⟹ *REL-IS-ID R* **by** *simp*
lemma *REL-FORCE-ID-trigger*: *R=Id* ⟹ *REL-FORCE-ID R* **by** *simp*

declaration ‹*Tagged-Solver.add-triggers*
Relators.relator-props-solver @{*thms REL-IS-ID-trigger*}›

declaration ‹*Tagged-Solver.add-triggers*
Relators.force-relator-props-solver @{*thms REL-FORCE-ID-trigger*}›

abbreviation *PREFER-id R* ≡ *PREFER REL-IS-ID R*

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of fun-rel i-fun*]

2.2.2 Booleans

consts

i-bool :: *interface*

lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of bool-rel i-bool*]

lemma [*autoref-itype*]:
True ::$_i$ *i-bool*
False ::$_i$ *i-bool*
conj ::$_i$ *i-bool* →$_i$ *i-bool* →$_i$ *i-bool*
(⟷) ::$_i$ *i-bool* →$_i$ *i-bool* →$_i$ *i-bool*
(⟶) ::$_i$ *i-bool* →$_i$ *i-bool* →$_i$ *i-bool*
disj ::$_i$ *i-bool* →$_i$ *i-bool* →$_i$ *i-bool*
Not ::$_i$ *i-bool* →$_i$ *i-bool*
case-bool ::$_i$ *I* →$_i$ *I* →$_i$ *i-bool* →$_i$ *I*
old.rec-bool ::$_i$ *I* →$_i$ *I* →$_i$ *i-bool* →$_i$ *I*
by *auto*

lemma *autoref-bool*[*autoref-rules*]:
(*x*,*x*)∈*bool-rel*
(*conj*,*conj*)∈*bool-rel*→*bool-rel*→*bool-rel*
(*disj*,*disj*)∈*bool-rel*→*bool-rel*→*bool-rel*
(*Not*,*Not*)∈*bool-rel*→*bool-rel*
(*case-bool*,*case-bool*)∈*R*→*R*→*bool-rel*→*R*
(*old.rec-bool*,*old.rec-bool*)∈*R*→*R*→*bool-rel*→*R*
((⟷), (⟷))∈*bool-rel*→*bool-rel*→*bool-rel*
((⟶), (⟶))∈*bool-rel*→*bool-rel*→*bool-rel*
by (*auto split*: *bool.split simp*: *rec-bool-is-case*)

2.2.3 Standard Type Classes

context begin interpretation *autoref-syn* **.**

We allow these operators for all interfaces.

lemma [*autoref-itype*]:
(<) ::$_i$ *I* →$_i$ *I* →$_i$ *i-bool*
(≤) ::$_i$ *I* →$_i$ *I* →$_i$ *i-bool*
(=) ::$_i$ *I* →$_i$ *I* →$_i$ *i-bool*
(+) ::$_i$ *I* →$_i$ *I* →$_i$ *I*
(−) ::$_i$ *I* →$_i$ *I* →$_i$ *I*
(*div*) ::$_i$ *I* →$_i$ *I* →$_i$ *I*
(*mod*) ::$_i$ *I* →$_i$ *I* →$_i$ *I*
(∗) ::$_i$ *I* →$_i$ *I* →$_i$ *I*
0 ::$_i$ *I*
1 ::$_i$ *I*
numeral x ::$_i$ *I*
uminus ::$_i$ *I* →$_i$ *I*
by *auto*

lemma *pat-num-generic*[*autoref-op-pat*]:
0 ≡ *OP 0* :::$_i$ *I*

1 ≡ *OP 1* :::$_i$ *I*
numeral x ≡ (*OP* (*numeral x*) :::$_i$ *I*)
by *simp-all*

lemma [*autoref-rules*]:
assumes *PRIO-TAG-GEN-ALGO*
shows ((<), (<)) ∈ *Id*→*Id*→*bool-rel*
and ((≤), (≤)) ∈ *Id*→*Id*→*bool-rel*
and ((=), (=)) ∈ *Id*→*Id*→*bool-rel*
and (*numeral x*,*OP* (*numeral x*) ::: *Id*) ∈ *Id*
and (*uminus*,*uminus*) ∈ *Id* → *Id*
and (*0*,*0*) ∈ *Id*
and (*1*,*1*) ∈ *Id*
by *auto*

2.2.4 Functional Combinators

lemma [*autoref-itype*]: *id* ::$_i$ *I* →$_i$ *I* **by** *simp*
lemma *autoref-id*[*autoref-rules*]: (*id*,*id*)∈*R*→*R* **by** *auto*

term (*o*)
lemma [*autoref-itype*]: (∘) ::$_i$ (*Ia*→$_i$*Ib*) →$_i$ (*Ic* →$_i$ *Ia*) →$_i$ *Ic* →$_i$ *Ib*
by *simp*
lemma *autoref-comp*[*autoref-rules*]:
((*o*), (*o*)) ∈ (*Ra* → *Rb*) → (*Rc* → *Ra*) → *Rc* → *Rb*
by (*auto dest*: *fun-relD*)

lemma [*autoref-itype*]: *If* ::$_i$ *i-bool* →$_i$ *I* →$_i$ *I* →$_i$ *I* **by** *simp*
lemma *autoref-If*[*autoref-rules*]: (*If*,*If*)∈*Id*→*R*→*R*→*R* **by** *auto*
lemma *autoref-If-cong*[*autoref-rules*]:
assumes (*c'*,*c*)∈*Id*
assumes *REMOVE-INTERNAL c* ⟹ (*t'*,*t*)∈*R*
assumes ¬ *REMOVE-INTERNAL c* ⟹ (*e'*,*e*)∈*R*
shows (*If c' t' e'*,(*OP If* ::: *Id*→*R*→*R*→*R*)$*c*$*t*$*e*)∈*R*
using *assms* **by** (*auto*)

lemma [*autoref-itype*]: *Let* ::$_i$ *Ix* →$_i$ (*Ix*→$_i$*Iy*) →$_i$ *Iy* **by** *auto*
lemma *autoref-Let*:
(*Let*,*Let*)∈*Ra* → (*Ra*→*Rr*) → *Rr*
by (*auto dest*: *fun-relD*)

lemma *autoref-Let-cong*[*autoref-rules*]:
assumes (*x'*,*x*)∈*Ra*
assumes ⋀*y y'*. *REMOVE-INTERNAL* (*x*=*y*) ⟹ (*y'*,*y*)∈*Ra* ⟹ (*f' y'*, *f*$*y*)∈*Rr*
shows (*Let x' f'*,(*OP Let* ::: *Ra* → (*Ra* → *Rr*) → *Rr*)$*x*$*f*)∈*Rr*
using *assms* **by** (*auto*)

end

2.2.5 Unit

consts *i-unit* :: *interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of unit-rel i-unit*]

lemma [*autoref-rules*]: ((),())∈*unit-rel* **by** *simp*

2.2.6 Nat

consts *i-nat* :: *interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of nat-rel i-nat*]

context begin interpretation *autoref-syn* **.**
lemma *pat-num-nat*[*autoref-op-pat*]:
0::*nat* ≡ *OP 0* :::$_i$ *i-nat*
1::*nat* ≡ *OP 1* :::$_i$ *i-nat*
(*numeral x*)::*nat* ≡ (*OP* (*numeral x*) :::$_i$ *i-nat*)
by *simp-all*

lemma *autoref-nat*[*autoref-rules*]:
(*0*, *0*::*nat*) ∈ *nat-rel*
(*Suc*, *Suc*) ∈ *nat-rel* → *nat-rel*
(*1*, *1*::*nat*) ∈ *nat-rel*
(*numeral n*::*nat*,*numeral n*::*nat*) ∈ *nat-rel*
((<), (<) ::*nat* ⇒ -) ∈ *nat-rel* → *nat-rel* → *bool-rel*
((≤), (≤) ::*nat* ⇒ -) ∈ *nat-rel* → *nat-rel* → *bool-rel*
((=), (=) ::*nat* ⇒ -) ∈ *nat-rel* → *nat-rel* → *bool-rel*
((+) ::*nat*⇒-,(+))∈*nat-rel*→*nat-rel*→*nat-rel*
((−) ::*nat*⇒-,(−))∈*nat-rel*→*nat-rel*→*nat-rel*
((*div*) ::*nat*⇒-,(*div*))∈*nat-rel*→*nat-rel*→*nat-rel*
((∗), (∗))∈*nat-rel*→*nat-rel*→*nat-rel*
((*mod*), (*mod*))∈*nat-rel*→*nat-rel*→*nat-rel*
by *auto*

lemma *autoref-case-nat*[*autoref-rules*]:
(*case-nat*,*case-nat*)∈*Ra* → (*Id* → *Ra*) → *Id* → *Ra*
apply (*intro fun-relI*)
apply (*auto split*: *nat.split dest*: *fun-relD*)
done

lemma *autoref-rec-nat*: (*rec-nat*,*rec-nat*) ∈ *R* → (*Id* → *R* → *R*) → *Id* → *R*
apply (*intro fun-relI*)
proof *goal-cases*
case (*1 s s′ f f′ n n′*) **thus** *?case*
apply (*induct n′ arbitrary*: *n s s′*)
apply (*fastforce simp*: *fun-rel-def*)+
done
qed
end

2.2.7 Int

consts *i-int* :: *interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of int-rel i-int*]

context begin interpretation *autoref-syn* **.**
lemma *pat-num-int*[*autoref-op-pat*]:
0::*int* ≡ *OP 0* :::$_i$ *i-int*
1::*int* ≡ *OP 1* :::$_i$ *i-int*
(*numeral x*)::*int* ≡ (*OP* (*numeral x*) :::$_i$ *i-int*)
by *simp-all*

lemma *autoref-int*[*autoref-rules* (**overloaded**)]:
(*0*, *0*::*int*) ∈ *int-rel*
(*1*, *1*::*int*) ∈ *int-rel*
(*numeral n*::*int*,*numeral n*::*int*) ∈ *int-rel*
((<), (<) ::*int* ⇒ -) ∈ *int-rel* → *int-rel* → *bool-rel*
((≤), (≤) ::*int* ⇒ -) ∈ *int-rel* → *int-rel* → *bool-rel*
((=), (=) ::*int* ⇒ -) ∈ *int-rel* → *int-rel* → *bool-rel*
((+) ::*int*⇒-,(+))∈*int-rel*→*int-rel*→*int-rel*
((−) ::*int*⇒-,(−))∈*int-rel*→*int-rel*→*int-rel*
((*div*) ::*int*⇒-,(*div*))∈*int-rel*→*int-rel*→*int-rel*
(*uminus*,*uminus*)∈*int-rel*→*int-rel*
((∗), (∗))∈*int-rel*→*int-rel*→*int-rel*
((*mod*), (*mod*))∈*int-rel*→*int-rel*→*int-rel*
by *auto*
end

2.2.8 Product

consts *i-prod* :: *interface* ⇒ *interface* ⇒ *interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of prod-rel i-prod*]

context begin interpretation *autoref-syn* **.**

lemma *prod-refine*[*autoref-rules*]:
(*Pair*,*Pair*)∈*Ra* → *Rb* → ⟨*Ra*,*Rb*⟩*prod-rel*
(*case-prod*,*case-prod*) ∈ (*Ra* → *Rb* → *Rr*) → ⟨*Ra*,*Rb*⟩*prod-rel* → *Rr*
(*old.rec-prod*,*old.rec-prod*) ∈ (*Ra* → *Rb* → *Rr*) → ⟨*Ra*,*Rb*⟩*prod-rel* → *Rr*
(*fst*,*fst*)∈⟨*Ra*,*Rb*⟩*prod-rel* → *Ra*
(*snd*,*snd*)∈⟨*Ra*,*Rb*⟩*prod-rel* → *Rb*
by (*auto dest*: *fun-relD split*: *prod.split*
simp: *prod-rel-def rec-prod-is-case*)

definition *prod-eq eqa eqb x1 x2* ≡
case x1 of (*a1*,*b1*) ⇒ *case x2 of* (*a2*,*b2*) ⇒ *eqa a1 a2* ∧ *eqb b1 b2*

lemma *prod-eq-autoref*[*autoref-rules* (**overloaded**)]:
⟦GEN-OP eqa (=) (Ra→Ra→Id); GEN-OP eqb (=) (Rb→Rb→Id)⟧
⟹ (prod-eq eqa eqb,(=)) ∈ ⟨Ra,Rb⟩prod-rel → ⟨Ra,Rb⟩prod-rel → Id
unfolding *prod-eq-def*[*abs-def*]
by (*fastforce dest*: *fun-relD*)

lemma *prod-eq-expand*[*autoref-struct-expand*]: *(=) = prod-eq (=) (=)*
unfolding *prod-eq-def*[*abs-def*]
by (*auto intro*!: *ext*)
end

2.2.9 Option

consts *i-option* :: *interface ⇒ interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of option-rel i-option*]

context begin interpretation *autoref-syn* **.**

lemma *autoref-opt*[*autoref-rules*]:
(None,None)∈⟨R⟩option-rel
(Some,Some)∈R → ⟨R⟩option-rel
(case-option,case-option)∈Rr→(R → Rr)→⟨R⟩option-rel → Rr
(rec-option,rec-option)∈Rr→(R → Rr)→⟨R⟩option-rel → Rr
by (*auto split*: *option.split*
simp: *option-rel-def case-option-def*[*symmetric*]
dest: *fun-relD*)

lemma *autoref-the*[*autoref-rules*]:
assumes *SIDE-PRECOND (x≠None)*
assumes *(x′,x)∈⟨R⟩option-rel*
shows (*the x′*, (*OP the* ::: *⟨R⟩option-rel → R*)$*x*) ∈ *R*
using *assms*
by (*auto simp*: *option-rel-def*)

lemma *autoref-the-default*[*autoref-rules*]:
(the-default, the-default) ∈ R → ⟨R⟩option-rel → R
by *parametricity*

definition [*simp*]: *is-None a ≡ case a of None ⇒ True | - ⇒ False*
lemma *pat-isNone*[*autoref-op-pat*]:
a=None ≡ (OP is-None :::$_i$ ⟨I⟩$_i$i-option →$_i$ i-bool)$a
None=a ≡ (OP is-None :::$_i$ ⟨I⟩$_i$i-option →$_i$ i-bool)$a
by (*auto intro*!: *eq-reflection split*: *option.splits*)
lemma *autoref-is-None*[*param*,*autoref-rules*]:
(is-None,is-None)∈⟨R⟩option-rel → Id
by (*auto split*: *option.splits*)

lemma *fold-is-None*: *x=None ⟷ is-None x* **by** (*cases x*) *auto*

definition *option-eq eq v1 v2* ≡
case (*v1*,*v2*) *of*
(*None*,*None*) ⇒ *True*
| (*Some x1*, *Some x2*) ⇒ *eq x1 x2*
| - ⇒ *False*

lemma *option-eq-autoref*[*autoref-rules* (**overloaded**)]:
⟦*GEN-OP eq* (=) (*R*→*R*→*Id*)⟧
⟹ (*option-eq eq*,(=)) ∈ ⟨*R*⟩*option-rel* → ⟨*R*⟩*option-rel* → *Id*
unfolding *option-eq-def*[*abs-def*]
by (*auto dest*: *fun-relD split*: *option.splits elim*!: *option-relE*)

lemma *option-eq-expand*[*autoref-struct-expand*]:
(=) = *option-eq* (=)
by (*auto intro*!: *ext simp*: *option-eq-def split*: *option.splits*)
end

2.2.10 Sum-Types

consts *i-sum* :: *interface* ⇒ *interface* ⇒ *interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of sum-rel i-sum*]

context begin interpretation *autoref-syn* **.**

lemma *autoref-sum*[*autoref-rules*]:
(*Inl*,*Inl*) ∈ *Rl* → ⟨*Rl*,*Rr*⟩*sum-rel*
(*Inr*,*Inr*) ∈ *Rr* → ⟨*Rl*,*Rr*⟩*sum-rel*
(*case-sum*,*case-sum*) ∈ (*Rl* → *R*) → (*Rr* → *R*) → ⟨*Rl*,*Rr*⟩*sum-rel* → *R*
(*old.rec-sum*,*old.rec-sum*) ∈ (*Rl* → *R*) → (*Rr* → *R*) → ⟨*Rl*,*Rr*⟩*sum-rel* → *R*
by (*fastforce split*: *sum.split dest*: *fun-relD*
simp: *rec-sum-is-case*)+

definition *sum-eq eql eqr s1 s2* ≡
case (*s1*,*s2*) *of*
(*Inl x1*, *Inl x2*) ⇒ *eql x1 x2*
| (*Inr x1*, *Inr x2*) ⇒ *eqr x1 x2*
| - ⇒ *False*

lemma *sum-eq-autoref*[*autoref-rules* (**overloaded**)]:
⟦*GEN-OP eql* (=) (*Rl*→*Rl*→*Id*); *GEN-OP eqr* (=) (*Rr*→*Rr*→*Id*)⟧
⟹ (*sum-eq eql eqr*,(=)) ∈ ⟨*Rl*,*Rr*⟩*sum-rel* → ⟨*Rl*,*Rr*⟩*sum-rel* → *Id*
unfolding *sum-eq-def*[*abs-def*]
by (*fastforce dest*: *fun-relD elim*!: *sum-relE*)

lemma *sum-eq-expand*[*autoref-struct-expand*]: (=) = *sum-eq* (=) (=)
by (*auto intro*!: *ext simp*: *sum-eq-def split*: *sum.splits*)

lemmas [*autoref-rules*] = *is-Inl-param is-Inr-param*

lemma *autoref-sum-Projl*[*autoref-rules*]:
⟦*SIDE-PRECOND* (*is-Inl s*); (*s'*,*s*)∈⟨*Ra*,*Rb*⟩*sum-rel*⟧
⟹ (*Sum-Type.sum.projl s'*, (*OP Sum-Type.sum.projl* ::: ⟨*Ra*,*Rb*⟩*sum-rel* → *Ra*)$*s*)∈*Ra*
by *simp parametricity*

lemma *autoref-sum-Projr*[*autoref-rules*]:
⟦*SIDE-PRECOND* (*is-Inr s*); (*s'*,*s*)∈⟨*Ra*,*Rb*⟩*sum-rel*⟧
⟹ (*Sum-Type.sum.projr s'*, (*OP Sum-Type.sum.projr* ::: ⟨*Ra*,*Rb*⟩*sum-rel* → *Rb*)$*s*)∈*Rb*
by *simp parametricity*

end

2.2.11 List

consts *i-list* :: *interface* ⇒ *interface*
lemmas [*autoref-rel-intf*] = *REL-INTFI*[*of list-rel i-list*]

context begin interpretation *autoref-syn* **.**

lemma *autoref-append*[*autoref-rules*]:
(*append*, *append*)∈⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel*
by (*auto simp*: *list-rel-def list-all2-appendI*)

lemma *refine-list*[*autoref-rules*]:
(*Nil*,*Nil*)∈⟨*R*⟩*list-rel*
(*Cons*,*Cons*)∈*R* → ⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel*
(*case-list*,*case-list*)∈*Rr*→(*R*→⟨*R*⟩*list-rel*→*Rr*)→⟨*R*⟩*list-rel*→*Rr*
apply (*force dest*: *fun-relD split*: *list.split*)+
done

lemma *autoref-rec-list*[*autoref-rules*]: (*rec-list*,*rec-list*)
∈ *Ra* → (*Rb* → ⟨*Rb*⟩*list-rel* → *Ra* → *Ra*) → ⟨*Rb*⟩*list-rel* → *Ra*
proof (*intro fun-relI*, *goal-cases*)
case *prems*: (*1 a a' f f' l l'*)
from *prems*(*3*) **show** *?case*
using *prems*(*1*,*2*)
apply (*induct arbitrary*: *a a'*)
apply *simp*
apply (*fastforce dest*: *fun-relD*)
done
qed

lemma *refine-map*[*autoref-rules*]:
(*map*,*map*)∈(*R1*→*R2*) → ⟨*R1*⟩*list-rel* → ⟨*R2*⟩*list-rel*

by (*fact param-map*)

lemma *refine-fold*[*autoref-rules*]:
(*fold*,*fold*)∈(*Re*→*Rs*→*Rs*) → ⟨*Re*⟩*list-rel* → *Rs* → *Rs*
(*foldl*,*foldl*)∈(*Rs*→*Re*→*Rs*) → *Rs* → ⟨*Re*⟩*list-rel* → *Rs*
(*foldr*,*foldr*)∈(*Re*→*Rs*→*Rs*) → ⟨*Re*⟩*list-rel* → *Rs* → *Rs*
unfolding *List.fold-def List.foldr-def List.foldl-def*
by (*autoref*)+

schematic-goal *autoref-take*[*autoref-rules*]: (*take*,*take*)∈(*?R*::(-×-) *set*)
unfolding *take-def* **by** *autoref*
schematic-goal *autoref-drop*[*autoref-rules*]: (*drop*,*drop*)∈(*?R*::(-×-) *set*)
unfolding *drop-def* **by** *autoref*
schematic-goal *autoref-length*[*autoref-rules*]:
(*length*,*length*)∈(*?R*::(-×-) *set*)
unfolding *size-list-overloaded-def size-list-def*
by (*autoref*)

lemma *autoref-nth*[*autoref-rules*]:
assumes (*l*,*l'*)∈⟨*R*⟩*list-rel*
assumes (*i*,*i'*)∈*Id*
assumes *SIDE-PRECOND* (*i'* < *length l'*)
shows (*nth l i*,(*OP nth* ::: ⟨*R*⟩*list-rel* → *Id* → *R*)$*l'*$*i'*)∈*R*
unfolding *ANNOT-def*
using *assms*
apply (*induct arbitrary*: *i i'*)
apply *simp*
apply (*case-tac i'*)
apply *auto*
done

fun *list-eq* :: ('*a* ⇒ '*a* ⇒ *bool*) ⇒ '*a list* ⇒ '*a list* ⇒ *bool* **where**
list-eq eq [] [] ⟷ *True*
| *list-eq eq* (*a*#*l*) (*a'*#*l'*)
⟷ (*if eq a a' then list-eq eq l l' else False*)
| *list-eq* - - - ⟷ *False*

lemma *autoref-list-eq-aux*:
(*list-eq*,*list-eq*) ∈
(*R* → *R* → *Id*) → ⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel* → *Id*
proof (*intro fun-relI*, *goal-cases*)
case (*1 eq eq' l1 l1' l2 l2'*)
thus *?case*
apply −
apply (*induct eq' l1' l2' arbitrary*: *l1 l2 rule*: *list-eq.induct*)
apply *simp*
apply (*case-tac l1*)
apply *simp*
apply (*case-tac l2*)

```
    apply (simp)
    apply (auto dest: fun-relD) []
    apply (case-tac l1)
    apply simp
    apply simp
    apply (case-tac l2)
    apply simp
    apply simp
    done
qed

lemma list-eq-expand[autoref-struct-expand]: (=) = (list-eq (=))
proof (intro ext)
  fix l1 l2 :: 'a list
  show (l1 = l2) ⟷ list-eq (=) l1 l2
    apply (induct (=) :: 'a ⇒ - l1 l2 rule: list-eq.induct)
    apply simp-all
    done
qed

lemma autoref-list-eq[autoref-rules (overloaded)]:
  GEN-OP eq (=) (R→R→Id) ⟹ (list-eq eq, (=))
  ∈ ⟨R⟩list-rel → ⟨R⟩list-rel → Id
  unfolding autoref-tag-defs
  apply (subst list-eq-expand)
  apply (parametricity add: autoref-list-eq-aux)
  done

lemma autoref-hd[autoref-rules]:
  ⟦ SIDE-PRECOND (l'≠[]); (l,l') ∈ ⟨R⟩list-rel ⟧ ⟹
    (hd l,(OP hd ::: ⟨R⟩list-rel → R)$l') ∈ R
  apply (simp add: ANNOT-def)
  apply (cases l')
  apply simp
  apply (cases l)
  apply auto
  done

lemma autoref-tl[autoref-rules]:
  (tl,tl) ∈ ⟨R⟩list-rel → ⟨R⟩list-rel
  unfolding tl-def[abs-def]
  by autoref

definition [simp]: is-Nil a ≡ case a of [] ⇒ True | - ⇒ False

lemma is-Nil-pat[autoref-op-pat]:
  a=[] ≡ (OP is-Nil :::ᵢ ⟨I⟩ᵢi-list →ᵢ i-bool)$a
  []=a ≡ (OP is-Nil :::ᵢ ⟨I⟩ᵢi-list →ᵢ i-bool)$a
  by (auto intro!: eq-reflection split: list.splits)
```

lemma *autoref-is-Nil*[*param*,*autoref-rules*]:
(*is-Nil*,*is-Nil*)∈⟨*R*⟩*list-rel* → *bool-rel*
by (*auto split*: *list.splits*)

lemma *conv-to-is-Nil*:
l=[] ⟷ *is-Nil l*
[]=*l* ⟷ *is-Nil l*
unfolding *is-Nil-def* **by** (*auto split*: *list.split*)

lemma *autoref-butlast*[*param*, *autoref-rules*]:
(*butlast*,*butlast*) ∈ ⟨*R*⟩*list-rel* → ⟨*R*⟩*list-rel*
unfolding *butlast-def conv-to-is-Nil*
by *parametricity*

definition [*simp*]: *op-list-singleton x* ≡ [*x*]
lemma *op-list-singleton-pat*[*autoref-op-pat*]:
[*x*] ≡ (*OP op-list-singleton* :::$_i$ *I* →$_i$ ⟨*I*⟩$_i$*i-list*)$*x* **by** *simp*
lemma *autoref-list-singleton*[*autoref-rules*]:
(λ*a*. [*a*],*op-list-singleton*) ∈ *R* → ⟨*R*⟩*list-rel*
by *auto*

definition [*simp*]: *op-list-append-elem s x* ≡ *s*@[*x*]

lemma *pat-list-append-elem*[*autoref-op-pat*]:
s@[*x*] ≡ (*OP op-list-append-elem* :::$_i$ ⟨*I*⟩$_i$*i-list* →$_i$ *I* →$_i$ ⟨*I*⟩$_i$*i-list*)$*s*$*x*
by (*simp add*: *relAPP-def*)

lemma *autoref-list-append-elem*[*autoref-rules*]:
(λ*s x*. *s*@[*x*], *op-list-append-elem*) ∈ ⟨*R*⟩*list-rel* → *R* → ⟨*R*⟩*list-rel*
unfolding *op-list-append-elem-def*[*abs-def*] **by** *parametricity*

declare *param-rev*[*autoref-rules*]

declare *param-all-interval-nat'*[*autoref-rules*]

declare *param-all-interval-nat*[*autoref-rules*]

lemma [*autoref-op-pat*]:
(∀ *i*<*u*. *P i*) ≡ *OP Param-HOL.all-interval-nat P 0 u*
(∀ *i*≤*u*. *P i*) ≡ *OP List.all-interval P 0 u*
(∀ *i*<*u*. *l*≤*i* ⟶ *P i*) ≡ *OP Param-HOL.all-interval-nat P l u*
(∀ *i*≤*u*. *l*≤*i* ⟶ *P i*) ≡ *OP List.all-interval P l u*
by (*auto intro*!: *eq-reflection split*: *nat.splits*)

lemmas [*autoref-rules*] = *param-dropWhile param-takeWhile*

end

2.2.12 Examples

Be careful to make the concrete type a schematic type variable. The default behaviour of *schematic-lemma* makes it a fixed variable, that will not unify with the infered term!

schematic-goal
(*?f*::*?'c*,[*1*,*2*,*3*]@[*4*::*nat*])∈*?R*
by *autoref*

schematic-goal
(*?f*::*?'c*,[*1*::*nat*,
2,*3*,*4*,*5*,*6*,*7*,*8*,*9*,*0*,*1*,*43*,*5*,*5*,*435*,*5*,*1*,*5*,*6*,*5*,*6*,*5*,*63*,*56*
]
)∈*?R*
apply (*autoref*)
done

schematic-goal
(*?f*::*?'c*,[*1*,*2*,*3*] = [])∈*?R*
by *autoref*

When specifying custom refinement rules on the fly, be careful with the type-inference between *notes* and *shows*. It's too easy to „decouple" the type *'a* in the autoref-rule and the actual goal, as shown below!

schematic-goal
notes [*autoref-rules*] = *IdI*[**where** *'a*=*'a*]
notes [*autoref-itype*] = *itypeI*[**where** *'t*=*'a*::*numeral* **and** *I*=*i-std*]
shows (*?f*::*?'c*, *hd* [*a*,*b*,*c*::*'a*::*numeral*])∈*?R*

The autoref-rule is bound with type *'a*::*typ*, while the goal statement has *'a*::*numeral*!

apply (*autoref* (*keep-goal*))

We get an unsolved goal, as it finds no rule to translate *a*

oops

Here comes the correct version. Note the duplicate sort annotation of type *'a*:

schematic-goal
notes [*autoref-rules-raw*] = *IdI*[**where** *'a*=*'a*::*numeral*]
notes [*autoref-itype*] = *itypeI*[**where** *'t*=*'a*::*numeral* **and** *I*=*i-std*]
shows (*?f*::*?'c*, *hd* [*a*,*b*,*c*::*'a*::*numeral*])∈*?R*
by (*autoref*)

Special cases of equality: Note that we do not require equality on the element type!

schematic-goal

assumes [*autoref-rules*]: $(ai,a)\in\langle R\rangle$*option-rel*
shows $(?f::?'c,\ a = None)\in ?R$
apply (*autoref* (*keep-goal*))
done

schematic-goal

assumes [*autoref-rules*]: $(ai,a)\in\langle R\rangle$*list-rel*
shows $(?f::?'c,\ [] = a)\in ?R$
apply (*autoref* (*keep-goal*))
done

schematic-goal
shows $(?f::?'c,\ [1,2] = [2,3::nat])\in ?R$
apply (*autoref* (*keep-goal*))
done

end

2.3 Entry Point for the Automatic Refinement Tool

theory *Automatic-Refinement*
imports
Tool/Autoref-Tool
Autoref-Bindings-HOL
begin

The automatic refinement tool should be used by importing this theory

2.3.1 Convenience

The following lemmas can be used to add tags to theorems

lemma *PREFER-I*: $P\ x \Longrightarrow PREFER\ P\ x$ **by** *simp*
lemma *PREFER-D*: $PREFER\ P\ x \Longrightarrow P\ x$ **by** *simp*

lemmas *PREFER-sv-D* = *PREFER-D*[*of single-valued*]
lemma *PREFER-id-D*: $PREFER\text{-}id\ R \Longrightarrow R{=}Id$ **by** *simp*

abbreviation $PREFER\text{-}RUNIV \equiv PREFER\ (\lambda R.\ Range\ R = UNIV)$
lemmas *PREFER-RUNIV-D* = *PREFER-D*[*of* $(\lambda R.\ Range\ R = UNIV)$]

lemma *SIDE-GEN-ALGO-D*: $SIDE\text{-}GEN\text{-}ALGO\ P \Longrightarrow P$ **by** *simp*

lemma *GEN-OP-D*: $GEN\text{-}OP\ c\ a\ R \Longrightarrow (c,a)\in R$
by *simp*

lemma *MINOR-PRIO-TAG-I*: $P \Longrightarrow (MINOR\text{-}PRIO\text{-}TAG\ p \Longrightarrow P)$ **by** *auto*
lemma *MAJOR-PRIO-TAG-I*: $P \Longrightarrow (MAJOR\text{-}PRIO\text{-}TAG\ p \Longrightarrow P)$ **by** *auto*
lemma *PRIO-TAG-I*: $P \Longrightarrow (PRIO\text{-}TAG\ ma\ mi \Longrightarrow P)$ **by** *auto*

end