

The Akra–Bazzi theorem and the Master theorem

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Abstract

This article contains a formalisation of the Akra–Bazzi method [1] based on a proof by Leighton [2]. It is a generalisation of the well-known Master Theorem for analysing the complexity of Divide & Conquer algorithms. We also include a generalised version of the Master theorem based on the Akra–Bazzi theorem, which is easier to apply than the Akra–Bazzi theorem itself.

Some proof methods that facilitate applying the Master theorem are also included. For a more detailed explanation of the formalisation and the proof methods, see the accompanying paper (publication forthcoming).

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1 Auxiliary lemmas

theory *Akra-Bazzi-Library*

imports

Complex-Main

Landau-Symbols.Landau-More

Pure-ex.Guess

begin

lemma *ln-mono*: $0 < x \implies 0 < y \implies x \leq y \implies \ln (x::\text{real}) \leq \ln y$
by (*subst ln-le-cancel-iff*) *simp-all*

lemma *ln-mono-strict*: $0 < x \implies 0 < y \implies x < y \implies \ln (x::\text{real}) < \ln y$
by (*subst ln-less-cancel-iff*) *simp-all*

declare *DERIV-powr*[*THEN DERIV-chain2, derivative-intros*]

lemma *sum-pos'*:

assumes *finite I*

assumes $\exists x \in I. f x > (0 :: - :: \text{linordered-ab-group-add})$

assumes $\bigwedge x. x \in I \implies f x \geq 0$

shows $\text{sum } f I > 0$

proof –

from *assms(2)* **guess** *x* **by** (*elim bexE*) **note** *x = this*

from *x* **have** *I = insert x I* **by** *blast*

also from *assms(1)* **have** $\text{sum } f \dots = f x + \text{sum } f (I - \{x\})$ **by** (*rule sum.insert-remove*)

also from *x assms* **have** $\dots > 0$ **by** (*intro add-pos-nonneg sum-nonneg*) *simp-all*

finally show *?thesis* .

qed

lemma *min-mult-left*:

assumes $(x::\text{real}) > 0$

shows $x * \text{min } y z = \text{min } (x*y) (x*z)$

using *assms* **by** (*auto simp add: min-def algebra-simps*)

lemma *max-mult-left*:

assumes $(x::\text{real}) > 0$

shows $x * \text{max } y z = \text{max } (x*y) (x*z)$

using *assms* **by** (*auto simp add: max-def algebra-simps*)

lemma *DERIV-nonneg-imp-mono*:

assumes $\bigwedge t. t \in \{x..y\} \implies (f \text{ has-field-derivative } f' t) (at t)$

assumes $\bigwedge t. t \in \{x..y\} \implies f' t \geq 0$

assumes $(x::real) \leq y$

shows $(f x :: real) \leq f y$

proof (*cases x y rule: linorder-cases*)

assume $xy: x < y$

hence $\exists z. x < z \wedge z < y \wedge f y - f x = (y - x) * f' z$

by (*rule MVT2*) (*insert assms(1), simp*)

then guess z **by** (*elim exE conjE*) **note** $z = \text{this}$

from $z(1,2)$ *assms(2)* xy **have** $0 \leq (y - x) * f' z$ **by** (*intro mult-nonneg-nonneg*) *simp-all*

also note $z(3)$ [*symmetric*]

finally show $f x \leq f y$ **by** *simp*

qed (*insert assms(3), simp-all*)

lemma *eventually-conjE*: $\text{eventually } (\lambda x. P x \wedge Q x) F \implies (\text{eventually } P F \implies \text{eventually } Q F \implies R) \implies R$

apply (*frule eventually-rev-mp[of - - P], simp*)

apply (*drule eventually-rev-mp[of - - Q], simp*)

apply *assumption*

done

lemma *real-natfloor-nat*: $x \in \mathbf{N} \implies \text{real } (\text{nat } \lfloor x \rfloor) = x$ **by** (*elim Nats-cases*) *simp*

lemma *eventually-natfloor*:

assumes $\text{eventually } P (at\text{-top} :: \text{nat filter})$

shows $\text{eventually } (\lambda x. P (\text{nat } \lfloor x \rfloor)) (at\text{-top} :: \text{real filter})$

proof –

from *assms* **obtain** N **where** $N: \bigwedge n. n \geq N \implies P n$ **using** *eventually-at-top-linorder*

by *blast*

have $\forall n \geq \text{real } N. P (\text{nat } \lfloor n \rfloor)$ **by** (*intro allI impI N le-nat-floor*) *simp-all*

thus *?thesis* **using** *eventually-at-top-linorder* **by** *blast*

qed

lemma *tendsto-0-smallo-1*: $f \in o(\lambda x. 1 :: \text{real}) \implies (f \longrightarrow 0) \text{ at-top}$

by (*drule smalloD-tendsto*) *simp*

lemma *smallo-1-tendsto-0*: $(f \longrightarrow 0) \text{ at-top} \implies f \in o(\lambda x. 1 :: \text{real})$

by (*rule smalloI-tendsto*) *simp-all*

lemma *filterlim-at-top-smallomega-1*:

assumes $f \in \omega[F](\lambda x. 1 :: \text{real}) \text{ eventually } (\lambda x. f x > 0) F$

shows *filterlim* f *at-top* F

proof –

from *assms* **have** *filterlim* $(\lambda x. \text{norm } (f x / 1))$ *at-top* F

by (*intro smallomegaD-filterlim-at-top-norm*) (*auto elim: eventually-mono*)

also have *?this* \longleftrightarrow *?thesis*

using *assms* by (*intro filterlim-cong refl*) (*auto elim!*: *eventually-mono*)
 finally show *?thesis* .
 qed

lemma *smallo-imp-abs-less-real*:
 assumes $f \in o[F](g)$ eventually $(\lambda x. g\ x > (0::real))\ F$
 shows eventually $(\lambda x. |f\ x| < g\ x)\ F$
proof –
 have $1/2 > (0::real)$ by *simp*
 from *landau-o.smallD[OF assms(1) this] assms(2)* show *?thesis*
 by *eventually-elim auto*
 qed

lemma *smallo-imp-less-real*:
 assumes $f \in o[F](g)$ eventually $(\lambda x. g\ x > (0::real))\ F$
 shows eventually $(\lambda x. f\ x < g\ x)\ F$
 using *smallo-imp-abs-less-real[OF assms]* by *eventually-elim simp*

lemma *smallo-imp-le-real*:
 assumes $f \in o[F](g)$ eventually $(\lambda x. g\ x \geq (0::real))\ F$
 shows eventually $(\lambda x. f\ x \leq g\ x)\ F$
 using *landau-o.smallD[OF assms(1) zero-less-one] assms(2)* by *eventually-elim simp*

lemma *filterlim-at-right*:
 $filterlim\ f\ (at-right\ a)\ F \longleftrightarrow eventually\ (\lambda x. f\ x > a)\ F \wedge filterlim\ f\ (nhds\ a)\ F$
 by (*subst filterlim-at*) (*auto elim!*: *eventually-mono*)

lemma *one-plus-x-powr-approx-ex*:
 assumes $x: abs\ (x::real) \leq 1/2$
 obtains t where $abs\ t < 1/2$ $(1 + x)^{powr\ p} =$
 $1 + p * x + p * (p - 1) * (1 + t)^{powr\ (p - 2)} / 2 * x^2$
proof (*cases x = 0*)
 assume $x': x \neq 0$
 let $?f = \lambda x. (1 + x)^{powr\ p}$
 let $?f' = \lambda x. p * (1 + x)^{powr\ (p - 1)}$
 let $?f'' = \lambda x. p * (p - 1) * (1 + x)^{powr\ (p - 2)}$
 let $?fs = (!) [?f, ?f', ?f'']$

 have $A: \forall m\ t. m < 2 \wedge t \geq -0.5 \wedge t \leq 0.5 \longrightarrow (?fs\ m\ has-real-derivative\ ?fs$
 $(Suc\ m)\ t)\ (at\ t)$
proof (*clarify*)
 fix $m :: nat$ and $t :: real$ assume $m: m < 2$ and $t: t \geq -0.5\ t \leq 0.5$
 thus $(?fs\ m\ has-real-derivative\ ?fs\ (Suc\ m)\ t)\ (at\ t)$
 using m by (*cases m*) (*force intro: derivative-eq-intros algebra-simps*)+
 qed
 have $\exists t. (if\ x < 0\ then\ x < t \wedge t < 0\ else\ 0 < t \wedge t < x) \wedge$

```

      (1 + x) powr p = (∑ m<2. ?fs m 0 / (fact m) * (x - 0) ^ m) +
      ?fs 2 t / (fact 2) * (x - 0)^2
    using assms x' by (intro Taylor[OF - - A]) simp-all
    then guess t by (elim exE conjE)
    note t = this
    with assms have abs t < 1/2 by (auto split: if-split-asm)
    moreover from t(2) have (1 + x) powr p = 1 + p * x + p * (p - 1) * (1 +
t) powr (p - 2) / 2 * x ^ 2
      by (simp add: numeral-2-eq-2 of-nat-Suc)
    ultimately show ?thesis by (rule that)
next
  assume x = 0
  with that[of 0] show ?thesis by simp
qed

```

```

lemma powr-lower-bound: [(l::real) > 0; l ≤ x; x ≤ u] ⇒ min (l powr z) (u powr
z) ≤ x powr z
apply (cases z ≥ 0)
apply (rule order.trans[OF min.cobounded1 powr-mono2], simp-all) []
apply (rule order.trans[OF min.cobounded2 powr-mono2'], simp-all) []
done

```

```

lemma powr-upper-bound: [(l::real) > 0; l ≤ x; x ≤ u] ⇒ max (l powr z) (u
powr z) ≥ x powr z
apply (cases z ≥ 0)
apply (rule order.trans[OF powr-mono2 max.cobounded2], simp-all) []
apply (rule order.trans[OF powr-mono2' max.cobounded1], simp-all) []
done

```

lemma one-plus-x-powr-Taylor2:

```

  obtains k where ∧x. abs (x::real) ≤ 1/2 ⇒ abs ((1 + x) powr p - 1 - p*x)
≤ k*x^2
proof -
  define k where k = |p*(p - 1)| * max ((1/2) powr (p - 2)) ((3/2) powr (p
- 2)) / 2
  show ?thesis
  proof (rule that[of k])
    fix x :: real assume abs x ≤ 1/2
    from one-plus-x-powr-approx-ex[OF this, of p] guess t . note t = this
    from t have abs ((1 + x) powr p - 1 - p*x) = |p*(p - 1)| * (1 + t) powr
(p - 2) / 2 * x^2
      by (simp add: abs-mult)
    also from t(1) have (1 + t) powr (p - 2) ≤ max ((1/2) powr (p - 2))
((3/2) powr (p - 2))
      by (intro powr-upper-bound) simp-all
    finally show abs ((1 + x) powr p - 1 - p*x) ≤ k*x^2
      by (simp add: mult-left-mono mult-right-mono k-def)
  qed
qed

```

lemma *one-plus-x-powr-Taylor2-bigo*:

assumes *lim*: $(f \longrightarrow 0) F$

shows $(\lambda x. (1 + f x) \text{ powr } (p::\text{real}) - 1 - p * f x) \in O[F](\lambda x. f x \wedge 2)$

proof –

from *one-plus-x-powr-Taylor2*[of *p*] **guess** *k* .

moreover from *tendstoD*[OF *lim*, of $1/2$]

have *eventually* $(\lambda x. \text{abs } (f x) < 1/2) F$ **by** (*simp add: dist-real-def*)

ultimately have *eventually* $(\lambda x. \text{norm } ((1 + f x) \text{ powr } p - 1 - p * f x) \leq k * \text{norm } (f x \wedge 2)) F$

by (*auto elim!: eventually-mono*)

thus *?thesis* **by** (*rule bigoI*)

qed

lemma *one-plus-x-powr-Taylor1-bigo*:

assumes *lim*: $(f \longrightarrow 0) F$

shows $(\lambda x. (1 + f x) \text{ powr } (p::\text{real}) - 1) \in O[F](\lambda x. f x)$

proof –

from *assms* **have** $(\lambda x. (1 + f x) \text{ powr } p - 1 - p * f x) \in O[F](\lambda x. (f x)^2)$

by (*rule one-plus-x-powr-Taylor2-bigo*)

also from *assms* **have** $f \in O[F](\lambda \cdot. 1)$ **by** (*intro bigoI-tendsto simp-all*)

from *landau-o.big.mult*[of *f F f*, OF - *this*] **have** $(\lambda x. (f x) \wedge 2) \in O[F](\lambda x. f x)$

by (*simp add: power2-eq-square*)

finally have *A*: $(\lambda x. (1 + f x) \text{ powr } p - 1 - p * f x) \in O[F](f)$.

have *B*: $(\lambda x. p * f x) \in O[F](f)$ **by** *simp*

from *sum-in-bigo*(1)[OF *A B*] **show** *?thesis* **by** *simp*

qed

lemma *x-times-x-minus-1-nonneg*: $x \leq 0 \vee x \geq 1 \implies (x:::\text{linordered-idom}) * (x - 1) \geq 0$

proof (*elim disjE*)

assume *x*: $x \leq 0$

also have $0 \leq x \wedge 2$ **by** *simp*

finally show $x * (x - 1) \geq 0$ **by** (*simp add: power2-eq-square algebra-simps*)

qed *simp*

lemma *x-times-x-minus-1-nonpos*: $x \geq 0 \implies x \leq 1 \implies (x:::\text{linordered-idom}) * (x - 1) \leq 0$

by (*intro mult-nonneg-nonpos simp-all*)

lemma *real-powr-at-bot*:

assumes $(a::\text{real}) > 1$

shows $((\lambda x. a \text{ powr } x) \longrightarrow 0)$ *at-bot*

proof –

from *assms* **have** *filterlim* $(\lambda x. \ln a * x)$ *at-bot at-bot*

by (*intro filterlim-tendsto-pos-mult-at-bot*[OF *tendsto-const - filterlim-ident*])

auto

hence $((\lambda x. \exp (x * \ln a)) \longrightarrow 0)$ *at-bot*

by (*intro filterlim-compose*[OF *exp-at-bot*]) (*simp add: algebra-simps*)

thus *?thesis* **using** *assms* **unfolding** *powr-def* **by** *simp*
qed

lemma *real-powr-at-bot-neg*:

assumes $(a::\text{real}) > 0 \ a < 1$

shows $\text{filterlim } (\lambda x. a \text{ powr } x) \text{ at-top at-bot}$

proof –

from *assms* **have** $\text{LIM } x \text{ at-bot. } \ln (\text{inverse } a) * -x \text{ :> at-top}$

by (*intro filterlim-tendsto-pos-mult-at-top*[*OF tendsto-const*] *filterlim-uminus-at-top-at-bot*)
(simp-all add: ln-inverse)

with *assms* **have** $\text{LIM } x \text{ at-bot. } x * \ln a \text{ :> at-top}$

by (*subst (asm) ln-inverse*) *(simp-all add: mult.commute)*

hence $\text{LIM } x \text{ at-bot. } \exp (x * \ln a) \text{ :> at-top}$

by (*intro filterlim-compose*[*OF exp-at-top*]) *simp*

thus *?thesis* **using** *assms* **unfolding** *powr-def* **by** *simp*

qed

lemma *real-powr-at-top-neg*:

assumes $(a::\text{real}) > 0 \ a < 1$

shows $((\lambda x. a \text{ powr } x) \longrightarrow 0) \text{ at-top}$

proof –

from *assms* **have** $\text{LIM } x \text{ at-top. } \ln (\text{inverse } a) * x \text{ :> at-top}$

by (*intro filterlim-tendsto-pos-mult-at-top*[*OF tendsto-const*])
(simp-all add: filterlim-ident field-simps)

with *assms* **have** $\text{LIM } x \text{ at-top. } \ln a * x \text{ :> at-bot}$

by (*subst filterlim-uminus-at-bot*) *(simp add: ln-inverse)*

hence $((\lambda x. \exp (x * \ln a)) \longrightarrow 0) \text{ at-top}$

by (*intro filterlim-compose*[*OF exp-at-bot*]) *(simp-all add: mult.commute)*

with *assms* **show** *?thesis* **unfolding** *powr-def* **by** *simp*

qed

lemma *eventually-nat-real*:

assumes *eventually* $P \text{ (at-top :: real filter)}$

shows *eventually* $(\lambda x. P \text{ (real } x)) \text{ (at-top :: nat filter)}$

using *assms* *filterlim-real-sequentially*

unfolding *filterlim-def le-filter-def eventually-filtermap* **by** *auto*

end

2 Asymptotic bounds

theory *Akra-Bazzi-Asymptotics*

imports

Complex-Main

Akra-Bazzi-Library

HOL-Library.Landau-Symbols

begin

locale *akra-bazzi-asymptotics-bep* =

fixes $b e p hb :: real$
assumes $bep: b > 0 b < 1 e > 0 hb > 0$
begin

context
begin

Functions that are negligible w.r.t. $\ln (b * x) \text{ powr } (e / 2 + 1)$.

private abbreviation $(input) \text{ negl} :: (real \Rightarrow real) \Rightarrow bool$ **where**
 $\text{negl } f \equiv f \in o(\lambda x. \ln (b*x) \text{ powr } (-(e/2 + 1)))$

private lemma $\text{neglD}: \text{negl } f \Longrightarrow c > 0 \Longrightarrow \text{eventually } (\lambda x. |f x| \leq c / \ln (b*x) \text{ powr } (e/2+1)) \text{ at-top}$
by $(drule (1) \text{ landau-o.smallD}, \text{subst } (asm) \text{ powr-minus})$ $(simp \text{ add: field-simps})$

private lemma $\text{negl-mult}: \text{negl } f \Longrightarrow \text{negl } g \Longrightarrow \text{negl } (\lambda x. f x * g x)$
by $(erule \text{ landau-o.small-1-mult}, \text{rule } \text{landau-o.small-imp-big}, \text{erule } \text{landau-o.small-trans})$
 $(insert \text{ bep}, \text{simp})$

private lemma ev4 :
assumes $g: \text{negl } g$
shows $\text{eventually } (\lambda x. \ln (b*x) \text{ powr } (-e/2) - \ln x \text{ powr } (-e/2) \geq g x) \text{ at-top}$
proof $(\text{rule } \text{smallo-imp-le-real})$
define $h1$ **where** $[\text{abs-def}]$:
 $h1 \ x = (1 + \ln b / \ln x) \text{ powr } (-e/2) - 1 + e/2 * (\ln b / \ln x)$ **for** x
define $h2$ **where** $[\text{abs-def}]$:
 $h2 \ x = \ln x \text{ powr } (-e/2) * ((1 + \ln b / \ln x) \text{ powr } (-e/2) - 1)$ **for** x
from bep **have** $(\lambda x. \ln b / \ln x \longrightarrow 0) \text{ at-top}$
by $(simp \text{ add: tendsto-0-smallo-1})$
note $\text{one-plus-x-powr-Taylor2-bigo}$ $[OF \text{ this}, \text{of } -e/2]$
also have $(\lambda x. (1 + \ln b / \ln x) \text{ powr } (-e/2) - 1 - e/2 * (\ln b / \ln x))$
 $= h1$
by $(simp \text{ add: h1-def})$
finally have $h1 \in o(\lambda x. 1 / \ln x)$
by $(\text{rule } \text{landau-o.big-small-trans})$ $(insert \text{ bep}, \text{simp add: power2-eq-square})$
with bep **have** $(\lambda x. h1 \ x - e/2 * (\ln b / \ln x)) \in \Theta(\lambda x. 1 / \ln x)$ **by** simp
also have $(\lambda x. h1 \ x - e/2 * (\ln b / \ln x)) = (\lambda x. (1 + \ln b / \ln x) \text{ powr } (-e/2) - 1)$
by $(\text{rule } \text{ext})$ $(simp \text{ add: h1-def})$
finally have $h2 \in \Theta(\lambda x. \ln x \text{ powr } (-e/2) * (1 / \ln x))$ **unfolding** $h2\text{-def}$
by $(\text{intro } \text{landau-theta.mult}) \text{ simp-all}$
also have $(\lambda x. \ln x \text{ powr } (-e/2) * (1 / \ln x)) \in \Theta(\lambda x. \ln x \text{ powr } (-(e/2+1)))$
by simp
also from $g \text{ bep}$ **have** $(\lambda x. \ln x \text{ powr } (-(e/2+1))) \in \omega(g)$ **by** $(simp \text{ add: small-lomega-iff-smallo})$
finally have $g \in o(h2)$ **by** $(simp \text{ add: smallomega-iff-smallo})$
also have $\text{eventually } (\lambda x. h2 \ x = \ln (b*x) \text{ powr } (-e/2) - \ln x \text{ powr } (-e/2))$
 at-top
using $\text{eventually-gt-at-top}[of \ 1::real] \text{ eventually-gt-at-top}[of \ 1/b]$

by *eventually-elim* (use *bep in* $\langle \text{simp add: ln-mult powr-diff h2-def powr-minus powr-divide field-simps} \rangle$)
 hence $h2 \in \Theta(\lambda x. \ln (b*x) \text{ powr } (-e/2) - \ln x \text{ powr } (-e/2))$ by (rule *bigth-etaI-cong*)
 finally show $g \in o(\lambda x. \ln (b * x) \text{ powr } (- e / 2) - \ln x \text{ powr } (- e / 2))$.
 next
 show *eventually* $(\lambda x. \ln (b*x) \text{ powr } (-e/2) - \ln x \text{ powr } (-e/2) \geq 0)$ at-top
 using *eventually-gt-at-top*[of $1/b$] *eventually-gt-at-top*[of $1::\text{real}$]
 by *eventually-elim* (use *bep in* $\langle \text{auto intro!: powr-mono2' simp: field-simps simp flip: ln-mult} \rangle$)
 qed

private lemma *ev1*:

negl $(\lambda x. (1 + c * \text{inverse } b * \ln x \text{ powr } (-(1+e))) \text{ powr } p - 1)$
proof –
 from *bep* have $((\lambda x. c * \text{inverse } b * \ln x \text{ powr } (-(1+e))) \longrightarrow 0)$ at-top
 by (*simp add: tendsto-0-smallo-1*)
 have $(\lambda x. (1 + c * \text{inverse } b * \ln x \text{ powr } (-(1+e))) \text{ powr } p - 1)$
 $\in O(\lambda x. c * \text{inverse } b * \ln x \text{ powr } - (1 + e))$
 using *bep* by (*intro one-plus-x-powr-Taylor1-bigo*) (*simp add: tendsto-0-smallo-1*)
 also from *bep* have *negl* $(\lambda x. c * \text{inverse } b * \ln x \text{ powr } - (1 + e))$ by *simp*
 finally show *?thesis* .
 qed

private lemma *ev2-aux*:

defines $f \equiv \lambda x. (1 + 1/\ln (b*x) * \ln (1 + hb / b * \ln x \text{ powr } (-1-e))) \text{ powr } (-e/2)$
obtains *h* where *eventually* $(\lambda x. f x \geq 1 + h x)$ at-top $h \in o(\lambda x. 1 / \ln x)$
proof (rule *that*[of $\lambda x. f x - 1$])
define *g* where [abs-def]: $g x = 1/\ln (b*x) * \ln (1 + hb / b * \ln x \text{ powr } (-1-e))$
for *x*
 have *lim*: $((\lambda x. \ln (1 + hb / b * \ln x \text{ powr } (- 1 - e))) \longrightarrow 0)$ at-top
 by (rule *tendsto-eq-rhs*[OF *tendsto-ln*[OF *tendsto-add*[OF *tendsto-const*, of -
 0]]])
 (*insert bep, simp-all add: tendsto-0-smallo-1*)
 hence *lim'*: $(g \longrightarrow 0)$ at-top **unfolding** *g-def*
 by (*intro tendsto-mult-zero*) (*insert bep, simp add: tendsto-0-smallo-1*)
from *one-plus-x-powr-Taylor2-bigo*[OF *this*, of $-e/2$]
 have $(\lambda x. (1 + g x) \text{ powr } (-e/2) - 1 - - e/2 * g x) \in O(\lambda x. (g x)^2)$.
also from *lim'* have $(\lambda x. g x \wedge 2) \in o(\lambda x. g x * 1)$ **unfolding** *power2-eq-square*
 by (*intro landau-o.big-small-mult smalloI-tendsto*) *simp-all*
also have $o(\lambda x. g x * 1) = o(g)$ by *simp*
also have $(\lambda x. (1 + g x) \text{ powr } (-e/2) - 1 - - e/2 * g x) = (\lambda x. f x - 1 + e/2 * g x)$
 by (*simp add: f-def g-def*)
finally have $A: (\lambda x. f x - 1 + e/2 * g x) \in O(g)$ by (rule *landau-o.small-imp-big*)
hence $(\lambda x. f x - 1 + e/2 * g x - e/2 * g x) \in O(g)$
 by (rule *sum-in-bigo*) (*insert bep, simp*)
also have $(\lambda x. f x - 1 + e/2 * g x - e/2 * g x) = (\lambda x. f x - 1)$ by *simp*

finally have $(\lambda x. f x - 1) \in O(g)$.
also from *bep lim* **have** $g \in o(\lambda x. 1 / \ln x)$ **unfolding** *g-def*
by (*auto intro!: smallo-1-tendsto-0*)
finally show $(\lambda x. f x - 1) \in o(\lambda x. 1 / \ln x)$.
qed *simp-all*

private lemma *ev2*:

defines $f \equiv \lambda x. \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (-e/2)$
obtains *h* **where**
negl h
eventually $(\lambda x. f x \geq \ln (b * x) \text{ powr } (-e/2) + h x)$ *at-top*
eventually $(\lambda x. |\ln (b * x) \text{ powr } (-e/2) + h x| < 1)$ *at-top*
proof –
define f'
where $f' x = (1 + 1 / \ln (b*x) * \ln (1 + hb / b * \ln x \text{ powr } (-1-e))) \text{ powr } (-e/2)$ **for** x
from *ev2-aux* **obtain** *g* **where** *g*: *eventually* $(\lambda x. 1 + g x \leq f' x)$ *at-top* $g \in o(\lambda x. 1 / \ln x)$
unfolding *f'-def* .
define *h* **where** [*abs-def*]: $h x = \ln (b*x) \text{ powr } (-e/2) * g x$ **for** x
show *?thesis*
proof (*rule that[of h]*)
from *bep g* **show** *negl h* **unfolding** *h-def*
by (*auto simp: powr-diff elim: landau-o.small-big-trans*)
next
from *g(2)* **have** $g \in o(\lambda x. 1)$ **by** (*rule landau-o.small-big-trans*) *simp*
with *bep* **have** *eventually* $(\lambda x. |\ln (b*x) \text{ powr } (-e/2) * (1 + g x)| < 1)$ *at-top*
by (*intro smallo-imp-abs-less-real*) *simp-all*
thus *eventually* $(\lambda x. |\ln (b*x) \text{ powr } (-e/2) + h x| < 1)$ *at-top*
by (*simp add: algebra-simps h-def*)
next
from *eventually-gt-at-top[of 1/b]* **and** *g(1)*
show *eventually* $(\lambda x. f x \geq \ln (b*x) \text{ powr } (-e/2) + h x)$ *at-top*
proof *eventually-elim*
case (*elim x*)
from *bep* **have** $b * x + hb * x / \ln x \text{ powr } (1 + e) = b*x * (1 + hb / b * \ln x \text{ powr } (-1 - e))$
by (*simp add: field-simps powr-diff powr-add powr-minus*)
also from *elim(1)* *bep*
have $\ln \dots = \ln (b*x) * (1 + 1/\ln (b*x) * \ln (1 + hb / b * \ln x \text{ powr } (-1-e)))$
by (*subst ln-mult-pos*) (*simp-all add: add-pos-nonneg field-simps*)
also from *elim(1)* *bep* **have** $\dots \text{ powr } (-e/2) = \ln (b*x) \text{ powr } (-e/2) * f' x$
by (*subst powr-mult*) (*simp-all add: field-simps f'-def*)
also from *elim* **have** $\dots \geq \ln (b*x) \text{ powr } (-e/2) * (1 + g x)$
by (*intro mult-left-mono*) *simp-all*
finally show $f x \geq \ln (b*x) \text{ powr } (-e/2) + h x$
by (*simp add: f-def h-def algebra-simps*)
qed

qed
qed

private lemma ev21:

obtains g where

negl g

*eventually $(\lambda x. 1 + \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (-e/2) \geq$*

*1 + \ln (b * x) \text{ powr } (-e/2) + g x) \text{ at-top}*

*eventually $(\lambda x. 1 + \ln (b * x) \text{ powr } (-e/2) + g x > 0) \text{ at-top}$*

proof –

from ev2 guess g . note $g = \text{this}$

from $g(3)$ have eventually $(\lambda x. 1 + \ln (b * x) \text{ powr } (-e/2) + g x > 0) \text{ at-top}$

by eventually-elim simp

with $g(1,2)$ show ?thesis by (intro that[of g]) simp-all

qed

private lemma ev22:

obtains g where

negl g

*eventually $(\lambda x. 1 - \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (-e/2) \leq$*

*1 - \ln (b * x) \text{ powr } (-e/2) - g x) \text{ at-top}*

*eventually $(\lambda x. 1 - \ln (b * x) \text{ powr } (-e/2) - g x > 0) \text{ at-top}$*

proof –

from ev2 guess g . note $g = \text{this}$

from $g(2)$ have eventually $(\lambda x. 1 - \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (-e/2) \leq$

*1 - \ln (b * x) \text{ powr } (-e/2) - g x) \text{ at-top}*

by eventually-elim simp

moreover from $g(3)$ have eventually $(\lambda x. 1 - \ln (b * x) \text{ powr } (-e/2) - g x > 0) \text{ at-top}$

by eventually-elim simp

ultimately show ?thesis using $g(1)$ by (intro that[of g]) simp-all

qed

lemma asymptotics1:

shows eventually $(\lambda x.$

*(1 + c * inverse b * ln x powr -(1+e)) powr p **

*(1 + ln (b * x + hb * x / ln x powr (1 + e)) powr (- e / 2)) \geq*

1 + (ln x powr (-e/2))) \text{ at-top}

proof –

let ?f = $\lambda x. (1 + c * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p$

let ?g = $\lambda x. 1 + \ln (b * x + hb * x / \ln x \text{ powr } (1 + e)) \text{ powr } (- e / 2)$

define f where [abs-def]: $f x = 1 - ?f x$ for x

from ev1[of c] have negl f unfolding f -def

by (subst landau-o.small.uminus-in-iff [symmetric]) simp

from landau-o.smallD[OF this zero-less-one]

have f : eventually $(\lambda x. f x \leq \ln (b*x) \text{ powr } -(e/2+1)) \text{ at-top}$

by eventually-elim (simp add: f -def)

from *ev21* **guess** g . **note** $g = \text{this}$
define h **where** [*abs-def*]: $h\ x = -g\ x + f\ x + f\ x * \ln\ (b*x)\ \text{powr}\ (-e/2) + f\ x * g\ x$ **for** x

have A : *eventually* $(\lambda x. ?f\ x * ?g\ x \geq 1 + \ln\ (b*x)\ \text{powr}\ (-e/2) - h\ x)$ *at-top*
using $g(2,3)$ f
proof *eventually-elim*
case (*elim* x)
let $?t = \ln\ (b*x)\ \text{powr}\ (-e/2)$
have $1 + ?t - h\ x = (1 - f\ x) * (1 + \ln\ (b*x)\ \text{powr}\ (-e/2) + g\ x)$
by (*simp add: algebra-simps h-def*)
also from *elim* **have** $?f\ x * ?g\ x \geq (1 - f\ x) * (1 + \ln\ (b*x)\ \text{powr}\ (-e/2) + g\ x)$
by (*intro mult-mono[OF - elim(1)] (simp-all add: algebra-simps f-def)*)
finally show $?f\ x * ?g\ x \geq 1 + \ln\ (b*x)\ \text{powr}\ (-e/2) - h\ x$.
qed
from *bep* $\langle \text{negl } f \rangle g(1)$ **have** *negl* h **unfolding** *h-def*
by (*fastforce intro!: sum-in-smalllo landau-o.small.mult simp: powr-diff intro: landau-o.small-trans*)
from *ev4*[*OF this*] A **show** *?thesis* **by** *eventually-elim simp*
qed

lemma *asymptotics2*:

shows *eventually* $(\lambda x.$

$$\begin{aligned} & (1 + c * \text{inverse } b * \ln\ x\ \text{powr}\ -(1+e))\ \text{powr } p * \\ & (1 - \ln\ (b * x + hb * x / \ln\ x\ \text{powr}\ (1 + e))\ \text{powr}\ (-e / 2)) \leq \\ & 1 - (\ln\ x\ \text{powr}\ (-e/2)) \text{ at-top} \end{aligned}$$

proof –

let $?f = \lambda x. (1 + c * \text{inverse } b * \ln\ x\ \text{powr}\ -(1+e))\ \text{powr } p$

let $?g = \lambda x. 1 - \ln\ (b * x + hb * x / \ln\ x\ \text{powr}\ (1 + e))\ \text{powr}\ (-e / 2)$

define f **where** [*abs-def*]: $f\ x = 1 - ?f\ x$ **for** x

from *ev1*[*of c*] **have** *negl* f **unfolding** *f-def*

by (*subst landau-o.small.uminus-in-iff [symmetric]*) *simp*

from *landau-o.smallD*[*OF this zero-less-one*]

have f : *eventually* $(\lambda x. f\ x \leq \ln\ (b*x)\ \text{powr}\ -(e/2+1))$ *at-top*

by *eventually-elim (simp add: f-def)*

from *ev22* **guess** g . **note** $g = \text{this}$

define h **where** [*abs-def*]: $h\ x = -g\ x - f\ x + f\ x * \ln\ (b*x)\ \text{powr}\ (-e/2) + f\ x * g\ x$ **for** x

have $((\lambda x. \ln\ (b * x + hb * x / \ln\ x\ \text{powr}\ (1 + e))\ \text{powr}\ -(e / 2)) \longrightarrow 0)$ *at-top*

apply (*insert bep, intro tendsto-neg-powr, simp*)

apply (*rule filterlim-compose[OF ln-at-top]*)

apply (*rule filterlim-at-top-smallomega-1, simp*)

using *eventually-gt-at-top*[*of max 1 (1/b)*]

apply (*auto elim!: eventually-mono intro!: add-pos-nonneg simp: field-simps*)

apply (*smt* (*z3*) *divide-nonneg-nonneg mult-neg-pos mult-nonneg-nonneg powr-non-neg*)
done
hence *ev-g*: *eventually* ($\lambda x. |1 - ?g x| < 1$) *at-top*
by (*intro smallo-imp-abs-less-real smalloI-tendsto*) *simp-all*

have *A*: *eventually* ($\lambda x. ?f x * ?g x \leq 1 - \ln (b*x) \text{ powr } (-e/2) + h x$) *at-top*
using *g(2,3) ev-g f*
proof *eventually-elim*
case (*elim x*)
let *?t* = $\ln (b*x) \text{ powr } (-e/2)$
from *elim* **have** $?f x * ?g x \leq (1 - f x) * (1 - \ln (b*x) \text{ powr } (-e/2) - g x)$
by (*intro mult-mono*) (*simp-all add: f-def*)
also **have** $\dots = 1 - ?t + h x$ **by** (*simp add: algebra-simps h-def*)
finally **show** $?f x * ?g x \leq 1 - \ln (b*x) \text{ powr } (-e/2) + h x$.
qed
from *bep* $\langle \text{negl } f \rangle g(1)$ **have** *negl h* **unfolding** *h-def*
by (*fastforce intro!: sum-in-smallo landau-o.small.mult simp: powr-diff*
intro: landau-o.small-trans)
from *ev4[OF this]* *A* **show** *?thesis* **by** *eventually-elim simp*
qed

lemma *asymptotics3*: *eventually* ($\lambda x. (1 + (\ln x \text{ powr } (-e/2))) / 2 \leq 1$) *at-top*
(is eventually ($\lambda x. ?f x \leq 1$) *-)*
proof (*rule eventually-mp[OF always-eventually]*, *clarify*)
from *bep* **have** ($?f \longrightarrow 1/2$) *at-top*
by (*force intro: tendsto-eq-intros tendsto-neg-powr ln-at-top*)
hence $\bigwedge e. e > 0 \implies \text{eventually } (\lambda x. |?f x - 0.5| < e)$ *at-top*
by (*subst (asm) tendsto-iff*) (*simp add: dist-real-def*)
from *this[of 0.5]* **show** *eventually* ($\lambda x. |?f x - 0.5| < 0.5$) *at-top* **by** *simp*
fix *x* **assume** $|?f x - 0.5| < 0.5$
thus $?f x \leq 1$ **by** *simp*
qed

lemma *asymptotics4*: *eventually* ($\lambda x. (1 - (\ln x \text{ powr } (-e/2))) * 2 \geq 1$) *at-top*
(is eventually ($\lambda x. ?f x \geq 1$) *-)*
proof (*rule eventually-mp[OF always-eventually]*, *clarify*)
from *bep* **have** ($?f \longrightarrow 2$) *at-top*
by (*force intro: tendsto-eq-intros tendsto-neg-powr ln-at-top*)
hence $\bigwedge e. e > 0 \implies \text{eventually } (\lambda x. |?f x - 2| < e)$ *at-top*
by (*subst (asm) tendsto-iff*) (*simp add: dist-real-def*)
from *this[of 1]* **show** *eventually* ($\lambda x. |?f x - 2| < 1$) *at-top* **by** *simp*
fix *x* **assume** $|?f x - 2| < 1$
thus $?f x \geq 1$ **by** *simp*
qed

lemma *asymptotics5*: *eventually* ($\lambda x. \ln (b*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2) < 1$) *at-top*
proof -
from *bep* **have** ($(\lambda x. b - hb * \ln x \text{ powr } -(1+e)) \longrightarrow b - 0$) *at-top*

by (*intro tendsto-intros tendsto-mult-right-zero tendsto-neg-powr ln-at-top*) *simp-all*
hence *LIM x at-top. (b - hb * ln x powr -(1+e)) * x :> at-top*
by (*rule filterlim-tendsto-pos-mult-at-top[OF - - filterlim-ident]*, *insert bep*)
simp-all
also have $(\lambda x. (b - hb * \ln x \text{ powr } -(1+e)) * x) = (\lambda x. b*x - hb*x*\ln x \text{ powr } -(1+e))$
by (*intro ext*) (*simp add: algebra-simps*)
finally have *filterlim ... at-top at-top .*
with bep have $((\lambda x. \ln (b*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } -(e/2)) \longrightarrow 0)$
at-top
by (*intro tendsto-neg-powr filterlim-compose[OF ln-at-top]*) *simp-all*
hence *eventually* $(\lambda x. |\ln (b*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } -(e/2)| < 1)$
at-top
by (*subst (asm) tendsto-iff*) (*simp add: dist-real-def*)
thus *?thesis by simp*
qed

lemma asymptotics6: *eventually* $(\lambda x. hb / \ln x \text{ powr } (1+e) < b/2)$ *at-top*
and *asymptotics7:* *eventually* $(\lambda x. hb / \ln x \text{ powr } (1+e) < (1-b)/2)$ *at-top*
and *asymptotics8:* *eventually* $(\lambda x. x*(1-b-hb/\ln x \text{ powr } (1+e)) > 1)$ *at-top*

proof–

from bep have $A: (\lambda x. hb / \ln x \text{ powr } (1+e)) \in o(\lambda-. 1)$ **by** *simp*
from bep have $B: b/3 > 0$ **and** $C: (1-b)/3 > 0$ **by** *simp-all*
from *landau-o.smallD[OF A B]* **show** *eventually* $(\lambda x. hb / \ln x \text{ powr } (1+e) < b/2)$ *at-top*
by *eventually-elim (insert bep, simp)*
from *landau-o.smallD[OF A C]* **show** *eventually* $(\lambda x. hb / \ln x \text{ powr } (1+e) < (1-b)/2)$ *at-top*
by *eventually-elim (insert bep, simp)*

from bep have $(\lambda x. hb / \ln x \text{ powr } (1+e)) \in o(\lambda-. 1)$ $(1-b)/2 > 0$ **by** *simp-all*

from *landau-o.smallD[OF this]* *eventually-gt-at-top[of 1::real]*
have $A: \text{eventually } (\lambda x. 1 - b - hb / \ln x \text{ powr } (1+e) > 0)$ *at-top*
by *eventually-elim (insert bep, simp add: field-simps)*
from bep have $(\lambda x. x * (1 - b - hb / \ln x \text{ powr } (1+e))) \in \omega(\lambda-. 1)$ $(0::\text{real})$
 < 2 **by** *simp-all*
from *landau-omega.smallD[OF this]* *A eventually-gt-at-top[of 0::real]*
show *eventually* $(\lambda x. x*(1-b-hb/\ln x \text{ powr } (1+e)) > 1)$ *at-top*
by *eventually-elim (simp-all add: abs-mult)*

qed

end

end

definition akra-bazzi-asymptotic1 $b hb e p x \longleftrightarrow$

$(1 - hb * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (b*x + hb*x/\ln x$

$\text{powr } (1+e) \text{ powr } (-e/2)$
 $\geq 1 + (\ln x \text{ powr } (-e/2) :: \text{real})$

definition *akra-bazzi-asymptotic1'* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $(1 + \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (b*x + \text{hb}*x/\ln x$
 $\text{powr } (1+e) \text{ powr } (-e/2))$
 $\geq 1 + (\ln x \text{ powr } (-e/2) :: \text{real})$

definition *akra-bazzi-asymptotic2* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $(1 + \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (b*x + \text{hb}*x/\ln x$
 $\text{powr } (1+e) \text{ powr } (-e/2))$
 $\leq 1 - \ln x \text{ powr } (-e/2) :: \text{real}$

definition *akra-bazzi-asymptotic2'* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $(1 - \text{hb} * \text{inverse } b * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (b*x + \text{hb}*x/\ln x$
 $\text{powr } (1+e) \text{ powr } (-e/2))$
 $\leq 1 - \ln x \text{ powr } (-e/2) :: \text{real}$

definition *akra-bazzi-asymptotic3* $e \text{ x} \longleftrightarrow (1 + (\ln x \text{ powr } (-e/2))) / 2 \leq$
 $(1 :: \text{real})$

definition *akra-bazzi-asymptotic4* $e \text{ x} \longleftrightarrow (1 - (\ln x \text{ powr } (-e/2))) * 2 \geq$
 $(1 :: \text{real})$

definition *akra-bazzi-asymptotic5* $b \text{ hb } e \text{ x} \longleftrightarrow$
 $\ln (b*x - \text{hb}*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2 :: \text{real}) < 1$

definition *akra-bazzi-asymptotic6* $b \text{ hb } e \text{ x} \longleftrightarrow \text{hb} / \ln x \text{ powr } (1 + e :: \text{real}) <$
 $b/2$

definition *akra-bazzi-asymptotic7* $b \text{ hb } e \text{ x} \longleftrightarrow \text{hb} / \ln x \text{ powr } (1 + e :: \text{real}) <$
 $(1 - b) / 2$

definition *akra-bazzi-asymptotic8* $b \text{ hb } e \text{ x} \longleftrightarrow x*(1 - b - \text{hb} / \ln x \text{ powr } (1 +$
 $e :: \text{real})) > 1$

definition *akra-bazzi-asymptotics* $b \text{ hb } e \text{ p } x \longleftrightarrow$
 $\text{akra-bazzi-asymptotic1 } b \text{ hb } e \text{ p } x \wedge \text{akra-bazzi-asymptotic1}' b \text{ hb } e \text{ p } x \wedge$
 $\text{akra-bazzi-asymptotic2 } b \text{ hb } e \text{ p } x \wedge \text{akra-bazzi-asymptotic2}' b \text{ hb } e \text{ p } x \wedge$
 $\text{akra-bazzi-asymptotic3 } e \text{ x} \wedge \text{akra-bazzi-asymptotic4 } e \text{ x} \wedge \text{akra-bazzi-asymptotic5}$
 $b \text{ hb } e \text{ x} \wedge$
 $\text{akra-bazzi-asymptotic6 } b \text{ hb } e \text{ x} \wedge \text{akra-bazzi-asymptotic7 } b \text{ hb } e \text{ x} \wedge$
 $\text{akra-bazzi-asymptotic8 } b \text{ hb } e \text{ x}$

lemmas *akra-bazzi-asymptotic-defs* =
 $\text{akra-bazzi-asymptotic1-def } \text{akra-bazzi-asymptotic1}'\text{-def}$
 $\text{akra-bazzi-asymptotic2-def } \text{akra-bazzi-asymptotic2}'\text{-def } \text{akra-bazzi-asymptotic3-def}$
 $\text{akra-bazzi-asymptotic4-def } \text{akra-bazzi-asymptotic5-def } \text{akra-bazzi-asymptotic6-def}$
 $\text{akra-bazzi-asymptotic7-def } \text{akra-bazzi-asymptotic8-def } \text{akra-bazzi-asymptotics-def}$

lemma *akra-bazzi-asymptotics*:
assumes $\bigwedge b. b \in \text{set } bs \implies b \in \{0 < .. < 1\}$
assumes $\text{hb} > 0 \text{ e} > 0$
shows *eventually* $(\lambda x. \forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ hb } e \text{ p } x)$ *at-top*
proof (*intro eventually-ball-finite ballI*)
fix b **assume** $b \in \text{set } bs$
with *assms* **interpret** *akra-bazzi-asymptotics-bep* $b \text{ e } p \text{ hb}$ **by** *unfold-locales auto*

```

show eventually ( $\lambda x. \text{akra-bazzi-asymptotics } b \text{ hb } e \text{ p } x$ ) at-top
unfolding akra-bazzi-asymptotic-defs
using asymptotics1[of  $-c$  for  $c$ ] asymptotics2[of  $-c$  for  $c$ ]
by (intro eventually-conj asymptotics1 asymptotics2 asymptotics3
      asymptotics4 asymptotics5 asymptotics6 asymptotics7 asymptotics8)
simp-all
qed simp

end

```

3 The continuous Akra-Bazzi theorem

```

theory Akra-Bazzi-Real
imports
  Complex-Main
  Akra-Bazzi-Asymptotics
begin

```

We want to be generic over the integral definition used; we fix some arbitrary notions of integrability and integral and assume just the properties we need. The user can then instantiate the theorems with any desired integral definition.

```

locale akra-bazzi-integral =
  fixes integrable :: ( $real \Rightarrow real$ )  $\Rightarrow real \Rightarrow real \Rightarrow bool$ 
  and integral :: ( $real \Rightarrow real$ )  $\Rightarrow real \Rightarrow real \Rightarrow real$ 
  assumes integrable-const:  $c \geq 0 \Longrightarrow integrable (\lambda-. c) a b$ 
  and integral-const:  $c \geq 0 \Longrightarrow a \leq b \Longrightarrow integral (\lambda-. c) a b = (b - a) * c$ 
  and integrable-subinterval:
     $integrable f a b \Longrightarrow a \leq a' \Longrightarrow b' \leq b \Longrightarrow integrable f a' b'$ 
  and integral-le:
     $integrable f a b \Longrightarrow integrable g a b \Longrightarrow (\bigwedge x. x \in \{a..b\} \Longrightarrow f x \leq g x)$ 
 $\Longrightarrow$ 
     $integral f a b \leq integral g a b$ 
  and integral-combine:
     $a \leq c \Longrightarrow c \leq b \Longrightarrow integrable f a b \Longrightarrow$ 
     $integral f a c + integral f c b = integral f a b$ 
begin
lemma integral-nonneg:
   $a \leq b \Longrightarrow integrable f a b \Longrightarrow (\bigwedge x. x \in \{a..b\} \Longrightarrow f x \geq 0) \Longrightarrow integral f a b \geq 0$ 
  using integral-le[OF integrable-const[of  $0$ ], of  $f a b$ ] by (simp add: integral-const)
end

```

```

declare sum.cong[fundef-cong]

```

```

lemma strict-mono-imp-ex1-real:
  fixes  $f :: real \Rightarrow real$ 

```

```

assumes lim-neg-inf: LIM x at-bot. f x :> at-top
assumes lim-inf: (f  $\longrightarrow$  z) at-top
assumes mono:  $\bigwedge a b. a < b \implies f b < f a$ 
assumes cont:  $\bigwedge x. \text{isCont } f x$ 
assumes y-greater-z:  $z < y$ 
shows  $\exists! x. f x = y$ 
proof (rule ex-ex1I)
  fix a b assume  $f a = y \wedge f b = y$ 
  thus  $a = b$  by (cases rule: linorder-cases[of a b]) (auto dest: mono)
next
  from lim-neg-inf have eventually ( $\lambda x. y \leq f x$ ) at-bot by (subst (asm) filter-lim-at-top) simp
  then obtain l where  $\bigwedge x. x \leq l \implies y \leq f x$  by (subst (asm) eventually-at-bot-linorder) auto

  from order-tendstoD(2)[OF lim-inf y-greater-z]
  obtain u where  $\bigwedge x. x \geq u \implies f x < y$  by (subst (asm) eventually-at-top-linorder)
  auto
  define a where  $a = \min l u$ 
  define b where  $b = \max l u$ 
  have  $a: f a \geq y$  unfolding a-def by (intro l) simp
  moreover have  $b: f b < y$  unfolding b-def by (intro u) simp
  moreover have a-le-b:  $a \leq b$  by (simp add: a-def b-def)
  ultimately have  $\exists x \geq a. x \leq b \wedge f x = y$  using cont by (intro IVT2) auto
  thus  $\exists x. f x = y$  by blast
qed

```

The parameter p in the Akra-Bazzi theorem always exists and is unique.

definition *akra-bazzi-exponent* :: *real list* \Rightarrow *real list* \Rightarrow *real* **where**
akra-bazzi-exponent as bs \equiv (*THE* $p. (\sum i < \text{length } as. as!i * bs!i \text{ powr } p) = 1$)

```

locale akra-bazzi-params =
  fixes k :: nat and as bs :: real list
  assumes length-as:  $\text{length } as = k$ 
  and length-bs:  $\text{length } bs = k$ 
  and k-not-0:  $k \neq 0$ 
  and a-ge-0:  $a \in \text{set } as \implies a \geq 0$ 
  and b-bounds:  $b \in \text{set } bs \implies b \in \{0 < .. < 1\}$ 
begin

```

abbreviation p :: *real* **where** $p \equiv \text{akra-bazzi-exponent } as \text{ } bs$

lemma *p-def*: $p = (\text{THE } p. (\sum i < k. as!i * bs!i \text{ powr } p) = 1)$
by (*simp add: akra-bazzi-exponent-def length-as*)

lemma *b-pos*: $b \in \text{set } bs \implies b > 0$ **and** *b-less-1*: $b \in \text{set } bs \implies b < 1$
using *b-bounds* **by** *simp-all*

lemma *as-nonempty* [*simp*]: $as \neq []$ **and** *bs-nonempty* [*simp*]: $bs \neq []$

```

using length-as length-bs k-not-0 by auto

lemma a-in-as[intro, simp]:  $i < k \implies as ! i \in set\ as$ 
  by (rule nth-mem) (simp add: length-as)

lemma b-in-bs[intro, simp]:  $i < k \implies bs ! i \in set\ bs$ 
  by (rule nth-mem) (simp add: length-bs)

end

locale akra-bazzi-params-nonzero =
  fixes  $k :: nat$  and  $as\ bs :: real\ list$ 
  assumes length-as:  $length\ as = k$ 
  and length-bs:  $length\ bs = k$ 
  and a-ge-0:  $a \in set\ as \implies a \geq 0$ 
  and ex-a-pos:  $\exists a \in set\ as. a > 0$ 
  and b-bounds:  $b \in set\ bs \implies b \in \{0 < .. < 1\}$ 
begin

sublocale akra-bazzi-params  $k\ as\ bs$ 
  by unfold-locales (insert length-as length-bs a-ge-0 ex-a-pos b-bounds, auto)

lemma akra-bazzi-p-strict-mono:
  assumes  $x < y$ 
  shows  $(\sum_{i < k}. as!i * bs!i\ powr\ y) < (\sum_{i < k}. as!i * bs!i\ powr\ x)$ 
proof (intro sum-strict-mono-ex1 ballI)
  from ex-a-pos obtain  $a$  where  $a \in set\ as\ a > 0$  by blast
  then obtain  $i$  where  $i < k\ as!i > 0$  by (force simp: in-set-conv-nth length-as)
  with b-bounds  $\langle x < y \rangle$  have  $as!i * bs!i\ powr\ y < as!i * bs!i\ powr\ x$ 
  by (intro mult-strict-left-mono powr-less-mono') auto
  with  $\langle i < k \rangle$  show  $\exists i \in \{.. < k\}. as!i * bs!i\ powr\ y < as!i * bs!i\ powr\ x$  by blast
next
  fix  $i$  assume  $i \in \{.. < k\}$ 
  with a-ge-0 b-bounds[of bs!i]  $\langle x < y \rangle$  show  $as!i * bs!i\ powr\ y \leq as!i * bs!i\ powr\ x$ 
  by (intro mult-left-mono powr-mono') simp-all
qed simp-all

lemma akra-bazzi-p-mono:
  assumes  $x \leq y$ 
  shows  $(\sum_{i < k}. as!i * bs!i\ powr\ y) \leq (\sum_{i < k}. as!i * bs!i\ powr\ x)$ 
apply (cases  $x < y$ )
using akra-bazzi-p-strict-mono[of x y] assms apply simp-all
done

lemma akra-bazzi-p-unique:
   $\exists ! p. (\sum_{i < k}. as!i * bs!i\ powr\ p) = 1$ 

```

proof (*rule strict-mono-imp-ex1-real*)
from *as-nonempty* **have** [*simp*]: $k > 0$ **by** (*auto simp: length-as[symmetric]*)
have [*simp*]: $\bigwedge i. i < k \implies as!i \geq 0$ **by** (*rule a-ge-0*) *simp*
from *ex-a-pos* **obtain** *a* **where** $a \in \text{set } as \ a > 0$ **by** *blast*
then obtain *i* **where** $i < k \ as!i > 0$ **by** (*force simp: in-set-conv-nth length-as*)

hence *LIM p at-bot. as!i * bs!i powr p := at-top using b-bounds i*
by (*intro filterlim-tendsto-pos-mult-at-top[OF tendsto-const] real-powr-at-bot-neg*)
simp-all
moreover have $\forall p. as!i * bs!i \text{ powr } p \leq (\sum i \in \{..<k\}. as!i * bs!i \text{ powr } p)$
proof
fix *p :: real*
from *a-ge-0 b-bounds* **have** $(\sum i \in \{..<k\} - \{i\}. as!i * bs!i \text{ powr } p) \geq 0$
by (*intro sum-nonneg mult-nonneg-nonneg*) *simp-all*
also have $as!i * bs!i \text{ powr } p + \dots = (\sum i \in \text{insert } i \ \{..<k\}. as!i * bs!i \text{ powr } p)$
p)
by (*simp add: sum.insert-remove*)
also from *i* **have** $\text{insert } i \ \{..<k\} = \{..<k\}$ **by** *blast*
finally show $as!i * bs!i \text{ powr } p \leq (\sum i \in \{..<k\}. as!i * bs!i \text{ powr } p)$ **by** *simp*
qed
ultimately show *LIM p at-bot. $\sum i < k. as!i * bs!i \text{ powr } p := at-top$*
by (*rule filterlim-at-top-mono[OF - always-eventually]*)

next
from *b-bounds* **show** $(\lambda x. \sum i < k. as!i * bs!i \text{ powr } x) \longrightarrow (\sum i < k. 0)$
at-top
by (*intro tendsto-sum tendsto-mult-right-zero real-powr-at-top-neg*) *simp-all*
next
fix *x*
from *b-bounds* **have** $A: \bigwedge i. i < k \implies bs!i > 0$ **by** *simp*
show *isCont* $(\lambda x. \sum i < k. as!i * bs!i \text{ powr } x)$ *x*
using *b-bounds*[*OF nth-mem*] **by** (*intro continuous-intros*) (*auto dest: A*)
qed (*simp-all add: akra-bazzi-p-strict-mono*)

lemma *p-props*: $(\sum i < k. as!i * bs!i \text{ powr } p) = 1$
and *p-unique*: $(\sum i < k. as!i * bs!i \text{ powr } p') = 1 \implies p = p'$
proof –
from *theI'*[*OF akra-bazzi-p-unique*] *the1-equality*[*OF akra-bazzi-p-unique*]
show $(\sum i < k. as!i * bs!i \text{ powr } p) = 1 \ (\sum i < k. as!i * bs!i \text{ powr } p') = 1 \implies p = p'$
unfolding *p-def* **by** – *blast+*
qed

lemma *p-greaterI*: $1 < (\sum i < k. as!i * bs!i \text{ powr } p') \implies p' < p$
by (*rule disjE*[*OF le-less-linear, of p p'*], *drule akra-bazzi-p-mono, subst (asm)*)
p-props, simp-all)

lemma *p-lessI*: $1 > (\sum i < k. as!i * bs!i \text{ powr } p') \implies p' > p$
by (*rule disjE*[*OF le-less-linear, of p' p*], *drule akra-bazzi-p-mono, subst (asm)*)
p-props, simp-all)

lemma *p-geI*: $1 \leq (\sum i < k. as!i * bs!i \text{ powr } p') \implies p' \leq p$
by (*rule disjE*[*OF le-less-linear*, of $p' p$], *simp*, *drule akra-bazzi-p-strict-mono*,
subst (asm) p-props, *simp-all*)

lemma *p-leI*: $1 \geq (\sum i < k. as!i * bs!i \text{ powr } p') \implies p' \geq p$
by (*rule disjE*[*OF le-less-linear*, of $p p'$], *simp*, *drule akra-bazzi-p-strict-mono*,
subst (asm) p-props, *simp-all*)

lemma *p-boundsI*: $(\sum i < k. as!i * bs!i \text{ powr } x) \leq 1 \wedge (\sum i < k. as!i * bs!i \text{ powr } y) \geq 1 \implies p \in \{y..x\}$
by (*elim conjE*, *drule p-leI*, *drule p-geI*, *simp*)

lemma *p-boundsI'*: $(\sum i < k. as!i * bs!i \text{ powr } x) < 1 \wedge (\sum i < k. as!i * bs!i \text{ powr } y) > 1 \implies p \in \{y < .. < x\}$
by (*elim conjE*, *drule p-lessI*, *drule p-greaterI*, *simp*)

lemma *p-nonneg*: *sum-list as* $\geq 1 \implies p \geq 0$
proof (*rule p-geI*)
assume *sum-list as* ≥ 1
also have ... = $(\sum i < k. as!i)$ **by** (*simp add: sum-list-sum-nth length-as atLeast0LessThan*)
also {
fix *i* **assume** $i < k$
with *b-bounds* **have** $bs!i > 0$ **by** *simp*
hence $as!i * bs!i \text{ powr } 0 = as!i$ **by** *simp*
}
hence $(\sum i < k. as!i) = (\sum i < k. as!i * bs!i \text{ powr } 0)$ **by** (*intro sum.cong*) *simp-all*
finally show $1 \leq (\sum i < k. as!i * bs!i \text{ powr } 0)$.
qed

end

locale *akra-bazzi-real-recursion* =
fixes *as bs* :: *real list* **and** *hs* :: (*real* \implies *real*) *list* **and** *k* :: *nat* **and** *x0 x1 hb e p*
:: *real*
assumes *length-as*: *length as* = *k*
and *length-bs*: *length bs* = *k*
and *length-hs*: *length hs* = *k*
and *k-not-0*: $k \neq 0$
and *a-ge-0*: $a \in \text{set } as \implies a \geq 0$
and *b-bounds*: $b \in \text{set } bs \implies b \in \{0 < .. < 1\}$

and *x0-ge-1*: $x_0 \geq 1$
and *x0-le-x1*: $x_0 \leq x_1$
and *x1-ge*: $b \in \text{set } bs \implies x_1 \geq 2 * x_0 * \text{inverse } b$

and *e-pos*: $e > 0$

and *h-bounds*: $x \geq x_1 \implies h \in \text{set } hs \implies |h x| \leq hb * x / \ln x \text{ powr } (1 + e)$

and *asymptotics*: $x \geq x_0 \implies b \in \text{set } bs \implies \text{akra-bazzi-asymptotics } b \text{ hb } e \text{ p } x$
begin

sublocale *akra-bazzi-params* *k as bs*

using *length-as length-bs k-not-0 a-ge-0 b-bounds* **by** *unfold-locales*

lemma *h-in-hs*[*intro, simp*]: $i < k \implies hs ! i \in \text{set } hs$
by (*rule nth-mem*) (*simp add: length-hs*)

lemma *x1-gt-1*: $x_1 > 1$

proof –

from *bs-nonempty* **obtain** *b* **where** $b \in \text{set } bs$ **by** (*cases bs*) *auto*

from *b-pos*[*OF this*] *b-less-1*[*OF this*] *x0-ge-1* **have** $1 < 2 * x_0 * \text{inverse } b$
by (*simp add: field-simps*)

also from *x1-ge* **and** $\langle b \in \text{set } bs \rangle$ **have** $\dots \leq x_1$ **by** *simp*

finally show *?thesis* .

qed

lemma *x1-ge-1*: $x_1 \geq 1$ **using** *x1-gt-1* **by** *simp*

lemma *x1-pos*: $x_1 > 0$ **using** *x1-ge-1* **by** *simp*

lemma *bx-le-x*: $x \geq 0 \implies b \in \text{set } bs \implies b * x \leq x$

using *b-pos*[*of b*] *b-less-1*[*of b*] **by** (*intro mult-left-le-one-le*) (*simp-all*)

lemma *x0-pos*: $x_0 > 0$ **using** *x0-ge-1* **by** *simp*

lemma

assumes $x \geq x_0$ $b \in \text{set } bs$

shows *x0-hb-bound0*: $hb / \ln x \text{ powr } (1 + e) < b/2$

and *x0-hb-bound1*: $hb / \ln x \text{ powr } (1 + e) < (1 - b) / 2$

and *x0-hb-bound2*: $x*(1 - b - hb / \ln x \text{ powr } (1 + e)) > 1$

using *asymptotics*[*OF assms*] **unfolding** *akra-bazzi-asymptotic-defs* **by** *blast+*

lemma *step-diff*:

assumes $i < k$ $x \geq x_1$

shows $bs ! i * x + (hs ! i) x + 1 < x$

proof –

have $bs ! i * x + (hs ! i) x + 1 \leq bs ! i * x + |(hs ! i) x| + 1$ **by** *simp*

also from *assms* **have** $|(hs ! i) x| \leq hb * x / \ln x \text{ powr } (1 + e)$ **by** (*simp add: h-bounds*)

also from *assms* *x0-le-x1* **have** $x*(1 - bs ! i - hb / \ln x \text{ powr } (1 + e)) > 1$

by (*simp add: x0-hb-bound2*)

hence $bs ! i * x + hb * x / \ln x \text{ powr } (1 + e) + 1 < x$ **by** (*simp add: algebra-simps*)

finally show *?thesis* **by** *simp*

qed

lemma *step-le-x*: $i < k \implies x \geq x_1 \implies bs ! i * x + (hs ! i) x \leq x$
by (*drule* (1) *step-diff*) *simp*

lemma *x0-hb-bound0'*: $\bigwedge x b. x \geq x_0 \implies b \in \text{set } bs \implies hb / \ln x \text{ powr } (1 + e) < b$
by (*drule* (1) *x0-hb-bound0*, *erule less-le-trans*) (*simp add: b-pos*)

lemma *step-pos*:

assumes $i < k$ $x \geq x_1$

shows $bs ! i * x + (hs ! i) x > 0$

proof –

from *assms x0-le-x1* have $hb / \ln x \text{ powr } (1 + e) < bs ! i$ by (*simp add: x0-hb-bound0'*)

with *assms x0-pos x0-le-x1* have $x * 0 < x * (bs ! i - hb / \ln x \text{ powr } (1 + e))$
by *simp*

also have $\dots = bs ! i * x - hb * x / \ln x \text{ powr } (1 + e)$

by (*simp add: algebra-simps*)

also from *assms* have $-hb * x / \ln x \text{ powr } (1 + e) \leq -|(hs ! i) x|$ by (*simp add: h-bounds*)

hence $bs ! i * x - hb * x / \ln x \text{ powr } (1 + e) \leq bs ! i * x + -|(hs ! i) x|$ by *simp*

also have $-|(hs ! i) x| \leq (hs ! i) x$ by *simp*

finally show $bs ! i * x + (hs ! i) x > 0$ by *simp*

qed

lemma *step-nonneg*: $i < k \implies x \geq x_1 \implies bs ! i * x + (hs ! i) x \geq 0$
by (*drule* (1) *step-pos*) *simp*

lemma *step-nonneg'*: $i < k \implies x \geq x_1 \implies bs ! i + (hs ! i) x / x \geq 0$
by (*frule* (1) *step-nonneg*, *insert x0-pos x0-le-x1*) (*simp-all add: field-simps*)

lemma *hb-nonneg*: $hb \geq 0$

proof –

from *k-not-0* and *length-hs* have $hs \neq []$ by *auto*

then obtain *h* where $h: h \in \text{set } hs$ by (*cases hs*) *auto*

have $0 \leq |h x_1|$ by *simp*

also from *h* have $|h x_1| \leq hb * x_1 / \ln x_1 \text{ powr } (1+e)$ by (*intro h-bounds*)

simp-all

finally have $0 \leq hb * x_1 / \ln x_1 \text{ powr } (1 + e)$.

hence $0 \leq \dots * (\ln x_1 \text{ powr } (1 + e) / x_1)$

by (*rule mult-nonneg-nonneg*) (*intro divide-nonneg-nonneg*, *insert x1-pos*, *simp-all*)

also have $\dots = hb$ using *x1-gt-1* by (*simp add: field-simps*)

finally show *?thesis*.

qed

lemma *x0-hb-bound3*:

assumes $x \geq x_1$ $i < k$

shows $x - (bs ! i * x + (hs ! i) x) \leq x$
proof –
have $-(hs ! i) x \leq |(hs ! i) x|$ **by** *simp*
also from *assms* **have** $\dots \leq hb * x / \ln x \text{ powr } (1 + e)$ **by** (*simp add: h-bounds*)
also have $\dots = x * (hb / \ln x \text{ powr } (1 + e))$ **by** *simp*
also from *assms* $x0\text{-pos } x0\text{-le-}x1$ **have** $\dots < x * bs ! i$
by (*intro mult-strict-left-mono x0-hb-bound0'*) *simp-all*
finally show *?thesis* **by** (*simp add: algebra-simps*)
qed

lemma *x0-hb-bound4*:
assumes $x \geq x_1 \ i < k$
shows $(bs ! i + (hs ! i) x / x) > bs ! i / 2$
proof –
from *assms* $x0\text{-le-}x1$ **have** $hb / \ln x \text{ powr } (1 + e) < bs ! i / 2$ **by** (*intro x0-hb-bound0*) *simp-all*
with *assms* $x0\text{-pos } x0\text{-le-}x1$ **have** $(-bs ! i / 2) * x < (-hb / \ln x \text{ powr } (1 + e)) * x$
by (*intro mult-strict-right-mono*) *simp-all*
also from *assms* $x0\text{-pos}$ **have** $\dots \leq -|(hs ! i) x|$ **using** *h-bounds* **by** *simp*
also have $\dots \leq (hs ! i) x$ **by** *simp*
finally show *?thesis* **using** *assms* $x1\text{-pos}$ **by** (*simp add: field-simps*)
qed

lemma *x0-hb-bound4'*: $x \geq x_1 \implies i < k \implies (bs ! i + (hs ! i) x / x) \geq bs ! i / 2$
by (*drule* (1) *x0-hb-bound4*) *simp*

lemma *x0-hb-bound5*:
assumes $x \geq x_1 \ i < k$
shows $(bs ! i + (hs ! i) x / x) \leq bs ! i * 3/2$
proof –
have $(hs ! i) x \leq |(hs ! i) x|$ **by** *simp*
also from *assms* **have** $\dots \leq hb * x / \ln x \text{ powr } (1 + e)$ **by** (*simp add: h-bounds*)
also have $\dots = x * (hb / \ln x \text{ powr } (1 + e))$ **by** *simp*
also from *assms* $x0\text{-pos } x0\text{-le-}x1$ **have** $\dots < x * (bs ! i / 2)$
by (*intro mult-strict-left-mono x0-hb-bound0*) *simp-all*
finally show *?thesis* **using** *assms* $x1\text{-pos}$ **by** (*simp add: field-simps*)
qed

lemma *x0-hb-bound6*:
assumes $x \geq x_1 \ i < k$
shows $x * ((1 - bs ! i) / 2) \leq x - (bs ! i * x + (hs ! i) x)$
proof –
from *assms* $x0\text{-le-}x1$ **have** $hb / \ln x \text{ powr } (1 + e) < (1 - bs ! i) / 2$ **using** *x0-hb-bound1* **by** *simp*
with *assms* $x1\text{-pos}$ **have** $x * ((1 - bs ! i) / 2) \leq x * (1 - (bs ! i + hb / \ln x \text{ powr } (1 + e)))$
by (*intro mult-left-mono*) (*simp-all add: field-simps*)
also have $\dots = x - bs ! i * x + -hb * x / \ln x \text{ powr } (1 + e)$ **by** (*simp add:*

algebra-simps)
also from *h-bounds* *assms* **have** $-hb * x / \ln x \text{ powr } (1 + e) \leq -|(hs ! i) x|$
by (*simp add: length-hs*)
also have $\dots \leq -(hs ! i) x$ **by** *simp*
finally show *?thesis* **by** (*simp add: algebra-simps*)
qed

lemma *x0-hb-bound7*:

assumes $x \geq x_1$ $i < k$
shows $bs!i*x + (hs!i) x > x_0$

proof –

from *assms* *x0-le-x1* **have** $x' : x \geq x_0$ **by** *simp*
from *x1-ge* *assms* **have** $2 * x_0 * \text{inverse } (bs!i) \leq x_1$ **by** *simp*
with *assms* *b-pos* **have** $x_0 \leq x_1 * (bs!i / 2)$ **by** (*simp add: field-simps*)
also from *assms* x' **have** $bs!i/2 < bs!i + (hs!i) x / x$ **by** (*intro x0-hb-bound4*)
also from *assms* *step-nonneg' x'* **have** $x_1 * \dots \leq x * \dots$ **by** (*intro mult-right-mono*)
(*simp-all*)
also from *assms* *x1-pos* **have** $x * (bs!i + (hs!i) x / x) = bs!i*x + (hs!i) x$
by (*simp add: field-simps*)
finally show *?thesis* **using** *x1-pos* **by** *simp*
qed

lemma *x0-hb-bound7'*: $x \geq x_1 \implies i < k \implies bs!i*x + (hs!i) x > 1$
by (*rule le-less-trans[OF - x0-hb-bound7]*) (*insert x0-le-x1 x0-ge-1, simp-all*)

lemma *x0-hb-bound8*:

assumes $x \geq x_1$ $i < k$
shows $bs!i*x - hb * x / \ln x \text{ powr } (1+e) > x_0$

proof –

from *assms* **have** $2 * x_0 * \text{inverse } (bs!i) \leq x_1$ **by** (*intro x1-ge*) *simp-all*
with *b-pos* *assms* **have** $x_0 \leq x_1 * (bs!i/2)$ **by** (*simp add: field-simps*)
also from *assms* *b-pos* **have** $\dots \leq x * (bs!i/2)$ **by** *simp*
also from *assms* *x0-le-x1* **have** $hb / \ln x \text{ powr } (1+e) < bs!i/2$ **by** (*intro x0-hb-bound0*) *simp-all*
with *assms* **have** $bs!i/2 < bs!i - hb / \ln x \text{ powr } (1+e)$ **by** (*simp add: field-simps*)
also have $x * \dots = bs!i*x - hb * x / \ln x \text{ powr } (1+e)$ **by** (*simp add: algebra-simps*)
finally show *?thesis* **using** *assms* *x1-pos* **by** (*simp add: field-simps*)
qed

lemma *x0-hb-bound8'*:

assumes $x \geq x_1$ $i < k$
shows $bs!i*x + hb * x / \ln x \text{ powr } (1+e) > x_0$

proof –

from *assms* **have** $x_0 < bs!i*x - hb * x / \ln x \text{ powr } (1+e)$ **by** (*rule x0-hb-bound8*)
also from *assms* *hb-nonneg* *x1-pos* **have** $hb * x / \ln x \text{ powr } (1+e) \geq 0$
by (*intro mult-nonneg-nonneg divide-nonneg-nonneg*) *simp-all*
hence $bs!i*x - hb * x / \ln x \text{ powr } (1+e) \leq bs!i*x + hb * x / \ln x \text{ powr } (1+e)$
by *simp*

finally show *?thesis* .
qed

lemma

assumes $x \geq x_0$
shows *asymptotics1*: $i < k \implies 1 + \ln x \text{ powr } (-e/2) \leq (1 - hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$
and *asymptotics2*: $i < k \implies 1 - \ln x \text{ powr } (-e/2) \geq (1 + hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$
and *asymptotics1'*: $i < k \implies 1 + \ln x \text{ powr } (-e/2) \leq (1 + hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$
and *asymptotics2'*: $i < k \implies 1 - \ln x \text{ powr } (-e/2) \geq (1 - hb * \text{inverse } (bs!i) * \ln x \text{ powr } -(1+e)) \text{ powr } p * (1 - \ln (bs!i*x + hb*x/\ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$
and *asymptotics3*: $(1 + \ln x \text{ powr } (-e/2)) / 2 \leq 1$
and *asymptotics4*: $(1 - \ln x \text{ powr } (-e/2)) * 2 \geq 1$
and *asymptotics5*: $i < k \implies \ln (bs!i*x - hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } (-e/2) < 1$
apply -
using *assms asymptotics*[of x $bs!i$] **unfolding** *akra-bazzi-asymptotic-defs*
apply *simp-all*[4]
using *assms asymptotics*[of x $bs!0$] **unfolding** *akra-bazzi-asymptotic-defs*
apply *simp-all*[2]
using *assms asymptotics*[of x $bs!i$] **unfolding** *akra-bazzi-asymptotic-defs*
apply *simp-all*
done

lemma *x0-hb-bound9*:

assumes $x \geq x_1$ $i < k$
shows $\ln (bs!i*x + (hs!i) x) \text{ powr } -(e/2) < 1$
proof-
from *b-pos assms* **have** $0 < bs!i/2$ **by** *simp*
also from *assms x0-le-x1* **have** $\dots < bs!i + (hs!i) x / x$ **by** (*intro x0-hb-bound4*)
simp-all
also from *assms x1-pos* **have** $x * \dots = bs!i*x + (hs!i) x$ **by** (*simp add: field-simps*)
finally have *pos*: $bs!i*x + (hs!i) x > 0$ **using** *assms x1-pos* **by** *simp*
from *x0-hb-bound8*[*OF assms*] *x0-ge-1* **have** *pos'*: $bs!i*x - hb * x / \ln x \text{ powr } (1+e) > 1$ **by** *simp*

from *assms* **have** $-(hb * x / \ln x \text{ powr } (1+e)) \leq -|(hs!i) x|$
by (*intro le-imp-neg-le h-bounds*) *simp-all*
also have $\dots \leq (hs!i) x$ **by** *simp*
finally have $\ln (bs!i*x - hb * x / \ln x \text{ powr } (1+e)) \leq \ln (bs!i*x + (hs!i) x)$
using *assms b-pos x0-pos pos'* **by** (*intro ln-mono mult-pos-pos pos*) *simp-all*
hence $\ln (bs!i*x + (hs!i) x) \text{ powr } -(e/2) \leq \ln (bs!i*x - hb * x / \ln x \text{ powr } (1+e))$

$(1+e)$ *powr* $-(e/2)$
using *assms e-pos asymptotics5*[*of x*] *pos'* **by** (*intro powr-mono2' ln-gt-zero*)
simp-all
also have $\dots < 1$ **using** *asymptotics5*[*of x i*] *assms x0-le-x1*
by (*subst (asm) powr-minus*) (*simp-all add: field-simps*)
finally show *?thesis* .
qed

definition *akra-bazzi-measure* :: *real* \Rightarrow *nat* **where**
akra-bazzi-measure $x = \text{nat } \lceil x \rceil$

lemma *akra-bazzi-measure-decreases*:

assumes $x \geq x_1$ $i < k$
shows *akra-bazzi-measure* ($bs!i * x + (hs!i) x$) $<$ *akra-bazzi-measure* x
proof –
from *step-diff assms* **have** ($bs!i * x + (hs!i) x$) $+ 1 < x$ **by** (*simp add: algebra-simps*)
hence $\lceil (bs!i * x + (hs!i) x) + 1 \rceil \leq \lceil x \rceil$ **by** (*intro ceiling-mono*) *simp*
hence $\lceil (bs!i * x + (hs!i) x) \rceil < \lceil x \rceil$ **by** *simp*
with *assms x1-pos* **have** $\text{nat } \lceil (bs!i * x + (hs!i) x) \rceil < \text{nat } \lceil x \rceil$ **by** (*subst nat-mono-iff*) *simp-all*
thus *?thesis* **unfolding** *akra-bazzi-measure-def* .
qed

lemma *akra-bazzi-induct*[*consumes 1, case-names base rec*]:

assumes $x \geq x_0$
assumes *base*: $\bigwedge x. x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow P x$
assumes *rec*: $\bigwedge x. x > x_1 \Longrightarrow (\bigwedge i. i < k \Longrightarrow P (bs!i * x + (hs!i) x)) \Longrightarrow P x$
shows $P x$
proof (*insert* $\langle x \geq x_0 \rangle$, *induction akra-bazzi-measure x arbitrary: x rule: less-induct*)
case *less*
show *?case*
proof (*cases* $x \leq x_1$)
case *True*
with *base* **and** $\langle x \geq x_0 \rangle$ **show** *?thesis* .
next
case *False*
hence $x > x_1$ **by** *simp*
thus *?thesis*
proof (*rule rec*)
fix i **assume** $i < k$
from *x0-hb-bound7*[*OF - i, of x*] x **have** $bs!i * x + (hs!i) x \geq x_0$ **by** *simp*
with $i x$ **show** $P (bs!i * x + (hs!i) x)$
by (*intro less akra-bazzi-measure-decreases*) *simp-all*
qed
qed
qed

end

locale *akra-bazzi-real* = *akra-bazzi-real-recursion* +
 fixes *integrable integral*
 assumes *integral: akra-bazzi-integral integrable integral*
 fixes *f :: real ⇒ real*
 and *g :: real ⇒ real*
 and *C :: real*
 assumes *p-props:* $(\sum i < k. as!i * bs!i \text{ powr } p) = 1$
 and *f-base:* $x \geq x_0 \implies x \leq x_1 \implies f x \geq 0$
 and *f-rec:* $x > x_1 \implies f x = g x + (\sum i < k. as!i * f (bs!i * x + (hs!i) x))$
 and *g-nonneg:* $x \geq x_0 \implies g x \geq 0$
 and *C-bound:* $b \in \text{set } bs \implies x \geq x_1 \implies C * x \leq b * x - hb * x / \ln x \text{ powr } (1 + e)$
 and *g-integrable:* $x \geq x_0 \implies \text{integrable } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x$
begin

interpretation *akra-bazzi-integral integrable integral* **by** (*rule integral*)

lemma *akra-bazzi-integrable:*

$a \geq x_0 \implies a \leq b \implies \text{integrable } (\lambda x. g x / x \text{ powr } (p + 1)) a b$
by (*rule integrable-subinterval[OF g-integrable, of b]*) *simp-all*

definition *g-approx :: nat ⇒ real ⇒ real* **where**

*g-approx i x = x powr p * integral* $(\lambda u. g u / u \text{ powr } (p + 1)) (bs!i * x + (hs!i) x) x$

lemma *f-nonneg:* $x \geq x_0 \implies f x \geq 0$

proof (*induction x rule: akra-bazzi-induct*)

case (*base x*)

with *f-base[of x]* **show** ?*case* **by** *simp*

next

case (*rec x*)

with *x0-le-x1* **have** $g x \geq 0$ **by** (*intro g-nonneg*) *simp-all*

moreover {

fix *i* **assume** *i: i < k*

with *rec.IH* **have** $f (bs!i * x + (hs!i) x) \geq 0$ **by** *simp*

with *i* **have** $as!i * f (bs!i * x + (hs!i) x) \geq 0$

by (*intro mult-nonneg-nonneg[OF a-ge-0]*) *simp-all*

 }

hence $(\sum i < k. as!i * f (bs!i * x + (hs!i) x)) \geq 0$ **by** (*intro sum-nonneg*) *blast*

ultimately show $f x \geq 0$ **using** *rec.hyps* **by** (*subst f-rec*) *simp-all*

qed

definition *f-approx :: real ⇒ real* **where**

$f\text{-approx } x = x \text{ powr } p * (1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x)$

lemma *f-approx-aux*:

assumes $x \geq x_0$

shows $1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x \geq 1$

proof –

from *assms* **have** $\text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x \geq 0$

by (*intro integral-nonneg ballI g-nonneg divide-nonneg-nonneg g-integrable*)

simp-all

thus *?thesis* **by** *simp*

qed

lemma *f-approx-pos*: $x \geq x_0 \implies f\text{-approx } x > 0$

unfolding *f-approx-def* **by** (*intro mult-pos-pos, insert x0-pos, simp, drule f-approx-aux, simp*)

lemma *f-approx-nonneg*: $x \geq x_0 \implies f\text{-approx } x \geq 0$

using *f-approx-pos[of x]* **by** *simp*

lemma *f-approx-bounded-below*:

obtains c **where** $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies f\text{-approx } x \geq c$ $c > 0$

proof –

{

fix x **assume** $x \geq x_0$ $x \leq x_1$

with *x0-pos* **have** $x \text{ powr } p \geq \min (x_0 \text{ powr } p) (x_1 \text{ powr } p)$

by (*intro powr-lower-bound*) *simp-all*

with x **have** $f\text{-approx } x \geq \min (x_0 \text{ powr } p) (x_1 \text{ powr } p) * 1$

unfolding *f-approx-def* **by** (*intro mult-mono f-approx-aux*) *simp-all*

}

from *this* *x0-pos* *x1-pos* **show** *?thesis* **by** (*intro that[of min (x_0 powr p) (x_1 powr p)]*) *auto*

qed

lemma *asymptotics-aux*:

assumes $x \geq x_1$ $i < k$

assumes $s \equiv (\text{if } p \geq 0 \text{ then } 1 \text{ else } -1)$

shows $(bs!i*x - s*hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } p \leq (bs!i*x + (hs!i) x) \text{ powr } p$ (*is ?thesis1*)

and $(bs!i*x + (hs!i) x) \text{ powr } p \leq (bs!i*x + s*hb*x*\ln x \text{ powr } -(1+e)) \text{ powr } p$ (*is ?thesis2*)

proof –

from *assms* *x1-gt-1* **have** *ln-x-pos*: $\ln x > 0$ **by** *simp*

from *assms* *x1-pos* **have** *x-pos*: $x > 0$ **by** *simp*

from *assms* *x0-le-x1* **have** $hb / \ln x \text{ powr } (1+e) < bs!i/2$ **by** (*intro x0-hb-bound0*) *simp-all*

with *hb-nonneg* *ln-x-pos* **have** $(bs!i - hb * \ln x \text{ powr } -(1+e)) > 0$

by (*subst powr-minus*) (*simp-all add: field-simps*)

with * **have** $0 < x * (bs!i - hb * \ln x \text{ powr } -(1+e))$ **using** $x\text{-pos}$
by $(\text{subst } (asm) \text{ powr-minus, intro mult-pos-pos})$
hence $A: 0 < bs!i*x - hb * x * \ln x \text{ powr } -(1+e)$ **by** $(\text{simp add: algebra-simps})$

from $assms$ **have** $-(hb*x*\ln x \text{ powr } -(1+e)) \leq -|(hs!i) x|$
using $h\text{-bounds}[of x hs!i]$ **by** $(\text{subst neg-le-iff-le, subst powr-minus})$ $(\text{simp add: field-simps})$
also **have** $\dots \leq (hs!i) x$ **by** simp
finally **have** $B: bs!i*x - hb*x*\ln x \text{ powr } -(1+e) \leq bs!i*x + (hs!i) x$ **by** simp

have $(hs!i) x \leq |(hs!i) x|$ **by** simp
also **from** $assms$ **have** $\dots \leq (hb*x*\ln x \text{ powr } -(1+e))$
using $h\text{-bounds}[of x hs!i]$ **by** $(\text{subst powr-minus})$ $(\text{simp-all add: field-simps})$
finally **have** $C: bs!i*x + hb*x*\ln x \text{ powr } -(1+e) \geq bs!i*x + (hs!i) x$ **by** simp

from $A B C$ **show** $?thesis1$
by $(\text{cases } p \geq 0)$ $(\text{auto intro: powr-mono2 powr-mono2' simp: assms}(3))$
from $A B C$ **show** $?thesis2$
by $(\text{cases } p \geq 0)$ $(\text{auto intro: powr-mono2 powr-mono2' simp: assms}(3))$
qed

lemma $asymptotics1'$:

assumes $x \geq x_1$ $i < k$
shows $(bs!i*x) \text{ powr } p * (1 + \ln x \text{ powr } (-e/2)) \leq$
 $(bs!i*x + (hs!i) x) \text{ powr } p * (1 + \ln (bs!i*x + (hs!i) x) \text{ powr } (-e/2))$

proof –

from $assms$ $x0\text{-le-}x1$ **have** $x: x \geq x_0$ **by** simp
from $b\text{-pos}[of bs!i]$ $assms$ **have** $b\text{-pos: } bs!i > 0$ $bs!i \neq 0$ **by** simp-all
from $b\text{-less-1}[of bs!i]$ $assms$ **have** $b\text{-less-1: } bs!i < 1$ **by** simp
from $x1\text{-gt-1}$ $assms$ **have** $\ln\text{-}x\text{-pos: } \ln x > 0$ **by** simp
have $\text{mono: } \bigwedge a b. a \leq b \implies (bs!i*x) \text{ powr } p * a \leq (bs!i*x) \text{ powr } p * b$
by $(\text{rule mult-left-mono})$ simp-all

define $s :: \text{real}$ **where** $[abs\text{-def}]: s = (\text{if } p \geq 0 \text{ then } 1 \text{ else } -1)$

have $1 + \ln x \text{ powr } (-e/2) \leq$
 $(1 - s*hb*\text{inverse}(bs!i)*\ln x \text{ powr } -(1+e)) \text{ powr } p * (1 + \ln (bs!i*x + hb * x / \ln x \text{ powr } (1+e)) \text{ powr } (-e/2))$ $(\text{is } - \leq ?A * ?B)$

using $assms$ x **unfolding** $s\text{-def}$ **using** $asymptotics1[OF x assms(2)]$ $asymptotics1'[OF x assms(2)]$

by simp

also **have** $(bs!i*x) \text{ powr } p * \dots = (bs!i*x) \text{ powr } p * ?A * ?B$ **by** simp

also **from** $x0\text{-hb-bound0}[OF x, of bs!i]$ $hb\text{-nonneg } x \ln\text{-}x\text{-pos}$ $assms$

have $s*hb * \ln x \text{ powr } -(1 + e) < bs!i$

by $(\text{subst powr-minus})$ $(\text{simp-all add: field-simps } s\text{-def})$

hence $(bs!i*x) \text{ powr } p * ?A = (bs!i*x*(1 - s*hb*\text{inverse}(bs!i)*\ln x \text{ powr } -(1+e))) \text{ powr } p$

using $b\text{-pos}$ $assms$ x $x0\text{-pos}$ $b\text{-less-1}$ $\ln\text{-}x\text{-pos}$

by $(\text{subst powr-mult}[symmetric])$ $(\text{simp-all add: } s\text{-def field-simps})$

also have $bs!i*x*(1 - s*hb*inverse (bs!i)*ln x powr -(1+e)) = bs!i*x - s*hb*x*ln x powr -(1+e)$
using *b-pos assms* **by** (*simp add: algebra-simps*)
also have $?B = 1 + ln (bs!i*x + hb*x*ln x powr -(1+e)) powr (-e/2)$
by (*subst powr-minus*) (*simp add: field-simps*)

also {
from *x assms* **have** $(bs!i*x - s*hb*x*ln x powr -(1+e)) powr p \leq (bs!i*x + (hs!i) x) powr p$
using *asymptotics-aux(1)[OF assms(1,2) s-def]* **by** *blast*
moreover {
have $(hs!i) x \leq |(hs!i) x|$ **by** *simp*
also from *assms* **have** $|(hs!i) x| \leq hb * x / ln x powr (1+e)$ **by** (*intro h-bounds*) *simp-all*
finally have $(hs ! i) x \leq hb * x * ln x powr -(1 + e)$
by (*subst powr-minus*) (*simp-all add: field-simps*)
moreover from *x hb-nonneg x0-pos* **have** $hb * x * ln x powr -(1+e) \geq 0$
by (*intro mult-nonneg-nonneg*) *simp-all*
ultimately have $1 + ln (bs!i*x + hb * x * ln x powr -(1+e)) powr (-e/2)$
 \leq
 $1 + ln (bs!i*x + (hs!i) x) powr (-e/2)$ **using** *assms x e-pos b-pos x0-pos*
by (*intro add-left-mono powr-mono2' ln-mono ln-gt-zero step-pos x0-hb-bound7' add-pos-nonneg mult-pos-pos*) *simp-all*
}
ultimately have $(bs!i*x - s*hb*x*ln x powr -(1+e)) powr p * (1 + ln (bs!i*x + hb * x * ln x powr -(1+e)) powr (-e/2))$
 $\leq (bs!i*x + (hs!i) x) powr p * (1 + ln (bs!i*x + (hs!i) x) powr (-e/2))$
by (*rule mult-mono*) *simp-all*
}
finally show *?thesis* **by** (*simp-all add: mono*)
qed

lemma *asymptotics2'*:

assumes $x \geq x_1$ $i < k$

shows $(bs!i*x + (hs!i) x) powr p * (1 - ln (bs!i*x + (hs!i) x) powr (-e/2))$
 \leq
 $(bs!i*x) powr p * (1 - ln x powr (-e/2))$

proof –

define $s :: real$ **where** $s = (if p \geq 0 then 1 else -1)$

from *assms x0-le-x1* **have** $x \geq x_0$ **by** *simp*

from *assms x1-gt-1* **have** *ln-x-pos: ln x > 0* **by** *simp*

from *b-pos[of bs!i]* *assms* **have** *b-pos: bs!i > 0* *bs!i ≠ 0* **by** *simp-all*

from *b-pos hb-nonneg* **have** *pos: 1 + s * hb * (inverse (bs!i) * ln x powr -(1+e)) > 0*

using *x0-hb-bound0'[OF x, of bs!i]* *b-pos assms ln-x-pos*

by (*subst powr-minus*) (*simp add: field-simps s-def*)

have *mono: $\bigwedge a b. a \leq b \implies (bs!i*x) powr p * a \leq (bs!i*x) powr p * b$*

by (rule mult-left-mono) simp-all

let $?A = (1 + s*hb*inverse(bs!i)*ln x powr -(1+e)) powr p$
let $?B = 1 - ln (bs!i*x + (hs!i) x) powr (-e/2)$
let $?B' = 1 - ln (bs!i*x + hb * x / ln x powr (1+e)) powr (-e/2)$

from *assms* *x* have $(bs!i*x + (hs!i) x) powr p \leq (bs!i*x + s*hb*x*ln x powr -(1+e)) powr p$
by (intro *asymptotics-aux*(2)) (simp-all add: *s-def*)
moreover from *x0-hb-bound9*[*OF assms*(1,2)] have $?B \geq 0$ by (simp add: *field-simps*)
ultimately have $(bs!i*x + (hs!i) x) powr p * ?B \leq (bs!i*x + s*hb*x*ln x powr -(1+e)) powr p * ?B$ by (rule *mult-right-mono*)
also from *assms e-pos pos* have $?B \leq ?B'$
proof –
from *x0-hb-bound8'*[*OF assms*(1,2)] *x0-hb-bound8*[*OF assms*(1,2)] *x0-ge-1*
have $*: bs ! i * x + s*hb * x / ln x powr (1 + e) > 1$ by (simp add: *s-def*)
moreover from $*$ have $\dots > 0$ by *simp*
moreover from *x0-hb-bound7*[*OF assms*(1,2)] *x0-ge-1* have $bs ! i * x + (hs ! i) x > 1$ by *simp*
moreover {
have $(hs!i) x \leq |(hs!i) x|$ by *simp*
also from *assms x0-le-x1* have $\dots \leq hb*x/ln x powr (1+e)$ by (intro *h-bounds*)
simp-all
finally have $bs!i*x + (hs!i) x \leq bs!i*x + hb*x/ln x powr (1+e)$ by *simp*
}
ultimately show $?B \leq ?B'$ using *assms e-pos x step-pos*
by (intro *diff-left-mono powr-mono2' ln-mono ln-gt-zero*) *simp-all*
qed
hence $(bs!i*x + s*hb*x*ln x powr -(1+e)) powr p * ?B \leq (bs!i*x + s*hb*x*ln x powr -(1+e)) powr p * ?B'$ by (intro *mult-left-mono*) *simp-all*
also have $bs!i*x + s*hb*x*ln x powr -(1+e) = bs!i*x*(1 + s*hb*inverse (bs!i)*ln x powr -(1+e))$
using *b-pos* by (simp-all add: *field-simps*)
also have $\dots powr p = (bs!i*x) powr p * ?A$
using *powr-mult* by *force*
also have $(bs!i*x) powr p * ?A * ?B' = (bs!i*x) powr p * (?A * ?B')$ by *simp*
also have $?A * ?B' \leq 1 - ln x powr (-e/2)$ using *assms x*
using *asymptotics2*[*OF x assms*(2)] *asymptotics2'*[*OF x assms*(2)] by (simp add: *s-def*)
finally show *?thesis* by (simp-all add: *mono*)
qed

lemma *Cx-le-step*:
assumes $i < k$ $x \geq x_1$
shows $C*x \leq bs!i*x + (hs!i) x$
proof –

from *assms* **have** $C*x \leq bs!i*x - hb*x/\ln x \text{ powr } (1+e)$ **by** (*intro C-bound*)
simp-all
also from *assms* **have** $-(hb*x/\ln x \text{ powr } (1+e)) \leq -|(hs!i) x|$
by (*subst neg-le-iff-le, intro h-bounds*) *simp-all*
hence $bs!i*x - hb*x/\ln x \text{ powr } (1+e) \leq bs!i*x + -|(hs!i) x|$ **by** *simp*
also have $-|(hs!i) x| \leq (hs!i) x$ **by** *simp*
finally show *?thesis* **by** *simp*
qed
end

locale *akra-bazzi-nat-to-real* = *akra-bazzi-real-recursion* +
fixes $f :: \text{nat} \Rightarrow \text{real}$
and $g :: \text{real} \Rightarrow \text{real}$
assumes *f-base*: $\text{real } x \geq x_0 \Longrightarrow \text{real } x \leq x_1 \Longrightarrow f x \geq 0$
and *f-rec*: $\text{real } x > x_1 \Longrightarrow$
 $f x = g (\text{real } x) + (\sum i < k. as!i * f (\text{nat } \lfloor bs!i * x + (hs!i)$
 $(\text{real } x) \rfloor))$
and *x0-int*: $\text{real } (\text{nat } \lfloor x_0 \rfloor) = x_0$
begin

function $f' :: \text{real} \Rightarrow \text{real}$ **where**
 $x \leq x_1 \Longrightarrow f' x = f (\text{nat } \lfloor x \rfloor)$
 $| x > x_1 \Longrightarrow f' x = g x + (\sum i < k. as!i * f' (bs!i * x + (hs!i) x))$
by (*force, simp-all*)
termination by (*relation Wellfounded.measure akra-bazzi-measure*)
(simp-all add: akra-bazzi-measure-decreases)

lemma *f'-base*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f' x \geq 0$
apply (*subst f'.simps(1), assumption*)
apply (*rule f-base*)
apply (*rule order.trans[of - real (nat floor x)], simp add: x0-int*)
apply (*subst of-nat-le-iff, intro nat-mono floor-mono, assumption*)
using *x0-pos* **apply** *linarith*
done

lemmas *f'-rec* = *f'.simps(2)*

end

locale *akra-bazzi-real-lower* = *akra-bazzi-real* +
fixes $fb2 \text{ gb2 } c2 :: \text{real}$
assumes *f-base2*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow f x \geq fb2$
and *fb2-pos*: $fb2 > 0$
and *g-growth2*: $\forall x \geq x_1. \forall u \in \{C*x..x\}. c2 * g x \geq g u$
and *c2-pos*: $c2 > 0$
and *g-bounded*: $x \geq x_0 \Longrightarrow x \leq x_1 \Longrightarrow g x \leq gb2$

begin

interpretation *akra-bazzi-integral integrable integral* **by** (*rule integral*)

lemma *gb2-nonneg*: $gb2 \geq 0$ **using** *g-bounded*[*of x₀*] *x0-le-x1* *x0-pos* *g-nonneg*[*of x₀*] **by** *simp*

lemma *g-growth2'*:

assumes $x \geq x_1$ $i < k$ $u \in \{bs!i*x+(hs!i) x..x\}$

shows $c2 * g x \geq g u$

proof –

from *assms* **have** $C*x \leq bs!i*x+(hs!i) x$ **by** (*intro Cx-le-step*)

with *assms* **have** $u \in \{C*x..x\}$ **by** *auto*

with *assms* *g-growth2* **show** *?thesis* **by** *simp*

qed

lemma *g-bounds2*:

obtains c_4 **where** $\bigwedge x i. x \geq x_1 \implies i < k \implies g\text{-approx } i x \leq c_4 * g x$ $c_4 > 0$

proof –

define c_4

where $c_4 = \text{Max } \{c2 / \min 1 (\min ((b/2) \text{powr } (p+1)) ((b*3/2) \text{powr } (p+1)))$

$|b. b \in \text{set } bs\}$

{

from *bs-nonempty* **obtain** b **where** $b: b \in \text{set } bs$ **by** (*cases bs*) *auto*

let $?m = \min 1 (\min ((b/2) \text{powr } (p+1)) ((b*3/2) \text{powr } (p+1)))$

from b *b-pos* **have** $?m > 0$ **unfolding** *min-def* **by** (*auto simp: not-le*)

with b *b-pos* *c2-pos* **have** $c2 / ?m > 0$ **by** (*simp-all add: field-simps*)

with b **have** $c_4 > 0$ **unfolding** *c4-def* **by** (*subst Max-gr-iff*) (*simp, simp,*

blast)

}

{

fix $x i$ **assume** $i: i < k$ **and** $x: x \geq x_1$

have *powr-negD*: $a \text{powr } b \leq 0 \implies a = 0$

for $a b :: \text{real}$ **unfolding** *powr-def* **by** (*simp split: if-split-asm*)

let $?m = \min 1 (\min ((bs!i/2) \text{powr } (p+1)) ((bs!i*3/2) \text{powr } (p+1)))$

have $\min 1 ((bs!i + (hs ! i) x / x) \text{powr } (p+1)) \geq \min 1 (\min ((bs!i/2) \text{powr } (p+1)) ((bs!i*3/2) \text{powr } (p+1)))$

apply (*insert x i x0-le-x1 x1-pos step-pos b-pos[OF b-in-bs[OF i]],*

rule min.mono, simp, cases p + 1 ≥ 0)

apply (*rule order.trans[OF min.cobounded1 powr-mono2[OF - - x0-hb-bound4 '], simp-all add: field-simps]*) []

apply (*rule order.trans[OF min.cobounded2 powr-mono2'[OF - - x0-hb-bound5]], simp-all add: field-simps]*) []

done

with i *b-pos*[*of bs!i*] **have** $c2 / \min 1 ((bs!i + (hs ! i) x / x) \text{powr } (p+1)) \leq c2 / ?m$ **using** *c2-pos*

unfolding *min-def* **by** (*intro divide-left-mono*) (*auto intro!: mult-pos-pos dest!:*

powr-negD)

```

also from  $i$  have  $\dots \leq c_4$  unfolding  $c_4$ -def by (intro Max.coboundedI) auto
finally have  $c_2 / \min 1 ((bs!i + (hs!i) x / x) \text{ powr } (p+1)) \leq c_4$  .
} note  $c_4 = \text{this}$ 

{
  fix  $x :: \text{real}$  and  $i :: \text{nat}$ 
  assume  $x: x \geq x_1$  and  $i: i < k$ 
  from  $x$   $x_1$ -pos have  $x$ -pos:  $x > 0$  by simp
  let  $?x' = bs!i * x + (hs!i) x$ 
  let  $?x'' = bs!i + (hs!i) x / x$ 
  from  $x$   $x_1$ -ge-1  $i$   $g$ -growth2'  $x_0$ -le- $x_1$   $c_2$ -pos
    have  $c_2: c_2 > 0 \forall u \in \{?x'..x\}. g u \leq c_2 * g x$  by auto

  from  $x_0$ -le- $x_1$   $x$   $i$  have  $x'$ -le- $x: ?x' \leq x$  by (intro step-le- $x$ ) simp-all
  let  $?m = \min (?x' \text{ powr } (p + 1)) (x \text{ powr } (p + 1))$ 
  define  $m'$  where  $m' = \min 1 (?x'' \text{ powr } (p + 1))$ 
  have [simp]:  $bs!i > 0$  by (intro b-pos nth-mem) (simp add: i length-bs)
  from  $x_0$ -le- $x_1$   $x$   $i$  have [simp]:  $?x' > 0$  by (intro step-pos) simp-all

  {
    fix  $u$  assume  $u: u \geq ?x' u \leq x$ 
    have  $?m \leq u \text{ powr } (p + 1)$  using  $x$   $u$  by (intro powr-lower-bound mult-pos-pos)
  } simp-all
  moreover from  $c_2$  and  $u$  have  $g u \leq c_2 * g x$  by simp
  ultimately have  $g u * ?m \leq c_2 * g x * u \text{ powr } (p + 1)$  using  $c_2$   $x$   $x_1$ -pos
 $x_0$ -le- $x_1$ 
    by (intro mult-mono mult-nonneg-nonneg  $g$ -nonneg) auto
  }
  hence  $\text{integral } (\lambda u. g u / u \text{ powr } (p+1)) ?x' x \leq \text{integral } (\lambda u. c_2 * g x / ?m)$ 
 $?x' x$ 
    using  $x$ -pos  $step$ -pos[OF  $i$   $x$ ]  $x_0$ -hb-bound7[OF  $x$   $i$ ]  $c_2$   $x$   $x_0$ -le- $x_1$ 
    by (intro integral-le  $x'$ -le- $x$  akra-bazzi-integrable ballI integrable-const)
    (auto simp: field-simps intro!: mult-nonneg-nonneg  $g$ -nonneg)

  also from  $x_0$ -pos  $x$   $x_0$ -le- $x_1$   $x'$ -le- $x$   $c_2$  have  $\dots = (x - ?x') * (c_2 * g x / ?m)$ 
    by (subst integral-const) (simp-all add:  $g$ -nonneg)
  also from  $c_2$   $x$ -pos  $x$   $x_0$ -le- $x_1$  have  $c_2 * g x \geq 0$ 
    by (intro mult-nonneg-nonneg  $g$ -nonneg) simp-all
  with  $x$   $i$   $x_0$ -le- $x_1$  have  $(x - ?x') * (c_2 * g x / ?m) \leq x * (c_2 * g x / ?m)$ 
    by (intro  $x_0$ -hb-bound3 mult-right-mono) (simp-all add: field-simps)

  also have  $x \text{ powr } (p + 1) = x \text{ powr } (p + 1) * 1$  by simp
  also have  $(bs!i * x + (hs!i) x) \text{ powr } (p + 1) =$ 
     $(bs!i + (hs!i) x / x) \text{ powr } (p + 1) * x \text{ powr } (p + 1)$ 
    using  $x$   $x_1$ -pos  $step$ -pos[OF  $i$   $x$ ]
    by (simp add: ring-class.ring-distrib flip: powr-mult)

```

also have $\dots = x \text{ powr } (p + 1) * (bs ! i + (hs ! i) x / x) \text{ powr } (p + 1)$ **by**
simp
also have $\min \dots (x \text{ powr } (p + 1) * 1) = x \text{ powr } (p + 1) * m'$ **unfolding**
m'-def **using** *x-pos*
by (*subst min.commute, intro min-mult-left[symmetric]*) *simp*

also from *x-pos* **have** $x * (c2 * g x / (x \text{ powr } (p + 1) * m^{\wedge})) = (c2/m^{\wedge}) * (g$
 $x / x \text{ powr } p)$
by (*simp add: field-simps powr-add*)
also from *x i g-nonneg x0-le-x1 x1-pos* **have** $\dots \leq c4 * (g x / x \text{ powr } p)$
unfolding *m'-def*
by (*intro mult-right-mono c4*) (*simp-all add: field-simps*)
finally have $g \text{ approx } i x \leq c4 * g x$
unfolding *g-approx-def* **using** *x-pos* **by** (*simp add: field-simps*)
}
thus *?thesis* **using** *that <c4 > 0* **by** *blast*
qed

lemma *f-approx-bounded-above*:

obtains *c* **where** $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies f \text{ approx } x \leq c \ c > 0$

proof–

let *?m1* = $\max (x_0 \text{ powr } p) (x_1 \text{ powr } p)$
let *?m2* = $\max (x_0 \text{ powr } (-(p+1))) (x_1 \text{ powr } (-(p+1)))$
let *?m3* = *gb2* * *?m2*
let *?m4* = $1 + (x_1 - x_0) * ?m3$
let *?int* = $\lambda x. \text{ integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x$
{
fix *x* **assume** $x: x \geq x_0 \ x \leq x_1$
with *x0-pos* **have** $x \text{ powr } p \leq ?m1 \ ?m1 \geq 0$ **by** (*intro powr-upper-bound*)
(*simp-all add: max-def*)
moreover **{**
fix *u* **assume** $u: u \in \{x_0..x\}$
have $g u / u \text{ powr } (p + 1) = g u * u \text{ powr } (-(p+1))$
by (*subst powr-minus*) (*simp add: field-simps*)
also from *u x x0-pos* **have** $u \text{ powr } (-(p+1)) \leq ?m2$
by (*intro powr-upper-bound*) *simp-all*
hence $g u * u \text{ powr } (-(p+1)) \leq g u * ?m2$
using *u g-nonneg x0-pos* **by** (*intro mult-left-mono*) *simp-all*
also from *x u x0-pos* **have** $g u \leq gb2$ **by** (*intro g-bounded*) *simp-all*
hence $g u * ?m2 \leq gb2 * ?m2$ **by** (*intro mult-right-mono*) (*simp-all add:*
max-def)
finally have $g u / u \text{ powr } (p + 1) \leq ?m3$.
} **note** *A = this*
{
from *A x gb2-nonneg* **have** $?int x \leq \text{ integral } (\lambda-. ?m3) x_0 x$
by (*intro integral-le akra-bazzi-integrable integrable-const mult-nonneg-nonneg*)
(*simp-all add: le-max-iff-disj*)
also from *x gb2-nonneg* **have** $\dots \leq (x - x_0) * ?m3$
by (*subst integral-const*) (*simp-all add: le-max-iff-disj*)

also from x *gb2-nonneg* **have** $\dots \leq (x_1 - x_0) * ?m3$
by (*intro mult-right-mono mult-nonneg-nonneg*) (*simp-all add: max-def*)
finally have $1 + ?int\ x \leq ?m4$ **by** *simp*
}
moreover from x *g-nonneg x0-pos* **have** $?int\ x \geq 0$
by (*intro integral-nonneg akra-bazzi-integrable*) (*simp-all add: powr-def field-simps*)
hence $1 + ?int\ x \geq 0$ **by** *simp*
ultimately have $f\text{-approx}\ x \leq ?m1 * ?m4$
unfolding *f-approx-def* **by** (*intro mult-mono*)
hence $f\text{-approx}\ x \leq \max\ 1\ (?m1 * ?m4)$ **by** *simp*
}
from *that[OF this]* **show** $?thesis$ **by** *auto*
qed

lemma *f-bounded-below*:

assumes $c': c' > 0$
obtains c **where** $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies 2 * (c * f\text{-approx}\ x) \leq f\ x\ c \leq c'$
 $c > 0$
proof –
obtain c **where** $c: \bigwedge x. x_0 \leq x \implies x \leq x_1 \implies f\text{-approx}\ x \leq c\ c > 0$
by (*rule f-approx-bounded-above*) *blast*
{
fix x **assume** $x: x_0 \leq x\ x \leq x_1$
with c **have** $inverse\ c * f\text{-approx}\ x \leq 1$ **by** (*simp add: field-simps*)
moreover from x *f-base2 x0-pos* **have** $f\ x \geq fb2$ **by** *auto*
ultimately have $inverse\ c * f\text{-approx}\ x * fb2 \leq 1 * f\ x$ **using** *fb2-pos*
by (*intro mult-mono*) *simp-all*
hence $inverse\ c * fb2 * f\text{-approx}\ x \leq f\ x$ **by** (*simp add: field-simps*)
moreover have $min\ c'\ (inverse\ c * fb2) * f\text{-approx}\ x \leq inverse\ c * fb2 * f\text{-approx}\ x$
using *f-approx-nonneg x c*
by (*intro mult-right-mono f-approx-nonneg*) (*simp-all add: field-simps*)
ultimately have $2 * (min\ c'\ (inverse\ c * fb2) / 2 * f\text{-approx}\ x) \leq f\ x$ **by** *simp*
}
moreover from c' **have** $min\ c'\ (inverse\ c * fb2) / 2 \leq c'$ **by** *simp*
moreover have $min\ c'\ (inverse\ c * fb2) / 2 > 0$
using *c fb2-pos c'* **by** *simp*
ultimately show $?thesis$ **by** (*rule that*)
qed

lemma *akra-bazzi-lower*:

obtains $c5$ **where** $\bigwedge x. x \geq x_0 \implies f\ x \geq c5 * f\text{-approx}\ x\ c5 > 0$
proof –
obtain $c4$ **where** $c4: \bigwedge x\ i. x \geq x_1 \implies i < k \implies g\text{-approx}\ i\ x \leq c4 * g\ x\ c4 > 0$
by (*rule g-bounds2*) *blast*
hence $inverse\ c4 / 2 > 0$ **by** *simp*
then obtain $c5$ **where** $c5: \bigwedge x. x \geq x_0 \implies x \leq x_1 \implies 2 * (c5 * f\text{-approx}\ x) \leq f\ x$

$c5 \leq \text{inverse } c4 / 2 \ c5 > 0$

by (rule f-bounded-below) blast

{

fix x :: real assume x: $x \geq x_0$

from c5 x have $c5 * 1 * f\text{-approx } x \leq c5 * (1 + \ln x \text{ powr } (-e/2)) * f\text{-approx } x$

by (intro mult-right-mono mult-left-mono f-approx-nonneg) simp-all

also from x have $c5 * (1 + \ln x \text{ powr } (-e/2)) * f\text{-approx } x \leq f x$

proof (induction x rule: akra-bazzi-induct)

case (base x)

have $1 + \ln x \text{ powr } (-e/2) \leq 2$ using asymptotics3 base by simp

hence $(1 + \ln x \text{ powr } (-e/2)) * (c5 * f\text{-approx } x) \leq 2 * (c5 * f\text{-approx } x)$

using c5 f-approx-nonneg base x0-ge-1 by (intro mult-right-mono mult-nonneg-nonneg) simp-all

also from base have $2 * (c5 * f\text{-approx } x) \leq f x$ by (intro c5) simp-all

finally show ?case by (simp add: algebra-simps)

next

case (rec x)

let ?a = $\lambda i. as!i$ and ?b = $\lambda i. bs!i$ and ?h = $\lambda i. hs!i$

let ?int = $\text{integral } (\lambda u. g u / u \text{ powr } (p+1)) x_0 x$

let ?int1 = $\lambda i. \text{integral } (\lambda u. g u / u \text{ powr } (p+1)) x_0 (?b i*x + ?h i x)$

let ?int2 = $\lambda i. \text{integral } (\lambda u. g u / u \text{ powr } (p+1)) (?b i*x + ?h i x) x$

let ?l = $\ln x \text{ powr } (-e/2)$ and ?l' = $\lambda i. \ln (?b i*x + ?h i x) \text{ powr } (-e/2)$

from rec and x0-le-x1 x0-ge-1 have x: $x \geq x_0$ and x-gt-1: $x > 1$ by simp-all

with x0-pos have x-pos: $x > 0$ and x-nonneg: $x \geq 0$ by simp-all

from c5 c4 have $c5 * c4 \leq 1/2$ by (simp add: field-simps)

moreover from asymptotics3 x have $(1 + ?l) \leq 2$ by (simp add: field-simps)

ultimately have $(c5*c4)*(1 + ?l) \leq (1/2) * 2$ by (rule mult-mono) simp-all

hence $0 \leq 1 - c5*c4*(1 + ?l)$ by simp

with g-nonneg[OF x] have $0 \leq g x * \dots$ by (intro mult-nonneg-nonneg) simp-all

hence $c5 * (1 + ?l) * f\text{-approx } x \leq c5 * (1 + ?l) * f\text{-approx } x + g x - c5*c4*(1 + ?l) * g x$

by (simp add: algebra-simps)

also from x-gt-1 have $\dots = c5 * x \text{ powr } p * (1 + ?l) * (1 + ?int - c4*g x/x \text{ powr } p) + g x$

by (simp add: field-simps f-approx-def powr-minus)

also have $c5 * x \text{ powr } p * (1 + ?l) * (1 + ?int - c4*g x/x \text{ powr } p) =$
 $(\sum i < k. (?a i * ?b i \text{ powr } p) * (c5 * x \text{ powr } p * (1 + ?l) * (1 + ?int - c4*g x/x \text{ powr } p)))$

by (subst sum-distrib-right[symmetric]) (simp add: p-props)

also have $\dots \leq (\sum i < k. ?a i * f (?b i*x + ?h i x))$

proof (intro sum-mono, clarify)

fix i assume i: $i < k$

let ?f = $c5 * ?a i * (?b i * x) \text{ powr } p$

from rec.hyps i have $x_0 < bs ! i * x + (hs ! i) x$ by (intro x0-hb-bound7) simp-all

hence $1 + ?int1 i \geq 1$ by (intro f-approx-aux x0-hb-bound7) simp-all

hence $int\text{-nonneg}: 1 + ?int1\ i \geq 0$ **by** *simp*

have $(?a\ i * ?b\ i\ powr\ p) * (c5 * x\ powr\ p * (1 + ?l) * (1 + ?int - c4 * g\ x/x\ powr\ p)) =$
 $?f * (1 + ?l) * (1 + ?int - c4 * g\ x/x\ powr\ p)$ (**is** $?expr = ?A * ?B$)
using $x\text{-pos}\ b\text{-pos}[of\ bs!i]$ **by** (*subst powr-mult*) *simp-all*
also from *rec.hyps* **have** $g\text{-approx}\ i\ x \leq c4 * g\ x$ **by** (*intro c4*) *simp-all*
hence $c4 * g\ x/x\ powr\ p \geq ?int2\ i$ **unfolding** $g\text{-approx-def}$ **using** $x\text{-pos}$
by (*simp add: field-simps*)
hence $?A * ?B \leq ?A * (1 + (?int - ?int2\ i))$ **using** $i\ c5\ a\text{-ge-0}$
by (*intro mult-left-mono mult-nonneg-nonneg*) *simp-all*
also from *rec.hyps* **have** $x_0 < bs!\ i * x + (hs!\ i)\ x$ **by** (*intro x0-hb-bound7*)
simp-all
hence $?int - ?int2\ i = ?int1\ i$
apply (*subst diff-eq-eq, subst eq-commute*)
apply (*intro integral-combine akra-bazzi-integrable*)
apply (*insert rec.hyps step-le-x[OF i, of x], simp-all*)
done
also have $?A * (1 + ?int1\ i) = (c5 * ?a\ i * (1 + ?int1\ i)) * ((?b\ i * x)\ powr\ p * (1 + ?l))$
by (*simp add: algebra-simps*)
also have $\dots \leq (c5 * ?a\ i * (1 + ?int1\ i)) * ((?b\ i * x + ?h\ i\ x)\ powr\ p * (1 + ?l'\ i))$
using *rec.hyps* $i\ c5\ a\text{-ge-0}\ int\text{-nonneg}$
by (*intro mult-left-mono asymptotics1' mult-nonneg-nonneg*) *simp-all*
also have $\dots = ?a\ i * (c5 * (1 + ?l'\ i) * f\text{-approx}\ (?b\ i * x + ?h\ i\ x))$
by (*simp add: algebra-simps f-approx-def*)
also from i **have** $\dots \leq ?a\ i * f\ (?b\ i * x + ?h\ i\ x)$
by (*intro mult-left-mono a-ge-0 rec.IH*) *simp-all*
finally show $?expr \leq ?a\ i * f\ (?b\ i * x + ?h\ i\ x)$.
qed
also have $\dots + g\ x = f\ x$ **using** $f\text{-rec}[of\ x]$ *rec.hyps* $x0\text{-le-x1}$ **by** *simp*
finally show $?case$ **by** *simp*
qed
finally have $c5 * f\text{-approx}\ x \leq f\ x$ **by** *simp*
}
from *this* **and** $c5(3)$ **show** $?thesis$ **by** (*rule that*)
qed

lemma *akra-bazzi-bigomega*:
 $f \in \Omega(\lambda x. x\ powr\ p * (1 + \text{integral}(\lambda u. g\ u / u\ powr\ (p + 1))\ x_0\ x))$
apply (*fold f-approx-def, rule akra-bazzi-lower, erule landau-omega.bigI*)
apply (*subst eventually-at-top-linorder, rule exI[of - x0]*)
apply (*simp add: f-nonneg f-approx-nonneg*)
done

end

locale *akra-bazzi-real-upper* = *akra-bazzi-real* +
fixes *fb1 c1* :: *real*
assumes *f-base1*: $x \geq x_0 \implies x \leq x_1 \implies f x \leq fb1$
and *g-growth1*: $\forall x \geq x_1. \forall u \in \{C*x..x\}. c1 * g x \leq g u$
and *c1-pos*: $c1 > 0$
begin

interpretation *akra-bazzi-integral integrable integral* **by** (*rule integral*)

lemma *g-growth1'*:
assumes $x \geq x_1$ $i < k$ $u \in \{bs!i*x+(hs!i) x..x\}$
shows $c1 * g x \leq g u$
proof –
from *assms* **have** $C*x \leq bs!i*x+(hs!i) x$ **by** (*intro Cx-le-step*)
with *assms* **have** $u \in \{C*x..x\}$ **by** *auto*
with *assms* *g-growth1* **show** *?thesis* **by** *simp*
qed

lemma *g-bounds1*:
obtains *c3* **where**
 $\bigwedge x i. x \geq x_1 \implies i < k \implies c3 * g x \leq g\text{-approx } i x c3 > 0$
proof –
define *c3* **where** *c3* =
 $Min \{c1*((1-b)/2) / max 1 (max ((b/2) powr (p+1)) ((b*3/2) powr (p+1)))$
 $| b. b \in set bs\}$

{
fix *b* **assume** *b*: $b \in set bs$
let *?x* = $max 1 (max ((b/2) powr (p+1)) ((b*3/2) powr (p+1)))$
have $?x \geq 1$ **by** *simp*
hence $?x > 0$ **by** (*rule less-le-trans[OF zero-less-one]*)
with *b* *b-less-1* *c1-pos* **have** $c1*((1-b)/2) / ?x > 0$
by (*intro divide-pos-pos mult-pos-pos*) (*simp-all add: algebra-simps*)
}
hence $c3 > 0$ **unfolding** *c3-def* **by** (*subst Min-gr-iff*) *auto*

{
fix *x i* **assume** *i*: $i < k$ **and** *x*: $x \geq x_1$
with *b-less-1* **have** *b-less-1'*: $bs ! i < 1$ **by** *simp*
let *?m* = $max 1 (max ((bs!i/2) powr (p+1)) ((bs!i*3/2) powr (p+1)))$
from *i x* **have** $c3 \leq c1*((1-bs!i)/2) / ?m$ **unfolding** *c3-def* **by** (*intro Min.coboundedI*) *auto*
also **have** $max 1 ((bs!i + (hs ! i) x / x) powr (p+1)) \leq max 1 (max ((bs!i/2) powr (p+1)) ((bs!i*3/2) powr (p+1)))$
apply (*insert x i x0-le-x1 x1-pos step-pos[OF i x] b-pos[OF b-in-bs[OF i]]*,
rule max.mono, simp, cases p + 1 ≥ 0)
apply (*rule order.trans[OF powr-mono2[OF -- x0-hb-bound5] max.cobounded2]*,
simp-all add: field-simps) []
apply (*rule order.trans[OF powr-mono2'[OF -- x0-hb-bound4] max.cobounded1]*,

```

simp-all add: field-simps) []
  done
  with b-less-1' c1-pos have c1*((1-bs!i)/2) / ?m ≤
    c1*((1-bs!i)/2) / max 1 ((bs!i + (hs ! i) x / x) powr (p+1))
  by (intro divide-left-mono mult-nonneg-nonneg) (simp-all add: algebra-simps)
  finally have c3 ≤ c1*((1-bs!i)/2) / max 1 ((bs!i + (hs ! i) x / x) powr
(p+1)) .
} note c3 = this

{
  fix x :: real and i :: nat
  assume x: x ≥ x1 and i: i < k
  from x x1-pos have x-pos: x > 0 by simp
  let ?x' = bs ! i * x + (hs ! i) x
  let ?x'' = bs ! i + (hs ! i) x / x
  from x x1-ge-1 x0-le-x1 i c1-pos g-growth1'
    have c1: c1 > 0 ∀ u ∈ {?x'..x}. g u ≥ c1 * g x by auto
  define b' where b' = (1 - bs!i)/2

  from x x0-le-x1 i have x'-le-x: ?x' ≤ x by (intro step-le-x) simp-all
  let ?m = max (?x' powr (p + 1)) (x powr (p + 1))
  define m' where m' = max 1 (?x'' powr (p + 1))
  have [simp]: bs ! i > 0 by (intro b-pos nth-mem) (simp add: i length-bs)
  from x x0-le-x1 i have x'-pos: ?x' > 0 by (intro step-pos) simp-all
  have m-pos: ?m > 0 unfolding max-def using x-pos step-pos[OF i x] by auto
  with x x0-le-x1 c1 have c1-g-m-nonneg: c1 * g x / ?m ≥ 0
    by (intro mult-nonneg-nonneg divide-nonneg-pos g-nonneg) simp-all

  from x i g-nonneg x0-le-x1 have c3 * (g x / x powr p) ≤ (c1*b'/m') * (g x /
x powr p)
  unfolding m'-def b'-def by (intro mult-right-mono c3) (simp-all add: field-simps)
  also from x-pos have ... = (x * b') * (c1 * g x / (x powr (p + 1) * m'))
    by (simp add: field-simps powr-add)
  also from x i c1-pos x1-pos x0-le-x1
    have ... ≤ (x - ?x') * (c1 * g x / (x powr (p + 1) * m'))
  unfolding b'-def m'-def by (intro x0-hb-bound6 mult-right-mono mult-nonneg-nonneg
divide-nonneg-nonneg g-nonneg) simp-all
  also have x powr (p + 1) * m' =
    max (x powr (p + 1) * (bs ! i + (hs ! i) x / x) powr (p + 1)) (x
powr (p + 1) * 1)
  unfolding m'-def using x-pos by (subst max.commute, intro max-mult-left)
simp
  also have (x powr (p + 1) * (bs ! i + (hs ! i) x / x) powr (p + 1)) =
    (bs ! i + (hs ! i) x / x) powr (p + 1) * x powr (p + 1) by simp
  also have ... = (bs ! i * x + (hs ! i) x) powr (p + 1)
    using x x1-pos by (simp add: ring-class.ring-distrib flip: powr-mult)
  also have x powr (p + 1) * 1 = x powr (p + 1) by simp
  also have (x - ?x') * (c1 * g x / ?m) = integral (λ-. c1 * g x / ?m) ?x' x
    using x'-le-x by (subst integral-const[OF c1-g-m-nonneg]) auto

```

```

also {
  fix u assume u: u ≥ ?x' u ≤ x
  have u powr (p + 1) ≤ ?m using x u x'-pos by (intro powr-upper-bound
mult-pos-pos) simp-all
  moreover from x'-pos u have u ≥ 0 by simp
  moreover from c1 and u have c1 * g x ≤ g u by simp
  ultimately have c1 * g x * u powr (p + 1) ≤ g u * ?m using c1 x u
x0-hb-bound7[OF x i]
  by (intro mult-mono g-nonneg) auto
  with m-pos u step-pos[OF i x]
  have c1 * g x / ?m ≤ g u / u powr (p + 1) by (simp add: field-simps)
}
hence integral (λ-. c1 * g x / ?m) ?x' x ≤ integral (λu. g u / u powr (p + 1))
?x' x
  using x0-hb-bound7[OF x i] x'-le-x
  by (intro integral-le ballI akra-bazzi-integrable integrable-const c1-g-m-nonneg)
simp-all
  finally have c3 * g x ≤ g-approx i x using x-pos
  unfolding g-approx-def by (simp add: field-simps)
}
thus ?thesis using that ⟨c3 > 0⟩ by blast
qed

```

lemma *f-bounded-above*:

```

assumes c': c' > 0
  obtains c where ∧x. x ≥ x0 ⇒ x ≤ x1 ⇒ f x ≤ (1/2) * (c * f-approx x) c
≥ c' c > 0
proof -
  obtain c where c: ∧x. x0 ≤ x ⇒ x ≤ x1 ⇒ f-approx x ≥ c c > 0
  by (rule f-approx-bounded-below) blast
  have fb1-nonneg: fb1 ≥ 0 using f-base1[of x0] f-nonneg[of x0] x0-le-x1 by simp
  {
    fix x assume x: x ≥ x0 x ≤ x1
    with f-base1 x0-pos have f x ≤ fb1 by simp
    moreover from c and x have f-approx x ≥ c by blast
    ultimately have f x * c ≤ fb1 * f-approx x using c fb1-nonneg by (intro
mult-mono) simp-all
    also from f-approx-nonneg x have ... ≤ (fb1 + 1) * f-approx x by (simp add:
algebra-simps)
    finally have f x ≤ ((fb1+1) / c) * f-approx x by (simp add: field-simps c)
    also have ... ≤ max ((fb1+1) / c) c' * f-approx x
    by (intro mult-right-mono) (simp-all add: f-approx-nonneg x)
    finally have f x ≤ 1/2 * (max ((fb1+1) / c) c' * 2 * f-approx x) by simp
  }
  moreover have max ((fb1+1) / c) c' * 2 ≥ max ((fb1+1) / c) c'
  by (subst mult-le-cancel-left1) (insert c', simp)
  hence max ((fb1+1) / c) c' * 2 ≥ c' by (rule order.trans[OF max.cobounded2])
  moreover from fb1-nonneg and c have (fb1+1) / c > 0 by simp

```

hence $\max((fb1+1) / c) c' * 2 > 0$ by *simp*
ultimately show *?thesis* by (rule that)
qed

lemma *akra-bazzi-upper*:

obtains *c6* where $\bigwedge x. x \geq x_0 \implies f x \leq c6 * f\text{-approx } x \ c6 > 0$

proof –

obtain *c3* where *c3*: $\bigwedge x i. x \geq x_1 \implies i < k \implies c3 * g x \leq g\text{-approx } i \ x \ c3 > 0$

by (rule *g-bounds1*) *blast*

hence $2 / c3 > 0$ by *simp*

then obtain *c6* where *c6*: $\bigwedge x. x \geq x_0 \implies x \leq x_1 \implies f x \leq 1/2 * (c6 * f\text{-approx } x)$

$$c6 \geq 2 / c3 \ c6 > 0$$

by (rule *f-bounded-above*) *blast*

{

fix *x* :: *real* assume *x*: $x \geq x_0$

hence $f x \leq c6 * (1 - \ln x \text{ powr } (-e/2)) * f\text{-approx } x$

proof (induction *x* rule: *akra-bazzi-induct*)

case (*base x*)

from *base* have $f x \leq 1/2 * (c6 * f\text{-approx } x)$ by (intro *c6*) *simp-all*

also have $1 - \ln x \text{ powr } (-e/2) \geq 1/2$ using *asymptotics4 base* by *simp*

hence $(1 - \ln x \text{ powr } (-e/2)) * (c6 * f\text{-approx } x) \geq 1/2 * (c6 * f\text{-approx } x)$

using *c6 f-approx-nonneg base x0-ge-1* by (intro *mult-right-mono mult-nonneg-nonneg*)

simp-all

finally show *?case* by (*simp add: algebra-simps*)

next

case (*rec x*)

let *?a* = $\lambda i. a!i$ and *?b* = $\lambda i. b!i$ and *?h* = $\lambda i. h!i$

let *?int* = $\text{integral } (\lambda u. g u / u \text{ powr } (p+1)) \ x_0 \ x$

let *?int1* = $\lambda i. \text{integral } (\lambda u. g u / u \text{ powr } (p+1)) \ x_0 \ (?b \ i*x + ?h \ i \ x)$

let *?int2* = $\lambda i. \text{integral } (\lambda u. g u / u \text{ powr } (p+1)) \ (?b \ i*x + ?h \ i \ x) \ x$

let *?l* = $\ln x \text{ powr } (-e/2)$ and *?l'* = $\lambda i. \ln (?b \ i*x + ?h \ i \ x) \text{ powr } (-e/2)$

from *rec* and *x0-le-x1* have *x*: $x \geq x_0$ by *simp*

with *x0-pos* have *x-pos*: $x > 0$ and *x-nonneg*: $x \geq 0$ by *simp-all*

from *c6 c3* have $c6 * c3 \geq 2$ by (*simp add: field-simps*)

have $f x = (\sum i < k. ?a \ i * f (?b \ i*x + ?h \ i \ x)) + g x$ (*is - = ?sum + -*)

using *f-rec[of x rec.hyps x0-le-x1]* by *simp*

also have $?sum \leq (\sum i < k. (?a \ i * ?b \ i \text{ powr } p) * (c6 * x \text{ powr } p * (1 - ?l) * (1 + ?int - c3 * g x / x \text{ powr } p)))$ (*is - ≤ ?sum'*)

proof (rule *sum-mono, clarify*)

fix *i* assume *i*: $i < k$

from *rec.hyps i* have $x_0 < b!i * x + (h!i) x$ by (intro *x0-hb-bound7*)

simp-all

hence $1 + ?int1 \ i \geq 1$ by (intro *f-approx-aux x0-hb-bound7*) *simp-all*

hence *int-nonneg*: $1 + ?int1 \ i \geq 0$ by *simp*

have $l-le-1: \ln x \text{ powr } -(e/2) \leq 1$ **using** *asymptotics3*[*OF x*] **by** (*simp add: field-simps*)

from i **have** $f (?b i*x + ?h i x) \leq c6 * (1 - ?l' i) * f\text{-approx } (?b i*x + ?h i x)$

by (*rule rec.IH*)

hence $?a i * f (?b i*x + ?h i x) \leq ?a i * \dots$ **using** *a-ge-0 i*

by (*intro mult-left-mono*) *simp-all*

also have $\dots = (c6 * ?a i * (1 + ?int1 i)) * ((?b i*x + ?h i x) \text{ powr } p * (1 - ?l' i))$

unfolding *f-approx-def* **by** (*simp add: algebra-simps*)

also from *rec.hyps c6 a-ge-0*

have $\dots \leq (c6 * ?a i * (1 + ?int1 i)) * ((?b i*x) \text{ powr } p * (1 - ?l))$

by (*intro mult-left-mono asymptotics2' mult-nonneg-nonneg int-nonneg*)

simp-all

also have $\dots = (1 + ?int1 i) * (c6 * ?a i * (?b i*x) \text{ powr } p * (1 - ?l))$

by (*simp add: algebra-simps*)

also from *rec.hyps i* **have** $x_0 < bs ! i * x + (hs ! i) x$ **by** (*intro x0-hb-bound7*)

simp-all

hence $?int1 i = ?int - ?int2 i$

apply (*subst eq-diff-eq*)

apply (*intro integral-combine akra-bazzi-integrable*)

apply (*insert rec.hyps step-le-x*[*OF i, of x*], *simp-all*)

done

also from *rec.hyps i* **have** $c3 * g x \leq g\text{-approx } i x$ **by** (*intro c3*) *simp-all*

hence $?int2 i \geq c3 * g x / x \text{ powr } p$ **unfolding** *g-approx-def* **using** *x-pos*

by (*simp add: field-simps*)

hence $(1 + (?int - ?int2 i)) * (c6 * ?a i * (?b i*x) \text{ powr } p * (1 - ?l)) \leq$
 $(1 + ?int - c3 * g x / x \text{ powr } p) * (c6 * ?a i * (?b i*x) \text{ powr } p * (1 - ?l))$

using *i c6 a-ge-0 l-le-1*

by (*intro mult-right-mono mult-nonneg-nonneg*) (*simp-all add: field-simps*)

also have $\dots = (?a i * ?b i \text{ powr } p) * (c6 * x \text{ powr } p * (1 - ?l)) * (1 + ?int - c3 * g x / x \text{ powr } p)$

using *b-pos*[*of bs!i*] *x x0-pos i* **by** (*subst powr-mult*) (*simp-all add: algebra-simps*)

finally show $?a i * f (?b i*x + ?h i x) \leq \dots$.

qed

hence $?sum + g x \leq ?sum' + g x$ **by** *simp*

also have $\dots = c6 * x \text{ powr } p * (1 - ?l) * (1 + ?int - c3 * g x / x \text{ powr } p) + g x$

by (*simp add: sum-distrib-right*[*symmetric*] *p-props*)

also have $\dots = c6 * (1 - ?l) * f\text{-approx } x - (c6 * c3 * (1 - ?l) - 1) * g x$

unfolding *f-approx-def* **using** *x-pos* **by** (*simp add: field-simps*)

also {

from *c6 c3* **have** $c6 * c3 \geq 2$ **by** (*simp add: field-simps*)

moreover have $(1 - ?l) \geq 1/2$ **using** *asymptotics4*[*OF x*] **by** *simp*

ultimately have $c6 * c3 * (1 - ?l) \geq 2 * (1/2)$ **by** (*intro mult-mono*) *simp-all*

with *x x-pos* **have** $(c6 * c3 * (1 - ?l) - 1) * g x \geq 0$

```

    by (intro mult-nonneg-nonneg g-nonneg) simp-all
  hence  $c6 * (1 - ?l) * f\text{-approx } x - (c6 * c3 * (1 - ?l) - 1) * g x \leq$ 
        $c6 * (1 - ?l) * f\text{-approx } x$  by (simp add: algebra-simps)
}
finally show ?case .
qed
also from  $x c6$  have ...  $\leq c6 * 1 * f\text{-approx } x$ 
  by (intro mult-left-mono mult-right-mono f-approx-nonneg) simp-all
finally have  $f x \leq c6 * f\text{-approx } x$  by simp
}
from this and  $c6(3)$  show ?thesis by (rule that)
qed

```

```

lemma akra-bazzi-bigo:
   $f \in O(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g u / u \text{ powr } (p + 1)) x_0 x))$ 
  apply (fold f-approx-def, rule akra-bazzi-upper, erule landau-o.bigI)
  apply (subst eventually-at-top-linorder, rule exI[of -  $x_0$ ])
  apply (simp add: f-nonneg f-approx-nonneg)
done

```

end

end

4 The discrete Akra-Bazzi theorem

```

theory Akra-Bazzi

```

```

imports

```

```

  Complex-Main

```

```

  HOL-Library.Landau-Symbols

```

```

  Akra-Bazzi-Real

```

```

begin

```

```

lemma ex-mono:  $(\exists x. P x) \implies (\bigwedge x. P x \implies Q x) \implies (\exists x. Q x)$  by blast

```

```

lemma x-over-ln-mono:

```

```

  assumes  $(e::real) > 0$ 

```

```

  assumes  $x > \exp e$ 

```

```

  assumes  $x \leq y$ 

```

```

  shows  $x / \ln x \text{ powr } e \leq y / \ln y \text{ powr } e$ 

```

```

proof (rule DERIV-nonneg-imp-mono[of - -  $\lambda x. x / \ln x \text{ powr } e$ ])

```

```

  fix  $t$  assume  $t: t \in \{x..y\}$ 

```

```

  from  $\text{assms}(1)$  have  $1 < \exp e$  by simp

```

```

  from this and  $\text{assms}(2)$  have  $x > 1$  by (rule less-trans)

```

```

  with  $t$  have  $t': t > 1$  by simp

```

```

  from  $\langle x > \exp e \rangle$  and  $t$  have  $t > \exp e$  by simp

```

```

  with  $t'$  have  $\ln t > \ln (\exp e)$  by (subst ln-less-cancel-iff) simp-all

```

```

  hence  $t': \ln t > e$  by simp

```

```

  show  $((\lambda x. x / \ln x \text{ powr } e)$  has-real-derivative

```

$(\ln t - e) / \ln t \text{ powr } (e+1)$ (at t) **using** *assms t t' t''*
by (*force intro!*: *derivative-eq-intros simp*: *powr-diff field-simps powr-add*)
from t'' **show** $(\ln t - e) / \ln t \text{ powr } (e + 1) \geq 0$ **by** (*intro divide-nonneg-nonneg*)
simp-all
qed (*simp-all add: assms*)

definition *akra-bazzi-term* :: $\text{nat} \Rightarrow \text{nat} \Rightarrow \text{real} \Rightarrow (\text{nat} \Rightarrow \text{nat}) \Rightarrow \text{bool}$ **where**
akra-bazzi-term $x_0 x_1 b t =$
 $(\exists e h. e > 0 \wedge (\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+e))) \wedge$
 $(\forall x \geq x_1. t x \geq x_0 \wedge t x < x \wedge b*x + h x = \text{real } (t x))$

lemma *akra-bazzi-termI* [*intro?*]:
assumes $e > 0 (\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+e))$
 $\bigwedge x. x \geq x_1 \implies t x \geq x_0 \bigwedge x. x \geq x_1 \implies t x < x$
 $\bigwedge x. x \geq x_1 \implies b*x + h x = \text{real } (t x)$
shows *akra-bazzi-term* $x_0 x_1 b t$
using *assms unfolding akra-bazzi-term-def by blast*

lemma *akra-bazzi-term-imp-less*:
assumes *akra-bazzi-term* $x_0 x_1 b t x \geq x_1$
shows $t x < x$
using *assms unfolding akra-bazzi-term-def by blast*

lemma *akra-bazzi-term-imp-less'*:
assumes *akra-bazzi-term* $x_0 (\text{Suc } x_1) b t x > x_1$
shows $t x < x$
using *assms unfolding akra-bazzi-term-def by force*

locale *akra-bazzi-recursion* =
fixes $x_0 x_1 k :: \text{nat}$ **and** *as bs* :: *real list* **and** *ts* :: $(\text{nat} \Rightarrow \text{nat})$ *list* **and** *f* :: $\text{nat} \Rightarrow \text{real}$
assumes *k-not-0*: $k \neq 0$
and *length-as*: $\text{length } as = k$
and *length-bs*: $\text{length } bs = k$
and *length-ts*: $\text{length } ts = k$
and *a-ge-0*: $a \in \text{set } as \implies a \geq 0$
and *b-bounds*: $b \in \text{set } bs \implies b \in \{0 <..<1\}$
and *ts*: $i < \text{length } bs \implies \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i)$
begin

sublocale *akra-bazzi-params* $k as bs$
using *length-as length-bs k-not-0 a-ge-0 b-bounds by unfold-locales*

lemma *ts-nonempty*: $ts \neq []$ **using** *length-ts k-not-0 by (cases ts) simp-all*

definition *e-hs* :: $\text{real} \times (\text{nat} \Rightarrow \text{real})$ *list* **where**
 $e\text{-hs} = (\text{SOME } (e,hs)).$

$e > 0 \wedge \text{length } hs = k \wedge (\forall h \in \text{set } hs. (\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+e))) \wedge$
 $(\forall t \in \text{set } ts. \forall x \geq x_1. t x \geq x_0 \wedge t x < x) \wedge$
 $(\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i) x = \text{real } ((ts!i) x))$
 $)$

definition $e = (\text{case } e\text{-hs of } (e,-) \Rightarrow e)$

definition $hs = (\text{case } e\text{-hs of } (-,hs) \Rightarrow hs)$

lemma *filterlim-powr-zero-cong*:

filterlim $(\lambda x. P (x::\text{real}) (x \text{ powr } (0::\text{real}))) F \text{ at-top} = \text{filterlim } (\lambda x. P x 1) F \text{ at-top}$

apply (*rule filterlim-cong[OF refl refl]*)

using *eventually-gt-at-top[of 0::real]* **by** *eventually-elim simp-all*

lemma *e-hs-aux*:

$0 < e \wedge \text{length } hs = k \wedge$
 $(\forall h \in \text{set } hs. (\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1 + e))) \wedge$
 $(\forall t \in \text{set } ts. \forall x \geq x_1. x_0 \leq t x \wedge t x < x) \wedge$
 $(\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i) x = \text{real } ((ts!i) x))$

proof–

have $Ex (\lambda(e,hs). e > 0 \wedge \text{length } hs = k \wedge$
 $(\forall h \in \text{set } hs. (\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+e))) \wedge$
 $(\forall t \in \text{set } ts. \forall x \geq x_1. t x \geq x_0 \wedge t x < x) \wedge$
 $(\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i) x = \text{real } ((ts!i) x)))$

proof–

from ts **have** $A: \forall i \in \{..<k\}. \text{akra-bazzi-term } x_0 x_1 (bs!i) (ts!i)$ **by** (*auto simp: length-bs*)

hence $\exists e h. (\forall i < k. e i > 0 \wedge$
 $(\lambda x. h i x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+e i))) \wedge$
 $(\forall x \geq x_1. (ts!i) x \geq x_0 \wedge (ts!i) x < x) \wedge$
 $(\forall i < k. \forall x \geq x_1. (bs!i)*\text{real } x + h i x = \text{real } ((ts!i) x)))$

unfolding *akra-bazzi-term-def*

by (*subst (asm) bchoice-iff, subst (asm) bchoice-iff*) *blast*

then guess $ee :: - \Rightarrow \text{real}$ **and** $hh :: - \Rightarrow \text{nat} \Rightarrow \text{real}$

by (*elim exE conjE*) **note** $eh = \text{this}$

define e **where** $e = \text{Min } \{ee i \mid i. i < k\}$

define hs **where** $hs = \text{map } hh (\text{upt } 0 k)$

have $e\text{-pos}: e > 0$ **unfolding** $e\text{-def}$ **using** eh $k\text{-not-0}$ **by** (*subst Min-gr-iff*)

auto

moreover have $\text{length } hs = k$ **unfolding** $hs\text{-def}$ **by** (*simp-all add: length-ts*)

moreover have $hs\text{-growth}: \forall h \in \text{set } hs. (\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+e))$

proof

fix h **assume** $h \in \text{set } hs$

then obtain i **where** $t: i < k$ $h = hh i$ **unfolding** $hs\text{-def}$ **by force**

hence $(\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+ee i))$ **using** eh **by**

blast

also from t *k-not-0* **have** $e \leq ee$ i **unfolding** e -def **by** (*subst Min-le-iff*)
auto
hence $(\lambda x::nat. real\ x / \ln\ (real\ x)\ powr\ (1+ee\ i)) \in O(\lambda x. real\ x / \ln\ (real\ x)\ powr\ (1+e))$
by (*intro bigo-real-nat-transfer*) *auto*
finally show $(\lambda x. h\ x) \in O(\lambda x. real\ x / \ln\ (real\ x)\ powr\ (1+e))$.
qed
moreover have $\forall t \in set\ ts. (\forall x \geq x_1. t\ x \geq x_0 \wedge t\ x < x)$
proof (*rule ballI*)
fix t **assume** $t \in set\ ts$
then obtain i **where** $i < k$ $t = ts!i$ **using** $length$ - ts **by** (*subst (asm)*)
in-set-conv-nth) *auto*
with eh **show** $\forall x \geq x_1. t\ x \geq x_0 \wedge t\ x < x$ **unfolding** hs -def **by** *force*
qed
moreover have $\forall i < k. \forall x \geq x_1. (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x)$
proof (*rule allI, rule impI*)
fix i **assume** $i < k$
with eh **show** $\forall x \geq x_1. (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x)$
using $length$ - ts **unfolding** hs -def **by** *fastforce*
qed
ultimately show *?thesis* **by** *blast*
qed
from *someI-ex[OF this, folded e-hs-def]* **show** *?thesis*
unfolding e -def hs -def **by** (*intro conjI*) *fastforce*+
qed

lemma

e-pos: $e > 0$ **and** *length-hs*: $length\ hs = k$ **and**
hs-growth: $\bigwedge h. h \in set\ hs \implies (\lambda x. h\ x) \in O(\lambda x. real\ x / \ln\ (real\ x)\ powr\ (1 + e))$ **and**
step-ge-x0: $\bigwedge t\ x. t \in set\ ts \implies x \geq x_1 \implies x_0 \leq t\ x$ **and**
step-less: $\bigwedge t\ x. t \in set\ ts \implies x \geq x_1 \implies t\ x < x$ **and**
decomp: $\bigwedge i\ x. i < k \implies x \geq x_1 \implies (bs!i)*x + (hs!i)\ x = real\ ((ts!i)\ x)$
by (*insert e-hs-aux*) *simp-all*

lemma *h-in-hs* [*intro, simp*]: $i < k \implies hs\ !\ i \in set\ hs$
by (*rule nth-mem*) (*simp add: length-hs*)

lemma *t-in-ts* [*intro, simp*]: $i < k \implies ts\ !\ i \in set\ ts$
by (*rule nth-mem*) (*simp add: length-ts*)

lemma *x0-less-x1*: $x_0 < x_1$ **and** *x0-le-x1*: $x_0 \leq x_1$

proof –

from *ts-nonempty* **have** $x_0 \leq hd\ ts\ x_1$ **using** *step-ge-x0* [*of hd ts x1*] **by** *simp*
also from *ts-nonempty* **have** $\dots < x_1$ **by** (*intro step-less*) *simp-all*
finally show $x_0 < x_1$ **by** *simp*
thus $x_0 \leq x_1$ **by** *simp*
qed

lemma *akra-bazzi-induct* [*consumes 1, case-names base rec*]:
assumes $x \geq x_0$
assumes *base*: $\bigwedge x. x \geq x_0 \implies x < x_1 \implies P x$
assumes *rec*: $\bigwedge x. x \geq x_1 \implies (\bigwedge t. t \in \text{set } ts \implies P (t x)) \implies P x$
shows $P x$
proof (*insert assms(1), induction x rule: less-induct*)
case (*less x*)
with *assms step-less step-ge-x0* **show** $P x$ **by** (*cases x < x₁*) *auto*
qed

end

locale *akra-bazzi-function* = *akra-bazzi-recursion* +
fixes *integrable integral*
assumes *integral*: *akra-bazzi-integral integrable integral*
fixes $g :: \text{nat} \Rightarrow \text{real}$
assumes *f-nonneg-base*: $x \geq x_0 \implies x < x_1 \implies f x \geq 0$
and *f-rec*: $x \geq x_1 \implies f x = g x + (\sum_{i < k. as!i} * f ((ts!i) x))$
and *g-nonneg*: $x \geq x_1 \implies g x \geq 0$
and *ex-pos-a*: $\exists a \in \text{set } as. a > 0$
begin

lemma *ex-pos-a'*: $\exists i < k. as!i > 0$
using *ex-pos-a* **by** (*auto simp: in-set-conv-nth length-as*)

sublocale *akra-bazzi-params-nonzero*
using *length-as length-bs ex-pos-a a-ge-0 b-bounds* **by** *unfold-locales*

definition *g-real* :: $\text{real} \Rightarrow \text{real}$ **where** *g-real* $x = g (\text{nat } \lfloor x \rfloor)$

lemma *g-real-real*[*simp*]: *g-real* ($\text{real } x$) = $g x$ **unfolding** *g-real-def* **by** *simp*

lemma *f-nonneg*: $x \geq x_0 \implies f x \geq 0$
proof (*induction x rule: akra-bazzi-induct*)
case (*base x*)
with *f-nonneg-base* **show** $f x \geq 0$ **by** *simp*
next
case (*rec x*)
from *rec.hyps* **have** $g x \geq 0$ **by** (*intro g-nonneg*) *simp*
moreover **have** $(\sum_{i < k. as!i} * f ((ts!i) x)) \geq 0$ **using** *rec.hyps length-ts length-as*
by (*intro sum-nonneg ballI mult-nonneg-nonneg[OF a-ge-0 rec.IH]*) *simp-all*
ultimately **show** $f x \geq 0$ **using** *rec.hyps* **by** (*simp add: f-rec*)
qed

definition *hs'* = *map* ($\lambda h x. h (\text{nat } \lfloor x \rfloor)$) *hs*

lemma *length-hs'*: *length* *hs'* = k **unfolding** *hs'-def* **by** (*simp add: length-hs*)

lemma *hs'-real*: $i < k \implies (hs'!i) (\text{real } x) = (hs!i) x$
unfolding *hs'-def* **by** (*simp add: length-hs*)

lemma *h-bound*:

obtains *hb* **where** $hb > 0$ **and**

eventually $(\lambda x. \forall h \in \text{set } hs'. |h x| \leq hb * x / \ln x \text{ powr } (1 + e))$ *at-top*

proof –

have $\forall h \in \text{set } hs. \exists c > 0. \text{eventually } (\lambda x. |h x| \leq c * \text{real } x / \ln (\text{real } x) \text{ powr } (1 + e))$ *at-top*

proof

fix *h* **assume** $h: h \in \text{set } hs$

hence $(\lambda x. h x) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1 + e))$ **by** (*rule hs-growth*)

thus $\exists c > 0. \text{eventually } (\lambda x. |h x| \leq c * x / \ln x \text{ powr } (1 + e))$ *at-top*

unfolding *bigo-def* **by** *auto*

qed

from *bchoice[OF this]* **obtain** *hb* **where** *hb*:

$\forall h \in \text{set } hs. hb h > 0 \wedge \text{eventually } (\lambda x. |h x| \leq hb h * \text{real } x / \ln (\text{real } x) \text{ powr } (1 + e))$ *at-top* **by** *blast*

define *hb'* **where** $hb' = \max 1 (\text{Max } \{hb h \mid h. h \in \text{set } hs\})$

have $hb' > 0$ **unfolding** *hb'-def* **by** *simp*

moreover **have** $\forall h \in \text{set } hs. \text{eventually } (\lambda x. |h (\text{nat } \lfloor x \rfloor)| \leq hb' * x / \ln x \text{ powr } (1 + e))$ *at-top*

proof (*intro ballI, rule eventually-mp[OF always-eventually eventually-conj], clarify*)

fix *h* **assume** $h: h \in \text{set } hs$

with *hb* **have** *hb-pos*: $hb h > 0$ **by** *auto*

show *eventually* $(\lambda x. x > \exp (1 + e) + 1)$ *at-top* **by** (*rule eventually-gt-at-top*)

from *h hb* **have** *e*: *eventually* $(\lambda x. |h (\text{nat } \lfloor x \rfloor)| \leq$

$hb h * \text{real } (\text{nat } \lfloor x \rfloor) / \ln (\text{real } (\text{nat } \lfloor x \rfloor)) \text{ powr } (1 + e))$ *at-top*

by (*intro eventually-natfloor*) *blast*

show *eventually* $(\lambda x. |h (\text{nat } \lfloor x \rfloor)| \leq hb' * x / \ln x \text{ powr } (1 + e))$ *at-top*

using *e eventually-gt-at-top*

proof *eventually-elim*

fix $x :: \text{real}$ **assume** $x: x > \exp (1 + e) + 1$

have $x': x > 0$ **by** (*rule le-less-trans[OF - x]*) *simp-all*

assume $|h (\text{nat } \lfloor x \rfloor)| \leq hb h * \text{real } (\text{nat } \lfloor x \rfloor) / \ln (\text{real } (\text{nat } \lfloor x \rfloor)) \text{ powr } (1 + e)$

also {

from *x* **have** $\exp (1 + e) < \text{real } (\text{nat } \lfloor x \rfloor)$ **by** *linarith*

moreover **have** $x > 0$ **by** (*rule le-less-trans[OF - x]*) *simp-all*

hence $\text{real } (\text{nat } \lfloor x \rfloor) \leq x$ **by** *simp*

ultimately **have** $\text{real } (\text{nat } \lfloor x \rfloor) / \ln (\text{real } (\text{nat } \lfloor x \rfloor)) \text{ powr } (1 + e) \leq x / \ln x$ *powr (1+e)*

using *e-pos* **by** (*intro x-over-ln-mono*) *simp-all*

from *hb-pos mult-left-mono[OF this, of hb h]*

have $hb h * \text{real } (\text{nat } \lfloor x \rfloor) / \ln (\text{real } (\text{nat } \lfloor x \rfloor)) \text{ powr } (1 + e) \leq hb h * x / \ln x \text{ powr } (1 + e)$

by (*simp add: algebra-simps*)

```

}
also from h have hb h ≤ hb'
unfolding hb'-def f-rec by (intro order.trans[OF Max.coboundedI max.cobounded2])
auto
with x' have hb h*x/ln x powr (1+e) ≤ hb'*x/ln x powr (1+e)
  by (intro mult-right-mono divide-right-mono) simp-all
finally show |h (nat [x])| ≤ hb' * x / ln x powr (1 + e) .
qed
qed
hence ∀ h ∈ set hs'. eventually (λx. |h x| ≤ hb' * x / ln x powr (1 + e)) at-top
  by (auto simp: hs'-def)
hence eventually (λx. ∀ h ∈ set hs'. |h x| ≤ hb' * x / ln x powr (1 + e)) at-top
  by (intro eventually-ball-finite) simp-all
ultimately show ?thesis by (rule that)
qed

lemma C-bound:
  assumes ∧ b. b ∈ set bs ⇒ C < b hb > 0
  shows eventually (λx::real. ∀ b ∈ set bs. C*x ≤ b*x - hb*x/ln x powr (1+e))
at-top
proof -
  from e-pos have ((λx. hb * ln x powr -(1+e)) → 0) at-top
  by (intro tendsto-mult-right-zero tendsto-neg-powr ln-at-top) simp-all
  with assms have ∀ b ∈ set bs. eventually (λx. |hb * ln x powr -(1+e)| < b - C)
at-top
  by (force simp: tendsto-iff dist-real-def)
  hence eventually (λx. ∀ b ∈ set bs. |hb * ln x powr -(1+e)| < b - C) at-top
  by (intro eventually-ball-finite) simp-all
  note A = eventually-conj[OF this eventually-gt-at-top]
  show ?thesis using A apply eventually-elim
  proof clarify
    fix x b :: real assume x: x > 0 and b: b ∈ set bs
    assume A: ∀ b ∈ set bs. |hb * ln x powr -(1+e)| < b - C
    from b A assms have hb * ln x powr -(1+e) < b - C by simp
    with x have x * (hb * ln x powr -(1+e)) < x * (b - C) by (intro mult-strict-left-mono)
    thus C*x ≤ b*x - hb*x / ln x powr (1+e)
    by (subst (asm) powr-minus) (simp-all add: field-simps)
  qed
qed
qed
end

```

```

locale akra-bazzi-lower = akra-bazzi-function +
  fixes g' :: real ⇒ real
  assumes f-pos: eventually (λx. f x > 0) at-top
  and g-growth2: ∃ C c2. c2 > 0 ∧ C < Min (set bs) ∧
    eventually (λx. ∀ u ∈ {C*x..x}. g' u ≤ c2 * g' x) at-top

```

and g' -integrable: $\exists a. \forall b \geq a. \text{integrable } (\lambda u. g' u / u \text{ powr } (p + 1)) a b$
and g' -bounded: *eventually* $(\lambda a::\text{real}. (\forall b > a. \exists c. \forall x \in \{a..b\}. g' x \leq c))$ *at-top*
and g -bigomega: $g \in \Omega(\lambda x. g' (\text{real } x))$
and g' -nonneg: *eventually* $(\lambda x. g' x \geq 0)$ *at-top*
begin

definition $gc2 \equiv \text{SOME } gc2. gc2 > 0 \wedge \text{eventually } (\lambda x. g x \geq gc2 * g' (\text{real } x))$ *at-top*

lemma $gc2$: $gc2 > 0$ *eventually* $(\lambda x. g x \geq gc2 * g' (\text{real } x))$ *at-top*

proof –

from g -bigomega **guess** c **by** (*elim landau-omega.bigE*) **note** $c = \text{this}$
from g' -nonneg **have** *eventually* $(\lambda x::\text{nat}. g' (\text{real } x) \geq 0)$ *at-top* **by** (*rule eventually-nat-real*)
with $c(2)$ **have** *eventually* $(\lambda x. g x \geq c * g' (\text{real } x))$ *at-top*
using *eventually-ge-at-top[of x_1]* **by** *eventually-elim (insert g -nonneg, simp-all)*
with $c(1)$ **have** $\exists gc2. gc2 > 0 \wedge \text{eventually } (\lambda x. g x \geq gc2 * g' (\text{real } x))$ *at-top*
by *blast*
from *someI-ex[OF this]* **show** $gc2 > 0$ *eventually* $(\lambda x. g x \geq gc2 * g' (\text{real } x))$ *at-top*
unfolding $gc2$ -def **by** *blast+*
qed

definition $gx0 \equiv \max x_1 (\text{SOME } gx0. \forall x \geq gx0. g x \geq gc2 * g' (\text{real } x) \wedge f x > 0 \wedge g' (\text{real } x) \geq 0)$

definition $gx1 \equiv \max gx0 (\text{SOME } gx1. \forall x \geq gx1. \forall i < k. (ts!i) x \geq gx0)$

lemma $gx0$:

assumes $x \geq gx0$
shows $g x \geq gc2 * g' (\text{real } x) f x > 0 g' (\text{real } x) \geq 0$
proof –
from *eventually-conj[OF $gc2(2)$ eventually-conj[OF f -pos eventually-nat-real[OF g' -nonneg]]]*
have $\exists gx0. \forall x \geq gx0. g x \geq gc2 * g' (\text{real } x) \wedge f x > 0 \wedge g' (\text{real } x) \geq 0$
by (*simp add: eventually-at-top-linorder*)
note *someI-ex[OF this]*
moreover **have** $x \geq (\text{SOME } gx0. \forall x \geq gx0. g x \geq gc2 * g' (\text{real } x) \wedge f x > 0 \wedge g' (\text{real } x) \geq 0)$
using *assms* **unfolding** $gx0$ -def **by** *simp*
ultimately show $g x \geq gc2 * g' (\text{real } x) f x > 0 g' (\text{real } x) \geq 0$ **unfolding** $gx0$ -def **by** *blast+*
qed

lemma $gx1$:

assumes $x \geq gx1 i < k$
shows $(ts!i) x \geq gx0$
proof –
define mb **where** $mb = \text{Min } (\text{set } bs) / 2$
from b -bounds bs -nonempty **have** mb -pos: $mb > 0$ **unfolding** mb -def **by** *simp*

from *h-bound* **guess** *hb* . **note** *hb = this*
from *e-pos* **have** $((\lambda x. hb * \ln x \text{ powr } -(1 + e)) \longrightarrow 0)$ *at-top*
by (*intro tendsto-mult-right-zero tendsto-neg-powr ln-at-top*) *simp-all*
moreover **note** *mb-pos*
ultimately **have** $\text{eventually } (\lambda x. hb * \ln x \text{ powr } -(1 + e) < mb)$ *at-top* **using**
hb(1)
by (*subst (asm) tendsto-iff*) (*simp-all add: dist-real-def*)

from *eventually-nat-real[OF hb(2)] eventually-nat-real[OF this]*
eventually-ge-at-top eventually-ge-at-top
have $\text{eventually } (\lambda x. \forall i < k. (ts!i) x \geq gx0)$ *at-top* **apply** *eventually-elim*
proof *clarify*
fix *i x :: nat* **assume** *A: hb * ln (real x) powr -(1+e) < mb* **and** *i: i < k*
assume *B: $\forall h \in \text{set } hs'. |h \text{ (real } x)| \leq hb * \text{real } x / \ln \text{ (real } x) \text{ powr } (1+e)$*
with *i* **have** *B': $|hs!i \text{ (real } x)| \leq hb * \text{real } x / \ln \text{ (real } x) \text{ powr } (1+e)$*
using *length-hs'[symmetric]* **by** *auto*
assume *C: $x \geq \text{nat } [gx0/mb]$*
hence *C': $\text{real } gx0/mb \leq \text{real } x$* **by** *linarith*
assume *D: $x \geq x_1$*

from *mb-pos* **have** $\text{real } gx0 = mb * (\text{real } gx0/mb)$ **by** *simp*
also **from** *i bs-nonempty* **have** $mb \leq bs!i/2$ **unfolding** *mb-def* **by** *simp*
hence $mb * (\text{real } gx0/mb) \leq bs!i/2 * x$
using *C' i b-bounds[of bs!i] mb-pos* **by** (*intro mult-mono*) *simp-all*
also **have** $\dots = bs!i * x + -bs!i/2 * x$ **by** *simp*
also {
have $-(hs!i) x \leq |(hs!i) x|$ **by** *simp*
also **from** *i B' length-hs* **have** $|hs!i x| \leq hb * \text{real } x / \ln \text{ (real } x) \text{ powr } (1+e)$
(1+e)
by (*simp add: hs'-def*)
also **from** *A* **have** $hb / \ln x \text{ powr } (1+e) \leq mb$
by (*subst (asm) powr-minus*) (*simp add: field-simps*)
hence $hb / \ln x \text{ powr } (1+e) * x \leq mb * x$ **by** (*intro mult-right-mono*) *simp-all*
hence $hb * x / \ln x \text{ powr } (1+e) \leq mb * x$ **by** *simp*
also **from** *i* **have** $\dots \leq (bs!i/2) * x$ **unfolding** *mb-def* **by** (*intro mult-right-mono*)
simp-all
finally **have** $-bs!i/2 * x \leq (hs!i) x$ **by** *simp*
}
also **have** $bs!i * \text{real } x + (hs!i) x = \text{real } ((ts!i) x)$ **using** *i D decomp* **by** *simp*
finally **show** $(ts!i) x \geq gx0$ **by** *simp*

qed
hence $\exists gx1. \forall x \geq gx1. \forall i < k. gx0 \leq (ts!i) x$ (**is** *Ex ?P*)
by (*simp add: eventually-at-top-linorder*)
from *someI-ex[OF this]* **have** *?P (SOME x. ?P x)* .
moreover **have** $\bigwedge x. x \geq gx1 \implies x \geq (\text{SOME } x. ?P x)$ **unfolding** *gx1-def* **by**
simp
ultimately **have** *?P gx1* **by** *blast*
with *assms* **show** *?thesis* **by** *blast*
qed

lemma $gx0\text{-}ge\text{-}x1$: $gx0 \geq x1$ **unfolding** $gx0\text{-}def$ **by** $simp$

lemma $gx0\text{-}le\text{-}gx1$: $gx0 \leq gx1$ **unfolding** $gx1\text{-}def$ **by** $simp$

function $f2' :: nat \Rightarrow real$ **where**

$x < gx1 \implies f2' x = \max 0 (f x / gc2)$

$| x \geq gx1 \implies f2' x = g' (real x) + (\sum i < k. as!i * f2' ((ts!i) x))$

using $le\text{-}less\text{-}linear$ **by** ($blast$, $simp\text{-}all$)

termination **by** ($relation$ $Wellfounded.measure$ $(\lambda x. x)$)

($insert$ $gx0\text{-}le\text{-}gx1$ $gx0\text{-}ge\text{-}x1$, $simp\text{-}all$ add : $step\text{-}less$)

lemma $f2'\text{-}nonneg$: $x \geq gx0 \implies f2' x \geq 0$

by ($induction$ x $rule$: $f2'.induct$)

($auto$ $intro!$: $add\text{-}nonneg\text{-}nonneg$ $sum\text{-}nonneg$ $gx0$ $gx1$ $mult\text{-}nonneg\text{-}nonneg$ [OF $a\text{-}ge\text{-}0$])

lemma $f2'\text{-}le\text{-}f$: $x \geq x0 \implies gc2 * f2' x \leq f x$

proof ($induction$ $rule$: $f2'.induct$)

case ($1 x$)

with $gc2$ $f\text{-}nonneg$ **show** $?case$ **by** ($simp$ add : $max\text{-}def$ $field\text{-}simps$)

next

case $prems$: ($2 x$)

with $gx0$ $gx0\text{-}le\text{-}gx1$ **have** $gc2 * g' (real x) \leq g x$ **by** $force$

moreover **from** $step\text{-}ge\text{-}x0$ $prems(1)$ $gx0\text{-}ge\text{-}x1$ $gx0\text{-}le\text{-}gx1$

have $\bigwedge i. i < k \implies x0 \leq (ts!i) x$ **by** $simp$

hence $\bigwedge i. i < k \implies as!i * (gc2 * f2' ((ts!i) x)) \leq as!i * f ((ts!i) x)$

using $prems(1)$ **by** ($intro$ $mult\text{-}left\text{-}mono$ $a\text{-}ge\text{-}0$ $prems(2)$) $auto$

hence $gc2 * (\sum i < k. as!i * f2' ((ts!i) x)) \leq (\sum i < k. as!i * f ((ts!i) x))$

by ($subst$ $sum\text{-}distrib\text{-}left$, $intro$ $sum\text{-}mono$) ($simp\text{-}all$ add : $algebra\text{-}simps$)

ultimately **show** $?case$ **using** $prems(1)$ $gx0\text{-}ge\text{-}x1$ $gx0\text{-}le\text{-}gx1$

by ($simp\text{-}all$ add : $algebra\text{-}simps$ $f\text{-}rec$)

qed

lemma $f2'\text{-}pos$: $eventually$ $(\lambda x. f2' x > 0)$ $at\text{-}top$

proof ($subst$ $eventually\text{-}at\text{-}top\text{-}linorder$, $intro$ exI $allI$ $impI$)

fix $x :: nat$ **assume** $x \geq gx0$

thus $f2' x > 0$

proof ($induction$ x $rule$: $f2'.induct$)

case ($1 x$)

with $gc2$ $gx0(2)$ [of x] **show** $?case$ **by** ($simp$ add : $max\text{-}def$ $field\text{-}simps$)

next

case $prems$: ($2 x$)

have $(\sum i < k. as!i * f2' ((ts!i) x)) > 0$

proof ($rule$ $sum\text{-}pos'$)

from $ex\text{-}pos\text{-}a'$ **guess** i **by** ($elim$ exE $conjE$) **note** $i = this$

with $prems(1)$ $gx0$ $gx1$ **have** $as!i * f2' ((ts!i) x) > 0$

by ($intro$ $mult\text{-}pos\text{-}pos$ $prems(2)$) $simp\text{-}all$

with i **show** $\exists i \in \{..<k\}. as!i * f2' ((ts!i) x) > 0$ **by** $blast$

next

fix i **assume** $i: i \in \{..<k\}$
with $\text{prems}(1)$ **have** $f2' ((ts!i) x) > 0$ **by** ($\text{intro } \text{prems}(2) \text{ } gx1$) simp-all
with i **show** $as!i * f2' ((ts!i) x) \geq 0$ **by** ($\text{intro } \text{mult-nonneg-nonneg}[OF$
 $a\text{-ge-}0]$) simp-all
qed simp-all
with $\text{prems}(1) \text{ } gx0\text{-le-}gx1$ **show** $?case$ **by** ($\text{auto } \text{intro!}: \text{add-nonneg-pos } gx0$)
qed
qed

lemma bigomega-f-aux :

obtains a **where** $a \geq A \forall a' \geq a. a' \in \mathbf{N} \longrightarrow$
 $f \in \Omega(\lambda x. x \text{ powr } p *(1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a' x))$
proof –
from g' -integrable **guess** $a0$ **by** ($\text{elim } exE$) **note** $a0 = \text{this}$
from h -bound **guess** hb . **note** $hb = \text{this}$
moreover from g -growth2 **guess** $C \text{ } c2$ **by** ($\text{elim } conjE \text{ } exE$) **note** $C = \text{this}$
hence eventually $(\lambda x. \forall b \in \text{set } bs. C*x \leq b*x - hb*x/\ln x \text{ powr } (1 + e))$ at-top
using $hb(1)$ bs -nonempty **by** ($\text{intro } C\text{-bound}$) simp-all
moreover from b -bounds $hb(1)$ e -pos
have eventually $(\lambda x. \forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x)$ at-top
by ($\text{rule } \text{akra-bazzi-asymptotics}$)
moreover note g' -bounded $C(3)$ g' -nonneg eventually-natfloor[$OF \text{ } f2'\text{-pos}$] eventually-natfloor[$OF \text{ } gc2(2)$]
ultimately have eventually $(\lambda x. (\forall h \in \text{set } hs'. |h x| \leq hb*x/\ln x \text{ powr } (1+e)) \wedge$
 $(\forall b \in \text{set } bs. C*x \leq b*x - hb*x/\ln x \text{ powr } (1+e)) \wedge$
 $(\forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x) \wedge$
 $(\forall b > x. \exists c. \forall x \in \{x..b\}. g' x \leq c) \wedge f2' (\text{nat } \lfloor x \rfloor) > 0 \wedge$
 $(\forall u \in \{C * x..x\}. g' u \leq c2 * g' x) \wedge$
 $g' x \geq 0)$ at-top
by ($\text{intro } \text{eventually-conj}$) ($\text{force } \text{elim!}: \text{eventually-conjE}$)+
then have $\exists X. (\forall x \geq X. (\forall h \in \text{set } hs'. |h x| \leq hb*x/\ln x \text{ powr } (1+e)) \wedge$
 $(\forall b \in \text{set } bs. C*x \leq b*x - hb*x/\ln x \text{ powr } (1+e)) \wedge$
 $(\forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x) \wedge$
 $(\forall b > x. \exists c. \forall x \in \{x..b\}. g' x \leq c) \wedge$
 $(\forall u \in \{C * x..x\}. g' u \leq c2 * g' x) \wedge$
 $f2' (\text{nat } \lfloor x \rfloor) > 0 \wedge g' x \geq 0)$
by ($\text{subst } (asm) \text{ } \text{eventually-at-top-linorder}$) ($\text{erule } ex\text{-mono}, \text{blast}$)
then guess X **by** ($\text{elim } exE \text{ } conjE$) **note** $X = \text{this}$

define x_0' -min **where** $x_0'\text{-min} = \max A (\max X (\max a0 (\max gx1 (\max 1 (\text{real } x_1 + 1))))))$

fix $x_0' :: \text{real}$ **assume** $x0'\text{-props}: x_0' \geq x_0'\text{-min } x_0' \in \mathbf{N}$
hence $x0'\text{-ge-}x1: x_0' \geq \text{real } (x_1+1)$ **and** $x0'\text{-ge-}1: x_0' \geq 1$ **and** $x0'\text{-ge-}X: x_0' \geq X$

unfolding x_0' -min-def **by** linarith+

hence $x0'\text{-pos}: x_0' > 0$ **and** $x0'\text{-nonneg}: x_0' \geq 0$ **by** simp-all
have $x0': \forall x \geq x_0'. (\forall h \in \text{set } hs'. |h x| \leq hb*x/\ln x \text{ powr } (1+e))$

$\forall x \geq x_0'. (\forall b \in \text{set } bs. C * x \leq b * x - hb * x / \ln x \text{ powr } (1+e))$
 $\forall x \geq x_0'. (\forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ hb } e \text{ p } x)$
 $\forall a \geq x_0'. \forall b > a. \exists c. \forall x \in \{a..b\}. g' x \leq c$
 $\forall x \geq x_0'. \forall u \in \{C * x..x\}. g' u \leq c2 * g' x$
 $\forall x \geq x_0'. f2' (\text{nat } \lfloor x \rfloor) > 0 \forall x \geq x_0'. g' x \geq 0$

using $X \text{ } x_0' \text{-ge-} X$ **by** *auto*
from $x_0' \text{-props}(2)$ **have** $x_0' \text{-int}: \text{real } (\text{nat } \lfloor x_0' \rfloor) = x_0'$ **by** (*rule real-natfloor-nat*)
from $x_0' \text{-props}$ **have** $x_0' \text{-ge-gx1}: x_0' \geq gx1$ **and** $x_0' \text{-ge-a0}: x_0' \geq a0$
unfolding $x_0' \text{-min-def}$ **by** *simp-all*
with $gx0 \text{-le-gx1}$ **have** $f2' \text{-nonneg}: \bigwedge x. x \geq x_0' \implies f2' x \geq 0$ **by** (*force intro!: f2'-nonneg*)

define bm **where** $bm = \text{Min } (\text{set } bs)$
define x_1' **where** $x_1' = 2 * x_0' * \text{inverse } bm$
define $fb2$ **where** $fb2 = \text{Min } \{f2' x \mid x. x \in \{x_0'..x_1'\}\}$
define $gb2$ **where** $gb2 = (\text{SOME } c. \forall x \in \{x_0'..x_1'\}. g' x \leq c)$

from $b \text{-bounds } bs \text{-nonempty}$ **have** $bm > 0 \text{ } bm < 1$ **unfolding** $bm \text{-def}$ **by** *auto*
hence $1 < 2 * \text{inverse } bm$ **by** (*simp add: field-simps*)
from $\text{mult-strict-left-mono}$ [*OF this x0'-pos*]
have $x_0' \text{-lt-} x_1': x_0' < x_1'$ **and** $x_0' \text{-le-} x_1': x_0' \leq x_1'$ **unfolding** $x_1' \text{-def}$ **by** *simp-all*

from $x_0 \text{-le-} x_1 \text{ } x_0' \text{-ge-} x_1$ **have** $ge \text{-} x_0' D: \bigwedge x. x_0' \leq \text{real } x \implies x_0 \leq x$ **by** *simp*
from $x_0' \text{-ge-} x_1 \text{ } x_0' \text{-le-} x_1'$ **have** $gt \text{-} x_1' D: \bigwedge x. x_1' < \text{real } x \implies x_1 \leq x$ **by** *simp*

have $x_0' \text{-} x_1': \forall b \in \text{set } bs. 2 * x_0' * \text{inverse } b \leq x_1'$
proof
fix b **assume** $b: b \in \text{set } bs$
hence $bm \leq b$ **by** (*simp add: bm-def*)
moreover from $b \text{ } bs \text{-nonempty } b \text{-bounds}$ **have** $bm > 0 \text{ } b > 0$ **unfolding** $bm \text{-def}$
by *auto*
ultimately have $\text{inverse } b \leq \text{inverse } bm$ **by** *simp*
with $x_0' \text{-nonneg}$ **show** $2 * x_0' * \text{inverse } b \leq x_1'$
unfolding $x_1' \text{-def}$ **by** (*intro mult-left-mono*) *simp-all*
qed

note $f \text{-nonneg}' = f \text{-nonneg}$
have $\bigwedge x. \text{real } x \geq x_0' \implies x \geq \text{nat } \lfloor x_0' \rfloor \bigwedge x. \text{real } x \leq x_1' \implies x \leq \text{nat } \lceil x_1 \rceil$ **by** *linarith+*
hence $\{x \mid x. \text{real } x \in \{x_0'..x_1'\}\} \subseteq \{x \mid x. x \in \{\text{nat } \lfloor x_0' \rfloor .. \text{nat } \lceil x_1 \rceil\}\}$ **by** *auto*
hence *finite* $\{x \mid x::\text{nat}. \text{real } x \in \{x_0'..x_1'\}\}$ **by** (*rule finite-subset*) *auto*
hence *fin: finite* $\{f2' x \mid x::\text{nat}. \text{real } x \in \{x_0'..x_1'\}\}$ **by** *force*

note $\text{facts} = \text{hs}' \text{-real } e \text{-pos } \text{length-} \text{hs}' \text{ } \text{length-} \text{as } \text{length-} \text{bs } k \text{-not-} 0 \text{ } a \text{-ge-} 0 \text{ } p \text{-props}$
 $x_0' \text{-ge-} 1$
 $f2' \text{-nonneg } f \text{-rec}$ [*OF gt-x1'D*] $x_0' \text{ } x_0' \text{-int } x_0' \text{-} x_1' \text{ } gc2(1)$ *decomp*
from $b \text{-bounds } x_0' \text{-le-} x_1' \text{ } x_0' \text{-ge-} gx1 \text{ } gx0 \text{-le-} gx1 \text{ } x_0' \text{-ge-} x_1$
interpret *abr: akra-bazzi-nat-to-real as bs hs' k x_0' x_1' hb e p f2' g'*

by (unfold-locales) (auto simp: facts simp del: f2'.simps intro!: f2'.simps(2))

have f'-nat: $\bigwedge x::\text{nat}. \text{abr}.f'(\text{real } x) = f2' x$

proof -

fix x :: nat show $\text{abr}.f'(\text{real } (x::\text{nat})) = f2' x$

proof (induction real x arbitrary: x rule: abr.f'.induct)

case (2 x)

note x = this(1) and IH = this(2)

from x have $\text{abr}.f'(\text{real } x) = g'(\text{real } x) + (\sum i < k. \text{as}!i * \text{abr}.f'(\text{bs}!i * \text{real } x + (\text{hs}!i) x))$

by (auto simp: gt-x1'D hs'-real g-real-def intro!: sum.cong)

also have $(\sum i < k. \text{as}!i * \text{abr}.f'(\text{bs}!i * \text{real } x + (\text{hs}!i) x)) = (\sum i < k. \text{as}!i * f2'((\text{ts}!i) x))$

proof (rule sum.cong, simp, clarify)

fix i assume i: $i < k$

from i x $x0'-le-x1' x0'-ge-x1$ have *: $\text{bs}!i * \text{real } x + (\text{hs}!i) x = \text{real } ((\text{ts}!i) x)$

by (intro decomp) simp-all

also from i * have $\text{abr}.f' \dots = f2'((\text{ts}!i) x)$

by (subst IH[of i]) (simp-all add: hs'-real)

finally show $\text{as}!i * \text{abr}.f'(\text{bs}!i * \text{real } x + (\text{hs}!i) x) = \text{as}!i * f2'((\text{ts}!i) x)$ by simp

qed

also have $g' x + \dots = f2' x$ using x $x0'-ge-gx1 x0'-le-x1'$

by (intro f2'.simps(2)[symmetric] gt-x1'D) simp-all

finally show ?case .

qed simp

qed

interpret akra-bazzi-integral integrable integral by (rule integral)

interpret akra-bazzi-real-lower as bs hs' k $x_0' x_1' hb e p$

integrable integral abr.f' g' C fb2 gb2 c2

proof unfold-locales

fix x assume $x \geq x_0' x \leq x_1'$

thus $\text{abr}.f' x \geq 0$ by (intro abr.f'-base) simp-all

next

fix x assume $x \geq x_0'$

show integrable $(\lambda x. g' x / x \text{ powr } (p + 1)) x_0' x$

by (rule integrable-subinterval[of - a0 x]) (insert a0 $x0'-ge-a0 x$, auto)

next

fix x assume $x: x \geq x_0' x \leq x_1'$

have $x_0' = \text{real } (\text{nat } \lfloor x_0' \rfloor)$ by (simp add: $x0'-int$)

also from x have $\dots \leq \text{real } (\text{nat } \lfloor x \rfloor)$ by (auto intro!: nat-mono floor-mono)

finally have $x_0' \leq \text{real } (\text{nat } \lfloor x \rfloor)$.

moreover have $\text{real } (\text{nat } \lfloor x \rfloor) \leq x_1'$ using x $x0'-ge-1$ by linarith

ultimately have $f2'(\text{nat } \lfloor x \rfloor) \in \{f2' x \mid x. \text{real } x \in \{x_0'..x_1'\}\}$ by force

from fin and this have $f2'(\text{nat } \lfloor x \rfloor) \geq \text{fb2}$ unfolding fb2-def by (rule Min-le)

with x show $\text{abr}.f' x \geq \text{fb2}$ by simp

next

from $x0'-int x0'-le-x1'$ have $\exists x::\text{nat}. \text{real } x \geq x_0' \wedge \text{real } x \leq x_1'$

by (intro exI[of - nat $\lfloor x_0' \rfloor$]) simp-all

```

moreover {
  fix  $x :: \text{nat}$  assume  $\text{real } x \geq x_0' \wedge \text{real } x \leq x_1'$ 
  with  $x_0'(6)$  have  $f_2'(\text{nat } \lfloor \text{real } x \rfloor) > 0$  by blast
  hence  $f_2' x > 0$  by simp
}
ultimately show  $fb_2 > 0$  unfolding fb_2-def using fin by (subst Min-gr-iff)
auto
next
  fix  $x$  assume  $x: x_0' \leq x \leq x_1'$ 
  with  $x_0'(4)$   $x_0'-lt-x_1'$  have  $\exists c. \forall x \in \{x_0'..x_1'\}. g' x \leq c$  by force
  from someI-ex[OF this]  $x$  show  $g' x \leq gb_2$  unfolding gb_2-def by simp
qed (insert g-nonneg integral x_0'(2) C x_0'-le-x_1' x_0'-ge-x_1, simp-all add: facts)

from akra-bazzi-lower guess  $c_5$  . note  $c_5 = \text{this}$ 
have eventually ( $\lambda x. |f x| \geq gc_2 * c_5 * |f\text{-approx } (\text{real } x)|$ ) at-top
proof (unfold eventually-at-top-linorder, intro exI allI impI)
  fix  $x :: \text{nat}$  assume  $x \geq \text{nat } \lceil x_0 \rceil$ 
  hence  $x: \text{real } x \geq x_0'$  by linarith
  note  $c_5(1)[OF x]$ 
  also have  $\text{abr}.f'(\text{real } x) = f_2' x$  by (rule f'-nat)
  also have  $gc_2 * \dots \leq f x$  using  $x_0'-ge-x_1$   $x_0'-le-x_1$  by (intro f_2'-le-f) simp-all
  also have  $f x = |f x|$  using  $x$  f-nonneg'  $x_0'-ge-x_1$   $x_0'-le-x_1$  by simp
  finally show  $gc_2 * c_5 * |f\text{-approx } (\text{real } x)| \leq |f x|$ 
    using  $gc_2$  f-approx-nonneg[OF x] by (simp add: algebra-simps)
qed
hence  $f \in \Omega(\lambda x. f\text{-approx } (\text{real } x))$  using  $gc_2(1)$  f-nonneg' f-approx-nonneg
  by (intro landau-omega.bigI[of gc_2 * c_5] eventually-conj
    mult-pos-pos c_5 eventually-nat-real) (auto simp: eventually-at-top-linorder)
note this[unfolded f-approx-def]
}
moreover have  $x_0'\text{-min} \geq A$  unfolding  $x_0'\text{-min-def}$   $gx_0\text{-ge-x_1}$  by simp
ultimately show ?thesis by (intro that) auto
qed

lemma bigomega-f:
  obtains  $a$  where  $a \geq A$   $f \in \Omega(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p+1)) a x))$ 
proof–
  from bigomega-f-aux[of  $A$ ] guess  $a$  . note  $a = \text{this}$ 
  define  $a'$  where  $a' = \text{real } (\text{max } (\text{nat } \lceil a \rceil) 0) + 1$ 
  note  $a$ 
  moreover have  $a' \in \mathbb{N}$  by (auto simp: max-def a'-def)
  moreover have  $*$ :  $a' \geq a + 1$  unfolding  $a'\text{-def}$  by linarith
  moreover from  $*$  and  $a$  have  $a' \geq A$  by simp
  ultimately show ?thesis by (intro that[of a']) auto
qed

end

```

locale *akra-bazzi-upper* = *akra-bazzi-function* +
fixes $g' :: \text{real} \Rightarrow \text{real}$
assumes *g'-integrable*: $\exists a. \forall b \geq a. \text{integrable } (\lambda u. g' u / u \text{ powr } (p + 1)) a b$
and *g-growth1*: $\exists C c1. c1 > 0 \wedge C < \text{Min } (\text{set } bs) \wedge$
 $\text{eventually } (\lambda x. \forall u \in \{C * x..x\}. g' u \geq c1 * g' x) \text{ at-top}$
and *g-bigo*: $g \in O(g')$
and *g'-nonneg*: $\text{eventually } (\lambda x. g' x \geq 0) \text{ at-top}$
begin

definition *gc1* $\equiv \text{SOME } gc1. gc1 > 0 \wedge \text{eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x))$
at-top

lemma *gc1*: $gc1 > 0 \text{ eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x)) \text{ at-top}$

proof –

from *g-bigo* **guess** c **by** (*elim landau-o.bigE*) **note** $c = \text{this}$
from *g'-nonneg* **have** $\text{eventually } (\lambda x :: \text{nat}. g' (\text{real } x) \geq 0) \text{ at-top}$ **by** (*rule eventually-nat-real*)
with $c(2)$ **have** $\text{eventually } (\lambda x. g x \leq c * g' (\text{real } x)) \text{ at-top}$
using *eventually-ge-at-top[of x₁]* **by** *eventually-elim (insert g-nonneg, simp-all)*
with $c(1)$ **have** $\exists gc1. gc1 > 0 \wedge \text{eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x)) \text{ at-top}$
by *blast*
from *someI-ex[OF this]* **show** $gc1 > 0 \text{ eventually } (\lambda x. g x \leq gc1 * g' (\text{real } x))$
at-top
unfolding *gc1-def* **by** *blast+*
qed

definition *gx3* $\equiv \text{max } x_1 (\text{SOME } gx0. \forall x \geq gx0. g x \leq gc1 * g' (\text{real } x))$

lemma *gx3*:

assumes $x \geq gx3$
shows $g x \leq gc1 * g' (\text{real } x)$

proof –

from *gc1(2)* **have** $\exists gx3. \forall x \geq gx3. g x \leq gc1 * g' (\text{real } x)$ **by** (*simp add: eventually-at-top-linorder*)
note *someI-ex[OF this]*
moreover **have** $x \geq (\text{SOME } gx0. \forall x \geq gx0. g x \leq gc1 * g' (\text{real } x))$
using *assms* **unfolding** *gx3-def* **by** *simp*
ultimately **show** $g x \leq gc1 * g' (\text{real } x)$ **unfolding** *gx3-def* **by** *blast*
qed

lemma *gx3-ge-x1*: $gx3 \geq x_1$ **unfolding** *gx3-def* **by** *simp*

function $f' :: \text{nat} \Rightarrow \text{real}$ **where**

$x < gx3 \implies f' x = \text{max } 0 (f x / gc1)$
 $| x \geq gx3 \implies f' x = g' (\text{real } x) + (\sum_{i < k}. \text{as!i} * f' ((\text{ts!i}) x))$
using *le-less-linear* **by** (*blast, simp-all*)

termination by (*relation Wellfounded.measure* ($\lambda x. x$))
(*insert gx3-ge-x1, simp-all add: step-less*)

lemma f' -ge-f: $x \geq x_0 \implies gc1 * f' x \geq f x$

proof (*induction rule: f'.induct*)

case (1 x)

with $gc1$ f -nonneg **show** ?case **by** (*simp add: max-def field-simps*)

next

case *prems*: (2 x)

with $gx3$ **have** $gc1 * g'(\text{real } x) \geq g x$ **by** *force*

moreover from *step-ge-x0 prems(1) gx3-ge-x1*

have $\bigwedge i. i < k \implies x_0 \leq \text{nat } \lfloor (ts!i) x \rfloor$ **by** (*intro le-nat-floor*) *simp*

hence $\bigwedge i. i < k \implies as!i * (gc1 * f'((ts!i) x)) \geq as!i * f((ts!i) x)$

using *prems(1)* **by** (*intro mult-left-mono a-ge-0 prems(2)*) *auto*

hence $gc1 * (\sum_{i < k}. as!i * f'((ts!i) x)) \geq (\sum_{i < k}. as!i * f((ts!i) x))$

by (*subst sum-distrib-left, intro sum-mono*) (*simp-all add: algebra-simps*)

ultimately show ?case **using** *prems(1) gx3-ge-x1*

by (*simp-all add: algebra-simps f-rec*)

qed

lemma *big-o-f-aux*:

obtains a **where** $a \geq A \forall a' \geq a. a' \in \mathbb{N} \longrightarrow$

$f \in O(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a' x))$

proof –

from g' -integrable **guess** $a0$ **by** (*elim exE*) **note** $a0 = \text{this}$

from h -bound **guess** hb . **note** $hb = \text{this}$

moreover from g -growth1 **guess** $C c1$ **by** (*elim conjE exE*) **note** $C = \text{this}$

hence *eventually* ($\lambda x. \forall b \in \text{set } bs. C * x \leq b * x - hb * x / \ln x \text{ powr } (1 + e)$) *at-top*

using $hb(1)$ bs -nonempty **by** (*intro C-bound*) *simp-all*

moreover from b -bounds $hb(1)$ e -pos

have *eventually* ($\lambda x. \forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x$) *at-top*

by (*rule akra-bazzi-asymptotics*)

moreover note $gc1(2)$ $C(3)$ g' -nonneg

ultimately have *eventually* ($\lambda x. (\forall h \in \text{set } hs'. |h x| \leq hb * x / \ln x \text{ powr } (1 + e)) \wedge$

$(\forall b \in \text{set } bs. C * x \leq b * x - hb * x / \ln x \text{ powr } (1 + e)) \wedge$

$(\forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x) \wedge$

$(\forall u \in \{C * x..x\}. g' u \geq c1 * g' x) \wedge g' x \geq 0$) *at-top*

by (*intro eventually-conj*) (*force elim!: eventually-conjE*) +

then have $\exists X. (\forall x \geq X. (\forall h \in \text{set } hs'. |h x| \leq hb * x / \ln x \text{ powr } (1 + e)) \wedge$

$(\forall b \in \text{set } bs. C * x \leq b * x - hb * x / \ln x \text{ powr } (1 + e)) \wedge$

$(\forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ } hb \text{ } e \text{ } p \text{ } x) \wedge$

$(\forall u \in \{C * x..x\}. g' u \geq c1 * g' x) \wedge g' x \geq 0$)

by (*subst (asm) eventually-at-top-linorder*) *fast*

then guess X **by** (*elim exE conjE*) **note** $X = \text{this}$

define x_0' -min **where** x_0' -min = $\max A (\max X (\max 1 (\max a0 (\max gx3 (\text{real } x_1 + 1))))))$

{

fix $x_0' :: \text{real}$ **assume** x_0' -props: $x_0' \geq x_0'$ -min $x_0' \in \mathbb{N}$

hence $x0'-ge-x1: x0' \geq \text{real } (x1+1)$ and $x0'-ge-1: x0' \geq 1$ and $x0'-ge-X: x0' \geq X$

unfolding $x0'-min-def$ by *linarith+*

hence $x0'-pos: x0' > 0$ and $x0'-nonneg: x0' \geq 0$ by *simp-all*

have $x0': \forall x \geq x0'. (\forall h \in \text{set } hs'. |h x| \leq hb*x/\ln x \text{ powr } (1+e))$

$\forall x \geq x0'. (\forall b \in \text{set } bs. C*x \leq b*x - hb*x/\ln x \text{ powr } (1+e))$

$\forall x \geq x0'. (\forall b \in \text{set } bs. \text{akra-bazzi-asymptotics } b \text{ hb } e \text{ p } x)$

$\forall x \geq x0'. \forall u \in \{C*x..x\}. g' u \geq c1 * g' x \forall x \geq x0'. g' x \geq 0$

using X $x0'-ge-X$ by *auto*

from $x0'-props(2)$ have $x0'-int: \text{real } (\text{nat } \lfloor x0' \rfloor) = x0'$ by (rule *real-natfloor-nat*)

from $x0'-props$ have $x0'-ge-gx0: x0' \geq gx3$ and $x0'-ge-a0: x0' \geq a0$

unfolding $x0'-min-def$ by *simp-all*

hence $f'-nonneg: \bigwedge x. x \geq x0' \implies f' x \geq 0$

using *order.trans*[*OF f'-nonneg f'-ge-f*] *gc1(1)* $x0'-ge-x1$ $x0'-le-x1$

by (*simp add: zero-le-mult-iff del: f'.simps*)

define bm where $bm = \text{Min } (\text{set } bs)$

define $x1'$ where $x1' = 2 * x0' * \text{inverse } bm$

define $fb1$ where $fb1 = \text{Max } \{f' x \mid x. x \in \{x0'..x1'\}\}$

from b -bounds bs -nonempty have $bm > 0$ $bm < 1$ unfolding $bm-def$ by *auto*

hence $1 < 2 * \text{inverse } bm$ by (*simp add: field-simps*)

from *mult-strict-left-mono*[*OF this x0'-pos*]

have $x0'-lt-x1': x0' < x1'$ and $x0'-le-x1': x0' \leq x1'$ unfolding $x1'-def$ by *simp-all*

from $x0'-le-x1$ $x0'-ge-x1$ have $ge-x0'D: \bigwedge x. x0' \leq \text{real } x \implies x0 \leq x$ by *simp*

from $x0'-ge-x1$ $x0'-le-x1'$ have $gt-x1'D: \bigwedge x. x1' < \text{real } x \implies x1 \leq x$ by *simp*

have $x0'-x1': \forall b \in \text{set } bs. 2 * x0' * \text{inverse } b \leq x1'$

proof

fix b assume $b: b \in \text{set } bs$

hence $bm \leq b$ by (*simp add: bm-def*)

moreover from b b -bounds bs -nonempty have $bm > 0$ $b > 0$ unfolding $bm-def$

by *auto*

ultimately have $\text{inverse } b \leq \text{inverse } bm$ by *simp*

with $x0'-nonneg$ show $2 * x0' * \text{inverse } b \leq x1'$

unfolding $x1'-def$ by (*intro mult-left-mono*) *simp-all*

qed

note $f'-nonneg' = f'-nonneg$

have $\bigwedge x. \text{real } x \geq x0' \implies x \geq \text{nat } \lfloor x0' \rfloor \bigwedge x. \text{real } x \leq x1' \implies x \leq \text{nat } \lceil x1' \rceil$ by *linarith+*

hence $\{x \mid x. \text{real } x \in \{x0'..x1'\}\} \subseteq \{x \mid x. x \in \{\text{nat } \lfloor x0' \rfloor.. \text{nat } \lceil x1' \rceil\}\}$ by *auto*

hence *finite* $\{x \mid x::\text{nat}. \text{real } x \in \{x0'..x1'\}\}$ by (rule *finite-subset*) *auto*

hence *fin: finite* $\{f' x \mid x::\text{nat}. \text{real } x \in \{x0'..x1'\}\}$ by *force*

note $\text{facts} = \text{hs'-real } e\text{-pos } \text{length-hs'} \text{ length-as } \text{length-bs } k\text{-not-0 } a\text{-ge-0 } p\text{-props } x0'-ge-1$

f' -nonneg f -rec[OF gt - x_1' D] x_0' x_0' -int x_0' - x_1' $gc_1(1)$ *decomp*
from b -bounds x_0' -le- x_1' x_0' -ge- gx_0 x_0' -ge- x_1
interpret abr : $akra$ - $bazzi$ - nat -to- $real$ as bs hs' k x_0' x_1' hb e p f' g'
by (*unfold-locales*) (*auto simp add: facts simp del: f' .simps intro!: f' .simps(2))*

have f' - nat : $\bigwedge x::nat. abr.f' (real\ x) = f' x$
proof–
fix $x :: nat$ **show** $abr.f' (real\ (x::nat)) = f' x$
proof (*induction real x arbitrary: x rule: $abr.f'$.induct)
case (2 x)
note $x = this(1)$ **and** $IH = this(2)$
from x **have** $abr.f' (real\ x) = g' (real\ x) + (\sum i < k. as!i * abr.f' (bs!i * real\ x + (hs!i)\ x))$
by (*auto simp: gt - x_1' D hs' - $real$ intro!: $sum.cong$*)
also have $(\sum i < k. as!i * abr.f' (bs!i * real\ x + (hs!i)\ x)) = (\sum i < k. as!i * f' ((ts!i)\ x))$
proof (*rule $sum.cong$, $simp$, $clarify$*)
fix i **assume** $i : i < k$
from $i\ x\ x_0'$ -le- x_1' x_0' -ge- x_1 **have** $*$: $bs!i * real\ x + (hs!i)\ x = real\ ((ts!i)\ x)$
by (*intro $decomp$ $simp$ -all*)
also from $i *$ **have** $abr.f' \dots = f' ((ts!i)\ x)$
by (*subst $IH[of\ i]$ ($simp$ -all add: hs' - $real$)*)
finally show $as!i * abr.f' (bs!i * real\ x + (hs!i)\ x) = as!i * f' ((ts!i)\ x)$ **by** *simp*
qed
also from x **have** $g' x + \dots = f' x$ **using** x_0' -le- x_1' x_0' -ge- gx_0 **by** *simp*
finally show *?case* .
qed *simp*
qed*

interpret $akra$ - $bazzi$ - $integral$ *integrable integral* **by** (*rule integral*)
interpret $akra$ - $bazzi$ - $real$ - $upper$ as bs hs' k x_0' x_1' hb e p *integrable integral $abr.f'$ g' C fb_1 c_1*
proof (*unfold-locales*)
fix x **assume** $x \geq x_0'$ $x \leq x_1'$
thus $abr.f' x \geq 0$ **by** (*intro $abr.f'$ -base $simp$ -all*)
next
fix x **assume** $x : x \geq x_0'$
show *integrable* $(\lambda x. g' x / x^{p+1}) x_0' x$
by (*rule integrable-subinterval[$of - a_0 x$]) (*insert a_0 x_0' -ge- $a_0 x$, $auto$*)
next
fix x **assume** $x : x \geq x_0' x \leq x_1'$
have $x_0' = real\ (nat\ \lfloor x_0' \rfloor)$ **by** (*simp add: x_0' -int)
also from x **have** $\dots \leq real\ (nat\ \lfloor x \rfloor)$ **by** (*auto intro!: nat -mono $floor$ -mono)
finally have $x_0' \leq real\ (nat\ \lfloor x \rfloor)$.
moreover have $real\ (nat\ \lfloor x \rfloor) \leq x_1'$ **using** $x\ x_0'$ -ge-1 **by** *linarith*
ultimately have $f' (nat\ \lfloor x \rfloor) \in \{f' x \mid x. real\ x \in \{x_0'..x_1'\}\}$ **by** *force*
from *fin* **and** *this* **have** $f' (nat\ \lfloor x \rfloor) \leq fb_1$ **unfolding** fb_1 -*def* **by** (*rule Max -ge)
with x **show** $abr.f' x \leq fb_1$ **by** *simp*****

```

qed (insert  $x_0'(2)$   $x_0'-le-x_1'$   $x_0'-ge-x_1$   $C$ , simp-all add: facts)

from akra-bazzi-upper guess  $c_6$  . note  $c_6 = this$ 
{
  fix  $x :: nat$  assume  $x \geq nat \lceil x_0 \rceil$ 
  hence  $x: real$   $x \geq x_0'$  by linarith
  have  $f x \leq gc1 * f' x$  using  $x$   $x_0'-ge-x_1$   $x_0-le-x_1$  by (intro  $f'-ge-f$ ) simp-all
  also have  $f' x = abr.f' (real x)$  by (simp add:  $f'-nat$ )
  also note  $c_6(1)[OF x]$ 
  also from  $f-nonneg'$   $x$   $x_0'-ge-x_1$   $x_0-le-x_1$  have  $f x = |f x|$  by simp
  also from  $f-approx-nonneg$   $x$  have  $f-approx (real x) = |f-approx (real x)|$  by
simp
  finally have  $gc1 * c_6 * |f-approx (real x)| \geq |f x|$  using  $gc1$  by (simp add:
algebra-simps)
}
hence eventually ( $\lambda x. |f x| \leq gc1 * c_6 * |f-approx (real x)|$ ) at-top
using eventually-ge-at-top[of  $nat \lceil x_0 \rceil$ ] by (auto elim!: eventually-mono)
hence  $f \in O(\lambda x. f-approx (real x))$  using  $gc1(1)$   $f-nonneg'$   $f-approx-nonneg$ 
by (intro landau-o.bigI[of  $gc1 * c_6$ ] eventually-conj
mult-pos-pos  $c_6$  eventually-nat-real) (auto simp: eventually-at-top-linorder)
note this[unfolded  $f-approx-def$ ]
}
moreover have  $x_0'-min \geq A$  unfolding  $x_0'-min-def$   $gx3-ge-x_1$  by simp
ultimately show ?thesis by (intro that) auto
qed

```

lemma bigo-f:

obtains a **where** $a > A$ $f \in O(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a x))$

proof–

```

from bigo-f-aux[of  $A$ ] guess  $a$  . note  $a = this$ 
define  $a'$  where  $a' = real (max (nat \lceil a \rceil) 0) + 1$ 
note  $a$ 
moreover have  $a' \in \mathbb{N}$  by (auto simp: max-def  $a'-def$ )
moreover have  $*$ :  $a' \geq a + 1$  unfolding  $a'-def$  by linarith
moreover from  $*$  and  $a$  have  $a' > A$  by simp
ultimately show ?thesis by (intro that[of  $a'$ ]) auto

```

qed

end

locale akra-bazzi = akra-bazzi-function +

fixes $g' :: real \Rightarrow real$

assumes $f\text{-pos}$: eventually ($\lambda x. f x > 0$) at-top

and $g'\text{-nonneg}$: eventually ($\lambda x. g' x \geq 0$) at-top

assumes $g'\text{-integrable}$: $\exists a. \forall b \geq a. \text{integrable } (\lambda u. g' u / u \text{ powr } (p + 1)) a b$

and $g\text{-growth1}$: $\exists C c1. c1 > 0 \wedge C < \text{Min } (set bs) \wedge$

eventually ($\lambda x. \forall u \in \{C*x..x\}. g' u \geq c1 * g' x$) at-top

and $g\text{-growth2}$: $\exists C c2. c2 > 0 \wedge C < \text{Min } (set bs) \wedge$

eventually $(\lambda x. \forall u \in \{C * x..x\}. g' u \leq c2 * g' x)$ at-top
and g -bounded: eventually $(\lambda a::real. (\forall b > a. \exists c. \forall x \in \{a..b\}. g' x \leq c))$ at-top
and g -bigheta: $g \in \Theta(g')$
begin

sublocale *akra-bazzi-lower* **using** f -pos g -growth2 g -bounded
bighetaD2[*OF* g -bigheta] g' -nonneg g' -integrable **by** *unfold-locales*
sublocale *akra-bazzi-upper* **using** g -growth1 *bighetaD1*[*OF* g -bigheta]
 g' -nonneg g' -integrable **by** *unfold-locales*

lemma *bigheta-f*:

obtains a **where** $a > A$ $f \in \Theta(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) a x))$

proof –

from *bigo-f-aux*[*of* A] **guess** a . **note** $a = \text{this}$
moreover from *bigomega-f-aux*[*of* A] **guess** b . **note** $b = \text{this}$
let $?a = \text{real } (\max (\max (\text{nat } \lceil a \rceil) (\text{nat } \lceil b \rceil)) 0) + 1$
have $?a \in \mathbb{N}$ **by** (*auto simp: max-def*)
moreover have $?a \geq a$ $?a \geq b$ **by** *linarith+*
ultimately have $f \in \Theta(\lambda x. x \text{ powr } p * (1 + \text{integral } (\lambda u. g' u / u \text{ powr } (p + 1)) ?a x))$
using a b **by** (*intro bighetaI*) *blast+*
moreover from a b **have** $?a > A$ **by** *linarith*
ultimately show *?thesis* **by** (*intro that*[*of* $?a$]) *simp-all*
qed

end

named-theorems *akra-bazzi-term-intros* *introduction rules for Akra–Bazzi terms*

lemma *akra-bazzi-term-floor-add* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0$ $b < 1$ $\text{real } x_0 \leq b * \text{real } x_1 + c$ $c < (1 - b) * \text{real } x_1$ $x_1 > 0$

shows *akra-bazzi-term* x_0 x_1 b $(\lambda x. \text{nat } \lfloor b * \text{real } x + c \rfloor)$

proof (*rule akra-bazzi-termI*[*OF* *zero-less-one*])

fix x **assume** $x: x \geq x_1$

from *assms* x **have** $\text{real } x_0 \leq b * \text{real } x_1 + c$ **by** *simp*

also from x *assms* **have** $\dots \leq b * \text{real } x + c$ **by** *auto*

finally have *step-ge-x0*: $b * \text{real } x + c \geq \text{real } x_0$ **by** *simp*

thus $\text{nat } \lfloor b * \text{real } x + c \rfloor \geq x_0$ **by** (*subst le-nat-iff*) (*simp-all add: le-floor-iff*)

from *assms* x **have** $c < (1 - b) * \text{real } x_1$ **by** *simp*

also from *assms* x **have** $\dots \leq (1 - b) * \text{real } x$ **by** (*intro mult-left-mono*) *simp-all*

finally show $\text{nat } \lfloor b * \text{real } x + c \rfloor < x$ **using** *assms step-ge-x0*

by (*subst nat-less-iff*) (*simp-all add: floor-less-iff algebra-simps*)

from *step-ge-x0* **have** $\text{real-of-int } \lfloor c + b * \text{real } x \rfloor = \text{real-of-int } (\text{nat } \lfloor c + b * \text{real } x \rfloor)$ **by** *linarith*

thus $(b * \text{real } x) + (\lfloor b * \text{real } x + c \rfloor - (b * \text{real } x)) =$
 $\text{real } (\text{nat } \lfloor b * \text{real } x + c \rfloor)$ **by** *linarith*

next
have $(\lambda x :: \text{nat}. \text{real-of-int } \lfloor b * \text{real } x + c \rfloor - b * \text{real } x) \in O(\lambda -. |c| + 1)$
by (*intro landau-o.big-mono always-eventually allI, unfold real-norm-def*) *linarith*

also have $(\lambda :: \text{nat}. |c| + 1) \in O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1 + 1))$ **by force**
finally show $(\lambda x :: \text{nat}. \text{real-of-int } \lfloor b * \text{real } x + c \rfloor - b * \text{real } x) \in$
 $O(\lambda x. \text{real } x / \ln (\text{real } x) \text{ powr } (1+1))$.

qed

lemma *akra-bazzi-term-floor-add'* [*akra-bazzi-term-intros*]:
assumes $(b :: \text{real}) > 0 \ b < 1 \ \text{real } x_0 \leq b * \text{real } x_1 + \text{real } c \ \text{real } c < (1 - b) * \text{real } x_1 \ x_1 > 0$
shows *akra-bazzi-term* $x_0 \ x_1 \ b \ (\lambda x. \text{nat } \lfloor b * \text{real } x \rfloor + c)$
proof –
from *assms* **have** *akra-bazzi-term* $x_0 \ x_1 \ b \ (\lambda x. \text{nat } \lfloor b * \text{real } x + \text{real } c \rfloor)$
by (*rule akra-bazzi-term-floor-add*)
also have $(\lambda x. \text{nat } \lfloor b * \text{real } x + \text{real } c \rfloor) = (\lambda x :: \text{nat}. \text{nat } \lfloor b * \text{real } x \rfloor + c)$
proof
fix $x :: \text{nat}$
have $\lfloor b * \text{real } x + \text{real } c \rfloor = \lfloor b * \text{real } x \rfloor + \text{int } c$ **by** *linarith*
also from *assms* **have** $\text{nat } \dots = \text{nat } \lfloor b * \text{real } x \rfloor + c$ **by** (*simp add: nat-add-distrib*)
finally show $\text{nat } \lfloor b * \text{real } x + \text{real } c \rfloor = \text{nat } \lfloor b * \text{real } x \rfloor + c$.
qed
finally show *?thesis* .
qed

lemma *akra-bazzi-term-floor-subtract* [*akra-bazzi-term-intros*]:
assumes $(b :: \text{real}) > 0 \ b < 1 \ \text{real } x_0 \leq b * \text{real } x_1 - c \ 0 < c + (1 - b) * \text{real } x_1 \ x_1 > 0$
shows *akra-bazzi-term* $x_0 \ x_1 \ b \ (\lambda x. \text{nat } \lfloor b * \text{real } x - c \rfloor)$
by (*subst diff-conv-add-uminus, rule akra-bazzi-term-floor-add, insert assms*) *simp-all*

lemma *akra-bazzi-term-floor-subtract'* [*akra-bazzi-term-intros*]:
assumes $(b :: \text{real}) > 0 \ b < 1 \ \text{real } x_0 \leq b * \text{real } x_1 - \text{real } c \ 0 < \text{real } c + (1 - b) * \text{real } x_1 \ x_1 > 0$
shows *akra-bazzi-term* $x_0 \ x_1 \ b \ (\lambda x. \text{nat } \lfloor b * \text{real } x \rfloor - c)$
proof –
from *assms* **have** *akra-bazzi-term* $x_0 \ x_1 \ b \ (\lambda x. \text{nat } \lfloor b * \text{real } x - \text{real } c \rfloor)$
by (*intro akra-bazzi-term-floor-subtract*) *simp-all*
also have $(\lambda x. \text{nat } \lfloor b * \text{real } x - \text{real } c \rfloor) = (\lambda x :: \text{nat}. \text{nat } \lfloor b * \text{real } x \rfloor - c)$
proof
fix $x :: \text{nat}$
have $\lfloor b * \text{real } x - \text{real } c \rfloor = \lfloor b * \text{real } x \rfloor - \text{int } c$ **by** *linarith*
also from *assms* **have** $\text{nat } \dots = \text{nat } \lfloor b * \text{real } x \rfloor - c$ **by** (*simp add: nat-diff-distrib*)
finally show $\text{nat } \lfloor b * \text{real } x - \text{real } c \rfloor = \text{nat } \lfloor b * \text{real } x \rfloor - c$.
qed

finally show *?thesis* .
qed

lemma *akra-bazzi-term-floor* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1$ $0 < (1 - b) * real\ x_1$ $x_1 > 0$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ \lceil b * real\ x \rceil)$
using *assms akra-bazzi-term-floor-add* [**where** $c = 0$] **by** *simp*

lemma *akra-bazzi-term-ceiling-add* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0$ $b < 1$ $real\ x_0 \leq b * real\ x_1 + c$ $c + 1 \leq (1 - b) * x_1$
shows *akra-bazzi-term* $x_0\ x_1\ b\ (\lambda x. nat\ \lceil b * real\ x + c \rceil)$
proof (*rule akra-bazzi-termI* [*OF zero-less-one*])
fix x **assume** $x: x \geq x_1$
have $0 \leq real\ x_0$ **by** *simp*
also from *assms* **have** $real\ x_0 \leq b * real\ x_1 + c$ **by** *simp*
also from *assms* x **have** $b * real\ x_1 \leq b * real\ x$ **by** (*intro mult-left-mono*)
simp-all
hence $b * real\ x_1 + c \leq b * real\ x + c$ **by** *simp*
also have $b * real\ x + c \leq real\text{-of-int}\ \lceil b * real\ x + c \rceil$ **by** *linarith*
finally have *bx-nonneg*: $real\text{-of-int}\ \lceil b * real\ x + c \rceil \geq 0$.

have $c + 1 \leq (1 - b) * x_1$ **by** *fact*
also have $(1 - b) * x_1 \leq (1 - b) * x$ **using** *assms* x **by** (*intro mult-left-mono*)
simp-all
finally have $b * real\ x + c + 1 \leq real\ x$ **using** *assms* **by** (*simp add: algebra-simps*)
with *bx-nonneg* **show** $nat\ \lceil b * real\ x + c \rceil < x$ **by** (*subst nat-less-iff*) (*simp-all add: ceiling-less-iff*)

have $real\ x_0 \leq b * real\ x_1 + c$ **by** *fact*
also have $\dots \leq real\text{-of-int}\ \lceil \dots \rceil$ **by** *linarith*
also have $x_1 \leq x$ **by** *fact*
finally show $x_0 \leq nat\ \lceil b * real\ x + c \rceil$ **using** *assms* **by** (*force simp: ceiling-mono*)

show $b * real\ x + (\lceil b * real\ x + c \rceil - b * real\ x) = real\ (nat\ \lceil b * real\ x + c \rceil)$
using *assms bx-nonneg* **by** *simp*

next

have $(\lambda x::nat. real\text{-of-int}\ \lceil b * real\ x + c \rceil - b * real\ x) \in O(\lambda-. |c| + 1)$
by (*intro landau-o.big-mono always-eventually allI, unfold real-norm-def*) *linarith*

also have $(\lambda-::nat. |c| + 1) \in O(\lambda x. real\ x / \ln (real\ x)\ powr (1 + 1))$ **by** *force*
finally show $(\lambda x::nat. real\text{-of-int}\ \lceil b * real\ x + c \rceil - b * real\ x) \in O(\lambda x. real\ x / \ln (real\ x)\ powr (1+1))$.

qed

lemma *akra-bazzi-term-ceiling-add'* [*akra-bazzi-term-intros*]:

assumes $(b::real) > 0 \ b < 1 \ real \ x_0 \leq b * real \ x_1 + real \ c \ real \ c + 1 \leq (1 - b)$
 $* \ x_1$
shows $akra-bazzi-term \ x_0 \ x_1 \ b \ (\lambda x. \ nat \ \lceil b * real \ x \rceil + c)$
proof –
from *assms* **have** $akra-bazzi-term \ x_0 \ x_1 \ b \ (\lambda x. \ nat \ \lceil b * real \ x + real \ c \rceil)$
by (*rule akra-bazzi-term-ceiling-add*)
also have $(\lambda x. \ nat \ \lceil b * real \ x + real \ c \rceil) = (\lambda x::nat. \ nat \ \lceil b * real \ x \rceil + c)$
proof
fix $x :: nat$
from *assms* **have** $0 \leq b * real \ x$ **by** *simp*
also have $b * real \ x \leq real-of-int \ \lceil b * real \ x \rceil$ **by** *linarith*
finally have $bx-nonneg: \ \lceil b * real \ x \rceil \geq 0$ **by** *simp*

have $\lceil b * real \ x + real \ c \rceil = \lceil b * real \ x \rceil + int \ c$ **by** *linarith*
also from *assms* $bx-nonneg$ **have** $nat \ \dots = nat \ \lceil b * real \ x \rceil + c$
by (*subst nat-add-distrib*) *simp-all*
finally show $nat \ \lceil b * real \ x + real \ c \rceil = nat \ \lceil b * real \ x \rceil + c$.
qed
finally show *?thesis* .
qed

lemma *akra-bazzi-term-ceiling-subtract* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0 \ b < 1 \ real \ x_0 \leq b * real \ x_1 - c \ 1 \leq c + (1 - b) * x_1$
shows $akra-bazzi-term \ x_0 \ x_1 \ b \ (\lambda x. \ nat \ \lceil b * real \ x - c \rceil)$
by (*subst diff-conv-add-uminus*, *rule akra-bazzi-term-ceiling-add*, *insert assms*)
simp-all

lemma *akra-bazzi-term-ceiling-subtract'* [*akra-bazzi-term-intros*]:
assumes $(b::real) > 0 \ b < 1 \ real \ x_0 \leq b * real \ x_1 - real \ c \ 1 \leq real \ c + (1 - b)$
 $* \ x_1$
shows $akra-bazzi-term \ x_0 \ x_1 \ b \ (\lambda x. \ nat \ \lceil b * real \ x \rceil - c)$
proof –
from *assms* **have** $akra-bazzi-term \ x_0 \ x_1 \ b \ (\lambda x. \ nat \ \lceil b * real \ x - real \ c \rceil)$
by (*intro akra-bazzi-term-ceiling-subtract*) *simp-all*
also have $(\lambda x. \ nat \ \lceil b * real \ x - real \ c \rceil) = (\lambda x::nat. \ nat \ \lceil b * real \ x \rceil - c)$
proof
fix $x :: nat$
from *assms* **have** $0 \leq b * real \ x$ **by** *simp*
also have $b * real \ x \leq real-of-int \ \lceil b * real \ x \rceil$ **by** *linarith*
finally have $bx-nonneg: \ \lceil b * real \ x \rceil \geq 0$ **by** *simp*

have $\lceil b * real \ x - real \ c \rceil = \lceil b * real \ x \rceil - int \ c$ **by** *linarith*
also from *assms* $bx-nonneg$ **have** $nat \ \dots = nat \ \lceil b * real \ x \rceil - c$ **by** *simp*
finally show $nat \ \lceil b * real \ x - real \ c \rceil = nat \ \lceil b * real \ x \rceil - c$.
qed
finally show *?thesis* .
qed

lemma *akra-bazzi-term-ceiling* [*akra-bazzi-term-intros*]:

```

assumes (b::real) > 0 b < 1 real x0 ≤ b * real x1 1 ≤ (1 - b) * x1
shows akra-bazzi-term x0 x1 b (λx. nat [b*real x])
using assms akra-bazzi-term-ceiling-add[where c = 0] by simp

```

end

5 The Master theorem

theory Master-Theorem

imports

HOL-Analysis.Equivalence-Lebesgue-Henstock-Integration

Akra-Bazzi-Library

Akra-Bazzi

begin

lemma *fundamental-theorem-of-calculus-real*:

$a \leq b \implies \forall x \in \{a..b\}. (f \text{ has-real-derivative } f' x) \text{ (at } x \text{ within } \{a..b\}) \implies$
 $(f' \text{ has-integral } (f b - f a)) \{a..b\}$

by (*intro fundamental-theorem-of-calculus ballI*)

(*simp-all add: has-real-derivative-iff-has-vector-derivative[symmetric]*)

lemma *integral-powr*:

$y \neq -1 \implies a \leq b \implies a > 0 \implies \text{integral } \{a..b\} (\lambda x. x \text{ powr } y :: \text{real}) =$
 $\text{inverse } (y + 1) * (b \text{ powr } (y + 1) - a \text{ powr } (y + 1))$

by (*subst right-diff-distrib, intro integral-unique fundamental-theorem-of-calculus-real*)

(*auto intro!: derivative-eq-intros*)

lemma *integral-ln-powr-over-x*:

$y \neq -1 \implies a \leq b \implies a > 1 \implies \text{integral } \{a..b\} (\lambda x. \ln x \text{ powr } y / x :: \text{real}) =$
 $\text{inverse } (y + 1) * (\ln b \text{ powr } (y + 1) - \ln a \text{ powr } (y + 1))$

by (*subst right-diff-distrib, intro integral-unique fundamental-theorem-of-calculus-real*)

(*auto intro!: derivative-eq-intros*)

lemma *integral-one-over-x-ln-x*:

$a \leq b \implies a > 1 \implies \text{integral } \{a..b\} (\lambda x. \text{inverse } (x * \ln x) :: \text{real}) = \ln (\ln b)$
 $- \ln (\ln a)$

by (*intro integral-unique fundamental-theorem-of-calculus-real*)

(*auto intro!: derivative-eq-intros simp: field-simps*)

lemma *akra-bazzi-integral-kurzweil-henstock*:

akra-bazzi-integral (λf a b. f *integrable-on* {a..b}) (λf a b. *integral* {a..b} f)

apply *unfold-locales*

apply (*rule integrable-const-ivl*)

apply *simp*

apply (*erule integrable-subinterval-real, simp*)

apply (*blast intro!: integral-le*)

apply (*rule integral-combine, simp-all*) []

done

```

locale master-theorem-function = akra-bazzi-recursion +
  fixes g :: nat ⇒ real
  assumes f-nonneg-base:  $x \geq x_0 \implies x < x_1 \implies f x \geq 0$ 
  and f-rec:  $x \geq x_1 \implies f x = g x + (\sum_{i < k. as!i} * f ((ts!i) x))$ 
  and g-nonneg:  $x \geq x_1 \implies g x \geq 0$ 
  and ex-pos-a:  $\exists a \in \text{set } as. a > 0$ 
begin

interpretation akra-bazzi-integral  $\lambda f a b. f$  integrable-on {a..b}  $\lambda f a b. \text{integral}$ 
{a..b} f
  by (rule akra-bazzi-integral-kurzweil-henstock)

sublocale akra-bazzi-function  $x_0 x_1 k as bs ts f \lambda f a b. f$  integrable-on {a..b}
 $\lambda f a b. \text{integral } \{a..b\} f g$ 
  using f-nonneg-base f-rec g-nonneg ex-pos-a by unfold-locales

context
begin

private lemma g-nonneg': eventually ( $\lambda x. g x \geq 0$ ) at-top
  using g-nonneg by (force simp: eventually-at-top-linorder)

private lemma g-pos:
  assumes  $g \in \Omega(h)$ 
  assumes eventually ( $\lambda x. h x > 0$ ) at-top
  shows eventually ( $\lambda x. g x > 0$ ) at-top
proof –
  from landau-omega.bigE-nonneg-real[OF assms(1) g-nonneg'] guess c . note c
= this
  from assms(2) c(2) show ?thesis
  by eventually-elim (rule less-le-trans[OF mult-pos-pos[OF c(1)]], simp-all)
qed

private lemma f-pos:
  assumes  $g \in \Omega(h)$ 
  assumes eventually ( $\lambda x. h x > 0$ ) at-top
  shows eventually ( $\lambda x. f x > 0$ ) at-top
  using g-pos[OF assms(1,2)] eventually-ge-at-top[of x1]
  by (eventually-elim) (subst f-rec, insert step-ge-x0,
    auto intro!: add-pos-nonneg sum-nonneg mult-nonneg-nonneg[OF a-ge-0]
    f-nonneg)

lemma bs-lower-bound:  $\exists C > 0. \forall b \in \text{set } bs. C < b$ 
proof (intro exI conjI ballI)
  from b-pos show A:  $\text{Min } (\text{set } bs) / 2 > 0$  by auto
  fix b assume b:  $b \in \text{set } bs$ 
  from A have  $\text{Min } (\text{set } bs) / 2 < \text{Min } (\text{set } bs)$  by simp

```

also from b have $\dots \leq b$ by *simp*
 finally show $\text{Min}(\text{set } bs) / 2 < b$.
 qed

private lemma *powr-growth2*:

$\exists C \ c2. 0 < c2 \wedge C < \text{Min}(\text{set } bs) \wedge$
eventually $(\lambda x. \forall u \in \{C * x..x\}. c2 * x \text{ powr } p' \geq u \text{ powr } p')$ *at-top*

proof (*intro exI conjI allI ballI*)

define C **where** $C = \text{Min}(\text{set } bs) / 2$

from *b-bounds bs-nonempty* **have** $C\text{-pos}: C > 0$ **unfolding** $C\text{-def}$ **by** *auto*

thus $C < \text{Min}(\text{set } bs)$ **unfolding** $C\text{-def}$ **by** *simp*

show $\max(C \text{ powr } p') 1 > 0$ **by** *simp*

show *eventually* $(\lambda x. \forall u \in \{C * x..x\}.$

$\max((\text{Min}(\text{set } bs)/2) \text{ powr } p') 1 * x \text{ powr } p' \geq u \text{ powr } p')$ *at-top*

using *eventually-gt-at-top[of 0::real]* **apply** *eventually-elim*

proof *clarify*

fix $x \ u$ **assume** $x: x > 0$ **and** $u \in \{C * x..x\}$

hence $u: u \geq C * x \ u \leq x$ **unfolding** $C\text{-def}$ **by** *simp-all*

from u **have** $u \text{ powr } p' \leq \max((C * x) \text{ powr } p') (x \text{ powr } p')$ **using** $C\text{-pos}$ x

by (*intro powr-upper-bound mult-pos-pos*) *simp-all*

also from $u \ x \ C\text{-pos}$ **have** $\max((C * x) \text{ powr } p') (x \text{ powr } p') = x \text{ powr } p' * \max(C \text{ powr } p') 1$

by (*subst max-mult-left*) (*simp-all add: powr-mult algebra-simps*)

finally show $u \text{ powr } p' \leq \max((\text{Min}(\text{set } bs)/2) \text{ powr } p') 1 * x \text{ powr } p'$

by (*simp add: C-def algebra-simps*)

qed

qed

private lemma *powr-growth1*:

$\exists C \ c1. 0 < c1 \wedge C < \text{Min}(\text{set } bs) \wedge$

eventually $(\lambda x. \forall u \in \{C * x..x\}. c1 * x \text{ powr } p' \leq u \text{ powr } p')$ *at-top*

proof (*intro exI conjI allI ballI*)

define C **where** $C = \text{Min}(\text{set } bs) / 2$

from *b-bounds bs-nonempty* **have** $C\text{-pos}: C > 0$ **unfolding** $C\text{-def}$ **by** *auto*

thus $C < \text{Min}(\text{set } bs)$ **unfolding** $C\text{-def}$ **by** *simp*

from $C\text{-pos}$ **show** $\min(C \text{ powr } p') 1 > 0$ **by** *simp*

show *eventually* $(\lambda x. \forall u \in \{C * x..x\}.$

$\min((\text{Min}(\text{set } bs)/2) \text{ powr } p') 1 * x \text{ powr } p' \leq u \text{ powr } p')$ *at-top*

using *eventually-gt-at-top[of 0::real]* **apply** *eventually-elim*

proof *clarify*

fix $x \ u$ **assume** $x: x > 0$ **and** $u \in \{C * x..x\}$

hence $u: u \geq C * x \ u \leq x$ **unfolding** $C\text{-def}$ **by** *simp-all*

from $u \ x \ C\text{-pos}$ **have** $x \text{ powr } p' * \min(C \text{ powr } p') 1 = \min((C * x) \text{ powr } p') (x \text{ powr } p')$

by (*subst min-mult-left*) (*simp-all add: powr-mult algebra-simps*)

also from u **have** $u \text{ powr } p' \geq \min((C * x) \text{ powr } p') (x \text{ powr } p')$ **using** $C\text{-pos}$ x

by (*intro powr-lower-bound mult-pos-pos*) *simp-all*

finally show $u \text{ powr } p' \geq \min((\text{Min}(\text{set } bs)/2) \text{ powr } p') 1 * x \text{ powr } p'$

by (*simp add: C-def algebra-simps*)

qed
qed

private lemma *powr-ln-powr-lower-bound*:

$a > 1 \implies a \leq x \implies x \leq b \implies$
 $\min (a \text{ powr } p) (b \text{ powr } p) * \min (\ln a \text{ powr } p') (\ln b \text{ powr } p') \leq x \text{ powr } p * \ln$
 $x \text{ powr } p'$
by (*intro mult-mono powr-lower-bound*) (*auto intro: min.coboundedI1*)

private lemma *powr-ln-powr-upper-bound*:

$a > 1 \implies a \leq x \implies x \leq b \implies$
 $\max (a \text{ powr } p) (b \text{ powr } p) * \max (\ln a \text{ powr } p') (\ln b \text{ powr } p') \geq x \text{ powr } p * \ln$
 $x \text{ powr } p'$
by (*intro mult-mono powr-upper-bound*) (*auto intro: max.coboundedI1*)

private lemma *powr-ln-powr-upper-bound'*:

eventually ($\lambda a. \forall b > a. \exists c. \forall x \in \{a..b\}. x \text{ powr } p * \ln x \text{ powr } p' \leq c$) *at-top*
by (*subst eventually-at-top-dense*) (*force intro: powr-ln-powr-upper-bound*)

private lemma *powr-upper-bound'*:

eventually ($\lambda a::\text{real}. \forall b > a. \exists c. \forall x \in \{a..b\}. x \text{ powr } p' \leq c$) *at-top*
by (*subst eventually-at-top-dense*) (*force intro: powr-upper-bound*)

lemmas *bounds* =

powr-ln-powr-lower-bound powr-ln-powr-upper-bound powr-ln-powr-upper-bound'
powr-upper-bound'

private lemma *eventually-ln-const*:

assumes ($C::\text{real}$) > 0
shows *eventually* ($\lambda x. \ln (C*x) / \ln x > 1/2$) *at-top*
proof –
from *tendstoD*[*OF tendsto-ln-over-ln*[*of C 1*], *of 1/2*] *assms*
have *eventually* ($\lambda x. |\ln (C*x) / \ln x - 1| < 1/2$) *at-top* **by** (*simp add:*
dist-real-def)
thus *?thesis* **by** *eventually-elim linarith*
qed

private lemma *powr-ln-powr-growth1*: $\exists C c1. 0 < c1 \wedge C < \text{Min} (\text{set } bs) \wedge$

eventually ($\lambda x. \forall u \in \{C * x..x\}. c1 * (x \text{ powr } r * \ln x \text{ powr } r') \leq u \text{ powr } r * \ln$
 $u \text{ powr } r')$ *at-top*

proof (*intro exI conjI*)

let $?C = \text{Min} (\text{set } bs) / 2$ **and** $?f = \lambda x. x \text{ powr } r * \ln x \text{ powr } r'$

define C **where** $C = ?C$

from *b-bounds* **have** $C\text{-pos}$: $C > 0$ **unfolding** $C\text{-def}$ **by** *simp*

let $?T = \min (C \text{ powr } r) (1 \text{ powr } r) * \min ((1/2) \text{ powr } r') (1 \text{ powr } r')$

from $C\text{-pos}$ **show** $?T > 0$ **unfolding** min-def **by** (*auto split: if-split*)

from *bs-nonempty b-bounds* **have** $C\text{-pos}$: $C > 0$ **unfolding** $C\text{-def}$ **by** *simp*

thus $C < \text{Min} (\text{set } bs)$ **by** (*simp add: C-def*)

show *eventually* $(\lambda x. \forall u \in \{C * x..x\}. ?T * ?f x \leq ?f u)$ *at-top*
using *eventually-gt-at-top*[of *max 1 (inverse C)*] *eventually-ln-const*[OF *C-pos*]
apply *eventually-elim*
proof *clarify*
fix $x u$ **assume** $x: x > \max 1$ (*inverse C*) **and** $u: u \in \{C * x..x\}$
hence $x': x > 1$ **by** (*simp add: field-simps*)
with *C-pos* **have** $x\text{-pos}: x > 0$ **by** (*simp add: field-simps*)
from $x u$ *C-pos* **have** $u': u > 1$ **by** (*simp add: field-simps*)
assume $A: \ln (C * x) / \ln x > 1/2$
have $\min (C \text{ powr } r) (1 \text{ powr } r) \leq (u/x) \text{ powr } r$
using $x u u' C\text{-pos}$ **by** (*intro powr-lower-bound*) (*simp-all add: field-simps*)
moreover {
note A
also from *C-pos* $x' u u'$ **have** $\ln (C * x) \leq \ln u$ **by** (*subst ln-le-cancel-iff*)
simp-all
with x' **have** $\ln (C * x) / \ln x \leq \ln u / \ln x$ **by** (*simp add: field-simps*)
finally have $\min ((1/2) \text{ powr } r') (1 \text{ powr } r') \leq (\ln u / \ln x) \text{ powr } r'$
using $x u u' C\text{-pos} A$ **by** (*intro powr-lower-bound*) *simp-all*
}
ultimately have $?T \leq (u/x) \text{ powr } r * (\ln u / \ln x) \text{ powr } r'$
using $x\text{-pos}$ **by** (*intro mult-mono*) *simp-all*
also from $x u u'$ **have** $\dots = ?f u / ?f x$ **by** (*simp add: powr-divide*)
finally show $?T * ?f x \leq ?f u$ **using** x' **by** (*simp add: field-simps*)
qed
qed

private lemma *powr-ln-powr-growth2*: $\exists C c1. 0 < c1 \wedge C < \text{Min} (\text{set } bs) \wedge$
eventually $(\lambda x. \forall u \in \{C * x..x\}. c1 * (x \text{ powr } r * \ln x \text{ powr } r') \geq u \text{ powr } r * \ln$
 $u \text{ powr } r')$ *at-top*
proof (*intro exI conjI*)
let $?C = \text{Min} (\text{set } bs) / 2$ **and** $?f = \lambda x. x \text{ powr } r * \ln x \text{ powr } r'$
define C **where** $C = ?C$
let $?T = \max (C \text{ powr } r) (1 \text{ powr } r) * \max ((1/2) \text{ powr } r') (1 \text{ powr } r')$
show $?T > 0$ **by** *simp*
from *b-bounds bs-nonempty* **have** $C\text{-pos}: C > 0$ **unfolding** $C\text{-def}$ **by** *simp*
thus $C < \text{Min} (\text{set } bs)$ **by** (*simp add: C-def*)

show *eventually* $(\lambda x. \forall u \in \{C * x..x\}. ?T * ?f x \geq ?f u)$ *at-top*
using *eventually-gt-at-top*[of *max 1 (inverse C)*] *eventually-ln-const*[OF *C-pos*]
apply *eventually-elim*
proof *clarify*
fix $x u$ **assume** $x: x > \max 1$ (*inverse C*) **and** $u: u \in \{C * x..x\}$
hence $x': x > 1$ **by** (*simp add: field-simps*)
with *C-pos* **have** $x\text{-pos}: x > 0$ **by** (*simp add: field-simps*)
from $x u$ *C-pos* **have** $u': u > 1$ **by** (*simp add: field-simps*)
assume $A: \ln (C * x) / \ln x > 1/2$
from $x u u'$ **have** $?f u / ?f x = (u/x) \text{ powr } r * (\ln u / \ln x) \text{ powr } r'$ **by** (*simp*
add: powr-divide)

```

also {
  have (u/x) powr r ≤ max (C powr r) (1 powr r)
    using x u u' C-pos by (intro powr-upper-bound) (simp-all add: field-simps)
  moreover {
    note A
    also from C-pos x' u u' have ln (C*x) ≤ ln u by (subst ln-le-cancel-iff)
  simp-all
    with x' have ln (C*x) / ln x ≤ ln u / ln x by (simp add: field-simps)
    finally have (ln u / ln x) powr r' ≤ max ((1/2) powr r') (1 powr r')
      using x u u' C-pos A by (intro powr-upper-bound) simp-all
  } ultimately have (u/x) powr r * (ln u / ln x) powr r' ≤ ?T
    using x-pos by (intro mult-mono) simp-all
  }
  finally show ?T * ?f x ≥ ?f u using x' by (simp add: field-simps)
qed
qed

```

lemmas *growths* = *powr-growth1 powr-growth2 powr-ln-powr-growth1 powr-ln-powr-growth2*

private lemma *master-integrable*:

```

∃ a::real. ∀ b ≥ a. (λu. u powr r * ln u powr s / u powr t) integrable-on {a..b}
∃ a::real. ∀ b ≥ a. (λu. u powr r / u powr s) integrable-on {a..b}
by (rule exI[of - 2], force intro!: integrable-continuous-real continuous-intros)+

```

private lemma *master-integral*:

```

fixes a p p' :: real
assumes p: p ≠ p' and a: a > 0
obtains c d where c ≠ 0 p > p' → d ≠ 0
  (λx::nat. x powr p * (1 + integral {a..x} (λu. u powr p' / u powr (p+1)))) ∈
  Θ(λx::nat. d * x powr p + c * x powr p')

```

proof –

```

define e where e = a powr (p' - p)
from assms have e: e ≥ 0 by (simp add: e-def)
define c where c = inverse (p' - p)
define d where d = 1 - inverse (p' - p) * e
have c ≠ 0 and p > p' → d ≠ 0
  using e p a unfolding c-def d-def by (auto simp: field-simps)
thus ?thesis
  apply (rule that) apply (rule bigtheta-real-nat-transfer, rule bigthetaI-cong)
  using eventually-ge-at-top[of a]
proof eventually-elim
  fix x assume x: x ≥ a
  hence integral {a..x} (λu. u powr p' / u powr (p+1)) =
    integral {a..x} (λu. u powr (p' - (p + 1)))
  by (intro Henstock-Kurzweil-Integration.integral-cong) (simp-all add: powr-diff
[symmetric])
  also have ... = inverse (p' - p) * (x powr (p' - p) - a powr (p' - p))
    using p x0-less-x1 a x by (simp add: integral-powr)

```

also have $x \text{ powr } p * (1 + \dots) = d * x \text{ powr } p + c * x \text{ powr } p'$
using p **unfolding** c -def d -def **by** (*simp add: algebra-simps powr-diff e-def*)
finally show $x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p' / u \text{ powr } (p+1)))$
 =
 $d * x \text{ powr } p + c * x \text{ powr } p'$.

qed
qed

private lemma *master-integral'*:

fixes $a p p' :: \text{real}$
assumes $p': p' \neq 0$ **and** $a: a > 1$
obtains $c d :: \text{real}$ **where** $p' < 0 \longrightarrow c \neq 0 d \neq 0$
 $(\lambda x::\text{nat. } x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } (p'-1) / u \text{ powr } (p+1)))) \in$
 $\Theta(\lambda x::\text{nat. } c * x \text{ powr } p + d * x \text{ powr } p * \ln x \text{ powr } p')$

proof –

define e **where** $e = \ln a \text{ powr } p'$
from *assms* **have** $e: e > 0$ **by** (*simp add: e-def*)
define c **where** $c = 1 - \text{inverse } p' * e$
define d **where** $d = \text{inverse } p'$
from *assms* e **have** $p' < 0 \longrightarrow c \neq 0 d \neq 0$ **unfolding** c -def d -def **by** (*auto simp: field-simps*)
thus *?thesis*
apply (*rule that*) **apply** (*rule landau-real-nat-transfer, rule bighetaI-cong*)
using *eventually-ge-at-top*[of a]

proof *eventually-elim*

fix $x :: \text{real}$ **assume** $x: x \geq a$
have $\text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } (p' - 1) / u \text{ powr } (p + 1)) =$
 $\text{integral } \{a..x\} (\lambda u. \ln u \text{ powr } (p' - 1) / u)$ **using** x *a x0-less-x1*
by (*intro Henstock-Kurzweil-Integration.integral-cong*) (*simp-all add: powr-add*)
also have $\dots = \text{inverse } p' * (\ln x \text{ powr } p' - \ln a \text{ powr } p')$
using p' *x0-less-x1 a(1) x* **by** (*simp add: integral-ln-powr-over-x*)
also have $x \text{ powr } p * (1 + \dots) = c * x \text{ powr } p + d * x \text{ powr } p * \ln x \text{ powr } p'$
using p' **by** (*simp add: algebra-simps c-def d-def e-def*)
finally show $x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } (p'-1) / u \text{ powr } (p+1))) =$
 $c * x \text{ powr } p + d * x \text{ powr } p * \ln x \text{ powr } p'$.

qed
qed

private lemma *master-integral''*:

fixes $a p p' :: \text{real}$
assumes $a: a > 1$
shows $(\lambda x::\text{nat. } x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } - 1 / u \text{ powr } (p+1)))) \in$
 $\Theta(\lambda x::\text{nat. } x \text{ powr } p * \ln (\ln x))$

proof (*rule landau-real-nat-transfer*)

have $(\lambda x::\text{real. } x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } - 1 / u \text{ powr } (p+1)))) \in$

$\Theta(\lambda x::\text{real}. (1 - \ln(\ln a)) * x^{\text{powr } p} + x^{\text{powr } p} * \ln(\ln x))$ (is ?f ∈ -)
apply (rule *bighetaI-cong*) **using** *eventually-ge-at-top*[of a]
proof *eventually-elim*
fix x **assume** x: x ≥ a
have *integral {a..x} (λu. u^{powr p} * ln u^{powr -1} / u^{powr (p+1)}) =*
*integral {a..x} (λu. inverse (u * ln u))* **using** x a *x0-less-x1*
by (intro *Henstock-Kurzweil-Integration.integral-cong*) (*simp-all add: powr-add*
powr-minus field-simps)
also have ... = ln(ln x) - ln(ln a)
using *x0-less-x1 a(1) x* **by** (*subst integral-one-over-x-ln-x simp-all*)
also have x^{powr p} * (1 + ...) = (1 - ln(ln a)) * x^{powr p} + x^{powr p} * ln
(ln x)
by (*simp add: algebra-simps*)
finally show x^{powr p} * (1 + *integral {a..x} (λu. u^{powr p} * ln u^{powr -1} /*
u^{powr (p+1)})) =
(1 - ln(ln a)) * x^{powr p} + x^{powr p} * ln(ln x) .
qed
also have (λx. (1 - ln(ln a)) * x^{powr p} + x^{powr p} * ln(ln x)) ∈
 $\Theta(\lambda x. x^{\text{powr } p} * \ln(\ln x))$ **by** *simp*
finally show ?f ∈ $\Theta(\lambda a. a^{\text{powr } p} * \ln(\ln a))$.
qed

lemma *master1-bigo*:

assumes *g-bigo*: g ∈ O(λx. real x^{powr p'})
assumes *less-p'*: (∑ i<k. as!i * bs!i^{powr p'}) > 1
shows f ∈ O(λx. real x^{powr p})
proof -
interpret *akra-bazzi-upper* x₀ x₁ k as bs ts f
λf a b. f *integrable-on* {a..b} λf a b. *integral {a..b} f g* λx. x^{powr p'}
using *assms growths g-bigo master-integrable* **by** *unfold-locales (assumption |*
simp)+
from *less-p'* **have** *less-p*: p' < p **by** (*rule p-greaterI*)
from *bigo-f[of 0]* **guess** a . **note** a = *this*
note a(2)
also from a(1) *less-p x0-less-x1* **have** p ≠ p' **by** *simp-all*
from *master-integral[OF this a(1)]* **guess** c d . **note** cd = *this*
note cd(3)
also from cd(1,2) *less-p*
have (λx::nat. d * real x^{powr p} + c * real x^{powr p'}) ∈ $\Theta(\lambda x. \text{real } x^{\text{powr } p})$
by force
finally show f ∈ O(λx::nat. x^{powr p}) .
qed

lemma *master1*:

assumes *g-bigo*: g ∈ O(λx. real x^{powr p'})
assumes *less-p'*: (∑ i<k. as!i * bs!i^{powr p'}) > 1

assumes $f\text{-pos}$: *eventually* $(\lambda x. f\ x > 0)$ *at-top*
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p)$
proof (*rule bigthetaI*)
interpret *akra-bazzi-lower* $x_0\ x_1\ k$ *as* $bs\ ts\ f$
 $\lambda f\ a\ b. f\ \text{integrable-on}\ \{a..b\}\ \lambda f\ a\ b. \text{integral}\ \{a..b\}\ f\ g\ \lambda-. 0$
using *assms(1,3)* *bs-lower-bound* **by** *unfold-locales* (*auto intro: always-eventually*)
from *bigomega-f* **show** $f \in \Omega(\lambda x. \text{real } x \text{ powr } p)$ **by** *force*
qed (*fact master1-bigo[OF g-bigo less-p']*)

lemma *master2-3*:

assumes $g\text{-bigtheta}$: $g \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln(\text{real } x) \text{ powr } (p' - 1))$
assumes p' : $p' > 0$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln(\text{real } x) \text{ powr } p')$
proof –
have *eventually* $(\lambda x::\text{real}. x \text{ powr } p * \ln\ x \text{ powr } (p' - 1) > 0)$ *at-top*
using *eventually-gt-at-top[of 1::real]* **by** *eventually-elim simp*
hence *eventually* $(\lambda x. f\ x > 0)$ *at-top*
by (*rule f-pos[OF bigthetaD2[OF g-bigtheta] eventually-nat-real]*)
then interpret *akra-bazzi* $x_0\ x_1\ k$ *as* $bs\ ts\ f$
 $\lambda f\ a\ b. f\ \text{integrable-on}\ \{a..b\}\ \lambda f\ a\ b. \text{integral}\ \{a..b\}\ f\ g\ \lambda x. x \text{ powr } p * \ln\ x \text{ powr } (p' - 1)$
using *assms growths bounds master-integrable* **by** *unfold-locales* (*assumption | simp*)
from *bigtheta-f[of 1]* **guess** a . **note** $a = \text{this}$
note $a(2)$
also from $a(1)$ p' **have** $p' \neq 0$ **by** *simp-all*
from *master-integral'[OF this a(1), of p]* **guess** $c\ d$. **note** $cd = \text{this}$
note $cd(3)$
also have $(\lambda x::\text{nat}. c * \text{real } x \text{ powr } p + d * \text{real } x \text{ powr } p * \ln(\text{real } x) \text{ powr } p') \in \Theta(\lambda x::\text{nat}. x \text{ powr } p * \ln\ x \text{ powr } p')$ **using** $cd(1,2)$ p' **by** *force*
finally show $f \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln(\text{real } x) \text{ powr } p')$.
qed

lemma *master2-1*:

assumes $g\text{-bigtheta}$: $g \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln(\text{real } x) \text{ powr } p')$
assumes p' : $p' < -1$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p)$
proof –
have *eventually* $(\lambda x::\text{real}. x \text{ powr } p * \ln\ x \text{ powr } p' > 0)$ *at-top*
using *eventually-gt-at-top[of 1::real]* **by** *eventually-elim simp*
hence *eventually* $(\lambda x. f\ x > 0)$ *at-top*
by (*rule f-pos[OF bigthetaD2[OF g-bigtheta] eventually-nat-real]*)
then interpret *akra-bazzi* $x_0\ x_1\ k$ *as* $bs\ ts\ f$
 $\lambda f\ a\ b. f\ \text{integrable-on}\ \{a..b\}\ \lambda f\ a\ b. \text{integral}\ \{a..b\}\ f\ g\ \lambda x. x \text{ powr } p * \ln\ x \text{ powr } p'$
using *assms growths bounds master-integrable* **by** *unfold-locales* (*assumption | simp*)
from *bigtheta-f[of 1]* **guess** a . **note** $a = \text{this}$
note $a(2)$

also from $a(1)$ p' **have** $A: p' + 1 \neq 0$ **by** *simp-all*
obtain $c d :: \text{real}$ **where** $cd: c \neq 0 d \neq 0$ **and**
 $(\lambda x::\text{nat. } x \text{ powr } p * (1 + \text{integral } \{a..x\} (\lambda u. u \text{ powr } p * \ln u \text{ powr } p' / u \text{ powr } (p+1)))) \in$
 $\Theta(\lambda x::\text{nat. } c * x \text{ powr } p + d * x \text{ powr } p * \ln x \text{ powr } (p' + 1))$
by (*rule master-integral'[OF A a(1), of p]*) (*insert p', simp*)
note *this(3)*
also have $(\lambda x::\text{nat. } c * \text{real } x \text{ powr } p + d * \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } (p' + 1)) \in$
 $\Theta(\lambda x::\text{nat. } x \text{ powr } p)$ **using** $cd(1,2)$ p' **by** *force*
finally show $f \in \Theta(\lambda x::\text{nat. } x \text{ powr } p)$.
qed

lemma *master2-2*:

assumes $g\text{-bigtheta}: g \in \Theta(\lambda x. \text{real } x \text{ powr } p / \ln (\text{real } x))$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\ln (\text{real } x)))$
proof –
have *eventually* $(\lambda x::\text{real. } x \text{ powr } p / \ln x > 0)$ *at-top*
using *eventually-gt-at-top[of 1::real]* **by** *eventually-elim simp*
hence *eventually* $(\lambda x. f x > 0)$ *at-top*
by (*rule f-pos[OF bigthetaD2[OF g-bigtheta] eventually-nat-real]*)
moreover from $g\text{-bigtheta}$ **have** $g\text{-bigtheta}' : g \in \Theta(\lambda x. \text{real } x \text{ powr } p * \ln (\text{real } x) \text{ powr } -1)$
by (*rule landau-theta.trans, intro landau-real-nat-transfer*) *simp*
ultimately interpret *akra-bazzi* $x_0 x_1 k$ *as* $bs ts f$
 $\lambda f a b. f \text{ integrable-on } \{a..b\} \lambda f a b. \text{integral } \{a..b\} f g \lambda x. x \text{ powr } p * \ln x \text{ powr } -1$
using *assms growths bounds master-integrable* **by** *unfold-locales (assumption | simp)+*
from *bigtheta-f[of 1]* **guess** a . **note** $a = \text{this}$
note $a(2)$
also note *master-integral''[OF a(1)]*
finally show $f \in \Theta(\lambda x::\text{nat. } x \text{ powr } p * \ln (\ln x))$.
qed

lemma *master3*:

assumes $g\text{-bigtheta}: g \in \Theta(\lambda x. \text{real } x \text{ powr } p')$
assumes $p'\text{-greater}' : (\sum i < k. as!i * bs!i \text{ powr } p') < 1$
shows $f \in \Theta(\lambda x. \text{real } x \text{ powr } p')$
proof –
have *eventually* $(\lambda x::\text{real. } x \text{ powr } p' > 0)$ *at-top*
using *eventually-gt-at-top[of 1::real]* **by** *eventually-elim simp*
hence *eventually* $(\lambda x. f x > 0)$ *at-top*
by (*rule f-pos[OF bigthetaD2[OF g-bigtheta] eventually-nat-real]*)
then interpret *akra-bazzi* $x_0 x_1 k$ *as* $bs ts f$
 $\lambda f a b. f \text{ integrable-on } \{a..b\} \lambda f a b. \text{integral } \{a..b\} f g \lambda x. x \text{ powr } p'$
using *assms growths bounds master-integrable* **by** *unfold-locales (assumption | simp)+*
from $p'\text{-greater}'$ **have** $p'\text{-greater}: p' > p$ **by** (*rule p-lessI*)

```

from bigheta-f[of 0] guess a . note a = this
note a(2)
also from p'-greater have  $p \neq p'$  by simp
from master-integral[OF this a(1)] guess c d . note cd = this
note cd(3)
also have  $(\lambda x::nat. d * x \text{ powr } p + c * x \text{ powr } p') \in \Theta(\lambda x::real. x \text{ powr } p')$ 
using p'-greater cd(1,2) by force
finally show  $f \in \Theta(\lambda x. real \ x \text{ powr } p')$  .
qed

end
end

end

```

6 Evaluating expressions with rational numerals

```

theory Eval-Numeral
imports
  Complex-Main
begin

```

```

lemma real-numeral-to-Ratreal:
   $(0::real) = Ratreal (Frct (0, 1))$ 
   $(1::real) = Ratreal (Frct (1, 1))$ 
   $(numeral \ x :: real) = Ratreal (Frct (numeral \ x, 1))$ 
   $(1::int) = numeral \ Num.One$ 
by (simp-all add: rat-number-collapse)

```

```

lemma real-equals-code:  $Ratreal \ x = Ratreal \ y \longleftrightarrow x = y$ 
by simp

```

```

lemma Rat-normalize-idempotent:  $Rat.normalize (Rat.normalize \ x) = Rat.normalize \ x$ 
apply (cases Rat.normalize \ x)
using Rat.normalize-stable[OF normalize-denom-pos normalize-coprime] apply auto
done

```

```

lemma uminus-pow-Numeral1:  $(-(x:::monoid-mult)) \wedge Numeral1 = -x$  by simp

```

```

lemmas power-numeral-simps = power-0 uminus-pow-Numeral1 power-minus-Bit0
power-minus-Bit1

```

```

lemma Fract-normalize:  $Fract (fst (Rat.normalize (x,y))) (snd (Rat.normalize (x,y))) = Fract \ x \ y$ 
by (rule quotient-of-inject) (simp add: quotient-of-Fract Rat-normalize-idempotent)

```

```

lemma Frct-add:  $Frct (a, numeral \ b) + Frct (c, numeral \ d) =$ 
   $Frct (Rat.normalize (a * numeral \ d + c * numeral \ b), numeral \ d)$ 

```

($b*d$))
by (*auto simp: rat-number-collapse Fract-normalize*)

lemma *Frct-uminus*: $-(\text{Frct } (a,b)) = \text{Frct } (-a,b)$ **by** *simp*

lemma *Frct-diff*: $\text{Frct } (a, \text{numeral } b) - \text{Frct } (c, \text{numeral } d) =$
 $\text{Frct } (\text{Rat.normalize } (a * \text{numeral } d - c * \text{numeral } b, \text{numeral } (b*d)))$
by (*auto simp: rat-number-collapse Fract-normalize*)

lemma *Frct-mult*: $\text{Frct } (a, \text{numeral } b) * \text{Frct } (c, \text{numeral } d) = \text{Frct } (a*c, \text{numeral } (b*d))$
by *simp*

lemma *Frct-inverse*: $\text{inverse } (\text{Frct } (a, b)) = \text{Frct } (b, a)$ **by** *simp*

lemma *Frct-divide*: $\text{Frct } (a, \text{numeral } b) / \text{Frct } (c, \text{numeral } d) = \text{Frct } (a*\text{numeral } d, \text{numeral } b * c)$
by *simp*

lemma *Frct-pow*: $\text{Frct } (a, \text{numeral } b) ^ c = \text{Frct } (a ^ c, \text{numeral } b ^ c)$
by (*induction c*) (*simp-all add: rat-number-collapse*)

lemma *Frct-less*: $\text{Frct } (a, \text{numeral } b) < \text{Frct } (c, \text{numeral } d) \longleftrightarrow a * \text{numeral } d < c * \text{numeral } b$
by *simp*

lemma *Frct-le*: $\text{Frct } (a, \text{numeral } b) \leq \text{Frct } (c, \text{numeral } d) \longleftrightarrow a * \text{numeral } d \leq c * \text{numeral } b$
by *simp*

lemma *Frct-equals*: $\text{Frct } (a, \text{numeral } b) = \text{Frct } (c, \text{numeral } d) \longleftrightarrow a * \text{numeral } d = c * \text{numeral } b$
apply (*intro iffI antisym*)
apply (*subst Frct-le[symmetric], simp*)
apply (*subst Frct-le, simp*)
done

lemma *real-power-code*: $(\text{Ratreal } x) ^ y = \text{Ratreal } (x ^ y)$ **by** (*simp add: of-rat-power*)

lemmas *real-arith-code* =
real-plus-code real-minus-code real-times-code real-uminus-code real-inverse-code
real-divide-code real-power-code real-less-code real-less-eq-code real-equals-code

lemmas *rat-arith-code* =
Frct-add Frct-uminus Frct-diff Frct-mult Frct-inverse Frct-divide Frct-pow
Frct-less Frct-le Frct-equals

lemma *gcd-numeral-red*: $\text{gcd} (\text{numeral } x::\text{int}) (\text{numeral } y) = \text{gcd} (\text{numeral } y)$
 $(\text{numeral } x \bmod \text{numeral } y)$

by (*fact gcd-red-int*)

lemma *divmod-one*:

$\text{divmod} (\text{Num.One}) (\text{Num.One}) = (\text{Numeral1}, 0)$

$\text{divmod} (\text{Num.One}) (\text{Num.Bit0 } x) = (0, \text{Numeral1})$

$\text{divmod} (\text{Num.One}) (\text{Num.Bit1 } x) = (0, \text{Numeral1})$

$\text{divmod } x (\text{Num.One}) = (\text{numeral } x, 0)$

unfolding *divmod-def* **by** *simp-all*

lemmas *divmod-numeral-simps* =

div-0 div-by-0 mod-0 mod-by-0

fst-divmod [symmetric]

snd-divmod [symmetric]

divmod-cancel

divmod-steps [simplified rel-simps if-True] divmod-trivial

rel-simps

lemma *Suc-0-to-numeral*: $\text{Suc } 0 = \text{Numeral1}$ **by** *simp*

lemmas *Suc-to-numeral* = *Suc-0-to-numeral Num.Suc-1 Num.Suc-numeral*

lemma *rat-powr*:

$0 \text{ powr } y = 0$

$x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct } (0, \text{Numeral1})) = \text{Ratreal} (\text{Frct} (\text{Numeral1}, \text{Numeral1}))$

$x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct} (\text{numeral } a, \text{Numeral1})) = x \wedge \text{numeral } a$

$x > 0 \implies x \text{ powr } \text{Ratreal} (\text{Frct} (-\text{numeral } a, \text{Numeral1})) = \text{inverse} (x \wedge \text{numeral } a)$

by (*simp-all add: rat-number-collapse powr-minus*)

lemmas *eval-numeral-simps* =

real-numeral-to-Ratreal real-arith-code rat-arith-code Num.arith-simps

Rat.normalize-def fst-conv snd-conv gcd-0-int gcd-0-left-int gcd.bottom-right-bottom gcd.bottom-left-bottom

gcd-neg1-int gcd-neg2-int gcd-numeral-red zmod-numeral-Bit0 zmod-numeral-Bit1 power-numeral-simps

divmod-numeral-simps numeral-One [symmetric] Groups.Let-0 Num.Let-numeral Suc-to-numeral power-numeral

greaterThanLessThan-iff atLeastAtMost-iff atLeastLessThan-iff greaterThanAtMost-iff rat-powr

Num.pow.simps Num.sqr.simps Product-Type.split of-int-numeral of-int-neg-numeral of-nat-numeral

ML <

signature EVAL-NUMERAL =

sig

val eval-numeral-tac : Proof.context -> int -> tactic

end

```
structure Eval-Numeral : EVAL-NUMERAL =
struct
```

```
fun eval-numeral-tac ctxt =
```

```
  let
    val ctxt' = put-simpset HOL-ss ctxt addsimps @ { thms eval-numeral-simps }
  in
    SELECT-GOAL (SOLVE (Simplifier.simp-tac ctxt' 1))
  end
```

```
end
```

```
>
```

```
lemma 21254387548659589512*314213523632464357453884361*2342523623324234*56432743858724173474
      12561712738645824362329316482973164398214286 powr 2 /
      (1130246312978423123+231212374631082764842731842*122474378389424362347451251263)
>
      (12313244512931247243543279768645745929475829310651205623844 :: real)
by (tactic ‹Eval-Numeral.eval-numeral-tac @ {context} 1›)
```

```
end
```

7 The proof methods

7.1 Master theorem and termination

```
theory Akra-Bazzi-Method
```

```
imports
```

```
  Complex-Main
```

```
  Akra-Bazzi
```

```
  Master-Theorem
```

```
  Eval-Numeral
```

```
begin
```

```
lemma landau-symbol-ge-3-cong:
```

```
  assumes landau-symbol L L' Lr
```

```
  assumes  $\bigwedge x::'a::\text{linordered-semidom}. x \geq 3 \implies f x = g x$ 
```

```
  shows L at-top (f) = L at-top (g)
```

```
apply (rule landau-symbol.cong[OF assms(1)])
```

```
apply (subst eventually-at-top-linorder, rule exI[of - 3], simp add: assms(2))
```

```
done
```

```
lemma exp-1-lt-3: exp (1::real) < 3
```

```
proof -
```

```
  from Taylor-up[of 3 λ-. exp exp 0 1 0]
```

```
  obtain t :: real where t > 0 t < 1 exp 1 = 5/2 + exp t / 6 by (auto simp:
```

```
eval-nat-numeral)
```

```
  note this(3)
```

also from $\langle t < 1 \rangle$ have $\text{exp } t < \text{exp } 1$ by *simp*
 finally show $\text{exp } (1::\text{real}) < 3$ by (*simp add: field-simps*)
qed

lemma *ln-ln-pos*:

assumes $(x::\text{real}) \geq 3$

shows $\ln (\ln x) > 0$

proof (*subst ln-gt-zero-iff*)

from *assms exp-1-lt-3* have $\ln x > \ln (\text{exp } 1)$ by (*intro ln-mono-strict*) *simp-all*

thus $\ln x > 0$ $\ln x > 1$ by *simp-all*

qed

definition *akra-bazzi-terms where*

akra-bazzi-terms x_0 x_1 bs $ts = (\forall i < \text{length } bs. \text{akra-bazzi-term } x_0$ x_1 $(bs!i)$ $(ts!i))$

lemma *akra-bazzi-termsI*:

$(\bigwedge i. i < \text{length } bs \implies \text{akra-bazzi-term } x_0$ x_1 $(bs!i)$ $(ts!i)) \implies \text{akra-bazzi-terms}$
 x_0 x_1 bs ts

unfolding *akra-bazzi-terms-def* by *blast*

lemma *master-theorem-functionI*:

assumes $\forall x \in \{x_0..<x_1\}. f x \geq 0$

assumes $\forall x \geq x_1. f x = g x + (\sum i < k. as ! i * f ((ts ! i) x))$

assumes $\forall x \geq x_1. g x \geq 0$

assumes $\forall a \in \text{set } as. a \geq 0$

assumes *list-ex* $(\lambda a. a > 0)$ as

assumes $\forall b \in \text{set } bs. b \in \{0 < .. < 1\}$

assumes $k \neq 0$

assumes $\text{length } as = k$

assumes $\text{length } bs = k$

assumes $\text{length } ts = k$

assumes *akra-bazzi-terms* x_0 x_1 bs ts

shows *master-theorem-function* x_0 x_1 k as bs ts f g

using *assms unfolding akra-bazzi-terms-def* by *unfold-locales (auto simp: list-ex-iff)*

lemma *akra-bazzi-term-measure*:

$x \geq x_1 \implies \text{akra-bazzi-term } 0$ x_1 b $t \implies (t x, x) \in \text{Wellfounded.measure } (\lambda n::\text{nat.}$
 $n)$

$x > x_1 \implies \text{akra-bazzi-term } 0$ $(\text{Suc } x_1)$ b $t \implies (t x, x) \in \text{Wellfounded.measure}$
 $(\lambda n::\text{nat. } n)$

unfolding *akra-bazzi-term-def* by *auto*

lemma *measure-prod-conv*:

$((a, b), (c, d)) \in \text{Wellfounded.measure } (\lambda x. t (fst x)) \longleftrightarrow (a, c) \in \text{Wellfounded.measure}$
 t

$((e, f), (g, h)) \in \text{Wellfounded.measure } (\lambda x. t (snd x)) \longleftrightarrow (f, h) \in \text{Well-}$
founded.measure t

by *simp-all*

lemmas *measure-prod-conv'* = *measure-prod-conv*[**where** $t = \lambda x. x$]

lemma *akra-bazzi-termination-simps*:

fixes $x :: \text{nat}$

shows $a * \text{real } x / b = a/b * \text{real } x$ $\text{real } x / b = 1/b * \text{real } x$

by *simp-all*

lemma *akra-bazzi-params-nonzeroI*:

$\text{length } as = \text{length } bs \implies$

$(\forall a \in \text{set } as. a \geq 0) \implies (\forall b \in \text{set } bs. b \in \{0 <..< 1\}) \implies (\exists a \in \text{set } as. a > 0) \implies$
akra-bazzi-params-nonzero ($\text{length } as$) as bs **by** (*unfold-locales, simp-all*) []

lemmas *akra-bazzi-p-rel-intros* =

akra-bazzi-params-nonzero.p-lessI[*rotated, OF - akra-bazzi-params-nonzeroI*]

akra-bazzi-params-nonzero.p-greaterI[*rotated, OF - akra-bazzi-params-nonzeroI*]

akra-bazzi-params-nonzero.p-leI[*rotated, OF - akra-bazzi-params-nonzeroI*]

akra-bazzi-params-nonzero.p-geI[*rotated, OF - akra-bazzi-params-nonzeroI*]

akra-bazzi-params-nonzero.p-boundsI[*rotated, OF - akra-bazzi-params-nonzeroI*]

akra-bazzi-params-nonzero.p-boundsI'[*rotated, OF - akra-bazzi-params-nonzeroI*]

lemma *eval-length*: $\text{length } [] = 0$ $\text{length } (x \# xs) = \text{Suc } (\text{length } xs)$ **by** *simp-all*

lemma *eval-akra-bazzi-sum*:

$(\sum i < 0. as!i * bs!i \text{ powr } x) = 0$

$(\sum i < \text{Suc } 0. (a\#as)!i * (b\#bs)!i \text{ powr } x) = a * b \text{ powr } x$

$(\sum i < \text{Suc } k. (a\#as)!i * (b\#bs)!i \text{ powr } x) = a * b \text{ powr } x + (\sum i < k. as!i * bs!i$
 $\text{ powr } x)$

apply *simp*

apply *simp*

apply (*induction k arbitrary: a as b bs*)

apply *simp-all*

done

lemma *eval-akra-bazzi-sum'*:

$(\sum i < 0. as!i * f ((ts!i) x)) = 0$

$(\sum i < \text{Suc } 0. (a\#as)!i * f (((t\#ts)!i) x)) = a * f (t x)$

$(\sum i < \text{Suc } k. (a\#as)!i * f (((t\#ts)!i) x)) = a * f (t x) + (\sum i < k. as!i * f ((ts!i)$
 $x))$

apply *simp*

apply *simp*

apply (*induction k arbitrary: a as t ts*)

apply (*simp-all add: algebra-simps*)

done

lemma *akra-bazzi-termsI'*:

akra-bazzi-terms x_0 x_1 [] []

akra-bazzi-term x_0 x_1 b $t \implies$ *akra-bazzi-terms* x_0 x_1 bs $ts \implies$ *akra-bazzi-terms*
 x_0 x_1 $(b\#bs)$ $(t\#ts)$

unfolding *akra-bazzi-terms-def* **using** *less-Suc-eq-0-disj* **by** *auto*

lemma *ball-set-intros*: $(\forall x \in \text{set } []. P x) P x \implies (\forall x \in \text{set } xs. P x) \implies (\forall x \in \text{set } (x \# xs). P x)$
by *auto*

lemma *ball-set-simps*: $(\forall x \in \text{set } []. P x) = \text{True} (\forall x \in \text{set } (x \# xs). P x) = (P x \wedge (\forall x \in \text{set } xs. P x))$
by *auto*

lemma *ball-set-simps*: $(\exists x \in \text{set } []. P x) = \text{False} (\exists x \in \text{set } (x \# xs). P x) = (P x \vee (\exists x \in \text{set } xs. P x))$
by *auto*

lemma *eval-akra-bazzi-le-list-ex*:
 $\text{list-ex } P (x \# y \# xs) \longleftrightarrow P x \vee \text{list-ex } P (y \# xs)$
 $\text{list-ex } P [x] \longleftrightarrow P x$
 $\text{list-ex } P [] \longleftrightarrow \text{False}$
by (*auto simp: list-ex-iff*)

lemma *eval-akra-bazzi-le-sum-list*:
 $x \leq \text{sum-list } [] \longleftrightarrow x \leq 0$
 $x \leq \text{sum-list } (y \# ys) \longleftrightarrow x \leq y + \text{sum-list } ys$
 $x \leq z + \text{sum-list } [] \longleftrightarrow x \leq z$
 $x \leq z + \text{sum-list } (y \# ys) \longleftrightarrow x \leq z + y + \text{sum-list } ys$
by (*simp-all add: algebra-simps*)

lemma *atLeastLessThanE*: $x \in \{a..<b\} \implies (x \geq a \implies x < b \implies P) \implies P$ **by** *simp*

lemma *master-theorem-preprocess*:
 $\Theta(\lambda n::\text{nat}. 1) = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0)$
 $\Theta(\lambda n::\text{nat}. \text{real } n) = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1)$
 $O(\lambda n::\text{nat}. 1) = O(\lambda n::\text{nat}. \text{real } n \text{ powr } 0)$
 $O(\lambda n::\text{nat}. \text{real } n) = O(\lambda n::\text{nat}. \text{real } n \text{ powr } 1)$
 $\Theta(\lambda n::\text{nat}. \ln (\ln (\text{real } n))) = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0 * \ln (\ln (\text{real } n)))$
 $\Theta(\lambda n::\text{nat}. \text{real } n * \ln (\ln (\text{real } n))) = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1 * \ln (\ln (\text{real } n)))$
 $\Theta(\lambda n::\text{nat}. \ln (\text{real } n)) = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0 * \ln (\text{real } n) \text{ powr } 1)$
 $\Theta(\lambda n::\text{nat}. \text{real } n * \ln (\text{real } n)) = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1 * \ln (\text{real } n) \text{ powr } 1)$
 $\Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } p * \ln (\text{real } n)) = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } p * \ln (\text{real } n) \text{ powr } 1)$
 $\Theta(\lambda n::\text{nat}. \ln (\text{real } n) \text{ powr } p') = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 0 * \ln (\text{real } n) \text{ powr } p')$
 $\Theta(\lambda n::\text{nat}. \text{real } n * \ln (\text{real } n) \text{ powr } p') = \Theta(\lambda n::\text{nat}. \text{real } n \text{ powr } 1 * \ln (\text{real } n) \text{ powr } p')$
apply (*simp-all*)
apply (*simp-all cong: landau-symbols[THEN landau-symbol-ge-3-cong]*)?
done

lemma *akra-bazzi-term-imp-size-less*:

$x_1 \leq x \implies \text{akra-bazzi-term } 0 \ x_1 \ b \ t \implies \text{size } (t \ x) < \text{size } x$
 $x_1 < x \implies \text{akra-bazzi-term } 0 \ (\text{Suc } x_1) \ b \ t \implies \text{size } (t \ x) < \text{size } x$
by (*simp-all add: akra-bazzi-term-imp-less*)

definition *CLAMP* ($f :: \text{nat} \Rightarrow \text{real}$) $x = (\text{if } x < 3 \text{ then } 0 \text{ else } f \ x)$
definition *CLAMP'* ($f :: \text{nat} \Rightarrow \text{real}$) $x = (\text{if } x < 3 \text{ then } 0 \text{ else } f \ x)$
definition *MASTER-BOUND* $a \ b \ c \ x = \text{real } x \ \text{powr } a * \ln (\text{real } x) \ \text{powr } b * \ln (\ln (\text{real } x)) \ \text{powr } c$
definition *MASTER-BOUND'* $a \ b \ x = \text{real } x \ \text{powr } a * \ln (\text{real } x) \ \text{powr } b$
definition *MASTER-BOUND''* $a \ x = \text{real } x \ \text{powr } a$

lemma *ln-1-imp-less-3*:

$\ln \ x = (1 :: \text{real}) \implies x < 3$

proof –

assume $\ln \ x = 1$

also have $(1 :: \text{real}) \leq \ln (\text{exp } 1)$ **by** *simp*

finally have $\ln \ x \leq \ln (\text{exp } 1)$ **by** *simp*

hence $x \leq \text{exp } 1$

by (*cases* $x > 0$) (*force simp del: ln-exp simp add: not-less intro: order.trans*) +

also have $\dots < 3$ **by** (*rule exp-1-lt-3*)

finally show *?thesis* .

qed

lemma *ln-1-imp-less-3'*: $\ln (\text{real } (x :: \text{nat})) = 1 \implies x < 3$ **by** (*drule ln-1-imp-less-3*) *simp*

lemma *ln-ln-nonneg*: $x \geq (3 :: \text{real}) \implies \ln (\ln \ x) \geq 0$ **using** *ln-ln-pos*[*of* x] **by** *simp*

lemma *ln-ln-nonneg'*: $x \geq (3 :: \text{nat}) \implies \ln (\ln (\text{real } x)) \geq 0$ **using** *ln-ln-pos*[*of* $\text{real } x$] **by** *simp*

lemma *MASTER-BOUND-postproc*:

$\text{CLAMP } (\text{MASTER-BOUND}' \ a \ 0) = \text{CLAMP } (\text{MASTER-BOUND}'' \ a)$

$\text{CLAMP } (\text{MASTER-BOUND}' \ a \ 1) = \text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \ x * \text{CLAMP } (\lambda x. \ln (\text{real } x)) \ x)$

$\text{CLAMP } (\text{MASTER-BOUND}' \ a \ (\text{numeral } n)) =$

$\text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \ x * \text{CLAMP } (\lambda x. \ln (\text{real } x)) \ x) \wedge \text{numeral } n \ x)$

$\text{CLAMP } (\text{MASTER-BOUND}' \ a \ (-1)) =$

$\text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \ x / \text{CLAMP } (\lambda x. \ln (\text{real } x)) \ x)$

$\text{CLAMP } (\text{MASTER-BOUND}' \ a \ (-\text{numeral } n)) =$

$\text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \ x / \text{CLAMP } (\lambda x. \ln (\text{real } x)) \ x) \wedge \text{numeral } n \ x)$

$\text{CLAMP } (\text{MASTER-BOUND}' \ a \ b) =$

$\text{CLAMP } (\lambda x. \text{CLAMP } (\text{MASTER-BOUND}'' \ a) \ x * \text{CLAMP } (\lambda x. \ln (\text{real } x)) \ \text{powr } b \ x)$

$\text{CLAMP } (\text{MASTER-BOUND}'' \ 0) = \text{CLAMP } (\lambda x. 1)$

$\text{CLAMP } (\text{MASTER-BOUND}'' \ 1) = \text{CLAMP } (\lambda x. (\text{real } x))$

$CLAMP (MASTER-BOUND'' (numeral n)) = CLAMP (\lambda x. (real x) \hat{\ } numeral n)$
 $CLAMP (MASTER-BOUND'' (-1)) = CLAMP (\lambda x. 1 / (real x))$
 $CLAMP (MASTER-BOUND'' (-numeral n)) = CLAMP (\lambda x. 1 / (real x) \hat{\ } numeral n)$
 $CLAMP (MASTER-BOUND'' a) = CLAMP (\lambda x. (real x) powr a)$

and *MASTER-BOUND-UNCLAMP*:

$CLAMP (\lambda x. CLAMP f x * CLAMP g x) = CLAMP (\lambda x. f x * g x)$

$CLAMP (\lambda x. CLAMP f x / CLAMP g x) = CLAMP (\lambda x. f x / g x)$

$CLAMP (CLAMP f) = CLAMP f$

unfolding *CLAMP-def[abs-def]* *MASTER-BOUND'-def[abs-def]* *MASTER-BOUND''-def[abs-def]*
by (*simp-all add: powr-minus divide-inverse fun-eq-iff*)

context

begin

private lemma *CLAMP-*:

$landau-symbol L L' Lr \implies L at-top (f::nat \implies real) \equiv L at-top (\lambda x. CLAMP f x)$

unfolding *CLAMP-def[abs-def]*

by (*intro landau-symbol.cong eq-reflection*)

(*auto intro: eventually-mono[OF eventually-ge-at-top[of 3::nat]]*)

private lemma *UNCLAMP'-*:

$landau-symbol L L' Lr \implies L at-top (CLAMP' (MASTER-BOUND a b c)) \equiv L at-top (MASTER-BOUND a b c)$

unfolding *CLAMP'-def[abs-def]* *CLAMP-def[abs-def]*

by (*intro landau-symbol.cong eq-reflection*)

(*auto intro: eventually-mono[OF eventually-ge-at-top[of 3::nat]]*)

private lemma *UNCLAMP-*:

$landau-symbol L L' Lr \implies L at-top (CLAMP f) \equiv L at-top (f)$

using *eventually-ge-at-top[of 3::nat]* **unfolding** *CLAMP'-def[abs-def]* *CLAMP-def[abs-def]*

by (*intro landau-symbol.cong eq-reflection*)

(*auto intro: eventually-mono[OF eventually-ge-at-top[of 3::nat]]*)

lemmas *CLAMP = landau-symbols[THEN CLAMP-]*

lemmas *UNCLAMP' = landau-symbols[THEN UNCLAMP'-]*

lemmas *UNCLAMP = landau-symbols[THEN UNCLAMP-]*

end

lemma *propagate-CLAMP*:

$CLAMP (\lambda x. f x * g x) = CLAMP' (\lambda x. CLAMP f x * CLAMP g x)$

$CLAMP (\lambda x. f x / g x) = CLAMP' (\lambda x. CLAMP f x / CLAMP g x)$

$CLAMP (\lambda x. inverse (f x)) = CLAMP' (\lambda x. inverse (CLAMP f x))$

$CLAMP (\lambda x. real x) = CLAMP' (MASTER-BOUND 1 0 0)$

$CLAMP (\lambda x. real x powr a) = CLAMP' (MASTER-BOUND a 0 0)$

$CLAMP (\lambda x. \text{real } x \hat{=} a') = CLAMP' (MASTER-BOUND (\text{real } a') 0 0)$
 $CLAMP (\lambda x. \ln (\text{real } x)) = CLAMP' (MASTER-BOUND 0 1 0)$
 $CLAMP (\lambda x. \ln (\text{real } x) \text{ powr } b) = CLAMP' (MASTER-BOUND 0 b 0)$
 $CLAMP (\lambda x. \ln (\text{real } x) \hat{=} b') = CLAMP' (MASTER-BOUND 0 (\text{real } b') 0)$
 $CLAMP (\lambda x. \ln (\ln (\text{real } x))) = CLAMP' (MASTER-BOUND 0 0 1)$
 $CLAMP (\lambda x. \ln (\ln (\text{real } x)) \text{ powr } c) = CLAMP' (MASTER-BOUND 0 0 c)$
 $CLAMP (\lambda x. \ln (\ln (\text{real } x)) \hat{=} c') = CLAMP' (MASTER-BOUND 0 0 (\text{real } c'))$
 $CLAMP' (CLAMP f) = CLAMP' f$
 $CLAMP' (\lambda x. CLAMP' (MASTER-BOUND a1 b1 c1) x * CLAMP' (MASTER-BOUND a2 b2 c2) x) =$
 $CLAMP' (MASTER-BOUND (a1+a2) (b1+b2) (c1+c2))$
 $CLAMP' (\lambda x. CLAMP' (MASTER-BOUND a1 b1 c1) x / CLAMP' (MASTER-BOUND a2 b2 c2) x) =$
 $CLAMP' (MASTER-BOUND (a1-a2) (b1-b2) (c1-c2))$
 $CLAMP' (\lambda x. \text{inverse } (MASTER-BOUND a1 b1 c1) x) = CLAMP' (MASTER-BOUND (-a1) (-b1) (-c1))$
by (*insert ln-1-imp-less-3' ln-ln-nonneg'*)
(rule ext, simp add: CLAMP-def CLAMP'-def MASTER-BOUND-def
powr-realpow powr-one[OF ln-ln-nonneg'] powr-realpow[OF ln-ln-pos] powr-add
powr-diff powr-minus)+

lemma numeral-assoc-simps:

$((a::\text{real}) + \text{numeral } b) + \text{numeral } c = a + \text{numeral } (b + c)$
 $(a + \text{numeral } b) - \text{numeral } c = a + \text{neg-numeral-class.sub } b c$
 $(a - \text{numeral } b) + \text{numeral } c = a + \text{neg-numeral-class.sub } c b$
 $(a - \text{numeral } b) - \text{numeral } c = a - \text{numeral } (b + c)$ **by** *simp-all*

lemmas CLAMP-aux =

arith-simps numeral-assoc-simps of-nat-power of-nat-mult of-nat-numeral
one-add-one numeral-One [symmetric]

lemmas CLAMP-postproc = numeral-One

context master-theorem-function

begin

lemma master1-bigo-automation:

assumes $g \in O(\lambda x. \text{real } x \text{ powr } p')$ $1 < (\sum i < k. as ! i * bs ! i \text{ powr } p')$
shows $f \in O(MASTER-BOUND p 0 0)$

proof –

have $MASTER-BOUND p 0 0 \in \Theta(\lambda x::\text{nat}. x \text{ powr } p)$ **unfolding** *MASTER-BOUND-def[abs-def]*

by (*intro landau-real-nat-transfer bigthetaI-cong*

eventually-mono[OF eventually-ge-at-top[of 3::real]]) (auto dest!: ln-1-imp-less-3)

from *landau-o.big.cong-bigtheta[OF this] master1-bigo[OF assms] show ?thesis*

by *simp*

qed

lemma master1-automation:

assumes $g \in O(MASTER-BOUND'' p')$ $1 < (\sum i < k. as ! i * bs ! i \text{ powr } p')$

eventually ($\lambda x. f x > 0$) *at-top*
shows $f \in \Theta(\text{MASTER-BOUND } p \ 0 \ 0)$
proof –
have $A: \text{MASTER-BOUND } p \ 0 \ 0 \in \Theta(\lambda x::\text{nat. } x \ \text{powr } p)$ **unfolding** $\text{MASTER-BOUND-def}[abs-def]$
by (*intro landau-real-nat-transfer bigthetaI-cong*
eventually-mono[OF eventually-ge-at-top[of 3::real]]) (*auto dest!: ln-1-imp-less-3*)
have $B: O(\text{MASTER-BOUND'' } p') = O(\lambda x::\text{nat. } \text{real } x \ \text{powr } p')$
using *eventually-ge-at-top[of 2::nat]*
by (*intro landau-o.big.cong*) (*auto elim!: eventually-mono simp: MASTER-BOUND''-def*)
from *landau-theta.cong-bigtheta[OF A] B assms(1) master1[OF - assms(2-)]*
show *?thesis* **by** *simp*
qed

lemma *master2-1-automation:*

assumes $g \in \Theta(\text{MASTER-BOUND}' p \ p')$ $p' < -1$
shows $f \in \Theta(\text{MASTER-BOUND } p \ 0 \ 0)$
proof –
have $A: \text{MASTER-BOUND } p \ 0 \ 0 \in \Theta(\lambda x::\text{nat. } x \ \text{powr } p)$ **unfolding** $\text{MASTER-BOUND-def}[abs-def]$
by (*intro landau-real-nat-transfer bigthetaI-cong*
eventually-mono[OF eventually-ge-at-top[of 3::real]]) (*auto dest!: ln-1-imp-less-3*)
have $B: \Theta(\text{MASTER-BOUND}' p \ p') = \Theta(\lambda x::\text{nat. } \text{real } x \ \text{powr } p * \ln(\text{real } x) \ \text{powr } p')$
powr } p')
by (*subst CLAMP, (subst MASTER-BOUND-postproc MASTER-BOUND-UNCLAMP)+, simp only: UNCLAMP*)
from *landau-theta.cong-bigtheta[OF A] B assms(1) master2-1[OF - assms(2-)]*
show *?thesis* **by** *simp*
qed

lemma *master2-2-automation:*

assumes $g \in \Theta(\text{MASTER-BOUND}' p \ (-1))$
shows $f \in \Theta(\text{MASTER-BOUND } p \ 0 \ 1)$
proof –
have $A: \text{MASTER-BOUND } p \ 0 \ 1 \in \Theta(\lambda x::\text{nat. } x \ \text{powr } p * \ln(\ln x))$ **unfolding** $\text{MASTER-BOUND-def}[abs-def]$
using *eventually-ge-at-top[of 3::real]*
apply (*intro landau-real-nat-transfer, intro bigthetaI-cong*)
apply (*elim eventually-mono, subst powr-one[OF ln-ln-nonneg]*)
apply *simp-all*
done
have $B: \Theta(\text{MASTER-BOUND}' p \ (-1)) = \Theta(\lambda x::\text{nat. } \text{real } x \ \text{powr } p / \ln(\text{real } x))$
by (*subst CLAMP, (subst MASTER-BOUND-postproc MASTER-BOUND-UNCLAMP)+, simp only: UNCLAMP*)
from *landau-theta.cong-bigtheta[OF A] B assms(1) master2-2* **show** *?thesis* **by** *simp*
qed

lemma *master2-3-automation:*

```

assumes  $g \in \Theta(\text{MASTER-BOUND}' p (p' - 1)) p' > 0$ 
shows  $f \in \Theta(\text{MASTER-BOUND} p p' 0)$ 
proof –
  have  $A: \text{MASTER-BOUND} p p' 0 \in \Theta(\lambda x::\text{nat. } x \text{ powr } p * \ln x \text{ powr } p')$  un-
folding  $\text{MASTER-BOUND-def}[abs-def]$ 
  using  $\text{eventually-ge-at-top}[of\ 3::\text{real}]$ 
  apply  $(\text{intro } \text{landau-real-nat-transfer}, \text{intro } \text{bigthetaI-cong})$ 
  apply  $(\text{elim } \text{eventually-mono}, \text{auto } \text{dest: } \text{ln-1-imp-less-3})$ 
  done
  have  $B: \Theta(\text{MASTER-BOUND}' p (p' - 1)) = \Theta(\lambda x::\text{nat. } \text{real } x \text{ powr } p * \ln x$ 
 $\text{ powr } (p' - 1))$ 
  by  $(\text{subst } \text{CLAMP}, (\text{subst } \text{MASTER-BOUND-postproc } \text{MASTER-BOUND-UNCLAMP})+,$ 
 $\text{simp } \text{only: } \text{UNCLAMP})$ 
  from  $\text{landau-theta.cong-bigtheta}[OF\ A] B \text{ assms}(1) \text{ master2-3}[OF - \text{assms}(2-)]$ 
show  $?thesis$  by  $\text{simp}$ 
qed

```

lemma *master3-automation*:

```

assumes  $g \in \Theta(\text{MASTER-BOUND}'' p') 1 > (\sum_{i < k.} \text{as } ! i * \text{bs } ! i \text{ powr } p')$ 
shows  $f \in \Theta(\text{MASTER-BOUND} p' 0 0)$ 
proof –
  have  $A: \text{MASTER-BOUND} p' 0 0 \in \Theta(\lambda x::\text{nat. } x \text{ powr } p')$  unfolding  $\text{MAS-}$ 
 $\text{TER-BOUND-def}[abs-def]$ 
  using  $\text{eventually-ge-at-top}[of\ 3::\text{real}]$ 
  apply  $(\text{intro } \text{landau-real-nat-transfer}, \text{intro } \text{bigthetaI-cong})$ 
  apply  $(\text{elim } \text{eventually-mono}, \text{auto } \text{dest: } \text{ln-1-imp-less-3})$ 
  done
  have  $B: \Theta(\text{MASTER-BOUND}'' p') = \Theta(\lambda x::\text{nat. } \text{real } x \text{ powr } p')$ 
  by  $(\text{subst } \text{CLAMP}, (\text{subst } \text{MASTER-BOUND-postproc})+, \text{simp } \text{only: } \text{UN-}$ 
 $\text{CLAMP})$ 
  from  $\text{landau-theta.cong-bigtheta}[OF\ A] B \text{ assms}(1) \text{ master3}[OF - \text{assms}(2-)]$ 
show  $?thesis$  by  $\text{simp}$ 
qed

```

lemmas *master-automation =*

```

master1-automation master2-1-automation master2-2-automation
master2-2-automation master3-automation

```

ML <

fun *generalize-master-thm* *ctxt* *thm* =

```

let
  val  $([p'], \text{ctxt}') = \text{Variable.variant-fixes } [p'] \text{ ctxt}$ 
  val  $p' = \text{Free } (p', \text{HOLogic.realT})$ 
  val  $a = @\{\text{term } \text{nth } \text{as}\} \$ \text{Bound } 0$ 
  val  $b = @\{\text{term } \text{Transcendental.powr} :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real}\} \$$ 
 $(@\{\text{term } \text{nth } \text{bs}\} \$ \text{Bound } 0) \$ p'$ 
  val  $f = \text{Abs } (i, \text{HOLogic.natT}, @\{\text{term } (*) :: \text{real} \Rightarrow \text{real} \Rightarrow \text{real}\} \$ a \$ b)$ 

```

```

    val sum = @{\term sum :: (nat => real) => nat set => real} $ f $ @{\term
{..<k}}
    val prop = HOLogic.mk-Trueprop (HOLogic.mk-eq (sum, @{\term 1::real}))
    val cprop = Thm.cterm-of ctxt' prop
  in
    thm
    |> Local-Defs.unfold ctxt' [Thm.assume cprop RS @{\thm p-unique}]
    |> Thm.implies-intr cprop
    |> rotate-prems 1
    |> singleton (Variable.export ctxt' ctxt)
  end

```

```

fun generalize-master-thm' (binding, thm) ctxt =
  Local-Theory.note ((binding, []), [generalize-master-thm ctxt thm]) ctxt |> snd

```

```

local-setup <
  fold generalize-master-thm'
    [(@{\binding master1-automation'}, @{\thm master1-automation}),
    (@{\binding master1-bigo-automation'}, @{\thm master1-bigo-automation}),
    (@{\binding master2-1-automation'}, @{\thm master2-1-automation}),
    (@{\binding master2-2-automation'}, @{\thm master2-2-automation}),
    (@{\binding master2-3-automation'}, @{\thm master2-3-automation}),
    (@{\binding master3-automation'}, @{\thm master3-automation})]

```

end

definition *arith-consts* ($x :: \text{real}$) ($y :: \text{nat}$) =
 (if $\neg (-x) + 3 / x * 5 - 1 \leq x \wedge \text{True} \vee \text{True} \longrightarrow \text{True}$ then
 $x < \text{inverse } 3 \text{ powr } 21$ else $x = \text{real } (\text{Suc } 0 \wedge 2 +$
 (if $42 - x \leq 1 \wedge 1 \text{ div } y = y \text{ mod } 2 \vee y < \text{Numeral1}$ then 0 else 0)) + Numeral1)

ML-file <akra-bazzi.ML>

hide-const *arith-consts*

method-setup *master-theorem* = <
 Akra-Bazzi.setup-master-theorem
 > automatically apply the Master theorem for recursive functions

method-setup *akra-bazzi-termination* = <
 Scan.succeed (fn ctxt => SIMPLE-METHOD' (Akra-Bazzi.akra-bazzi-termination-tac
 ctxt))
 > prove termination of Akra–Bazzi functions

hide-const CLAMP CLAMP' MASTER-BOUND MASTER-BOUND' MASTER-BOUND''

```

end
theory Akra-Bazzi-Approximation
imports
  Complex-Main
  Akra-Bazzi-Method
  HOL-Decision-Procs.Approximation
begin

```

```

context akra-bazzi-params-nonzero
begin

```

```

lemma sum-alt:  $(\sum_{i < k}. as!i * bs!i \text{ powr } p^\wedge) = (\sum_{i < k}. as!i * \text{exp } (p' * \text{ln } (bs!i)))$ 
proof (intro sum.cong)
  fix i assume  $i \in \{..<k\}$ 
  with b-bounds have  $bs!i > 0$  by simp
  thus  $as!i * bs!i \text{ powr } p' = as!i * \text{exp } (p' * \text{ln } (bs!i))$  by (simp add: powr-def)
qed simp

```

```

lemma akra-bazzi-p-rel-intros-aux:
   $1 < (\sum_{i < k}. as!i * \text{exp } (p' * \text{ln } (bs!i))) \implies p' < p$ 
   $1 > (\sum_{i < k}. as!i * \text{exp } (p' * \text{ln } (bs!i))) \implies p' > p$ 
   $1 \leq (\sum_{i < k}. as!i * \text{exp } (p' * \text{ln } (bs!i))) \implies p' \leq p$ 
   $1 \geq (\sum_{i < k}. as!i * \text{exp } (p' * \text{ln } (bs!i))) \implies p' \geq p$ 
   $(\sum_{i < k}. as!i * \text{exp } (x * \text{ln } (bs!i))) \leq 1 \wedge (\sum_{i < k}. as!i * \text{exp } (y * \text{ln } (bs!i))) \geq 1 \implies p \in \{y..x\}$ 
   $(\sum_{i < k}. as!i * \text{exp } (x * \text{ln } (bs!i))) < 1 \wedge (\sum_{i < k}. as!i * \text{exp } (y * \text{ln } (bs!i))) > 1 \implies p \in \{y<..
  using p-lessI p-greaterI p-leI p-geI p-boundsI p-boundsI' by (simp-all only: sum-alt)
end$ 
```

```

lemmas akra-bazzi-p-rel-intros-exp =
  akra-bazzi-params-nonzero.akra-bazzi-p-rel-intros-aux[rotated, OF - akra-bazzi-params-nonzeroI]

```

```

lemma eval-akra-bazzi-sum:
   $(\sum_{i < 0}. as!i * \text{exp } (x * \text{ln } (bs!i))) = 0$ 
   $(\sum_{i < \text{Suc } 0}. (a\#as)!i * \text{exp } (x * \text{ln } ((b\#bs)!i))) = a * \text{exp } (x * \text{ln } b)$ 
   $(\sum_{i < \text{Suc } k}. (a\#as)!i * \text{exp } (x * \text{ln } ((b\#bs)!i))) = a * \text{exp } (x * \text{ln } b) + (\sum_{i < k}. as!i * \text{exp } (x * \text{ln } (bs!i)))$ 
  apply simp
  apply simp
  apply (induction k arbitrary: a as b bs)
  apply simp-all
  done

```

```

ML ‹
signature AKRA-BAZZI-APPROXIMATION =
sig
  val akra-bazzi-approximate-tac : int -> Proof.context -> int -> tactic
end

structure Akra-Bazzi-Approximation: AKRA-BAZZI-APPROXIMATION =
struct

fun akra-bazzi-approximate-tac prec ctxt =
  let
    val_simps = @{thms eval-length eval-akra-bazzi-sum add-0-left add-0-right
                  mult-1-left mult-1-right}
  in
    SELECT-GOAL (
      resolve-tac ctxt @{thms akra-bazzi-p-rel-intros-exp} 1
      THEN ALLGOALS (fn i =>
        if i > 1 then
          SELECT-GOAL (
            Local-Defs.unfold-tac ctxt
            @{thms bez-set-simps ball-set-simps greaterThanLessThan-iff eval-length}
            THEN TRY (SOLVE (Eval-Numeral.eval-numeral-tac ctxt 1))
          ) i
        else
          SELECT-GOAL (Local-Defs.unfold-tac ctxt_simps) i
          THEN Approximation.approximation-tac prec [] NONE ctxt i
        )
      )
    )
  end

end;
›

method-setup akra-bazzi-approximate = ‹
  Scan.lift Parse.nat >>
  (fn prec => fn ctxt =>
    SIMPLE-METHOD' (Akra-Bazzi-Approximation.akra-bazzi-approximate-tac
      prec ctxt))
› approximate transcendental Akra–Bazzi parameters

end

```

8 Examples

```

theory Master-Theorem-Examples
imports
  Complex-Main
  Akra-Bazzi-Method

```

Akra-Bazzi-Approximation

begin

8.1 Merge sort

function *merge-sort-cost* :: (nat \Rightarrow real) \Rightarrow nat \Rightarrow real **where**

merge-sort-cost *t* 0 = 0

| *merge-sort-cost* *t* 1 = 1

| $n \geq 2 \implies$ *merge-sort-cost* *t* *n* =

merge-sort-cost *t* (nat \lfloor real *n* / 2 \rfloor) + *merge-sort-cost* *t* (nat \lceil real *n* / 2 \rceil) + *t*

n

by *force simp-all*

termination **by** *akra-bazzi-termination simp-all*

lemma *merge-sort-nonneg*[*simp*]: ($\bigwedge n. t\ n \geq 0$) \implies *merge-sort-cost* *t* *x* ≥ 0

by (*induction t x rule: merge-sort-cost.induct*) (*simp-all del: One-nat-def*)

lemma $t \in \Theta(\lambda n. \text{real } n) \implies (\bigwedge n. t\ n \geq 0) \implies$ *merge-sort-cost* *t* $\in \Theta(\lambda n. \text{real } n * \ln(\text{real } n))$

by (*master-theorem 2.3*) *simp-all*

8.2 Karatsuba multiplication

function *karatsuba-cost* :: nat \Rightarrow real **where**

karatsuba-cost 0 = 0

| *karatsuba-cost* 1 = 1

| $n \geq 2 \implies$ *karatsuba-cost* *n* =

3 * *karatsuba-cost* (nat \lceil real *n* / 2 \rceil) + real *n*

by *force simp-all*

termination **by** *akra-bazzi-termination simp-all*

lemma *karatsuba-cost-nonneg*[*simp*]: *karatsuba-cost* *n* ≥ 0

by (*induction n rule: karatsuba-cost.induct*) (*simp-all del: One-nat-def*)

lemma *karatsuba-cost* $\in O(\lambda n. \text{real } n \text{ powr } \log 2\ 3)$

by (*master-theorem 1 p': 1*) (*simp-all add: powr-divide*)

lemma *karatsuba-cost-pos*: $n \geq 1 \implies$ *karatsuba-cost* *n* > 0

by (*induction n rule: karatsuba-cost.induct*) (*auto intro!: add-nonneg-pos simp del: One-nat-def*)

lemma *karatsuba-cost* $\in \Theta(\lambda n. \text{real } n \text{ powr } \log 2\ 3)$

using *karatsuba-cost-pos*

by (*master-theorem 1 p': 1*) (*auto simp add: powr-divide eventually-at-top-linorder*)

8.3 Strassen matrix multiplication

function *strassen-cost* :: nat \Rightarrow real **where**

strassen-cost 0 = 0

| *strassen-cost* 1 = 1

| $n \geq 2 \implies \text{strassen-cost } n = 7 * \text{strassen-cost } (\text{nat } \lceil \text{real } n / 2 \rceil) + \text{real } (n^{\wedge}2)$
by *force simp-all*
termination *by akra-bazzi-termination simp-all*

lemma *strassen-cost-nonneg[simp]: strassen-cost $n \geq 0$*
by (*induction n rule: strassen-cost.induct*) (*simp-all del: One-nat-def*)

lemma *strassen-cost $\in O(\lambda n. \text{real } n \text{ powr } \log 2 7)$*
by (*master-theorem 1 p': 2*) (*auto simp: powr-divide eventually-at-top-linorder*)

lemma *strassen-cost-pos: $n \geq 1 \implies \text{strassen-cost } n > 0$*
by (*cases n rule: strassen-cost.cases*) (*simp-all add: add-nonneg-pos del: One-nat-def*)

lemma *strassen-cost $\in \Theta(\lambda n. \text{real } n \text{ powr } \log 2 7)$*
using *strassen-cost-pos*
by (*master-theorem 1 p': 2*) (*auto simp: powr-divide eventually-at-top-linorder*)

8.4 Deterministic select

function *select-cost :: nat \Rightarrow real where*
 $n \leq 20 \implies \text{select-cost } n = 0$
| $n > 20 \implies \text{select-cost } n =$
 $\text{select-cost } (\text{nat } \lfloor \text{real } n / 5 \rfloor) + \text{select-cost } (\text{nat } \lfloor 7 * \text{real } n / 10 \rfloor + 6) + 12$
 $* \text{real } n / 5$
by *force simp-all*
termination *by akra-bazzi-termination simp-all*

lemma *select-cost $\in \Theta(\lambda n. \text{real } n)$*
by (*master-theorem 3*) *auto*

8.5 Decreasing function

function *dec-cost :: nat \Rightarrow real where*
 $n \leq 2 \implies \text{dec-cost } n = 1$
| $n > 2 \implies \text{dec-cost } n = 0.5 * \text{dec-cost } (\text{nat } \lfloor \text{real } n / 2 \rfloor) + 1 / \text{real } n$
by *force simp-all*
termination *by akra-bazzi-termination simp-all*

lemma *dec-cost $\in \Theta(\lambda x::\text{nat}. \ln x / x)$*
by (*master-theorem 2.3*) *simp-all*

8.6 Example taken from Drmota and Szpakowski

function *drmot1 :: nat \Rightarrow real where*
 $n < 20 \implies \text{drmot1 } n = 1$
| $n \geq 20 \implies \text{drmot1 } n = 2 * \text{drmot1 } (\text{nat } \lfloor \text{real } n / 2 \rfloor) + 8/9 * \text{drmot1 } (\text{nat } \lfloor 3 * \text{real } n / 4 \rfloor) + \text{real } n^{\wedge}2 / \ln (\text{real } n)$
by *force simp-all*
termination *by akra-bazzi-termination simp-all*

lemma *drmot1* $\in \Theta(\lambda n::\text{real}. n^{\wedge}2 * \ln (\ln n))$
by (*master-theorem 2.2*) (*simp-all add: power-divide*)

function *drmot2* $:: \text{nat} \Rightarrow \text{real}$ **where**
 $n < 20 \implies \text{drmot2 } n = 1$
 $| n \geq 20 \implies \text{drmot2 } n = 1/3 * \text{drmot2 } (\text{nat } \lfloor \text{real } n/3 + 1/2 \rfloor) + 2/3 * \text{drmot2 } (\text{nat } \lfloor 2 * \text{real } n/3 - 1/2 \rfloor) + 1$
by force *simp-all*
termination by *akra-bazzi-termination simp-all*

lemma *drmot2* $\in \Theta(\lambda x. \ln (\text{real } x))$
by *master-theorem simp-all*

lemma *boncelet-phrase-length*:
fixes $p \delta :: \text{real}$ **assumes** $p: p > 0 \ p < 1$ **and** $\delta: \delta > 0 \ \delta < 1$ $2 * p + \delta < 2$
fixes $d :: \text{nat} \Rightarrow \text{real}$
defines $q \equiv 1 - p$
assumes *d-nonneg*: $\bigwedge n. d \ n \geq 0$
assumes *d-rec*: $\bigwedge n. n \geq 2 \implies d \ n = 1 + p * d (\text{nat } \lfloor p * \text{real } n + \delta \rfloor) + q * d (\text{nat } \lfloor q * \text{real } n - \delta \rfloor)$
shows $d \in \Theta(\lambda x. \ln x)$
using *assms* **by** (*master-theorem recursion: d-rec, simp-all*)

8.7 Transcendental exponents

function *foo-cost* $:: \text{nat} \Rightarrow \text{real}$ **where**
 $n < 200 \implies \text{foo-cost } n = 0$
 $| n \geq 200 \implies \text{foo-cost } n = \text{foo-cost } (\text{nat } \lfloor \text{real } n / 3 \rfloor) + \text{foo-cost } (\text{nat } \lfloor 3 * \text{real } n / 4 \rfloor + 42) + \text{real } n$
by force *simp-all*
termination by *akra-bazzi-termination simp-all*

lemma *foo-cost-nonneg* [*simp*]: $\text{foo-cost } n \geq 0$
by (*induction n rule: foo-cost.induct*) *simp-all*

lemma *foo-cost* $\in \Theta(\lambda n. \text{real } n \text{ powr } \text{akra-bazzi-exponent } [1,1] [1/3,3/4])$

proof (*master-theorem 1 p': 1*)
have $\forall n \geq 200. \text{foo-cost } n > 0$ **by** (*simp add: add-nonneg-pos*)
thus eventually ($\lambda n. \text{foo-cost } n > 0$) **at-top** **unfolding** *eventually-at-top-linorder*
by *blast*
qed *simp-all*

lemma *akra-bazzi-exponent* $[1,1] [1/3,3/4] \in \{1.1519623..1.1519624\}$
by (*akra-bazzi-approximate 29*)

8.8 Functions in locale contexts

locale *det-select* =

fixes $b :: \text{real}$
assumes $b: b > 0 \ b < 7/10$
begin

function $\text{select-cost}' :: \text{nat} \Rightarrow \text{real}$ **where**
 $n \leq 20 \implies \text{select-cost}' \ n = 0$
 $| \ n > 20 \implies \text{select-cost}' \ n =$
 $\text{select-cost}' \ (\text{nat} \lfloor \text{real } n / 5 \rfloor) + \text{select-cost}' \ (\text{nat} \lfloor b * \text{real } n \rfloor + 6) + 6 * \text{real}$
 $n + 5$
by force simp-all
termination using b **by** $\text{akra-bazzi-termination simp-all}$

lemma $a \geq 0 \implies \text{select-cost}' \in \Theta(\lambda n. \text{real } n)$
using b **by** $(\text{master-theorem } 3, \text{force+})$

end

8.9 Non-curried functions

function $\text{baz-cost} :: \text{nat} \times \text{nat} \Rightarrow \text{real}$ **where**
 $n \leq 2 \implies \text{baz-cost} \ (a, n) = 0$
 $| \ n > 2 \implies \text{baz-cost} \ (a, n) = 3 * \text{baz-cost} \ (a, \text{nat} \lfloor \text{real } n / 2 \rfloor) + \text{real } a$
by force simp-all
termination by $\text{akra-bazzi-termination simp-all}$

lemma $\text{baz-cost-nonneg} [\text{simp}]: a \geq 0 \implies \text{baz-cost} \ (a, n) \geq 0$
by $(\text{induction } a \ n \ \text{rule: baz-cost.induct}[\text{split-format} \ (\text{complete})]) \ \text{simp-all}$

lemma
assumes $a > 0$
shows $(\lambda x. \text{baz-cost} \ (a, x)) \in \Theta(\lambda x. x \ \text{powr} \ \log \ 2 \ 3)$
proof $(\text{master-theorem } 1 \ p': 0)$
from assms **have** $\forall x \geq 3. \text{baz-cost} \ (a, x) > 0$ **by** $(\text{auto intro: add-nonneg-pos})$
thus $\text{eventually} \ (\lambda x. \text{baz-cost} \ (a, x) > 0)$ **at-top** **by** $(\text{force simp: eventually-at-top-linorder})$
qed $(\text{insert assms, simp-all add: powr-divide})$

function $\text{bar-cost} :: \text{nat} \times \text{nat} \Rightarrow \text{real}$ **where**
 $n \leq 2 \implies \text{bar-cost} \ (a, n) = 0$
 $| \ n > 2 \implies \text{bar-cost} \ (a, n) = 3 * \text{bar-cost} \ (2 * a, \text{nat} \lfloor \text{real } n / 2 \rfloor) + \text{real } a$
by force simp-all
termination by $\text{akra-bazzi-termination simp-all}$

8.10 Ham-sandwich trees

function $\text{ham-sandwich-cost} :: \text{nat} \Rightarrow \text{real}$ **where**
 $n < 4 \implies \text{ham-sandwich-cost} \ n = 1$
 $| \ n \geq 4 \implies \text{ham-sandwich-cost} \ n =$
 $\text{ham-sandwich-cost} \ (\text{nat} \lfloor n/4 \rfloor) + \text{ham-sandwich-cost} \ (\text{nat} \lfloor n/2 \rfloor) + 1$
by force simp-all

termination by akra-bazzi-termination simp-all

lemma *ham-sandwich-cost-pos* [simp]: *ham-sandwich-cost* $n > 0$
by (*induction n rule: ham-sandwich-cost.induct*) *simp-all*

The golden ratio

definition $\varphi = ((1 + \text{sqrt } 5) / 2 :: \text{real})$

lemma $\varphi\text{-pos}$ [simp]: $\varphi > 0$ **and** $\varphi\text{-nonneg}$ [simp]: $\varphi \geq 0$ **and** $\varphi\text{-nonzero}$ [simp]:
 $\varphi \neq 0$

proof –

show $\varphi > 0$ **unfolding** $\varphi\text{-def}$ **by** (*simp add: add-pos-nonneg*)

thus $\varphi \geq 0$ $\varphi \neq 0$ **by** *simp-all*

qed

lemma *ham-sandwich-cost* $\in \Theta(\lambda n. n \text{ powr } (\log 2 \varphi))$

proof (*master-theorem 1 p': 0*)

have $(1 / 4) \text{ powr } \log 2 \varphi + (1 / 2) \text{ powr } \log 2 \varphi =$

$\text{inverse } (2 \text{ powr } \log 2 \varphi)^2 + \text{inverse } (2 \text{ powr } \log 2 \varphi)$

by (*simp add: powr-divide field-simps powr-powr power2-eq-square powr-mult[symmetric]*
del: powr-log-cancel)

also have $\dots = \text{inverse } (\varphi^2) + \text{inverse } \varphi$ **by** (*simp add: power2-eq-square*)

also have $\varphi + 1 = \varphi * \varphi$ **by** (*simp add: \varphi-def field-simps*)

hence $\text{inverse } (\varphi^2) + \text{inverse } \varphi = 1$ **by** (*simp add: field-simps power2-eq-square*)

finally show $(1 / 4) \text{ powr } \log 2 \varphi + (1 / 2) \text{ powr } \log 2 \varphi = 1$ **by** *simp*

qed *simp-all*

end

References

- [1] M. Akra and L. Bazzi. On the solution of linear recurrence equations. *Computational Optimization and Applications*, 10(2):195–210, 1998.
- [2] T. Leighton. Notes on better Master theorems for divide-and-conquer recurrences. 1996.