# Verified Approximation Algorithms

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**Abstract**

We present the first formal verifications of approximation algorithms for NP-complete optimization problems: vertex cover, set cover, independent set, center selection, load balancing, and bin packing. The proofs correct incompletnesses in existing proofs and improve the approximation ratio in one case. A detailed description of our work (excluding center selection) has been published in the proceedings of *IJCAR 2020* [3].

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# 1 Vertex Cover

**theory** *Approx-VC-Hoare*
**imports** *HOL−Hoare.Hoare-Logic*
**begin**

The algorithm is classical, the proof is based on and augments the one by Berghammer and Müller-Olm [1].

## 1.1 Graph

A graph is simply a set of edges, where an edge is a 2-element set.

**definition** *vertex-cover* :: *'a set set* ⇒ *'a set* ⇒ *bool* **where**
*vertex-cover E C* = (∀ *e* ∈ *E*. *e* ∩ *C* ≠ {})

**abbreviation** *matching* :: *'a set set* ⇒ *bool* **where**
*matching M* ≡ *pairwise disjnt M*

**lemma** *card-matching-vertex-cover*:
⟦ *finite C*; *matching M*; *M* ⊆ *E*; *vertex-cover E C* ⟧ ⟹ *card M* ≤ *card C*
**apply**(*erule card-le-if-inj-on-rel*[**where** *r* = λ*e v*. *v* ∈ *e*])
**apply** (*meson disjnt-def disjnt-iff vertex-cover-def subsetCE*)
**by** (*meson disjnt-iff pairwise-def*)

## 1.2 The Approximation Algorithm

Formulated using a simple(!) predefined Hoare-logic. This leads to a streamlined proof based on standard invariant reasoning.

The nondeterministic selection of an element from a set $F$ is simulated by *SOME x*. *x* ∈ *F*. The *SOME* operator is built into HOL: *SOME x*. *P x* denotes some $x$ that satisfies $P$ if such an $x$ exists; otherwise it denotes an arbitrary element. Note that there is no actual nondeterminism involved: *SOME x*. *P x* is some fixed element but in general we don't know which one. Proofs about *SOME* are notoriously tedious. Typically it involves showing first that ∃ *x*. *P x*. Then ∃ *x*. *?P x* ⟹ *?P* (*SOME x*. *?P x*) implies *P* (*SOME x*. *P x*). There are a number of (more) useful related theorems: just click on ∃ *x*. *?P x* ⟹ *?P* (*SOME x*. *?P x*) to be taken there.

Convenient notation for choosing an arbitrary element from a set:

**abbreviation** *some* $A \equiv SOME\ x.\ x \in A$

**locale** *Edges* =
  **fixes** $E :: 'a\ set\ set$
  **assumes** *finE*: *finite E*
  **assumes** *edges2*: $e \in E \Longrightarrow card\ e = 2$
**begin**

The invariant:

**definition** *inv-matching C F M* =
  (*matching M* $\wedge$ $M \subseteq E \wedge$ *card C* $\leq 2 *$ *card M* $\wedge$ $(\forall e \in M.\ \forall f \in F.\ e \cap f = \{\})$)

**definition** *invar* :: $'a\ set \Rightarrow 'a\ set\ set \Rightarrow$ *bool* **where**
*invar C F* = ($F \subseteq E \wedge$ *vertex-cover* $(E-F)$ *C* $\wedge$ *finite C* $\wedge$ $(\exists M.$ *inv-matching C F M*))

Preservation of the invariant by the loop body:

**lemma** *invar-step*:
  **assumes** $F \neq \{\}$ *invar C F*
  **shows** *invar* $(C \cup$ *some F*$)$ $(F - \{e' \in F.$ *some F* $\cap\ e' \neq \{\}\})$
**proof** −
  **from** *assms*(*2*) **obtain** *M* **where** $F \subseteq E$ **and** *vc*: *vertex-cover* $(E-F)$ *C* **and** *fC*: *finite C*
    **and** *m*: *matching M M* $\subseteq E$ **and** *card*: *card C* $\leq 2 *$ *card M*
    **and** *disj*: $\forall e \in M.\ \forall f \in F.\ e \cap f = \{\}$
  **by** (*auto simp*: *invar-def inv-matching-def*)
  **let** *?e* = *SOME e*. $e \in F$
  **have** *?e* $\in F$ **using** ‹$F \neq \{\}$› **by** (*simp add*: *some-in-eq*)
  **hence** *fe'*: *finite ?e* **using** ‹$F \subseteq E$› *edges2* **by**(*intro card-ge-0-finite*) *auto*
  **have** *?e* $\notin M$ **using** *edges2* ‹*?e* $\in F$› *disj* ‹$F \subseteq E$› **by** *fastforce*
  **have** *card'*: *card* $(C \cup$ *?e*$) \leq 2 *$ *card* (*insert ?e M*)
    **using** ‹*?e* $\in F$› ‹*?e* $\notin M$› *card-Un-le*[*of C ?e*] ‹$F \subseteq E$› *edges2 card finite-subset*[*OF m*(*2*) *finE*]
    **by** *fastforce*
  **let** *?M* = $M \cup \{$*?e*$\}$
  **have** *vc'*: *vertex-cover* $(E - (F - \{e' \in F.$ *?e* $\cap\ e' \neq \{\}\}))$ $(C \cup$ *?e*$)$
    **using** *vc* **by**(*auto simp*: *vertex-cover-def*)
  **have** *m'*: *inv-matching* $(C \cup$ *?e*$)$ $(F - \{e' \in F.$ *?e* $\cap\ e' \neq \{\}\})$ *?M*
    **using** *m card'* ‹$F \subseteq E$› ‹*?e* $\in F$› *disj*
    **by**(*auto simp*: *inv-matching-def Int-commute disjnt-def pairwise-insert*)
  **show** *?thesis* **using** ‹$F \subseteq E$› *vc' fC fe' m'* **by**(*auto simp add*: *invar-def Let-def*)
**qed**

**lemma** *approx-vertex-cover*:
*VARS C F*
  {*True*}

```
  C := {};
  F := E;
  WHILE F ≠ {}
  INV {invar C F}
  DO C := C ∪ some F;
     F := F − {e' ∈ F. some F ∩ e' ≠ {}}
  OD
  {vertex-cover E C ∧ (∀ C'. finite C' ∧ vertex-cover E C' ⟶ card C ≤ 2 * card
C')}
proof (vcg, goal-cases)
  case (1 C F)
  have inv-matching {} E {} by (auto simp add: inv-matching-def)
  with 1 show ?case by (auto simp add: invar-def vertex-cover-def)
next
  case (2 C F)
  thus ?case using invar-step[of F C] by(auto simp: Let-def)
next
  case (3 C F)
  then obtain M :: 'a set set where
    post: vertex-cover E C matching M M ⊆ E card C ≤ 2 * card M
    by(auto simp: invar-def inv-matching-def)

  have opt: card C ≤ 2 * card C' if C': finite C' vertex-cover E C' for C'
  proof −
    note post(4)
    also have 2 * card M ≤ 2 * card C'
    using card-matching-vertex-cover[OF C'(1) post(2,3) C'(2)] by simp
    finally show card C ≤ 2 * card C' .
  qed

  show ?case using post(1) opt by auto
qed

end
```

## 1.3 Version for Hypergraphs

Almost the same. We assume that the degree of every edge is bounded.

```
locale Bounded-Hypergraph =
  fixes E :: 'a set set
  fixes k :: nat
  assumes finE: finite E
  assumes edge-bnd: e ∈ E ⟹ finite e ∧ card e ≤ k
  assumes E1: {} ∉ E
begin

definition inv-matching C F M =
  (matching M ∧ M ⊆ E ∧ card C ≤ k * card M ∧ (∀ e ∈ M. ∀ f ∈ F. e ∩ f =
{}))
```

**definition** *invar* :: *'a set ⇒ 'a set set ⇒ bool* **where**
*invar C F* = (*F* ⊆ *E* ∧ *vertex-cover* (*E*−*F*) *C* ∧ *finite C* ∧ (∃ *M*. *inv-matching C F M*))

**lemma** *invar-step*:
**assumes** *F* ≠ {} *invar C F*
**shows** *invar* (*C* ∪ *some F*) (*F* − {*e'* ∈ *F*. *some F* ∩ *e'* ≠ {}})
**proof** −
**from** *assms*(2) **obtain** *M* **where** *F* ⊆ *E* **and** *vc*: *vertex-cover* (*E*−*F*) *C* **and** *fC*: *finite C*
**and** *m*: *matching M M* ⊆ *E* **and** *card*: *card C* ≤ *k* ∗ *card M*
**and** *disj*: ∀ *e* ∈ *M*. ∀ *f* ∈ *F*. *e* ∩ *f* = {}
**by** (*auto simp*: *invar-def inv-matching-def*)
**let** *?e* = *SOME e*. *e* ∈ *F*
**have** *?e* ∈ *F* **using** ‹*F* ≠ {}› **by** (*simp add*: *some-in-eq*)
**hence** *fe'*: *finite ?e* **using** ‹*F* ⊆ *E*› *assms*(2) *edge-bnd* **by** *blast*
**have** *?e* ∉ *M* **using** *E1* ‹*?e* ∈ *F*› *disj* ‹*F* ⊆ *E*› **by** *fastforce*
**have** *card'*: *card* (*C* ∪ *?e*) ≤ *k* ∗ *card* (*insert ?e M*)
**using** ‹*?e* ∈ *F*› ‹*?e* ∉ *M*› *card-Un-le*[*of C ?e*] ‹*F* ⊆ *E*› *edge-bnd card finite-subset*[*OF m*(2) *finE*]
**by** *fastforce*
**let** *?M* = *M* ∪ {*?e*}
**have** *vc'*: *vertex-cover* (*E* − (*F* − {*e'* ∈ *F*. *?e* ∩ *e'* ≠ {}})) (*C* ∪ *?e*)
**using** *vc* **by**(*auto simp*: *vertex-cover-def*)
**have** *m'*: *inv-matching* (*C* ∪ *?e*) (*F* − {*e'* ∈ *F*. *?e* ∩ *e'* ≠ {}}) *?M*
**using** *m card'* ‹*F* ⊆ *E*› ‹*?e* ∈ *F*› *disj*
**by**(*auto simp*: *inv-matching-def Int-commute disjnt-def pairwise-insert*)
**show** *?thesis* **using** ‹*F* ⊆ *E*› *vc' fC fe' m'* **by**(*auto simp add*: *invar-def Let-def*)
**qed**

**lemma** *approx-vertex-cover-bnd*:
*VARS C F*
{*True*}
*C* := {};
*F* := *E*;
*WHILE F* ≠ {}
*INV* {*invar C F*}
*DO C* := *C* ∪ *some F*;
*F* := *F* − {*e'* ∈ *F*. *some F* ∩ *e'* ≠ {}}
*OD*
{*vertex-cover E C* ∧ (∀ *C'*. *finite C'* ∧ *vertex-cover E C'* ⟶ *card C* ≤ *k* ∗ *card C'*)}
**proof** (*vcg*, *goal-cases*)
**case** (*1 C F*)
**have** *inv-matching* {} *E* {} **by** (*auto simp add*: *inv-matching-def*)
**with** *1* **show** *?case* **by** (*auto simp add*: *invar-def vertex-cover-def*)
**next**

**case** (*2 C F*)
**thus** *?case* **using** *invar-step*[*of F C*] **by**(*auto simp*: *Let-def*)
**next**
**case** (*3 C F*)
**then obtain** *M* :: *'a set set* **where**
*post*: *vertex-cover E C matching M M* ⊆ *E card C* ≤ *k* ∗ *card M*
**by**(*auto simp*: *invar-def inv-matching-def*)

**have** *opt*: *card C* ≤ *k* ∗ *card C'* **if** *C'*: *finite C' vertex-cover E C'* **for** *C'*
**proof** −
**note** *post*(*4*)
**also have** *k* ∗ *card M* ≤ *k* ∗ *card C'*
**using** *card-matching-vertex-cover*[*OF C'*(*1*) *post*(*2*,*3*) *C'*(*2*)] **by** *simp*
**finally show** *card C* ≤ *k* ∗ *card C'* **.**
**qed**

**show** *?case* **using** *post*(*1*) *opt* **by** *auto*
**qed**

**end**

**end**

# 2 Set Cover

**theory** *Approx-SC-Hoare*
**imports**
*HOL−Hoare.Hoare-Logic*
*Complex-Main*
**begin**

This is a formalization of the set cover algorithm and proof in the book by Kleinberg and Tardos [4].

**definition** *harm* :: *nat* ⇒ *'a* :: *real-normed-field* **where**
*harm n* = (∑ *k*=*1*..*n*. *inverse* (*of-nat k*))

**locale** *Set-Cover* =
**fixes** *w* :: *nat* ⇒ *real*
**and** *m* :: *nat*
**and** *S* :: *nat* ⇒ *'a set*
**assumes** *S-finite*: ∀ *i* ∈ {*1*..*m*}. *finite* (*S i*)
**and** *w-nonneg*: ∀ *i*. *0* ≤ *w i*
**begin**

**definition** *U* :: *'a set* **where**
*U* = (⋃ *i* ∈ {*1*..*m*}. *S i*)

**lemma** *S-subset*: ∀ *i* ∈ {*1*..*m*}. *S i* ⊆ *U*

**using** *U-def* **by** *blast*

**lemma** *U-finite*: *finite U*
**unfolding** *U-def* **using** *S-finite* **by** *blast*

**lemma** *empty-cover*: $m = 0 \Longrightarrow U = \{\}$
**using** *U-def* **by** *simp*

**definition** *sc* :: *nat set* ⇒ *'a set* ⇒ *bool* **where**
$sc\ C\ X \longleftrightarrow C \subseteq \{1..m\} \wedge (\bigcup i \in C.\ S\ i) = X$

**definition** *cost* :: *'a set* ⇒ *nat* ⇒ *real* **where**
$cost\ R\ i = w\ i\ /\ card\ (S\ i \cap R)$

**lemma** *cost-nonneg*: $0 \leq cost\ R\ i$
**using** *w-nonneg* **by** (*simp add*: *cost-def*)

$cost\ R\ i = 0$ if $card\ (S\ i \cap R) = 0$! Needs to be accounted for separately in *min-arg*.

**fun** *min-arg* :: *'a set* ⇒ *nat* ⇒ *nat* **where**
*min-arg R 0 = 1*
| *min-arg R (Suc x) =*
(*let j = min-arg R x*
*in if S j ∩ R = {} ∨ (S (Suc x) ∩ R ≠ {} ∧ cost R (Suc x) < cost R j) then (Suc x) else j*)

**lemma** *min-in-range*: $k > 0 \Longrightarrow min\text{-}arg\ R\ k \in \{1..k\}$
**by** (*induction k*) (*force simp*: *Let-def*)+

**lemma** *min-empty*: $S\ (min\text{-}arg\ R\ k) \cap R = \{\} \Longrightarrow \forall i \in \{1..k\}.\ S\ i \cap R = \{\}$
**proof** (*induction k*)
**case** (*Suc k*)
**from** *Suc.prems* **have** *prem*: $S\ (min\text{-}arg\ R\ k) \cap R = \{\}$ **by** (*auto simp*: *Let-def split*: *if-splits*)
**with** *Suc.IH* **have** *IH*: $\forall i \in \{1..k\}.\ S\ i \cap R = \{\}$ **.**
**show** *?case* **proof fix** *i* **assume** $i \in \{1..Suc\ k\}$ **show** $S\ i \cap R = \{\}$
**proof** (*cases* ‹$i = Suc\ k$›)
**case** *True* **with** *Suc.prems prem* **show** *?thesis* **by** *simp*
**next**
**case** *False* **with** *IH* ‹$i \in \{1..Suc\ k\}$› **show** *?thesis* **by** *simp*
**qed**
**qed**
**qed** *simp*

**lemma** *min-correct*: $[\![\ i \in \{1..k\};\ S\ i \cap R \neq \{\}\ ]\!] \Longrightarrow cost\ R\ (min\text{-}arg\ R\ k) \leq cost\ R\ i$
**proof** (*induction k*)
**case** (*Suc k*)
**show** *?case* **proof** (*cases* ‹$i = Suc\ k$›)

**case** *True* **with** *Suc.prems* **show** *?thesis* **by** (*auto simp*: *Let-def*)
**next**
**case** *False* **with** *Suc.prems Suc.IH* **have** *IH*: *cost R* (*min-arg R k*) ≤ *cost R i*
**by** *simp*
**from** *Suc.prems False min-empty*[*of R k*] **have** *S* (*min-arg R k*) ∩ *R* ≠ {} **by**
*force*
**with** *IH* **show** *?thesis* **by** (*auto simp*: *Let-def*)
**qed**
**qed** *simp*

Correctness holds quite trivially for both m = 0 and m > 0 (assuming a set cover can be found at all, otherwise algorithm would not terminate).

**lemma** *set-cover-correct*:
*VARS* (*R* :: *'a set*) (*C* :: *nat set*) (*i* :: *nat*)
{*True*}
*R* := *U*; *C* := {};
*WHILE R* ≠ {} *INV* {*R* ⊆ *U* ∧ *sc C* (*U* − *R*)} *DO*
*i* := *min-arg R m*;
*R* := *R* − *S i*;
*C* := *C* ∪ {*i*}
*OD*
{*sc C U*}
**proof** (*vcg*, *goal-cases*)
**case** *2* **show** *?case* **proof** (*cases m*)
**case** *0*
**from** *empty-cover*[*OF this*] *2* **show** *?thesis* **by** (*auto simp*: *sc-def*)
**next**
**case** *Suc* **then have** *m* > *0* **by** *simp*
**from** *min-in-range*[*OF this*] *2* **show** *?thesis* **using** *S-subset* **by** (*auto simp*:
*sc-def*)
**qed**
**qed** (*auto simp*: *sc-def*)

**definition** *c-exists* :: *nat set* ⇒ *'a set* ⇒ *bool* **where**
*c-exists C R* = (∃ *c*. *sum w C* = *sum c* (*U* − *R*) ∧ (∀ *i*. *0* ≤ *c i*)
∧ (∀ *k* ∈ {*1..m*}. *sum c* (*S k* ∩ (*U* − *R*))
≤ (∑ *j* = *card* (*S k* ∩ *R*) + *1..card* (*S k*). *inverse j*) ∗ *w k*))

**definition** *inv* :: *nat set* ⇒ *'a set* ⇒ *bool* **where**
*inv C R* ⟷ *sc C* (*U* − *R*) ∧ *R* ⊆ *U* ∧ *c-exists C R*

**lemma** *invI*:
**assumes** *sc C* (*U* − *R*) *R* ⊆ *U*
∃ *c*. *sum w C* = *sum c* (*U* − *R*) ∧ (∀ *i*. *0* ≤ *c i*)
∧ (∀ *k* ∈ {*1..m*}. *sum c* (*S k* ∩ (*U* − *R*))
≤ (∑ *j* = *card* (*S k* ∩ *R*) + *1..card* (*S k*). *inverse j*) ∗ *w k*)
**shows** *inv C R* **using** *assms* **by** (*auto simp*: *inv-def c-exists-def*)

**lemma** *invD*:

```
  assumes inv C R
  shows sc C (U − R) R ⊆ U
      ∃ c. sum w C = sum c (U − R) ∧ (∀ i. 0 ≤ c i)
    ∧ (∀ k ∈ {1..m}. sum c (S k ∩ (U − R))
                ≤ (∑ j = card (S k ∩ R) + 1..card (S k). inverse j) * w k)
  using assms by (auto simp: inv-def c-exists-def)

lemma inv-init: inv {} U
proof (rule invI, goal-cases)
  case 3
  let ?c = (λ-. 0) :: 'a ⇒ real
  have sum w {} = sum ?c (U − U) by simp
  moreover {
    have ∀ k ∈ {1..m}. 0 ≤ (∑ j = card (S k ∩ U) + 1..card (S k). inverse j) *
w k
      by (simp add: sum-nonneg w-nonneg)
    then have (∀ k∈{1..m}. sum ?c (S k ∩ (U − U))
          ≤ (∑ j = card (S k ∩ U) + 1..card (S k). inverse j) * w k) by simp
  }
  ultimately show ?case by blast
qed (simp-all add: sc-def)

lemma inv-step:
  assumes inv C R R ≠ {}
  defines [simp]: i ≡ min-arg R m
  shows inv (C ∪ {i}) (R − (S i))
proof (cases m)
  case 0
  from empty-cover[OF this] invD(2)[OF assms(1)] have R = {} by blast
  then show ?thesis using assms(2) by simp
next
  case Suc then have 0 < m by simp
  note hyp = invD[OF assms(1)]
  show ?thesis proof (rule invI, goal-cases)
      — Correctness
    case 1 have i ∈ {1..m} using min-in-range[OF ‹0 < m›] by simp
    with hyp(1) S-subset show ?case by (auto simp: sc-def)
  next
    case 2 from hyp(2) show ?case by auto
  next
    case 3
      — Set Cover grows
    have ∃ i ∈ {1..m}. S i ∩ R ≠ {}
      using assms(2) U-def hyp(2) by blast
    then have S i ∩ R ≠ {} using min-empty by auto
    then have 0 < card (S i ∩ R)
      using S-finite min-in-range[OF ‹0 < m›] by auto

      — Proving properties of cost function
```

**from** *hyp*(*3*) **obtain** *c* **where** *sum w C* = *sum c* (*U* − *R*) ∀ *i*. *0* ≤ *c i* **and**
*SUM*: ∀ *k*∈{*1..m*}. *sum c* (*S k* ∩ (*U* − *R*))
≤ (∑ *j* = *card* (*S k* ∩ *R*) + *1..card* (*S k*). *inverse j*) ∗ *w k* **by** *blast*
**let** *?c* = (λ*x*. *if x* ∈ *S i* ∩ *R then cost R i else c x*)

— Proof of Lemma 11.9
**have** *finite* (*U* − *R*) *finite* (*S i* ∩ *R*) (*U* − *R*) ∩ (*S i* ∩ *R*) = {}
**using** *U-finite S-finite min-in-range*[*OF* ‹*0* < *m*›] **by** *auto*
**then have** *sum ?c* (*U* − *R* ∪ (*S i* ∩ *R*)) = *sum ?c* (*U* − *R*) + *sum ?c* (*S i* ∩ *R*)
**by** (*rule sum.union-disjoint*)
**moreover have** *U-split*: *U* − (*R* − *S i*) = *U* − *R* ∪ (*S i* ∩ *R*) **using** *hyp*(*2*) **by** *blast*
**moreover {**
**have** *sum ?c* (*S i* ∩ *R*) = *card* (*S i* ∩ *R*) ∗ *cost R i* **by** *simp*
**also have** ... = *w i* **unfolding** *cost-def* **using** ‹*0* < *card* (*S i* ∩ *R*)› **by** *simp*
**finally have** *sum ?c* (*S i* ∩ *R*) = *w i* **.**
**}**
**ultimately have** *sum ?c* (*U* − (*R* − *S i*)) = *sum ?c* (*U* − *R*) + *w i* **by** *simp*
**moreover {**
**have** *C* ∩ {*i*} = {} **using** *hyp*(*1*) ‹*S i* ∩ *R* ≠ {}› **by** (*auto simp*: *sc-def*)
**from** *sum.union-disjoint*[*OF* - - *this*] **have** *sum w* (*C* ∪ {*i*}) = *sum w C* + *w i*
**using** *hyp*(*1*) **by** (*auto simp*: *sc-def intro*: *finite-subset*)
**}**
**ultimately have** *1*: *sum w* (*C* ∪ {*i*}) = *sum ?c* (*U* − (*R* − *S i*)) — Lemma 11.9
**using** ‹*sum w C* = *sum c* (*U* − *R*)› **by** *simp*

**have** *2*: ∀ *i*. *0* ≤ *?c i* **using** ‹∀ *i*. *0* ≤ *c i*› *cost-nonneg* **by** *simp*

— Proof of Lemma 11.10
**have** *3*: ∀ *k*∈{*1..m*}. *sum ?c* (*S k* ∩ (*U* − (*R* − *S i*)))
≤ (∑ *j* = *card* (*S k* ∩ (*R* − *S i*)) + *1..card* (*S k*). *inverse j*) ∗ *w k*
**proof**
**fix** *k* **assume** *k* ∈ {*1..m*}
**let** *?rem* = *S k* ∩ *R* — Remaining elements to be covered
**let** *?add* = *S k* ∩ *S i* ∩ *R* — Elements that will be covered in this step
**let** *?cov* = *S k* ∩ (*U* − *R*) — Covered elements

— Transforming left and right sides
**have** *sum ?c* (*S k* ∩ (*U* − (*R* − *S i*))) = *sum ?c* (*S k* ∩ (*U* − *R* ∪ (*S i* ∩ *R*)))
**unfolding** *U-split* **..**
**also have** ... = *sum ?c* (*?cov* ∪ *?add*)
**by** (*simp add*: *Int-Un-distrib Int-assoc*)
**also have** ... = *sum ?c ?cov* + *sum ?c ?add*
**by** (*rule sum.union-disjoint*) (*insert S-finite* ‹*k* ∈ -›, *auto*)
**finally have** *lhs*:

*sum ?c (S k ∩ (U − (R − S i))) = sum ?c ?cov + sum ?c ?add* **.**
**have** *S k ∩ (R − S i) = ?rem − ?add* **by** *blast*
**then have** *card (S k ∩ (R − S i)) = card (?rem − ?add)* **by** *simp*
**also have** *... = card ?rem − card ?add*
**using** *S-finite ‹k ∈ -›* **by** (*auto intro*: *card-Diff-subset*)
**finally have** *rhs*:
*card (S k ∩ (R − S i)) + 1 = card ?rem − card ?add + 1* **by** *simp*

— The apparent complexity of the remaining proof is deceiving. Much of this is just about convincing Isabelle that these sum transformations are allowed.
**have** *sum ?c ?add = card ?add ∗ cost R i* **by** *simp*
**also have** *... ≤ card ?add ∗ cost R k*
**proof** (*cases ?rem = {}*)
**case** *True*
**then have** *card ?add = 0* **by** (*auto simp*: *card-eq-0-iff*)
**then show** *?thesis* **by** *simp*
**next**
**case** *False*
**from** *min-correct*[*OF ‹k ∈ -› this*] **have** *cost R i ≤ cost R k* **by** *simp*
**then show** *?thesis* **by** (*simp add*: *mult-left-mono*)
**qed**
**also have** *... = card ?add ∗ inverse (card ?rem) ∗ w k*
**by** (*simp add*: *cost-def divide-inverse-commute*)
**also have** *... = (∑ j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse (card ?rem)) ∗ w k*
**proof** −
**have** *card ?add ≤ card ?rem*
**using** *S-finite ‹k ∈ -›* **by** (*blast intro*: *card-mono*)
**then show** *?thesis* **by** (*simp add*: *sum-distrib-left*)
**qed**
**also have** *... ≤ (∑ j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse j) ∗ w k*
**proof** −
**have** *∀ j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse (card ?rem) ≤ inverse j*
**by** *force*
**then have** *(∑ j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse (card ?rem))*
*≤ (∑ j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse j)*
**by** (*blast intro*: *sum-mono*)
**with** *w-nonneg* **show** *?thesis* **by** (*blast intro*: *mult-right-mono*)
**qed**
**finally have** *sum ?c ?add*
*≤ (∑ j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse j) ∗ w k* **.**
**moreover from** *SUM* **have** *sum ?c ?cov*
*≤ (∑ j ∈ {card ?rem + 1 .. card (S k)}. inverse j) ∗ w k*
**using** *‹k ∈ {1..m}›* **by** *simp*
**ultimately have** *sum ?c (S k ∩ (U − (R − S i)))*

```
            ≤ ((∑j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse j) +
                (∑j ∈ {card ?rem + 1 .. card (S k)}. inverse j)) ∗ w k
      unfolding lhs by argo
    also have ... = (∑j ∈ {card ?rem − card ?add + 1 .. card (S k)}. inverse
j) ∗ w k
      proof −
        have sum-split: b ∈ {a .. c} ⟹ sum f {a .. c} = sum f {a .. b} + sum f
{Suc b .. c}
          for f :: nat ⇒ real and a b c :: nat
        proof −
          assume b ∈ {a .. c}
          then have {a .. b} ∪ {Suc b .. c} = {a .. c} by force
          moreover have {a .. b} ∩ {Suc b .. c} = {}
            using ‹b ∈ {a .. c}› by auto
        ultimately show ?thesis by (metis finite-atLeastAtMost sum.union-disjoint)
        qed

        have (∑j ∈ {card ?rem − card ?add + 1 .. card (S k)}. inverse j)
            = (∑j ∈ {card ?rem − card ?add + 1 .. card ?rem}. inverse j)
            + (∑j ∈ {card ?rem + 1 .. card (S k)}. inverse j)
          proof (cases ‹?add = {}›)
            case False
            then have 0 < card ?add 0 < card ?rem
              using S-finite ‹k ∈ -› by fastforce+
            then have Suc (card ?rem − card ?add) ≤ card ?rem by simp
            moreover have card ?rem ≤ card (S k)
              using S-finite ‹k ∈ -› by (simp add: card-mono)
            ultimately show ?thesis by (auto intro: sum-split)
          qed simp
        then show ?thesis by algebra
      qed
    finally show sum ?c (S k ∩ (U − (R − S i)))
          ≤ (∑j ∈ {card (S k ∩ (R − S i)) + 1 .. card (S k)}. inverse j) ∗ w k
      unfolding rhs .
  qed

  from 1 2 3 show ?case by blast
 qed
qed

lemma cover-sum:
 fixes c :: 'a ⇒ real
 assumes sc C V ∀i. 0 ≤ c i
 shows sum c V ≤ (∑i ∈ C. sum c (S i))
proof −
 from assms(1) have finite C by (auto simp: sc-def finite-subset)
 then show ?thesis using assms(1)
 proof (induction C arbitrary: V rule: finite-induct)
  case (insert i C)
```

```
    have V-split: (⋃ (S ‘ insert i C)) = (⋃ (S ‘ C)) ∪ S i by auto
    have finite: finite (⋃ (S ‘ C)) finite (S i)
      using insert S-finite by (auto simp: sc-def)

    have sum c (S i) − sum c (⋃ (S ‘ C) ∩ S i) ≤ sum c (S i)
      using assms(2) by (simp add: sum-nonneg)
    then have sum c (⋃ (S ‘ insert i C)) ≤ sum c (⋃ (S ‘ C)) + sum c (S i)
      unfolding V-split using sum-Un[OF finite, of c] by linarith
    moreover have (∑ i∈insert i C. sum c (S i)) = (∑ i ∈ C. sum c (S i)) +
sum c (S i)
      by (simp add: insert.hyps)
    ultimately show ?case using insert by (fastforce simp: sc-def)
  qed (simp add: sc-def)
qed

abbreviation H :: nat ⇒ real where H ≡ harm

definition d-star :: nat (‹d*›) where d* ≡ Max (card ‘ (S ‘ {1..m}))

lemma set-cover-bound:
  assumes inv C {} sc C′ U
  shows sum w C ≤ H d* ∗ sum w C′
proof −
  from invD(3)[OF assms(1)] obtain c where
    sum w C = sum c U ∀ i. 0 ≤ c i and H-bound:
    ∀ k ∈ {1..m}. sum c (S k) ≤ H (card (S k)) ∗ w k — Lemma 11.10
    by (auto simp: harm-def Int-absorb2 S-subset)

  have ∀ k ∈ {1..m}. card (S k) ≤ d* by (auto simp: d-star-def)
  then have ∀ k ∈ {1..m}. H (card (S k)) ≤ H d* by (auto simp: harm-def intro!:
sum-mono2)
  with H-bound have ∀ k ∈ {1..m}. sum c (S k) ≤ H d* ∗ w k
    by (metis atLeastAtMost-iff atLeastatMost-empty-iff empty-iff mult-right-mono
w-nonneg)
  moreover have C′ ⊆ {1..m} using assms(2) by (simp add: sc-def)
  ultimately have ∀ i ∈ C′. sum c (S i) ≤ H d* ∗ w i by blast
  then have (∑ i ∈ C′. sum c (S i)) ≤ H d* ∗ sum w C′
    by (auto simp: sum-distrib-left intro: sum-mono)

  have sum w C = sum c U by fact — Lemma 11.9
  also have ... ≤ (∑ i ∈ C′. sum c (S i)) by (rule cover-sum[OF assms(2)]) fact
  also have ... ≤ H d* ∗ sum w C′ by fact
  finally show ?thesis .
qed

theorem set-cover-approx:
VARS (R :: 'a set) (C :: nat set) (i :: nat)
  {True}
  R := U; C := {};
```

*WHILE* $R \neq \{\}$ *INV* $\{inv\ C\ R\}$ *DO*
$i := min\text{-}arg\ R\ m$;
$R := R - S\ i$;
$C := C \cup \{i\}$
*OD*
$\{sc\ C\ U \wedge (\forall C'.\ sc\ C'\ U \longrightarrow sum\ w\ C \leq H\ d^* * sum\ w\ C')\}$
**proof** (*vcg*, *goal-cases*)
**case** *1* **show** *?case* **by** (*rule inv-init*)
**next**
**case** *2* **thus** *?case* **using** *inv-step* **..**
**next**
**case** (*3 R C i*)
**then have** *sc C U* **unfolding** *inv-def* **by** *auto*
**with** *3* **show** *?case* **by** (*auto intro*: *set-cover-bound*)
**qed**

**end**

**end**

# 3 Independent Set

**theory** *Approx-MIS-Hoare*
**imports**
*HOL−Hoare.Hoare-Logic*
*HOL−Library.Disjoint-Sets*
**begin**

The algorithm is classical, the proofs are inspired by the ones by Berghammer and Müller-Olm [1]. In particular the approximation ratio is improved from $\Delta+1$ to $\Delta$.

## 3.1 Graph

A set set is simply a set of edges, where an edge is a 2-element set.

**definition** *independent-vertices* :: *'a set set* ⇒ *'a set* ⇒ *bool* **where**
*independent-vertices E S* ⟷ $S \subseteq \bigcup E \wedge (\forall v1\ v2.\ v1 \in S \wedge v2 \in S \longrightarrow \{v1, v2\} \notin E)$

**locale** *Graph-E* =
**fixes** *E* :: *'a set set*
**assumes** *finite-E*: *finite E*
**assumes** *edges2*: $e \in E \Longrightarrow card\ e = 2$
**begin**

**fun** *vertices* :: *'a set set* ⇒ *'a set* **where**
$vertices\ G = \bigcup G$

**abbreviation** $V :: 'a\ set$ **where**
$V \equiv vertices\ E$

**definition** *approximation-miv* :: $nat \Rightarrow 'a\ set \Rightarrow bool$ **where**
*approximation-miv* $n\ S \longleftrightarrow$ *independent-vertices* $E\ S \wedge (\forall S'.$ *independent-vertices* $E\ S' \longrightarrow$ *card* $S' \leq$ *card* $S * n)$

**fun** *neighbors* :: $'a \Rightarrow 'a\ set$ **where**
*neighbors* $v = \{u.\ \{u,v\} \in E\}$

**fun** *degree-vertex* :: $'a \Rightarrow nat$ **where**
*degree-vertex* $v =$ *card* (*neighbors* $v$)

**abbreviation** $\Delta :: nat$ **where**
$\Delta \equiv Max\{$*degree-vertex* $u | u.\ u \in V\}$

**lemma** *finite-edges*: $e \in E \Longrightarrow$ *finite* $e$
**using** *card-ge-0-finite* **and** *edges2* **by** *force*

**lemma** *finite-V*: *finite* $V$
**using** *finite-edges* **and** *finite-E* **by** *auto*

**lemma** *finite-neighbors*: *finite* (*neighbors* $u$)
**using** *finite-V* **and** *rev-finite-subset* [*of* $V$ *neighbors* $u$] **by** *auto*

**lemma** *independent-vertices-finite*: *independent-vertices* $E\ S \Longrightarrow$ *finite* $S$
**by** (*metis rev-finite-subset independent-vertices-def vertices.simps finite-V*)

**lemma** *edge-ex-vertices*: $e \in E \Longrightarrow \exists\, u\ v.\ u \neq v \wedge e = \{u,\ v\}$
**proof** −
**assume** $e \in E$
**then have** *card* $e =$ *Suc* (*Suc 0*) **using** *edges2* **by** *auto*
**then show** $\exists\, u\ v.\ u \neq v \wedge e = \{u,\ v\}$
**by** (*metis card-eq-SucD insertI1*)
**qed**

**lemma** $\Delta$*-pos* [*simp*]: $E = \{\} \vee 0 < \Delta$
**proof** *cases*
**assume** $E = \{\}$
**then show** $E = \{\} \vee 0 < \Delta$ **by** *auto*
**next**
**assume** *1*: $E \neq \{\}$
**then have** $V \neq \{\}$ **using** *edges2* **by** *fastforce*
**moreover have** *finite* $\{$*degree-vertex* $u\ |u.\ u \in V\}$
**by** (*metis finite-V finite-imageI Setcompr-eq-image*)
**ultimately have** *2*: $\Delta \in \{$*degree-vertex* $u\ |u.\ u \in V\}$ **using** *Max-in* **by** *auto*
**have** $\Delta \neq 0$
**proof**
**assume** $\Delta = 0$

**with** *2* **obtain** $u$ **where** *3*: $u \in V$ **and** *4*: *degree-vertex* $u = 0$ **by** *auto*
**from** *3* **obtain** $e$ **where** *5*: $e \in E$ **and** $u \in e$ **by** *auto*
**moreover with** *4* **have** $\forall v.\ \{u, v\} \neq e$ **using** *finite-neighbors insert-absorb*
**by** *fastforce*
**ultimately show** *False* **using** *edge-ex-vertices* **by** *auto*
**qed**
**then show** $E = \{\} \vee 0 < \Delta$ **by** *auto*
**qed**

**lemma** $\Delta$-*max-degree*: $u \in V \Longrightarrow$ *degree-vertex* $u \leq \Delta$
**proof** −
**assume** *H*: $u \in V$
**have** *finite* $\{$*degree-vertex* $u$ $|u.\ u \in V\}$
**by** (*metis finite-V finite-imageI Setcompr-eq-image*)
**with** *H* **show** *degree-vertex* $u \leq \Delta$ **using** *Max-ge* **by** *auto*
**qed**

## 3.2 Wei's algorithm: ($\Delta$+*1*)-approximation

The 'functional' part of the invariant, used to prove that the algorithm produces an independent set of vertices.

**definition** *inv-iv* :: *'a set* ⇒ *'a set* ⇒ *bool* **where**
*inv-iv S X* ⟷ *independent-vertices E S*
∧ $X \subseteq V$
∧ $(\forall\, v1 \in (V - X).\ \forall\, v2 \in S.\ \{v1, v2\} \notin E)$
∧ $S \subseteq X$

Strenghten the invariant with an approximation ratio $r$:

**definition** *inv-approx* :: *'a set* ⇒ *'a set* ⇒ *nat* ⇒ *bool* **where**
*inv-approx S X r* ⟷ *inv-iv S X* ∧ *card X* ≤ *card S* ∗ *r*

Preservation of the functional invariant:

**lemma** *inv-preserv*:
**fixes** *S* :: *'a set*
**and** *X* :: *'a set*
**and** *x* :: *'a*
**assumes** *inv*: *inv-iv S X*
**and** *x-def*: $x \in V - X$
**shows** *inv-iv* (*insert x S*) $(X \cup$ *neighbors* $x \cup \{x\})$
**proof** −
**have** *inv1*: *independent-vertices E S*
**and** *inv2*: $X \subseteq V$
**and** *inv3*: $S \subseteq X$
**and** *inv4*: $\forall\, v1\ v2.\ v1 \in (V - X) \wedge v2 \in S \longrightarrow \{v1, v2\} \notin E$
**using** *inv* **unfolding** *inv-iv-def* **by** *auto*
**have** *finite-S*: *finite S* **using** *inv1* **and** *independent-vertices-finite* **by** *auto*
**have** *S1*: $\forall\, y \in S.\ \{x, y\} \notin E$ **using** *inv4* **and** *x-def* **by** *blast*
**have** *S2*: $\forall\, x \in S.\ \forall\, y \in S.\ \{x, y\} \notin E$ **using** *inv1* **unfolding** *independent-vertices-def*
**by** *metis*

```
  have S3: v1 ∈ insert x S ⟹ v2 ∈ insert x S ⟹ {v1, v2} ∉ E for v1 v2
    proof −
      assume v1 ∈ insert x S and v2 ∈ insert x S
      then consider
          (a) v1 = x and v2 = x
        | (b) v1 = x and v2 ∈ S
        | (c) v1 ∈ S and v2 = x
        | (d) v1 ∈ S and v2 ∈ S
        by auto
      then show {v1, v2} ∉ E
      proof cases
        case a then show ?thesis using edges2 by force
      next
        case b then show ?thesis using S1 by auto
      next
        case c then show ?thesis using S1 by (metis doubleton-eq-iff)
      next
        case d then show ?thesis using S2 by auto
      qed
    qed

  have independent-vertices E (insert x S)
    using S3 and inv1 and x-def unfolding independent-vertices-def by auto

  moreover have X ∪ neighbors x ∪ {x} ⊆ V
  proof
    fix xa
    assume xa ∈ X ∪ neighbors x ∪ {x}
    then consider (a) xa ∈ X | (b) xa ∈ neighbors x | (c) xa = x by auto
    then show xa ∈ V
    proof cases
      case a
      then show ?thesis using inv2 by blast
    next
      case b
      then show ?thesis by auto
    next
      case c
      then show ?thesis using x-def by blast
    qed
  qed

  moreover have insert x S ⊆ X ∪ neighbors x ∪ {x} using inv3 by auto

  moreover have v1 ∈ V − (X ∪ neighbors x ∪ {x}) ⟹ v2 ∈ insert x S ⟹
{v1, v2} ∉ E for v1 v2
  proof −
    assume H: v1 ∈ V − (X ∪ neighbors x ∪ {x}) and v2 ∈ insert x S
    then consider (a) v2 = x | (b) v2 ∈ S by auto
```

```
    then show {v1, v2} ∉ E
    proof cases
      case a
      with H have v1 ∉ neighbors v2 by blast
      then show ?thesis by auto
    next
      case b
      from H have v1 ∈ V − X by blast
      with b and inv4 show ?thesis by blast
    qed
  qed

   ultimately show inv-iv (insert x S) (X ∪ neighbors x ∪ {x}) unfolding
inv-iv-def by blast
qed

lemma inv-approx-preserv:
  assumes inv: inv-approx S X (Δ + 1)
      and x-def: x ∈ V − X
    shows inv-approx (insert x S) (X ∪ neighbors x ∪ {x}) (Δ + 1)
proof −
  have finite-S: finite S using inv and independent-vertices-finite
    unfolding inv-approx-def inv-iv-def by auto
  have Sx: x ∉ S using inv and x-def unfolding inv-approx-def inv-iv-def by
blast

  from inv have inv-iv S X unfolding inv-approx-def by auto
  with x-def have inv-iv (insert x S) (X ∪ neighbors x ∪ {x})
  proof (intro inv-preserv, auto) qed

  moreover have card (X ∪ neighbors x ∪ {x}) ≤ card (insert x S) ∗ (Δ + 1)
  proof −
    have degree-vertex x ≤ Δ using Δ-max-degree and x-def by auto
    then have card (neighbors x ∪ {x}) ≤ Δ + 1 using card-Un-le [of neighbors
x {x}] by auto
    then have card (X ∪ neighbors x ∪ {x}) ≤ card X + Δ + 1 using card-Un-le
[of X neighbors x ∪ {x}] by auto
   also have ... ≤ card S ∗ (Δ + 1) + Δ + 1 using inv unfolding inv-approx-def
by auto
    also have ... = card (insert x S) ∗ (Δ + 1) using finite-S and Sx by auto
    finally show ?thesis .
  qed

  ultimately show inv-approx (insert x S) (X ∪ neighbors x ∪ {x}) (Δ + 1)
    unfolding inv-approx-def by auto
qed


lemma inv-approx: independent-vertices E S ⟹ card V ≤ card S ∗ r ⟹ ap-
```

```
proximation-miv r S
proof −
  assume 1: independent-vertices E S and 2: card V ≤ card S * r
  have independent-vertices E S' ⟹ card S' ≤ card S * r for S'
  proof −
    assume  independent-vertices E S'
    then have S' ⊆ V unfolding independent-vertices-def by auto
    then have card S' ≤ card V using finite-V and card-mono by auto
    also have ... ≤ card S * r using 2 by auto
    finally show card S' ≤ card S * r .
  qed
  with 1 show approximation-miv r S unfolding approximation-miv-def by auto
qed

theorem wei-approx-Δ-plus-1:
VARS (S :: 'a set) (X :: 'a set) (x :: 'a)
  { True }
  S := {};
  X := {};
  WHILE X ≠ V
  INV { inv-approx S X (Δ + 1) }
  DO x := (SOME x. x ∈ V − X);
     S := insert x S;
     X := X ∪ neighbors x ∪ {x}
  OD
  { approximation-miv (Δ + 1) S }
proof (vcg, goal-cases)
  case (1 S X x)
  then show ?case unfolding inv-approx-def inv-iv-def independent-vertices-def
by auto
next
  case (2 S X x)

  let ?x = (SOME x. x ∈ V − X)
  have V − X ≠ {} using 2 unfolding inv-approx-def inv-iv-def by blast
  then have ?x ∈ V − X using some-in-eq by metis
  with 2 show ?case using inv-approx-preserv by auto
next
  case (3 S X x)
  then show ?case using inv-approx unfolding inv-approx-def inv-iv-def by auto
qed
```

### 3.3 Wei's algorithm: Δ-approximation

The previous approximation uses very little information about the optimal solution (it has at most as many vertices as the set itself). With some extra effort we can improve the ratio to Δ instead of Δ+*1*. In order to do that we must show that among the vertices removed in each iteration, at most Δ could belong to an optimal solution. This requires carrying around a set $P$

(via a ghost variable) which records the vertices deleted in each iteration.

**definition** *inv-partition* :: *'a set* ⇒ *'a set* ⇒ *'a set set* ⇒ *bool* **where**
*inv-partition S X P* ⟷ *inv-iv S X*
∧ ⋃ *P* = *X*
∧ (∀ *p* ∈ *P*. ∃ *s* ∈ *V*. *p* = {*s*} ∪ *neighbors s*)
∧ *card P* = *card S*
∧ *finite P*

**lemma** *inv-partition-preserv*:
**assumes** *inv*: *inv-partition S X P*
**and** *x-def*: *x* ∈ *V* − *X*
**shows** *inv-partition* (*insert x S*) (*X* ∪ *neighbors x* ∪ {*x*}) (*insert* ({*x*} ∪ *neighbors x*) *P*)
**proof** −
**have** *finite-S*: *finite S* **using** *inv* **and** *independent-vertices-finite*
**unfolding** *inv-partition-def inv-iv-def* **by** *auto*
**have** *Sx*: *x* ∉ *S* **using** *inv* **and** *x-def* **unfolding** *inv-partition-def inv-iv-def* **by** *blast*

**from** *inv* **have** *inv-iv S X* **unfolding** *inv-partition-def* **by** *auto*
**with** *x-def* **have** *inv-iv* (*insert x S*) (*X* ∪ *neighbors x* ∪ {*x*})
**proof** (*intro inv-preserv*, *auto*) **qed**

**moreover have** ⋃(*insert* ({*x*} ∪ *neighbors x*) *P*) = *X* ∪ *neighbors x* ∪ {*x*}
**using** *inv* **unfolding** *inv-partition-def* **by** *auto*

**moreover have** (∀ *p*∈*insert* ({*x*} ∪ *neighbors x*) *P*. ∃ *s* ∈ *V*. *p* = {*s*} ∪ *neighbors s*)
**using** *inv* **and** *x-def* **unfolding** *inv-partition-def* **by** *auto*

**moreover have** *card* (*insert* ({*x*} ∪ *neighbors x*) *P*) = *card* (*insert x S*)
**proof** −
**from** *x-def* **and** *inv* **have** *x* ∉ ⋃ *P* **unfolding** *inv-partition-def* **by** *auto*
**then have** {*x*} ∪ *neighbors x* ∉ *P* **by** *auto*
**then have** *card* (*insert* ({*x*} ∪ *neighbors x*) *P*) = *card P* + *1* **using** *inv* **unfolding** *inv-partition-def* **by** *auto*
**moreover have** *card* (*insert x S*) = *card S* + *1* **using** *Sx* **and** *finite-S* **by** *auto*
**ultimately show** *?thesis* **using** *inv* **unfolding** *inv-partition-def* **by** *auto*
**qed**

**moreover have** *finite* (*insert* ({*x*} ∪ *neighbors x*) *P*)
**using** *inv* **unfolding** *inv-partition-def* **by** *auto*

**ultimately show** *inv-partition* (*insert x S*) (*X* ∪ *neighbors x* ∪ {*x*}) (*insert* ({*x*} ∪ *neighbors x*) *P*)
**unfolding** *inv-partition-def* **by** *auto*
**qed**

**lemma** *card-Union-le-sum-card*:

**fixes** $U :: 'a\ set\ set$
**assumes** $\forall\, u \in U.\ finite\ u$
**shows** $card\ (\bigcup U) \leq sum\ card\ U$
**proof** (*cases finite U*)
**case** *False*
**then show** $card\ (\bigcup U) \leq sum\ card\ U$
**using** *card-eq-0-iff finite-UnionD* **by** *auto*
**next**
**case** *True*
**then show** $card\ (\bigcup U) \leq sum\ card\ U$
**proof** (*induct U rule*: *finite-induct*)
**case** *empty*
**then show** *?case* **by** *auto*
**next**
**case** (*insert x F*)
**then have** $card(\bigcup (insert\ x\ F)) \leq card(x) + card\ (\bigcup F)$ **using** *card-Un-le* **by** *auto*
**also have** $... \leq card(x) + sum\ card\ F$ **using** *insert.hyps* **by** *auto*
**also have** $... = sum\ card\ (insert\ x\ F)$ **using** *sum.insert-if* **and** *insert.hyps* **by** *auto*
**finally show** *?case* **.**
**qed**
**qed**

**lemma** *sum-card*:
**fixes** $U :: 'a\ set\ set$
**and** $n :: nat$
**assumes** $\forall\, S \in U.\ card\ S \leq n$
**shows** $sum\ card\ U \leq card\ U * n$
**proof** *cases*
**assume** $infinite\ U \vee U = \{\}$
**then have** $sum\ card\ U = 0$ **using** *sum.infinite* **by** *auto*
**then show** $sum\ card\ U \leq card\ U * n$ **by** *auto*
**next**
**assume** $\neg(infinite\ U \vee U = \{\})$
**with** *assms* **have** $finite\ U$ **and** $U \neq \{\}$**and** $\forall\, S \in U.\ card\ S \leq n$ **by** *auto*
**then show** $sum\ card\ U \leq card\ U * n$
**proof** (*induct U rule*: *finite-ne-induct*)
**case** (*singleton x*)
**then show** *?case* **by** *auto*
**next**
**case** (*insert x F*)
**assume** $\forall\, S{\in}insert\ x\ F.\ card\ S \leq n$
**then have** *1*:$card\ x \leq n$ **and** *2*:$sum\ card\ F \leq card\ F * n$ **using** *insert.hyps* **by** *auto*
**then have** $sum\ card\ (insert\ x\ F) = card\ x + sum\ card\ F$ **using** *sum.insert-if* **and** *insert.hyps* **by** *auto*
**also have** $... \leq n + card\ F * n$ **using** *1* **and** *2* **by** *auto*

**also have** ... = *card* (*insert x F*) ∗ *n* **using** *card-insert-if* **and** *insert.hyps* **by** *auto*
**finally show** *?case* **.**
**qed**
**qed**

**lemma** *x-or-neighbors*:
**fixes** *P* :: *'a set set*
**and** *S* :: *'a set*
**assumes** *inv*: ∀ *p*∈*P*. ∃ *s* ∈ *V*. *p* = {*s*} ∪ *neighbors s*
**and** *ivS*: *independent-vertices E S*
**shows** ∀ *p* ∈ *P*. *card* (*S* ∩ *p*) ≤ Δ
**proof**
**fix** *p*
**assume** *p* ∈ *P*
**then obtain** *s* **where** *1*: *s* ∈ *V* ∧ *p* = {*s*} ∪ *neighbors s* **using** *inv* **by** *blast*
**then show** *card* (*S* ∩ *p*) ≤ Δ
**proof** *cases*
**assume** *s* ∈ *S*
**then have** *S* ∩ *neighbors s* = {} **using** *ivS* **unfolding** *independent-vertices-def* **by** *auto*
**then have** *S* ∩ *p* ⊆ {*s*} **using** *1* **by** *auto*
**then have** *2*: *card* (*S* ∩ *p*) ≤ *1* **using** *subset-singletonD* **by** *fastforce*
**consider** (*a*) *E* = {} | (*b*) *0* < Δ **using** Δ*-pos* **by** *auto*
**then show** *card* (*S* ∩ *p*) ≤ Δ
**proof** *cases*
**case** *a*
**then have** *S* = {} **using** *ivS* **unfolding** *independent-vertices-def* **by** *auto*
**then show** *?thesis* **by** *auto*
**next**
**case** *b*
**then show** *?thesis* **using** *2* **by** *auto*
**qed**
**next**
**assume** *s* ∉ *S*
**with** *1* **have** *S* ∩ *p* ⊆ *neighbors s* **by** *auto*
**then have** *card* (*S* ∩ *p*) ≤ *degree-vertex s* **using** *card-mono* **and** *finite-neighbors* **by** *auto*
**then show** *card* (*S* ∩ *p*) ≤ Δ **using** *1* **and** Δ*-max-degree* [*of s*] **by** *auto*
**qed**
**qed**

**lemma** *inv-partition-approx*: *inv-partition S V P* ⟹ *approximation-miv* Δ *S*
**proof** −
**assume** *H1*: *inv-partition S V P*
**then have** *independent-vertices E S* **unfolding** *inv-partition-def inv-iv-def* **by** *auto*

```
  moreover have independent-vertices E S' ⟹ card S' ≤ card S * Δ for S'
  proof −
    let ?I = {S' ∩ p | p. p ∈ P}

    assume H2: independent-vertices E S'
     then have S' ⊆ V unfolding independent-vertices-def using vertices.simps
by blast
    with H1 have S' = S' ∩ ⋃P unfolding inv-partition-def by auto
    then have S' = (⋃p ∈ P. S' ∩ p) using Int-Union by auto
    then have S' = ⋃ ?I by blast
    moreover have finite S' using H2 and independent-vertices-finite by auto
    then have p ∈ P ⟹ finite (S' ∩ p) for p by auto
    ultimately have card S' ≤ sum card ?I using card-Union-le-sum-card [of ?I]
by auto
    also have ... ≤ card ?I * Δ
      using x-or-neighbors [of P S']
        and sum-card [of ?I Δ]
        and H1 and H2 unfolding inv-partition-def by auto
    also have ... ≤ card P * Δ
    proof −
      have finite P using H1 unfolding inv-partition-def by auto
      then have card ?I ≤ card P
        using Setcompr-eq-image [of λp. S' ∩ p P]
          and card-image-le unfolding inv-partition-def by auto
      then show ?thesis by auto
    qed
    also have ... = card S * Δ using H1 unfolding inv-partition-def by auto
    ultimately show card S' ≤ card S * Δ by auto
  qed
  ultimately show approximation-miv Δ S unfolding approximation-miv-def by
auto
qed

theorem wei-approx-Δ:
VARS (S :: 'a set) (X :: 'a set) (x :: 'a)
  { True }
  S := {};
  X := {};
  WHILE X ≠ V
  INV { ∃P. inv-partition S X P }
  DO x := (SOME x. x ∈ V − X);
     S := insert x S;
     X := X ∪ neighbors x ∪ {x}
  OD
  { approximation-miv Δ S }
proof (vcg, goal-cases)
  case (1 S X x)

  have inv-partition {} {} {} unfolding inv-partition-def inv-iv-def independent-vertices-def
```

**by** *auto*
**then show** *?case* **by** *auto*
**next**
**case** (*2 S X x*)

**let** *?x* = (*SOME x*. *x* ∈ *V* − *X*)
**from** *2* **obtain** *P* **where** *I*: *inv-partition S X P* **by** *auto*
**then have** *V* − *X* ≠ {} **using** *2* **unfolding** *inv-partition-def* **by** *auto*
**then have** *?x* ∈ *V* − *X* **using** *some-in-eq* **by** *metis*

**with** *I* **have** *inv-partition* (*insert ?x S*) (*X* ∪ *neighbors ?x* ∪ {*?x*}) (*insert* ({*?x*} ∪ *neighbors ?x*) *P*)
**using** *inv-partition-preserv* **by** *blast*
**then show** *?case* **by** *auto*
**next**
**case** (*3 S X x*)
**then show** *?case* **using** *inv-partition-approx* **unfolding** *inv-approx-def* **by** *auto*
**qed**

## 3.4 Wei's algorithm with dynamically computed approximation ratio

In this subsection, we augment the algorithm with a variable used to compute the effective approximation ratio of the solution. In addition, the vertex of smallest degree is picked. With this heuristic, the algorithm achieves an approximation ratio of (Δ+*2*)/*3*, but this is not proved here.

**definition** *vertex-heuristic* :: *'a set* ⇒ *'a* ⇒ *bool* **where**
*vertex-heuristic X v* = (∀ *u* ∈ *V* − *X*. *card* (*neighbors v* − *X*) ≤ *card* (*neighbors u* − *X*))

**lemma** *ex-min-finite-set*:
**fixes** *S* :: *'a set*
**and** *f* :: *'a* ⇒ *nat*
**shows** *finite S* ⟹ *S* ≠ {} ⟹ ∃ *x*. *x* ∈ *S* ∧ (∀ *y* ∈ *S*. *f x* ≤ *f y*)
(**is** *?P1* ⟹ *?P2* ⟹ ∃ *x*. *?minf S x*)
**proof** (*induct S rule*: *finite-ne-induct*)
**case** (*singleton x*)
**have** *?minf* {*x*} *x* **by** *auto*
**then show** *?case* **by** *auto*
**next**
**case** (*insert x F*)
**from** *insert*(*4*) **obtain** *y* **where** *Py*: *?minf F y* **by** *auto*
**show** ∃ *z*. *?minf* (*insert x F*) *z*
**proof** *cases*
**assume** *f x* < *f y*
**then have** *?minf* (*insert x F*) *x* **using** *Py* **by** *auto*
**then show** *?case* **by** *auto*

**next**
**assume** $\neg f\ x < f\ y$
**then have** *?minf* (*insert x F*) *y* **using** *Py* **by** *auto*
**then show** *?case* **by** *auto*
**qed**
**qed**

**lemma** *inv-approx-preserv2*:
**fixes** $S :: 'a\ set$
**and** $X :: 'a\ set$
**and** $s :: nat$
**and** $x :: 'a$
**assumes** *inv*: *inv-approx S X s*
**and** *x-def*: $x \in V - X$
**shows** *inv-approx* (*insert x S*) ($X \cup$ *neighbors x* $\cup \{x\}$) (*max* (*card* (*neighbors x* $\cup \{x\} - X$)) *s*)
**proof** −
**have** *finite-S*: *finite S* **using** *inv* **and** *independent-vertices-finite* **unfolding** *inv-approx-def inv-iv-def* **by** *auto*
**have** *Sx*: $x \notin S$ **using** *inv* **and** *x-def* **unfolding** *inv-approx-def inv-iv-def* **by** *blast*

**from** *inv* **have** *inv-iv S X* **unfolding** *inv-approx-def* **by** *auto*
**with** *x-def* **have** *inv-iv* (*insert x S*) ($X \cup$ *neighbors x* $\cup \{x\}$)
**proof** (*intro inv-preserv*, *auto*) **qed**

**moreover have** *card* ($X \cup$ *neighbors x* $\cup \{x\}$) $\leq$ *card* (*insert x S*) $*$ *max* (*card* (*neighbors x* $\cup \{x\} - X$)) *s*
**proof** −
**let** *?N* = *neighbors x* $\cup \{x\} - X$
**have** *card* ($X \cup$ *?N*) $\leq$ *card X* + *card ?N* **using** *card-Un-le* [*of X ?N*] **by** *auto*
**also have** ... $\leq$ *card S* $*$ *s* + *card ?N* **using** *inv* **unfolding** *inv-approx-def* **by** *auto*
**also have** ... $\leq$ *card S* $*$ *max* (*card ?N*) *s* + *card ?N* **by** *auto*
**also have** ... $\leq$ *card S* $*$ *max* (*card ?N*) *s* + *max* (*card ?N*) *s* **by** *auto*
**also have** ... = *card* (*insert x S*) $*$ *max* (*card ?N*) *s* **using** *Sx* **and** *finite-S* **by** *auto*
**finally show** *?thesis* **by** *auto*
**qed**

**ultimately show** *inv-approx* (*insert x S*) ($X \cup$ *neighbors x* $\cup \{x\}$) (*max* (*card* (*neighbors x* $\cup \{x\} - X$)) *s*)
**unfolding** *inv-approx-def* **by** *auto*
**qed**

**theorem** *wei-approx-min-degree-heuristic*:
*VARS* ($S :: 'a\ set$) ($X :: 'a\ set$) ($x :: 'a$) ($r :: nat$)
{ *True* }
$S := \{\}$;

```
  X := {};
  r := 0;
  WHILE X ≠ V
  INV { inv-approx S X r }
  DO x := (SOME x. x ∈ V − X ∧ vertex-heuristic X x);
     S := insert x S;
     r := max (card (neighbors x ∪ {x} − X)) r;
     X := X ∪ neighbors x ∪ {x}
  OD
  { approximation-miv r S }
proof (vcg, goal-cases)
  case (1 S X x r)
   then show ?case unfolding inv-approx-def inv-iv-def independent-vertices-def
by auto
next
  case (2 S X x r)

  let ?x = (SOME x. x ∈ V − X ∧ vertex-heuristic X x)
  have V − X ≠ {} using 2 unfolding inv-approx-def inv-iv-def by blast
  moreover have finite (V − X) using 2 and finite-V by auto
  ultimately have ∃x. x ∈ V − X ∧ vertex-heuristic X x
    using ex-min-finite-set [where ?f = λx. card (neighbors x − X)]
    unfolding vertex-heuristic-def by auto
  then have x-def: ?x ∈ V − X ∧ vertex-heuristic X ?x
    using someI-ex [where ?P = λx. x ∈ V − X ∧ vertex-heuristic X x] by auto
  with 2 show ?case using inv-approx-preserv2 by auto
next
  case (3 S X x r)
 then show ?case using inv-approx unfolding inv-approx-def inv-iv-def by auto
qed

end
end
```

# 4 Load Balancing

```
theory Approx-LB-Hoare
  imports Complex-Main HOL−Hoare.Hoare-Logic
begin
```

This is a formalization of the load balancing algorithms and proofs in the book by Kleinberg and Tardos [4].

```
hide-const (open) sorted



lemma sum-le-card-Max: ⟦ finite A; A ≠ {} ⟧ ⟹ sum f A ≤ card A * Max (f ‘
A)
proof(induction A rule: finite-ne-induct)
```

**case** (*singleton x*)
**then show** *?case* **by** *simp*
**next**
**case** (*insert x F*)
**then show** *?case* **by** (*auto simp*: *max-def order.trans*[*of sum f F card F * Max (f ‘ F)*])
**qed**

**lemma** *Max-const*[*simp*]: ⟦ *finite A*; *A* ≠ {} ⟧ ⟹ *Max* ((λ-. *c*) ‘ *A*) = *c*
**using** *Max-in image-is-empty* **by** *blast*

**abbreviation** *Max*$_0$ :: *nat set* ⇒ *nat* **where**
*Max*$_0$ *N* ≡ (*if N*={} *then 0 else Max N*)

**fun** *f-Max*$_0$ :: (*nat* ⇒ *nat*) ⇒ *nat* ⇒ *nat* **where**
*f-Max*$_0$ *f 0* = *0*
| *f-Max*$_0$ *f* (*Suc x*) = *max* (*f* (*Suc x*)) (*f-Max*$_0$ *f x*)

**lemma** *f-Max*$_0$*-equiv*: *f-Max*$_0$ *f n* = *Max*$_0$ (*f* ‘ {*1..n*})
**by** (*induction n*) (*auto simp*: *not-le atLeastAtMostSuc-conv*)

**lemma** *f-Max*$_0$*-correct*:
∀ *x* ∈ {*1..m*}. *T x* ≤ *f-Max*$_0$ *T m*
*m* > *0* ⟹ ∃ *x* ∈ {*1..m*}. *T x* = *f-Max*$_0$ *T m*
**apply** (*induction m*)
**apply** *simp-all*
**apply** (*metis atLeastAtMost-iff le-Suc-eq max.cobounded1 max.coboundedI2*)
**subgoal for** *m* **by** (*cases* ‹*m* = *0*›) (*auto simp*: *max-def*)
**done**

**lemma** *f-Max*$_0$*-mono*:
*y* ≤ *T x* ⟹ *f-Max*$_0$ (*T* (*x* := *y*)) *m* ≤ *f-Max*$_0$ *T m*
*T x* ≤ *y* ⟹ *f-Max*$_0$ *T m* ≤ *f-Max*$_0$ (*T* (*x* := *y*)) *m*
**by** (*induction m*) *auto*

**lemma** *f-Max*$_0$*-out-of-range* [*simp*]:
*x* ∉ {*1..k*} ⟹ *f-Max*$_0$ (*T* (*x* := *y*)) *k* = *f-Max*$_0$ *T k*
**by** (*induction k*) *auto*

**lemma** *fun-upd-f-Max*$_0$:
**assumes** *x* ∈ {*1..m*} *T x* ≤ *y*
**shows** *f-Max*$_0$ (*T* (*x* := *y*)) *m* = *max y* (*f-Max*$_0$ *T m*)
**using** *assms* **by** (*induction m*) *auto*

**locale** *LoadBalancing* =
**fixes** *t* :: *nat* ⇒ *nat*
**and** *m* :: *nat*
**and** *n* :: *nat*
**assumes** *m-gt-0*: *m* > *0*

**begin**

## 4.1 Formalization of a Correct Load Balancing

### 4.1.1 Definition

**definition** *lb* :: (*nat* ⇒ *nat*) ⇒ (*nat* ⇒ *nat set*) ⇒ *nat* ⇒ *bool* **where**
*lb T A j* = ((∀ *x* ∈ {*1..m*}. ∀ *y* ∈ {*1..m*}. *x* ≠ *y* ⟶ *A x* ∩ *A y* = {}) — No job is assigned to more than one machine
∧ (⋃ *x* ∈ {*1..m*}. *A x*) = {*1..j*} — Every job is assigned
∧ (∀ *x* ∈ {*1..m*}. (∑ *j* ∈ *A x*. *t j*) = *T x*) — The processing times sum up to the correct load)

**abbreviation** *makespan* :: (*nat* ⇒ *nat*) ⇒ *nat* **where**
*makespan T* ≡ *f-Max*$_0$ *T m*

**lemma** *makespan-def'*: *makespan T* = *Max* (*T* ‘ {*1..m*})
**using** *m-gt-0* **by** (*simp add*: *f-Max*$_0$*-equiv*)

**lemma** *makespan-correct*:
∀ *x* ∈ {*1..m*}. *T x* ≤ *makespan T*
∃ *x* ∈ {*1..m*}. *T x* = *makespan T*
**using** *f-Max*$_0$*-correct m-gt-0* **by** *auto*

**lemma** *lbE*:
**assumes** *lb T A j*
**shows** ∀ *x* ∈ {*1..m*}. ∀ *y* ∈ {*1..m*}. *x* ≠ *y* ⟶ *A x* ∩ *A y* = {}
(⋃ *x* ∈ {*1..m*}. *A x*) = {*1..j*}
∀ *x* ∈ {*1..m*}. (∑ *y* ∈ *A x*. *t y*) = *T x*
**using** *assms* **unfolding** *lb-def* **by** *blast+*

**lemma** *lbI*:
**assumes** ∀ *x* ∈ {*1..m*}. ∀ *y* ∈ {*1..m*}. *x* ≠ *y* ⟶ *A x* ∩ *A y* = {}
(⋃ *x* ∈ {*1..m*}. *A x*) = {*1..j*}
∀ *x* ∈ {*1..m*}. (∑ *y* ∈ *A x*. *t y*) = *T x*
**shows** *lb T A j* **using** *assms* **unfolding** *lb-def* **by** *blast*

**lemma** *A-lb-finite* [*simp*]:
**assumes** *lb T A j x* ∈ {*1..m*}
**shows** *finite* (*A x*)
**by** (*metis lbE*(*2*) *assms finite-UN finite-atLeastAtMost*)

If *A x* is pairwise disjoint for all *x* ∈ {*1..m*}, then the the sum over the sums of the individual *A x* is equal to the sum over the union of all *A x*.

**lemma** *sum-sum-eq-sum-Un*:
**fixes** *A* :: *nat* ⇒ *nat set*
**assumes** ∀ *x* ∈ {*1..m*}. ∀ *y* ∈ {*1..m*}. *x* ≠ *y* ⟶ *A x* ∩ *A y* = {}
**and** ∀ *x* ∈ {*1..m*}. *finite* (*A x*)
**shows** (∑ *x* ∈ {*1..m*}. (∑ *y* ∈ *A x*. *t y*)) = (∑ *x* ∈ (⋃ *y* ∈ {*1..m*}. *A y*). *t x*)

**using** *assms*
**proof** (*induction m*)
**case** (*Suc m*)
**have** *FINITE*: *finite* ($\bigcup x \in \{1..m\}$. *A x*) *finite* (*A* (*Suc m*))
**using** *Suc.prems*(2) **by** *auto*
**have** $\forall x \in \{1..m\}$. *A x* $\cap$ *A* (*Suc m*) = {}
**using** *Suc.prems*(1) **by** *simp*
**then have** *DISJNT*: ($\bigcup x \in \{1..m\}$. *A x*) $\cap$ (*A* (*Suc m*)) = {} **using** *Union-disjoint*
**by** *blast*
**have** ($\sum x \in (\bigcup y \in \{1..m\}$. *A y*). *t x*) + ($\sum x \in$ *A* (*Suc m*). *t x*)
= ($\sum x \in ((\bigcup y \in \{1..m\}$. *A y*) $\cup$ *A* (*Suc m*)). *t x*)
**using** *sum.union-disjoint*[*OF FINITE DISJNT*, *symmetric*] **.**
**also have** ... = ($\sum x \in (\bigcup y \in \{1..Suc\ m\}$. *A y*). *t x*)
**by** (*metis UN-insert image-Suc-lessThan image-insert inf-sup-aci*(5) *lessThan-Suc*)
**finally show** *?case* **using** *Suc* **by** *auto*
**qed** *simp*

If $T$ and $A$ are a correct load balancing for $j$ jobs and $m$ machines, then the sum of the loads has to be equal to the sum of the processing times of the jobs

**lemma** *lb-impl-job-sum*:
**assumes** *lb T A j*
**shows** ($\sum x \in \{1..m\}$. *T x*) = ($\sum x \in \{1..j\}$. *t x*)
**proof** −
**note** *lbrules* = *lbE*[*OF assms*]
**from** *assms* **have** *FINITE*: $\forall x \in \{1..m\}$. *finite* (*A x*) **by** *simp*
**have** ($\sum x \in \{1..m\}$. *T x*) = ($\sum x \in \{1..m\}$. ($\sum y \in$ *A x*. *t y*))
**using** *lbrules*(3) **by** *simp*
**also have** ... = ($\sum x \in \{1..j\}$. *t x*)
**using** *sum-sum-eq-sum-Un*[*OF lbrules*(1) *FINITE*]
**unfolding** *lbrules*(2) **.**
**finally show** *?thesis* **.**
**qed**

### 4.1.2 Lower Bounds for the Makespan

If $T$ and $A$ are a correct load balancing for $j$ jobs and $m$ machines, then the processing time of any job $x \in \{1..j\}$ is a lower bound for the load of some machine $y \in \{1..m\}$

**lemma** *job-lower-bound-machine*:
**assumes** *lb T A j x* $\in \{1..j\}$
**shows** $\exists y \in \{1..m\}$. *t x* $\leq$ *T y*
**proof** −
**note** *lbrules* = *lbE*[*OF assms*(1)]
**have** $\exists y \in \{1..m\}$. *x* $\in$ *A y* **using** *lbrules*(2) *assms*(2) **by** *blast*
**then obtain** *y* **where** *y-def*: *y* $\in \{1..m\}$ *x* $\in$ *A y* **..**
**moreover have** *finite* (*A y*) **using** *assms*(1) *y-def*(1) **by** *simp*
**ultimately have** *t x* $\leq$ ($\sum x \in$ *A y*. *t x*) **using** *lbrules*(1) *member-le-sum* **by**
*fast*

**also have** ... = *T y* **using** *lbrules*(*3*) *y-def*(*1*) **by** *blast*
**finally show** *?thesis* **using** *y-def*(*1*) **by** *blast*
**qed**

As the load of any machine is a lower bound for the makespan, the processing time of any job $x \in \{1..j\}$ has to also be a lower bound for the makespan. Follows from *job-lower-bound-machine* and *makespan-correct*.

**lemma** *job-lower-bound-makespan*:
**assumes** *lb T A j x* ∈ {*1..j*}
**shows** *t x* ≤ *makespan T*
**by** (*meson job-lower-bound-machine*[*OF assms*] *makespan-correct*(*1*) *le-trans*)

The makespan over $j$ jobs is a lower bound for the makespan of any correct load balancing for $j$ jobs.

**lemma** *max-job-lower-bound-makespan*:
**assumes** *lb T A j*
**shows** *Max*$_0$ (*t* ‘ {*1..j*}) ≤ *makespan T*
**using** *job-lower-bound-makespan*[*OF assms*] **by** *fastforce*

**lemma** *job-dist-lower-bound-makespan*:
**assumes** *lb T A j*
**shows** (∑ *x* ∈ {*1..j*}. *t x*) / *m* ≤ *makespan T*
**proof** −
**have** (∑ *x* ∈ {*1..j*}. *t x*) ≤ *m* ∗ *makespan T*
**using** *assms lb-impl-job-sum*[*symmetric*]
**and** *sum-le-card-Max*[*of* {*1..m*}] *m-gt-0* **by** (*simp add*: *makespan-def′*)
**then have** *real* (∑ *x* ∈ {*1..j*}. *t x*) ≤ *real m* ∗ *real* (*makespan T*)
**using** *of-nat-mono* **by** *fastforce*
**then show** *?thesis* **by** (*simp add*: *field-simps m-gt-0*)
**qed**

## 4.2 The Greedy Approximation Algorithm

This function will perform a linear scan from $k \in \{1..m\}$ and return the index of the machine with minimum load assuming $m > 0$

**fun** *min-arg* :: (*nat* ⇒ *nat*) ⇒ *nat* ⇒ *nat* **where**
*min-arg T 0* = *1*
| *min-arg T* (*Suc x*) =
(*let k* = *min-arg T x*
*in if T* (*Suc x*) < *T k then* (*Suc x*) *else k*)

**lemma** *min-correct*:
∀ *x* ∈ {*1..m*}. *T* (*min-arg T m*) ≤ *T x*
**by** (*induction m*) (*auto simp*: *Let-def le-Suc-eq*, *force*)

**lemma** *min-in-range*:
*k* > *0* ⟹ (*min-arg T k*) ∈ {*1..k*}
**by** (*induction k*) (*auto simp*: *Let-def*, *force*+)

**lemma** *add-job*:
**assumes** *lb T A j x* ∈ {*1..m*}
**shows** *lb* (*T* (*x* := *T x* + *t* (*Suc j*))) (*A* (*x* := *A x* ∪ {*Suc j*})) (*Suc j*)
(**is** ‹*lb ?T ?A* -›)
**proof** −
**note** *lbrules* = *lbE*[*OF assms*(*1*)]

— Rule 1: *A*(*x* := *A x* ∪ {*Suc j*}) pairwise disjoint
**have** *NOTIN*: ∀ *i* ∈ {*1..m*}. *Suc j* ∉ *A i* **using** *lbrules*(*2*) *assms*(*2*) **by** *force*
**with** *lbrules*(*1*) **have** ∀ *i* ∈ {*1..m*}. *i* ≠ *x* ⟶ *A i* ∩ (*A x* ∪ {*Suc j*}) = {}
**using** *assms*(*2*) **by** *blast*
**then have** *1*: ∀ *x* ∈ {*1..m*}. ∀ *y* ∈ {*1..m*}. *x* ≠ *y* ⟶ *?A x* ∩ *?A y* = {}
**using** *lbrules*(*1*) **by** *simp*

— Rule 2: *A*(*x* := *A x* ∪ {*Suc j*}) contains all jobs
**have** (⋃ *y* ∈ {*1..m*}. *?A y*) = (⋃ *y* ∈ {*1..m*}. *A y*) ∪ {*Suc j*}
**using** *UNION-fun-upd assms*(*2*) **by** *auto*
**also have** ... = {*1..Suc j*} **unfolding** *lbrules*(*2*) **by** *auto*
**finally have** *2*: (⋃ *y* ∈ {*1..m*}. *?A y*) = {*1..Suc j*} **.**

— Rule 3: *A*(*x* := *A x* ∪ {*Suc j*}) sums to *T*(*x* := *T x* + *t* (*Suc j*))
**have** (∑ *i* ∈ *?A x*. *t i*) = (∑ *i* ∈ *A x* ∪ {*Suc j*}. *t i*) **by** *simp*
**moreover have** *A x* ∩ {*Suc j*} = {} **using** *NOTIN assms*(*2*) **by** *blast*
**moreover have** *finite* (*A x*) *finite* {*Suc j*} **using** *assms* **by** *simp*+
**ultimately have** (∑ *i* ∈ *?A x*. *t i*) = (∑ *i* ∈ *A x*. *t i*) + (∑ *i* ∈ {*Suc j*}. *t i*)
**using** *sum.union-disjoint* **by** *simp*
**also have** ... = *T x* + *t* (*Suc j*) **using** *lbrules*(*3*) *assms*(*2*) **by** *simp*
**finally have** (∑ *i* ∈ *?A x*. *t i*) = *?T x* **by** *simp*
**then have** *3*: ∀ *i* ∈ {*1..m*}. (∑ *j* ∈ *?A i*. *t j*) = *?T i*
**using** *lbrules*(*3*) *assms*(*2*) **by** *simp*

**from** *lbI*[*OF 1 2 3*] **show** *?thesis* **.**
**qed**

**lemma** *makespan-mono*:
*y* ≤ *T x* ⟹ *makespan* (*T* (*x* := *y*)) ≤ *makespan T*
*T x* ≤ *y* ⟹ *makespan T* ≤ *makespan* (*T* (*x* := *y*))
**using** *f-Max*$_0$*-mono* **by** *auto*

**lemma** *smaller-optimum*:
**assumes** *lb T A* (*Suc j*)
**shows** ∃ *T′ A′*. *lb T′ A′ j* ∧ *makespan T′* ≤ *makespan T*
**proof** −
**note** *lbrules* = *lbE*[*OF assms*]
**have** ∃ *x* ∈ {*1..m*}. *Suc j* ∈ *A x* **using** *lbrules*(*2*) **by** *auto*
**then obtain** *x* **where** *x-def*: *x* ∈ {*1..m*} *Suc j* ∈ *A x* **..**
**let** *?T* = *T* (*x* := *T x* − *t* (*Suc j*))
**let** *?A* = *A* (*x* := *A x* − {*Suc j*})

— Rule 1: $A(x := A\ x - \{Suc\ j\})$ pairwise disjoint
**from** *lbrules(1)* **have** $\forall i \in \{1..m\}.\ i \neq x \longrightarrow A\ i \cap (A\ x - \{Suc\ j\}) = \{\}$
**using** *x-def(1)* **by** *blast*
**then have** *1*: $\forall x \in \{1..m\}.\ \forall y \in \{1..m\}.\ x \neq y \longrightarrow ?A\ x \cap ?A\ y = \{\}$
**using** *lbrules(1)* **by** *auto*

— Rule 2: $A(x := A\ x - \{Suc\ j\})$ contains all jobs
**have** *NOTIN*: $\forall i \in \{1..m\}.\ i \neq x \longrightarrow Suc\ j \notin A\ i$ **using** *lbrules(1) x-def* **by** *blast*
**then have** $(\bigcup y \in \{1..m\}.\ ?A\ y) = (\bigcup y \in \{1..m\}.\ A\ y) - \{Suc\ j\}$
**using** *UNION-fun-upd x-def* **by** *auto*
**also have** ... $= \{1..j\}$ **unfolding** *lbrules(2)* **by** *auto*
**finally have** *2*: $(\bigcup y \in \{1..m\}.\ ?A\ y) = \{1..j\}$ **.**

— Rule 3: $A(x := A\ x - \{Suc\ j\})$ sums to $T(x := T\ x - t\ (Suc\ j))$
**have** $(\sum i \in A\ x - \{Suc\ j\}.\ t\ i) = (\sum i \in A\ x.\ t\ i) - t\ (Suc\ j)$
**by** (*simp add*: *sum-diff1-nat x-def(2)*)
**also have** ... $= T\ x - t\ (Suc\ j)$ **using** *lbrules(3) x-def(1)* **by** *simp*
**finally have** $(\sum i \in ?A\ x.\ t\ i) = ?T\ x$ **by** *simp*
**then have** *3*: $\forall i \in \{1..m\}.\ (\sum j \in ?A\ i.\ t\ j) = ?T\ i$
**using** *lbrules(3) x-def(1)* **by** *simp*

— *makespan* is not larger
**have** *lb ?T ?A j* $\wedge$ *makespan ?T* $\leq$ *makespan T*
**using** *lbI*[*OF 1 2 3*] *makespan-mono(1)* **by** *force*
**then show** *?thesis* **by** *blast*
**qed**

If the processing time $y$ does not contribute to the makespan, we can ignore it.

**lemma** *remove-small-job*:
**assumes** *makespan* $(T\ (x := T\ x + y)) \neq T\ x + y$
**shows** *makespan* $(T\ (x := T\ x + y)) =$ *makespan T*
**proof** −
**let** *?T* $= T\ (x := T\ x + y)$
**have** *NOT-X*: *makespan ?T* $\neq$ *?T x* **using** *assms(1)* **by** *simp*
**then have** $\exists i \in \{1..m\}.$ *makespan ?T* $=$ *?T i* $\wedge$ $i \neq x$
**using** *makespan-correct(2)* **by** *metis*
**then obtain** *i* **where** *i-def*: $i \in \{1..m\}$ *makespan ?T* $=$ *?T i* $i \neq x$ **by** *blast*
**then have** *?T i* $=$ *T i* **using** *NOT-X* **by** *simp*
**moreover from** *this* **have** *makespan T* $=$ *T i*
**by** (*metis i-def*(*1*,*2*) *antisym-conv le-add1 makespan-mono(2) makespan-correct(1)*)
**ultimately show** *?thesis* **using** *i-def(2)* **by** *simp*
**qed**

**lemma** *greedy-makespan-no-jobs* [*simp*]:
*makespan* $(\lambda$-. *0*$) = 0$
**using** *m-gt-0* **by** (*simp add*: *makespan-def'*)

**lemma** *min-avg*: $m * T\ (\textit{min-arg}\ T\ m) \leq (\sum i \in \{1..m\}.\ T\ i)$
(**is** ‹- * ?T ≤ ?S›)
**proof** −
**have** $(\sum \text{-} \in \{1..m\}.\ ?T) \leq ?S$
**using** *sum-mono*[*of* ‹$\{1..m\}$› ‹$\lambda$-. ?T› $T$]
**and** *min-correct* **by** *blast*
**then show** *?thesis* **by** *simp*
**qed**

**definition** $\textit{inv}_1 :: (\textit{nat} \Rightarrow \textit{nat}) \Rightarrow (\textit{nat} \Rightarrow \textit{nat set}) \Rightarrow \textit{nat} \Rightarrow \textit{bool}$ **where**
$\textit{inv}_1\ T\ A\ j = (\textit{lb}\ T\ A\ j \wedge j \leq n \wedge (\forall T'\ A'.\ \textit{lb}\ T'\ A'\ j \longrightarrow \textit{makespan}\ T \leq 2 * \textit{makespan}\ T'))$

**lemma** $\textit{inv}_1 E$:
**assumes** $\textit{inv}_1\ T\ A\ j$
**shows** $\textit{lb}\ T\ A\ j\ j \leq n$
$\textit{lb}\ T'\ A'\ j \Longrightarrow \textit{makespan}\ T \leq 2 * \textit{makespan}\ T'$
**using** *assms* **unfolding** $\textit{inv}_1$*-def* **by** *blast+*

**lemma** $\textit{inv}_1 I$:
**assumes** $\textit{lb}\ T\ A\ j\ j \leq n\ \forall T'\ A'.\ \textit{lb}\ T'\ A'\ j \longrightarrow \textit{makespan}\ T \leq 2 * \textit{makespan}\ T'$
**shows** $\textit{inv}_1\ T\ A\ j$ **using** *assms* **unfolding** $\textit{inv}_1$*-def* **by** *blast*

**lemma** $\textit{inv}_1$*-step*:
**assumes** $\textit{inv}_1\ T\ A\ j\ j < n$
**shows** $\textit{inv}_1\ (T\ ((\textit{min-arg}\ T\ m) := T\ (\textit{min-arg}\ T\ m) + t\ (\textit{Suc}\ j)))$
$(A\ ((\textit{min-arg}\ T\ m) := A\ (\textit{min-arg}\ T\ m) \cup \{\textit{Suc}\ j\}))\ (\textit{Suc}\ j)$
(**is** ‹$\textit{inv}_1$ ?T ?A -›)
**proof** −
**note** *invrules* = $\textit{inv}_1 E$[*OF assms*(*1*)]
— Greedy is correct
**have** *LB*: *lb ?T ?A* (*Suc j*)
**using** *add-job*[*OF invrules*(*1*) *min-in-range*[*OF m-gt-0*]] **by** *blast*
— Greedy maintains approximation factor
**have** *MK*: $\forall T'\ A'.\ \textit{lb}\ T'\ A'\ (\textit{Suc}\ j) \longrightarrow \textit{makespan}\ ?T \leq 2 * \textit{makespan}\ T'$
**proof** *rule+*
**fix** $T_1\ A_1$ **assume** $\textit{lb}\ T_1\ A_1\ (\textit{Suc}\ j)$
**from** *smaller-optimum*[*OF this*]
**obtain** $T_0\ A_0$ **where** $\textit{lb}\ T_0\ A_0\ j\ \textit{makespan}\ T_0 \leq \textit{makespan}\ T_1$ **by** *blast*
**then have** *IH*: $\textit{makespan}\ T \leq 2 * \textit{makespan}\ T_1$
**using** *invrules*(*3*) **by** *force*
**show** $\textit{makespan}\ ?T \leq 2 * \textit{makespan}\ T_1$
**proof** (*cases* ‹*makespan ?T* = *T* (*min-arg T m*) + *t* (*Suc j*)›)
**case** *True*
**have** $m * T\ (\textit{min-arg}\ T\ m) \leq (\sum i \in \{1..m\}.\ T\ i)$ **by** (*rule min-avg*)
**also have** ... = $(\sum i \in \{1..j\}.\ t\ i)$ **by** (*rule lb-impl-job-sum*[*OF invrules*(*1*)])
**finally have** *real* $m * T\ (\textit{min-arg}\ T\ m) \leq (\sum i \in \{1..j\}.\ t\ i)$
**by** (*auto dest*: *of-nat-mono*)

```
    with m-gt-0 have T (min-arg T m) ≤ (∑ i ∈ {1..j}. t i) / m
      by (simp add: field-simps)
    then have T (min-arg T m) ≤ makespan T₁
      using job-dist-lower-bound-makespan[OF ‹lb T₀ A₀ j›]
        and ‹makespan T₀ ≤ makespan T₁› by linarith
    moreover have t (Suc j) ≤ makespan T₁
      using job-lower-bound-makespan[OF ‹lb T₁ A₁ (Suc j)›] by simp
    ultimately show ?thesis unfolding True by simp
   next
    case False show ?thesis using remove-small-job[OF False] IH by simp
   qed
  qed
  from inv₁I[OF LB - MK] show ?thesis using assms(2) by simp
qed

lemma simple-greedy-approximation:
VARS T A i j
{True}
T := (λ-. 0);
A := (λ-. {});
j := 0;
WHILE j < n INV {inv₁ T A j} DO
  i := min-arg T m;
  j := (Suc j);
  A := A (i := A(i) ∪ {j});
  T := T (i := T(i) + t j)
OD
{lb T A n ∧ (∀ T' A'. lb T' A' n ⟶ makespan T ≤ 2 * makespan T')}
proof (vcg, goal-cases)
  case (1 T A i j)
  then show ?case by (simp add: lb-def inv₁-def)
next
  case (2 T A i j)
  then show ?case using inv₁-step by simp
next
  case (3 T A i j)
  then show ?case unfolding inv₁-def by force
qed

definition sorted :: nat ⇒ bool where
  sorted j = (∀ x ∈ {1..j}. ∀ y ∈ {1..x}. t x ≤ t y)

lemma sorted-smaller [simp]: ⟦ sorted j; j ≥ j' ⟧ ⟹ sorted j'
  unfolding sorted-def by simp

lemma j-gt-m-pigeonhole:
  assumes lb T A j j > m
  shows ∃ x ∈ {1..j}. ∃ y ∈ {1..j}. ∃ z ∈ {1..m}. x ≠ y ∧ x ∈ A z ∧ y ∈ A z
proof −
```

```
  have ∀ x ∈ {1..j}. ∃ y ∈ {1..m}. x ∈ A y
    using lbE(2)[OF assms(1)] by blast
  then have ∃ f. ∀ x ∈ {1..j}. x ∈ A (f x) ∧ f x ∈ {1..m} by metis
  then obtain f where f-def: ∀ x ∈ {1..j}. x ∈ A (f x) ∧ f x ∈ {1..m} ..
  then have card (f ‘ {1..j}) ≤ card {1..m}
    by (meson card-mono finite-atLeastAtMost image-subset-iff)
  also have ... < card {1..j} using assms(2) by simp
  finally have card (f ‘ {1..j}) < card {1..j} .
  then have ¬ inj-on f {1..j} using pigeonhole by blast
  then have ∃ x ∈ {1..j}. ∃ y ∈ {1..j}. x ≠ y ∧ f x = f y
    unfolding inj-on-def by blast
  then show ?thesis using f-def by metis
qed
```

If $T$ and $A$ are a correct load balancing for $j$ jobs and $m$ machines with $j > m$, and the jobs are sorted in descending order, then there exists a machine $x \in \{1..m\}$ whose load is at least twice as large as the processing time of job $j$.

```
lemma sorted-job-lower-bound-machine:
  assumes lb T A j j > m sorted j
  shows ∃ x ∈ {1..m}. 2 * t j ≤ T x
proof −
  — Step 1: Obtaining the jobs
  note lbrules = lbE[OF assms(1)]
  obtain j₁ j₂ x where *:
    j₁ ∈ {1..j} j₂ ∈ {1..j} x ∈ {1..m} j₁ ≠ j₂ j₁ ∈ A x j₂ ∈ A x
    using j-gt-m-pigeonhole[OF assms(1,2)] by blast

  — Step 2: Jobs contained in sum
  have finite (A x) using assms(1) *(3) by simp
  then have SUM: (∑ i ∈ A x. t i) = t j₁ + t j₂ + (∑ i ∈ A x − {j₁} − {j₂}. t
i)
    using *(4−6) by (simp add: sum.remove)

  — Step 3: Proof of lower bound
  have t j ≤ t j₁ t j ≤ t j₂
    using assms(3) *(1−2) unfolding sorted-def by auto
  then have 2 * t j ≤ t j₁ + t j₂ by simp
  also have ... ≤ (∑ i ∈ A x. t i) unfolding SUM by simp
  finally have 2 * t j ≤ T x using lbrules(3) *(3) by simp
  then show ?thesis using *(3) by blast
qed
```

Reasoning analogous to *job-lower-bound-makespan*.

```
lemma sorted-job-lower-bound-makespan:
  assumes lb T A j j > m sorted j
  shows 2 * t j ≤ makespan T
proof −
  obtain x where x-def: x ∈ {1..m} 2 * t j ≤ T x
```

```
    using sorted-job-lower-bound-machine[OF assms] ..
  with makespan-correct(1) have T x ≤ makespan T by blast
  with x-def(2) show ?thesis by simp
qed

lemma min-zero:
  assumes x ∈ {1..k} T x = 0
  shows T (min-arg T k) = 0
  using assms(1)
proof (induction k)
  case (Suc k)
  show ?case proof (cases ‹x = Suc k›)
    case True
    then show ?thesis using assms(2) by (simp add: Let-def)
  next
    case False
    with Suc have T (min-arg T k) = 0 by simp
    then show ?thesis by simp
  qed
qed simp

lemma min-zero-index:
  assumes x ∈ {1..k} T x = 0
  shows min-arg T k ≤ x
  using assms(1)
proof (induction k)
  case (Suc k)
  show ?case proof (cases ‹x = Suc k›)
    case True
    then show ?thesis using min-in-range[of Suc k] by simp
  next
    case False
    with Suc.prems have x ∈ {1..k} by simp
    from min-zero[OF this, of T] assms(2) Suc.IH[OF this]
    show ?thesis by simp
  qed
qed simp

definition inv₂ :: (nat ⇒ nat) ⇒ (nat ⇒ nat set) ⇒ nat ⇒ bool where
  inv₂ T A j = (lb T A j ∧ j ≤ n
              ∧ (∀ T' A'. lb T' A' j ⟶ makespan T ≤ 3 / 2 * makespan T')
              ∧ (∀ x > j. T x = 0)
              ∧ (j ≤ m ⟶ makespan T = Max₀ (t ‘ {1..j}))))

lemma inv₂E:
  assumes inv₂ T A j
  shows lb T A j j ≤ n
        lb T' A' j ⟹ makespan T ≤ 3 / 2 * makespan T'
        ∀ x > j. T x = 0 j ≤ m ⟹ makespan T = Max₀ (t ‘ {1..j})
```

```
  using assms unfolding inv₂-def by blast+

lemma inv₂I:
  assumes lb T A j j ≤ n
        ∀ T' A'. lb T' A' j ⟶ makespan T ≤ 3 / 2 * makespan T'
        ∀ x > j. T x = 0
        j ≤ m ⟹ makespan T = Max₀ (t ` {1..j})
  shows inv₂ T A j
  unfolding inv₂-def using assms by blast

lemma inv₂-step:
  assumes sorted n inv₂ T A j j < n
  shows inv₂ (T (min-arg T m := T(min-arg T m) + t(Suc j)))
            (A (min-arg T m := A(min-arg T m) ∪ {Suc j})) (Suc j)
    (is ‹inv₂ ?T ?A -›)
proof (cases ‹Suc j > m›)
  case True note invrules = inv₂E[OF assms(2)]
  — Greedy is correct
  have LB: lb ?T ?A (Suc j)
    using add-job[OF invrules(1) min-in-range[OF m-gt-0]] by blast
  — Greedy maintains approximation factor
  have MK: ∀ T' A'. lb T' A' (Suc j) ⟶ makespan ?T ≤ 3 / 2 * makespan T'
  proof rule+
    fix T₁ A₁ assume lb T₁ A₁ (Suc j)
    from smaller-optimum[OF this]
    obtain T₀ A₀ where lb T₀ A₀ j makespan T₀ ≤ makespan T₁ by blast
    then have IH: makespan T ≤ 3 / 2 * makespan T₁
      using invrules(3) by force
    show makespan ?T ≤ 3 / 2 * makespan T₁
    proof (cases ‹makespan ?T = T (min-arg T m) + t (Suc j)›)
      case True
      have m * T (min-arg T m) ≤ (∑ i ∈ {1..m}. T i) by (rule min-avg)
      also have ... = (∑ i ∈ {1..j}. t i) by (rule lb-impl-job-sum[OF invrules(1)])
      finally have real m * T (min-arg T m) ≤ (∑ i ∈ {1..j}. t i)
        by (auto dest: of-nat-mono)
      with m-gt-0 have T (min-arg T m) ≤ (∑ i ∈ {1..j}. t i) / m
        by (simp add: field-simps)
      then have T (min-arg T m) ≤ makespan T₁
        using job-dist-lower-bound-makespan[OF ‹lb T₀ A₀ j›]
          and ‹makespan T₀ ≤ makespan T₁› by linarith
      moreover have 2 * t (Suc j) ≤ makespan T₁
       using sorted-job-lower-bound-makespan[OF ‹lb T₁ A₁ (Suc j)› ‹Suc j > m›]
          and assms(1,3) by simp
      ultimately show ?thesis unfolding True by simp
    next
      case False show ?thesis using remove-small-job[OF False] IH by simp
    qed
  qed
  have ∀ x > Suc j. ?T x = 0
```

```
    using invrules(4) min-in-range[OF m-gt-0, of T] True by simp
  with inv_2I[OF LB - MK] show ?thesis using assms(3) True by simp
next
  case False
  then have IN-RANGE: Suc j ∈ {1..m} by simp
  note invrules = inv_2E[OF assms(2)]
  then have T (Suc j) = 0 by blast

  — Greedy is correct
  have LB: lb ?T ?A (Suc j)
    using add-job[OF invrules(1) min-in-range[OF m-gt-0]] by blast

  — Greedy is trivially optimal
  from IN-RANGE ‹T (Suc j) = 0› have min-arg T m ≤ Suc j
    using min-zero-index by blast
  with invrules(4) have EMPTY: ∀ x > Suc j. ?T x = 0 by simp
  from IN-RANGE ‹T (Suc j) = 0› have T (min-arg T m) = 0
    using min-zero by blast
  with fun-upd-f-Max_0[OF min-in-range[OF m-gt-0]] invrules(5) False
 have TRIV: makespan ?T = Max_0 (t ` {1..Suc j}) unfolding f-Max_0-equiv[symmetric]
by simp
  have MK: ∀ T' A'. lb T' A' (Suc j) ⟶ makespan ?T ≤ 3 / 2 * makespan T'
    by (auto simp: TRIV[folded f-Max_0-equiv]
          dest!: max-job-lower-bound-makespan[folded f-Max_0-equiv])

  from inv_2I[OF LB - MK EMPTY TRIV] show ?thesis using assms(3) by simp
qed

lemma sorted-greedy-approximation:
sorted n ⟹ VARS T A i j
{True}
T := (λ-. 0);
A := (λ-. {});
j := 0;
WHILE j < n INV {inv_2 T A j} DO
  i := min-arg T m;
  j := (Suc j);
  A := A (i := A(i) ∪ {j});
  T := T (i := T(i) + t j)
OD
{lb T A n ∧ (∀ T' A'. lb T' A' n ⟶ makespan T ≤ 3 / 2 * makespan T')}
proof (vcg, goal-cases)
  case (1 T A i j)
  then show ?case by (simp add: lb-def inv_2-def)
next
  case (2 T A i j)
  then show ?case using inv_2-step by simp
next
  case (3 T A i j)
```

**then show** *?case* **unfolding** $inv_2$*-def* **by** *force*
**qed**

**end**

**end**

# 5 Bin Packing

**theory** *Approx-BP-Hoare*
**imports** *Complex-Main HOL−Hoare.Hoare-Logic HOL−Library.Disjoint-Sets*
**begin**

The algorithm and proofs are based on the work by Berghammer and Reuter [2].

## 5.1 Formalization of a Correct Bin Packing

Definition of the unary operator $[\![\cdot]\!]$ from the article. $B$ will only be wrapped into a set if it is non-empty.

**definition** *wrap* :: *'a set* ⇒ *'a set set* **where**
*wrap B* = (*if B* = {} *then* {} *else* {*B*})

**lemma** *wrap-card*:
*card* (*wrap B*) ≤ *1*
**unfolding** *wrap-def* **by** *auto*

If $M$ and $N$ are pairwise disjoint with $V$ and not yet contained in V, then the union of $M$ and $N$ is also pairwise disjoint with $V$.

**lemma** *pairwise-disjnt-Un*:
**assumes** *pairwise disjnt* ({*M*} ∪ {*N*} ∪ *V*) *M* ∉ *V N* ∉ *V*
**shows** *pairwise disjnt* ({*M* ∪ *N*} ∪ *V*)
**using** *assms* **unfolding** *pairwise-def* **by** *auto*

A Bin Packing Problem is defined like in the article:

**locale** *BinPacking* =
**fixes** *U* :: *'a set* — A finite, non-empty set of objects
**and** *w* :: *'a* ⇒ *real* — A mapping from objects to their respective weights (positive real numbers)
**and** *c* :: *nat* — The maximum capacity of a bin (a natural number)
**and** *S* :: *'a set* — The set of *small* objects (weight no larger than *1/2* of *c*)
**and** *L* :: *'a set* — The set of *large* objects (weight larger than *1/2* of *c*)
**assumes** *weight*: ∀ *u* ∈ *U*. *0* < *w*(*u*) ∧ *w*(*u*) ≤ *c*
**and** *U-Finite*: *finite U*
**and** *U-NE*: *U* ≠ {}
**and** *S-def*: *S* = {*u* ∈ *U*. *w*(*u*) ≤ *c* / *2*}
**and** *L-def*: *L* = *U* − *S*
**begin**

In the article, this is defined as $w$ as well. However, to avoid ambiguity, we will abbreviate the weight of a bin as $W$.

**abbreviation** $W :: 'a\ set \Rightarrow real$ **where**
$W\ B \equiv (\sum u \in B.\ w(u))$

$P$ constitutes as a correct bin packing if $P$ is a partition of $U$ (as defined in *partition-on-def*) and the weights of the bins do not exceed their maximum capacity $c$.

**definition** $bp :: 'a\ set\ set \Rightarrow bool$ **where**
$bp\ P \longleftrightarrow partition\text{-}on\ U\ P \wedge (\forall B \in P.\ W(B) \leq c)$

**lemma** *bpE*:
**assumes** *bp P*
**shows** *pairwise disjnt P* $\{\} \notin P$ $\bigcup P = U$ $\forall B \in P.\ W(B) \leq c$
**using** *assms* **unfolding** *bp-def partition-on-def* **by** *blast+*

**lemma** *bpI*:
**assumes** *pairwise disjnt P* $\{\} \notin P$ $\bigcup P = U$ $\forall B \in P.\ W(B) \leq c$
**shows** *bp P*
**using** *assms* **unfolding** *bp-def partition-on-def* **by** *blast*

Although we assume the $S$ and $L$ sets as given, manually obtaining them from $U$ is trivial and can be achieved in linear time. Proposed by the article [2].

**lemma** *S-L-set-generation*:
*VARS S L W u*
$\{True\}$
$S := \{\}$; $L := \{\}$; $W := U$;
*WHILE* $W \neq \{\}$
*INV* $\{W \subseteq U \wedge S = \{v \in U - W.\ w(v) \leq c\ /\ 2\} \wedge L = \{v \in U - W.\ w(v) > c\ /\ 2\}\}$ *DO*
$u := (SOME\ u.\ u \in W)$;
*IF* $2 * w(u) \leq c$
*THEN* $S := S \cup \{u\}$
*ELSE* $L := L \cup \{u\}$ *FI*;
$W := W - \{u\}$
*OD*
$\{S = \{v \in U.\ w(v) \leq c\ /\ 2\} \wedge L = \{v \in U.\ w(v) > c\ /\ 2\}\}$
**by** *vcg* (*auto simp*: *some-in-eq*)

## 5.2 The Proposed Approximation Algorithm

### 5.2.1 Functional Correctness

According to the article, $inv_1$ holds if $P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\}$ is a correct solution for the bin packing problem [2]. However, various assumptions made in the article seem to suggest that more information is demanded from this invariant and, indeed, mere correctness

(as defined in *bp-def*) does not appear to suffice. To amend this, four additional conjuncts have been added to this invariant, whose necessity will be explained in the following proofs. It should be noted that there may be other (shorter) ways to amend this invariant. This approach, however, makes for rather straight-forward proofs, as these conjuncts can be utilized and proved in relatively few steps.

**definition** $inv_1 :: 'a\ set\ set \Rightarrow 'a\ set\ set \Rightarrow 'a\ set \Rightarrow 'a\ set \Rightarrow 'a\ set \Rightarrow bool$ **where**
$inv_1\ P_1\ P_2\ B_1\ B_2\ V \longleftrightarrow bp\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\})$ — A correct solution to the bin packing problem
$\wedge \bigcup(P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) = U - V$ — The partial solution does not contain objects that have not yet been assigned
$\wedge\ B_1 \notin (P_1 \cup P_2 \cup wrap\ B_2)$ — $B_1$ is distinct from all the other bins
$\wedge\ B_2 \notin (P_1 \cup wrap\ B_1 \cup P_2)$ — $B_2$ is distinct from all the other bins
$\wedge\ (P_1 \cup wrap\ B_1) \cap (P_2 \cup wrap\ B_2) = \{\}$ — The first and second partial solutions are disjoint from each other.

**lemma** $inv_1E$:
**assumes** $inv_1\ P_1\ P_2\ B_1\ B_2\ V$
**shows** $bp\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\})$
**and** $\bigcup(P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) = U - V$
**and** $B_1 \notin (P_1 \cup P_2 \cup wrap\ B_2)$
**and** $B_2 \notin (P_1 \cup wrap\ B_1 \cup P_2)$
**and** $(P_1 \cup wrap\ B_1) \cap (P_2 \cup wrap\ B_2) = \{\}$
**using** *assms* **unfolding** $inv_1$*-def* **by** *auto*

**lemma** $inv_1I$:
**assumes** $bp\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\})$
**and** $\bigcup(P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) = U - V$
**and** $B_1 \notin (P_1 \cup P_2 \cup wrap\ B_2)$
**and** $B_2 \notin (P_1 \cup wrap\ B_1 \cup P_2)$
**and** $(P_1 \cup wrap\ B_1) \cap (P_2 \cup wrap\ B_2) = \{\}$
**shows** $inv_1\ P_1\ P_2\ B_1\ B_2\ V$
**using** *assms* **unfolding** $inv_1$*-def* **by** *blast*

**lemma** *wrap-Un* [*simp*]: $wrap\ (M \cup \{x\}) = \{M \cup \{x\}\}$ **unfolding** *wrap-def* **by** *simp*
**lemma** *wrap-empty* [*simp*]: $wrap\ \{\} = \{\}$ **unfolding** *wrap-def* **by** *simp*
**lemma** *wrap-not-empty* [*simp*]: $M \neq \{\} \longleftrightarrow wrap\ M = \{M\}$ **unfolding** *wrap-def* **by** *simp*

If $inv_1$ holds for the current partial solution, and the weight of an object $u \in V$ added to $B_1$ does not exceed its capacity, then $inv_1$ also holds if $B_1$ and $\{u\}$ are replaced by $B_1 \cup \{u\}$.

**lemma** $inv_1$*-stepA*:
**assumes** $inv_1\ P_1\ P_2\ B_1\ B_2\ V\ u \in V\ W(B_1) + w(u) \leq c$
**shows** $inv_1\ P_1\ P_2\ (B_1 \cup \{u\})\ B_2\ (V - \{u\})$

**proof** −
**note** *invrules* = *inv*$_1$*E*[*OF assms*(*1*)] **and** *bprules* = *bpE*[*OF invrules*(*1*)]

In the proof for *Theorem 3.2* of the article it is erroneously argued that if $P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\}$ is a partition of $U$, then the same holds if $B_1$ is replaced by $B_1 \cup \{u\}$. This is, however, not necessarily the case if $B_1$ or $\{u\}$ are already contained in the partial solution. Suppose $P_1$ contains the non-empty bin $B_1$, then $P_1 \cup wrap\ B_1$ would still be pairwise disjoint, provided $P_1$ was pairwise disjoint before, as the union simply ignores the duplicate $B_1$. Now, if the algorithm modifies $B_1$ by adding an element from $V$ such that $B_1$ becomes some non-empty $B_1{'}$ with $B_1 \cap B_1{'} \neq \emptyset$ and $B_1{'} \notin P_1$, one can see that this property would no longer be preserved. To avoid such a situation, we will use the first additional conjunct in *inv*$_1$ to ensure that $\{u\}$ is not yet contained in the partial solution, and the second additional conjunct to ensure that $B_1$ is not yet contained in the partial solution.

**have** *NOTIN*: $\forall M \in P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ u \notin M$
**using** *invrules*(*2*) *assms*(*2*) **by** *blast*
**have** $\{\{v\}\ |v.\ v \in V\} = \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
**using** *assms*(*2*) **by** *blast*
**then have** *pairwise disjnt* $(P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup (\{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$
**using** *bprules*(*1*) *assms*(*2*) **by** *simp*
**then have** *pairwise disjnt* $(wrap\ B_1 \cup \{\{u\}\} \cup P_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$ **by** (*simp add*: *Un-commute*)
**then have** *assm*: *pairwise disjnt* $(wrap\ B_1 \cup \{\{u\}\} \cup (P_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$ **by** (*simp add*: *Un-assoc*)
**have** *pairwise disjnt* $(\{B_1 \cup \{u\}\} \cup (P_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$
**proof** (*cases* ‹$B_1 = \{\}$›)
**case** *True* **with** *assm* **show** *?thesis* **by** *simp*
**next**
**case** *False*
**with** *assm* **have** *assm*: *pairwise disjnt* $(\{B_1\} \cup \{\{u\}\} \cup (P_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$ **by** *simp*
**from** *NOTIN* **have** $\{u\} \notin P_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$ **by** *blast*
**from** *pairwise-disjnt-Un*[*OF assm - this*] *invrules*(*2*,*3*) **show** *?thesis*
**using** *False* **by** *auto*
**qed**
**then have** *1*: *pairwise disjnt* $(P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$
**unfolding** *wrap-Un* **by** *simp*

— Rule 2: No empty sets
**from** *bprules*(*2*) **have** *2*: $\{\} \notin P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$

**unfolding** *wrap-def* **by** *simp*

— Rule 3: Union preserved
**from** *bprules*(*3*) **have** $\bigcup\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}) = U$
**using** *assms*(*2*) **by** *blast*
**then have** *3*: $\bigcup\ (P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}) = U$
**unfolding** *wrap-def* **by** *force*

— Rule 4: Weights below capacity
**have** $0 < w\ u$ **using** *weight assms*(*2*) *bprules*(*3*) **by** *blast*
**have** *finite* $B_1$ **using** *bprules*(*3*) *U-Finite* **by** (*cases* ‹$B_1 = \{\}$›) *auto*
**then have** $W\ (B_1 \cup \{u\}) \leq W\ B_1 + w\ u$ **using** ‹$0 < w\ u$› **by** (*cases* ‹$u \in B_1$›) (*auto simp*: *insert-absorb*)
**also have** $... \leq c$ **using** *assms*(*3*) **.**
**finally have** $W\ (B_1 \cup \{u\}) \leq c$ **.**
**then have** $\forall\, B \in wrap\ (B_1 \cup \{u\}).\ W\ B \leq c$ **unfolding** *wrap-Un* **by** *blast*
**moreover have** $\forall\, B \in P_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ W\ B \leq c$
**using** *bprules*(*4*) **by** *blast*
**ultimately have** *4*: $\forall\, B \in P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ W\ B \leq c$ **by** *blast*
**from** *bpI*[*OF 1 2 3 4*] **have** *1*: *bp* $(P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$ **.**

— Auxiliary information is preserved
**have** $u \in U$ **using** *assms*(*2*) *bprules*(*3*) **by** *blast*
**then have** *R*: $U - (V - \{u\}) = U - V \cup \{u\}$ **by** *blast*
**have** *L*: $\bigcup\ (P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2) = \bigcup\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) \cup \{u\}$
**unfolding** *wrap-def* **using** *NOTIN* **by** *auto*
**have** *2*: $\bigcup\ (P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2) = U - (V - \{u\})$
**unfolding** *L R invrules*(*2*) **..**
**have** *3*: $B_1 \cup \{u\} \notin P_1 \cup P_2 \cup wrap\ B_2$
**using** *NOTIN* **by** *auto*
**have** *4*: $B_2 \notin P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2$
**using** *invrules*(*4*) *NOTIN* **unfolding** *wrap-def* **by** *fastforce*
**have** *5*: $(P_1 \cup wrap\ (B_1 \cup \{u\})) \cap (P_2 \cup wrap\ B_2) = \{\}$
**using** *invrules*(*5*) *NOTIN* **unfolding** *wrap-Un* **by** *auto*

**from** *inv*$_1$*I*[*OF 1 2 3 4 5*] **show** *?thesis* **.**
**qed**

If $inv_1$ holds for the current partial solution, and the weight of an object $u \in V$ added to $B_2$ does not exceed its capacity, then $inv_1$ also holds if $B_2$ and $\{u\}$ are replaced by $B_2 \cup \{u\}$.

**lemma** *inv*$_1$*-stepB*:
**assumes** *inv*$_1$ $P_1\ P_2\ B_1\ B_2\ V\ u \in V\ W\ B_2 + w\ u \leq c$
**shows** *inv*$_1$ $(P_1 \cup wrap\ B_1)\ P_2\ \{\}\ (B_2 \cup \{u\})\ (V - \{u\})$

**proof** −
**note** *invrules* = *inv*$_1$*E*[*OF assms*(*1*)] **and** *bprules* = *bpE*[*OF invrules*(*1*)]

The argumentation here is similar to the one in *inv*$_1$-*stepA* with $B_1$ replaced with $B_2$ and using the first and third additional conjuncts of *inv*$_1$ to amend the issue, instead of the first and second.

**have** *NOTIN*: $\forall M \in P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ u \notin M$
**using** *invrules*(*2*) *assms*(*2*) **by** *blast*
**have** $\{\{v\}\ |v.\ v \in V\} = \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
**using** *assms*(*2*) **by** *blast*
**then have** *pairwise disjnt* $(P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$
**using** *bprules*(*1*) *assms*(*2*) **by** *simp*
**then have** *assm*: *pairwise disjnt* $(wrap\ B_2 \cup \{\{u\}\} \cup (P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$
**by** (*simp add*: *Un-assoc Un-commute*)
**have** *pairwise disjnt* $(\{B_2 \cup \{u\}\} \cup (P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$
**proof** (*cases* ‹$B_2 = \{\}$›)
**case** *True* **with** *assm* **show** *?thesis* **by** *simp*
**next**
**case** *False*
**with** *assm* **have** *assm*: *pairwise disjnt* $(\{B_2\} \cup \{\{u\}\} \cup (P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$ **by** *simp*
**from** *NOTIN* **have** $\{u\} \notin P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$ **by** *blast*
**from** *pairwise-disjnt-Un*[*OF assm - this*] *invrules*(*2*,*4*) **show** *?thesis*
**using** *False* **by** *auto*
**qed**
**then have** *1*: *pairwise disjnt* $(P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ (B_2 \cup \{u\}) \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$
**unfolding** *wrap-Un* **by** *simp*

— Rule 2: No empty sets
**from** *bprules*(*2*) **have** *2*: $\{\} \notin P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ (B_2 \cup \{u\}) \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
**unfolding** *wrap-def* **by** *simp*

— Rule 3: Union preserved
**from** *bprules*(*3*) **have** $\bigcup\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}) = U$
**using** *assms*(*2*) **by** *blast*
**then have** *3*: $\bigcup\ (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ (B_2 \cup \{u\}) \cup \{\{v\}\ |v.\ v \in V - \{u\}\}) = U$
**unfolding** *wrap-def* **by** *force*

— Rule 4: Weights below capacity
**have** *0* < *w u* **using** *weight assms*(*2*) *bprules*(*3*) **by** *blast*

**have** *finite* $B_2$ **using** *bprules(3) U-Finite* **by** (*cases* ‹$B_2 = \{\}$›) *auto*
**then have** $W\ (B_2 \cup \{u\}) \leq W\ B_2 + w\ u$ **using** ‹$0 < w\ u$› **by** (*cases* ‹$u \in B_2$›) (*auto simp*: *insert-absorb*)
**also have** $... \leq c$ **using** *assms(3)* **.**
**finally have** $W\ (B_2 \cup \{u\}) \leq c$ **.**
**then have** $\forall B \in wrap\ (B_2 \cup \{u\}).\ W\ B \leq c$ **unfolding** *wrap-Un* **by** *blast*
**moreover have** $\forall B \in P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ W\ B \leq c$
**using** *bprules(4)* **by** *blast*
**ultimately have** *4*: $\forall B \in P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ (B_2 \cup \{u\}) \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ W\ B \leq c$
**by** *auto*
**from** *bpI*[*OF 1 2 3 4*] **have** *1*: $bp\ (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ (B_2 \cup \{u\}) \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$ **.**

— Auxiliary information is preserved
**have** $u \in U$ **using** *assms(2) bprules(3)* **by** *blast*
**then have** *R*: $U - (V - \{u\}) = U - V \cup \{u\}$ **by** *blast*
**have** *L*: $\bigcup\ (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ (B_2 \cup \{u\})) = \bigcup\ (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ B_2) \cup \{u\}$
**unfolding** *wrap-def* **using** *NOTIN* **by** *auto*
**have** *2*: $\bigcup\ (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2 \cup wrap\ (B_2 \cup \{u\})) = U - (V - \{u\})$
**unfolding** *L R* **using** *invrules(2)* **by** *simp*
**have** *3*: $\{\} \notin P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ (B_2 \cup \{u\})$
**using** *bpE(2)*[*OF 1*] **by** *simp*
**have** *4*: $B_2 \cup \{u\} \notin P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup P_2$
**using** *NOTIN* **by** *auto*
**have** *5*: $(P_1 \cup wrap\ B_1 \cup wrap\ \{\}) \cap (P_2 \cup wrap\ (B_2 \cup \{u\})) = \{\}$
**using** *invrules(5) NOTIN* **unfolding** *wrap-empty wrap-Un* **by** *auto*

**from** *inv*$_1$*I*[*OF 1 2 3 4 5*] **show** *?thesis* **.**
**qed**

If $inv_1$ holds for the current partial solution, then $inv_1$ also holds if $B_1$ and $B_2$ are added to $P_1$ and $P_2$ respectively, $B_1$ is emptied and $B_2$ initialized with $u \in V$.

**lemma** *inv*$_1$*-stepC*:
**assumes** $inv_1\ P_1\ P_2\ B_1\ B_2\ V\ u \in V$
**shows** $inv_1\ (P_1 \cup wrap\ B_1)\ (P_2 \cup wrap\ B_2)\ \{\}\ \{u\}\ (V - \{u\})$
**proof** −
**note** *invrules* = *inv*$_1$*E*[*OF assms(1)*]
— Rule 1-4: Correct Bin Packing
**have** $P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup (P_2 \cup wrap\ B_2) \cup wrap\ \{u\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
$= P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
**by** (*metis* (*no-types*, *lifting*) *Un-assoc Un-empty-right insert-not-empty wrap-empty wrap-not-empty*)
**also have** $... = P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\}$
**using** *assms(2)* **by** *auto*

**finally have** *EQ*: $P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup (P_2 \cup wrap\ B_2) \cup wrap\ \{u\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
$= P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\}$ **.**
**from** *invrules(1)* **have** *1*: *bp* $(P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup (P_2 \cup wrap\ B_2) \cup wrap\ \{u\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$
**unfolding** *EQ* **.**

— Auxiliary information is preserved
**have** *NOTIN*: $\forall M \in P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ u \notin M$
**using** *invrules(2)* *assms(2)* **by** *blast*
**have** $u \in U$ **using** *assms(2)* *bpE(3)*[*OF invrules(1)*] **by** *blast*
**then have** *R*: $U - (V - \{u\}) = U - V \cup \{u\}$ **by** *blast*
**have** *L*: $\bigcup (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup (P_2 \cup wrap\ B_2) \cup wrap\ \{u\}) = \bigcup (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup (P_2 \cup wrap\ B_2)) \cup \{u\}$
**unfolding** *wrap-def* **using** *NOTIN* **by** *auto*
**have** *2*: $\bigcup (P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup (P_2 \cup wrap\ B_2) \cup wrap\ \{u\}) = U - (V - \{u\})$
**unfolding** *L R* **using** *invrules(2)* **by** *auto*
**have** *3*: $\{\} \notin P_1 \cup wrap\ B_1 \cup (P_2 \cup wrap\ B_2) \cup wrap\ \{u\}$
**using** *bpE(2)*[*OF 1*] **by** *simp*
**have** *4*: $\{u\} \notin P_1 \cup wrap\ B_1 \cup wrap\ \{\} \cup (P_2 \cup wrap\ B_2)$
**using** *NOTIN* **by** *auto*
**have** *5*: $(P_1 \cup wrap\ B_1 \cup wrap\ \{\}) \cap (P_2 \cup wrap\ B_2 \cup wrap\ \{u\}) = \{\}$
**using** *invrules(5)* *NOTIN* **unfolding** *wrap-def* **by** *force*

**from** $inv_1 I$[*OF 1 2 3 4 5*] **show** *?thesis* **.**
**qed**

A simplified version of the bin packing algorithm proposed in the article. It serves as an introduction into the approach taken, and, while it does not provide the desired approximation factor, it does ensure that $P$ is a correct solution of the bin packing problem.

**lemma** *simple-bp-correct*:
*VARS P* $P_1$ $P_2$ $B_1$ $B_2$ *V u*
{*True*}
$P_1 := \{\}$; $P_2 := \{\}$; $B_1 := \{\}$; $B_2 := \{\}$; $V := U$;
*WHILE* $V \cap S \neq \{\}$ *INV* {$inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ *V*} *DO*
$u :=$ (*SOME u*. $u \in V$); $V := V - \{u\}$;
*IF* $W(B_1) + w(u) \leq c$
*THEN* $B_1 := B_1 \cup \{u\}$
*ELSE IF* $W(B_2) + w(u) \leq c$
*THEN* $B_2 := B_2 \cup \{u\}$
*ELSE* $P_2 := P_2 \cup wrap\ B_2$; $B_2 := \{u\}$ *FI*;
$P_1 := P_1 \cup wrap\ B_1$; $B_1 := \{\}$ *FI*
*OD*;
$P := P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\} \mid v.\ v \in V\}$
{*bp P*}
**proof** (*vcg*, *goal-cases*)

**case** (*1 P* $P_1$ $P_2$ $B_1$ $B_2$ *V u*)
**show** *?case*
**unfolding** *bp-def partition-on-def pairwise-def wrap-def* $inv_1$*-def*
**using** *weight* **by** *auto*
**next**
**case** (*2 P* $P_1$ $P_2$ $B_1$ $B_2$ *V u*)
**then have** *INV*: $inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ *V* **..**
**from** *2* **have** $V \neq \{\}$ **by** *blast*
**then have** *IN*: (*SOME u*. $u \in V$) $\in V$ **by** (*simp add*: *some-in-eq*)
**from** $inv_1$*-stepA*[*OF INV IN*] $inv_1$*-stepB*[*OF INV IN*] $inv_1$*-stepC*[*OF INV IN*]
**show** *?case* **by** *blast*
**next**
**case** (*3 P* $P_1$ $P_2$ $B_1$ $B_2$ *V u*)
**then show** *?case* **unfolding** $inv_1$*-def* **by** *blast*
**qed**

### 5.2.2 Lower Bounds for the Bin Packing Problem

**lemma** *bp-bins-finite* [*simp*]:
**assumes** *bp P*
**shows** $\forall B \in P$. *finite B*
**using** *bpE*(*3*)[*OF assms*] *U-Finite* **by** (*meson Sup-upper finite-subset*)

**lemma** *bp-sol-finite* [*simp*]:
**assumes** *bp P*
**shows** *finite P*
**using** *bpE*(*3*)[*OF assms*] *U-Finite* **by** (*simp add*: *finite-UnionD*)

If $P$ is a solution of the bin packing problem, then no bin in $P$ may contain more than one large object.

**lemma** *only-one-L-per-bin*:
**assumes** *bp P B* $\in P$
**shows** $\forall x \in B. \forall y \in B. x \neq y \longrightarrow x \notin L \vee y \notin L$
**proof** (*rule ccontr*, *simp*)
**assume** $\exists x{\in}B. \exists y{\in}B. x \neq y \wedge x \in L \wedge y \in L$
**then obtain** *x y* **where** $*$: $x \in B$ $y \in B$ $x \neq y$ $x \in L$ $y \in L$ **by** *blast*
**then have** $c < w\ x + w\ y$ **using** *L-def S-def* **by** *force*
**have** *finite B* **using** *assms* **by** *simp*
**have** $y \in B - \{x\}$ **using** $*$(*2*,*3*) **by** *blast*
**have** $W\ B = W\ (B - \{x\}) + w\ x$
**using** $*$(*1*) ‹*finite B*› **by** (*simp add*: *sum.remove*)
**also have** ... $= W\ (B - \{x\} - \{y\}) + w\ x + w\ y$
**using** ‹$y \in B - \{x\}$› ‹*finite B*› **by** (*simp add*: *sum.remove*)
**finally have** $*$: $W\ B = W\ (B - \{x\} - \{y\}) + w\ x + w\ y$ **.**
**have** $\forall u \in B.\ 0 < w\ u$ **using** *bpE*(*3*)[*OF assms*(*1*)] *assms*(*2*) *weight* **by** *blast*
**then have** $0 \leq W\ (B - \{x\} - \{y\})$ **by** (*smt* (*verit*) *DiffD1 sum-nonneg*)
**with** $*$ **have** $c < W\ B$ **using** ‹$c < w\ x + w\ y$› **by** *simp*
**then show** *False* **using** *bpE*(*4*)[*OF assms*(*1*)] *assms*(*2*) **by** *fastforce*
**qed**

If $P$ is a solution of the bin packing problem, then the amount of large objects is a lower bound for the amount of bins in P.

**lemma** *L-lower-bound-card*:
**assumes** *bp P*
**shows** *card L* $\leq$ *card P*
**proof** −
**have** $\forall x \in L.\ \exists B \in P.\ x \in B$
**using** *bpE*(*3*)[*OF assms*] *L-def* **by** *blast*
**then obtain** *f* **where** *f-def*: $\forall u \in L.\ u \in f\ u \wedge f\ u \in P$ **by** *metis*
**then have** *inj-on f L*
**unfolding** *inj-on-def* **using** *only-one-L-per-bin*[*OF assms*] **by** *blast*
**then have** *card-eq*: *card L* = *card* (*f* ‘ *L*) **by** (*simp add*: *card-image*)
**have** *f* ‘ *L* $\subseteq$ *P* **using** *f-def* **by** *blast*
**moreover have** *finite P* **using** *assms* **by** *simp*
**ultimately have** *card* (*f* ‘ *L*) $\leq$ *card P* **by** (*simp add*: *card-mono*)
**then show** *?thesis* **unfolding** *card-eq* **.**
**qed**

If $P$ is a solution of the bin packing problem, then the amount of bins of a subset of P in which every bin contains a large object is a lower bound on the amount of large objects.

**lemma** *subset-bp-card*:
**assumes** *bp P M* $\subseteq$ *P* $\forall B \in M.\ B \cap L \neq \{\}$
**shows** *card M* $\leq$ *card L*
**proof** −
**have** $\forall B \in M.\ \exists u \in L.\ u \in B$ **using** *assms*(*3*) **by** *fast*
**then have** $\exists f.\ \forall B \in M.\ f\ B \in L \wedge f\ B \in B$ **by** *metis*
**then obtain** *f* **where** *f-def*: $\forall B \in M.\ f\ B \in L \wedge f\ B \in B$ **..**
**have** *inj-on f M*
**proof** (*rule ccontr*)
**assume** ¬ *inj-on f M*
**then have** $\exists x \in M.\ \exists y \in M.\ x \neq y \wedge f\ x = f\ y$ **unfolding** *inj-on-def* **by** *blast*
**then obtain** *x y* **where** ∗: $x \in M\ y \in M\ x \neq y\ f\ x = f\ y$ **by** *blast*
**then have** $\exists u.\ u \in x \wedge u \in y$ **using** *f-def* **by** *metis*
**then have** $x \cap y \neq \{\}$ **by** *blast*
**moreover have** *pairwise disjnt M* **using** *pairwise-subset*[*OF bpE*(*1*)[*OF assms*(*1*)] *assms*(*2*)] **.**
**ultimately show** *False* **using** ∗ **unfolding** *pairwise-def disjnt-def* **by** *simp*
**qed**
**moreover have** *finite L* **using** *L-def U-Finite* **by** *blast*
**moreover have** *f* ‘ *M* $\subseteq$ *L* **using** *f-def* **by** *blast*
**ultimately show** *?thesis* **using** *card-inj-on-le* **by** *blast*
**qed**

If $P$ is a correct solution of the bin packing problem, $inv_1$ holds for the partial solution, and every bin in $P_1 \cup wrap\ B_1$ contains a large object, then the amount of bins in $P_1 \cup wrap\ B_1 \cup \{\{v\}\ |v.\ v \in V \cap L\}$ is a lower bound for the amount of bins in $P$.

**lemma** *L-bins-lower-bound-card*:

**assumes** *bp P inv*$_1$ $P_1$ $P_2$ $B_1$ $B_2$ $V$ $\forall B \in P_1 \cup wrap\ B_1.\ B \cap L \neq \{\}$
**shows** *card* $(P_1 \cup wrap\ B_1 \cup \{\{v\}\ |v.\ v \in V \cap L\}) \leq card\ P$
**proof** −
**note** *invrules* = *inv*$_1$*E*[*OF assms*(*2*)]
**have** $\forall B \in \{\{v\}\ |v.\ v \in V \cap L\}.\ B \cap L \neq \{\}$ **by** *blast*
**with** *assms*(*3*) **have**
$P_1 \cup wrap\ B_1 \cup \{\{v\}\ |v.\ v \in V \cap L\} \subseteq P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\}$
$\forall B \in P_1 \cup wrap\ B_1 \cup \{\{v\}\ |v.\ v \in V \cap L\}.\ B \cap L \neq \{\}$ **by** *blast*+
**from** *subset-bp-card*[*OF invrules*(*1*) *this*] **show** *?thesis*
**using** *L-lower-bound-card*[*OF assms*(*1*)] **by** *linarith*
**qed**

If $P$ is a correct solution of the bin packing problem, then the sum of the weights of the objects is equal to the sum of the weights of the bins in $P$.

**lemma** *sum-Un-eq-sum-sum*:
**assumes** *bp P*
**shows** $(\sum u \in U.\ w\ u) = (\sum B \in P.\ W\ B)$
**proof** −
**have** *FINITE*: $\forall B \in P.$ *finite B* **using** *assms* **by** *simp*
**have** *DISJNT*: $\forall A \in P.\ \forall B \in P.\ A \neq B \longrightarrow A \cap B = \{\}$
**using** *bpE*(*1*)[*OF assms*] **unfolding** *pairwise-def disjnt-def* **.**
**have** $(\sum u \in (\bigcup P).\ w\ u) = (\sum B \in P.\ W\ B)$
**using** *sum.Union-disjoint*[*OF FINITE DISJNT*] **by** *auto*
**then show** *?thesis* **unfolding** *bpE*(*3*)[*OF assms*] **.**
**qed**

If $P$ is a correct solution of the bin packing problem, then the sum of the weights of the items is a lower bound of amount of bins in $P$ multiplied by their maximum capacity.

**lemma** *sum-lower-bound-card*:
**assumes** *bp P*
**shows** $(\sum u \in U.\ w\ u) \leq c * card\ P$
**proof** −
**have** $*$: $\forall B \in P.\ 0 < W\ B \wedge W\ B \leq c$
**using** *bpE*(*2*−*4*)[*OF assms*] *weight* **by** (*metis UnionI assms bp-bins-finite sum-pos*)
**have** $(\sum u \in U.\ w\ u) = (\sum B \in P.\ W\ B)$
**using** *sum-Un-eq-sum-sum*[*OF assms*] **.**
**also have** ... $\leq (\sum B \in P.\ c)$ **using** *sum-mono* $*$ **by** *fastforce*
**also have** ... $= c * card\ P$ **by** *simp*
**finally show** *?thesis* **.**
**qed**

**lemma** *bp-NE*:
**assumes** *bp P*
**shows** $P \neq \{\}$
**using** *U-NE bpE*(*3*)[*OF assms*] **by** *blast*

**lemma** *sum-Un-ge*:
**fixes** $f :: - \Rightarrow real$
**assumes** *finite M finite N* $\forall B \in M \cup N.\ 0 < f\ B$
**shows** *sum f M* $\leq$ *sum f* $(M \cup N)$
**proof** −
**have** $0 \leq$ *sum f N* − *sum f* $(M \cap N)$
**using** *assms* **by** (*smt* (*verit*) *DiffD1 inf.cobounded2 UnCI sum-mono2*)
**then have** *sum f M* $\leq$ *sum f M* + *sum f N* − *sum f* $(M \cap N)$
**by** *simp*
**also have** ... = *sum f* $(M \cup N)$
**using** *sum-Un*[*OF assms*(*1*,*2*), *symmetric*] **.**
**finally show** *?thesis* **.**
**qed**

If *bij-exists* holds, one can obtain a function which is bijective between the bins in $P$ and the objects in $V$ such that an object returned by the function would cause the bin to exceed its capacity.

**definition** *bij-exists* :: *'a set set* ⇒ *'a set* ⇒ *bool* **where**
*bij-exists P V* = $(\exists f.$ *bij-betw f P V* $\wedge$ $(\forall B \in P.\ W\ B + w\ (f\ B) > c))$

If $P$ is a functionally correct solution of the bin packing problem, $inv_1$ holds for the partial solution, and such a bijective function exists between the bins in $P_1$ and the objects in $P_2 \cup$ *wrap* $B_2$, the following strict lower bound can be shown:

**lemma** $P_1$*-lower-bound-card*:
**assumes** *bp P* $inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ *V bij-exists* $P_1$ $(\bigcup(P_2 \cup wrap\ B_2))$
**shows** *card* $P_1 + 1 \leq$ *card P*
**proof** (*cases* ‹$P_1 = \{\}$›)
**case** *True*
**have** *finite P* **using** *assms*(*1*) **by** *simp*
**then have** $1 \leq$ *card P* **using** *bp-NE*[*OF assms*(*1*)]
**by** (*metis Nat.add-0-right Suc-diff-1 Suc-le-mono card-gt-0-iff le0 mult-Suc-right nat-mult-1*)
**then show** *?thesis* **unfolding** *True* **by** *simp*
**next**
**note** *invrules* = $inv_1E$[*OF assms*(*2*)]
**case** *False*
**obtain** *f* **where** *f-def*: *bij-betw f* $P_1$ $(\bigcup(P_2 \cup wrap\ B_2))$ $\forall B \in P_1.\ W\ B + w\ (f\ B) > c$
**using** *assms*(*3*) **unfolding** *bij-exists-def* **by** *blast*
**have** *FINITE*: *finite* $P_1$ *finite* $(P_2 \cup wrap\ B_2)$ *finite* $(P_1 \cup P_2 \cup wrap\ B_2)$ *finite* $(wrap\ B_1 \cup \{\{v\}\ |v.\ v \in V\})$
**using** $inv_1E(1)$[*OF assms*(*2*)] *bp-sol-finite* **by** *blast+*

**have** *F*: $\forall B \in P_2 \cup wrap\ B_2.$ *finite B* **using** *invrules*(*1*) **by** *simp*
**have** *D*: $\forall A \in P_2 \cup wrap\ B_2.\ \forall B \in P_2 \cup wrap\ B_2.\ A \neq B \longrightarrow A \cap B = \{\}$
**using** $bpE(1)$[*OF invrules*(*1*)] **unfolding** *pairwise-def disjnt-def* **by** *auto*
**have** *sum-eq*: $W\ (\bigcup\ (P_2 \cup wrap\ B_2)) = (\sum B \in P_2 \cup wrap\ B_2.\ W\ B)$

**using** *sum.Union-disjoint*[*OF F D*] **by** *auto*

**have** $\forall B{\in}P_1 \cup \mathit{wrap}\ B_1 \cup P_2 \cup \mathit{wrap}\ B_2 \cup \{\{v\}\ |v.\ v \in V\}.\ 0 < W\ B$
**using** *bpE*(*2*,*3*)[*OF invrules*(*1*)] *weight* **by** (*metis* (*no-types*, *lifting*) *UnionI bp-bins-finite invrules*(*1*) *sum-pos*)
**then have** $(\sum B \in P_1 \cup P_2 \cup \mathit{wrap}\ B_2.\ W\ B) \leq (\sum B \in P_1 \cup P_2 \cup \mathit{wrap}\ B_2 \cup (\mathit{wrap}\ B_1 \cup \{\{v\}\ |v.\ v \in V\}).\ W\ B)$
**using** *sum-Un-ge*[*OF FINITE*(*3*,*4*), *of W*] **by** *blast*
**also have** $\ldots = (\sum B \in P_1 \cup \mathit{wrap}\ B_1 \cup P_2 \cup \mathit{wrap}\ B_2 \cup \{\{v\}\ |v.\ v \in V\}.\ W\ B)$ **by** (*smt* (*verit*) *Un-assoc Un-commute*)
**also have** $\ldots = W\ U$ **using** *sum-Un-eq-sum-sum*[*OF invrules*(*1*), *symmetric*] **.**
**finally have** $*$: $(\sum B \in P_1 \cup P_2 \cup \mathit{wrap}\ B_2.\ W\ B) \leq W\ U$ **.**

— This follows from the fourth and final additional conjunct of $inv_1$ and is necessary to combine the sums of the bins of the two partial solutions. This does not inherently follow from the union being a correct solution, as this need not be the case if $P_1$ and $P_2 \cup \mathit{wrap}\ B_2$ happened to be equal.
**have** *DISJNT*: $P_1 \cap (P_2 \cup \mathit{wrap}\ B_2) = \{\}$ **using** *invrules*(*5*) **by** *blast*

— This part of the proof is based on the proof on page 72 of the article [2].
**have** $c * \mathit{card}\ P_1 = (\sum B \in P_1.\ c)$ **by** *simp*
**also have** $\ldots < (\sum B \in P_1.\ W\ B + w\ (f\ B))$
**using** *f-def*(*2*) *sum-strict-mono*[*OF FINITE*(*1*) *False*] **by** *fastforce*
**also have** $\ldots = (\sum B \in P_1.\ W\ B) + (\sum B \in P_1.\ w\ (f\ B))$
**by** (*simp add*: *Groups-Big.comm-monoid-add-class.sum.distrib*)
**also have** $\ldots = (\sum B \in P_1.\ W\ B) + W\ (\bigcup\ (P_2 \cup \mathit{wrap}\ B_2))$ **unfolding** *sum.reindex-bij-betw*[*OF f-def*(*1*), *of w*] **..**
**also have** $\ldots = (\sum B \in P_1.\ W\ B) + (\sum B \in P_2 \cup \mathit{wrap}\ B_2.\ W\ B)$ **unfolding** *sum-eq* **..**
**also have** $\ldots = (\sum B \in P_1 \cup P_2 \cup \mathit{wrap}\ B_2.\ W\ B)$ **using** *sum.union-disjoint*[*OF FINITE*(*1*,*2*) *DISJNT*, *of W*] **by** (*simp add*: *Un-assoc*)
**also have** $\ldots \leq (\sum u \in U.\ w\ u)$ **using** $*$ **.**
**also have** $\ldots \leq c * \mathit{card}\ P$ **using** *sum-lower-bound-card*[*OF assms*(*1*)] **.**
**finally show** *?thesis*
**by** (*simp flip*: *of-nat-mult*)
**qed**

As $\mathit{card}\ (\mathit{wrap}\ ?B) \leq 1$ holds, it follows that the amount of bins in $P_1 \cup \mathit{wrap}\ B_1$ are a lower bound for the amount of bins in $P$.

**lemma** $P_1$-$B_1$-*lower-bound-card*:
**assumes** *bp P inv*$_1$ $P_1$ $P_2$ $B_1$ $B_2$ *V bij-exists* $P_1$ $(\bigcup(P_2 \cup \mathit{wrap}\ B_2))$
**shows** $\mathit{card}\ (P_1 \cup \mathit{wrap}\ B_1) \leq \mathit{card}\ P$
**proof** −
**have** $\mathit{card}\ (P_1 \cup \mathit{wrap}\ B_1) \leq \mathit{card}\ P_1 + \mathit{card}\ (\mathit{wrap}\ B_1)$
**using** *card-Un-le* **by** *blast*
**also have** $\ldots \leq \mathit{card}\ P_1 + 1$ **using** *wrap-card* **by** *simp*
**also have** $\ldots \leq \mathit{card}\ P$ **using** $P_1$-*lower-bound-card*[*OF assms*] **.**
**finally show** *?thesis* **.**
**qed**

If $inv_1$ holds, there are at most half as many bins in $P_2$ as there are objects in $P_2$, and we can again obtain a bijective function between the bins in $P_1$ and the objects of the second partial solution, then the amount of bins in the second partial solution are a strict lower bound for half the bins of the first partial solution.

**lemma** *$P_2$-$B_2$-lower-bound-$P_1$*:
  **assumes** *$inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ V 2 * card $P_2$ ≤ card ($\bigcup P_2$) bij-exists $P_1$ ($\bigcup$($P_2$ ∪ wrap $B_2$))*
  **shows** *2 * card ($P_2$ ∪ wrap $B_2$) ≤ card $P_1$ + 1*
**proof** −
  **note** *invrules = $inv_1$E[OF assms(1)]* **and** *bprules = bpE[OF invrules(1)]*

  **have** *pairwise disjnt ($P_2$ ∪ wrap $B_2$)*
    **using** *bprules(1) pairwise-subset* **by** *blast*
  **moreover have** *$B_2$ ∉ $P_2$* **using** *invrules(4)* **by** *simp*
  **ultimately have** *DISJNT*: $\bigcup P_2 \cap B_2 = \{\}$
    **by** (*auto*, *metis* (*no-types*, *opaque-lifting*) *sup-bot.right-neutral Un-insert-right disjnt-iff mk-disjoint-insert pairwise-insert wrap-Un*)

  **have** *finite* ($\bigcup P_2$) **using** *U-Finite bprules(3)* **by** *auto*
  **have** *finite $B_2$* **using** *bp-bins-finite[OF invrules(1)] wrap-not-empty* **by** *blast*
  **have** *finite $P_2$ finite (wrap $B_2$)* **using** *bp-sol-finite[OF invrules(1)]* **by** *blast+*
  **have** *DISJNT2*: $P_2 \cap$ *wrap* $B_2 = \{\}$ **unfolding** *wrap-def* **using** ‹$B_2 \notin P_2$› **by** *auto*
  **have** *card (wrap $B_2$) ≤ card $B_2$*
  **proof** (*cases* ‹$B_2 = \{\}$›)
    **case** *False*
    **then have** *1 ≤ card $B_2$* **by** (*simp add*: *leI* ‹*finite $B_2$*›)
    **then show** *?thesis* **using** *wrap-card[of $B_2$]* **by** *linarith*
  **qed** *simp*

  — This part of the proof is based on the proof on page 73 of the article [2].
  **from** *assms(2)* **have** *2 * card $P_2$ + 2 * card (wrap $B_2$) ≤ card ($\bigcup P_2$) + card (wrap $B_2$) + 1*
    **using** *wrap-card[of $B_2$]* **by** *linarith*
  **then have** *2 * (card $P_2$ + card (wrap $B_2$)) ≤ card ($\bigcup P_2$) + card $B_2$ + 1*
    **using** ‹*card (wrap $B_2$) ≤ card $B_2$*› **by** *simp*
  **then have** *2 * (card ($P_2$ ∪ wrap $B_2$)) ≤ card ($\bigcup P_2 \cup B_2$) + 1*
    **using** *card-Un-disjoint[OF* ‹*finite* ($\bigcup P_2$)› ‹*finite $B_2$*› *DISJNT]*
      **and** *card-Un-disjoint[OF* ‹*finite $P_2$*› ‹*finite (wrap $B_2$)*› *DISJNT2]* **by** *argo*
  **then have** *2 * (card ($P_2$ ∪ wrap $B_2$)) ≤ card ($\bigcup$($P_2$ ∪ wrap $B_2$)) + 1*
    **by** (*cases* ‹$B_2 = \{\}$›) (*auto simp*: *Un-commute*)
  **then show** *2 * (card ($P_2$ ∪ wrap $B_2$)) ≤ card $P_1$ + 1*
    **using** *assms(3) bij-betw-same-card* **unfolding** *bij-exists-def* **by** *metis*
**qed**

### 5.2.3 Proving the Approximation Factor

We define $inv_2$ as it is defined in the article. These conjuncts allow us to prove the desired approximation factor.

**definition** $inv_2 :: 'a\ set\ set \Rightarrow 'a\ set\ set \Rightarrow 'a\ set \Rightarrow 'a\ set \Rightarrow 'a\ set \Rightarrow bool$ **where**
$inv_2\ P_1\ P_2\ B_1\ B_2\ V \longleftrightarrow inv_1\ P_1\ P_2\ B_1\ B_2\ V$ — $inv_1$ holds for the partial solution
$\wedge\ (V \cap L \neq \{\} \longrightarrow (\forall B \in P_1 \cup wrap\ B_1.\ B \cap L \neq \{\}))$ — If there are still large objects left, then every bin of the first partial solution must contain a large object
$\wedge\ bij\text{-}exists\ P_1\ (\bigcup(P_2 \cup wrap\ B_2))$ — There exists a bijective function between the bins of the first partial solution and the objects of the second one
$\wedge\ (2 * card\ P_2 \leq card\ (\bigcup P_2))$ — There are at most twice as many bins in $P_2$ as there are objects in $P_2$

**lemma** $inv_2E$:
**assumes** $inv_2\ P_1\ P_2\ B_1\ B_2\ V$
**shows** $inv_1\ P_1\ P_2\ B_1\ B_2\ V$
**and** $V \cap L \neq \{\} \Longrightarrow \forall B \in P_1 \cup wrap\ B_1.\ B \cap L \neq \{\}$
**and** $bij\text{-}exists\ P_1\ (\bigcup(P_2 \cup wrap\ B_2))$
**and** $2 * card\ P_2 \leq card\ (\bigcup P_2)$
**using** *assms* **unfolding** $inv_2\text{-}def$ **by** *blast+*

**lemma** $inv_2I$:
**assumes** $inv_1\ P_1\ P_2\ B_1\ B_2\ V$
**and** $V \cap L \neq \{\} \Longrightarrow \forall B \in P_1 \cup wrap\ B_1.\ B \cap L \neq \{\}$
**and** $bij\text{-}exists\ P_1\ (\bigcup(P_2 \cup wrap\ B_2))$
**and** $2 * card\ P_2 \leq card\ (\bigcup P_2)$
**shows** $inv_2\ P_1\ P_2\ B_1\ B_2\ V$
**using** *assms* **unfolding** $inv_2\text{-}def$ **by** *blast*

If $P$ is a correct solution of the bin packing problem, $inv_2$ holds for the partial solution, and there are no more small objects left to be distributed, then the amount of bins of the partial solution is no larger than $3\ /\ 2$ of the amount of bins in $P$. This proof strongly follows the proof in *Theorem 4.1* of the article [2].

**lemma** *bin-packing-lower-bound-card*:
**assumes** $V \cap S = \{\}\ inv_2\ P_1\ P_2\ B_1\ B_2\ V\ bp\ P$
**shows** $card\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\}) \leq 3\ /\ 2 * card\ P$
**proof** (*cases* ‹$V = \{\}$›)
**note** $invrules = inv_2E[OF\ assms(2)]$
**case** *True*
**then have** $card\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\})$
$= card\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2)$ **by** *simp*
**also have** $... \leq card\ (P_1 \cup wrap\ B_1) + card\ (P_2 \cup wrap\ B_2)$
**using** $card\text{-}Un\text{-}le[of\ ‹P_1 \cup wrap\ B_1›]$ **by** (*simp add*: *Un-assoc*)
**also have** $... \leq card\ P + card\ (P_2 \cup wrap\ B_2)$

```
    using P1-B1-lower-bound-card[OF assms(3) invrules(1,3)] by simp
  also have ... ≤ card P + card P / 2
    using P2-B2-lower-bound-P1[OF invrules(1,4,3)]
      and P1-lower-bound-card[OF assms(3) invrules(1,3)] by linarith
  finally show ?thesis by linarith
next
  note invrules = inv2E[OF assms(2)]
  case False
  have U = S ∪ L using S-def L-def by blast
  then have ∗: V = V ∩ L
    using bpE(3)[OF inv1E(1)[OF invrules(1)]]
      and assms(1) by blast
  with False have NE: V ∩ L ≠ {} by simp
  have card (P1 ∪ wrap B1 ∪ P2 ∪ wrap B2 ∪ {{v} |v. v ∈ V})
      = card (P1 ∪ wrap B1 ∪ {{v} |v. v ∈ V ∩ L} ∪ P2 ∪ wrap B2)
    using ∗ by (simp add: Un-commute Un-assoc)
  also have ... ≤ card (P1 ∪ wrap B1 ∪ {{v} |v. v ∈ V ∩ L}) + card (P2 ∪ wrap
B2)
    using card-Un-le[of ‹P1 ∪ wrap B1 ∪ {{v} |v. v ∈ V ∩ L}›] by (simp add:
Un-assoc)
  also have ... ≤ card P + card (P2 ∪ wrap B2)
    using L-bins-lower-bound-card[OF assms(3) invrules(1) invrules(2)[OF NE]]
by linarith
  also have ... ≤ card P + card P / 2
    using P2-B2-lower-bound-P1[OF invrules(1,4,3)]
      and P1-lower-bound-card[OF assms(3) invrules(1,3)] by linarith
  finally show ?thesis by linarith
qed
```

We define $inv_3$ as it is defined in the article. This final conjunct allows us to prove that the invariant will be maintained by the algorithm.

```
definition inv3 :: 'a set set ⇒ 'a set set ⇒ 'a set ⇒ 'a set ⇒ 'a set ⇒ bool where
  inv3 P1 P2 B1 B2 V ⟷ inv2 P1 P2 B1 B2 V ∧ B2 ⊆ S

lemma inv3E:
  assumes inv3 P1 P2 B1 B2 V
  shows inv2 P1 P2 B1 B2 V and B2 ⊆ S
  using assms unfolding inv3-def by blast+

lemma inv3I:
  assumes inv2 P1 P2 B1 B2 V and B2 ⊆ S
  shows inv3 P1 P2 B1 B2 V
  using assms unfolding inv3-def by blast

lemma loop-init:
  inv3 {} {} {} {} U
proof −
  have ∗: inv1 {} {} {} {} U
    unfolding bp-def partition-on-def pairwise-def wrap-def inv1-def
```

**using** *weight* **by** *auto*
**have** *bij-exists* {} (⋃ ({} ∪ *wrap* {}))
**using** *bij-betwI'* **unfolding** *bij-exists-def* **by** *fastforce*
**from** $inv_2I$[*OF* ∗ - *this*] **have** $inv_2$ {} {} {} {} *U* **by** *auto*
**from** $inv_3I$[*OF this*] **show** *?thesis* **by** *blast*
**qed**

If $B_1$ is empty and there are no large objects left, then $inv_3$ will be maintained if $B_1$ is initialized with $u \in V \cap S$.

**lemma** *loop-stepA*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ V\ B_1 = \{\}\ V \cap L = \{\}\ u \in V \cap S$
**shows** $inv_3\ P_1\ P_2\ \{u\}\ B_2\ (V - \{u\})$
**proof** −
**note** *invrules* = $inv_2E$[*OF* $inv_3E(1)$[*OF assms*(*1*)]]
**have** *WEIGHT*: $W\ B_1 + w\ u \leq c$ **using** *S-def assms*(*2*,*4*) **by** *simp*
**from** *assms*(*4*) **have** $u \in V$ **by** *blast*
**from** $inv_1$-*stepA*[*OF invrules*(*1*) *this WEIGHT*] *assms*(*2*) **have** *1*: $inv_1\ P_1\ P_2\ \{u\}\ B_2\ (V - \{u\})$ **by** *simp*
**have** *2*: $(V - \{u\}) \cap L \neq \{\} \Longrightarrow \forall B{\in}P_1 \cup wrap\ \{u\}.\ B \cap L \neq \{\}$ **using** *assms*(*3*) **by** *blast*
**from** $inv_2I$[*OF 1 2*] *invrules* **have** $inv_2\ P_1\ P_2\ \{u\}\ B_2\ (V - \{u\})$ **by** *blast*
**from** $inv_3I$[*OF this*] **show** *?thesis* **using** $inv_3E(2)$[*OF assms*(*1*)] **.**
**qed**

If $B_1$ is empty and there are large objects left, then $inv_3$ will be maintained if $B_1$ is initialized with $u \in V \cap L$.

**lemma** *loop-stepB*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ V\ B_1 = \{\}\ u \in V \cap L$
**shows** $inv_3\ P_1\ P_2\ \{u\}\ B_2\ (V - \{u\})$
**proof** −
**note** *invrules* = $inv_2E$[*OF* $inv_3E(1)$[*OF assms*(*1*)]]
**have** *WEIGHT*: $W\ B_1 + w\ u \leq c$ **using** *L-def weight assms*(*2*,*3*) **by** *simp*
**from** *assms*(*3*) **have** $u \in V$ **by** *blast*
**from** $inv_1$-*stepA*[*OF invrules*(*1*) *this WEIGHT*] *assms*(*2*) **have** *1*: $inv_1\ P_1\ P_2\ \{u\}\ B_2\ (V - \{u\})$ **by** *simp*
**have** $\forall B{\in}P_1.\ B \cap L \neq \{\}$ **using** *assms*(*3*) *invrules*(*2*) **by** *blast*
**then have** *2*: $(V - \{u\}) \cap L \neq \{\} \Longrightarrow \forall B{\in}P_1 \cup wrap\ \{u\}.\ B \cap L \neq \{\}$
**using** *assms*(*3*) **by** (*metis Int-iff UnE empty-iff insertE singletonI wrap-not-empty*)
**from** $inv_2I$[*OF 1 2*] *invrules* **have** $inv_2\ P_1\ P_2\ \{u\}\ B_2\ (V - \{u\})$ **by** *blast*
**from** $inv_3I$[*OF this*] **show** *?thesis* **using** $inv_3E(2)$[*OF assms*(*1*)] **.**
**qed**

If $B_1$ is not empty and $u \in V \cap S$ does not exceed its maximum capacity, then $inv_3$ will be maintained if $B_1$ and $\{u\}$ are replaced with $B_1 \cup \{u\}$.

**lemma** *loop-stepC*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ V\ B_1 \neq \{\}\ u \in V \cap S\ W\ B_1 + w(u) \leq c$
**shows** $inv_3\ P_1\ P_2\ (B_1 \cup \{u\})\ B_2\ (V - \{u\})$
**proof** −
**note** *invrules* = $inv_2E$[*OF* $inv_3E(1)$[*OF assms*(*1*)]]

**from** *assms(3)* **have** $u \in V$ **by** *blast*
**from** *inv$_1$-stepA[OF invrules(1) this assms(4)]* **have** *1*: *inv$_1$* $P_1$ $P_2$ $(B_1 \cup \{u\})$ $B_2$ $(V - \{u\})$ **.**
**have** $(V - \{u\}) \cap L \neq \{\} \Longrightarrow \forall B{\in}P_1 \cup wrap\ B_1.\ B \cap L \neq \{\}$ **using** *invrules(2)* **by** *blast*
**then have** *2*: $(V - \{u\}) \cap L \neq \{\} \Longrightarrow \forall B{\in}P_1 \cup wrap\ (B_1 \cup \{u\}).\ B \cap L \neq \{\}$
**by** (*metis Int-commute Un-empty-right Un-insert-right assms(2) disjoint-insert(2) insert-iff wrap-not-empty*)
**from** *inv$_2$I[OF 1 2] invrules* **have** *inv$_2$* $P_1$ $P_2$ $(B_1 \cup \{u\})$ $B_2$ $(V - \{u\})$ **by** *blast*
**from** *inv$_3$I[OF this]* **show** *?thesis* **using** *inv$_3$E(2)[OF assms(1)]* **.**
**qed**

If $B_1$ is not empty and $u \in V \cap S$ does exceed its maximum capacity but not the capacity of $B_2$, then *inv$_3$* will be maintained if $B_1$ is added to $P_1$ and emptied, and $B_2$ and $\{u\}$ are replaced with $B_2 \cup \{u\}$.

**lemma** *loop-stepD*:
**assumes** *inv$_3$* $P_1$ $P_2$ $B_1$ $B_2$ $V$ $B_1 \neq \{\}$ $u \in V \cap S$ $W\ B_1 + w(u) > c$ $W\ B_2 + w(u) \leq c$
**shows** *inv$_3$* $(P_1 \cup wrap\ B_1)$ $P_2$ $\{\}$ $(B_2 \cup \{u\})$ $(V - \{u\})$
**proof** −
**note** *invrules* = *inv$_2$E[OF inv$_3$E(1)[OF assms(1)]]*
**from** *assms(3)* **have** $u \in V$ **by** *blast*
**from** *inv$_1$-stepB[OF invrules(1) this assms(5)]* **have** *1*: *inv$_1$* $(P_1 \cup wrap\ B_1)$ $P_2$ $\{\}$ $(B_2 \cup \{u\})$ $(V - \{u\})$ **.**

**have** *2*: $(V - \{u\}) \cap L \neq \{\} \Longrightarrow \forall B{\in}P_1 \cup wrap\ B_1 \cup wrap\ \{\}.\ B \cap L \neq \{\}$
**using** *invrules(2)* **unfolding** *wrap-empty* **by** *blast*

**from** *invrules(3)* **obtain** $f$ **where** *f-def*: *bij-betw f* $P_1$ $(\bigcup\ (P_2 \cup wrap\ B_2))$ $\forall B{\in}P_1.\ c < W\ B + w\ (f\ B)$ **unfolding** *bij-exists-def* **by** *blast*
**have** $B_1 \notin P_1$ **using** *inv$_1$E(3)[OF invrules(1)]* **by** *blast*
**have** $u \notin (\bigcup\ (P_2 \cup wrap\ B_2))$ **using** *inv$_1$E(2)[OF invrules(1)] assms(3)* **by** *blast*
**then have** $(\bigcup\ (P_2 \cup wrap\ (B_2 \cup \{u\}))) = (\bigcup\ (P_2 \cup wrap\ B_2 \cup \{\{u\}\}))$
**by** (*metis Sup-empty Un-assoc Union-Un-distrib ccpo-Sup-singleton wrap-empty wrap-not-empty*)
**also have** ... $= (\bigcup\ (P_2 \cup wrap\ B_2)) \cup \{u\}$ **by** *simp*
**finally have** *UN*: $(\bigcup\ (P_2 \cup wrap\ (B_2 \cup \{u\}))) = (\bigcup\ (P_2 \cup wrap\ B_2)) \cup \{u\}$ **.**
**have** *wrap* $B_1 = \{B_1\}$ **using** *wrap-not-empty[of* $B_1$*] assms(2)* **by** *simp*
**let** *?f* $= f\ (B_1 := u)$
**have** *BIJ*: *bij-betw ?f* $(P_1 \cup wrap\ B_1)$ $(\bigcup\ (P_2 \cup wrap\ (B_2 \cup \{u\})))$
**unfolding** *wrap-empty* ‹*wrap* $B_1 = \{B_1\}$› *UN* **using** *f-def(1)* ‹$B_1 \notin P_1$› ‹$u \notin (\bigcup\ (P_2 \cup wrap\ B_2))$›
**by** (*metis* (*no-types*, *lifting*) *bij-betw-cong fun-upd-other fun-upd-same notIn-Un-bij-betw3*)
**have** $c < W\ B_1 + w\ (?f\ B_1)$ **using** *assms(4)* **by** *simp*
**then have** $(\forall B{\in}P_1 \cup wrap\ B_1.\ c < W\ B + w\ (?f\ B))$
**unfolding** ‹*wrap* $B_1 = \{B_1\}$› **using** *f-def(2)* **by** *simp*

**with** *BIJ* **have** *bij-betw ?f* $(P_1 \cup wrap\ B_1)$ $(\bigcup\ (P_2 \cup wrap\ (B_2 \cup \{u\})))$
$\wedge\ (\forall B{\in}P_1 \cup wrap\ B_1.\ c < W\ B + w\ (?f\ B))$ **by** *blast*
**then have** *3*: *bij-exists* $(P_1 \cup wrap\ B_1)$ $(\bigcup\ (P_2 \cup wrap\ (B_2 \cup \{u\})))$
**unfolding** *bij-exists-def* **by** *blast*
**from** $inv_2 I[OF\ 1\ 2\ 3]$ **have** $inv_2\ (P_1 \cup wrap\ B_1)\ P_2\ \{\}\ (B_2 \cup \{u\})\ (V - \{u\})$
**using** *invrules*(*4*) **by** *blast*

**from** $inv_3 I[OF\ this]$ **show** *?thesis* **using** $inv_3 E(2)[OF\ assms(1)]$ *assms*(*3*) **by** *blast*
**qed**

If the maximum capacity of $B_2$ is exceeded by $u \in V \cap S$, then $B_2$ must contain at least two objects.

**lemma** *$B_2$-at-least-two-objects*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ V\ u \in V \cap S\ W\ B_2 + w(u) > c$
**shows** $2 \leq card\ B_2$
**proof** (*rule ccontr*, *simp add*: *not-le*)
**have** *FINITE*: *finite* $B_2$ **using** $inv_1 E(1)[OF\ inv_2 E(1)[OF\ inv_3 E(1)[OF\ assms(1)]]]$
**by** (*metis* (*no-types*, *lifting*) *Finite-Set.finite.simps U-Finite Union-Un-distrib bpE*(*3*) *ccpo-Sup-singleton finite-Un wrap-not-empty*)
**assume** $card\ B_2 < 2$
**then consider** (*0*) $card\ B_2 = 0$ | (*1*) $card\ B_2 = 1$ **by** *linarith*
**then show** *False* **proof** *cases*
**case** *0* **then have** $B_2 = \{\}$ **using** *FINITE* **by** *simp*
**then show** *?thesis* **using** *assms*(*2*,*3*) *S-def* **by** *simp*
**next**
**case** *1* **then obtain** *v* **where** $B_2 = \{v\}$
**using** *card-1-singletonE* **by** *auto*
**with** $inv_3 E(2)[OF\ assms(1)]$ **have** $2 * w\ v \leq c$ **using** *S-def* **by** *simp*
**moreover from** $\langle B_2 = \{v\} \rangle$ **have** $W\ B_2 = w\ v$ **by** *simp*
**ultimately show** *?thesis* **using** *assms*(*2*,*3*) *S-def* **by** *simp*
**qed**
**qed**

If $B_1$ is not empty and $u \in V \cap S$ exceeds the maximum capacity of both $B_1$ and $B_2$, then $inv_3$ will be maintained if $B_1$ and $B_2$ are added to $P_1$ and $P_2$ respectively, emptied, and $B_2$ initialized with $u$.

**lemma** *loop-stepE*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ V\ B_1 \neq \{\}\ u \in V \cap S\ W\ B_1 + w(u) > c\ W\ B_2 + w(u) > c$
**shows** $inv_3\ (P_1 \cup wrap\ B_1)\ (P_2 \cup wrap\ B_2)\ \{\}\ \{u\}\ (V - \{u\})$
**proof** −
**note** *invrules* = $inv_2 E[OF\ inv_3 E(1)[OF\ assms(1)]]$
**from** *assms*(*3*) **have** $u \in V$ **by** *blast*
**from** *$inv_1$-stepC*[*OF invrules*(*1*) *this*] **have** *1*: $inv_1\ (P_1 \cup wrap\ B_1)\ (P_2 \cup wrap\ B_2)\ \{\}\ \{u\}\ (V - \{u\})$ **.**

**have** *2*: $(V - \{u\}) \cap L \neq \{\} \Longrightarrow \forall B{\in}P_1 \cup wrap\ B_1 \cup wrap\ \{\}.\ B \cap L \neq \{\}$
**using** *invrules*(*2*) **unfolding** *wrap-empty* **by** *blast*

**from** *invrules(3)* **obtain** *f* **where** *f-def*: *bij-betw f* $P_1$ $(\bigcup\ (P_2 \cup wrap\ B_2))$ $\forall B{\in}P_1.\ c < W\ B + w\ (f\ B)$ **unfolding** *bij-exists-def* **by** *blast*
**have** $B_1 \notin P_1$ **using** $inv_1E(3)$[*OF invrules(1)*] **by** *blast*
**have** $u \notin (\bigcup\ (P_2 \cup wrap\ B_2))$ **using** $inv_1E(2)$[*OF invrules(1)*] *assms(3)* **by** *blast*
**have** $(\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\})) = (\bigcup\ (P_2 \cup wrap\ B_2 \cup \{\{u\}\}))$ **unfolding** *wrap-def* **by** *simp*
**also have** ... $= (\bigcup\ (P_2 \cup wrap\ B_2)) \cup \{u\}$ **by** *simp*
**finally have** *UN*: $(\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\})) = (\bigcup\ (P_2 \cup wrap\ B_2)) \cup \{u\}$ **.**
**have** *wrap* $B_1 = \{B_1\}$ **using** *wrap-not-empty*[*of* $B_1$] *assms(2)* **by** *simp*
**let** *?f* $= f\ (B_1 := u)$
**have** *BIJ*: *bij-betw ?f* $(P_1 \cup wrap\ B_1)$ $(\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\}))$
**unfolding** *wrap-empty* ‹*wrap* $B_1 = \{B_1\}$› *UN* **using** *f-def(1)* ‹$B_1 \notin P_1$› ‹$u \notin (\bigcup\ (P_2 \cup wrap\ B_2))$›
**by** (*metis* (*no-types*, *lifting*) *bij-betw-cong fun-upd-other fun-upd-same notIn-Un-bij-betw3*)
**have** $c < W\ B_1 + w$ (*?f* $B_1$) **using** *assms(4)* **by** *simp*
**then have** $(\forall B{\in}P_1 \cup wrap\ B_1.\ c < W\ B + w\ (?f\ B))$
**unfolding** ‹*wrap* $B_1 = \{B_1\}$› **using** *f-def(2)* **by** *simp*
**with** *BIJ* **have** *bij-betw ?f* $(P_1 \cup wrap\ B_1)$ $(\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\}))$
$\wedge\ (\forall B{\in}P_1 \cup wrap\ B_1.\ c < W\ B + w\ (?f\ B))$ **by** *blast*
**then have** *3*: *bij-exists* $(P_1 \cup wrap\ B_1)$ $(\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\}))$
**unfolding** *bij-exists-def* **by** *blast*

**have** *4*: $2 * card\ (P_2 \cup wrap\ B_2) \leq card\ (\bigcup\ (P_2 \cup wrap\ B_2))$
**proof** −
**note** *bprules* $= bpE$[*OF* $inv_1E(1)$[*OF invrules(1)*]]
**have** *pairwise disjnt* $(P_2 \cup wrap\ B_2)$
**using** *bprules(1) pairwise-subset* **by** *blast*
**moreover have** $B_2 \notin P_2$ **using** $inv_1E(4)$[*OF invrules(1)*] **by** *simp*
**ultimately have** *DISJNT*: $\bigcup P_2 \cap B_2 = \{\}$
**by** (*auto*, *metis* (*no-types*, *opaque-lifting*) *sup-bot.right-neutral Un-insert-right disjnt-iff mk-disjoint-insert pairwise-insert wrap-Un*)
**have** *finite* $(\bigcup P_2)$ **using** *U-Finite bprules(3)* **by** *auto*
**have** *finite* $B_2$ **using** $inv_1E(1)$[*OF invrules(1)*] *bp-bins-finite wrap-not-empty* **by** *blast*

**have** $2 * card\ (P_2 \cup wrap\ B_2) \leq 2 * (card\ P_2 + card\ (wrap\ B_2))$
**using** *card-Un-le*[*of* $P_2$ ‹*wrap* $B_2$›] **by** *simp*
**also have** ... $\leq 2 * card\ P_2 + 2$ **using** *wrap-card* **by** *auto*
**also have** ... $\leq card\ (\bigcup\ P_2) + 2$ **using** *invrules(4)* **by** *simp*
**also have** ... $\leq card\ (\bigcup\ P_2) + card\ B_2$ **using** $B_2$*-at-least-two-objects*[*OF assms(1,3,5)*] **by** *simp*
**also have** ... $= card\ (\bigcup\ (P_2 \cup \{B_2\}))$ **using** *DISJNT card-Un-disjoint*[*OF* ‹*finite* $(\bigcup P_2)$› ‹*finite* $B_2$›] **by** (*simp add*: *Un-commute*)
**also have** ... $= card\ (\bigcup\ (P_2 \cup wrap\ B_2))$ **by** (*cases* ‹$B_2 = \{\}$›) *auto*
**finally show** *?thesis* **.**
**qed**

**from** *inv₂I*[*OF 1 2 3 4*] **have** *inv₂* ($P_1 \cup$ *wrap* $B_1$) ($P_2 \cup$ *wrap* $B_2$) {} {$u$} ($V$ − {$u$}) **.**

**from** *inv₃I*[*OF this*] **show** *?thesis* **using** *assms*(*3*) **by** *blast*
**qed**

The bin packing algorithm as it is proposed in the article [2]. $P$ will not only be a correct solution of the bin packing problem, but the amount of bins will be a lower bound for *3 / 2* of the amount of bins of any correct solution $Q$, and thus guarantee an approximation factor of *3 / 2* for the optimum.

**lemma** *bp-approx*:
*VARS P* $P_1$ $P_2$ $B_1$ $B_2$ $V$ $u$
{*True*}
$P_1 := \{\}$; $P_2 := \{\}$; $B_1 := \{\}$; $B_2 := \{\}$; $V := U$;
*WHILE* $V \cap S \neq \{\}$ *INV* {*inv₃* $P_1$ $P_2$ $B_1$ $B_2$ $V$} *DO*
*IF* $B_1 \neq \{\}$
*THEN* $u := (SOME\ u.\ u \in V \cap S)$
*ELSE IF* $V \cap L \neq \{\}$
*THEN* $u := (SOME\ u.\ u \in V \cap L)$
*ELSE* $u := (SOME\ u.\ u \in V \cap S)$ *FI FI*;
$V := V - \{u\}$;
*IF* $W(B_1) + w(u) \leq c$
*THEN* $B_1 := B_1 \cup \{u\}$
*ELSE IF* $W(B_2) + w(u) \leq c$
*THEN* $B_2 := B_2 \cup \{u\}$
*ELSE* $P_2 := P_2 \cup$ *wrap* $B_2$; $B_2 := \{u\}$ *FI*;
$P_1 := P_1 \cup$ *wrap* $B_1$; $B_1 := \{\}$ *FI*
*OD*;
$P := P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V\}$
{*bp* $P \wedge (\forall Q.\ bp\ Q \longrightarrow card\ P \leq 3\ /\ 2 * card\ Q)$}
**proof** (*vcg*, *goal-cases*)
**case** (*1* $P$ $P_1$ $P_2$ $B_1$ $B_2$ $V$ $u$)
**then show** *?case* **by** (*simp add*: *loop-init*)
**next**
**case** (*2* $P$ $P_1$ $P_2$ $B_1$ $B_2$ $V$ $u$)
**then have** *INV*: *inv₃* $P_1$ $P_2$ $B_1$ $B_2$ $V$ **..**
**let** *?s* = *SOME u*. $u \in V \cap S$
**let** *?l* = *SOME u*. $u \in V \cap L$
**have** *LIN*: $V \cap L \neq \{\} \Longrightarrow$ *?l* $\in V \cap L$ **using** *some-in-eq* **by** *metis*
**then have** *LWEIGHT*: $V \cap L \neq \{\} \Longrightarrow w$ *?l* $\leq c$ **using** *L-def weight* **by** *blast*
**from** *2* **have** $V \cap S \neq \{\}$ **..**
**then have** *IN*: *?s* $\in V \cap S$ **using** *some-in-eq* **by** *metis*
**then have** $w$ *?s* $\leq c$ **using** *S-def* **by** *simp*

**then show** *?case*
**using** *LWEIGHT loop-stepA*[*OF INV - - IN*] *loop-stepB*[*OF INV - LIN*] *loop-stepC*[*OF INV - IN*]
**and** *loop-stepD*[*OF INV - IN*] *loop-stepE*[*OF INV - IN*] **by** (*cases* ‹$B_1 = \{\}$›,

*cases* ‹$V \cap L = \{\}$›) *auto*
**next**
**case** (*3 P* $P_1$ $P_2$ $B_1$ $B_2$ *V u*)
**then have** *INV*: $inv_3$ $P_1$ $P_2$ $B_1$ $B_2$ *V* **and** *EMPTY*: $V \cap S = \{\}$ **by** *blast+*
**from** $inv_1E(1)$[*OF* $inv_2E(1)$[*OF* $inv_3E(1)$[*OF INV*]]] **and** *bin-packing-lower-bound-card*[*OF EMPTY* $inv_3E(1)$[*OF INV*]]
**show** *?case* **by** *blast*
**qed**

**end**

## 5.3 The Full Linear Time Version of the Proposed Algorithm

Finally, we prove the Algorithm proposed on page 78 of the article [2]. This version generates the S and L sets beforehand and uses them directly to calculate the solution, thus removing the need for intersection operations, and ensuring linear time if we can perform *insertion*, *removal*, *and selection of an element*, *the union of two sets*, *and the emptiness test in constant time* [2].

**locale** *BinPacking-Complete* =
**fixes** *U* :: *'a set* — A finite, non-empty set of objects
**and** *w* :: *'a* ⇒ *real* — A mapping from objects to their respective weights (positive real numbers)
**and** *c* :: *nat* — The maximum capacity of a bin (as a natural number)
**assumes** *weight*: ∀ *u* ∈ *U*. *0* < *w*(*u*) ∧ *w*(*u*) ≤ *c*
**and** *U-Finite*: *finite U*
**and** *U-NE*: *U* ≠ {}
**begin**

The correctness proofs will be identical to the ones of the simplified algorithm.

**abbreviation** *W* :: *'a set* ⇒ *real* **where**
*W B* ≡ (∑ *u* ∈ *B*. *w*(*u*))

**definition** *bp* :: *'a set set* ⇒ *bool* **where**
*bp P* ⟷ *partition-on U P* ∧ (∀ *B* ∈ *P*. *W*(*B*) ≤ *c*)

**lemma** *bpE*:
**assumes** *bp P*
**shows** *pairwise disjnt P* {} ∉ *P* ⋃*P* = *U* ∀ *B* ∈ *P*. *W*(*B*) ≤ *c*
**using** *assms* **unfolding** *bp-def partition-on-def* **by** *blast+*

**lemma** *bpI*:
**assumes** *pairwise disjnt P* {} ∉ *P* ⋃*P* = *U* ∀ *B* ∈ *P*. *W*(*B*) ≤ *c*
**shows** *bp P*
**using** *assms* **unfolding** *bp-def partition-on-def* **by** *blast*

**definition** $inv_1$ :: *'a set set* ⇒ *'a set set* ⇒ *'a set* ⇒ *'a set* ⇒ *'a set* ⇒ *bool* **where**

$inv_1\ P_1\ P_2\ B_1\ B_2\ V \longleftrightarrow bp\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\})$ — A correct solution to the bin packing problem

$\wedge \bigcup (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) = U - V$ — The partial solution does not contain objects that have not yet been assigned

$\wedge\ B_1 \notin (P_1 \cup P_2 \cup wrap\ B_2)$ — $B_1$ is distinct from all the other bins

$\wedge\ B_2 \notin (P_1 \cup wrap\ B_1 \cup P_2)$ — $B_2$ is distinct from all the other bins

$\wedge\ (P_1 \cup wrap\ B_1) \cap (P_2 \cup wrap\ B_2) = \{\}$ — The first and second partial solutions are disjoint from each other.

**lemma** $inv_1E$:
  **assumes** $inv_1\ P_1\ P_2\ B_1\ B_2\ V$
  **shows** $bp\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\})$
    **and** $\bigcup (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) = U - V$
    **and** $B_1 \notin (P_1 \cup P_2 \cup wrap\ B_2)$
    **and** $B_2 \notin (P_1 \cup wrap\ B_1 \cup P_2)$
    **and** $(P_1 \cup wrap\ B_1) \cap (P_2 \cup wrap\ B_2) = \{\}$
  **using** *assms* **unfolding** $inv_1$*-def* **by** *auto*

**lemma** $inv_1I$:
  **assumes** $bp\ (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V\})$
    **and** $\bigcup (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) = U - V$
    **and** $B_1 \notin (P_1 \cup P_2 \cup wrap\ B_2)$
    **and** $B_2 \notin (P_1 \cup wrap\ B_1 \cup P_2)$
    **and** $(P_1 \cup wrap\ B_1) \cap (P_2 \cup wrap\ B_2) = \{\}$
  **shows** $inv_1\ P_1\ P_2\ B_1\ B_2\ V$
  **using** *assms* **unfolding** $inv_1$*-def* **by** *blast*

**lemma** *wrap-Un* [*simp*]: $wrap\ (M \cup \{x\}) = \{M \cup \{x\}\}$ **unfolding** *wrap-def* **by** *simp*
**lemma** *wrap-empty* [*simp*]: $wrap\ \{\} = \{\}$ **unfolding** *wrap-def* **by** *simp*
**lemma** *wrap-not-empty* [*simp*]: $M \neq \{\} \longleftrightarrow wrap\ M = \{M\}$ **unfolding** *wrap-def* **by** *simp*

**lemma** $inv_1$*-stepA*:
  **assumes** $inv_1\ P_1\ P_2\ B_1\ B_2\ V\ u \in V\ W(B_1) + w(u) \leq c$
  **shows** $inv_1\ P_1\ P_2\ (B_1 \cup \{u\})\ B_2\ (V - \{u\})$
**proof** −
  **note** $invrules = inv_1E[OF\ assms(1)]$ **and** $bprules = bpE[OF\ invrules(1)]$

  — Rule 1: Pairwise Disjoint
  **have** *NOTIN*: $\forall M \in P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ u \notin M$
    **using** $invrules(2)$ $assms(2)$ **by** *blast*
  **have** $\{\{v\}\ |v.\ v \in V\} = \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
    **using** $assms(2)$ **by** *blast*
  **then have** *pairwise disjnt* $(P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup (\{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$

**using** *bprules(1) assms(2)* **by** *simp*
**then have** *pairwise disjnt* (*wrap* $B_1 \cup \{\{u\}\} \cup P_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\})$ **by** (*simp add*: *Un-commute*)
**then have** *assm*: *pairwise disjnt* (*wrap* $B_1 \cup \{\{u\}\} \cup (P_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}))$ **by** (*simp add*: *Un-assoc*)
**have** *pairwise disjnt* $(\{B_1 \cup \{u\}\} \cup (P_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}))$
**proof** (*cases* ‹$B_1 = \{\}$›)
**case** *True* **with** *assm* **show** *?thesis* **by** *simp*
**next**
**case** *False*
**with** *assm* **have** *assm*: *pairwise disjnt* $(\{B_1\} \cup \{\{u\}\} \cup (P_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}))$ **by** *simp*
**from** *NOTIN* **have** $\{u\} \notin P_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}$ **by** *blast*
**from** *pairwise-disjnt-Un*[*OF assm - this*] *invrules(2,3)* **show** *?thesis*
**using** *False* **by** *auto*
**qed**
**then have** *1*: *pairwise disjnt* $(P_1 \cup$ *wrap* $(B_1 \cup \{u\}) \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\})$
**unfolding** *wrap-Un* **by** *simp*

— Rule 2: No empty sets
**from** *bprules(2)* **have** *2*: $\{\} \notin P_1 \cup$ *wrap* $(B_1 \cup \{u\}) \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}$
**unfolding** *wrap-def* **by** *simp*

— Rule 3: Union preserved
**from** *bprules(3)* **have** $\bigcup (P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{u\}\} \cup \{\{v\} \mid v.\ v \in V - \{u\}\}) = U$
**using** *assms(2)* **by** *blast*
**then have** *3*: $\bigcup (P_1 \cup$ *wrap* $(B_1 \cup \{u\}) \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}) = U$
**unfolding** *wrap-def* **by** *force*

— Rule 4: Weights below capacity
**have** $0 < w\ u$ **using** *weight assms(2) bprules(3)* **by** *blast*
**have** *finite* $B_1$ **using** *bprules(3) U-Finite* **by** (*cases* ‹$B_1 = \{\}$›) *auto*
**then have** $W\ (B_1 \cup \{u\}) \leq W\ B_1 + w\ u$ **using** ‹$0 < w\ u$› **by** (*cases* ‹$u \in B_1$›) (*auto simp*: *insert-absorb*)
**also have** $\ldots \leq c$ **using** *assms(3)* **.**
**finally have** $W\ (B_1 \cup \{u\}) \leq c$ **.**
**then have** $\forall B \in$ *wrap* $(B_1 \cup \{u\}).\ W\ B \leq c$ **unfolding** *wrap-Un* **by** *blast*
**moreover have** $\forall B \in P_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}.\ W\ B \leq c$
**using** *bprules(4)* **by** *blast*
**ultimately have** *4*: $\forall B \in P_1 \cup$ *wrap* $(B_1 \cup \{u\}) \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}.\ W\ B \leq c$ **by** *blast*
**from** *bpI*[*OF 1 2 3 4*] **have** *1*: *bp* $(P_1 \cup$ *wrap* $(B_1 \cup \{u\}) \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\})$ **.**

— Auxiliary information is preserved
**have** $u \in U$ **using** *assms(2) bprules(3)* **by** *blast*
**then have** *R*: $U - (V - \{u\}) = U - V \cup \{u\}$ **by** *blast*
**have** *L*: $\bigcup (P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2) = \bigcup (P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2) \cup \{u\}$
**unfolding** *wrap-def* **using** *NOTIN* **by** *auto*
**have** *2*: $\bigcup (P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2 \cup wrap\ B_2) = U - (V - \{u\})$
**unfolding** *L R invrules(2)* **..**
**have** *3*: $B_1 \cup \{u\} \notin P_1 \cup P_2 \cup wrap\ B_2$
**using** *NOTIN* **by** *auto*
**have** *4*: $B_2 \notin P_1 \cup wrap\ (B_1 \cup \{u\}) \cup P_2$
**using** *invrules(4) NOTIN* **unfolding** *wrap-def* **by** *fastforce*
**have** *5*: $(P_1 \cup wrap\ (B_1 \cup \{u\})) \cap (P_2 \cup wrap\ B_2) = \{\}$
**using** *invrules(5) NOTIN* **unfolding** *wrap-Un* **by** *auto*

**from** *inv*$_1$*I*[*OF 1 2 3 4 5*] **show** *?thesis* **.**
**qed**

**lemma** *inv*$_1$*-stepB*:
**assumes** *inv*$_1$ $P_1$ $P_2$ $B_1$ $B_2$ $V$ $u \in V$ $W\ B_2 + w\ u \leq c$
**shows** *inv*$_1$ $(P_1 \cup wrap\ B_1)$ $P_2$ $\{\}$ $(B_2 \cup \{u\})$ $(V - \{u\})$
**proof** −
**note** *invrules* = *inv*$_1$*E*[*OF assms(1)*] **and** *bprules* = *bpE*[*OF invrules(1)*]

**have** *NOTIN*: $\forall M \in P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}.\ u \notin M$
**using** *invrules(2) assms(2)* **by** *blast*
**have** $\{\{v\}\ |v.\ v \in V\} = \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$
**using** *assms(2)* **by** *blast*
**then have** *pairwise disjnt* $(P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{u\}\} \cup \{\{v\}\ |v.\ v \in V - \{u\}\})$
**using** *bprules(1) assms(2)* **by** *simp*
**then have** *assm*: *pairwise disjnt* $(wrap\ B_2 \cup \{\{u\}\} \cup (P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$
**by** (*simp add*: *Un-assoc Un-commute*)
**have** *pairwise disjnt* $(\{B_2 \cup \{u\}\} \cup (P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$
**proof** (*cases* ‹$B_2 = \{\}$›)
**case** *True* **with** *assm* **show** *?thesis* **by** *simp*
**next**
**case** *False*
**with** *assm* **have** *assm*: *pairwise disjnt* $(\{B_2\} \cup \{\{u\}\} \cup (P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}))$ **by** *simp*
**from** *NOTIN* **have** $\{u\} \notin P_1 \cup wrap\ B_1 \cup P_2 \cup \{\{v\}\ |v.\ v \in V - \{u\}\}$ **by** *blast*
**from** *pairwise-disjnt-Un*[*OF assm - this*] *invrules(2,4)* **show** *?thesis*
**using** *False* **by** *auto*
**qed**

**then have** *1*: *pairwise disjnt* ($P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\}) \cup \{\{v\} \mid v.\ v \in V - \{u\}\}$)
**unfolding** *wrap-Un* **by** *simp*

— Rule 2: No empty sets
**from** *bprules*(*2*) **have** *2*: $\{\} \notin P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\}) \cup \{\{v\} \mid v.\ v \in V - \{u\}\}$
**unfolding** *wrap-def* **by** *simp*

— Rule 3: Union preserved
**from** *bprules*(*3*) **have** $\bigcup$ $(P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{u\}\} \cup \{\{v\} \mid v.\ v \in V - \{u\}\}) = U$
**using** *assms*(*2*) **by** *blast*
**then have** *3*: $\bigcup$ $(P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\}) \cup \{\{v\} \mid v.\ v \in V - \{u\}\}) = U$
**unfolding** *wrap-def* **by** *force*

— Rule 4: Weights below capacity
**have** $0 < w\ u$ **using** *weight assms*(*2*) *bprules*(*3*) **by** *blast*
**have** *finite* $B_2$ **using** *bprules*(*3*) *U-Finite* **by** (*cases* ‹$B_2 = \{\}$›) *auto*
**then have** $W\ (B_2 \cup \{u\}) \leq W\ B_2 + w\ u$ **using** ‹$0 < w\ u$› **by** (*cases* ‹$u \in B_2$›) (*auto simp*: *insert-absorb*)
**also have** ... $\leq c$ **using** *assms*(*3*) **.**
**finally have** $W\ (B_2 \cup \{u\}) \leq c$ **.**
**then have** $\forall B \in$ *wrap* $(B_2 \cup \{u\}).\ W\ B \leq c$ **unfolding** *wrap-Un* **by** *blast*
**moreover have** $\forall B \in P_1 \cup$ *wrap* $B_1 \cup P_2 \cup \{\{v\} \mid v.\ v \in V - \{u\}\}.\ W\ B \leq c$
**using** *bprules*(*4*) **by** *blast*
**ultimately have** *4*: $\forall B \in P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\}) \cup \{\{v\} \mid v.\ v \in V - \{u\}\}.\ W\ B \leq c$
**by** *auto*
**from** *bpI*[*OF 1 2 3 4*] **have** *1*: *bp* ($P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\}) \cup \{\{v\} \mid v.\ v \in V - \{u\}\}$) **.**

— Auxiliary information is preserved
**have** $u \in U$ **using** *assms*(*2*) *bprules*(*3*) **by** *blast*
**then have** *R*: $U - (V - \{u\}) = U - V \cup \{u\}$ **by** *blast*
**have** *L*: $\bigcup$ $(P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\})) = \bigcup$ $(P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $B_2) \cup \{u\}$
**unfolding** *wrap-def* **using** *NOTIN* **by** *auto*
**have** *2*: $\bigcup$ $(P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\})) = U - (V - \{u\})$
**unfolding** *L R* **using** *invrules*(*2*) **by** *simp*
**have** *3*: $\{\} \notin P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $(B_2 \cup \{u\})$
**using** *bpE*(*2*)[*OF 1*] **by** *simp*
**have** *4*: $B_2 \cup \{u\} \notin P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\} \cup P_2$
**using** *NOTIN* **by** *auto*
**have** *5*: $(P_1 \cup$ *wrap* $B_1 \cup$ *wrap* $\{\}) \cap (P_2 \cup$ *wrap* $(B_2 \cup \{u\})) = \{\}$
**using** *invrules*(*5*) *NOTIN* **unfolding** *wrap-empty wrap-Un* **by** *auto*

**from** *inv$_1$I*[*OF 1 2 3 4 5*] **show** *?thesis* **.**
**qed**

**lemma** *inv$_1$-stepC*:
**assumes** *inv$_1$ P$_1$ P$_2$ B$_1$ B$_2$ V u* $\in$ *V*
**shows** *inv$_1$* (*P$_1$* $\cup$ *wrap B$_1$*) (*P$_2$* $\cup$ *wrap B$_2$*) {} {*u*} (*V* $-$ {*u*})
**proof** $-$
**note** *invrules* = *inv$_1$E*[*OF assms(1)*]
— Rule 1-4: Correct Bin Packing
**have** *P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {} $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*) $\cup$ *wrap* {*u*} $\cup$ {{*v*} |*v*. *v* $\in$ *V* $-$ {*u*}}
= *P$_1$* $\cup$ *wrap B$_1$* $\cup$ *P$_2$* $\cup$ *wrap B$_2$* $\cup$ {{*u*}} $\cup$ {{*v*} |*v*. *v* $\in$ *V* $-$ {*u*}}
**by** (*metis* (*no-types*, *lifting*) *Un-assoc Un-empty-right insert-not-empty wrap-empty wrap-not-empty*)
**also have** ... = *P$_1$* $\cup$ *wrap B$_1$* $\cup$ *P$_2$* $\cup$ *wrap B$_2$* $\cup$ {{*v*} |*v*. *v* $\in$ *V*}
**using** *assms(2)* **by** *auto*
**finally have** *EQ*: *P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {} $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*) $\cup$ *wrap* {*u*} $\cup$ {{*v*} |*v*. *v* $\in$ *V* $-$ {*u*}}
= *P$_1$* $\cup$ *wrap B$_1$* $\cup$ *P$_2$* $\cup$ *wrap B$_2$* $\cup$ {{*v*} |*v*. *v* $\in$ *V*} **.**
**from** *invrules(1)* **have** *1*: *bp* (*P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {} $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*) $\cup$ *wrap* {*u*} $\cup$ {{*v*} |*v*. *v* $\in$ *V* $-$ {*u*}})
**unfolding** *EQ* **.**

— Auxiliary information is preserved
**have** *NOTIN*: $\forall$ *M* $\in$ *P$_1$* $\cup$ *wrap B$_1$* $\cup$ *P$_2$* $\cup$ *wrap B$_2$* $\cup$ {{*v*} |*v*. *v* $\in$ *V* $-$ {*u*}}. *u* $\notin$ *M*
**using** *invrules(2)* *assms(2)* **by** *blast*
**have** *u* $\in$ *U* **using** *assms(2)* *bpE(3)*[*OF invrules(1)*] **by** *blast*
**then have** *R*: *U* $-$ (*V* $-$ {*u*}) = *U* $-$ *V* $\cup$ {*u*} **by** *blast*
**have** *L*: $\bigcup$ (*P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {} $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*) $\cup$ *wrap* {*u*}) = $\bigcup$ (*P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {} $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*)) $\cup$ {*u*}
**unfolding** *wrap-def* **using** *NOTIN* **by** *auto*
**have** *2*: $\bigcup$ (*P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {} $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*) $\cup$ *wrap* {*u*}) = *U* $-$ (*V* $-$ {*u*})
**unfolding** *L R* **using** *invrules(2)* **by** *auto*
**have** *3*: {} $\notin$ *P$_1$* $\cup$ *wrap B$_1$* $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*) $\cup$ *wrap* {*u*}
**using** *bpE(2)*[*OF 1*] **by** *simp*
**have** *4*: {*u*} $\notin$ *P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {} $\cup$ (*P$_2$* $\cup$ *wrap B$_2$*)
**using** *NOTIN* **by** *auto*
**have** *5*: (*P$_1$* $\cup$ *wrap B$_1$* $\cup$ *wrap* {}) $\cap$ (*P$_2$* $\cup$ *wrap B$_2$* $\cup$ *wrap* {*u*}) = {}
**using** *invrules(5)* *NOTIN* **unfolding** *wrap-def* **by** *force*

**from** *inv$_1$I*[*OF 1 2 3 4 5*] **show** *?thesis* **.**
**qed**

From this point onward, we will require a different approach for proving lower bounds. Instead of fixing and assuming the definitions of the $S$ and $L$ sets, we will introduce the abbreviations $S_U$ and $L_U$ for any occurrences of the original $S$ and $L$ sets. The union of $S$ and $L$ can be interpreted as

$V$. As a result, occurrences of $V \cap S$ become $(S \cup L) \cap S = S$, and $V \cap L$ become $(S \cup L) \cap L = L$. Occurrences of these sets will have to be replaced appropriately.

**abbreviation** $S_U$ **where**
  $S_U \equiv \{u \in U.\ w\ u \leq c\ /\ 2\}$

**abbreviation** $L_U$ **where**
  $L_U \equiv \{u \in U.\ c\ /\ 2 < w\ u\}$

As we will remove elements from $S$ and $L$, we will only be able to show that they remain subsets of $S_U$ and $L_U$ respectively.

**abbreviation** *SL* **where**
  $SL\ S\ L \equiv S \subseteq S_U \wedge L \subseteq L_U$

**lemma** *bp-bins-finite* [*simp*]:
  **assumes** *bp P*
  **shows** $\forall B \in P.$ *finite B*
  **using** *bpE*(*3*)[*OF assms*] *U-Finite* **by** (*meson Sup-upper finite-subset*)

**lemma** *bp-sol-finite* [*simp*]:
  **assumes** *bp P*
  **shows** *finite P*
  **using** *bpE*(*3*)[*OF assms*] *U-Finite* **by** (*simp add*: *finite-UnionD*)

**lemma** *only-one-L-per-bin*:
  **assumes** *bp P B* $\in P$
  **shows** $\forall x \in B.\ \forall y \in B.\ x \neq y \longrightarrow x \notin L_U \vee y \notin L_U$
**proof** (*rule ccontr*, *simp*)
  **assume** $\exists x{\in}B.\ \exists y{\in}B.\ x \neq y \wedge y \in U \wedge x \in U \wedge$ *real* $c < w\ x * 2 \wedge$ *real* $c < w\ y * 2$
  **then obtain** *x y* **where** $*$: $x \in B\ y \in B\ x \neq y\ x \in L_U\ y \in L_U$ **by** *auto*
  **then have** $c < w\ x + w\ y$ **by** *force*
  **have** *finite B* **using** *assms* **by** *simp*
  **have** $y \in B - \{x\}$ **using** $*$(*2*,*3*) **by** *blast*
  **have** $W\ B = W\ (B - \{x\}) + w\ x$
    **using** $*$(*1*) ‹*finite B*› **by** (*simp add*: *sum.remove*)
  **also have** $... = W\ (B - \{x\} - \{y\}) + w\ x + w\ y$
    **using** ‹$y \in B - \{x\}$› ‹*finite B*› **by** (*simp add*: *sum.remove*)
  **finally have** $*$: $W\ B = W\ (B - \{x\} - \{y\}) + w\ x + w\ y$ **.**
  **have** $\forall u \in B.\ 0 < w\ u$ **using** *bpE*(*3*)[*OF assms*(*1*)] *assms*(*2*) *weight* **by** *blast*
  **then have** $0 \leq W\ (B - \{x\} - \{y\})$ **by** (*smt* (*verit*) *DiffD1 sum-nonneg*)
  **with** $*$ **have** $c < W\ B$ **using** ‹$c < w\ x + w\ y$› **by** *simp*
  **then show** *False* **using** *bpE*(*4*)[*OF assms*(*1*)] *assms*(*2*) **by** *fastforce*
**qed**

**lemma** *L-lower-bound-card*:
  **assumes** *bp P*
  **shows** *card* $L_U \leq$ *card P*
**proof** −

**have** $\forall x \in L_U.\ \exists B \in P.\ x \in B$
**using** *bpE*(*3*)[*OF assms*] **by** *blast*
**then obtain** $f$ **where** *f-def*: $\forall u \in L_U.\ u \in f\ u \wedge f\ u \in P$ **by** *metis*
**then have** *inj-on f* $L_U$
**unfolding** *inj-on-def* **using** *only-one-L-per-bin*[*OF assms*] **by** *blast*
**then have** *card-eq*: *card* $L_U$ = *card* ($f$ ‘ $L_U$) **by** (*simp add*: *card-image*)
**have** $f$ ‘ $L_U \subseteq P$ **using** *f-def* **by** *blast*
**moreover have** *finite* $P$ **using** *assms* **by** *simp*
**ultimately have** *card* ($f$ ‘ $L_U$) $\leq$ *card* $P$ **by** (*simp add*: *card-mono*)
**then show** *?thesis* **unfolding** *card-eq* **.**
**qed**

**lemma** *subset-bp-card*:
**assumes** *bp* $P$ $M \subseteq P$ $\forall B \in M.\ B \cap L_U \neq \{\}$
**shows** *card* $M \leq$ *card* $L_U$
**proof** −
**have** $\forall B \in M.\ \exists u \in L_U.\ u \in B$ **using** *assms*(*3*) **by** *fast*
**then have** $\exists f.\ \forall B \in M.\ f\ B \in L_U \wedge f\ B \in B$ **by** *metis*
**then obtain** $f$ **where** *f-def*: $\forall B \in M.\ f\ B \in L_U \wedge f\ B \in B$ **..**
**have** *inj-on f M*
**proof** (*rule ccontr*)
**assume** ¬ *inj-on f M*
**then have** $\exists x \in M.\ \exists y \in M.\ x \neq y \wedge f\ x = f\ y$ **unfolding** *inj-on-def* **by** *blast*
**then obtain** $x\ y$ **where** $*$: $x \in M\ y \in M\ x \neq y\ f\ x = f\ y$ **by** *blast*
**then have** $\exists u.\ u \in x \wedge u \in y$ **using** *f-def* **by** *metis*
**then have** $x \cap y \neq \{\}$ **by** *blast*
**moreover have** *pairwise disjnt M* **using** *pairwise-subset*[*OF bpE*(*1*)[*OF assms*(*1*)] *assms*(*2*)] **.**
**ultimately show** *False* **using** $*$ **unfolding** *pairwise-def disjnt-def* **by** *simp*
**qed**
**moreover have** *finite* $L_U$ **using** *U-Finite* **by** *auto*
**moreover have** $f$ ‘ $M \subseteq L_U$ **using** *f-def* **by** *blast*
**ultimately show** *?thesis* **using** *card-inj-on-le* **by** *blast*
**qed**

**lemma** *L-bins-lower-bound-card*:
**assumes** *bp* $P$ *inv*$_1$ $P_1$ $P_2$ $B_1$ $B_2$ ($S \cup L$) $\forall B \in P_1 \cup$ *wrap* $B_1.\ B \cap L_U \neq \{\}$
**and** *SL-def*: *SL S L*
**shows** *card* ($P_1 \cup$ *wrap* $B_1 \cup \{\{v\}\ |v.\ v \in L\}$) $\leq$ *card* $P$
**proof** −
**note** *invrules* = *inv*$_1$*E*[*OF assms*(*2*)]
**have** $\forall B \in \{\{v\}\ |v.\ v \in L\}.\ B \cap L_U \neq \{\}$ **using** *SL-def* **by** *blast*
**with** *assms*(*3*) **have**
$P_1 \cup$ *wrap* $B_1 \cup \{\{v\}\ |v.\ v \in L\} \subseteq P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\}\ |v.\ v \in S \cup L\}$
$\forall B \in P_1 \cup$ *wrap* $B_1 \cup \{\{v\}\ |v.\ v \in L\}.\ B \cap L_U \neq \{\}$ **by** *blast*+
**from** *subset-bp-card*[*OF invrules*(*1*) *this*] **show** *?thesis*
**using** *L-lower-bound-card*[*OF assms*(*1*)] **by** *linarith*
**qed**

**lemma** *sum-Un-eq-sum-sum*:
**assumes** *bp P*
**shows** $(\sum u \in U.\ w\ u) = (\sum B \in P.\ W\ B)$
**proof** −
**have** *FINITE*: $\forall B \in P.$ *finite B* **using** *assms* **by** *simp*
**have** *DISJNT*: $\forall A \in P.\ \forall B \in P.\ A \neq B \longrightarrow A \cap B = \{\}$
**using** *bpE*(*1*)[*OF assms*] **unfolding** *pairwise-def disjnt-def* **.**
**have** $(\sum u \in (\bigcup P).\ w\ u) = (\sum B \in P.\ W\ B)$
**using** *sum.Union-disjoint*[*OF FINITE DISJNT*] **by** *auto*
**then show** *?thesis* **unfolding** *bpE*(*3*)[*OF assms*] **.**
**qed**

**lemma** *sum-lower-bound-card*:
**assumes** *bp P*
**shows** $(\sum u \in U.\ w\ u) \leq c * card\ P$
**proof** −
**have** $*$: $\forall B \in P.\ 0 < W\ B \wedge W\ B \leq c$
**using** *bpE*(*2*−*4*)[*OF assms*] *weight* **by** (*metis UnionI assms bp-bins-finite sum-pos*)
**have** $(\sum u \in U.\ w\ u) = (\sum B \in P.\ W\ B)$
**using** *sum-Un-eq-sum-sum*[*OF assms*] **.**
**also have** $... \leq (\sum B \in P.\ c)$ **using** *sum-mono* $*$ **by** *fastforce*
**also have** $... = c * card\ P$ **by** *simp*
**finally show** *?thesis* **.**
**qed**

**lemma** *bp-NE*:
**assumes** *bp P*
**shows** $P \neq \{\}$
**using** *U-NE bpE*(*3*)[*OF assms*] **by** *blast*

**lemma** *sum-Un-ge*:
**fixes** $f :: - \Rightarrow real$
**assumes** *finite M finite N* $\forall B \in M \cup N.\ 0 < f\ B$
**shows** $sum\ f\ M \leq sum\ f\ (M \cup N)$
**proof** −
**have** $0 \leq sum\ f\ N - sum\ f\ (M \cap N)$
**using** *assms* **by** (*smt* (*verit*) *DiffD1 inf.cobounded2 UnCI sum-mono2*)
**then have** $sum\ f\ M \leq sum\ f\ M + sum\ f\ N - sum\ f\ (M \cap N)$
**by** *simp*
**also have** $... = sum\ f\ (M \cup N)$
**using** *sum-Un*[*OF assms*(*1*,*2*), *symmetric*] **.**
**finally show** *?thesis* **.**
**qed**

**definition** *bij-exists* :: *'a set set* ⇒ *'a set* ⇒ *bool* **where**
*bij-exists P V* = $(\exists f.$ *bij-betw f P V* $\wedge$ $(\forall B \in P.\ W\ B + w\ (f\ B) > c))$

**lemma** *$P_1$-lower-bound-card*:
**assumes** *bp P inv$_1$ $P_1$ $P_2$ $B_1$ $B_2$ $(S \cup L)$ bij-exists $P_1$ $(\bigcup(P_2 \cup wrap\ B_2))$*
**shows** *card $P_1$ + 1 ≤ card P*
**proof** (*cases ‹$P_1 = \{\}$›*)
**case** *True*
**have** *finite P* **using** *assms(1)* **by** *simp*
**then have** *1 ≤ card P* **using** *bp-NE[OF assms(1)]*
**by** (*metis Nat.add-0-right Suc-diff-1 Suc-le-mono card-gt-0-iff le0 mult-Suc-right nat-mult-1*)
**then show** *?thesis* **unfolding** *True* **by** *simp*
**next**
**note** *invrules = inv$_1$E[OF assms(2)]*
**case** *False*
**obtain** *f* **where** *f-def*: *bij-betw f $P_1$ $(\bigcup(P_2 \cup wrap\ B_2))$ $\forall B \in P_1$. W B + w (f B) > c*
**using** *assms(3)* **unfolding** *bij-exists-def* **by** *blast*
**have** *FINITE*: *finite $P_1$ finite $(P_2 \cup wrap\ B_2)$ finite $(P_1 \cup P_2 \cup wrap\ B_2)$ finite $(wrap\ B_1 \cup \{\{v\}\ |v.\ v \in S \cup L\})$*
**using** *inv$_1$E(1)[OF assms(2)] bp-sol-finite* **by** *blast+*

**have** *F*: $\forall B \in P_2 \cup wrap\ B_2$. *finite B* **using** *invrules(1)* **by** *simp*
**have** *D*: $\forall A \in P_2 \cup wrap\ B_2.\ \forall B \in P_2 \cup wrap\ B_2.\ A \neq B \longrightarrow A \cap B = \{\}$
**using** *bpE(1)[OF invrules(1)]* **unfolding** *pairwise-def disjnt-def* **by** *auto*
**have** *sum-eq*: $W\ (\bigcup\ (P_2 \cup wrap\ B_2)) = (\sum B \in P_2 \cup wrap\ B_2.\ W\ B)$
**using** *sum.Union-disjoint[OF F D]* **by** *auto*

**have** $\forall B \in P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in S \cup L\}.\ 0 < W\ B$
**using** *bpE(2,3)[OF invrules(1)] weight* **by** (*metis* (*no-types*, *lifting*) *UnionI bp-bins-finite invrules(1) sum-pos*)
**then have** $(\sum B \in P_1 \cup P_2 \cup wrap\ B_2.\ W\ B) \leq (\sum B \in P_1 \cup P_2 \cup wrap\ B_2 \cup (wrap\ B_1 \cup \{\{v\}\ |v.\ v \in S \cup L\}).\ W\ B)$
**using** *sum-Un-ge[OF FINITE(3,4), of W]* **by** *blast*
**also have** ... $= (\sum B \in P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in S \cup L\}.\ W\ B)$ **by** (*smt* (*verit*) *Un-assoc Un-commute*)
**also have** ... $= W\ U$ **using** *sum-Un-eq-sum-sum[OF invrules(1), symmetric]* **.**
**finally have** $*$: $(\sum B \in P_1 \cup P_2 \cup wrap\ B_2.\ W\ B) \leq W\ U$ **.**
**have** *DISJNT*: $P_1 \cap (P_2 \cup wrap\ B_2) = \{\}$ **using** *invrules(5)* **by** *blast*

— This part of the proof is based on the proof on page 72 of the article [2].
**have** *c ∗ card $P_1$* $= (\sum B \in P_1.\ c)$ **by** *simp*
**also have** ... $< (\sum B \in P_1.\ W\ B + w\ (f\ B))$
**using** *f-def(2) sum-strict-mono[OF FINITE(1) False]* **by** *fastforce*
**also have** ... $= (\sum B \in P_1.\ W\ B) + (\sum B \in P_1.\ w\ (f\ B))$
**by** (*simp add*: *Groups-Big.comm-monoid-add-class.sum.distrib*)
**also have** ... $= (\sum B \in P_1.\ W\ B) + W\ (\bigcup\ (P_2 \cup wrap\ B_2))$ **unfolding** *sum.reindex-bij-betw[OF f-def(1), of w]* **..**
**also have** ... $= (\sum B \in P_1.\ W\ B) + (\sum B \in P_2 \cup wrap\ B_2.\ W\ B)$ **unfolding** *sum-eq* **..**
**also have** ... $= (\sum B \in P_1 \cup P_2 \cup wrap\ B_2.\ W\ B)$ **using** *sum.union-disjoint[OF*

*FINITE*(*1*,*2*) *DISJNT*, *of W*] **by** (*simp add*: *Un-assoc*)
**also have** ... $\leq (\sum u \in U.\ w\ u)$ **using** $*$ **.**
**also have** ... $\leq c * card\ P$ **using** *sum-lower-bound-card*[*OF assms*(*1*)] **.**
**finally show** *?thesis*
**by** (*simp flip*: *of-nat-mult*)
**qed**

**lemma** $P_1$-$B_1$-*lower-bound-card*:
**assumes** *bp P* $inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ $(S \cup L)$ *bij-exists* $P_1$ $(\bigcup (P_2 \cup wrap\ B_2))$
**shows** *card* $(P_1 \cup wrap\ B_1) \leq card\ P$
**proof** −
**have** *card* $(P_1 \cup wrap\ B_1) \leq card\ P_1 + card\ (wrap\ B_1)$
**using** *card-Un-le* **by** *blast*
**also have** ... $\leq card\ P_1 + 1$ **using** *wrap-card* **by** *simp*
**also have** ... $\leq card\ P$ **using** $P_1$-*lower-bound-card*[*OF assms*] **.**
**finally show** *?thesis* **.**
**qed**

**lemma** $P_2$-$B_2$-*lower-bound-*$P_1$:
**assumes** $inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ $(S \cup L)$ $2 * card\ P_2 \leq card\ (\bigcup P_2)$ *bij-exists* $P_1$ $(\bigcup (P_2 \cup wrap\ B_2))$
**shows** $2 * card\ (P_2 \cup wrap\ B_2) \leq card\ P_1 + 1$
**proof** −
**note** $invrules = inv_1E$[*OF assms*(*1*)] **and** *bprules* = *bpE*[*OF invrules*(*1*)]

**have** *pairwise disjnt* $(P_2 \cup wrap\ B_2)$
**using** *bprules*(*1*) *pairwise-subset* **by** *blast*
**moreover have** $B_2 \notin P_2$ **using** *invrules*(*4*) **by** *simp*
**ultimately have** *DISJNT*: $\bigcup P_2 \cap B_2 = \{\}$
**by** (*auto*, *metis* (*no-types*, *opaque-lifting*) *sup-bot.right-neutral Un-insert-right disjnt-iff mk-disjoint-insert pairwise-insert wrap-Un*)

**have** *finite* $(\bigcup P_2)$ **using** *U-Finite bprules*(*3*) **by** *auto*
**have** *finite* $B_2$ **using** *bp-bins-finite*[*OF invrules*(*1*)] *wrap-not-empty* **by** *blast*
**have** *finite* $P_2$ *finite* $(wrap\ B_2)$ **using** *bp-sol-finite*[*OF invrules*(*1*)] **by** *blast*+
**have** *DISJNT2*: $P_2 \cap wrap\ B_2 = \{\}$ **unfolding** *wrap-def* **using** ‹$B_2 \notin P_2$› **by** *auto*
**have** *card* $(wrap\ B_2) \leq card\ B_2$
**proof** (*cases* ‹$B_2 = \{\}$›)
**case** *False*
**then have** $1 \leq card\ B_2$ **by** (*simp add*: *leI* ‹*finite* $B_2$›)
**then show** *?thesis* **using** *wrap-card*[*of* $B_2$] **by** *linarith*
**qed** *simp*

— This part of the proof is based on the proof on page 73 of the article [2].
**from** *assms*(*2*) **have** $2 * card\ P_2 + 2 * card\ (wrap\ B_2) \leq card\ (\bigcup P_2) + card\ (wrap\ B_2) + 1$
**using** *wrap-card*[*of* $B_2$] **by** *linarith*
**then have** $2 * (card\ P_2 + card\ (wrap\ B_2)) \leq card\ (\bigcup P_2) + card\ B_2 + 1$

**using** ‹*card* (*wrap* $B_2$) ≤ *card* $B_2$› **by** *simp*
**then have** *2* ∗ (*card* ($P_2$ ∪ *wrap* $B_2$)) ≤ *card* (⋃ $P_2$ ∪ $B_2$) + *1*
**using** *card-Un-disjoint*[*OF* ‹*finite* (⋃ $P_2$)› ‹*finite* $B_2$› *DISJNT*]
**and** *card-Un-disjoint*[*OF* ‹*finite* $P_2$› ‹*finite* (*wrap* $B_2$)› *DISJNT2*] **by** *argo*
**then have** *2* ∗ (*card* ($P_2$ ∪ *wrap* $B_2$)) ≤ *card* (⋃ ($P_2$ ∪ *wrap* $B_2$)) + *1*
**by** (*cases* ‹$B_2$ = {}›) (*auto simp*: *Un-commute*)
**then show** *2* ∗ (*card* ($P_2$ ∪ *wrap* $B_2$)) ≤ *card* $P_1$ + *1*
**using** *assms*(*3*) *bij-betw-same-card* **unfolding** *bij-exists-def* **by** *metis*
**qed**

We add *SL S L* to $inv_2$ to ensure that the $S$ and $L$ sets only contain objects with correct weights.

**definition** $inv_2$ :: *'a set set* ⇒ *'a set set* ⇒ *'a set* ⇒ *'a set* ⇒ *'a set* ⇒ *'a set* ⇒ *bool* **where**
$inv_2$ $P_1$ $P_2$ $B_1$ $B_2$ $S$ $L$ ⟷ $inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ ($S$ ∪ $L$) — $inv_1$ holds for the partial solution
∧ ($L$ ≠ {} ⟶ (∀ $B$ ∈ $P_1$ ∪ *wrap* $B_1$. $B$ ∩ $L_U$ ≠ {})) — If there are still large objects left, then every bin of the first partial solution must contain a large object
∧ *bij-exists* $P_1$ (⋃ ($P_2$ ∪ *wrap* $B_2$)) — There exists a bijective function between the bins of the first partial solution and the objects of the second one
∧ (*2* ∗ *card* $P_2$ ≤ *card* (⋃ $P_2$)) — There are at most twice as many bins in $P_2$ as there are objects in $P_2$
∧ *SL S L* — $S$ and $L$ are subsets of $S_U$ and $L_U$

**lemma** $inv_2E$:
**assumes** $inv_2$ $P_1$ $P_2$ $B_1$ $B_2$ $S$ $L$
**shows** $inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ ($S$ ∪ $L$)
**and** $L$ ≠ {} ⟹ ∀ $B$ ∈ $P_1$ ∪ *wrap* $B_1$. $B$ ∩ $L_U$ ≠ {}
**and** *bij-exists* $P_1$ (⋃ ($P_2$ ∪ *wrap* $B_2$))
**and** *2* ∗ *card* $P_2$ ≤ *card* (⋃ $P_2$)
**and** *SL S L*
**using** *assms* **unfolding** $inv_2$*-def* **by** *blast*+

**lemma** $inv_2I$:
**assumes** $inv_1$ $P_1$ $P_2$ $B_1$ $B_2$ ($S$ ∪ $L$)
**and** $L$ ≠ {} ⟹ ∀ $B$ ∈ $P_1$ ∪ *wrap* $B_1$. $B$ ∩ $L_U$ ≠ {}
**and** *bij-exists* $P_1$ (⋃ ($P_2$ ∪ *wrap* $B_2$))
**and** *2* ∗ *card* $P_2$ ≤ *card* (⋃ $P_2$)
**and** *SL S L*
**shows** $inv_2$ $P_1$ $P_2$ $B_1$ $B_2$ $S$ $L$
**using** *assms* **unfolding** $inv_2$*-def* **by** *blast*

**lemma** *bin-packing-lower-bound-card*:
**assumes** $S$ = {} $inv_2$ $P_1$ $P_2$ $B_1$ $B_2$ $S$ $L$ *bp P*
**shows** *card* ($P_1$ ∪ *wrap* $B_1$ ∪ $P_2$ ∪ *wrap* $B_2$ ∪ {{$v$} |$v$. $v$ ∈ $S$ ∪ $L$}) ≤ *3* / *2* ∗ *card P*
**proof** (*cases* ‹$L$ = {}›)

**note** *invrules* = $inv_2E$[*OF assms*(*2*)]
**case** *True*
**then have** *card* ($P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in S \cup L\}$)
= *card* ($P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $B_2$) **using** *assms*(*1*) **by** *simp*
**also have** ... $\leq$ *card* ($P_1 \cup$ *wrap* $B_1$) + *card* ($P_2 \cup$ *wrap* $B_2$)
**using** *card-Un-le*[*of* ‹$P_1 \cup$ *wrap* $B_1$›] **by** (*simp add*: *Un-assoc*)
**also have** ... $\leq$ *card* $P$ + *card* ($P_2 \cup$ *wrap* $B_2$)
**using** $P_1$-$B_1$-*lower-bound-card*[*OF assms*(*3*) *invrules*(*1*,*3*)] **by** *simp*
**also have** ... $\leq$ *card* $P$ + *card* $P$ / *2*
**using** $P_2$-$B_2$-*lower-bound-*$P_1$[*OF invrules*(*1*,*4*,*3*)]
**and** $P_1$-*lower-bound-card*[*OF assms*(*3*) *invrules*(*1*,*3*)] **by** *linarith*
**finally show** *?thesis* **by** *linarith*
**next**
**note** *invrules* = $inv_2E$[*OF assms*(*2*)]
**case** *False*
**have** *card* ($P_1 \cup$ *wrap* $B_1 \cup P_2 \cup$ *wrap* $B_2 \cup \{\{v\} \mid v.\ v \in S \cup L\}$)
= *card* ($P_1 \cup$ *wrap* $B_1 \cup \{\{v\} \mid v.\ v \in L\} \cup P_2 \cup$ *wrap* $B_2$)
**using** *assms*(*1*) **by** (*simp add*: *Un-commute Un-assoc*)
**also have** ... $\leq$ *card* ($P_1 \cup$ *wrap* $B_1 \cup \{\{v\} \mid v.\ v \in L\}$) + *card* ($P_2 \cup$ *wrap* $B_2$)
**using** *card-Un-le*[*of* ‹$P_1 \cup$ *wrap* $B_1 \cup \{\{v\} \mid v.\ v \in L\}$›] **by** (*simp add*: *Un-assoc*)
**also have** ... $\leq$ *card* $P$ + *card* ($P_2 \cup$ *wrap* $B_2$)
**using** *L-bins-lower-bound-card*[*OF assms*(*3*) *invrules*(*1*) *invrules*(*2*)[*OF False*]
*invrules*(*5*)] **by** *linarith*
**also have** ... $\leq$ *card* $P$ + *card* $P$ / *2*
**using** $P_2$-$B_2$-*lower-bound-*$P_1$[*OF invrules*(*1*,*4*,*3*)]
**and** $P_1$-*lower-bound-card*[*OF assms*(*3*) *invrules*(*1*,*3*)] **by** *linarith*
**finally show** *?thesis* **by** *linarith*
**qed**

**definition** $inv_3$ :: *'a set set* $\Rightarrow$ *'a set set* $\Rightarrow$ *'a set* $\Rightarrow$ *'a set* $\Rightarrow$ *'a set* $\Rightarrow$ *'a set* $\Rightarrow$ *bool* **where**
$inv_3\ P_1\ P_2\ B_1\ B_2\ S\ L \longleftrightarrow inv_2\ P_1\ P_2\ B_1\ B_2\ S\ L \wedge B_2 \subseteq S_U$

**lemma** $inv_3E$:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ S\ L$
**shows** $inv_2\ P_1\ P_2\ B_1\ B_2\ S\ L$ **and** $B_2 \subseteq S_U$
**using** *assms* **unfolding** $inv_3$-*def* **by** *blast*+

**lemma** $inv_3I$:
**assumes** $inv_2\ P_1\ P_2\ B_1\ B_2\ S\ L$ **and** $B_2 \subseteq S_U$
**shows** $inv_3\ P_1\ P_2\ B_1\ B_2\ S\ L$
**using** *assms* **unfolding** $inv_3$-*def* **by** *blast*

**lemma** *loop-init*:
$inv_3$ {} {} {} {} $S_U$ $L_U$
**proof** −
**have** $S_U \cup L_U = U$ **by** *auto*
**then have** ∗: $inv_1$ {} {} {} {} ($S_U \cup L_U$)
**unfolding** *bp-def partition-on-def pairwise-def wrap-def* $inv_1$-*def*

**using** *weight* **by** *auto*
**have** *bij-exists* {} ($\bigcup$ ({} $\cup$ *wrap* {}))
**using** *bij-betwI'* **unfolding** *bij-exists-def* **by** *fastforce*
**from** $inv_2I[OF * \text{ - } this]$ **have** $inv_2$ {} {} {} {} $S_U$ $L_U$ **by** *auto*
**from** $inv_3I[OF\ this]$ **show** *?thesis* **by** *blast*
**qed**

**lemma** *loop-stepA*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ S\ L\ B_1 = \{\}\ L = \{\}\ u \in S$
**shows** $inv_3\ P_1\ P_2\ \{u\}\ B_2\ (S - \{u\})\ L$
**proof** −
**note** $invrules = inv_2E[OF\ inv_3E(1)[OF\ assms(1)]]$
**have** *WEIGHT*: $W\ B_1 + w\ u \leq c$ **using** *invrules*(5) *assms*(2,4) **by** *fastforce*
**from** *assms*(4) **have** $u \in S \cup L$ **by** *blast*
**from** $inv_1$*-stepA*[*OF invrules*(1) *this WEIGHT*] *assms*(2,3) **have** *1*: $inv_1\ P_1\ P_2$ $\{u\}\ B_2\ (S - \{u\} \cup L)$ **by** *simp*
**have** *2*: $L \neq \{\} \Longrightarrow \forall\, B{\in}P_1 \cup wrap\ \{u\}.\ B \cap L_U \neq \{\}$ **using** *assms*(3) **by** *blast*
**from** $inv_2I[OF\ 1\ 2]$ *invrules* **have** $inv_2\ P_1\ P_2\ \{u\}\ B_2\ (S - \{u\})\ L$ **by** *blast*
**from** $inv_3I[OF\ this]$ **show** *?thesis* **using** $inv_3E(2)[OF\ assms(1)]$ **.**
**qed**

**lemma** *loop-stepB*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ S\ L\ B_1 = \{\}\ u \in L$
**shows** $inv_3\ P_1\ P_2\ \{u\}\ B_2\ S\ (L - \{u\})$
**proof** −
**note** $invrules = inv_2E[OF\ inv_3E(1)[OF\ assms(1)]]$
**have** *WEIGHT*: $W\ B_1 + w\ u \leq c$ **using** *weight invrules*(5) *assms*(2,3) **by** *fastforce*

— This observation follows from the fact that the $S$ and $L$ sets have to be disjoint from each other, and allows us to reuse our proofs of the preservation of $inv_1$ by simply replacing $V$ with $S \cup L$
**have** $*$: $S \cup L - \{u\} = S \cup (L - \{u\})$ **using** *invrules*(5) *assms*(3) **by** *force*
**from** *assms*(3) **have** $u \in S \cup L$ **by** *blast*
**from** $inv_1$*-stepA*[*OF invrules*(1) *this WEIGHT*] *assms*(2) $*$ **have** *1*: $inv_1\ P_1\ P_2$ $\{u\}\ B_2\ (S \cup (L - \{u\}))$ **by** *simp*
**have** $\forall\, B{\in}P_1.\ B \cap L_U \neq \{\}\ \{u\} \cap L_U \neq \{\}$ **using** *assms*(3) *invrules*(2,5) **by** *blast+*
**then have** *2*: $L \neq \{\} \Longrightarrow \forall\, B{\in}P_1 \cup wrap\ \{u\}.\ B \cap L_U \neq \{\}$
**using** *assms*(3) **by** (*metis* (*full-types*) *Un-iff empty-iff insert-iff wrap-not-empty*)
**from** $inv_2I[OF\ 1\ 2]$ *invrules* **have** $inv_2\ P_1\ P_2\ \{u\}\ B_2\ S\ (L - \{u\})$ **by** *blast*
**from** $inv_3I[OF\ this]$ **show** *?thesis* **using** $inv_3E(2)[OF\ assms(1)]$ **.**
**qed**

**lemma** *loop-stepC*:
**assumes** $inv_3\ P_1\ P_2\ B_1\ B_2\ S\ L\ B_1 \neq \{\}\ u \in S\ W\ B_1 + w(u) \leq c$
**shows** $inv_3\ P_1\ P_2\ (B_1 \cup \{u\})\ B_2\ (S - \{u\})\ L$
**proof** −
**note** $invrules = inv_2E[OF\ inv_3E(1)[OF\ assms(1)]]$

— Same approach, but removing $\{u\}$ from $S$ instead of $L$
**have** $*$: $S \cup L - \{u\} = (S - \{u\}) \cup L$ **using** *invrules(5) assms(3)* **by** *force*
**from** *assms(3)* **have** $u \in S \cup L$ **by** *blast*
**from** *inv$_1$-stepA[OF invrules(1) this assms(4)]* $*$ **have** *1*: *inv$_1$* $P_1$ $P_2$ $(B_1 \cup \{u\})$ $B_2$ $(S - \{u\} \cup L)$ **by** *simp*
**have** $L \neq \{\} \Longrightarrow \forall\, B{\in}P_1 \cup \mathit{wrap}\ B_1.\ B \cap L_U \neq \{\}$ **using** *invrules(2)* **by** *blast*
**then have** *2*: $L \neq \{\} \Longrightarrow \forall\, B{\in}P_1 \cup \mathit{wrap}\ (B_1 \cup \{u\}).\ B \cap L_U \neq \{\}$
**by** (*smt* (*verit*) *Int-insert-left Un-empty-right Un-iff Un-insert-right assms(2) insert-not-empty singletonD singletonI wrap-def*)
**from** *inv$_2$I[OF 1 2] invrules* **have** *inv$_2$* $P_1$ $P_2$ $(B_1 \cup \{u\})$ $B_2$ $(S - \{u\})$ $L$ **by** *blast*
**from** *inv$_3$I[OF this]* **show** *?thesis* **using** *inv$_3$E(2)[OF assms(1)]* **.**
**qed**

**lemma** *loop-stepD*:
**assumes** *inv$_3$* $P_1$ $P_2$ $B_1$ $B_2$ $S$ $L$ $B_1 \neq \{\}$ $u \in S$ $W\ B_1 + w(u) > c$ $W\ B_2 + w(u) \leq c$
**shows** *inv$_3$* $(P_1 \cup \mathit{wrap}\ B_1)$ $P_2$ $\{\}$ $(B_2 \cup \{u\})$ $(S - \{u\})$ $L$
**proof** −
**note** *invrules* = *inv$_2$E[OF inv$_3$E(1)[OF assms(1)]]*
**have** $*$: $S \cup L - \{u\} = (S - \{u\}) \cup L$ **using** *invrules(5) assms(3)* **by** *force*
**from** *assms(3)* **have** $u \in S \cup L$ **by** *blast*
**from** *inv$_1$-stepB[OF invrules(1) this assms(5)]* $*$ **have** *1*: *inv$_1$* $(P_1 \cup \mathit{wrap}\ B_1)$ $P_2$ $\{\}$ $(B_2 \cup \{u\})$ $(S - \{u\} \cup L)$ **by** *simp*

**have** *2*: $L \neq \{\} \Longrightarrow \forall\, B{\in}P_1 \cup \mathit{wrap}\ B_1 \cup \mathit{wrap}\ \{\}.\ B \cap L_U \neq \{\}$
**using** *invrules(2)* **unfolding** *wrap-empty* **by** *blast*

**from** *invrules(3)* **obtain** *f* **where** *f-def*: *bij-betw f* $P_1$ $(\bigcup\ (P_2 \cup \mathit{wrap}\ B_2))$ $\forall\, B{\in}P_1.\ c < W\ B + w\ (f\ B)$ **unfolding** *bij-exists-def* **by** *blast*
**have** $B_1 \notin P_1$ **using** *inv$_1$E(3)[OF invrules(1)]* **by** *blast*
**have** $u \notin (\bigcup\ (P_2 \cup \mathit{wrap}\ B_2))$ **using** *inv$_1$E(2)[OF invrules(1)] assms(3)* **by** *blast*
**then have** $(\bigcup\ (P_2 \cup \mathit{wrap}\ (B_2 \cup \{u\}))) = (\bigcup\ (P_2 \cup \mathit{wrap}\ B_2 \cup \{\{u\}\}))$
**by** (*metis Sup-empty Un-assoc Union-Un-distrib ccpo-Sup-singleton wrap-empty wrap-not-empty*)
**also have** ... $= (\bigcup\ (P_2 \cup \mathit{wrap}\ B_2)) \cup \{u\}$ **by** *simp*
**finally have** *UN*: $(\bigcup\ (P_2 \cup \mathit{wrap}\ (B_2 \cup \{u\}))) = (\bigcup\ (P_2 \cup \mathit{wrap}\ B_2)) \cup \{u\}$ **.**
**have** *wrap* $B_1 = \{B_1\}$ **using** *wrap-not-empty[of* $B_1$*] assms(2)* **by** *simp*
**let** *?f* $= f\ (B_1 := u)$
**have** *BIJ*: *bij-betw ?f* $(P_1 \cup \mathit{wrap}\ B_1)$ $(\bigcup\ (P_2 \cup \mathit{wrap}\ (B_2 \cup \{u\})))$
**unfolding** *wrap-empty* ‹*wrap* $B_1 = \{B_1\}$› *UN* **using** *f-def(1)* ‹$B_1 \notin P_1$› ‹$u \notin (\bigcup\ (P_2 \cup \mathit{wrap}\ B_2))$›
**by** (*metis* (*no-types*, *lifting*) *bij-betw-cong fun-upd-other fun-upd-same notIn-Un-bij-betw3*)
**have** $c < W\ B_1 + w\ (?f\ B_1)$ **using** *assms(4)* **by** *simp*
**then have** $(\forall\, B{\in}P_1 \cup \mathit{wrap}\ B_1.\ c < W\ B + w\ (?f\ B))$
**unfolding** ‹*wrap* $B_1 = \{B_1\}$› **using** *f-def(2)* **by** *simp*
**with** *BIJ* **have** *bij-betw ?f* $(P_1 \cup \mathit{wrap}\ B_1)$ $(\bigcup\ (P_2 \cup \mathit{wrap}\ (B_2 \cup \{u\})))$

$\wedge$ ($\forall$ $B$∈$P_1$ $\cup$ *wrap* $B_1$. $c$ < $W$ $B$ + $w$ (*?f* $B$)) **by** *blast*
**then have** *3*: *bij-exists* ($P_1$ $\cup$ *wrap* $B_1$) ($\bigcup$ ($P_2$ $\cup$ *wrap* ($B_2$ $\cup$ {$u$})))
**unfolding** *bij-exists-def* **by** *blast*
**from** *inv*$_2$*I*[*OF 1 2 3*] **have** *inv*$_2$ ($P_1$ $\cup$ *wrap* $B_1$) $P_2$ {} ($B_2$ $\cup$ {$u$}) ($S$ − {$u$}) $L$ **using** *invrules*(*4*,*5*) **by** *blast*

**from** *inv*$_3$*I*[*OF this*] **show** *?thesis* **using** *inv*$_3$*E*(*2*)[*OF assms*(*1*)] *assms*(*3*) *invrules*(*5*) **by** *blast*
**qed**

**lemma** *B*$_2$*-at-least-two-objects*:
**assumes** *inv*$_3$ $P_1$ $P_2$ $B_1$ $B_2$ $S$ $L$ $u$ ∈ $S$ $W$ $B_2$ + $w$($u$) > $c$
**shows** *2* ≤ *card* $B_2$
**proof** (*rule ccontr*, *simp add*: *not-le*)
**have** *FINITE*: *finite* $B_2$ **using** *inv*$_1$*E*(*1*)[*OF inv*$_2$*E*(*1*)[*OF inv*$_3$*E*(*1*)[*OF assms*(*1*)]]]
**by** (*metis* (*no-types*, *lifting*) *Finite-Set.finite.simps U-Finite Union-Un-distrib bpE*(*3*) *ccpo-Sup-singleton finite-Un wrap-not-empty*)
**assume** *card* $B_2$ < *2*
**then consider** (*0*) *card* $B_2$ = *0* | (*1*) *card* $B_2$ = *1* **by** *linarith*
**then show** *False* **proof** *cases*
**case** *0* **then have** $B_2$ = {} **using** *FINITE* **by** *simp*
**then show** *?thesis* **using** *assms*(*2*,*3*) *inv*$_2$*E*(*5*)[*OF inv*$_3$*E*(*1*)[*OF assms*(*1*)]] **by** *force*
**next**
**case** *1* **then obtain** $v$ **where** $B_2$ = {$v$}
**using** *card-1-singletonE* **by** *auto*
**with** *inv*$_3$*E*(*2*)[*OF assms*(*1*)] **have** *2* ∗ $w$ $v$ ≤ $c$ **using** *inv*$_2$*E*(*5*)[*OF inv*$_3$*E*(*1*)[*OF assms*(*1*)]] **by** *simp*
**moreover from** ‹$B_2$ = {$v$}› **have** $W$ $B_2$ = $w$ $v$ **by** *simp*
**ultimately show** *?thesis* **using** *assms*(*2*,*3*) *inv*$_2$*E*(*5*)[*OF inv*$_3$*E*(*1*)[*OF assms*(*1*)]] **by** *force*
**qed**
**qed**

**lemma** *loop-stepE*:
**assumes** *inv*$_3$ $P_1$ $P_2$ $B_1$ $B_2$ $S$ $L$ $B_1$ ≠ {} $u$ ∈ $S$ $W$ $B_1$ + $w$($u$) > $c$ $W$ $B_2$ + $w$($u$) > $c$
**shows** *inv*$_3$ ($P_1$ $\cup$ *wrap* $B_1$) ($P_2$ $\cup$ *wrap* $B_2$) {} {$u$} ($S$ − {$u$}) $L$
**proof** −
**note** *invrules* = *inv*$_2$*E*[*OF inv*$_3$*E*(*1*)[*OF assms*(*1*)]]
**have** ∗: $S$ $\cup$ $L$ − {$u$} = ($S$ − {$u$}) $\cup$ $L$ **using** *invrules*(*5*) *assms*(*3*) **by** *force*
**from** *assms*(*3*) **have** $u$ ∈ $S$ $\cup$ $L$ **by** *blast*
**from** *inv*$_1$*-stepC*[*OF invrules*(*1*) *this*] ∗ **have** *1*: *inv*$_1$ ($P_1$ $\cup$ *wrap* $B_1$) ($P_2$ $\cup$ *wrap* $B_2$) {} {$u$} ($S$ − {$u$} $\cup$ $L$) **by** *simp*

**have** *2*: $L$ ≠ {} ⟹ $\forall$ $B$∈$P_1$ $\cup$ *wrap* $B_1$ $\cup$ *wrap* {}. $B$ $\cap$ $L_U$ ≠ {}
**using** *invrules*(*2*) **unfolding** *wrap-empty* **by** *blast*

**from** *invrules*(*3*) **obtain** $f$ **where** *f-def*: *bij-betw* $f$ $P_1$ ($\bigcup$ ($P_2$ $\cup$ *wrap* $B_2$))

$\forall B{\in}P_1.\ c < W\ B + w\ (f\ B)$ **unfolding** *bij-exists-def* **by** *blast*
**have** $B_1 \notin P_1$ **using** *inv*$_1$*E(3)[OF invrules(1)]* **by** *blast*
**have** $u \notin (\bigcup\ (P_2 \cup wrap\ B_2))$ **using** *inv*$_1$*E(2)[OF invrules(1)] assms(3)* **by** *blast*
**have** $(\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\})) = (\bigcup\ (P_2 \cup wrap\ B_2 \cup \{\{u\}\}))$ **unfolding** *wrap-def* **by** *simp*
**also have** $... = (\bigcup\ (P_2 \cup wrap\ B_2)) \cup \{u\}$ **by** *simp*
**finally have** *UN*: $(\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\})) = (\bigcup\ (P_2 \cup wrap\ B_2)) \cup \{u\}$ **.**
**have** $wrap\ B_1 = \{B_1\}$ **using** *wrap-not-empty[of* $B_1$*] assms(2)* **by** *simp*
**let** *?f* $= f\ (B_1 := u)$
**have** *BIJ*: *bij-betw ?f* $(P_1 \cup wrap\ B_1)\ (\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\}))$
**unfolding** *wrap-empty* ‹$wrap\ B_1 = \{B_1\}$› *UN* **using** *f-def(1)* ‹$B_1 \notin P_1$› ‹$u \notin (\bigcup\ (P_2 \cup wrap\ B_2))$›
**by** (*metis* (*no-types*, *lifting*) *bij-betw-cong fun-upd-other fun-upd-same notIn-Un-bij-betw3*)
**have** $c < W\ B_1 + w\ (?f\ B_1)$ **using** *assms(4)* **by** *simp*
**then have** $(\forall B{\in}P_1 \cup wrap\ B_1.\ c < W\ B + w\ (?f\ B))$
**unfolding** ‹$wrap\ B_1 = \{B_1\}$› **using** *f-def(2)* **by** *simp*
**with** *BIJ* **have** *bij-betw ?f* $(P_1 \cup wrap\ B_1)\ (\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\}))$
$\wedge\ (\forall B{\in}P_1 \cup wrap\ B_1.\ c < W\ B + w\ (?f\ B))$ **by** *blast*
**then have** *3*: *bij-exists* $(P_1 \cup wrap\ B_1)\ (\bigcup\ (P_2 \cup wrap\ B_2 \cup wrap\ \{u\}))$
**unfolding** *bij-exists-def* **by** *blast*

**have** *4*: $2 * card\ (P_2 \cup wrap\ B_2) \leq card\ (\bigcup\ (P_2 \cup wrap\ B_2))$
**proof** −
**note** *bprules* = *bpE*[*OF inv*$_1$*E(1)*[*OF invrules(1)*]]
**have** *pairwise disjnt* $(P_2 \cup wrap\ B_2)$
**using** *bprules(1) pairwise-subset* **by** *blast*
**moreover have** $B_2 \notin P_2$ **using** *inv*$_1$*E(4)*[*OF invrules(1)*] **by** *simp*
**ultimately have** *DISJNT*: $\bigcup P_2 \cap B_2 = \{\}$
**by** (*auto*, *metis* (*no-types*, *opaque-lifting*) *sup-bot.right-neutral Un-insert-right disjnt-iff mk-disjoint-insert pairwise-insert wrap-Un*)
**have** *finite* $(\bigcup P_2)$ **using** *U-Finite bprules(3)* **by** *auto*
**have** *finite* $B_2$ **using** *inv*$_1$*E(1)*[*OF invrules(1)*] *bp-bins-finite wrap-not-empty* **by** *blast*

**have** $2 * card\ (P_2 \cup wrap\ B_2) \leq 2 * (card\ P_2 + card\ (wrap\ B_2))$
**using** *card-Un-le*[*of* $P_2$ ‹$wrap\ B_2$›] **by** *simp*
**also have** $... \leq 2 * card\ P_2 + 2$ **using** *wrap-card* **by** *auto*
**also have** $... \leq card\ (\bigcup\ P_2) + 2$ **using** *invrules(4)* **by** *simp*
**also have** $... \leq card\ (\bigcup\ P_2) + card\ B_2$ **using** $B_2$*-at-least-two-objects*[*OF assms(1,3,5)*] **by** *simp*
**also have** $... = card\ (\bigcup\ (P_2 \cup \{B_2\}))$ **using** *DISJNT card-Un-disjoint*[*OF* ‹*finite* $(\bigcup P_2)$› ‹*finite* $B_2$›] **by** (*simp add*: *Un-commute*)
**also have** $... = card\ (\bigcup\ (P_2 \cup wrap\ B_2))$ **by** (*cases* ‹$B_2 = \{\}$›) *auto*
**finally show** *?thesis* **.**
**qed**
**from** *inv*$_2$*I*[*OF 1 2 3 4*] **have** *inv*$_2$ $(P_1 \cup wrap\ B_1)\ (P_2 \cup wrap\ B_2)\ \{\}\ \{u\}\ (S - \{u\})\ L$

**using** *invrules*(*5*) **by** *blast*

**from** $inv_3 I[OF\ this]$ **show** *?thesis* **using** *assms*(*3*) *invrules*(*5*) **by** *blast*
**qed**

The bin packing algorithm as it is proposed on page 78 of the article [2]. $P$ will not only be a correct solution of the bin packing problem, but the amount of bins will be a lower bound for *3 / 2* of the amount of bins of any correct solution $Q$, and thus guarantee an approximation factor of *3 / 2* for the optimum.

**lemma** *bp-approx*:
*VARS P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*
{*True*}
*S* := {}; *L*:= {}; *V* := *U*;
*WHILE V* ≠ {} *INV* {*V* ⊆ *U* ∧ *S* = {*u* ∈ *U* − *V*. *w*(*u*) ≤ *c / 2*} ∧ *L* = {*u* ∈ *U* − *V*. *c / 2* < *w*(*u*)}} *DO*
*u* := (*SOME u*. *u* ∈ *V*);
*IF w*(*u*) ≤ *c / 2*
*THEN S* := *S* ∪ {*u*}
*ELSE L* := *L* ∪ {*u*} *FI*;
*V* := *V* − {*u*}
*OD*;
$P_1$ := {}; $P_2$ := {}; $B_1$ := {}; $B_2$ := {};
*WHILE S* ≠ {} *INV* {$inv_3$ $P_1$ $P_2$ $B_1$ $B_2$ *S L*} *DO*
*IF* $B_1$ ≠ {}
*THEN u* := (*SOME u*. *u* ∈ *S*); *S* := *S* − {*u*}
*ELSE IF L* ≠ {}
*THEN u* := (*SOME u*. *u* ∈ *L*); *L* := *L* − {*u*}
*ELSE u* := (*SOME u*. *u* ∈ *S*); *S* := *S* − {*u*} *FI FI*;
*IF* $W(B_1)$ + *w*(*u*) ≤ *c*
*THEN* $B_1$ := $B_1$ ∪ {*u*}
*ELSE IF* $W(B_2)$ + *w*(*u*) ≤ *c*
*THEN* $B_2$ := $B_2$ ∪ {*u*}
*ELSE* $P_2$ := $P_2$ ∪ *wrap* $B_2$; $B_2$ := {*u*} *FI*;
$P_1$ := $P_1$ ∪ *wrap* $B_1$; $B_1$ := {} *FI*
*OD*;
*P* := $P_1$ ∪ *wrap* $B_1$ ∪ $P_2$ ∪ *wrap* $B_2$; *V* := *L*;
*WHILE V* ≠ {}
*INV* {*S* = {} ∧ $inv_3$ $P_1$ $P_2$ $B_1$ $B_2$ *S L* ∧ *V* ⊆ *L* ∧ *P* = $P_1$ ∪ *wrap* $B_1$ ∪ $P_2$ ∪ *wrap* $B_2$ ∪ {{*v*}|*v*. *v* ∈ *L* − *V*}} *DO*
*u* := (*SOME u*. *u* ∈ *V*); *P* := *P* ∪ {{*u*}}; *V* := *V* − {*u*}
*OD*
{*bp P* ∧ (∀ *Q*. *bp Q* ⟶ *card P* ≤ *3 / 2* ∗ *card Q*)}
**proof** (*vcg*, *goal-cases*)
**case** (*1 P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*)
**then show** *?case* **by** *blast*
**next**
**case** (*2 P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*)
**then show** *?case* **by** (*auto simp*: *some-in-eq*)

**next**
**case** (*3 P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*)
**then show** *?case* **using** *loop-init* **by** *force*
**next**
**case** (*4 P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*)
**then have** *INV*: $inv_3$ $P_1$ $P_2$ $B_1$ $B_2$ *S L* **..**
**let** *?s* = *SOME u*. $u \in S$
**let** *?l* = *SOME u*. $u \in L$
**note** *SL-def* = $inv_2E(5)$[*OF* $inv_3E(1)$[*OF INV*]]
**have** *LIN*: $L \neq \{\} \Longrightarrow$ *?l* $\in L$ **using** *some-in-eq* **by** *metis*
**then have** *LWEIGHT*: $L \neq \{\} \Longrightarrow w$ *?l* $\leq c$ **using** *weight SL-def* **by** *blast*
**from** *4* **have** $S \neq \{\}$ **..**
**then have** *IN*: *?s* $\in S$ **using** *some-in-eq* **by** *metis*
**then have** *w ?s* $\leq c$ **using** *SL-def* **by** *auto*
**then show** *?case*
**using** *LWEIGHT loop-stepA*[*OF INV - - IN*] *loop-stepB*[*OF INV - LIN*] *loop-stepC*[*OF INV - IN*]
**and** *loop-stepD*[*OF INV - IN*] *loop-stepE*[*OF INV - IN*] **by** (*cases* ‹$B_1 = \{\}$›, *cases* ‹$L = \{\}$›) *auto*
**next**
**case** (*5 P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*)
**then show** *?case* **by** *blast*
**next**
**case** (*6 P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*)
**then have** $*$: (*SOME u*. $u \in V$) $\in V$ (*SOME u*. $u \in V$) $\in L$ **by** (*auto simp add*: *some-in-eq*)
**then have** $P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in L - (V - \{SOME\ u.\ u \in V\})\}$
$= P_1 \cup wrap\ B_1 \cup P_2 \cup wrap\ B_2 \cup \{\{v\}\ |v.\ v \in L - V \cup \{SOME\ u.\ u \in V\}\}$
**by** *blast*
**with** *6* $*$ **show** *?case* **by** *blast*
**next**
**case** (*7 P* $P_1$ $P_2$ $B_1$ $B_2$ *V S L u*)
**then have** $*$: $inv_2$ $P_1$ $P_2$ $B_1$ $B_2$ *S L*
**using** $inv_3E(1)$ **by** *blast*
**from** $inv_1E(1)$[*OF* $inv_2E(1)$[*OF* $*$]] *7*
**have** *bp P* **by** *fastforce*
**with** *bin-packing-lower-bound-card*[*OF* - $*$] *7*
**show** *?case* **by** *fastforce*
**qed**

**end**

**end**

# 6 Center Selection

**theory** *Center-Selection*

**imports** *Complex-Main HOL−Hoare.Hoare-Logic*
**begin**

The Center Selection (or metric k-center) problem. Given a set of *sites* $S$ in a metric space, find a subset $C \subseteq S$ that minimizes the maximal distance from any $s \in S$ to some $c \in C$. This theory presents a verified 2-approximation algorithm. It is based on Section 11.2 in the book by Kleinberg and Tardos [4]. In contrast to the proof in the book, our proof is a standard invariant proof.

**locale** *Center-Selection* =
  **fixes** $S$ :: ('*a* :: *metric-space*) *set*
    **and** $k$ :: *nat*
  **assumes** *finite-sites*: *finite S*
  **and** *non-empty-sites*: $S \neq \{\}$
**and** *non-zero-k*: $k > 0$
**begin**

**definition** *distance* :: ('*a*::*metric-space*) *set* ⇒ ('*a*::*metric-space*) ⇒ *real* **where**
*distance C s* = *Min* (*dist s* ‘ *C*)

**definition** *radius* :: ('*a* :: *metric-space*) *set* ⇒ *real* **where**
*radius C* = *Max* (*distance C* ‘ *S*)

**lemma** *distance-mono*:
**assumes** $C_1 \subseteq C_2$ **and** $C_1 \neq \{\}$ **and** *finite* $C_2$
**shows** *distance* $C_1$ *s* ≥ *distance* $C_2$ *s*
**by** (*simp add*: *Min.subset-imp assms distance-def image-mono*)

**lemma** *finite-distances*: *finite* (*distance C* ‘ *S*)
  **using** *finite-sites* **by** *simp*

**lemma** *non-empty-distances*: *distance C* ‘ *S* ≠ {}
  **using** *non-empty-sites* **by** *simp*

**lemma** *radius-contained*: *radius C* ∈ *distance C* ‘ *S*
  **using** *finite-distances non-empty-distances Max-in radius-def* **by** *simp*

**lemma** *radius-def2*: ∃ *s* ∈ *S*. *distance C s* = *radius C*
  **using** *radius-contained image-iff* **by** *metis*

**lemma** *dist-lemmas-aux*:
  **assumes** *finite C*
    **and** *C* ≠ {}
  **shows** *finite* (*dist s* ‘ *C*)
    **and** *finite* (*dist s* ‘ *C*) ⟹ *distance C s* ∈ *dist s* ‘ *C*
    **and** *distance C s* ∈ *dist s* ‘ *C* ⟹ ∃ *c* ∈ *C*. *dist s c* = *distance C s*
**and** ∃ *c* ∈ *C*. *dist s c* = *distance C s* ⟹ *distance C s* ≥ *0*
**proof**
  **show** *finite C* **using** *assms*(*1*) **by** *simp*

**next**
**assume** *finite* (*dist s ‘ C*)
**then show** *distance C s* ∈ *dist s ‘ C* **using** *distance-def eq-Min-iff assms*(*2*) **by** *blast*
**next**
**assume** *distance C s* ∈ *dist s ‘ C*
**then show** ∃ *c* ∈ *C*. *dist s c* = *distance C s* **by** *auto*
**next**
**assume** ∃ *c* ∈ *C*. *dist s c* = *distance C s*
**then show** *distance C s* ≥ *0* **by** (*metis zero-le-dist*)
**qed**

**lemma** *dist-lemmas*:
**assumes** *finite C*
**and** *C* ≠ {}
**shows** *finite* (*dist s ‘ C*)
**and** *distance C s* ∈ *dist s ‘ C*
**and** ∃ *c* ∈ *C*. *dist s c* = *distance C s*
**and** *distance C s* ≥ *0*
**using** *dist-lemmas-aux assms* **by** *auto*

**lemma** *radius-max-prop*: (∀ *s* ∈ *S*. *distance C s* ≤ *r*) ⟹ (*radius C* ≤ *r*)
**by** (*metis image-iff radius-contained*)

**lemma** *dist-ins*:
**assumes** ∀ *c*$_1$ ∈ *C*. ∀ *c*$_2$ ∈ *C*. *c*$_1$ ≠ *c*$_2$ ⟶ *x* < *dist c*$_1$ *c*$_2$
**and** *distance C s* > *x*
**and** *finite C*
**and** *C* ≠ {}
**shows** ∀ *c*$_1$ ∈ (*C* ∪ {*s*}). ∀ *c*$_2$ ∈ (*C* ∪ {*s*}). *c*$_1$ ≠ *c*$_2$ ⟶ *x* < *dist c*$_1$ *c*$_2$
**proof** (*rule+*)
**fix** *c*$_1$ *c*$_2$
**assume** *local-assms*: *c*$_1$∈*C* ∪ {*s*} *c*$_2$∈*C* ∪ {*s*} *c*$_1$ ≠ *c*$_2$
**then have** *c*$_1$ ∈ *C* ∧ *c*$_2$ ∈ *C* ∨ *c*$_1$ ∈*C* ∧ *c*$_2$∈ {*s*} ∨ *c*$_2$∈*C* ∧ *c*$_1$ ∈ {*s*} ∨ *c*$_1$ ∈ {*s*} ∧ *c*$_2$∈ {*s*} **by** *auto*
**then show** *x* < *dist c*$_1$ *c*$_2$
**proof** (*elim disjE*)
**assume** *c*$_1$ ∈*C* ∧ *c*$_2$∈*C*
**then show** *?thesis* **using** *assms*(*1*) *local-assms*(*3*) **by** *simp*
**next**
**assume** *case-assm*: *c*$_1$ ∈ *C* ∧ *c*$_2$ ∈ {*s*}
**have** *x* < *distance C c*$_2$ **using** *assms*(*2*) *case-assm* **by** *simp*
**also have** ... ≤ *dist c*$_2$ *c*$_1$
**using** *Min.coboundedI distance-def assms*(*3*,*4*) *dist-lemmas*(*1*, *2*) *case-assm* **by** *simp*
**also have** ... = *dist c*$_1$ *c*$_2$ **using** *dist-commute* **by** *metis*
**finally show** *?thesis* **.**
**next**
**assume** *case-assm*: *c*$_2$ ∈ *C* ∧ *c*$_1$ ∈ {*s*}

**have** $x <$ *distance* $C\ c_1$ **using** *assms*(*2*) *case-assm* **by** *simp*
**also have** $\ldots \leq$ *dist* $c_1\ c_2$
**using** *Min.coboundedI distance-def assms*(*3*,*4*) *dist-lemmas*(*1*, *2*) *case-assm*
**by** *simp*
**finally show** *?thesis* **.**
**next**
**assume** $c_1 \in \{s\} \wedge c_2 \in \{s\}$
**then have** *False* **using** *local-assms* **by** *simp*
**then show** *?thesis* **by** *simp*
**qed**
**qed**

## 6.1 A Preliminary Algorithm and Proof

This subsection verifies an auxiliary algorithm by Kleinberg and Tardos. Our proof of the main algorithm does not does not rely on this auxiliary algorithm at all but we do reuse part off its invariant proof later on.

**definition** *inv* :: ('*a* :: *metric-space*) *set* ⇒ ('*a* :: *metric-space set*) ⇒ *real* ⇒ *bool*
**where**
*inv* $S'\ C\ r =$
$((\forall s \in (S - S').\ distance\ C\ s \leq 2{*}r) \wedge S' \subseteq S \wedge C \subseteq S \wedge$
$(\forall c \in C.\ \forall s \in S'.\ S' \neq \{\} \longrightarrow dist\ c\ s > 2 * r) \wedge (S' = S \vee C \neq \{\}) \wedge$
$(\forall c_1 \in C.\ \forall c_2 \in C.\ c_1 \neq c_2 \longrightarrow dist\ c_1\ c_2 > 2 * r))$

**lemma** *inv-init*: *inv* $S\ \{\}\ r$
**unfolding** *inv-def non-empty-sites* **by** *simp*
**lemma** *inv-step*:
**assumes** $S' \neq \{\}$
**and** *IH*: *inv* $S'\ C\ r$
**defines**[*simp*]: $s \equiv (SOME\ s.\ s \in S')$
**shows** *inv* $(S' - \{s' .\ s' \in S' \wedge dist\ s\ s' \leq 2{*}r\})\ (C \cup \{s\})\ r$
**proof** −
**have** *s-def*: $s \in S'$ **using** *assms*(*1*) *some-in-eq* **by** *auto*

**have** *finite* $(C \cup \{s\})$ **using** *IH finite-subset*[*OF - finite-sites*] **by** (*simp add*: *inv-def*)

**moreover**

**have** $(\forall s' \in (S - (S' - \{s' .\ s' \in S' \wedge dist\ s\ s' \leq 2{*}r\}))).\ distance\ (C \cup \{s\})$ $s' \leq 2{*}r)$
**proof**
**fix** $s''$
**assume** $s'' \in S - (S' - \{s' .\ s' \in S' \wedge dist\ s\ s' \leq 2{*}r\})$
**then have** $s'' \in S - S' \vee s'' \in \{s' .\ s' \in S' \wedge dist\ s\ s' \leq 2{*}r\}$ **by** *simp*
**then show** *distance* $(C \cup \{s\})\ s'' \leq 2 * r$
**proof** (*elim disjE*)
**assume** *local-assm*: $s'' \in S - S'$
**have** $S' = S \vee C \neq \{\}$ **using** *IH* **by** (*simp add*: *inv-def*)

**then show** *?thesis*
**proof** (*elim disjE*)
**assume** $S' = S$
**then have** $s'' \in \{\}$ **using** *local-assm* **by** *simp*
**then show** *?thesis* **by** *simp*
**next**
**assume** *C-not-empty*: $C \neq \{\}$
**have** *finite C* **using** *IH finite-subset*[*OF - finite-sites*] **by** (*simp add*: *inv-def*)
**then have** *distance* $(C \cup \{s\})$ $s'' \leq$ *distance C* $s''$
**using** *distance-mono C-not-empty* **by** (*meson Un-upper1 calculation*)
**also have** $\ldots \leq 2 * r$ **using** *IH local-assm inv-def* **by** *simp*
**finally show** *?thesis* **.**
**qed**
**next**
**assume** *local-assm*: $s'' \in \{s' \,.\, s' \in S' \wedge$ *dist s* $s' \leq 2{*}r\}$
**then have** *distance* $(C \cup \{s\})$ $s'' \leq$ *dist* $s''$ $s$
**using** *Min.coboundedI distance-def dist-lemmas calculation* **by** *auto*
**also have** $\ldots \leq 2 * r$ **using** *local-assm* **by** (*smt* (*verit*) *dist-self dist-triangle2 mem-Collect-eq*)
**finally show** *?thesis* **.**
**qed**
**qed**

**moreover**

**have** $S' - \{s' \,.\, s' \in S' \wedge$ *dist s* $s' \leq 2{*}r\} \subseteq S$ **using** *IH* **by** (*auto simp*: *inv-def*)

**moreover**
**{**
**have** $s \in S$ **using** *IH inv-def s-def* **by** *auto*
**then have** $C \cup \{s\} \subseteq S$ **using** *IH* **by** (*simp add*: *inv-def*)
**}**
**moreover**

**have** $(\forall\, c{\in}C \cup \{s\}.\ \forall\, c_2{\in}C \cup \{s\}.\ c \neq c_2 \longrightarrow 2 * r <$ *dist c* $c_2)$
**proof** (*rule+*)
**fix** $c_1$ $c_2$
**assume** *local-assms*: $c_1 \in C \cup \{s\}$ $c_2 \in C \cup \{s\}$ $c_1 \neq c_2$
**then have** $(c_1 \in C \wedge c_2 \in C) \vee (c_1 = s \wedge c_2 \in C) \vee (c_1 \in C \wedge c_2 = s) \vee (c_1 = s \wedge c_2 = s)$
**using** *assms* **by** *auto*
**then show** $2 * r <$ *dist* $c_1$ $c_2$
**proof** (*elim disjE*)
**assume** $c_1 \in C \wedge c_2 \in C$
**then show** $2 * r <$ *dist* $c_1$ $c_2$ **using** *IH inv-def local-assms* **by** *simp*
**next**
**assume** *case-assm*: $c_1 = s \wedge c_2 \in C$
**have** $(\forall\, c \in C.\ \forall\, s{\in}S'.\ S' \neq \{\} \longrightarrow 2 * r <$ *dist c s*$)$ **using** *IH inv-def* **by** *simp*

**then show** *?thesis* **by** (*smt* (*verit*) *case-assm s-def assms*(*1*) *dist-self dist-triangle3 singletonD*)
**next**
**assume** *case-assm*: $c_1 \in C \wedge c_2 = s$
**have** $(\forall\, c \in C.\ \forall\, s{\in}S'.\ S' \neq \{\} \longrightarrow 2 * r < dist\ c\ s)$ **using** *IH inv-def* **by** *simp*
**then show** *?thesis* **by** (*smt* (*verit*) *case-assm s-def assms*(*1*) *dist-self dist-triangle3 singletonD*)
**next**
**assume** $c_1 = s \wedge c_2 = s$
**then have** *False* **using** *local-assms*(*3*) **by** *simp*
**then show** *?thesis* **by** *simp*
**qed**
**qed**

**moreover**

**have** $(\forall\, c{\in}C \cup \{s\}.\ \forall\, s'' \in S' - \{s' \in S'.\ dist\ s\ s' \leq 2 * r\}.$
$S' - \{s' \in S'.\ dist\ s\ s' \leq 2 * r\} \neq \{\} \longrightarrow 2 * r < dist\ c\ s'')$
**using** *IH inv-def* **by** *fastforce*

**moreover**

**have** $(S' - \{s' \in S'.\ dist\ s\ s' \leq 2 * r\} = S \vee C \cup \{s\} \neq \{\})$ **by** *simp*

**ultimately show** *?thesis* **unfolding** *inv-def* **by** *blast*
**qed**

**lemma** *inv-last-1*:
**assumes** $\forall\, s \in (S - S').\ distance\ C\ s \leq 2{*}r$
**and** $S' = \{\}$
**shows** $radius\ C \leq 2{*}r$
**by** (*metis Diff-empty assms image-iff radius-contained*)

**lemma** *inv-last-2*:
**assumes** *finite C*
**and** $card\ C > n$
**and** $C \subseteq S$
**and** $\forall\, c_1 \in C.\ \forall\, c_2 \in C.\ c_1 \neq c_2 \longrightarrow dist\ c_1\ c_2 > 2{*}r$
**shows** $\forall\, C'.\ card\ C' \leq n \wedge card\ C' > 0 \longrightarrow radius\ C' > r$ (**is** *?P*)
**proof** (*rule ccontr*)
**assume** $\neg$ *?P*
**then obtain** $C'$ **where** *card-C'*: $card\ C' \leq n \wedge card\ C' > 0$ **and** *radius-C'*: $radius\ C' \leq r$ **by** *auto*
**have** $\forall\, c \in C.\ (\exists\, c'.\ c' \in C' \wedge dist\ c\ c' \leq r)$
**proof**
**fix** *c*
**assume** $c \in C$
**then have** $c \in S$ **using** *assms*(*3*) **by** *blast*

**then have** *distance* $C'$ $c \leq$ *radius* $C'$ **using** *finite-distances* **by** (*simp add*: *radius-def*)
**then have** *distance* $C'$ $c \leq r$ **using** *radius-*$C'$ **by** *simp*
**then show** $\exists\, c'.\ c' \in C' \wedge$ *dist* $c$ $c' \leq r$ **using** *dist-lemmas*
**by** (*metis card-*$C'$ *card-gt-0-iff*)
**qed**
**then obtain** $f$ **where** $f$: $\forall\, c{\in}C.\ f\ c \in C' \wedge$ *dist* $c$ $(f\ c) \leq r$ **by** *metis*
**have** $\neg$*inj-on* $f$ $C$
**proof**
**assume** *inj-on* $f$ $C$
**then have** *card* $C' \geq$ *card* $C$ **using** ‹*inj-on* $f$ $C$› *card-inj-on-le card-ge-0-finite card-*$C'$ $f$ **by** *blast*
**then show** *False* **using** *card-*$C'$ ‹$n <$ *card* $C$› **by** *linarith*
**qed**
**then obtain** *c1 c2* **where** *defs*: *c1* $\in$ $C \wedge$ *c2* $\in$ $C \wedge$ *c1* $\neq$ *c2* $\wedge$ $f$ *c1* $=$ $f$ *c2* **using** *inj-on-def* **by** *blast*
**then have** $*$: *dist c1* $(f$ *c1*$) \leq r \wedge$ *dist c2* $(f$ *c1*$) \leq r$ **using** $f$ **by** *auto*

**have** *2* $*$ $r <$ *dist c1 c2* **using** *assms defs* **by** *simp*
**also have** ... $\leq$ *dist c1* $(f$ *c1*$)$ $+$ *dist* $(f$ *c1*$)$ *c2* **by**(*rule dist-triangle*)
**also have** ... $=$ *dist c1* $(f$ *c1*$)$ $+$ *dist c2* $(f$ *c1*$)$ **using** *dist-commute* **by** *simp*
**also have** ... $\leq$ *2* $*$ $r$ **using** $*$ **by** *simp*
**finally show** *False* **by** *simp*
**qed**

**lemma** *inv-last*:
**assumes** *inv* {} $C$ $r$
**shows** (*card* $C \leq k \longrightarrow$ *radius* $C \leq$ *2*$*r$) $\wedge$ (*card* $C > k \longrightarrow$ ($\forall\, C'$. *card* $C' >$ *0* $\wedge$ *card* $C' \leq k \longrightarrow$ *radius* $C' > r$))
**using** *assms inv-def inv-last-1 inv-last-2 finite-subset*[*OF - finite-sites*] **by** *auto*

**theorem** *Center-Selection-r*:
*VARS* ($S'$ :: ('*a* :: *metric-space*) *set*) ($C$ :: ('*a* :: *metric-space*) *set*) ($r$ :: *real*) ($s$ :: '*a*)
{*True*}
$S' :=$ *S*;
$C :=$ {};
*WHILE* $S' \neq$ {} *INV* {*inv* $S'$ $C$ $r$} *DO*
$s :=$ (*SOME s*. $s \in S'$);
$C :=$ $C \cup$ {$s$};
$S' :=$ $S' -$ {$s'$ . $s' \in S' \wedge$ *dist* $s$ $s' \leq$ *2*$*r$}
*OD*
{(*card* $C \leq k \longrightarrow$ *radius* $C \leq$ *2*$*r$) $\wedge$ (*card* $C > k \longrightarrow$ ($\forall\, C'$. *card* $C' >$ *0* $\wedge$ *card* $C' \leq k \longrightarrow$ *radius* $C' > r$))}
**proof** (*vcg*, *goal-cases*)
**case** (*1* $S'$ $C$ $r$)
**then show** *?case* **using** *inv-init* **by** *simp*
**next**
**case** (*2* $S'$ $C$ $r$)

**then show** *?case* **using** *inv-step* **by** *simp*
**next**
**case** (*3 S′ C r*)
**then show** *?case* **using** *inv-last* **by** *blast*
**qed**

## 6.2 The Main Algorithm

**definition** *invar* :: ('*a* :: *metric-space*) *set* ⇒ *bool* **where**
*invar C* = (*C* ≠ {} ∧ *card C* ≤ *k* ∧ *C* ⊆ *S* ∧
(∀ *C′*. (∀ *c*$_1$ ∈ *C*. ∀ *c*$_2$ ∈ *C*. *c*$_1$ ≠ *c*$_2$ ⟶ *dist c*$_1$ *c*$_2$ > *2* ∗ *radius C′*)
∨ (∀ *s* ∈ *S*. *distance C s* ≤ *2* ∗ *radius C′*)))

**abbreviation** *some* **where** *some A* ≡ (*SOME s*. *s* ∈ *A*)

**lemma** *invar-init*: *invar* {*some S*}
**proof** −
**let** *?s* = *some S*
**have** *s-in-S*: *?s* ∈ *S* **using** *some-in-eq non-empty-sites* **by** *blast*

**have** {*?s*} ≠ {} **by** *simp*

**moreover**

**have** {*SOME s*. *s* ∈ *S*} ⊆ *S* **using** *s-in-S* **by** *simp*

**moreover**

**have** *card* {*SOME s*. *s* ∈ *S*} ≤ *k* **using** *non-zero-k* **by** *simp*

**ultimately show** *?thesis* **by** (*auto simp*: *invar-def*)
**qed**

**abbreviation** *furthest-from* **where**
*furthest-from C* ≡ (*SOME s*. *s* ∈ *S* ∧ *distance C s* = *Max* (*distance C* ‘ *S*))

**lemma** *invar-step*:
**assumes** *invar C*
**and** *card C* < *k*
**shows** *invar* (*C* ∪ {*furthest-from C*})
**proof** −
**have** *furthest-from-C-props*: *furthest-from C* ∈ *S* ∧ *distance C* (*furthest-from C*)
= *radius C*
**using** *someI-ex*[*of* λ*x*. *x* ∈ *S* ∧ *distance C x* = *radius C*] *radius-def2 radius-def*
**by** *auto*
**have** *C-props*: *finite C* ∧ *C* ≠ {}
**using** *finite-subset*[*OF - finite-sites*] *assms*(*1*) **unfolding** *invar-def* **by** *blast*
**{**
**have** *card* (*C* ∪ {*furthest-from C*}) ≤ *card C* + *1*

```
    using assms(1) C-props unfolding invar-def by (simp add: card-insert-if)
   then have card (C ∪ {furthest-from C}) < k + 1 using assms(2) by simp
   then have card (C ∪ {furthest-from C}) ≤ k by simp
  }
  moreover

  have C ∪ {furthest-from C} ≠ {} by simp

  moreover

  have (C ∪ {furthest-from C}) ⊆ S using assms(1) furthest-from-C-props unfolding invar-def by simp

  moreover

  have ∀ C'. (∀ s ∈ S. distance (C ∪ {furthest-from C}) s ≤ 2 * radius C')
        ∨ (∀ c1 ∈ C ∪ {furthest-from C}. ∀ c2 ∈ C ∪ {furthest-from C}. c1 ≠ c2
⟶ 2 * radius C' < dist c1 c2)
  proof
    fix C'
   have distance C (furthest-from C) > 2 * radius C' ∨ distance C (furthest-from
C) ≤ 2 * radius C' by auto
    then show (∀ s ∈ S. distance (C ∪ {furthest-from C}) s ≤ 2 * radius C')
          ∨ (∀ c1 ∈ C ∪ {furthest-from C}. ∀ c2 ∈ C ∪ {furthest-from C}. c1 ≠
c2 ⟶ 2 * radius C' < dist c1 c2)
    proof (elim disjE)
      assume asm: distance C (furthest-from C) > 2 * radius C'
     then have ¬(∀ s ∈ S. distance C s ≤ 2 * radius C') using furthest-from-C-props
by force
      then have IH: ∀ c1 ∈ C. ∀ c2 ∈ C. c1 ≠ c2 ⟶ 2 * radius C' < dist c1 c2
        using assms(1) unfolding invar-def by blast
      have (∀ c1 ∈ C ∪ {furthest-from C}. (∀ c2 ∈ C ∪ {furthest-from C}. c1 ≠ c2
⟶ 2 * radius C' < dist c1 c2))
        using dist-ins[of C 2 * radius C' furthest-from C] IH C-props asm by simp
      then show ?thesis by simp
    next
      assume main-assm: 2 * radius C' ≥ distance C (furthest-from C)
      have (∀ s ∈ S. distance (C ∪ {furthest-from C}) s ≤ 2 * radius C')
      proof
        fix s
        assume local-assm: s ∈ S
        then show distance (C ∪ {furthest-from C}) s ≤ 2 * radius C'
        proof −
          have distance (C ∪ {furthest-from C}) s ≤ distance C s
            using distance-mono[of C C ∪ {furthest-from C}] C-props by auto
          also have ... ≤ distance C (furthest-from C)
         using Max.coboundedI local-assm finite-distances radius-def furthest-from-C-props
by auto
          also have ... ≤ 2 * radius C' using main-assm by simp
```

```
        finally show ?thesis .
      qed
    qed
    then show ?thesis by blast
  qed
 qed

 ultimately show ?thesis unfolding invar-def by blast
qed

lemma invar-last:
assumes invar C and ¬card C < k
shows card C = k and card C' > 0 ∧ card C' ≤ k ⟶ radius C ≤ 2 * radius C'
proof −
 show card C = k using assms(1, 2) unfolding invar-def by simp
next
 have C-props: finite C ∧ C ≠ {} using finite-sites assms(1) unfolding invar-def
by (meson finite-subset)
 show card C' > 0 ∧ card C' ≤ k ⟶ radius C ≤ 2 * radius C'
 proof (rule impI)
  assume C'-assms: 0 < card (C' :: 'a set) ∧ card C' ≤ k
  let ?r = radius C'
  have (∀ c1 ∈ C. ∀ c2 ∈ C. c1 ≠ c2 ⟶ 2 * ?r < dist c1 c2) ∨ (∀ s ∈ S. distance
C s ≤ 2 * ?r)
   using assms(1) unfolding invar-def by simp
  then show radius C ≤ 2 * ?r
  proof
   assume case-assm: ∀ c1∈C. ∀ c2∈C. c1 ≠ c2 ⟶ 2 * ?r < dist c1 c2
   obtain s where s-def: radius C = distance C s ∧ s ∈ S using radius-def2
by metis
   show ?thesis
   proof (rule ccontr)
    assume contr-assm: ¬ radius C ≤ 2 * ?r
    then have s-prop: distance C s > 2 * ?r using s-def by simp
    then have ‹∀ c1 ∈ C ∪ {s}. ∀ c2 ∈ C ∪ {s}. c1 ≠ c2 ⟶ dist c1 c2 > 2 *
?r›
     using C-props dist-ins[of C 2*?r s] case-assm by blast
    moreover
    {
     have s ∉ C
     proof
      assume s ∈ C
      then have distance C s ≤ dist s s using Min.coboundedI[of distance C
‘ S dist s s]
       by (simp add: distance-def C-props)
      also have ... = 0 by simp
      finally have distance C s = 0 using dist-lemmas(4) by (smt (verit)
C-props)
      then have radius-le-zero: 2 * ?r < 0 using contr-assm s-def by simp
```

```
          obtain x where x-def: ?r = distance C' x using radius-def2 by metis
          obtain l where l-def: distance C' x = dist x l using dist-lemmas(3) by
(metis C'-assms card-gt-0-iff)
          then have dist x l = ?r by (simp add: x-def)
          also have ... < 0 using C'-assms radius-le-zero by simp
          finally show False by simp
         qed
          then have card (C ∪ {s}) > k using assms(1,2) C-props unfolding
invar-def by simp
       }
       moreover
         have C ∪ {s} ⊆ S using assms(1) s-def unfolding invar-def by simp
       moreover
          have finite (C ∪ {s}) using calculation(3) finite-subset finite-sites by
auto
       ultimately have ∀ C. card C ≤ k ∧ card C > 0 ⟶ radius C > ?r using
inv-last-2 by metis
       then have ?r > ?r using C'-assms by blast
       then show False by simp
     qed
   next
     assume ∀ s∈S. distance C s ≤ 2 * radius C'
     then show ?thesis by (metis image-iff radius-contained)
   qed
 qed
qed

theorem Center-Selection:
VARS (C :: ('a :: metric-space) set) (s :: ('a :: metric-space))
 {k ≤ card S}
 C := {some S};
 WHILE card C < k INV {invar C} DO
   C := C ∪ {furthest-from C}
 OD
 {card C = k ∧ (∀ C'. card C' > 0 ∧ card C' ≤ k ⟶ radius C ≤ 2 * radius
C')}
proof (vcg, goal-cases)
 case (1 C s)
 show ?case using invar-init by simp
next
 case (2 C s)
 then show ?case using invar-step by blast
next
 case (3 C s)
 then show ?case using invar-last by blast
qed

end
end
```

## References


[1] R. Berghammer and M. Müller-Olm. Formal development and verification of approximation algorithms using auxiliary variables. In M. Bruynooghe, editor, *Logic Based Program Synthesis and Transformation, LOPSTR 2003*, volume 3018 of *LNCS*, pages 59–74. Springer, 2003.

[2] R. Berghammer and F. Reuter. A linear approximation algorithm for bin packing with absolute approximation factor 3/2. *Sci. Comput. Program.*, 48(1):67–80, 2003.

[3] R. Eßmann, T. Nipkow, and S. Robillard. Verified approximation algorithms. In N. Peltier and V. Sofronie-Stokkermans, editors, *Automated Reasoning (IJCAR 2020)*, volume 12167 of *LNCS*, page 12167. Springer, 2020. https://doi.org/10.1007/978-3-030-51054-1_17.

[4] J. M. Kleinberg and É. Tardos. *Algorithm Design.* Addison-Wesley, 2006.