Actuarial Mathematics

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Abstract

Actuarial Mathematics is a theory in applied mathematics, which is mainly used for determining the prices of insurance products and evaluating the liability of a company associating with insurance contracts. It is related to calculus, probability theory and financial theory, etc.

In this entry, I formalize the very basic part of Actuarial Mathematics in Isabelle/HOL. It includes the theory of interest, survival model, and life table. The theory of interest deals with interest rates, present value factors, an annuity certain, etc. The survival model is a probabilistic model that represents the human mortality. The life table is based on the survival model and used for practical calculations.

I have already formalized the basic part of Actuarial Mathematics in Coq (https://github.com/Yosuke-Ito-345/Actuary) in a purely axiomatic manner. In contrast, Isabelle formalization is based on the probability theory and the survival model is developed as generally as possible. Such rigorous and general formulation seems very rare; at least I cannot find any similar documentation on the web.

This formalization in Isabelle is still at an early stage, and I cannot guarantee the backward compatibility in the future development. If you heavily depend on the "Actuarial Mathematics" library, please let me know.

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theory *Preliminaries*
imports *HOL−Library.Rewrite HOL−Library.Extended-Nonnegative-Real HOL−Library.Extended-Real HOL−Probability.Probability*
begin
declare [[*show-types*]]
notation *powr* (**infixr** ‹.^› *80*)

1 Preliminary Lemmas

lemma *Collect-conj-eq2*: $\{x \in A.\ P\ x \wedge Q\ x\} = \{x \in A.\ P\ x\} \cap \{x \in A.\ Q\ x\}$
⟨*proof*⟩

lemma *vimage-compl-atMost*:
fixes $f :: 'a \Rightarrow 'b$::*linorder*
shows $-(f\ -`\ \{..y\}) = f\ -`\ \{y<..\}$
⟨*proof*⟩

context *linorder*
begin

lemma *Icc-minus-Ico*:
fixes $a\ b$
assumes $a \leq b$
shows $\{a..b\} - \{a..<b\} = \{b\}$
⟨*proof*⟩

lemma *Icc-minus-Ioc*:
fixes $a\ b$
assumes $a \leq b$
shows $\{a..b\} - \{a<..b\} = \{a\}$
⟨*proof*⟩

lemma *Int-atLeastAtMost-Unbounded*[*simp*]: $\{a..\}\ Int\ \{..b\} = \{a..b\}$
⟨*proof*⟩

lemma *Int-greaterThanAtMost-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..b\} = \{a<..b\}$
⟨*proof*⟩

lemma *Int-atLeastLessThan-Unbounded*[*simp*]: $\{a..\}$ *Int* $\{..<b\} = \{a..<b\}$
⟨*proof*⟩

lemma *Int-greaterThanLessThan-Unbounded*[*simp*]: $\{a<..\}$ *Int* $\{..<b\} = \{a<..<b\}$
⟨*proof*⟩

end

lemma *Ico-real-nat-disjoint*:
disjoint-family $(\lambda n{::}nat.\ \{a + real\ n\ ..<\ a + real\ n + 1\})$ **for** $a{::}real$
⟨*proof*⟩

corollary *Ico-nat-disjoint*: *disjoint-family* $(\lambda n{::}nat.\ \{real\ n\ ..<\ real\ n + 1\})$
⟨*proof*⟩

lemma *Ico-real-nat-union*: $(\bigcup n{::}nat.\ \{a + real\ n\ ..<\ a + real\ n + 1\}) = \{a..\}$
for $a{::}real$
⟨*proof*⟩

corollary *Ico-nat-union*: $(\bigcup n{::}nat.\ \{real\ n\ ..<\ real\ n + 1\}) = \{0..\}$
⟨*proof*⟩

lemma *Ico-nat-union-finite*: $(\bigcup (n{::}nat){<}m.\ \{real\ n\ ..<\ real\ n + 1\}) = \{0..<m\}$
⟨*proof*⟩

lemma *seq-part-multiple*: **fixes** $m\ n :: nat$ **assumes** $m \neq 0$ **defines** $A \equiv \lambda i{::}nat.$
$\{i{*}m\ ..<\ (i{+}1){*}m\}$
shows $\forall\, i\ j.\ i \neq j \longrightarrow A\ i \cap A\ j = \{\}$ **and** $(\bigcup i{<}n.\ A\ i) = \{..<\ n{*}m\}$
⟨*proof*⟩

lemma *frontier-Icc-real*: *frontier* $\{a..b\} = \{a,\ b\}$ **if** $a \leq b$ **for** $a\ b :: real$
⟨*proof*⟩

lemma(**in** *field*) *divide-mult-cancel*[*simp*]: **fixes** $a\ b$ **assumes** $b \neq 0$
shows $a\ /\ b * b = a$
⟨*proof*⟩

lemma *inverse-powr*: $(1/a).\hat{}\,b = a.\hat{}\,{-}b$ **if** $a > 0$ **for** $a\ b :: real$
⟨*proof*⟩

lemma *powr-eq-one-iff-gen*[*simp*]: $a.\hat{}\,x = 1 \longleftrightarrow x = 0$ **if** $a > 0\ a \neq 1$ **for** $a\ x ::$
real
⟨*proof*⟩

lemma *powr-less-cancel2*: $0 < a \Longrightarrow 0 < x \Longrightarrow 0 < y \Longrightarrow x.\hat{}\,a < y.\hat{}\,a \Longrightarrow x <$

y
for *a x y* ::*real*
⟨*proof*⟩

lemma *geometric-increasing-sum-aux*: (*1*−*r*)^*2* * (∑ *k*<*n*. (*k*+*1*)∗*r*^*k*) = *1* − (*n*+*1*)∗*r*^*n* + *n*∗*r*^(*n*+*1*)
for *n*::*nat* **and** *r*::*real*
⟨*proof*⟩

lemma *geometric-increasing-sum*: (∑ *k*<*n*. (*k*+*1*)∗*r*^*k*) = (*1* − (*n*+*1*)∗*r*^*n* + *n*∗*r*^(*n*+*1*)) / (*1*−*r*)^*2*
if *r* ≠ *1* **for** *n*::*nat* **and** *r*::*real*
⟨*proof*⟩

lemma *Reals-UNIV*[*simp*]: ℝ = {*x*::*real*. *True*}
⟨*proof*⟩

lemma *Lim-cong*:
assumes ∀ *F* *x* *in* *F*. *f x* = *g x*
shows *Lim F f* = *Lim F g*
⟨*proof*⟩

lemma *LIM-zero-iff'*: ((λ*x*. *l* − *f x*) ⟶ *0*) *F* = (*f* ⟶ *l*) *F*
for *f* :: *'a* ⇒ *'b*::*real-normed-vector*
⟨*proof*⟩

lemma *antimono-onI*:
(⋀*r s*. *r* ∈ *A* ⟹ *s* ∈ *A* ⟹ *r* ≤ *s* ⟹ *f r* ≥ *f s*) ⟹ *antimono-on A f*
⟨*proof*⟩

lemma *antimono-onD*:
⟦*antimono-on A f*; *r* ∈ *A*; *s* ∈ *A*; *r* ≤ *s*⟧ ⟹ *f r* ≥ *f s*
⟨*proof*⟩

lemma *antimono-imp-mono-on*: *antimono f* ⟹ *antimono-on A f*
⟨*proof*⟩

lemma *antimono-on-subset*: *antimono-on A f* ⟹ *B* ⊆ *A* ⟹ *antimono-on B f*
⟨*proof*⟩

lemma *mono-on-antimono-on*:
fixes *f* :: *'a*::*order* ⇒ *'b*::*ordered-ab-group-add*
shows *mono-on A f* ⟷ *antimono-on A* (λ*r*. − *f r*)
⟨*proof*⟩

corollary *mono-antimono*:
fixes *f* :: *'a*::*order* ⇒ *'b*::*ordered-ab-group-add*
shows *mono f* ⟷ *antimono* (λ*r*. − *f r*)
⟨*proof*⟩

lemma *mono-on-at-top-le*:
fixes $a :: 'a{::}linorder$ **and** $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$
and $f :: 'a \Rightarrow 'b$
assumes *f-mono*: *mono-on* $\{a..\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-top*
shows $\bigwedge x.\ x \in \{a..\} \Longrightarrow f\ x \leq l$
⟨*proof*⟩

corollary *mono-at-top-le*:
fixes $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$ **and** $f :: 'a{::}linorder \Rightarrow 'b$
assumes *mono f* **and** $(f \longrightarrow b)$ *at-top*
shows $\bigwedge x.\ f\ x \leq b$
⟨*proof*⟩

lemma *mono-on-at-bot-ge*:
fixes $a :: 'a{::}linorder$ **and** $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$
and $f :: 'a \Rightarrow 'b$
assumes *f-mono*: *mono-on* $\{..a\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-bot*
shows $\bigwedge x.\ x \in \{..a\} \Longrightarrow f\ x \geq l$
⟨*proof*⟩

corollary *mono-at-bot-ge*:
fixes $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$ **and** $f :: 'a{::}linorder \Rightarrow 'b$
assumes *mono f* **and** $(f \longrightarrow b)$ *at-bot*
shows $\bigwedge x.\ f\ x \geq b$
⟨*proof*⟩

lemma *antimono-on-at-top-ge*:
fixes $a :: 'a{::}linorder$ **and** $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$
and $f :: 'a \Rightarrow 'b$
assumes *f-antimono*: *antimono-on* $\{a..\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-top*
shows $\bigwedge x.\ x \in \{a..\} \Longrightarrow f\ x \geq l$
⟨*proof*⟩

corollary *antimono-at-top-le*:
fixes $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$ **and** $f :: 'a{::}linorder \Rightarrow 'b$
assumes *antimono f* **and** $(f \longrightarrow b)$ *at-top*
shows $\bigwedge x.\ f\ x \geq b$
⟨*proof*⟩

lemma *antimono-on-at-bot-ge*:
fixes $a :: 'a{::}linorder$ **and** $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$
and $f :: 'a \Rightarrow 'b$
assumes *f-antimono*: *antimono-on* $\{..a\}$ f **and** *f-to-l*: $(f \longrightarrow l)$ *at-bot*
shows $\bigwedge x.\ x \in \{..a\} \Longrightarrow f\ x \leq l$
⟨*proof*⟩

corollary *antimono-at-bot-ge*:
fixes $b :: 'b{::}\{order\text{-}topology, linordered\text{-}ab\text{-}group\text{-}add\}$ **and** $f :: 'a{::}linorder \Rightarrow 'b$

assumes *antimono f* **and** (*f* $\longrightarrow$ *b*) *at-bot*
shows $\bigwedge x.\ f\ x \leq b$
⟨*proof*⟩

lemma *continuous-cdivide*:
fixes *c*::′*a*::*real-normed-field*
assumes $c \neq 0$ *continuous F f*
shows *continuous F* (λ*x*. *f x* / *c*)
⟨*proof*⟩

lemma *continuous-mult-left-iff*:
fixes *c*::′*a*::*real-normed-field*
assumes $c \neq 0$
shows *continuous F f* $\longleftrightarrow$ *continuous F* (λ*x*. *c* ∗ *f x*)
⟨*proof*⟩

lemma *continuous-mult-right-iff*:
fixes *c*::′*a*::*real-normed-field*
assumes $c \neq 0$
shows *continuous F f* $\longleftrightarrow$ *continuous F* (λ*x*. *f x* ∗ *c*)
⟨*proof*⟩

lemma *continuous-cdivide-iff*:
fixes *c*::′*a*::*real-normed-field*
assumes $c \neq 0$
shows *continuous F f* $\longleftrightarrow$ *continuous F* (λ*x*. *f x* / *c*)
⟨*proof*⟩

lemma *continuous-cong*:
assumes *eventually* (λ*x*. *f x* = *g x*) *F f* (*Lim F* (λ*x*. *x*)) = *g* (*Lim F* (λ*x*. *x*))
shows *continuous F f* $\longleftrightarrow$ *continuous F g*
⟨*proof*⟩

lemma *continuous-at-within-cong*:
assumes *f x* = *g x eventually* (λ*x*. *f x* = *g x*) (*at x within s*)
shows *continuous* (*at x within s*) *f* $\longleftrightarrow$ *continuous* (*at x within s*) *g*
⟨*proof*⟩

lemma *continuous-within-shift*:
fixes *a x* :: ′*a* :: {*topological-ab-group-add*, *t2-space*}
and *s* :: ′*a set*
and *f* :: ′*a* ⇒ ′*b*::*topological-space*
shows *continuous* (*at x within s*) (λ*x*. *f* (*x*+*a*)) $\longleftrightarrow$ *continuous* (*at* (*x*+*a*) *within plus a* ‘ *s*) *f*
⟨*proof*⟩

lemma *isCont-shift*:
fixes *a x* :: ′*a* :: {*topological-ab-group-add*, *t2-space*}
and *f* :: ′*a* ⇒ ′*b*::*topological-space*

shows *isCont* ($\lambda x.\ f\ (x+a)$) $x \longleftrightarrow$ *isCont* $f\ (x+a)$
⟨*proof*⟩

lemma *has-real-derivative-at-split*:
(f *has-real-derivative* D) (*at* x) $\longleftrightarrow$
(f *has-real-derivative* D) (*at-left* x) $\wedge$ (f *has-real-derivative* D) (*at-right* x)
⟨*proof*⟩

lemma *DERIV-cmult-iff*:
assumes $c \neq 0$
shows (f *has-field-derivative* D) (*at* x *within* s) $\longleftrightarrow$
(($\lambda x.\ c * f\ x$) *has-field-derivative* $c * D$) (*at* x *within* s)
⟨*proof*⟩

lemma *DERIV-cmult-right-iff*:
assumes $c \neq 0$
shows (f *has-field-derivative* D) (*at* x *within* s) $\longleftrightarrow$
(($\lambda x.\ f\ x * c$) *has-field-derivative* $D * c$) (*at* x *within* s)
⟨*proof*⟩

lemma *DERIV-cdivide-iff*:
assumes $c \neq 0$
shows (f *has-field-derivative* D) (*at* x *within* s) $\longleftrightarrow$
(($\lambda x.\ f\ x\ /\ c$) *has-field-derivative* $D\ /\ c$) (*at* x *within* s)
⟨*proof*⟩

lemma *DERIV-ln-divide-chain*:
fixes $f :: real \Rightarrow real$
assumes $f\ x > 0$ **and** (f *has-real-derivative* D) (*at* x *within* s)
shows (($\lambda x.\ ln\ (f\ x)$) *has-real-derivative* ($D\ /\ f\ x$)) (*at* x *within* s)
⟨*proof*⟩

lemma *inverse-fun-has-integral-ln*:
fixes $f :: real \Rightarrow real$ **and** $f' :: real \Rightarrow real$
assumes $a \leq b$ **and**
$\bigwedge x.\ x \in \{a..b\} \Longrightarrow f\ x > 0$ **and**
continuous-on $\{a..b\}$ f **and**
$\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow$ (f *has-real-derivative* $f'\ x$) (*at* x)
shows (($\lambda x.\ f'\ x\ /\ f\ x$) *has-integral* ($ln\ (f\ b) - ln\ (f\ a)$)) $\{a..b\}$
⟨*proof*⟩

lemma *DERIV-fun-powr2*:
fixes a::*real*
assumes *a-pos*: $a > 0$
and f: *DERIV* $f\ x$:> r
shows *DERIV* ($\lambda x.\ a\ powr\ (f\ x)$) x :> $a\ powr\ (f\ x) * r * ln\ a$
⟨*proof*⟩

lemma *has-real-derivative-powr2*:

assumes *a-pos*: $a > 0$
shows $((\lambda x.\ a.\hat{}x)$ *has-real-derivative* $a.\hat{}x * ln\ a)\ (at\ x)$
⟨*proof*⟩

lemma *field-differentiable-shift*:
$(f$ *field-differentiable* $(at\ (x + z))) = ((\lambda x.\ f\ (x + z))$ *field-differentiable* $(at\ x))$
⟨*proof*⟩

1.1 Lemmas on *indicator* for a Linearly Ordered Type

lemma *indicator-Icc-shift*:
fixes $a\ b\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{a..b\}\ x =$ *indicator* $\{t+a..t+b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Icc-shift-inverse*:
fixes $a\ b\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{a-t..b-t\}\ x =$ *indicator* $\{a..b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Ici-shift*:
fixes $a\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{a..\}\ x =$ *indicator* $\{t+a..\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Ici-shift-inverse*:
fixes $a\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{a-t..\}\ x =$ *indicator* $\{a..\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Iic-shift*:
fixes $b\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{..b\}\ x =$ *indicator* $\{..t+b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Iic-shift-inverse*:
fixes $b\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{..b-t\}\ x =$ *indicator* $\{..b\}\ (t+x)$
⟨*proof*⟩

lemma *indicator-Icc-reverse*:
fixes $a\ b\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{a..b\}\ x =$ *indicator* $\{t-b..t-a\}\ (t-x)$
⟨*proof*⟩

lemma *indicator-Icc-reverse-inverse*:
fixes $a\ b\ t\ x :: 'a$::*linordered-ab-group-add*
shows *indicator* $\{t-b..t-a\}\ x =$ *indicator* $\{a..b\}\ (t-x)$

⟨*proof*⟩

lemma *indicator-Ici-reverse*:
fixes *a t x* :: *'a*::*linordered-ab-group-add*
shows *indicator* {*a*..} *x* = *indicator* {..*t*−*a*} (*t*−*x*)
⟨*proof*⟩

lemma *indicator-Ici-reverse-inverse*:
fixes *b t x* :: *'a*::*linordered-ab-group-add*
shows *indicator* {*t*−*b*..} *x* = *indicator* {..*b*} (*t*−*x*)
⟨*proof*⟩

lemma *indicator-Iic-reverse*:
fixes *b t x* :: *'a*::*linordered-ab-group-add*
shows *indicator* {..*b*} *x* = *indicator* {*t*−*b*..} (*t*−*x*)
⟨*proof*⟩

lemma *indicator-Iic-reverse-inverse*:
fixes *a t x* :: *'a*::*linordered-field*
shows *indicator* {..*t*−*a*} *x* = *indicator* {*a*..} (*t*−*x*)
⟨*proof*⟩

lemma *indicator-Icc-affine-pos*:
fixes *a b c t x* :: *'a*::*linordered-field*
assumes *c* > *0*
shows *indicator* {*a*..*b*} *x* = *indicator* {*t*+*c*∗*a*..*t*+*c*∗*b*} (*t* + *c*∗*x*)
⟨*proof*⟩

lemma *indicator-Icc-affine-pos-inverse*:
fixes *a b c t x* :: *'a*::*linordered-field*
assumes *c* > *0*
shows *indicator* {(*a*−*t*)/*c*..(*b*−*t*)/*c*} *x* = *indicator* {*a*..*b*} (*t* + *c*∗*x*)
⟨*proof*⟩

lemma *indicator-Ici-affine-pos*:
fixes *a c t x* :: *'a*::*linordered-field*
assumes *c* > *0*
shows *indicator* {*a*..} *x* = *indicator* {*t*+*c*∗*a*..} (*t* + *c*∗*x*)
⟨*proof*⟩

lemma *indicator-Ici-affine-pos-inverse*:
fixes *a c t x* :: *'a*::*linordered-field*
assumes *c* > *0*
shows *indicator* {(*a*−*t*)/*c*..} *x* = *indicator* {*a*..} (*t* + *c*∗*x*)
⟨*proof*⟩

lemma *indicator-Iic-affine-pos*:
fixes *b c t x* :: *'a*::*linordered-field*
assumes *c* > *0*

shows *indicator* $\{..b\}$ x = *indicator* $\{..t+c*b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Iic-affine-pos-inverse*:
fixes b c t x :: $'a$::*linordered-field*
assumes $c > 0$
shows *indicator* $\{..(b-t)/c\}$ x = *indicator* $\{..b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Icc-affine-neg*:
fixes a b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{a..b\}$ x = *indicator* $\{t+c*b..t+c*a\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Icc-affine-neg-inverse*:
fixes a b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{(b-t)/c..(a-t)/c\}$ x = *indicator* $\{a..b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Ici-affine-neg*:
fixes a c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{a..\}$ x = *indicator* $\{..t+c*a\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Ici-affine-neg-inverse*:
fixes b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{(b-t)/c..\}$ x = *indicator* $\{..b\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Iic-affine-neg*:
fixes b c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{..b\}$ x = *indicator* $\{t+c*b..\}$ $(t + c*x)$
⟨*proof*⟩

lemma *indicator-Iic-affine-neg-inverse*:
fixes a c t x :: $'a$::*linordered-field*
assumes $c < 0$
shows *indicator* $\{..(a-t)/c\}$ x = *indicator* $\{a..\}$ $(t + c*x)$
⟨*proof*⟩

2 Additional Lemmas for the *HOL−Analysis* Library

lemma *differentiable-eq-field-differentiable-real*:
fixes f :: *real* ⇒ *real*

shows *f differentiable F* $\longleftrightarrow$ *f field-differentiable F*
⟨*proof*⟩

lemma *differentiable-on-eq-field-differentiable-real*:
fixes *f* :: *real* ⇒ *real*
shows *f differentiable-on s* $\longleftrightarrow$ (∀ *x*∈*s*. *f field-differentiable* (*at x within s*))
⟨*proof*⟩

lemma *differentiable-on-cong* :
assumes ⋀*x*. *x*∈*s* ⟹ *f x* = *g x* **and** *f differentiable-on s*
shows *g differentiable-on s*
⟨*proof*⟩

lemma *C1-differentiable-imp-deriv-continuous-on*:
f C1-differentiable-on S ⟹ *continuous-on S* (*deriv f*)
⟨*proof*⟩

lemma *deriv-shift*:
assumes *f field-differentiable at* (*x*+*a*)
shows *deriv f* (*x*+*a*) = *deriv* (λ*s*. *f* (*x*+*s*)) *a*
⟨*proof*⟩

lemma *piecewise-differentiable-on-cong*:
assumes *f piecewise-differentiable-on i*
and ⋀*x*. *x* ∈ *i* ⟹ *f x* = *g x*
shows *g piecewise-differentiable-on i*
⟨*proof*⟩

lemma *differentiable-piecewise*:
assumes *continuous-on i f*
and *f differentiable-on i*
shows *f piecewise-differentiable-on i*
⟨*proof*⟩

lemma *piecewise-differentiable-scaleR*:
assumes *f piecewise-differentiable-on S*
shows (λ*x*. *a* ∗$_R$ *f x*) *piecewise-differentiable-on S*
⟨*proof*⟩

lemma *differentiable-on-piecewise-compose*:
assumes *f piecewise-differentiable-on S*
and *g differentiable-on f* ‘ *S*
shows *g* ∘ *f piecewise-differentiable-on S*
⟨*proof*⟩

lemma *MVT-order-free*:
fixes *r h* :: *real*
defines *I* ≡ {*r*..*r*+*h*} ∪ {*r*+*h*..*r*}
assumes *continuous-on I f* **and** *f differentiable-on interior I*

obtains t **where** $t \in \{0<..<1\}$ **and** $f\ (r+h) - f\ r = h * deriv\ f\ (r + t*h)$
⟨*proof*⟩

lemma *integral-combine2*:
fixes f :: *real* ⇒ *'a*::*banach*
assumes $a \leq c\ c \leq b$
and f *integrable-on* $\{a..c\}$ f *integrable-on* $\{c..b\}$
shows *integral* $\{a..c\}$ f + *integral* $\{c..b\}$ f = *integral* $\{a..b\}$ f
⟨*proof*⟩

lemma *has-integral-null-interval*: **fixes** $a\ b$ :: *real* **and** f::*real* ⇒ *real* **assumes** $a \geq b$
shows (f *has-integral 0*) $\{a..b\}$
⟨*proof*⟩

lemma *has-integral-interval-reverse*: **fixes** f :: *real* ⇒ *real* **and** $a\ b$ :: *real*
assumes $a \leq b$
and *continuous-on* $\{a..b\}$ f
shows $((\lambda x.\ f\ (a+b-x))$ *has-integral* (*integral* $\{a..b\}$ $f))$ $\{a..b\}$
⟨*proof*⟩

lemma *FTC-real-deriv-has-integral*:
fixes F :: *real* ⇒ *real*
assumes $a \leq b$
and F *piecewise-differentiable-on* $\{a<..<b\}$
and *continuous-on* $\{a..b\}$ F
shows (*deriv* F *has-integral* $F\ b - F\ a$) $\{a..b\}$
⟨*proof*⟩

lemma *integrable-spike-cong*:
assumes *negligible* S $\bigwedge x.\ x \in T - S \Longrightarrow g\ x = f\ x$
shows f *integrable-on* $T \longleftrightarrow g$ *integrable-on* T
⟨*proof*⟩

lemma *has-integral-powr2-from-0*:
fixes $a\ c$ :: *real*
assumes *a-pos*: $a > 0$ **and** *a-neq-1*: $a \neq 1$ **and** *c-nneg*: $c \geq 0$
shows $((\lambda x.\ a.\hat{}x)$ *has-integral* $((a.\hat{}c - 1)\ /\ (ln\ a)))$ $\{0..c\}$
⟨*proof*⟩

lemma *integrable-on-powr2-from-0*:
fixes $a\ c$ :: *real*
assumes *a-pos*: $a > 0$ **and** *a-neq-1*: $a \neq 1$ **and** *c-nneg*: $c \geq 0$
shows $(\lambda x.\ a.\hat{}x)$ *integrable-on* $\{0..c\}$
⟨*proof*⟩

lemma *integrable-on-powr2-from-0-general*:
fixes $a\ c$ :: *real*
assumes *a-pos*: $a > 0$ **and** *c-nneg*: $c \geq 0$

shows $(\lambda x.\ a.\widehat{\ }x)$ *integrable-on* $\{0..c\}$
⟨*proof*⟩

lemma *has-bochner-integral-power*:
fixes $a\ b :: real$ **and** $k :: nat$
assumes $a \leq b$
shows *has-bochner-integral lborel* $(\lambda x.\ x\widehat{\ }k * indicator\ \{a..b\}\ x)\ ((b\widehat{\ }(k+1) - a\widehat{\ }(k+1))\ /\ (k+1))$
⟨*proof*⟩

corollary *integrable-power*: $(a::real) \leq b \Longrightarrow$ *integrable lborel* $(\lambda x.\ x\widehat{\ }k * indicator\ \{a..b\}\ x)$
⟨*proof*⟩

lemma *has-integral-set-integral-real*:
fixes $f::'a::euclidean\text{-}space \Rightarrow real$ **and** $A :: 'a\ set$
assumes f: *set-integrable lborel A f*
shows $(f\ has\text{-}integral\ (set\text{-}lebesgue\text{-}integral\ lborel\ A\ f))\ A$
⟨*proof*⟩

lemma *set-borel-measurable-lborel*:
set-borel-measurable lborel A f $\longleftrightarrow$ *set-borel-measurable borel A f*
⟨*proof*⟩

lemma *restrict-space-whole*[*simp*]: *restrict-space M* $(space\ M) = M$
⟨*proof*⟩

lemma *deriv-measurable-real*:
fixes $f :: real \Rightarrow real$
assumes *f differentiable-on S open S* $f \in$ *borel-measurable borel*
shows *set-borel-measurable borel S* $(deriv\ f)$
⟨*proof*⟩

lemma *piecewise-differentiable-on-deriv-measurable-real*:
fixes $f :: real \Rightarrow real$
assumes *f piecewise-differentiable-on S open S* $f \in$ *borel-measurable borel*
shows *set-borel-measurable borel S* $(deriv\ f)$
⟨*proof*⟩

lemma *borel-measurable-antimono*:
fixes $f :: real \Rightarrow real$
shows *antimono* $f \Longrightarrow f \in$ *borel-measurable borel*
⟨*proof*⟩

lemma *set-borel-measurable-restrict-space-iff*:
fixes $f :: 'a \Rightarrow 'b::real\text{-}normed\text{-}vector$
assumes Ω[*measurable*, *simp*]: $\Omega \cap space\ M \in sets\ M$

shows $f \in$ *borel-measurable* (*restrict-space* $M\ \Omega$) $\longleftrightarrow$ *set-borel-measurable* $M\ \Omega$ f
⟨*proof*⟩

lemma *set-integrable-restrict-space-iff*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes $A \in$ *sets* M
shows *set-integrable* $M\ A\ f \longleftrightarrow$ *integrable* (*restrict-space* $M\ A$) f
⟨*proof*⟩

lemma *set-lebesgue-integral-restrict-space*:
fixes $f :: 'a \Rightarrow 'b$::{*banach*, *second-countable-topology*}
assumes $A \in$ *sets* M
shows *set-lebesgue-integral* $M\ A\ f =$ *integral*^{L} (*restrict-space* $M\ A$) f
⟨*proof*⟩

lemma *distr-borel-lborel*: *distr* M *borel* $f =$ *distr* M *lborel* f
⟨*proof*⟩

lemma *AE-translation*:
assumes *AE* x *in lborel*. $P\ x$ **shows** *AE* x *in lborel*. $P\ (a+x)$
⟨*proof*⟩

lemma *set-AE-translation*:
assumes *AE* $x{\in}S$ *in lborel*. $P\ x$ **shows** *AE* $x \in$ *plus* $(-a)$ ‘ S *in lborel*. $P\ (a+x)$
⟨*proof*⟩

lemma *AE-scale-measure-iff*:
assumes $r > 0$
shows (*AE* x *in* (*scale-measure* $r\ M$). $P\ x$) $\longleftrightarrow$ (*AE* x *in* M. $P\ x$)
⟨*proof*⟩

lemma *nn-set-integral-cong2*:
assumes *AE* $x{\in}A$ *in* M. $f\ x = g\ x$
shows $(\int^+ x{\in}A.\ f\ x\ \partial M) = (\int^+ x{\in}A.\ g\ x\ \partial M)$
⟨*proof*⟩

lemma *set-lebesgue-integral-cong-AE2*:
assumes [*measurable*]: $A \in$ *sets* M *set-borel-measurable* $M\ A\ f$ *set-borel-measurable* $M\ A\ g$
assumes *AE* $x \in A$ *in* M. $f\ x = g\ x$
shows (*LINT* $x{:}A|M$. $f\ x$) = (*LINT* $x{:}A|M$. $g\ x$)
⟨*proof*⟩

proposition *set-nn-integral-eq-set-integral*:
assumes *AE* $x{\in}A$ *in* M. $0 \le f\ x$ *set-integrable* $M\ A\ f$
shows $(\int^+ x{\in}A.\ f\ x\ \partial M) = (\int x{\in}A.\ f\ x\ \partial M)$
⟨*proof*⟩

proposition *nn-integral-disjoint-family-on-finite*:
assumes [*measurable*]: $f \in$ *borel-measurable* M $\bigwedge$(*n*::*nat*). $n \in S \Longrightarrow B\ n \in$ *sets* M
and *disjoint-family-on* $B\ S$ *finite* S
shows $(\int^+ x \in (\bigcup n{\in}S.\ B\ n).\ f\ x\ \partial M) = (\sum n{\in}S.\ (\int^+ x \in B\ n.\ f\ x\ \partial M))$
⟨*proof*⟩

lemma *nn-integral-distr-set*:
assumes $T \in$ *measurable* $M\ M'$ **and** $f \in$ *borel-measurable* (*distr* $M\ M'\ T$)
and $A \in$ *sets* M' **and** $\bigwedge x.\ x \in$ *space* $M \Longrightarrow T\ x \in A$
shows *integral*^{N} (*distr* $M\ M'\ T$) $f =$ *set-nn-integral* (*distr* $M\ M'\ T$) $A\ f$
⟨*proof*⟩

lemma *measure-eqI-Ioc*:
fixes $M\ N$ :: *real measure*
assumes *sets*: *sets* $M =$ *sets borel sets* $N =$ *borel*
assumes *fin*: $\bigwedge a\ b.\ a \le b \Longrightarrow$ *emeasure* M $\{a<..b\} < \infty$
assumes *eq*: $\bigwedge a\ b.\ a \le b \Longrightarrow$ *emeasure* M $\{a<..b\} =$ *emeasure* N $\{a<..b\}$
shows $M = N$
⟨*proof*⟩

lemma (**in** *finite-measure*) *distributed-measure*:
assumes *distributed* $M\ N\ X\ f$
and $\bigwedge x.\ x \in$ *space* $N \Longrightarrow\ f\ x \ge 0$
and $A \in$ *sets* N
shows *measure* M $(X\ -`\ A \cap$ *space* $M) = (\int x.$ *indicator* $A\ x * f\ x\ \partial N)$
⟨*proof*⟩

lemma *set-integrable-const*[*simp*]:
$A \in$ *sets* $M \Longrightarrow$ *emeasure* $M\ A < \infty \Longrightarrow$ *set-integrable* $M\ A$ (λ-. c)
⟨*proof*⟩

lemma *set-integral-const*[*simp*]:
$A \in$ *sets* $M \Longrightarrow$ *emeasure* $M\ A < \infty \Longrightarrow$ *set-lebesgue-integral* $M\ A$ (λ-. c) = *measure* $M\ A *_R\ c$
⟨*proof*⟩

lemma *set-integral-empty-0*[*simp*]: *set-lebesgue-integral* M $\{\}$ $f = 0$
⟨*proof*⟩

lemma *set-integral-nonneg*[*simp*]:
fixes f :: $'a \Rightarrow$ *real* **and** A :: $'a$ *set*
shows $(\bigwedge x.\ x \in A \Longrightarrow 0 \le f\ x) \Longrightarrow 0 \le$ *set-lebesgue-integral* $M\ A\ f$
⟨*proof*⟩

lemma
fixes f :: $'a \Rightarrow 'b$::{*banach*, *second-countable-topology*} **and** w :: $'a \Rightarrow$ *real*

assumes $A \in$ *sets M set-borel-measurable M A f*
$\bigwedge i$. *set-borel-measurable M A* (*s i*) *set-integrable M A w*
assumes *lim*: *AE x∈A in M*. (λi. *s i x*) $\longrightarrow$ *f x*
assumes *bound*: $\bigwedge i$::*nat*. *AE x∈A in M*. *norm* (*s i x*) $\leq$ *w x*
shows *set-integrable-dominated-convergence*: *set-integrable M A f*
and *set-integrable-dominated-convergence2*: $\bigwedge i$. *set-integrable M A* (*s i*)
and *set-integral-dominated-convergence*:
(λi. *set-lebesgue-integral M A* (*s i*)) $\longrightarrow$ *set-lebesgue-integral M A f*
⟨*proof*⟩

lemma *absolutely-integrable-on-iff-set-integrable*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *real*
assumes *f* ∈ *borel-measurable lborel*
and *S* ∈ *sets lborel*
shows *set-integrable lborel S f* ⟷ *f absolutely-integrable-on S*
⟨*proof*⟩

corollary *integrable-on-iff-set-integrable-nonneg*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *real*
assumes $\bigwedge x$. *x* ∈ *S* ⟹ *f x* ≥ *0 f* ∈ *borel-measurable lborel*
and *S* ∈ *sets lborel*
shows *set-integrable lborel S f* ⟷ *f integrable-on S*
⟨*proof*⟩

lemma *integrable-on-iff-set-integrable-nonneg-AE*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *real*
assumes *AE x∈S in lborel*. *f x* ≥ *0 f* ∈ *borel-measurable lborel*
and *S* ∈ *sets lborel*
shows *set-integrable lborel S f* ⟷ *f integrable-on S*
⟨*proof*⟩

lemma *set-borel-integral-eq-integral-nonneg*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *real*
assumes $\bigwedge x$. *x* ∈ *S* ⟹ *f x* ≥ *0 f* ∈ *borel-measurable borel S* ∈ *sets borel*
shows (*LINT x* : *S* | *lborel*. *f x*) = *integral S f*
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

lemma *set-borel-integral-eq-integral-nonneg-AE*:
fixes *f* :: *'a*::*euclidean-space* ⇒ *real*
assumes *AE x∈S in lborel*. *f x* ≥ *0 f* ∈ *borel-measurable borel S* ∈ *sets borel*
shows (*LINT x* : *S* | *lborel*. *f x*) = *integral S f*
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

2.1 Set Lebesgue Integrability on Affine Transformation

lemma *set-integrable-Icc-affine-pos-iff*:
fixes *f* :: *real* ⇒ *'a*::{*banach*, *second-countable-topology*} **and** *a b c t* :: *real*

assumes $c > 0$
shows *set-integrable lborel* $\{(a-t)/c..(b-t)/c\}$ $(\lambda x.\ f\ (t + c*x))$
$\longleftrightarrow$ *set-integrable lborel* $\{a..b\}$ f
⟨*proof*⟩

corollary *set-integrable-Icc-shift*:
fixes f :: *real* $\Rightarrow$ $'a$::{*banach*, *second-countable-topology*} **and** a b t :: *real*
shows *set-integrable lborel* $\{a-t..b-t\}$ $(\lambda x.\ f\ (t+x)) \longleftrightarrow$ *set-integrable lborel* $\{a..b\}$ f
⟨*proof*⟩

lemma *set-integrable-Ici-affine-pos-iff*:
fixes f :: *real* $\Rightarrow$ $'a$::{*banach*, *second-countable-topology*} **and** a c t :: *real*
assumes $c > 0$
shows *set-integrable lborel* $\{(a-t)/c..\}$ $(\lambda x.\ f\ (t + c*x))$
$\longleftrightarrow$ *set-integrable lborel* $\{a..\}$ f
⟨*proof*⟩

corollary *set-integrable-Ici-shift*:
fixes f :: *real* $\Rightarrow$ $'a$::{*banach*, *second-countable-topology*} **and** a t :: *real*
shows *set-integrable lborel* $\{a-t..\}$ $(\lambda x.\ f\ (t+x)) \longleftrightarrow$ *set-integrable lborel* $\{a..\}$ f
⟨*proof*⟩

lemma *set-integrable-Iic-affine-pos-iff*:
fixes f :: *real* $\Rightarrow$ $'a$::{*banach*, *second-countable-topology*} **and** b c t :: *real*
assumes $c > 0$
shows *set-integrable lborel* $\{..(b-t)/c\}$ $(\lambda x.\ f\ (t + c*x))$
$\longleftrightarrow$ *set-integrable lborel* $\{..b\}$ f
⟨*proof*⟩

corollary *set-integrable-Iic-shift*:
fixes f :: *real* $\Rightarrow$ $'a$::{*banach*, *second-countable-topology*} **and** b t :: *real*
shows *set-integrable lborel* $\{..b-t\}$ $(\lambda x.\ f\ (t+x)) \longleftrightarrow$ *set-integrable lborel* $\{..b\}$ f
⟨*proof*⟩

lemma *set-integrable-Icc-affine-neg-iff*:
fixes f :: *real* $\Rightarrow$ $'a$::{*banach*, *second-countable-topology*} **and** a b c t :: *real*
assumes $c < 0$
shows *set-integrable lborel* $\{(b-t)/c..(a-t)/c\}$ $(\lambda x.\ f\ (t + c*x))$
$\longleftrightarrow$ *set-integrable lborel* $\{a..b\}$ f
⟨*proof*⟩

corollary *set-integrable-Icc-reverse*:
fixes f :: *real* $\Rightarrow$ $'a$::{*banach*, *second-countable-topology*} **and** a b t :: *real*
shows *set-integrable lborel* $\{t-b..t-a\}$ $(\lambda x.\ f\ (t-x)) \longleftrightarrow$ *set-integrable lborel* $\{a..b\}$ f
⟨*proof*⟩

lemma *set-integrable-Ici-affine-neg-iff*:

fixes $f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$ **and** $b\ c\ t :: real$
assumes $c < 0$
shows *set-integrable lborel* $\{(b-t)/c..\}$ $(\lambda x.\ f\ (t + c*x))$
$\longleftrightarrow$ *set-integrable lborel* $\{..b\}$ f
⟨*proof*⟩

corollary *set-integrable-Ici-reverse*:
fixes $f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$ **and** $b\ t :: real$
shows *set-integrable lborel* $\{t-b..\}$ $(\lambda x.\ f\ (t-x)) \longleftrightarrow$ *set-integrable lborel* $\{..b\}$ f
⟨*proof*⟩

lemma *set-integrable-Iic-affine-neg-iff*:
fixes $f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ c\ t :: real$
assumes $c < 0$
shows *set-integrable lborel* $\{..(a-t)/c\}$ $(\lambda x.\ f\ (t + c*x))$
$\longleftrightarrow$ *set-integrable lborel* $\{a..\}$ f
⟨*proof*⟩

corollary *set-integrable-Iic-reverse*:
fixes $f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ t :: real$
shows *set-integrable lborel* $\{..t-a\}$ $(\lambda x.\ f\ (t-x)) \longleftrightarrow$ *set-integrable lborel* $\{a..\}$ f
⟨*proof*⟩

2.2 Set Lebesgue Integral on Affine Transformation

lemma *lborel-set-integral-Icc-affine-pos*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ b\ c :: real$
assumes $c > 0$
shows $(\int x{\in}\{a..b\}.\ f\ x\ \partial lborel) = c *_R (\int x{\in}\{(a-t)/c..(b-t)/c\}.\ f\ (t + c*x)\ \partial lborel)$
⟨*proof*⟩

corollary *lborel-set-integral-Icc-shift*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ b :: real$
shows $(\int x{\in}\{a..b\}.\ f\ x\ \partial lborel) = (\int x{\in}\{a-t..b-t\}.\ f\ (t+x)\ \partial lborel)$
⟨*proof*⟩

lemma *lborel-set-integral-Ici-affine-pos*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ c :: real$
assumes $c > 0$
shows $(\int x{\in}\{a..\}.\ f\ x\ \partial lborel) = c *_R (\int x{\in}\{(a-t)/c..\}.\ f\ (t + c*x)\ \partial lborel)$
⟨*proof*⟩

corollary *lborel-set-integral-Ici-shift*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a::real$
shows $(\int x{\in}\{a..\}.\ f\ x\ \partial lborel) = (\int x{\in}\{a-t..\}.\ f\ (t+x)\ \partial lborel)$
⟨*proof*⟩

lemma *lborel-set-integral-Iic-affine-pos*:

fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $b\ c :: real$
assumes $c > 0$
shows $(\int x{\in}\{..b\}.\ f\ x\ \partial lborel) = c *_R (\int x{\in}\{..(b-t)/c\}.\ f\ (t + c*x)\ \partial lborel)$
⟨*proof*⟩

corollary *lborel-set-integral-Iic-shift*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $b::real$
shows $(\int x{\in}\{..b\}.\ f\ x\ \partial lborel) = (\int x{\in}\{..b-t\}.\ f\ (t+x)\ \partial lborel)$
⟨*proof*⟩

lemma *lborel-set-integral-Icc-affine-neg*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ b\ c :: real$
assumes $c < 0$
shows $(\int x{\in}\{a..b\}.\ f\ x\ \partial lborel) = -c *_R (\int x{\in}\{(b-t)/c..(a-t)/c\}.\ f\ (t + c*x)\ \partial lborel)$
⟨*proof*⟩

corollary *lborel-set-integral-Icc-reverse*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ b :: real$
shows $(\int x{\in}\{a..b\}.\ f\ x\ \partial lborel) = (\int x{\in}\{t-b..t-a\}.\ f\ (t-x)\ \partial lborel)$
⟨*proof*⟩

lemma *lborel-set-integral-Ici-affine-neg*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $b\ c :: real$
assumes $c < 0$
shows $(\int x{\in}\{..b\}.\ f\ x\ \partial lborel) = -c *_R (\int x{\in}\{(b-t)/c..\}.\ f\ (t + c*x)\ \partial lborel)$
⟨*proof*⟩

corollary *lborel-set-integral-Ici-reverse*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $b::real$
shows $(\int x{\in}\{..b\}.\ f\ x\ \partial lborel) = (\int x{\in}\{t-b..\}.\ f\ (t-x)\ \partial lborel)$
⟨*proof*⟩

lemma *lborel-set-integral-Iic-affine-neg*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ c :: real$
assumes $c < 0$
shows $(\int x{\in}\{a..\}.\ f\ x\ \partial lborel) = -c *_R (\int x{\in}\{..(a-t)/c\}.\ f\ (t + c*x)\ \partial lborel)$
⟨*proof*⟩

corollary *lborel-set-integral-Iic-reverse*:
fixes $f :: real \Rightarrow 'a :: \{banach, second\text{-}countable\text{-}topology\}$ **and** $a::real$
shows $(\int x{\in}\{a..\}.\ f\ x\ \partial lborel) = (\int x{\in}\{..t-a\}.\ f\ (t-x)\ \partial lborel)$
⟨*proof*⟩

lemma *set-integrable-Ici-equiv-aux*:
fixes $f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ b :: real$
assumes $\bigwedge c\ d.$ *set-integrable lborel* $\{c..d\}\ f\ a \le b$
shows *set-integrable lborel* $\{a..\}\ f \longleftrightarrow$ *set-integrable lborel* $\{b..\}\ f$
⟨*proof*⟩

corollary *set-integrable-Ici-equiv*:
fixes $f :: real \Rightarrow 'a::\{banach, second\text{-}countable\text{-}topology\}$ **and** $a\ b :: real$
assumes $\bigwedge c\ d.\ set\text{-}integrable\ lborel\ \{c..d\}\ f$
shows $set\text{-}integrable\ lborel\ \{a..\}\ f \longleftrightarrow set\text{-}integrable\ lborel\ \{b..\}\ f$
⟨*proof*⟩

lemma *set-integrable-Iic-equiv*:
fixes $f :: real \Rightarrow real$ **and** $a\ b :: real$
assumes $\bigwedge c\ d.\ set\text{-}integrable\ lborel\ \{c..d\}\ f$
shows $set\text{-}integrable\ lborel\ \{..a\}\ f \longleftrightarrow set\text{-}integrable\ lborel\ \{..b\}\ f$ (**is** *?LHS* $\longleftrightarrow$ *?RHS*)
⟨*proof*⟩

2.3 Alternative Integral Test

lemma *nn-integral-suminf-Ico-real-nat*:
fixes $a::real$ **and** $f :: real \Rightarrow ennreal$
assumes $f \in borel\text{-}measurable\ lborel$
shows $(\int^+ x{\in}\{a..\}.\ f\ x\ \partial lborel) = (\sum k.\ \int^+ x{\in}\{a+k..<a+k+1\}.\ f\ x\ \partial lborel)$
⟨*proof*⟩

lemma *set-integrable-iff-bounded*:
fixes $f :: 'a \Rightarrow 'b::\{banach, second\text{-}countable\text{-}topology\}$
assumes $A \in sets\ M$
shows $set\text{-}integrable\ M\ A\ f \longleftrightarrow set\text{-}borel\text{-}measurable\ M\ A\ f \wedge (\int^+ x{\in}A.\ norm\ (f\ x)\ \partial M) < \infty$
⟨*proof*⟩

theorem *set-integrable-iff-summable*:
fixes $a::real$ **and** $f :: real \Rightarrow real$
assumes $antimono\text{-}on\ \{a..\}\ f \bigwedge x.\ a \le x \Longrightarrow f\ x \ge 0\ f \in borel\text{-}measurable\ lborel$
shows $set\text{-}integrable\ lborel\ \{a..\}\ f \longleftrightarrow summable\ (\lambda k.\ f\ (a+k))$
⟨*proof*⟩

2.4 Interchange of Differentiation and Lebesgue Integration

definition *measurable-extension* :: $'a\ measure \Rightarrow 'b\ measure \Rightarrow ('a \Rightarrow 'b) \Rightarrow 'a \Rightarrow 'b$ **where**
measurable-extension M N f =
(*SOME g*. $g \in M \to_M N \wedge (\exists S{\in}(null\text{-}sets\ M).\ \{x \in space\ M.\ f\ x \neq g\ x\} \subseteq S)$)
— The term *measurable-extension* is proposed by Reynald Affeldt.
— This function is used to make an almost-everywhere-defined function measurable.

lemma
fixes $f\ g$
assumes $g \in M \to_M N\ S \in null\text{-}sets\ M\ \{x \in space\ M.\ f\ x \neq g\ x\} \subseteq S$

shows *measurable-extensionI*: *AE x in M. f x = measurable-extension M N f x*
and
measurable-extensionI2: *AE x in M. g x = measurable-extension M N f x* **and**
measurable-extension-measurable: *measurable-extension M N f ∈ measurable M N*
⟨*proof*⟩

corollary *measurable-measurable-extension-AE*:
fixes *f*
assumes *f ∈ M →M N*
shows *AE x in M. f x = measurable-extension M N f x*
⟨*proof*⟩

definition *borel-measurable-extension* ::
'a measure ⇒ ('a ⇒ 'b::topological-space) ⇒ 'a ⇒ 'b **where**
borel-measurable-extension M f = measurable-extension M borel f

lemma
fixes *f g*
assumes *g ∈ borel-measurable M S ∈ null-sets M {x ∈ space M. f x ≠ g x} ⊆ S*
shows *borel-measurable-extensionI*: *AE x in M. f x = borel-measurable-extension M f x* **and**
borel-measurable-extensionI2: *AE x in M. g x = borel-measurable-extension M f x* **and**
borel-measurable-extension-measurable: *borel-measurable-extension M f ∈ borel-measurable M*
⟨*proof*⟩

corollary *borel-measurable-measurable-extension-AE*:
fixes *f*
assumes *f ∈ borel-measurable M*
shows *AE x in M. f x = borel-measurable-extension M f x*
⟨*proof*⟩

definition *set-borel-measurable-extension* ::
'a measure ⇒ 'a set ⇒ ('a ⇒ 'b::topological-space) ⇒ 'a ⇒ 'b
where *set-borel-measurable-extension M A f = borel-measurable-extension (restrict-space M A) f*

lemma
fixes *f g :: 'a ⇒ 'b::real-normed-vector* **and** *A*
assumes *A ∈ sets M set-borel-measurable M A g S ∈ null-sets M {x ∈ A. f x ≠ g x} ⊆ S*
shows *set-borel-measurable-extensionI*:
AE x∈A in M. f x = set-borel-measurable-extension M A f x **and**
set-borel-measurable-extensionI2:
AE x∈A in M. g x = set-borel-measurable-extension M A f x **and**
set-borel-measurable-extension-measurable:
set-borel-measurable M A (set-borel-measurable-extension M A f)

⟨*proof*⟩

corollary *set-borel-measurable-measurable-extension-AE*:
fixes $f::'a \Rightarrow 'b::$*real-normed-vector* **and** A
assumes *set-borel-measurable M A f A* $\in$ *sets M*
shows *AE x*$\in$*A in M*. *f x* = *set-borel-measurable-extension M A f x*
⟨*proof*⟩

proposition *interchange-deriv-LINT-general*:
fixes *a b* :: *real* **and** *f* :: *real* $\Rightarrow$ $'a$ $\Rightarrow$ *real* **and** *g* :: $'a$ $\Rightarrow$ *real*
assumes *f-integ*: $\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ *integrable M* (*f r*) **and**
f-diff: *AE x in M*. (*λr*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
Df-bound: *AE x in M*. $\forall$ *r*$\in${*a*<..<*b*}. |*deriv* (*λr*. *f r x*) *r*| $\leq$ *g x integrable M g*
shows $\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ ((*λr*. $\int$ *x*. *f r x* ∂*M*) *has-real-derivative*
$\int$ *x*. *borel-measurable-extension M* (*λx*. *deriv* (*λr*. *f r x*) *r*) *x* ∂*M*) (*at r*)
⟨*proof*⟩

proposition *interchange-deriv-LINT*:
fixes *a b* :: *real* **and** *f* :: *real* $\Rightarrow$ $'a$ $\Rightarrow$ *real* **and** *g* :: $'a$ $\Rightarrow$ *real*
assumes $\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ *integrable M* (*f r*) **and**
AE x in M. (*λr*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
$\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ (*λx*. (*deriv* (*λr*. *f r x*) *r*)) $\in$ *borel-measurable M* **and**
AE x in M. $\forall$ *r*$\in${*a*<..<*b*}. |*deriv* (*λr*. *f r x*) *r*| $\leq$ *g x integrable M g*
shows $\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ ((*λr*. $\int$ *x*. *f r x* ∂*M*) *has-real-derivative*
$\int$ *x*. *deriv* (*λr*. *f r x*) *r* ∂*M*) (*at r*)
⟨*proof*⟩

proposition *interchange-deriv-LINT-set-general*:
fixes *a b* :: *real* **and** *f* :: *real* $\Rightarrow$ $'a$ $\Rightarrow$ *real* **and** *g* :: $'a$ $\Rightarrow$ *real* **and** *A* :: $'a$ *set*
assumes *A-msr*: *A* $\in$ *sets M* **and**
f-integ: $\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ *set-integrable M A* (*f r*) **and**
f-diff: *AE x*$\in$*A in M*. (*λr*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
Df-bound: *AE x*$\in$*A in M*. $\forall$ *r*$\in${*a*<..<*b*}. |*deriv* (*λr*. *f r x*) *r*| $\leq$ *g x set-integrable M A g*
shows $\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ ((*λr*. $\int$ *x*$\in$*A*. *f r x* ∂*M*) *has-real-derivative*
($\int$ *x*$\in$*A*. *set-borel-measurable-extension M A* (*λx*. *deriv* (*λr*. *f r x*) *r*) *x* ∂*M*))
(*at r*)
⟨*proof*⟩

proposition *interchange-deriv-LINT-set*:
fixes *a b* :: *real* **and** *f* :: *real* $\Rightarrow$ $'a$ $\Rightarrow$ *real* **and** *g* :: $'a$ $\Rightarrow$ *real* **and** *A* :: $'a$ *set*
assumes *A* $\in$ *sets M* **and**
$\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ *set-integrable M A* (*f r*) **and**
AE x$\in$*A in M*. (*λr*. *f r x*) *differentiable-on* {*a*<..<*b*} **and**
$\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ *set-borel-measurable M A* (*λx*. (*deriv* (*λr*. *f r x*) *r*)) **and**
AE x$\in$*A in M*. $\forall$ *r*$\in${*a*<..<*b*}. |*deriv* (*λr*. *f r x*) *r*| $\leq$ *g x set-integrable M A g*
shows $\bigwedge$*r*. *r*$\in${*a*<..<*b*} $\Longrightarrow$ ((*λr*. $\int$ *x*$\in$*A*. *f r x* ∂*M*) *has-real-derivative*
($\int$ *x*$\in$*A*. *deriv* (*λr*. *f r x*) *r* ∂*M*)) (*at r*)
⟨*proof*⟩

3 Additional Lemmas for the *HOL−Probability* Library

lemma (**in** *finite-borel-measure*)
  fixes $F :: real \Rightarrow real$
  assumes *nondecF* : $\bigwedge x\ y.\ x \leq y \Longrightarrow F\ x \leq F\ y$ **and**
    right-cont-F : $\bigwedge a.$ *continuous* (*at-right a*) F **and**
    lim-F-at-bot : $(F \longrightarrow 0)$ *at-bot* **and**
    lim-F-at-top : $(F \longrightarrow m)$ *at-top* **and**
    $m : 0 \leq m$
  shows *emeasure-interval-measure-Ioi*: *emeasure* (*interval-measure F*) $\{x<..\} = m - F\ x$
    and *measure-interval-measure-Ioi*: *measure* (*interval-measure F*) $\{x<..\} = m - F\ x$
⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-nonneg*[*simp*]: *cond-prob M P Q* ≥ 0
  ⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-whole-1*: *cond-prob M P P* $= 1$ **if** *prob* $\{\omega \in space\ M.\ P\ \omega\} \neq 0$
  ⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-0-null*: *cond-prob M P Q* $= 0$ **if** *prob* $\{\omega \in space\ M.\ Q\ \omega\} = 0$
  ⟨*proof*⟩

lemma (**in** *prob-space*) *cond-prob-AE-prob*:
  assumes $\{\omega \in space\ M.\ P\ \omega\} \in events$ $\{\omega \in space\ M.\ Q\ \omega\} \in events$
    and *AE* ω *in M*. $Q\ \omega$
  shows *cond-prob M P Q* $=$ *prob* $\{\omega \in space\ M.\ P\ \omega\}$
⟨*proof*⟩

3.1 More Properties of *cdf*'s

context *finite-borel-measure*
begin

lemma *cdf-diff-eq2*:
  assumes $x \leq y$
  shows *cdf M y* $-$ *cdf M x* $=$ *measure M* $\{x<..y\}$
⟨*proof*⟩

end

context *prob-space*
begin

lemma *cdf-distr-measurable* [*measurable*]:

assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel X*) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *cdf-distr-P*:
assumes *random-variable borel X*
shows *cdf* (*distr M borel X*) *x* = $\mathcal{P}$(ω *in M*. *X* ω ≤ *x*)
⟨*proof*⟩

lemma *cdf-continuous-distr-P-lt*:
assumes *random-variable borel X isCont* (*cdf* (*distr M borel X*)) *x*
shows *cdf* (*distr M borel X*) *x* = $\mathcal{P}$(ω *in M*. *X* ω < *x*)
⟨*proof*⟩

lemma *cdf-distr-diff-P*:
assumes *x* ≤ *y*
and *random-variable borel X*
shows *cdf* (*distr M borel X*) *y* − *cdf* (*distr M borel X*) *x* = $\mathcal{P}$(ω *in M*. *x* < *X* ω ∧ *X* ω ≤ *y*)
⟨*proof*⟩

lemma *cdf-distr-max*:
fixes *c*::*real*
assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel* (λ*x*. *max* (*X x*) *c*)) *u* = *cdf* (*distr M borel X*) *u* ∗ *indicator* {*c*..} *u*
⟨*proof*⟩

lemma *cdf-distr-min*:
fixes *c*::*real*
assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel* (λ*x*. *min* (*X x*) *c*)) *u* =
1 − (1 − *cdf* (*distr M borel X*) *u*) ∗ *indicator* {..<*c*} *u*
⟨*proof*⟩

lemma *cdf-distr-floor-P*:
fixes *X* :: 'a ⇒ *real*
assumes [*measurable*]: *random-variable borel X*
shows *cdf* (*distr M borel* (λ*x*. ⌊*X x*⌋)) *u* = $\mathcal{P}$(*x in M*. *X x* < ⌊*u*⌋ + 1)
⟨*proof*⟩

lemma *expectation-nonneg-tail*:
assumes [*measurable*]: *random-variable borel X*
and *X-nonneg*: ⋀*x*. *x* ∈ *space M* ⟹ *X x* ≥ 0
defines *F u* ≡ *cdf* (*distr M borel X*) *u*
shows (∫⁺*x*. *ennreal* (*X x*) ∂*M*) = (∫⁺*u*∈{0..}. *ennreal* (1 − *F u*) ∂*lborel*)
⟨*proof*⟩

lemma *expectation-nonpos-tail*:

assumes [*measurable*]: *random-variable borel X*
and *X-nonpos*: $\bigwedge x.\ x \in space\ M \Longrightarrow X\ x \leq 0$
defines $F\ u \equiv cdf\ (distr\ M\ borel\ X)\ u$
shows $(\int^+ x.\ ennreal\ (-\ X\ x)\ \partial M) = (\int^+ u{\in}\{..0\}.\ ennreal\ (F\ u)\ \partial lborel)$
⟨*proof*⟩

proposition *expectation-tail*:
assumes [*measurable*]: *integrable M X*
defines $F\ u \equiv cdf\ (distr\ M\ borel\ X)\ u$
shows $expectation\ X = (LBINT\ u{:}\{0..\}.\ 1\ -\ F\ u) - (LBINT\ u{:}\{..0\}.\ F\ u)$
⟨*proof*⟩

proposition *distributed-deriv-cdf*:
assumes [*measurable*]: *random-variable borel X*
defines $F\ u \equiv cdf\ (distr\ M\ borel\ X)\ u$
assumes *finite S* $\bigwedge x.\ x \notin S \Longrightarrow (F\ has\text{-}real\text{-}derivative\ f\ x)\ (at\ x)$
and *continuous-on UNIV F f* $\in$ *borel-measurable lborel*
shows *distributed M lborel X f*
⟨*proof*⟩

end

Define the conditional probability space. This is obtained by rescaling the restricted space of a probability space.

3.2 Conditional Probability Space

lemma *restrict-prob-space*:
assumes *measure-space* Ω *A* μ *a* $\in$ *A*
and $0 < \mu\ a\ \mu\ a < \infty$
shows *prob-space* (*scale-measure* (*1 / μ a*) (*restrict-space* (*measure-of* Ω *A* μ) *a*))
⟨*proof*⟩

definition *cond-prob-space* :: *'a measure* ⇒ *'a set* ⇒ *'a measure* (**infix** ‹⌊› *200*)
where $M \lfloor A \equiv scale\text{-}measure\ (1\ /\ emeasure\ M\ A)\ (restrict\text{-}space\ M\ A)$

context *prob-space*
begin

lemma *cond-prob-space-whole*[*simp*]: $M \lfloor space\ M = M$
⟨*proof*⟩

lemma *cond-prob-space-correct*:
assumes $A \in events\ prob\ A > 0$
shows *prob-space* $(M \lfloor A)$
⟨*proof*⟩

lemma *space-cond-prob-space*:

assumes $A \in$ *events*
shows *space* $(M\lfloor A) = A$
⟨*proof*⟩

lemma *sets-cond-prob-space*: *sets* $(M\lfloor A) = (\cap)\ A$ ‘ *events*
⟨*proof*⟩

lemma *measure-cond-prob-space*:
assumes $A \in$ *events* $B \in$ *events*
and *prob* $A > 0$
shows *measure* $(M\lfloor A)\ (B \cap A) =$ *prob* $(B \cap A)$ / *prob* A
⟨*proof*⟩

corollary *measure-cond-prob-space-subset*:
assumes $A \in$ *events* $B \in$ *events* $B \subseteq A$
and *prob* $A > 0$
shows *measure* $(M\lfloor A)\ B =$ *prob* B / *prob* A
⟨*proof*⟩

lemma *cond-cond-prob-space*:
assumes $A \in$ *events* $B \in$ *events* $B \subseteq A$ *prob* $B > 0$
shows $(M\lfloor A)\lfloor B = M\lfloor B$
⟨*proof*⟩

lemma *cond-prob-space-prob*:
assumes[*measurable*]: *Measurable.pred M P Measurable.pred M Q*
and $\mathcal{P}(x$ *in* $M.\ Q\ x) > 0$
shows *measure* $(M \lfloor \{x \in$ *space* $M.\ Q\ x\})\ \{x \in$ *space* $M.\ P\ x \wedge Q\ x\} = \mathcal{P}(x$ *in* $M.\ P\ x \mid Q\ x)$
⟨*proof*⟩

lemma *cond-prob-space-cond-prob*:
assumes [*measurable*]: *Measurable.pred M P Measurable.pred M Q*
and $\mathcal{P}(x$ *in* $M.\ Q\ x) > 0$
shows $\mathcal{P}(x$ *in* $M.\ P\ x \mid Q\ x) = \mathcal{P}(x$ *in* $(M \lfloor \{x \in$ *space* $M.\ Q\ x\}).\ P\ x)$
⟨*proof*⟩

lemma *cond-prob-neg*:
assumes[*measurable*]: *Measurable.pred M P Measurable.pred M Q*
and $\mathcal{P}(x$ *in* $M.\ Q\ x) > 0$
shows $\mathcal{P}(x$ *in* $M.\ \neg\ P\ x \mid Q\ x) = 1 - \mathcal{P}(x$ *in* $M.\ P\ x \mid Q\ x)$
⟨*proof*⟩

lemma *random-variable-cond-prob-space*:
assumes $A \in$ *events prob* $A > 0$
and [*measurable*]: *random-variable borel X*
shows $X \in$ *borel-measurable* $(M\lfloor A)$
⟨*proof*⟩

lemma *AE-cond-prob-space-iff*:
assumes $A \in$ *events prob* $A > 0$
shows $(AE\ x\ in\ M\lfloor A.\ P\ x) \longleftrightarrow (AE\ x\ in\ M.\ x \in A \longrightarrow P\ x)$
⟨*proof*⟩

lemma *integral-cond-prob-space-nn*:
assumes $A \in$ *events prob* $A > 0$
and [*measurable*]: *random-variable borel* X
and *nonneg*: $AE\ x\ in\ M.\ x \in A \longrightarrow 0 \leq X\ x$
shows $integral^L\ (M\lfloor A)\ X = expectation\ (\lambda x.\ indicator\ A\ x * X\ x)\ /\ prob\ A$
⟨*proof*⟩

end

Define the complementary cumulative distribution function, also known as tail distribution. The theory presented below is a slight modification of the subsection "Properties of cdf's" in the theory *Distribution-Functions*.

3.3 Complementary Cumulative Distribution Function

definition *ccdf* :: *real measure* ⇒ *real* ⇒ *real*
where $ccdf\ M \equiv \lambda x.\ measure\ M\ \{x<..\}$
— complementary cumulative distribution function (tail distribution)

lemma *ccdf-def2*: $ccdf\ M\ x = measure\ M\ \{x<..\}$
⟨*proof*⟩

context *finite-borel-measure*
begin

lemma *add-cdf-ccdf*: $cdf\ M\ x + ccdf\ M\ x = measure\ M\ (space\ M)$
⟨*proof*⟩

lemma *ccdf-cdf*: $ccdf\ M = (\lambda x.\ measure\ M\ (space\ M) - cdf\ M\ x)$
⟨*proof*⟩

lemma *cdf-ccdf*: $cdf\ M = (\lambda x.\ measure\ M\ (space\ M) - ccdf\ M\ x)$
⟨*proof*⟩

lemma *isCont-cdf-ccdf*: $isCont\ (cdf\ M)\ x \longleftrightarrow isCont\ (ccdf\ M)\ x$
⟨*proof*⟩

lemma *isCont-ccdf*: $isCont\ (ccdf\ M)\ x \longleftrightarrow measure\ M\ \{x\} = 0$
⟨*proof*⟩

lemma *continuous-cdf-ccdf*:
shows $continuous\ F\ (cdf\ M) \longleftrightarrow continuous\ F\ (ccdf\ M)$
(**is** *?LHS* ⟷ *?RHS*)
⟨*proof*⟩

lemma *has-real-derivative-cdf-ccdf*:
(*cdf M has-real-derivative D*) *F* ⟷ (*ccdf M has-real-derivative* −*D*) *F*
⟨*proof*⟩

lemma *differentiable-cdf-ccdf*: (*cdf M differentiable F*) ⟷ (*ccdf M differentiable F*)
⟨*proof*⟩

lemma *deriv-cdf-ccdf*:
assumes *cdf M differentiable at x*
shows *deriv* (*cdf M*) *x* = − *deriv* (*ccdf M*) *x*
⟨*proof*⟩

lemma *ccdf-diff-eq2*:
assumes *x* ≤ *y*
shows *ccdf M x* − *ccdf M y* = *measure M* {*x*<..*y*}
⟨*proof*⟩

lemma *ccdf-nonincreasing*: *x* ≤ *y* ⟹ *ccdf M x* ≥ *ccdf M y*
⟨*proof*⟩

lemma *ccdf-nonneg*: *ccdf M x* ≥ *0*
⟨*proof*⟩

lemma *ccdf-bounded*: *ccdf M x* ≤ *measure M* (*space M*)
⟨*proof*⟩

lemma *ccdf-lim-at-top*: (*ccdf M* ⟶ *0*) *at-top*
⟨*proof*⟩

lemma *ccdf-lim-at-bot*: (*ccdf M* ⟶ *measure M* (*space M*)) *at-bot*
⟨*proof*⟩

lemma *ccdf-is-right-cont*: *continuous* (*at-right a*) (*ccdf M*)
⟨*proof*⟩

end

context *prob-space*
begin

lemma *ccdf-distr-measurable* [*measurable*]:
assumes [*measurable*]: *random-variable borel X*
shows *ccdf* (*distr M borel X*) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *ccdf-distr-P*:
assumes *random-variable borel X*

shows *ccdf* (*distr M borel X*) $x = \mathcal{P}(\omega$ *in* $M.$ $X\ \omega > x)$
⟨*proof*⟩

lemma *ccdf-continuous-distr-P-ge*:
assumes *random-variable borel X isCont* (*ccdf* (*distr M borel X*)) x
shows *ccdf* (*distr M borel X*) $x = \mathcal{P}(\omega$ *in* $M.$ $X\ \omega \geq x)$
⟨*proof*⟩

lemma *ccdf-distr-diff-P*:
assumes $x \leq y$
and *random-variable borel X*
shows *ccdf* (*distr M borel X*) x − *ccdf* (*distr M borel X*) $y = \mathcal{P}(\omega$ *in* $M.$ $x < X\ \omega \wedge X\ \omega \leq y)$
⟨*proof*⟩

end

context *real-distribution*
begin

lemma *ccdf-bounded-prob*: $\bigwedge x.$ *ccdf M x* ≤ *1*
⟨*proof*⟩

lemma *ccdf-lim-at-bot-prob*: (*ccdf M* ⟶ *1*) *at-bot*
⟨*proof*⟩

end

Introduce the hazard rate. This notion will be used to define the force of mortality.

3.4 Hazard Rate

context *prob-space*
begin

definition *hazard-rate* :: (*'a* ⇒ *real*) ⇒ *real* ⇒ *real*
where *hazard-rate X t* ≡
Lim (*at-right 0*) (λ*dt*. $\mathcal{P}(x$ *in* $M.$ $t < X\ x \wedge X\ x \leq t + dt \mid X\ x > t)$ / *dt*)
— Here, X is supposed to be a random variable.

lemma *hazard-rate-0-ccdf-0*:
assumes [*measurable*]: *random-variable borel X*
and *ccdf* (*distr M borel X*) *t* = *0*
shows *hazard-rate X t* = *0*
— Note that division by *0* is calculated as *0* in Isabelle/HOL.
⟨*proof*⟩

lemma *hazard-rate-deriv-cdf*:

assumes [*measurable*]: *random-variable borel X*
and (*cdf* (*distr M borel X*)) *differentiable at t*
shows *hazard-rate X t* = *deriv* (*cdf* (*distr M borel X*)) *t* / *ccdf* (*distr M borel X*) *t*
⟨*proof*⟩

lemma *deriv-cdf-hazard-rate*:
assumes [*measurable*]: *random-variable borel X*
and (*cdf* (*distr M borel X*)) *differentiable at t*
shows *deriv* (*cdf* (*distr M borel X*)) *t* = *ccdf* (*distr M borel X*) *t* ∗ *hazard-rate X t*
⟨*proof*⟩

lemma *hazard-rate-deriv-ccdf*:
assumes [*measurable*]: *random-variable borel X*
and (*ccdf* (*distr M borel X*)) *differentiable at t*
shows *hazard-rate X t* = − *deriv* (*ccdf* (*distr M borel X*)) *t* / *ccdf* (*distr M borel X*) *t*
⟨*proof*⟩

lemma *deriv-ccdf-hazard-rate*:
assumes [*measurable*]: *random-variable borel X*
and (*ccdf* (*distr M borel X*)) *differentiable at t*
shows *deriv* (*ccdf* (*distr M borel X*)) *t* = − *ccdf* (*distr M borel X*) *t* ∗ *hazard-rate X t*
⟨*proof*⟩

lemma *hazard-rate-deriv-ln-ccdf*:
assumes [*measurable*]: *random-variable borel X*
and (*ccdf* (*distr M borel X*)) *differentiable at t*
and *ccdf* (*distr M borel X*) *t* ≠ *0*
shows *hazard-rate X t* = − *deriv* (λ*t*. *ln* (*ccdf* (*distr M borel X*) *t*)) *t*
⟨*proof*⟩

lemma *hazard-rate-has-integral*:
assumes [*measurable*]: *random-variable borel X*
and *t* ≤ *u*
and (*ccdf* (*distr M borel X*)) *piecewise-differentiable-on* {*t*<..<*u*}
and *continuous-on* {*t*..*u*} (*ccdf* (*distr M borel X*))
and ⋀*s*. *s* ∈ {*t*..*u*} ⟹ *ccdf* (*distr M borel X*) *s* ≠ *0*
shows
(*hazard-rate X has-integral ln* (*ccdf* (*distr M borel X*) *t* / *ccdf* (*distr M borel X*) *u*)) {*t*..*u*}
⟨*proof*⟩

corollary *hazard-rate-integrable*:
assumes [*measurable*]: *random-variable borel X*
and *t* ≤ *u*
and (*ccdf* (*distr M borel X*)) *piecewise-differentiable-on* {*t*<..<*u*}

and *continuous-on* $\{t..u\}$ (*ccdf* (*distr M borel X*))
and $\bigwedge s.\ s \in \{t..u\} \Longrightarrow$ *ccdf* (*distr M borel X*) $s \neq 0$
shows *hazard-rate X integrable-on* $\{t..u\}$
⟨*proof*⟩

lemma *ccdf-exp-cumulative-hazard*:
assumes [*measurable*]: *random-variable borel X*
and $t \leq u$
and (*ccdf* (*distr M borel X*)) *piecewise-differentiable-on* $\{t<..<u\}$
and *continuous-on* $\{t..u\}$ (*ccdf* (*distr M borel X*))
and $\bigwedge s.\ s \in \{t..u\} \Longrightarrow$ *ccdf* (*distr M borel X*) $s \neq 0$
shows *ccdf* (*distr M borel X*) *u* / *ccdf* (*distr M borel X*) *t* =
exp (− *integral* $\{t..u\}$ (*hazard-rate X*))
⟨*proof*⟩

lemma *hazard-rate-density-ccdf*:
assumes *distributed M lborel X f*
and $\bigwedge s.\ f\ s \geq 0\ t < u$ *continuous-on* $\{t..u\}$ *f*
shows *hazard-rate X t* = *f t* / *ccdf* (*distr M borel X*) *t*
⟨*proof*⟩

end

end
theory *Interest*
imports *Preliminaries*
begin

4 Theory of Interest

locale *interest* =
fixes *i* :: *real* — *i* stands for an interest rate.
assumes *v-futr-pos*: *1 + i > 0* — Assume that the future value is positive.
begin

definition *i-nom* :: *nat* ⇒ *real* (‹$*i*^{-}› [*0*] *200*)
where $*i*^{*m*} ≡ *m* ∗ ((*1*+*i*).^(*1*/*m*) − *1*) — nominal interest rate

definition *i-force* :: *real* (‹$δ› *200*)
where $δ ≡ *ln* (*1*+*i*) — force of interest

definition *d-nom* :: *nat* ⇒ *real* (‹$*d*^{-}› [*0*] *200*)
where $*d*^{*m*} ≡ $*i*^{*m*} / (*1* + $*i*^{*m*}/*m*) — discount rate

abbreviation *d-nom-yr* :: *real* (‹$*d*› *200*)
where $*d* ≡ $*d*^{*1*} — Post-fix *yr* stands for "year".

definition *v-pres* :: *real* (‹$*v*› *200*)
where $*v* ≡ *1* / (*1*+*i*) — present value factor

definition *ann* :: *nat* ⇒ *nat* ⇒ *real* (‹$a^{-}'--› [*0*,*101*] *200*)
  where $*a*^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. $*v*.^((*k*+*1*::*nat*)/*m*) / *m*
    — present value of an immediate annuity

abbreviation *ann-yr* :: *nat* ⇒ *real* (‹$*a*'--› [*101*] *200*)
  where $*a*-*n* ≡ $*a*^{*1*}-*n*

definition *acc* :: *nat* ⇒ *nat* ⇒ *real* (‹$*s*^{-}'--› [*0*,*101*] *200*)
  where $*s*^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. (*1*+*i*).^((*k*::*nat*)/*m*) / *m*
    — future value of an immediate annuity
    — The name *acc* stands for "accumulation".

abbreviation *acc-yr* :: *nat* ⇒ *real* (‹$*s*'--› *200*)
  where $*s*-*n* ≡ $*s*^{*1*}-*n*

definition *ann-due* :: *nat* ⇒ *nat* ⇒ *real* (‹$*a*''''^{-}'--› [*0*,*101*] *200*)
  where $*a*''^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. $*v*.^((*k*::*nat*)/*m*) / *m*
    — present value of an annuity-due

abbreviation *ann-due-yr* :: *nat* ⇒ *real* (‹$*a*''''--› [*101*] *200*)
  where $*a*''-*n* ≡ $*a*''^{*1*}-*n*

definition *acc-due* :: *nat* ⇒ *nat* ⇒ *real* (‹$*s*''''^{-}'--› [*0*,*101*] *200*)
  where $*s*''^{*m*}-*n* ≡ ∑ *k*<*n*∗*m*. (*1*+*i*).^((*k*+*1*::*nat*)/*m*) / *m*
    — future value of an annuity-due

abbreviation *acc-due-yr* :: *nat* ⇒ *real* (‹$*s*''''--› [*101*] *200*)
  where $*s*''-*n* ≡ $*s*''^{*1*}-*n*

definition *ann-cont* :: *real* ⇒ *real* (‹$*a*'''--› [*101*] *200*)
  where $*a*'-*n* ≡ *integral* {*0*..*n*} (λ*t*::*real*. $*v*.^*t*)
    — present value of a continuous annuity

definition *acc-cont* :: *real* ⇒ *real* (‹$*s*'''--› [*101*] *200*)
  where $*s*'-*n* ≡ *integral* {*0*..*n*} (λ*t*::*real*. (*1*+*i*).^*t*)
    — future value of a continuous annuity

definition *perp* :: *nat* ⇒ *real* (‹$*a*^{-}'-∞› [*0*] *200*)
  where $*a*^{*m*}-∞ ≡ *1* / $*i*^{*m*}
    — present value of a perpetual annuity

abbreviation *perp-yr* :: *real* (‹$*a*'-∞› *200*)
  where $*a*-∞ ≡ $*a*^{*1*}-∞

definition *perp-due* :: *nat* ⇒ *real* (‹$*a*''''^{-}'-∞› [*0*] *200*)
  where $*a*''^{*m*}-∞ ≡ *1* / $*d*^{*m*}
    — present value of a perpetual annuity-due

abbreviation *perp-due-yr* :: *real* (‹$a''''-∞› *200*)
  where $a''-∞ ≡ $a''^{1}-∞

definition *ann-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* (‹$'(I^{-}a')^{-}'--› [*0*,*0*,*101*] *200*)
  where $(I^{l}a)^{m}-n ≡ ∑k<n*m. $v.^((k+*1*::*nat*)/m) * ⌈l*(k+*1*::*nat*)/m⌉ / (l*m)
    — present value of an increasing annuity
    — This is my original definition.
    — Here, *l* represents the number of increments per unit time.

abbreviation *ann-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* (‹$'(Ia')^{-}'--› [*0*,*101*] *200*)
  where $(Ia)^{m}-n ≡ $(I^{1}a)^{m}-n
    — The post-fix *lvl* stands for "level".

abbreviation *ann-incr-yr* :: *nat* ⇒ *real* (‹$'(Ia')'--› [*101*] *200*)
  where $(Ia)-n ≡ $(Ia)^{1}-n

definition *acc-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* (‹$'(I^{-}s')^{-}'--› [*0*,*0*,*101*] *200*)
  where $(I^{l}s)^{m}-n ≡ ∑k<n*m. (*1*+i).^(n−(k+*1*::*nat*)/m) * ⌈l*(k+*1*::*nat*)/m⌉ / (l*m)
    — future value of an increasing annuity

abbreviation *acc-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* (‹$'(Is')^{-}'--› [*0*,*101*] *200*)
  where $(Is)^{m}-n ≡ $(I^{1}s)^{m}-n

abbreviation *acc-incr-yr* :: *nat* ⇒ *real* (‹$'(Is')'--› [*101*] *200*)
  where $(Is)-n ≡ $(Is)^{1}-n

definition *ann-due-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* (‹$'(I^{-}a'''')^{-}'--› [*0*,*0*,*101*] *200*)
  where $(I^{l}a'')^{m}-n ≡ ∑k<n*m. $v.^((k::*nat*)/m) * ⌈l*(k+*1*::*nat*)/m⌉ / (l*m)

abbreviation *ann-due-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* (‹$'(Ia'''')^{-}'--› [*0*,*101*] *200*)
  where $(Ia'')^{m}-n ≡ $(I^{1}a'')^{m}-n

abbreviation *ann-due-incr-yr* :: *nat* ⇒ *real* (‹$'(Ia'''')'--› [*101*] *200*)
  where $(Ia'')-n ≡ $(Ia'')^{1}-n

definition *acc-due-incr* :: *nat* ⇒ *nat* ⇒ *nat* ⇒ *real* (‹$'(I^{-}s'''')^{-}'--› [*0*,*0*,*101*] *200*)
  where $(I^{l}s'')^{m}-n ≡ ∑k<n*m. (*1*+i).^(n−(k::*nat*)/m) * ⌈l*(k+*1*::*nat*)/m⌉ / (l*m)

abbreviation *acc-due-incr-lvl* :: *nat* ⇒ *nat* ⇒ *real* (‹$'(Is'''')^{-}'--› [*0*,*101*] *200*)
  where $(Is'')^{m}-n ≡ $(I^{1}s'')^{m}-n

abbreviation *acc-due-incr-yr* :: *nat* ⇒ *real* (‹$′(*Is*′′′′)′--› [*101*] *200*)
  where $(*Is*′′)-*n* ≡ $(*Is*′′)^{*1*}-*n*

definition *perp-incr* :: *nat* ⇒ *nat* ⇒ *real* (‹$′(*I*^{-}*a*′)^{-}′-∞› [*0*,*0*] *200*)
  where $(*I*^{*l*}*a*)^{*m*}-∞ ≡ *lim* (λ*n*. $(*I*^{*l*}*a*)^{*m*}-*n*)

abbreviation *perp-incr-lvl* :: *nat* ⇒ *real* (‹$′(*Ia*′)^{-}′-∞› [*0*] *200*)
  where $(*Ia*)^{*m*}-∞ ≡ $(*I*^{*1*}*a*)^{*m*}-∞

abbreviation *perp-incr-yr* :: *real* (‹$′(*Ia*′)′-∞› *200*)
  where $(*Ia*)-∞ ≡ $(*Ia*)^{*1*}-∞

definition *perp-due-incr* :: *nat* ⇒ *nat* ⇒ *real* (‹$′(*I*^{-}*a*′′′′)^{-}′-∞› [*0*,*0*] *200*)
  where $(*I*^{*l*}*a*′′)^{*m*}-∞ ≡ *lim* (λ*n*. $(*I*^{*l*}*a*′′)^{*m*}-*n*)

abbreviation *perp-due-incr-lvl* :: *nat* ⇒ *real* (‹$′(*Ia*′′′′)^{-}′-∞› [*0*] *200*)
  where $(*Ia*′′)^{*m*}-∞ ≡ $(*I*^{*1*}*a*′′)^{*m*}-∞

abbreviation *perp-due-incr-yr* :: *real* (‹$′(*Ia*′′′′)′-∞› *200*)
  where $(*Ia*′′)-∞ ≡ $(*Ia*′′)^{*1*}-∞

lemma *v-futr-m-pos*: *1* + $*i*^{*m*}/*m* > *0* **if** *m* ≠ *0* **for** *m*::*nat*
  ⟨*proof*⟩

lemma *i-nom-1*[*simp*]: $*i*^{*1*} = *i*
  ⟨*proof*⟩

lemma *i-nom-eff*: (*1* + $*i*^{*m*}/*m*)^*m* = *1* + *i* **if** *m* ≠ *0* **for** *m*::*nat*
  ⟨*proof*⟩

lemma *i-nom-i*: *1* + $*i*^{*m*}/*m* = (*1*+*i*).^(*1*/*m*) **if** *m* ≠ *0* **for** *m*::*nat*
  ⟨*proof*⟩

lemma *i-nom-0-iff-i-0*: $*i*^{*m*} = *0* ⟷ *i* = *0* **if** *m* ≠ *0* **for** *m*::*nat*
⟨*proof*⟩

lemma *i-nom-pos-iff-i-pos*: $*i*^{*m*} > *0* ⟷ *i* > *0* **if** *m* ≠ *0* **for** *m*::*nat*
⟨*proof*⟩

lemma *e-delta*: *exp* ($δ) = *1* + *i*
  ⟨*proof*⟩

lemma *delta-0-iff-i-0*: $δ = *0* ⟷ *i* = *0*
⟨*proof*⟩

lemma *lim-i-nom*: (λ*m*. $*i*^{*m*}) ⟶ $δ
⟨*proof*⟩

lemma *d-nom-0-iff-i-0*: $*d*^{*m*} = *0* ⟷ *i* = *0* **if** *m* ≠ *0* **for** *m*::*nat*

⟨*proof*⟩

lemma *d-nom-pos-iff-i-pos*: $\$d\hat{}\{m\} > 0 \longleftrightarrow i > 0$ **if** $m \neq 0$ **for** m::*nat*
⟨*proof*⟩

lemma *d-nom-i-nom*: $1 - \$d\hat{}\{m\}/m = 1 \,/\, (1 + \$i\hat{}\{m\}/m)$ **if** $m \neq 0$ **for** m::*nat*
⟨*proof*⟩

lemma *lim-d-nom*: $(\lambda m.\ \$d\hat{}\{m\}) \longrightarrow \$\delta$
⟨*proof*⟩

lemma *v-pos*: $\$v > 0$
⟨*proof*⟩

lemma *v-1-iff-i-0*: $\$v = 1 \longleftrightarrow i = 0$
⟨*proof*⟩

lemma *v-lt-1-iff-i-pos*: $\$v < 1 \longleftrightarrow i > 0$
⟨*proof*⟩

lemma *v-i-nom*: $\$v = (1 + \$i\hat{}\{m\}/m).\hat{}-m$ **if** $m \neq 0$ **for** m::*nat*
⟨*proof*⟩

lemma *i-v*: $1 + i = \$v.\hat{}-1$
⟨*proof*⟩

lemma *i-v-powr*: $(1 + i).\hat{}a = \$v.\hat{}-a$ **for** a::*real*
⟨*proof*⟩

lemma *v-delta*: $ln\ (\$v) = -\ \$\delta$
⟨*proof*⟩

lemma *is-derive-vpow*: $DERIV\ (\lambda t.\ \$v.\hat{}t)\ t :> -\ \$\delta * \$v.\hat{}t$
⟨*proof*⟩

lemma *d-nom-v*: $\$d\hat{}\{m\} = m * (1 - \$v.\hat{}(1/m))$ **if** $m \neq 0$ **for** m::*nat*
⟨*proof*⟩

lemma *d-nom-i-nom-v*: $\$d\hat{}\{m\} = \$i\hat{}\{m\} * \$v.\hat{}(1/m)$ **if** $m \neq 0$ **for** m::*nat*
⟨*proof*⟩

lemma *a-calc*: $\$a\hat{}\{m\}\text{-}n = (1 - \$v\hat{}n) \,/\, \$i\hat{}\{m\}$ **if** $m \neq 0\ i \neq 0$ **for** $n\ m$ ::*nat*
⟨*proof*⟩

lemma *a-calc-i-0*: $\$a\hat{}\{m\}\text{-}n = n$ **if** $m \neq 0\ i = 0$ **for** $n\ m$ :: *nat*
⟨*proof*⟩

lemma *s-calc-i-0*: $\$s\hat{}\{m\}\text{-}n = n$ **if** $m \neq 0\ i = 0$ **for** $n\ m$ :: *nat*
⟨*proof*⟩

lemma *s-a*: $s^{m}-n = (1+i)^n * $a^{m}-n **if** $m \neq 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *s-calc*: $s^{m}-n = ((1+i)^n - 1) / $i^{m} **if** $m \neq 0\ i \neq 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *a''-a*: $a''^{m}-n = (1+i).^(1/m) * $a^{m}-n **if** $m \neq 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *a-a''*: $a^{m}-n = $v.^(1/m) * $a''^{m}-n **if** $m \neq 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *a''-calc-i-0*: $a''^{m}-n = n$ **if** $m \neq 0\ i = 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *s''-calc-i-0*: $s''^{m}-n = n$ **if** $m \neq 0\ i = 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *a''-calc*: $a''^{m}-n = (1 - $v^n) / $d^{m} **if** $m \neq 0\ i \neq 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *s''-s*: $s''^{m}-n = (1+i).^(1/m) * $s^{m}-n **if** $m \neq 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *s-s''*: $s^{m}-n = $v.^(1/m) * $s''^{m}-n **if** $m \neq 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *s''-calc*: $s''^{m}-n = ((1+i)^n - 1) / $d^{m} **if** $m \neq 0\ i \neq 0$ **for** $n\ m :: nat$
⟨*proof*⟩

lemma *s''-a''*: $s''^{m}-n = (1+i)^n * $a''^{m}-n **if** $m \neq 0$ **for** *m*::*nat*
⟨*proof*⟩

lemma *a'-calc*: $a'-n = (1 - $v.^n) / δ **if** $i \neq 0\ n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *a'-calc-i-0*: $a'-n = n$ **if** $i = 0\ n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *s'-calc*: $s'-n = ((1+i).^n - 1) / δ **if** $i \neq 0\ n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *s'-calc-i-0*: $s'-n = n$ **if** $i = 0\ n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *s′-a′*: $*s′-n* = (*1+i*).^*n* * $*a′-n* **if** *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *lim-m-a*: (λ*m*. $*a*^{*m*}-*n*) ⟶ $*a′-n* **for** *n*::*nat*
⟨*proof*⟩

lemma *lim-m-a′′*: (λ*m*. $*a′′*^{*m*}-*n*) ⟶ $*a′-n* **for** *n*::*nat*
⟨*proof*⟩

lemma *lim-m-s*: (λ*m*. $*s*^{*m*}-*n*) ⟶ $*s′-n* **for** *n*::*nat*
⟨*proof*⟩

lemma *lim-m-s′′*: (λ*m*. $*s′′*^{*m*}-*n*) ⟶ $*s′-n* **for** *n*::*nat*
⟨*proof*⟩

lemma *lim-n-a*: (λ*n*. $*a*^{*m*}-*n*) ⟶ $*a*^{*m*}-∞ **if** *m* ≠ *0 i* > *0* **for** *m*::*nat*
⟨*proof*⟩

lemma *lim-n-a′′*: (λ*n*. $*a′′*^{*m*}-*n*) ⟶ $*a′′*^{*m*}-∞ **if** *m* ≠ *0 i* > *0* **for** *m*::*nat*
⟨*proof*⟩

lemma *Ilsm-Ilam*: $(*I*^{*l*}*s*)^{*m*}-*n* = (*1+i*)^*n* * $(*I*^{*l*}*a*)^{*m*}-*n*
if *l* ≠ *0 m* ≠ *0* **for** *l n m* :: *nat*
⟨*proof*⟩

lemma *Iam-calc*: $(*Ia*)^{*m*}-*n* = (∑ *j*<*n*. (*j+1*)/*m* * (∑ *k*=*j**m*..<(*j+1*)**m*. $*v*.^((*k+1*)/*m*)))
if *m* ≠ *0* **for** *n m* :: *nat*
⟨*proof*⟩

lemma *Ism-calc*: $(*Is*)^{*m*}-*n* = (∑ *j*<*n*. (*j+1*)/*m* * (∑ *k*=*j**m*..<(*j+1*)**m*. (*1+i*).^(*n*−(*k+1*)/*m*)))
if *m* ≠ *0* **for** *n m* :: *nat*
⟨*proof*⟩

lemma *Imam-calc-aux*: $(*I*^{*m*}*a*)^{*m*}-*n* = (∑ *k*<*n***m*. $*v*.^((*k+1*)/*m*) * (*k+1*)
/ *m*^*2*)
if *m* ≠ *0* **for** *m*::*nat*
⟨*proof*⟩

lemma *Imam-calc*:
$(*I*^{*m*}*a*)^{*m*}-*n* = ($*v*.^(*1*/*m*) * (*1* − (*n***m*+*1*)*$*v*^*n* + *n***m**$*v*.^(*n*+*1*/*m*)))
/ (*m**(*1*−$*v*.^(*1*/*m*)))^*2*
if *i* ≠ *0 m* ≠ *0* **for** *n m* :: *nat*
⟨*proof*⟩

lemma *Imam-calc-i-0*: $(*I*^{*m*}*a*)^{*m*}-*n* = (*n***m*+*1*)**n* / (*2***m*) **if** *i* = *0 m* ≠ *0*
for *n m* :: *nat*
⟨*proof*⟩

lemma *Imsm-calc*:

$(I^{m}s)^{m}-n = ((1+i).^(n+1/m) - (n*m+1)*(1+i).^(1/m) + n*m) / (m*((1+i).^(1/m)-1))^2$
if $i \neq 0$ $m \neq 0$ **for** n m :: *nat*
⟨*proof*⟩

lemma *Imsm-calc-i-0*: $(I^{m}s)^{m}-n = (n*m+1)*n / (2*m)$ **if** $i = 0$ $m \neq 0$ **for** n m :: *nat*
⟨*proof*⟩

lemma *Ila''m-Ilam*: $(I^{l}a'')^{m}-n = (1+i).^(1/m) * (I^{l}a)^{m}-n$
if $l \neq 0$ $m \neq 0$ **for** l m n :: *nat*
⟨*proof*⟩

lemma *Ia''m-calc*: $(Ia'')^{m}-n = (\sum j<n.\ (j+1)/m * (\sum k=j*m..<(j+1)*m.\ v.^(k/m)))$
if $m \neq 0$ **for** n m :: *nat*
⟨*proof*⟩

lemma *Ima''m-calc-aux*: $(I^{m}a'')^{m}-n = (\sum k<n*m.\ v.^(k/m) * (k+1) / m^2)$
if $m \neq 0$ **for** m::*nat*
⟨*proof*⟩

lemma *Ima''m-calc*: $(I^{m}a'')^{m}-n = (1 - (n*m+1)*v^n + n*m*v.^(n+1/m)) / (m*(1-v.^(1/m)))^2$
if $i \neq 0$ $m \neq 0$ **for** n m :: *nat*
⟨*proof*⟩

lemma *Ils''m-Ilsm*: $(I^{l}s'')^{m}-n = (1+i).^(1/m) * (I^{l}s)^{m}-n$
if $l \neq 0$ $m \neq 0$ **for** l m n :: *nat*
⟨*proof*⟩

lemma *Ims''m-calc*:
$(I^{m}s'')^{m}-n =$
$(1+i).^(1/m) * ((1+i).^(n+1/m) - (n*m+1)*(1+i).^(1/m) + n*m) / (m*((1+i).^(1/m)-1))^2$
if $i \neq 0$ $m \neq 0$ **for** n m :: *nat*
⟨*proof*⟩

lemma *lim-Imam*: $(\lambda n.\ (I^{m}a)^{m}-n) \longrightarrow 1 / (i^{m}*d^{m})$ **if** $m \neq 0$ $i > 0$ **for** m::*nat*
⟨*proof*⟩

lemma *perp-incr-calc*: $(I^{m}a)^{m}-\infty = 1 / (i^{m}*d^{m})$ **if** $m \neq 0$ $i > 0$ **for** m::*nat*
⟨*proof*⟩

lemma *lim-Ima''m*: $(\lambda n.\ (I^{m}a'')^{m}-n) \longrightarrow 1 / (d^{m})^2$ **if** $m \neq 0$ $i > 0$ **for** m::*nat*

⟨*proof*⟩

lemma *perp-due-incr-calc*: $(I^{m}a'')^{m}-\infty = 1 / ($d^{m})^2$ **if** $m \neq 0$ $i > 0$ **for** m::*nat*
⟨*proof*⟩

end

end
theory *Survival-Model*
imports *HOL−Library.Rewrite HOL−Library.Extended-Nonnegative-Real HOL−Library.Extended-Real HOL−Probability.Probability Preliminaries*
begin

5 Survival Model

The survival model is built on the probability space $\mathfrak{M}$. Additionally, the random variable X : *space* $\mathfrak{M} \to \mathbb{R}$ is introduced, which represents the age at death.

locale *prob-space-actuary* = *MM-PS*: *prob-space* $\mathfrak{M}$ **for** $\mathfrak{M}$
— Since the letter M may be used as a commutation function, adopt the letter $\mathfrak{M}$ instead as a notation for the measure space.

locale *survival-model* = *prob-space-actuary* +
fixes X :: $'a \Rightarrow$ *real*
assumes *X-RV*[*simp*]: *MM-PS.random-variable* (*borel* :: *real measure*) X
and *X-pos-AE*[*simp*]: *AE* ξ *in* $\mathfrak{M}$. $X\ \xi > 0$
begin

5.1 General Theory of Survival Model

interpretation *distrX-RD*: *real-distribution distr* $\mathfrak{M}$ *borel X*
⟨*proof*⟩

lemma *X-le-event*[*simp*]: $\{\xi \in$ *space* $\mathfrak{M}$. $X\ \xi \leq x\} \in$ *MM-PS.events*
⟨*proof*⟩

lemma *X-gt-event*[*simp*]: $\{\xi \in$ *space* $\mathfrak{M}$. $X\ \xi > x\} \in$ *MM-PS.events*
⟨*proof*⟩

lemma *X-compl-le-gt*: *space* $\mathfrak{M} - \{\xi \in$ *space* $\mathfrak{M}$. $X\ \xi \leq x\} = \{\xi \in$ *space* $\mathfrak{M}$. $X\ \xi > x\}$ **for** x::*real*
⟨*proof*⟩

lemma *X-compl-gt-le*: *space* $\mathfrak{M} - \{\xi \in$ *space* $\mathfrak{M}$. $X\ \xi > x\} = \{\xi \in$ *space* $\mathfrak{M}$. $X\ \xi \leq x\}$ **for** x::*real*
⟨*proof*⟩

5.1.1 Introduction of Survival Function for X

Note that *ccdf* (*distr* $\mathfrak{M}$ *borel* X) is the survival (distributive) function for X.

lemma *ccdfX-0-1*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) $0 = 1$
⟨*proof*⟩

lemma *ccdfX-unborn-1*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) $x = 1$ **if** $x \leq 0$
⟨*proof*⟩

definition *death-pt* :: *ereal* (‹$\$\psi$›)
where $\$\psi \equiv Inf$ (*ereal* ' $\{x \in \mathbb{R}.$ *ccdf* (*distr* $\mathfrak{M}$ *borel* X) $x = 0\}$)
— This is my original notation, which is used to develop life insurance mathematics rigorously.

lemma *psi-nonneg*: $\$\psi \geq 0$
⟨*proof*⟩

lemma *ccdfX-beyond-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) $x = 0$ **if** $x > \$\psi$ **for** x::*real*
⟨*proof*⟩

lemma *ccdfX-psi-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) (*real-of-ereal* $\$\psi$) $= 0$ **if** $\$\psi < \infty$
⟨*proof*⟩

lemma *ccdfX-0-equiv*: *ccdf* (*distr* $\mathfrak{M}$ *borel* X) $x = 0 \longleftrightarrow x \geq \$\psi$ **for** x::*real*
⟨*proof*⟩

lemma *psi-pos*[*simp*]: $\$\psi > 0$
⟨*proof*⟩

corollary *psi-pos'*[*simp*]: $\$\psi >$ *ereal* 0
⟨*proof*⟩

5.1.2 Introdution of Future-Lifetime Random Variable $T(x)$

definition *alive* :: *real* ⇒ *'a set*
where *alive* $x \equiv \{\xi \in$ *space* $\mathfrak{M}.$ $X\ \xi > x\}$

lemma *alive-event*[*simp*]: *alive* $x \in$ *MM-PS.events* **for** x::*real*
⟨*proof*⟩

lemma *X-alivex-measurable*[*measurable*, *simp*]: $X \in$ *borel-measurable* ($\mathfrak{M}$ ⌊ *alive* x) **for** x::*real*
⟨*proof*⟩

definition *futr-life* :: *real* ⇒ (*'a* ⇒ *real*) (‹T›)
where $T\ x \equiv (\lambda\xi.\ X\ \xi - x)$
— Note that $T(x)$: *space* $\mathfrak{M} \to \mathbb{R}$ represents the time until death of a person aged x.

lemma *T0-eq-X*[*simp*]: $T\ 0 = X$
⟨*proof*⟩

lemma *Tx-measurable*[*measurable*, *simp*]: $T\ x \in$ *borel-measurable* $\mathfrak{M}$ **for** x::*real*
⟨*proof*⟩

lemma *Tx-alivex-measurable*[*measurable*, *simp*]: $T\ x \in$ *borel-measurable* $(\mathfrak{M} \restriction$ *alive* $x)$ **for** x::*real*
⟨*proof*⟩

lemma *alive-T*: *alive* $x = \{\xi \in$ *space* $\mathfrak{M}.\ T\ x\ \xi > 0\}$ **for** x::*real*
⟨*proof*⟩

lemma *alivex-Tx-pos*[*simp*]: $0 < T\ x\ \xi$ **if** $\xi \in$ *space* $(\mathfrak{M} \restriction$ *alive* $x)$ **for** x::*real*
⟨*proof*⟩

lemma *PT0-eq-PX-lborel*: $\mathcal{P}(\xi$ *in* $\mathfrak{M}.\ T\ 0\ \xi \in A \mid T\ 0\ \xi > 0) = \mathcal{P}(\xi$ *in* $\mathfrak{M}.\ X\ \xi \in A)$
if $A \in$ *sets lborel* **for** A :: *real set*
⟨*proof*⟩

5.1.3 Actuarial Notations on the Survival Model

definition *survive* :: *real* ⇒ *real* ⇒ *real* (‹$p′-{-&-}› [0,0] 200)
where $p-{t&x} ≡ *ccdf* (*distr* (\mathfrak{M} ⌊ *alive* x) *borel* (T x)) t
— the probability that a person aged x will survive for t years
— Note that the function $p-{·&x} is the survival function on (\mathfrak{M} ⌊ *alive* x) for the random variable $T(x)$.
— The parameter t is usually nonnegative, but theoretically it can be negative.

abbreviation *survive-1* :: *real* ⇒ *real* (‹$p′--› [101] 200)
where $p-x ≡ $p-{1&x}

definition *die* :: *real* ⇒ *real* ⇒ *real* (‹$q′-{-&-}› [0,0] 200)
where $q-{t&x} ≡ *cdf* (*distr* (\mathfrak{M} ⌊ *alive* x) *borel* (T x)) t
— the probability that a person aged x will die within t years
— Note that the function $q-{·&x} is the cumulative distributive function on (\mathfrak{M} ⌊ *alive* x) for the random variable $T(x)$.
— The parameter t is usually nonnegative, but theoretically it can be negative.

abbreviation *die-1* :: *real* ⇒ *real* (‹$q′--› [101] 200)
where $q-x ≡ $q-{1&x}

definition *die-defer* :: *real* ⇒ *real* ⇒ *real* ⇒ *real* (‹$q′-{-|-&-}› [0,0,0] 200)
where $q-{f|t&x} = |$q-{f+t&x} − $q-{f&x}|
— the probability that a person aged x will die within t years, deferred f years
— The parameters f and t are usually nonnegative, but theoretically they can be negative.

abbreviation *die-defer-1* :: *real* ⇒ *real* ⇒ *real* (‹$*q*′-{-|&-}› [*0*,*0*] *200*)
where $*q*-{*f*|&*x*} ≡ $*q*-{*f*|*1*&*x*}

definition *life-expect* :: *real* ⇒ *real* (‹$*e*‘∘-› [*101*] *200*)
where $*e*‘∘-*x* ≡ *integral*^{L} (𝔐 ⌊ *alive x*) (*T x*)
— complete life expectation
— Note that $*e*‘∘-*x* is calculated as *0* when *nn-integral* (𝔐 ⌊ *alve x*) (*T x*) = ∞.

definition *temp-life-expect* :: *real* ⇒ *real* ⇒*real* (‹$*e*‘∘′-{-:-}› [*0*,*0*] *200*)
where $*e*‘∘-{*x*:*n*} ≡ *integral*^{L} (𝔐 ⌊ *alive x*) (λξ. *min* (*T x* ξ) *n*)
— temporary complete life expectation

definition *curt-life-expect* :: *real* ⇒ *real* (‹$*e*′-› [*101*] *200*)
where $*e*-*x* ≡ *integral*^{L} (𝔐 ⌊ *alive x*) (λξ. ⌊*T x* ξ⌋)
— curtate life expectation
— Note that $*e*-*x* is calculated as *0* when *nn-integral* (𝔐 ⌊ *alive x*) ⌊*T x*⌋ = ∞.

definition *temp-curt-life-expect* :: *real* ⇒ *real* ⇒ *real* (‹$*e*′-{-:-}› [*0*,*0*] *200*)
where $*e*-{*x*:*n*} ≡ *integral*^{L} (𝔐 ⌊ *alive x*) (λξ. ⌊*min* (*T x* ξ) *n*⌋)
— temporary curtate life expectation
— In the definition *n* can be a real number, but in practice *n* is usually a natural number.

5.1.4 Properties of Survival Function for $T(x)$

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: *x* < $ψ
begin

lemma *PXx-pos*[*simp*]: 𝒫(ξ *in* 𝔐. *X* ξ > *x*) > *0*
⟨*proof*⟩

lemma *PTx-pos*[*simp*]: 𝒫(ξ *in* 𝔐. *T x* ξ > *0*) > *0*
⟨*proof*⟩

interpretation *alivex-PS*: *prob-space* 𝔐 ⌊ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* (𝔐 ⌊ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *ccdfTx-cond-prob*:
ccdf (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *t* = 𝒫(ξ *in* 𝔐. *T x* ξ > *t* | *T x* ξ > *0*) **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-0-1*: *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *0* = *1*

⟨*proof*⟩

lemma *ccdfTx-nonpos-1*: *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *t* = *1* **if** *t* ≤ *0* **for** *t* :: *real*
⟨*proof*⟩

lemma *ccdfTx-0-equiv*: *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *t* = *0* ⟷ *x*+*t* ≥ $*ψ* **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-continuous-on-nonpos*[*simp*]:
continuous-on {*..0*} (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

lemma *ccdfTx-differentiable-on-nonpos*[*simp*]:
(*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *differentiable-on* {*..0*}
⟨*proof*⟩

lemma *ccdfTx-has-real-derivative-0-at-neg*:
(*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *has-real-derivative 0*) (*at t*) **if** *t* < *0* **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-integrable-Icc*:
set-integrable lborel {*a..b*} (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) **for** *a b* :: *real*
⟨*proof*⟩

corollary *ccdfTx-integrable-on-Icc*:
ccdf (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *integrable-on* {*a..b*} **for** *a b* :: *real*
⟨*proof*⟩

lemma *ccdfTx-PX*:
ccdf (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *t* = 𝒫(*ξ in* 𝔐. *X ξ* > *x*+*t*) / 𝒫(*ξ in* 𝔐. *X ξ* > *x*)
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfTx-ccdfX*: *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *t* =
ccdf (*distr* 𝔐 *borel X*) (*x* + *t*) / *ccdf* (*distr* 𝔐 *borel X*) *x*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *ccdfT0-eq-ccdfX*: *ccdf* (*distr* (𝔐 ⌊ *alive 0*) *borel* (*T 0*)) = *ccdf* (*distr* 𝔐 *borel X*)
⟨*proof*⟩

lemma *continuous-ccdfX-ccdfTx*:
continuous (*at* (*x*+*t*) *within* {*x..*}) (*ccdf* (*distr* 𝔐 *borel X*)) ⟷
continuous (*at t within* {*0..*}) (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)))

if $t \geq 0$ **for** t::*real*
⟨*proof*⟩

lemma *isCont-ccdfX-ccdfTx*:
$isCont\ (ccdf\ (distr\ \mathfrak{M}\ borel\ X))\ (x+t) \longleftrightarrow$
$isCont\ (ccdf\ (distr\ (\mathfrak{M} \restriction alive\ x)\ borel\ (T\ x)))\ t$
if $t > 0$ **for** t::*real*
⟨*proof*⟩

lemma *has-real-derivative-ccdfX-ccdfTx*:
$((ccdf\ (distr\ \mathfrak{M}\ borel\ X))\ has\text{-}real\text{-}derivative\ D)\ (at\ (x+t)) \longleftrightarrow$
$((ccdf\ (distr\ (\mathfrak{M} \restriction alive\ x)\ borel\ (T\ x)))\ has\text{-}real\text{-}derivative\ (D\ /\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ X\ \xi > x)))\ (at\ t)$
if $t > 0$ **for** $t\ D$ :: *real*
⟨*proof*⟩

lemma *differentiable-ccdfX-ccdfTx*:
$(ccdf\ (distr\ \mathfrak{M}\ borel\ X))\ differentiable\ at\ (x+t) \longleftrightarrow$
$(ccdf\ (distr\ (\mathfrak{M} \restriction alive\ x)\ borel\ (T\ x)))\ differentiable\ at\ t$
if $t > 0$ **for** t::*real*
⟨*proof*⟩

5.1.5 Properties of $\$p\text{-}\{t\&x\}$

lemma *p-0-1*: $\$p\text{-}\{0\&x\} = 1$
⟨*proof*⟩

lemma *p-nonneg*[*simp*]: $\$p\text{-}\{t\&x\} \geq 0$ **for** t::*real*
⟨*proof*⟩

lemma *p-le-1*[*simp*]: $\$p\text{-}\{t\&x\} \leq 1$ **for** t::*real*
⟨*proof*⟩

lemma *p-0-equiv*: $\$p\text{-}\{t\&x\} = 0 \longleftrightarrow x+t \geq \$\psi$ **for** t::*real*
⟨*proof*⟩

lemma *p-PTx*: $\$p\text{-}\{t\&x\} = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi > t \mid T\ x\ \xi > 0)$ **for** t::*real*
⟨*proof*⟩

lemma *p-PX*: $\$p\text{-}\{t\&x\} = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ X\ \xi > x + t)\ /\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ X\ \xi > x)$ **if** $t \geq 0$ **for** t::*real*
⟨*proof*⟩

lemma *p-mult*: $\$p\text{-}\{t+t'\ \&\ x\} = \$p\text{-}\{t\&x\} * \$p\text{-}\{t'\ \&\ x+t\}$
if $t \geq 0\ t' \geq 0\ x+t < \$\psi$ **for** $t\ t'$ :: *real*
⟨*proof*⟩

lemma *p-PTx-ge-ccdf-isCont*: $\$p\text{-}\{t\&x\} = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi \geq t \mid T\ x\ \xi > 0)$
if $isCont\ (ccdf\ (distr\ \mathfrak{M}\ borel\ X))\ (x+t)\ t > 0$ **for** t::*real*

⟨*proof*⟩

end

5.1.6 Properties of Survival Function for X

lemma *ccdfX-continuous-unborn*[*simp*]: *continuous-on* {..*0*} (*ccdf* (*distr* 𝔐 *borel* *X*))
⟨*proof*⟩

lemma *ccdfX-differentiable-unborn*[*simp*]: (*ccdf* (*distr* 𝔐 *borel X*)) *differentiable-on* {..*0*}
⟨*proof*⟩

lemma *ccdfX-has-real-derivative-0-unborn*:
(*ccdf* (*distr* 𝔐 *borel X*) *has-real-derivative 0*) (*at x*) **if** $x < 0$ **for** *x*::*real*
⟨*proof*⟩

lemma *ccdfX-integrable-Icc*:
set-integrable lborel {*a*..*b*} (*ccdf* (*distr* 𝔐 *borel X*)) **for** *a b* :: *real*
⟨*proof*⟩

corollary *ccdfX-integrable-on-Icc*:
ccdf (*distr* 𝔐 *borel X*) *integrable-on* {*a*..*b*} **for** *a b* :: *real*
⟨*proof*⟩

lemma *ccdfX-p*: *ccdf* (*distr* 𝔐 *borel X*) *x* = $*p*-{*x*&*0*} **for** *x*::*real*
⟨*proof*⟩

5.1.7 Introduction of Cumulative Distributive Function for X

lemma *cdfX-0-0*: *cdf* (*distr* 𝔐 *borel X*) *0* = *0*
⟨*proof*⟩

lemma *cdfX-unborn-0*: *cdf* (*distr* 𝔐 *borel X*) *x* = *0* **if** $x \leq 0$
⟨*proof*⟩

lemma *cdfX-beyond-1*: *cdf* (*distr* 𝔐 *borel X*) *x* = *1* **if** $x > \$\psi$ **for** *x*::*real*
⟨*proof*⟩

lemma *cdfX-psi-1*: *cdf* (*distr* 𝔐 *borel X*) (*real-of-ereal* ψ) = *1* **if** $\$\psi < \infty$
⟨*proof*⟩

lemma *cdfX-1-equiv*: *cdf* (*distr* 𝔐 *borel X*) *x* = *1* $\longleftrightarrow$ $x \geq \$\psi$ **for** *x*::*real*
⟨*proof*⟩

5.1.8 Properties of Cumulative Distributive Function for $T(x)$

context
fixes *x*::*real*

assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

interpretation *alivex-PS*: *prob-space* $\mathfrak{M} \restriction$ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *cdfTx-cond-prob*:
cdf (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)) *t* = $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *T x* $\xi \leq t \mid T\ x\ \xi > 0)$ **for** *t*::*real*
⟨*proof*⟩

lemma *cdfTx-0-0*: *cdf* (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)) *0* = *0*
⟨*proof*⟩

lemma *cdfTx-nonpos-0*: *cdf* (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)) *t* = *0* **if** $t \leq 0$ **for** *t* :: *real*
⟨*proof*⟩

lemma *cdfTx-1-equiv*: *cdf* (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)) *t* = *1* $\longleftrightarrow$ *x*+*t* $\geq \$\psi$ **for** *t*::*real*
⟨*proof*⟩

lemma *cdfTx-continuous-on-nonpos*[*simp*]:
continuous-on {*..0*} (*cdf* (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

lemma *cdfTx-differentiable-on-nonpos*[*simp*]:
(*cdf* (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*))) *differentiable-on* {*..0*}
⟨*proof*⟩

lemma *cdfTx-has-real-derivative-0-at-neg*:
(*cdf* (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)) *has-real-derivative 0*) (*at t*) **if** *t* < *0* **for** *t*::*real*
⟨*proof*⟩

lemma *cdfTx-integrable-Icc*:
set-integrable lborel {*a..b*} (*cdf* (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*))) **for** *a b* :: *real*
⟨*proof*⟩

corollary *cdfTx-integrable-on-Icc*:
cdf (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)) *integrable-on* {*a..b*} **for** *a b* :: *real*
⟨*proof*⟩

lemma *cdfTx-PX*:
cdf (*distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*)) *t* = $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *x* < *X* $\xi \wedge$ *X* $\xi \leq$ *x*+*t*) / $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *X* ξ > *x*)
for *t*::*real*

⟨*proof*⟩

lemma *cdfT0-eq-cdfX*: *cdf* (*distr* (𝔐 ⌊ *alive* 0) *borel* (*T* 0)) = *cdf* (*distr* 𝔐 *borel* *X*)
⟨*proof*⟩

lemma *continuous-cdfX-cdfTx*:
continuous (*at* (*x*+*t*) *within* {*x*..}) (*cdf* (*distr* 𝔐 *borel* *X*)) ⟷
continuous (*at* *t* *within* {0..}) (*cdf* (*distr* (𝔐 ⌊ *alive* *x*) *borel* (*T* *x*)))
if *t* ≥ 0 **for** *t*::*real*
⟨*proof*⟩

lemma *isCont-cdfX-cdfTx*:
isCont (*cdf* (*distr* 𝔐 *borel* *X*)) (*x*+*t*) ⟷
isCont (*cdf* (*distr* (𝔐 ⌊ *alive* *x*) *borel* (*T* *x*))) *t*
if *t* > 0 **for** *t*::*real*
⟨*proof*⟩

lemma *has-real-derivative-cdfX-cdfTx*:
((*cdf* (*distr* 𝔐 *borel* *X*)) *has-real-derivative* *D*) (*at* (*x*+*t*)) ⟷
((*cdf* (*distr* (𝔐 ⌊ *alive* *x*) *borel* (*T* *x*))) *has-real-derivative* (*D* / 𝒫(*ξ* *in* 𝔐. *X* *ξ* > *x*))) (*at* *t*)
if *t* > 0 **for** *t* *D* :: *real*
⟨*proof*⟩

lemma *differentiable-cdfX-cdfTx*:
(*cdf* (*distr* 𝔐 *borel* *X*)) *differentiable at* (*x*+*t*) ⟷
(*cdf* (*distr* (𝔐 ⌊ *alive* *x*) *borel* (*T* *x*))) *differentiable at* *t*
if *t* > 0 **for** *t*::*real*
⟨*proof*⟩

5.1.9 Properties of $q-{t&x}

lemma *q-nonpos-0*: $q-{*t*&*x*} = 0 **if** *t* ≤ 0 **for** *t*::*real*
⟨*proof*⟩

corollary *q-0-0*: $q-{0&*x*} = 0
⟨*proof*⟩

lemma *q-nonneg*[*simp*]: $q-{*t*&*x*} ≥ 0 **for** *t*::*real*
⟨*proof*⟩

lemma *q-le-1*[*simp*]: $q-{*t*&*x*} ≤ 1 **for** *t*::*real*
⟨*proof*⟩

lemma *q-1-equiv*: $q-{*t*&*x*} = 1 ⟷ *x*+*t* ≥ $*ψ* **for** *t*::*real*
⟨*proof*⟩

lemma *q-PTx*: $q-{*t*&*x*} = 𝒫(*ξ* *in* 𝔐. *T* *x* *ξ* ≤ *t* | *T* *x* *ξ* > 0) **for** *t*::*real*

⟨*proof*⟩

lemma *q-PX*: $\$q\text{-}\{t\&x\} = \mathcal{P}(\xi \ in \ \mathfrak{M}.\ x < X\ \xi \wedge X\ \xi \leq x + t)\ /\ \mathcal{P}(\xi \ in \ \mathfrak{M}.\ X\ \xi > x)$
⟨*proof*⟩

lemma *q-defer-0-q*[*simp*]: $\$q\text{-}\{0|t\&x\} = \$q\text{-}\{t\&x\}$ **for** *t*::*real*
⟨*proof*⟩

lemma *q-defer-0-0*: $\$q\text{-}\{f|0\&x\} = 0$ **for** *f*::*real*
⟨*proof*⟩

lemma *q-defer-nonneg*[*simp*]: $\$q\text{-}\{f|t\&x\} \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-q*: $\$q\text{-}\{f|t\&x\} = \$q\text{-}\{f{+}t\ \&\ x\} - \$q\text{-}\{f\&x\}$ **if** $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

corollary *q-defer-le-1*[*simp*]: $\$q\text{-}\{f|t\&x\} \leq 1$ **if** $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-PTx*: $\$q\text{-}\{f|t\&x\} = \mathcal{P}(\xi \ in \ \mathfrak{M}.\ f < T\ x\ \xi \wedge T\ x\ \xi \leq f + t \mid T\ x\ \xi > 0)$
if $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-PX*: $\$q\text{-}\{f|t\&x\} = \mathcal{P}(\xi \ in \ \mathfrak{M}.\ x + f < X\ \xi \wedge X\ \xi \leq x + f + t)\ /\ \mathcal{P}(\xi \ in \ \mathfrak{M}.\ X\ \xi > x)$
if $f \geq 0\ t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-old-0*: $\$q\text{-}\{f|t\&x\} = 0$ **if** $x{+}f \geq \$\psi\ t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

end

5.1.10 Properties of Cumulative Distributive Function for X

lemma *cdfX-continuous-unborn*[*simp*]: *continuous-on* {..*0*} (*cdf* (*distr* $\mathfrak{M}$ *borel X*))
⟨*proof*⟩

lemma *cdfX-differentiable-unborn*[*simp*]: (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable-on* {..*0*}
⟨*proof*⟩

lemma *cdfX-has-real-derivative-0-unborn*:
(*cdf* (*distr* $\mathfrak{M}$ *borel X*) *has-real-derivative 0*) (*at x*) **if** $x < 0$ **for** *x*::*real*
⟨*proof*⟩

lemma *cdfX-integrable-Icc*:
set-integrable lborel $\{a..b\}$ (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) **for** *a b* :: *real*
⟨*proof*⟩

corollary *cdfX-integrable-on-Icc*:
cdf (*distr* $\mathfrak{M}$ *borel X*) *integrable-on* $\{a..b\}$ **for** *a b* :: *real*
⟨*proof*⟩

lemma *cdfX-q*: *cdf* (*distr* $\mathfrak{M}$ *borel X*) $x = \$q\text{-}\{x\&0\}$ **if** $x \geq 0$ **for** *x*::*real*
⟨*proof*⟩

5.1.11 Relations between $\$p\text{-}\{t\&x\}$ and $\$q\text{-}\{t\&x\}$

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

interpretation *alivex-PS*: *prob-space* $\mathfrak{M} \restriction$ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *p-q-1*: $\$p\text{-}\{t\&x\} + \$q\text{-}\{t\&x\} = 1$ **for** *t*::*real*
⟨*proof*⟩

lemma *q-defer-p*: $\$q\text{-}\{f|t\&x\} = \$p\text{-}\{f\&x\} - \$p\text{-}\{f{+}t\ \&\ x\}$ **if** $t \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

lemma *q-defer-p-q-defer*: $\$p\text{-}\{f\&x\} * \$q\text{-}\{f'|t\ \&\ x{+}f\} = \$q\text{-}\{f{+}f'|t\ \&\ x\}$
if $x{+}f < \$\psi$ $f \geq 0$ $f' \geq 0$ $t \geq 0$ **for** *f f' t* :: *real*
⟨*proof*⟩

lemma *q-defer-pq*: $\$q\text{-}\{f|t\&x\} = \$p\text{-}\{f\&x\} * \$q\text{-}\{t\ \&\ x{+}f\}$
if $x{+}f < \$\psi$ $t \geq 0$ $f \geq 0$ **for** *f t* :: *real*
⟨*proof*⟩

5.1.12 Properties of Life Expectation

lemma *e-nonneg*: $\$e\text{`}\circ\text{-}x \geq 0$
⟨*proof*⟩

lemma *e-P*: $\$e\text{`}\circ\text{-}x =$
MM-PS.expectation ($\lambda\xi$. *indicator* (*alive x*) ξ * *T x* ξ) / $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *T x* $\xi > 0)$
⟨*proof*⟩

proposition *nn-integral-T-p*:
$(\int^+\xi.$ *ennreal* (*T x* ξ) $\partial(\mathfrak{M} \restriction$ *alive x*)) = $(\int^+ t{\in}\{0..\}.$ *ennreal* ($\$p\text{-}\{t\&x\}$) *∂lborel*)
⟨*proof*⟩

lemma *nn-integral-T-pos*: $(\int^+\xi.\ ennreal\ (T\ x\ \xi)\ \partial(\mathfrak{M} \restriction alive\ x)) > 0$
⟨*proof*⟩

lemma *e-pos-Tx*: $\$e`\circ x > 0$ **if** *integrable* $(\mathfrak{M} \restriction alive\ x)\ (T\ x)$
⟨*proof*⟩

proposition *e-LBINT-p*: $\$e`\circ x = (LBINT\ t{:}\{0..\}.\ \$p\text{-}\{t\&x\})$
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-integral-p*: $\$e`\circ x = integral\ \{0..\}\ (\lambda t.\ \$p\text{-}\{t\&x\})$
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

lemma *e-pos*: $\$e`\circ x > 0$ **if** *set-integrable lborel* $\{0..\}\ (\lambda t.\ \$p\text{-}\{t\&x\})$
⟨*proof*⟩

corollary *e-pos'*: $\$e`\circ x > 0$ **if** $(\lambda t.\ \$p\text{-}\{t\&x\})$ *integrable-on* $\{0..\}$
⟨*proof*⟩

lemma *e-LBINT-p-Icc*: $\$e`\circ x = (LBINT\ t{:}\{0..n\}.\ \$p\text{-}\{t\&x\})$ **if** $x{+}n \geq \$\psi$ **for** *n*::*real*
⟨*proof*⟩

lemma *e-integral-p-Icc*: $\$e`\circ x = integral\ \{0..n\}\ (\lambda t.\ \$p\text{-}\{t\&x\})$ **if** $x{+}n \geq \$\psi$ **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-le-n*: $\$e`\circ\text{-}\{x{:}n\} \leq n$ **if** $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-P*: $\$e`\circ\text{-}\{x{:}n\} =$
MM-PS.expectation $(\lambda\xi.\ indicator\ (alive\ x)\ \xi * min\ (T\ x\ \xi)\ n)\ /\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi > 0)$
if $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-LBINT-p*: $\$e`\circ\text{-}\{x{:}n\} = (LBINT\ t{:}\{0..n\}.\ \$p\text{-}\{t\&x\})$ **if** $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-integral-p*: $\$e`\circ\text{-}\{x{:}n\} = integral\ \{0..n\}\ (\lambda t.\ \$p\text{-}\{t\&x\})$ **if** $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *e-eq-temp*: $\$e`\circ x = \$e`\circ\text{-}\{x{:}n\}$ **if** $n \geq 0\ x{+}n \geq \$\psi$ **for** *n*::*real*
⟨*proof*⟩

lemma *curt-e-P*: $\$e\text{-}x =$

MM-PS.expectation ($\lambda\xi$. *indicator* (*alive x*) ξ * $\lfloor$*T x* $\xi\rfloor$) / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ > *0*)
⟨*proof*⟩

lemma *curt-e-sum-P*: $e-x = ($\sum$ k. $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ ≥ *k* + *1* | *T x* ξ > *0*))
if *summable* (λk. $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ ≥ *k* + *1* | *T x* ξ > *0*))
⟨*proof*⟩

lemma *curt-e-sum-P-finite*: $e-x = ($\sum$ k<n. $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ ≥ *k* + *1* | *T x* ξ > *0*))
if *x*+*n*+*1* > ψ **for** *n*::*nat*
⟨*proof*⟩

lemma *curt-e-sum-p*: $e-x = ($\sum$ k. $p-{*k*+*1*&*x*})
if *summable* (λk. $p-{*k*+*1*&*x*}) $\bigwedge$k::*nat*. *isCont* (λt. $p-{*t*&*x*}) (*k*+*1*)
⟨*proof*⟩

lemma *curt-e-rec*: $e-x = $p-x * (*1* + $e-(*x*+*1*))
if *summable* (λk. $p-{*k*+*1*&*x*}) $\bigwedge$k::*nat*. *isCont* (λt. $p-{*t*&*x*}) (*real k* + *1*) *x*+*1* < ψ
⟨*proof*⟩

lemma *curt-e-sum-p-finite*: $e-x = ($\sum$ k<n. $p-{*k*+*1*&*x*})
if $\bigwedge$k::*nat*. *k* < *n* ⟹ *isCont* (λt. $p-{*t*&*x*}) (*real k* + *1*) *x*+*n*+*1* > ψ **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-P*: $e-{*x*:*n*} =
MM-PS.expectation ($\lambda\xi$. *indicator* (*alive x*) ξ * $\lfloor$*min* (*T x* ξ) *n*$\rfloor$) / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ > *0*)
if *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *temp-curt-e-sum-P*: $e-{*x*:*n*} = ($\sum$ k<n. $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ ≥ *k* + *1* | *T x* ξ > *0*)) **for** *n*::*nat*
⟨*proof*⟩

corollary *curt-e-eq-temp*: $e-x = $e-{*x*:*n*} **if** *x*+*n*+*1* > ψ **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-sum-p*: $e-{*x*:*n*} = ($\sum$ k<n. $p-{*k*+*1*&*x*})
if $\bigwedge$k::*nat*. *k* < *n* ⟹ *isCont* (λt. $p-{*t*&*x*}) (*real k* + *1*) **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-rec*: $e-{*x*:*n*} = $p-x * (*1* + $e-{*x*+*1*:*n*−*1*})
if $\bigwedge$k::*nat*. *k* < *n* ⟹ *isCont* (λt. $p-{*t*&*x*}) (*real k* + *1*) *x*+*1* < ψ *n* ≠ *0* **for** *n*::*nat*
⟨*proof*⟩

end

lemma *p-set-integrable-shift*:
set-integrable lborel {*0*..} (λ*t*. $*p*-{*t*&*0*}) ⟷ *set-integrable lborel* {*0*..} (λ*t*. $*p*-{*t*&*x*})
if *x* < $ψ **for** *x*::*real*
⟨*proof*⟩

lemma *e-p-e*: $*e*`∘-*x* = $*e*`∘-{*x*:*n*} + $*p*-{*n*&*x*} ∗ $*e*`∘-(*x*+*n*)
if *set-integrable lborel* {*0*..} (λ*t*. $*p*-{*t*&*x*}) *n* ≥ *0* *x*+*n* < $ψ **for** *x n* :: *real*
⟨*proof*⟩

proposition *x-ex-mono*: *x* + $*e*`∘-*x* ≤ *y* + $*e*`∘-*y* **if** *x* ≤ *y* *y* < $ψ **for** *x y* :: *real*
⟨*proof*⟩

proposition *x-ex-const-equiv*: *x* + $*e*`∘-*x* = *y* + $*e*`∘-*y* ⟷ $*q*-{*y*−*x*&*x*} = *0*
if *set-integrable lborel* {*0*..} (λ*t*. $*p*-{*t*&*0*}) *x* ≤ *y* *y* < $ψ **for** *x y* :: *real*
⟨*proof*⟩

end

5.2 Piecewise Differentiable Survival Function

locale *smooth-survival-function* = *survival-model* +
assumes *ccdfX-piecewise-differentiable*[*simp*]:
(*ccdf* (*distr* 𝔐 *borel X*)) *piecewise-differentiable-on UNIV*
begin

interpretation *distrX-RD*: *real-distribution distr* 𝔐 *borel X*
⟨*proof*⟩

5.2.1 Properties of Survival Function for X

lemma *ccdfX-continuous*[*simp*]: *continuous-on UNIV* (*ccdf* (*distr* 𝔐 *borel X*))
⟨*proof*⟩

corollary *ccdfX-borel-measurable*[*measurable*]: *ccdf* (*distr* 𝔐 *borel X*) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *ccdfX-nondifferentiable-finite-set*[*simp*]:
finite {*x*. ¬ *ccdf* (*distr* 𝔐 *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *ccdfX-differentiable-open-set*: *open* {*x*. *ccdf* (*distr* 𝔐 *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *ccdfX-differentiable-borel-set*[*measurable*, *simp*]:
{*x*. *ccdf* (*distr* 𝔐 *borel X*) *differentiable at x*} ∈ *sets borel*

⟨*proof*⟩

lemma *ccdfX-differentiable-AE*:
AE x in lborel. (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x*
⟨*proof*⟩

lemma *deriv-ccdfX-measurable*[*measurable*]: *deriv* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.2 Properties of Cumulative Distributive Function for X

lemma *cdfX-piecewise-differentiable*[*simp*]:
(*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *piecewise-differentiable-on UNIV*
⟨*proof*⟩

lemma *cdfX-continuous*[*simp*]: *continuous-on UNIV* (*cdf* (*distr* $\mathfrak{M}$ *borel X*))
⟨*proof*⟩

corollary *cdfX-borel-measurable*[*measurable*]: *cdf* (*distr* $\mathfrak{M}$ *borel X*) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *cdfX-nondifferentiable-finite-set*[*simp*]:
finite {*x*. ¬ *cdf* (*distr* $\mathfrak{M}$ *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *cdfX-differentiable-open-set*: *open* {*x*. *cdf* (*distr* $\mathfrak{M}$ *borel X*) *differentiable at x*}
⟨*proof*⟩

lemma *cdfX-differentiable-borel-set*[*measurable*, *simp*]:
{*x*. *cdf* (*distr* $\mathfrak{M}$ *borel X*) *differentiable at x*} ∈ *sets borel*
⟨*proof*⟩

lemma *cdfX-differentiable-AE*:
AE x in lborel. (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x*
⟨*proof*⟩

lemma *deriv-cdfX-measurable*[*measurable*]: *deriv* (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.3 Introduction of Probability Density Functions of X and $T(x)$

definition *pdfX* :: *real* ⇒ *real*
where *pdfX x* ≡ *if cdf* (*distr* $\mathfrak{M}$ *borel X*) *differentiable at x*
then deriv (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *x else 0*
— This function is defined to be always nonnegative for future application.

definition *pdfT* :: *real* ⇒ *real* ⇒ *real*
  where *pdfT x t* ≡ *if cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *differentiable at t*
    then deriv (*cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *t else 0*
    — This function is defined to be always nonnegative for future application.

lemma *pdfX-measurable*[*measurable*]: *pdfX* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *distributed-pdfX*: *distributed* 𝔐 *lborel X pdfX*
⟨*proof*⟩

lemma *pdfT0-X*: *pdfT 0* = *pdfX*
  ⟨*proof*⟩

5.2.4 Properties of Survival Function for $T(x)$

context
  fixes *x*::*real*
  assumes *x-lt-psi*[*simp*]: *x* < $ψ
begin

interpretation *alivex-PS*: *prob-space* 𝔐 ⌊ *alive x*
  ⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* (𝔐 ⌊ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *ccdfTx-continuous-on-nonneg*[*simp*]:
  continuous-on {*0*..} (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)))
  ⟨*proof*⟩

lemma *ccdfTx-continuous*[*simp*]: *continuous-on UNIV* (*ccdf* (*distr* (𝔐 ⌊ *alive x*)
borel (*T x*)))
⟨*proof*⟩

corollary *ccdfTx-borel-measurable*[*measurable*]:
  ccdf (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) ∈ *borel-measurable borel*
  ⟨*proof*⟩

lemma *ccdfTx-nondifferentiable-finite-set*[*simp*]:
  finite {*t*. ¬ *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *differentiable at t*}
⟨*proof*⟩

lemma *ccdfTx-differentiable-open-set*:
  open {*t*. *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *differentiable at t*}
  ⟨*proof*⟩

lemma *ccdfTx-differentiable-borel-set*[*measurable*, *simp*]:
  {*t*. *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *differentiable at t*} ∈ *sets borel*

⟨*proof*⟩

lemma *ccdfTx-differentiable-AE*:
AE t in lborel. (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *differentiable at t*
⟨*proof*⟩

lemma *ccdfTx-piecewise-differentiable*[*simp*]:
(*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *piecewise-differentiable-on UNIV*
⟨*proof*⟩

lemma *deriv-ccdfTx-measurable*[*measurable*]:
deriv (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.5 Properties of Cumulative Distributive Function for $T(x)$

lemma *cdfTx-continuous*[*simp*]:
continuous-on UNIV (*cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)))
⟨*proof*⟩

corollary *cdfTx-borel-measurable*[*measurable*]:
cdf (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *cdfTx-nondifferentiable-finite-set*[*simp*]:
finite {*t*. ¬ *cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *differentiable at t*}
⟨*proof*⟩

lemma *cdfTx-differentiable-open-set*:
open {*t*. *cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *differentiable at t*}
⟨*proof*⟩

lemma *cdfTx-differentiable-borel-set*[*measurable*, *simp*]:
{*t*. *cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *differentiable at t*} ∈ *sets borel*
⟨*proof*⟩

lemma *cdfTx-differentiable-AE*:
AE t in lborel. (*cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *differentiable at t*
⟨*proof*⟩

lemma *cdfTx-piecewise-differentiable*[*simp*]:
(*cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *piecewise-differentiable-on UNIV*
⟨*proof*⟩

lemma *deriv-cdfTx-measurable*[*measurable*]:
deriv (*cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) ∈ *borel-measurable borel*
⟨*proof*⟩

5.2.6 Properties of Probability Density Function of $T(x)$

lemma *pdfTx-nonneg*: *pdfT x t* $\geq$ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *pdfTx-neg-0*: *pdfT x t* = *0* **if** *t* < *0* **for** *t*::*real*
⟨*proof*⟩

lemma *pdfTx-0-0*: *pdfT x 0* = *0*
⟨*proof*⟩

lemma *pdfTx-nonpos-0*: *pdfT x t* = *0* **if** *t* $\leq$ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *pdfTx-beyond-0*: *pdfT x t* = *0* **if** *x*+*t* $\geq$ $\$\psi$ **for** *t*::*real*
⟨*proof*⟩

lemma *pdfTx-pdfX*: *pdfT x t* = *pdfX* (*x*+*t*) / $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *X* ξ > *x*) **if** *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *pdfTx-measurable*[*measurable*]: *pdfT x* $\in$ *borel-measurable borel*
⟨*proof*⟩

lemma *distributed-pdfTx*: *distributed* ($\mathfrak{M}$ $\downharpoonright$ *alive x*) *lborel* (*T x*) (*pdfT x*)
⟨*proof*⟩

lemma *nn-integral-pdfTx-1*: ($\int^+ s$. *pdfT x s* ∂*lborel*) = *1*
⟨*proof*⟩

corollary *has-bochner-integral-pdfTx-1*: *has-bochner-integral lborel* (*pdfT x*) *1*
⟨*proof*⟩

corollary *LBINT-pdfTx-1*: (*LBINT s*. *pdfT x s*) = *1*
⟨*proof*⟩

corollary *pdfTx-has-integral-1*: (*pdfT x has-integral 1*) *UNIV*
⟨*proof*⟩

lemma *set-nn-integral-pdfTx-1*: ($\int^+ s \in \{0..\}$. *pdfT x s* ∂*lborel*) = *1*
⟨*proof*⟩

corollary *has-bochner-integral-pdfTx-1-nonpos*:
has-bochner-integral lborel (λs. *pdfT x s* $*$ *indicator* $\{0..\}$ *s*) *1*
⟨*proof*⟩

corollary *set-LBINT-pdfTx-1*: (*LBINT s*:$\{0..\}$. *pdfT x s*) = *1*
⟨*proof*⟩

corollary *pdfTx-has-integral-1-nonpos*: (*pdfT x has-integral 1*) $\{0..\}$

⟨*proof*⟩

lemma *set-nn-integral-pdfTx-PTx*: $(\int^+ s{\in}A.\ pdfT\ x\ s\ \partial lborel) = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi \in A \mid T\ x\ \xi > 0)$
  if $A \in sets\ lborel$ **for** A :: *real set*
⟨*proof*⟩

lemma *pdfTx-set-integrable*: *set-integrable lborel A* (*pdfT x*) **if** $A \in sets\ lborel$
  ⟨*proof*⟩

lemma *set-integral-pdfTx-PTx*: $(LBINT\ s{:}A.\ pdfT\ x\ s) = \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi \in A \mid T\ x\ \xi > 0)$
  if $A \in sets\ lborel$ **for** A :: *real set*
  ⟨*proof*⟩

lemma *pdfTx-has-integral-PTx*: $(pdfT\ x\ has\text{-}integral\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ T\ x\ \xi \in A \mid T\ x\ \xi > 0))\ A$
  if $A \in sets\ lborel$ **for** A :: *real set*
⟨*proof*⟩

corollary *pdfTx-has-integral-PTx-Icc*:
  $(pdfT\ x\ has\text{-}integral\ \mathcal{P}(\xi\ in\ \mathfrak{M}.\ a \le T\ x\ \xi \wedge T\ x\ \xi \le b \mid T\ x\ \xi > 0))\ \{a..b\}$ **for** $a\ b$ :: *real*
  ⟨*proof*⟩

corollary *pdfTx-integrable-on-Icc*: *pdfT x integrable-on* $\{a..b\}$ **for** $a\ b$ :: *real*
  ⟨*proof*⟩

end

5.2.7 Properties of Probability Density Function of X

lemma *pdfX-nonneg*: $pdfX\ x \ge 0$ **for** *x*::*real*
  ⟨*proof*⟩

lemma *pdfX-nonpos-0*: $pdfX\ x = 0$ **if** $x \le 0$ **for** *x*::*real*
  ⟨*proof*⟩

lemma *pdfX-beyond-0*: $pdfX\ x = 0$ **if** $x \ge \$\psi$ **for** *x*::*real*
  ⟨*proof*⟩

lemma *nn-integral-pdfX-1*: $integral^N\ lborel\ pdfX = 1$
  ⟨*proof*⟩

corollary *has-bochner-integral-pdfX-1*: *has-bochner-integral lborel pdfX 1*
  ⟨*proof*⟩

corollary *LBINT-pdfX-1*: $(LBINT\ s.\ pdfX\ s) = 1$
  ⟨*proof*⟩

corollary *pdfX-has-integral-1*: (*pdfX has-integral 1*) *UNIV*
⟨*proof*⟩

lemma *set-nn-integral-pdfX-PX*: *set-nn-integral lborel A pdfX* = $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *X* ξ ∈ *A*)
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

lemma *pdfX-set-integrable*: *set-integrable lborel A pdfX* **if** *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

lemma *set-integral-pdfX-PX*: (*LBINT s*:*A*. *pdfX s*) = $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *X* ξ ∈ *A*)
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

lemma *pdfX-has-integral-PX*: (*pdfX has-integral* $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *X* ξ ∈ *A*)) *A*
if *A* ∈ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

corollary *pdfX-has-integral-PX-Icc*: (*pdfX has-integral* $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *a* ≤ *X* ξ ∧ *X* ξ ≤ *b*)) {*a*..*b*}
for *a b* :: *real*
⟨*proof*⟩

corollary *pdfX-integrable-on-Icc*: *pdfX integrable-on* {*a*..*b*} **for** *a b* :: *real*
⟨*proof*⟩

5.2.8 Relations between Life Expectation and Probability Density Function

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: *x* < $\$\psi$
begin

interpretation *alivex-PS*: *prob-space* $\mathfrak{M}$ ⌊ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*) ⟨*proof*⟩

proposition *nn-integral-T-pdfT*:
(∫$^+\xi$. *ennreal* (*g* (*T x* ξ)) ∂($\mathfrak{M}$ ⌊ *alive x*)) = (∫$^+$*s*∈{*0*..}. *ennreal* (*pdfT x s* ∗ *g s*) ∂*lborel*)
if *g* ∈ *borel-measurable lborel* **for** *g* :: *real* ⇒ *real*
⟨*proof*⟩

lemma *expectation-LBINT-pdfT-nonneg*:

alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = (*LBINT s*:{*0*..}. *pdfT x s* $*$ *g s*)
if $\bigwedge$*s*. *s* $\geq$ *0* $\Longrightarrow$ *g s* $\geq$ *0 g* $\in$ *borel-measurable lborel* **for** *g* :: *real* $\Rightarrow$ *real*
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

corollary *expectation-integral-pdfT-nonneg*:
alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = *integral* {*0*..} (λ*s*. *pdfT x s* $*$ *g s*)
if $\bigwedge$*s*. *s* $\geq$ *0* $\Longrightarrow$ *g s* $\geq$ *0 g* $\in$ *borel-measurable lborel* **for** *g* :: *real* $\Rightarrow$ *real*
— Note that *0* = *0* holds when the integral diverges.
⟨*proof*⟩

proposition *expectation-LBINT-pdfT*:
alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = (*LBINT s*:{*0*..}. *pdfT x s* $*$ *g s*)
if *set-integrable lborel* {*0*..} (λ*s*. *pdfT x s* $*$ *g s*) *g* $\in$ *borel-measurable lborel*
for *g* :: *real* $\Rightarrow$ *real*
⟨*proof*⟩

corollary *expectation-integral-pdfT*:
alivex-PS.expectation ($\lambda\xi$. *g* (*T x* ξ)) = *integral* {*0*..} (λ*s*. *pdfT x s* $*$ *g s*)
if (λ*s*. *pdfT x s* $*$ *g s*) *absolutely-integrable-on* {*0*..} *g* $\in$ *borel-measurable lborel*
for *g* :: *real* $\Rightarrow$ *real*
⟨*proof*⟩

corollary *e-LBINT-pdfT*: $*e*‘$\circ$-*x* = (*LBINT s*:{*0*..}. *pdfT x s* $*$ *s*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

corollary *e-integral-pdfT*: $*e*‘$\circ$-*x* = *integral* {*0*..} (λ*s*. *pdfT x s* $*$ *s*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

end

corollary *e-LBINT-pdfX*: $*e*‘$\circ$-*0* = (*LBINT x*:{*0*..}. *pdfX x* $*$ *x*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

corollary *e-integral-pdfX*: $*e*‘$\circ$-*0* = *integral* {*0*..} (λ*x*. *pdfX x* $*$ *x*)
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

5.2.9 Introduction of Force of Mortality

definition *force-mortal* :: *real* $\Rightarrow$ *real* (‹μ'--› [*101*] *200*)
where μ-*x* $\equiv$ *MM-PS.hazard-rate X x*

lemma *mu-pdfX*: μ-*x* = *pdfX x* / *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x*
if (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *mu-unborn-0*: $\$\mu\text{-}x = 0$ **if** $x < 0$ **for** x::*real*
⟨*proof*⟩

lemma *mu-beyond-0*: $\$\mu\text{-}x = 0$ **if** $x \geq \$\psi$ **for** x::*real*
— Note that division by *0* is defined as *0* in Isabelle/HOL.
⟨*proof*⟩

lemma *mu-nonneg-differentiable*: $\$\mu\text{-}x \geq 0$
if (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* **for** x::*real*
⟨*proof*⟩

lemma *mu-nonneg-AE*: *AE x in lborel*. $\$\mu\text{-}x \geq 0$
⟨*proof*⟩

lemma *mu-measurable*[*measurable*]: $(\lambda x.\ \$\mu\text{-}x) \in$ *borel-measurable borel*
⟨*proof*⟩

lemma *mu-deriv-ccdf*: $\$\mu\text{-}x = -$ *deriv* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *x* / *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x*
if (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* $x < \$\psi$ **for** x::*real*
⟨*proof*⟩

lemma *mu-deriv-ln*: $\$\mu\text{-}x = -$ *deriv* $(\lambda x.$ *ln* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x*)) *x*
if (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) *differentiable at x* $x < \$\psi$ **for** x::*real*
⟨*proof*⟩

lemma *p-exp-integral-mu*: $\$p\text{-}\{t\&x\} =$ *exp* $(-$ *integral* $\{x..x+t\}$ $(\lambda y.\ \$\mu\text{-}y))$
if $x \geq 0$ $t \geq 0$ $x+t < \$\psi$ **for** *x t* :: *real*
⟨*proof*⟩

corollary *ccdfX-exp-integral-mu*: *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x* = *exp* $(-$ *integral* $\{0..x\}$ $(\lambda y.\ \$\mu\text{-}y))$
if $0 \leq x \wedge x < \$\psi$ **for** x::*real*
⟨*proof*⟩

5.2.10 Properties of Force of Mortality

context
fixes x::*real*
assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

interpretation *alivex-PS*: *prob-space* $\mathfrak{M} \restriction$ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* ($\mathfrak{M} \restriction$ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *hazard-rate-Tx-mu*: *alivex-PS.hazard-rate* (*T x*) *t* = $\$\mu\text{-}(x+t)$

if $t \geq 0$ $x+t < \$\psi$ **for** t::*real*
⟨*proof*⟩

lemma *pdfTx-p-mu*: *pdfT x t* = $p-{t&x} * $μ-(x+t)
if (*cdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *differentiable at t t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *deriv-t-p-mu*: *deriv* (λ*s*. $p-{s&x}) *t* = − $p-{t&x} * $μ-(x+t)
if (λ*s*. $p-{s&x}) *differentiable at t t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *pdfTx-p-mu-AE*: *AE s in lborel*. *s* > *0* ⟶ *pdfT x s* = $p-{s&x} * $μ-(x+s)
⟨*proof*⟩

lemma *LBINT-p-mu-q-defer*: (*LBINT s*:{*f*<..*f*+*t*}. $p-{s&x} * $μ-(x+s)) = $q-{*f*|*t*&*x*}
if *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

corollary *LBINT-p-mu-q*: (*LBINT s*:{*0*<..*t*}. $p-{s&x} * $μ-(x+s)) = $q-{*t*&*x*}
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *set-integrable-p-mu*: *set-integrable lborel* {*f*<..*f*+*t*} (λ*s*. $p-{s&x} * $μ-(x+s))
if *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *p-mu-has-integral-q-defer-Ioc*:
((λ*s*. $p-{s&x} * $μ-(x+s)) *has-integral* $q-{*f*|*t*&*x*}) {*f*<..*f*+*t*}
if *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *p-mu-has-integral-q-defer-Icc*:
((λ*s*. $p-{s&x} * $μ-(x+s)) *has-integral* $q-{*f*|*t*&*x*}) {*f*..*f*+*t*} **if** *t* ≥ *0 f* ≥ *0* **for**
t f :: *real*
⟨*proof*⟩

corollary *p-mu-has-integral-q-Icc*:
((λ*s*. $p-{s&x} * $μ-(x+s)) *has-integral* $q-{*t*&*x*}) {*0*..*t*} **if** *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

corollary *p-mu-integrable-on-Icc*:
(λ*s*. $p-{s&x} * $μ-(x+s)) *integrable-on* {*0*..*t*} **if** *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *e-ennreal-p-mu*: (∫⁺ξ. *ennreal* (*T x* ξ) ∂(𝔐 ⌊ *alive x*)) =
(∫⁺*s*∈{*0*..}. *ennreal* ($p-{s&x} * $μ-(x+s) * *s*) ∂*lborel*)
⟨*proof*⟩

lemma *e-LBINT-p-mu*: $*e*‘○-*x* = (*LBINT s*:{*0*..}. $p-{s&x} * $μ-(x+s) * *s*)

— Note that $0 = 0$ holds when the life expectation diverges.
⟨*proof*⟩

lemma *e-integral-p-mu*: $\$e\text{`}\circ\text{-}x = \textit{integral}\ \{0..\}\ (\lambda s.\ \$p\text{-}\{s\&x\} * \$\mu\text{-}(x+s) * s)$
— Note that $0 = 0$ holds when the life expectation diverges.
⟨*proof*⟩

end

lemma *p-has-real-derivative-x-ccdfX*:
$((\lambda y.\ \$p\text{-}\{t\&y\})\ \textit{has-real-derivative}$
$((\textit{deriv svl}\ (x+t) * \textit{svl}\ x - \textit{svl}\ (x+t) * \textit{deriv svl}\ x)\ /\ (\textit{svl}\ x)^2))\ (\textit{at}\ x)$
if $\textit{svl} \equiv \textit{ccdf}\ (\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)$ *svl differentiable at x svl differentiable at* $(x+t)$
$t \geq 0\ x < \$\psi$ **for** $x\ t :: \textit{real}$
⟨*proof*⟩

lemma *p-has-real-derivative-x-p-mu*:
$((\lambda y.\ \$p\text{-}\{t\&y\})\ \textit{has-real-derivative}\ \$p\text{-}\{t\&x\} * (\$\mu\text{-}x - \$\mu\text{-}(x+t)))\ (\textit{at}\ x)$
if *ccdf* $(\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)$ *differentiable at x ccdf* $(\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)$ *differentiable at* $(x+t)$
$t \geq 0\ x < \$\psi$ **for** $x\ t :: \textit{real}$
⟨*proof*⟩

corollary *deriv-x-p-mu*: $\textit{deriv}\ (\lambda y.\ \$p\text{-}\{t\&y\})\ x = \$p\text{-}\{t\&x\} * (\$\mu\text{-}x - \$\mu\text{-}(x+t))$
if *ccdf* $(\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)$ *differentiable at x ccdf* $(\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)$ *differentiable at* $(x+t)$
$t \geq 0\ x < \$\psi$ **for** $x\ t :: \textit{real}$
⟨*proof*⟩

lemma *e-has-derivative-mu-e*: $((\lambda x.\ \$e\text{`}\circ\text{-}x)\ \textit{has-real-derivative}\ (\$\mu\text{-}x * \$e\text{`}\circ\text{-}x - 1))\ (\textit{at}\ x)$
if $\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow \textit{set-integrable lborel}\ \{x..\}\ (\textit{ccdf}\ (\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X))$
ccdf $(\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)$ *differentiable at x* $x \in \{a<..<b\}\ b \leq \$\psi$
for $a\ b\ x :: \textit{real}$
⟨*proof*⟩

corollary *e-has-derivative-mu-e'*: $((\lambda x.\ \$e\text{`}\circ\text{-}x)\ \textit{has-real-derivative}\ (\$\mu\text{-}x * \$e\text{`}\circ\text{-}x - 1))\ (\textit{at}\ x)$
if $\bigwedge x.\ x \in \{a<..<b\} \Longrightarrow \textit{ccdf}\ (\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)\ \textit{integrable-on}\ \{x..\}$
ccdf $(\textit{distr}\ \mathfrak{M}\ \textit{borel}\ X)$ *differentiable at x* $x \in \{a<..<b\}\ b \leq \$\psi$
for $a\ b\ x :: \textit{real}$
⟨*proof*⟩

5.2.11 Properties of Curtate Life Expectation

context
fixes $x :: \textit{real}$
assumes *x-lt-psi*[*simp*]: $x < \$\psi$
begin

lemma *isCont-p-nat*: *isCont* ($\lambda t.$ \$*p*-{*t*&*x*}) (*k* + (*1*::*real*)) **for** *k*::*nat*
⟨*proof*⟩

lemma *curt-e-sum-p-smooth*: \$*e*-*x* = ($\sum k.$ \$*p*-{*k*+*1*&*x*}) **if** *summable* ($\lambda k.$ \$*p*-{*k*+*1*&*x*})
⟨*proof*⟩

lemma *curt-e-rec-smooth*: \$*e*-*x* = \$*p*-*x* ∗ (*1* + \$*e*-(*x*+*1*)) **if** *summable* ($\lambda k.$ \$*p*-{*k*+*1*&*x*})
x+*1* < \$$\psi$
⟨*proof*⟩

lemma *curt-e-sum-p-finite-smooth*: \$*e*-*x* = ($\sum k$<*n*. \$*p*-{*k*+*1*&*x*}) **if** *x*+*n*+*1* > \$$\psi$ **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-sum-p-smooth*: \$*e*-{*x*:*n*} = ($\sum k$<*n*. \$*p*-{*k*+*1*&*x*}) **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-rec-smooth*: \$*e*-{*x*:*n*} = \$*p*-*x* ∗ (*1* + \$*e*-{*x*+*1*:*n*−*1*})
if *x*+*1* < \$$\psi$ *n* ≠ *0* **for** *n*::*nat*
⟨*proof*⟩

end

end

5.3 Limited Survival Function

locale *limited-survival-function* = *survival-model* +
assumes *psi-limited*[*simp*]: \$$\psi$ < ∞
begin

definition *ult-age* :: *nat* (‹\$$\omega$›)
where \$$\omega$ ≡ *LEAST x*::*nat*. *ccdf* (*distr* $\mathfrak{M}$ *borel X*) *x* = *0*
— the conventional notation for ultimate age

lemma *ccdfX-ceil-psi-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel X*) ⌈*real-of-ereal* \$$\psi$⌉ = *0*
⟨*proof*⟩

lemma *ccdfX-omega-0*: *ccdf* (*distr* $\mathfrak{M}$ *borel X*) \$$\omega$ = *0*
⟨*proof*⟩

corollary *psi-le-omega*: \$$\psi$ ≤ \$$\omega$
⟨*proof*⟩

corollary *omega-pos*: \$$\omega$ > *0*
⟨*proof*⟩

lemma *omega-ceil-psi*: \$$\omega$ = ⌈*real-of-ereal* \$$\psi$⌉

⟨*proof*⟩

lemma *ccdfX-0-equiv-nat*: *ccdf* (*distr* 𝔐 *borel X*) *x* = *0* ⟷ *x* ≥ $ω **for** *x*::*nat*
⟨*proof*⟩

lemma *psi-le-iff-omega-le*: $ψ ≤ *x* ⟷ $ω ≤ *x* **for** *x*::*nat*
⟨*proof*⟩

context
fixes *x*::*nat*
assumes *x-lt-omega*[*simp*]: *x* < $ω
begin

lemma *x-lt-psi*[*simp*]: *x* < $ψ
⟨*proof*⟩

lemma *p-0-1-nat*: $*p*-{*0*&*x*} = *1*
⟨*proof*⟩

lemma *p-0-equiv-nat*: $*p*-{*t*&*x*} = *0* ⟷ *x*+*t* ≥ $ω **for** *t*::*nat*
⟨*proof*⟩

lemma *q-0-0-nat*: $*q*-{*0*&*x*} = *0*
⟨*proof*⟩

lemma *q-1-equiv-nat*: $*q*-{*t*&*x*} = *1* ⟷ *x*+*t* ≥ $ω **for** *t*::*nat*
⟨*proof*⟩

lemma *q-defer-old-0-nat*: $*q*-{*f*|*t*&*x*} = *0* **if** $ω ≤ *x*+*f* **for** *f t* :: *nat*
⟨*proof*⟩

lemma *curt-e-sum-P-finite-nat*: $*e*-*x* = (∑ *k*<*n*. 𝒫(ξ *in* 𝔐. *T x* ξ ≥ *k* + *1* | *T x* ξ > *0*))
if *x*+*n* ≥ $ω **for** *n*::*nat*
⟨*proof*⟩

lemma *curt-e-sum-p-finite-nat*: $*e*-*x* = (∑ *k*<*n*. $*p*-{*k*+*1*&*x*})
if ⋀*k*::*nat*. *k* < *n* ⟹ *isCont* (λ*t*. $*p*-{*t*&*x*}) (*real k* + *1*) *x*+*n* ≥ $ω **for** *n*::*nat*
⟨*proof*⟩

end

lemma *q-omega-1*: $*q*-($ω−*1*) = *1*
⟨*proof*⟩

end

end
theory *Life-Table*

imports *Survival-Model*
begin

6 Life Table

Define a life table axiomatically.

locale *life-table* =
  fixes l :: *real* ⇒ *real* (‹$l'--› [*101*] *200*)
  assumes *l-0-pos*: $0 < l\ 0$
    and *l-neg-nil*: $\bigwedge x.\ x \leq 0 \Longrightarrow l\ x = l\ 0$
    and *l-PInfty-0*: $(l \longrightarrow 0)$ *at-top*
    and *l-antimono*: *antimono* l
    and *l-right-continuous*: $\bigwedge x.$ *continuous* (*at-right* x) l
begin

6.1 Basic Properties of Life Table

lemma *l-0-neq-0*[*simp*]: $\$l\text{-}0 \neq 0$
  ⟨*proof*⟩

lemma *l-nonneg*[*simp*]: $\$l\text{-}x \geq 0$ **for** x::*real*
  ⟨*proof*⟩

lemma *l-bounded*[*simp*]: $\$l\text{-}x \leq \$l\text{-}0$ **for** x::*real*
  ⟨*proof*⟩

lemma *l-measurable*[*measurable*, *simp*]: $l \in$ *borel-measurable borel*
  ⟨*proof*⟩

lemma *l-left-continuous-nonpos*: *continuous* (*at-left* x) l **if** $x \leq 0$ **for** x::*real*
⟨*proof*⟩

lemma *l-integrable-Icc*: *set-integrable lborel* $\{a..b\}$ l **for** $a\ b$ :: *real*
  ⟨*proof*⟩

corollary *l-integrable-on-Icc*: l *integrable-on* $\{a..b\}$ **for** $a\ b$ :: *real*
  ⟨*proof*⟩

lemma *l-integrable-Icc-shift*: *set-integrable lborel* $\{a..b\}$ $(\lambda t.\ \$l\text{-}(x+t))$ **for** $a\ b\ x$ :: *real*
  ⟨*proof*⟩

corollary *l-integrable-on-Icc-shift*: $(\lambda t.\ \$l\text{-}(x+t))$ *integrable-on* $\{a..b\}$ **for** $x\ a\ b$ :: *real*
  ⟨*proof*⟩

lemma *l-normal-antimono*: *antimono* $(\lambda x.\ \$l\text{-}x\ /\ \$l\text{-}0)$
  ⟨*proof*⟩

lemma *compl-l-normal-right-continuous*: *continuous* (*at-right* *x*) (λ*x*. *1* − $*l-x* / $*l-0*) **for** *x*::*real*
⟨*proof*⟩

lemma *compl-l-normal-NInfty-0*: ((λ*x*. *1* − $*l-x* / $*l-0*) ⟶ *0*) *at-bot*
⟨*proof*⟩

lemma *compl-l-normal-PInfty-1*: ((λ*x*. *1* − $*l-x* / $*l-0*) ⟶ *1*) *at-top*
⟨*proof*⟩

lemma *compl-l-real-distribution*: *real-distribution* (*interval-measure* (λ*x*. *1* − $*l-x* / $*l-0*))
⟨*proof*⟩

definition *total* :: *real* ⇒ *real* (‹$*T'*--› [*101*] *200*) **where** $*T-x* ≡ *LBINT* *y*:{*x*..}. $*l-y*
— the number of lives older than the ones aged *x*
— The parameter *x* must be nonnegative.

lemma *T-nonneg*[*simp*]: $*T-x* ≥ *0* **for** *x*::*real*
⟨*proof*⟩

definition *total-finite* ≡ *set-integrable lborel* {*0*..} *l*

lemma *total-finite-iff-set-integrable-Ici*:
total-finite ⟷ *set-integrable lborel* {*x*..} *l* **for** *x*::*real*
⟨*proof*⟩

lemma *total-finite-iff-integrable-on-Ici*: *total-finite* ⟷ *l* *integrable-on* {*x*..} **for** *x*::*real*
⟨*proof*⟩

lemma *total-finite-iff-summable*: *total-finite* ⟷ *summable* (λ*k*. $*l*-(*x*+*k*)) **for** *x*::*real*
⟨*proof*⟩

lemma *T-tendsto-0*: ((λ*x*. $*T-x*) ⟶ *0*) *at-top* **if** *total-finite*
⟨*proof*⟩

definition *lives* :: *real* ⇒ *real* ⇒ *real* (‹$*L'*-{-&-}› [*0*,*0*] *200*)
where $*L*-{*n*&*x*} ≡ *LBINT* *y*:{*x*..*x*+*n*}. $*l-y*
— the number of lives between ages *x* and *x*+*n*
— The parameter *x* must be nonnegative.
— The parameter *n* is usually nonnegative, but theoretically it can be negative.

abbreviation *lives-1* :: *real* ⇒ *real* (‹$*L'*--› [*101*] *200*)
where $*L-x* ≡ $*L*-{*1*&*x*}

lemma *l-has-integral-L*: (*l* *has-integral* $*L*-{*n*&*x*}) {*x*..*x*+*n*} **for** *x* *n* :: *real*

⟨*proof*⟩

lemma *L-neg-0*[*simp*]: $*L*-{*n*&*x*} = *0* **if** *n* < *0* **for** *x n* :: *real*
⟨*proof*⟩

lemma *L-nonneg*[*simp*]: $*L*-{*n*&*x*} ≥ *0* **for** *x n* :: *real*
⟨*proof*⟩

lemma *L-T*: $*L*-{*n*&*x*} = $*T*-*x* − $*T*-(*x*+*n*) **if** *total-finite n* ≥ *0* **for** *x n* :: *real*
⟨*proof*⟩

lemma *L-sums-T*: (λ*k*. $*L*-(*x*+*k*)) *sums* $*T*-*x* **if** *total-finite* **for** *x*::*real*
⟨*proof*⟩

definition *death* :: *real* ⇒ *real* ⇒ *real* (‹$*d*'-{-&-}› [*0*,*0*] *200*)
where $*d*-{*t*&*x*} ≡ *max 0* ($*l*-*x* − $*l*-(*x*+*t*))
— the number of deaths between ages *x* and *x*+*t*
— The parameter *t* is usually nonnegative, but theoretically it can be negative.

abbreviation *death1* :: *real* ⇒ *real* (‹$*d*'--› [*101*] *200*)
where $*d*-*x* ≡ $*d*-{*1*&*x*}

lemma *death-def-nonneg*: $*d*-{*t*&*x*} = $*l*-*x* − $*l*-(*x*+*t*) **if** *t* ≥ *0* **for** *t x* :: *real*
⟨*proof*⟩

lemma *d-nonpos-0*: $*d*-{*t*&*x*} = *0* **if** *t* ≤ *0* **for** *t x* :: *real*
⟨*proof*⟩

corollary *d-0-0*: $*d*-{*0*&*x*} = *0* **for** *x*::*real*
⟨*proof*⟩

lemma *d-nonneg*[*simp*]: $*d*-{*t*&*x*} ≥ *0* **for** *t x* :: *real*
⟨*proof*⟩

lemma *dx-l*: $*d*-*x* = $*l*-*x* − $*l*-(*x*+*1*) **for** *x*::*real*
⟨*proof*⟩

lemma *sum-dx-l*: (∑ *k*<*n*. $*d*-(*x*+*k*)) = $*l*-*x* − $*l*-(*x*+*n*) **for** *x*::*real* **and** *n*::*nat*
⟨*proof*⟩

corollary *d-sums-l*: (λ*k*. $*d*-(*x*+*k*)) *sums* $*l*-*x* **for** *x*::*real*
⟨*proof*⟩

lemma *add-d*: $*d*-{*t*&*x*} + $*d*-{*t'* & *x*+*t*} = $*d*-{*t*+*t'* & *x*} **if** *t* ≥ *0 t'* ≥ *0* **for** *t t'* :: *real*
⟨*proof*⟩

definition *die-central* :: *real* ⇒ *real* ⇒ *real* (‹$*m*'-{-&-}› [*0*,*0*] *200*)
where $*m*-{*n*&*x*} ≡ $*d*-{*n*&*x*} / $*L*-{*n*&*x*}

— central death rate

abbreviation *die-central-1* :: *real* ⇒ *real* (‹$*m*′--› [*101*] *200*)
where $*m*-*x* ≡ $*m*-{*1*&*x*}

6.2 Construction of Survival Model from Life Table

definition *life-table-measure* :: *real measure* (‹𝔐›)
where 𝔐 ≡ *interval-measure* (λ*x*. *1* − $*l*-*x* / $*l*-*0*)

lemma *prob-space-actuary-MM*: *prob-space-actuary* 𝔐
⟨*proof*⟩

definition *survival-model-X* :: *real* ⇒ *real* (‹*X*›) **where** *X* ≡ λ*x*. *x*

lemma *survival-model-MM-X*: *survival-model* 𝔐 *X*
⟨*proof*⟩

end

sublocale *life-table* ⊆ *survival-model* 𝔐 *X*
⟨*proof*⟩

context *life-table*
begin

interpretation *distrX-RD*: *real-distribution distr* 𝔐 *borel X*
⟨*proof*⟩

6.2.1 Relations between Life Table and Survival Function for *X*

lemma *ccdfX-l-normal*: *ccdf* (*distr* 𝔐 *borel X*) = (λ*x*. $*l*-*x* / $*l*-*0*)
⟨*proof*⟩

corollary *deriv-ccdfX-l*: *deriv* (*ccdf* (*distr* 𝔐 *borel X*)) *x* = *deriv l x* / $*l*-*0*
if *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

notation *death-pt* (‹$*ψ*›)

lemma *l-0-equiv*: $*l*-*x* = *0* ⟷ *x* ≥ $*ψ* **for** *x*::*real*
⟨*proof*⟩

lemma *d-old-0*: $*d*-{*t*&*x*} = *0* **if** *x* ≥ $*ψ* *t* ≥ *0* **for** *x t* :: *real*
⟨*proof*⟩

lemma *d-l-equiv*: $*d*-{*t*&*x*} = $*l*-*x* ⟷ *x*+*t* ≥ $*ψ* **if** *t* ≥ *0* **for** *x t* :: *real*
⟨*proof*⟩

lemma *continuous-ccdfX-l*: *continuous F* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) $\longleftrightarrow$ *continuous F l*
for *F* :: *real filter*
⟨*proof*⟩

lemma *has-real-derivative-ccdfX-l*:
(*ccdf* (*distr* $\mathfrak{M}$ *borel X*) *has-real-derivative D*) (*at x*) $\longleftrightarrow$
(*l has-real-derivative* $l-0 ∗ D) (*at x*)
for *D x* :: *real*
⟨*proof*⟩

corollary *differentiable-ccdfX-l*:
ccdf (*distr* $\mathfrak{M}$ *borel X*) *differentiable* (*at x*) $\longleftrightarrow$ *l differentiable* (*at x*)
for *D x* :: *real*
⟨*proof*⟩

lemma *PX-l-normal*: $\mathcal{P}(\xi$ *in* $\mathfrak{M}$. *X* $\xi > x) = \$l-x / \$l-0$ **for** *x*::*real*
⟨*proof*⟩

lemma *set-integrable-ccdfX-l*:
set-integrable lborel A (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) $\longleftrightarrow$ *set-integrable lborel A l*
if *A* $\in$ *sets lborel* **for** *A* :: *real set*
⟨*proof*⟩

corollary *integrable-ccdfX-l*: *integrable lborel* (*ccdf* (*distr* $\mathfrak{M}$ *borel X*)) $\longleftrightarrow$ *integrable lborel l*
⟨*proof*⟩

lemma *integrable-on-ccdfX-l*:
ccdf (*distr* $\mathfrak{M}$ *borel X*) *integrable-on A* $\longleftrightarrow$ *l integrable-on A* **for** *A* :: *real set*
⟨*proof*⟩

6.2.2 Relations between Life Table and Cumulative Distributive Function for X

lemma *cdfX-l-normal*: *cdf* (*distr* $\mathfrak{M}$ *borel X*) = (λx. *1* − $l-x / \$l-0$) **for** *x*::*real*
⟨*proof*⟩

lemma *deriv-cdfX-l*: *deriv* (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) *x* = − *deriv l x* / $l-0$
if *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *continuous-cdfX-l*: *continuous F* (*cdf* (*distr* $\mathfrak{M}$ *borel X*)) $\longleftrightarrow$ *continuous F l*
for *F* :: *real filter*
⟨*proof*⟩

lemma *has-real-derivative-cdfX-l*:
(*cdf* (*distr* $\mathfrak{M}$ *borel X*) *has-real-derivative D*) (*at x*) $\longleftrightarrow$

(*l has-real-derivative* − (*$l-0* ∗ *D*)) (*at x*)
for *D x* :: *real*
⟨*proof*⟩

lemma *differentiable-cdfX-l*:
cdf (*distr* 𝔐 *borel X*) *differentiable* (*at x*) ⟷ *l differentiable* (*at x*) **for** *D x* :: *real*
⟨*proof*⟩

lemma *PX-compl-l-normal*: 𝒫(ξ *in* 𝔐. *X* ξ ≤ *x*) = *1* − *$l-x* / *$l-0* **for** *x*::*real*
⟨*proof*⟩

6.2.3 Relations between Life Table and Survival Function for $T(x)$

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: *x* < *$ψ*
begin

notation *futr-life* (‹*T*›)

interpretation *alivex-PS*: *prob-space* 𝔐 ⌊ *alive x*
⟨*proof*⟩

interpretation *distrTx-RD*: *real-distribution distr* (𝔐 ⌊ *alive x*) *borel* (*T x*) ⟨*proof*⟩

lemma *lx-neq-0*[*simp*]: *$l-x* ≠ *0*
⟨*proof*⟩

corollary *lx-pos*[*simp*]: *$l-x* > *0*
⟨*proof*⟩

lemma *ccdfTx-l-normal*: *ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*)) *t* = *$l-*(*x*+*t*) / *$l-x*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *deriv-ccdfTx-l*:
deriv (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) *t* = *deriv* (λ*t*. *$l-*(*x*+*t*) / *$l-x*) *t*
if *t* > *0 l differentiable at* (*x*+*t*) **for** *t*::*real*
⟨*proof*⟩

lemma *continuous-at-within-ccdfTx-l*:
continuous (*at t within* {*0*..}) (*ccdf* (*distr* (𝔐 ⌊ *alive x*) *borel* (*T x*))) ⟷
continuous (*at* (*x*+*t*) *within* {*x*..}) *l*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *isCont-ccdfTx-l*:

isCont (*ccdf* (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*))) *t* ⟷ *isCont l* (*x*+*t*) **if** *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *has-real-derivative-ccdfTx-l*:
(*ccdf* (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*)) *has-real-derivative D*) (*at t*) ⟷
(*l has-real-derivative* $*l*-*x* ∗ *D*) (*at* (*x*+*t*))
if *t* > *0* **for** *t D* :: *real*
⟨*proof*⟩

lemma *differentiable-ccdfTx-l*:
ccdf (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*)) *differentiable at t* ⟷ *l differentiable* (*at* (*x*+*t*))
if *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *PTx-l-normal*: $\mathcal{P}$(ξ *in* $\mathfrak{M}$. *T x* ξ > *t* | *T x* ξ > *0*) = $*l*-(*x*+*t*) / $*l*-*x* **if** *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

6.2.4 Relations between Life Table and Cumulative Distributive Function for $T(x)$

lemma *cdfTx-compl-l-normal*: *cdf* (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*)) *t* = *1* − $*l*-(*x*+*t*) / $*l*-*x*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *deriv-cdfTx-l*:
deriv (*cdf* (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*))) *t* = − *deriv* (λ*t*. $*l*-(*x*+*t*) / $*l*-*x*) *t*
if *t* > *0 l differentiable at* (*x*+*t*) **for** *t*::*real*
⟨*proof*⟩

lemma *continuous-at-within-cdfTx-l*:
continuous (*at t within* {*0*..}) (*cdf* (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*))) ⟷
continuous (*at* (*x*+*t*) *within* {*x*..}) *l*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *isCont-cdfTx-l*:
isCont (*cdf* (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*))) *t* ⟷ *isCont l* (*x*+*t*) **if** *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *has-real-derivative-cdfTx-l*:
(*cdf* (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*)) *has-real-derivative D*) (*at t*) ⟷
(*l has-real-derivative* − $*l*-*x* ∗ *D*) (*at* (*x*+*t*))
if *t* > *0* **for** *t D* :: *real*
⟨*proof*⟩

lemma *differentiable-cdfTx-l*:
cdf (*distr* ($\mathfrak{M}$ ⌊ *alive x*) *borel* (*T x*)) *differentiable at t* ⟷ *l differentiable* (*at* (*x*+*t*))
if *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *PTx-compl-l-normal*: $\mathcal{P}$(*ξ in* $\mathfrak{M}$. *T x ξ* ≤ *t* | *T x ξ* > *0*) = *1*− $*l*-(*x*+*t*) / $*l*-*x*
if *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

6.2.5 Life Table and Actuarial Notations

notation *survive* (‹$*p*′-{-&-}› [*0*,*0*] *200*)
notation *survive-1* (‹$*p*′--› [*101*] *200*)
notation *die* (‹$*q*′-{-&-}› [*0*,*0*] *200*)
notation *die-1* (‹$*q*′--› [*101*] *200*)
notation *die-defer* (‹$*q*′-{-|-&-}› [*0*,*0*,*0*] *200*)
notation *die-defer-1* (‹$*q*′-{-|&-}› [*0*,*0*] *200*)
notation *life-expect* (‹$*e*‘○′--› [*101*] *200*)
notation *temp-life-expect* (‹$*e*‘○′-{-:-}› [*0*,*0*] *200*)
notation *curt-life-expect* (‹$*e*′--› [*101*] *200*)
notation *temp-curt-life-expect* (‹$*e*′-{-:-}› [*0*,*0*] *200*)

lemma *p-l*: $*p*-{*t*&*x*} = $*l*-(*x*+*t*) / $*l*-*x* **if** *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

corollary *p-1-l*: $*p*-*x* = $*l*-(*x*+*1*) / $*l*-*x*
⟨*proof*⟩

lemma *isCont-p-l*: *isCont* (λ*s*. $*p*-{*s*&*x*}) *t* ⟷ *isCont l* (*x*+*t*) **if** *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *total-finite-iff-p-set-integrable-Ici*:
total-finite ⟷ *set-integrable lborel* {*0*..} (λ*t*. $*p*-{*t*&*x*})
⟨*proof*⟩

lemma *p-PTx-ge-l-isCont*: $*p*-{*t*&*x*} = $\mathcal{P}$(*ξ in* $\mathfrak{M}$. *T x ξ* ≥ *t* | *T x ξ* > *0*)
if *isCont l* (*x*+*t*) *t* > *0* **for** *t*::*real*
⟨*proof*⟩

lemma *q-defer-l*: $*q*-{*f*|*t*&*x*} = ($*l*-(*x*+*f*) − $*l*-(*x*+*f*+*t*)) / $*l*-*x* **if** *f* ≥ *0 t* ≥ *0* **for** *f t* :: *real*
⟨*proof*⟩

corollary *q-defer-d-l*: $*q*-{*f*|*t*&*x*} = $*d*-{*t* & *x*+*f*} / $*l*-*x* **if** *f* ≥ *0 t* ≥ *0* **for** *f t* :: *real*
⟨*proof*⟩

corollary *q-defer-1-d-l*: $q-{f|&x} = $d-(x+f) / $l-x **if** $f \geq 0$ **for** *f*::*real*
⟨*proof*⟩

lemma *q-d-l*: $q-{t&x} = $d-{t&x} / $l-x **for** *t*::*real*
⟨*proof*⟩

corollary *q-1-d-l*: $q-x = $d-x / $l-x
⟨*proof*⟩

lemma *LBINT-p-l*: (*LBINT* *t*:*A*. $p-{t&x}) = (*LBINT* *t*:*A*. $l-(x+t)) / $l-x
if $A \subseteq \{0..\}$ $A \in$ *sets lborel* **for** *A* :: *real set*
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-LBINT-l*: $e`○-x = (*LBINT* *t*:{0..}. $l-(x+t)) / $l-x
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-LBINT-l-Icc*: $e`○-x = (*LBINT* *t*:{0..*n*}. $l-(x+t)) / $l-x **if** $x+n \geq$ ψ **for** *n*::*real*
⟨*proof*⟩

lemma *temp-e-LBINT-l*: $e`○-{x:n} = (*LBINT* *t*:{0..*n*}. $l-(x+t)) / $l-x **if** $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *integral-p-l*: *integral* *A* (λ*t*. $p-{t&x}) = (*integral* *A* (λ*t*. $l-(x+t))) / $l-x
if $A \subseteq \{0..\}$ $A \in$ *sets lborel* **for** *A* :: *real set*
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-integral-l*: $e`○-x = *integral* {0..} (λ*t*. $l-(x+t)) / $l-x
— Note that $0 = 0$ holds when the integral diverges.
⟨*proof*⟩

corollary *e-integral-l-Icc*:
$e`○-x = *integral* {0..*n*} (λ*t*. $l-(x+t)) / $l-x **if** $x+n \geq$ ψ **for** *n*::*real*
⟨*proof*⟩

lemma *e-pos-total-finite*: $e`○-x > 0$ **if** *total-finite*
⟨*proof*⟩

lemma *temp-e-integral-l*:
$e`○-{x:n} = *integral* {0..*n*} (λ*t*. $l-(x+t)) / $l-x **if** $n \geq 0$ **for** *n*::*real*
⟨*proof*⟩

lemma *curt-e-sum-l*: $e-x = (∑ *k*. $l-(x+k+1)) / $l-x **if** *total-finite* ⋀*k*::*nat*. *isCont* *l* (*x*+*k*+1)

⟨*proof*⟩

lemma *curt-e-sum-l-finite*: $e-x = (∑ k<n. $l-(x+k+1)) / $l-x
if ⋀k::*nat*. k < n ⟹ *isCont l* (x+k+1) x+n+1 > $ψ **for** n::*nat*
⟨*proof*⟩

lemma *temp-curt-e-sum-p*: $e-{x:n} = (∑ k<n. $l-(x+k+1)) / $l-x
if ⋀k::*nat*. k < n ⟹ *isCont l* (x+k+1) **for** n::*nat*
⟨*proof*⟩

lemma *e-T-l*: $e‘∘-x = $T-x / $l-x
⟨*proof*⟩

lemma *temp-e-L-l*: $e‘∘-{x:n} = $L-{n&x} / $l-x **if** n ≥ 0 **for** n::*real*
⟨*proof*⟩

lemma *m-q-e*: $m-{n&x} = $q-{n&x} / $e‘∘-{x:n} **if** n ≥ 0 **for** n::*real*
⟨*proof*⟩

end

lemma *l-p*: $l-x / $l-0 = $p-{x&0} **for** x::*real*
⟨*proof*⟩

lemma *e-p-e-total-finite*: $e‘∘-x = $e‘∘-{x:n} + $p-{n&x} ∗ $e‘∘-(x+n)
if *total-finite* n ≥ 0 x+n < $ψ **for** x n :: *real*
⟨*proof*⟩

proposition *x-ex-const-equiv-total-finite*: x + $e‘∘-x = y + $e‘∘-y ⟷ $q-{y−x&x} = 0
if *total-finite* x ≤ y y < $ψ **for** x y :: *real*
⟨*proof*⟩

corollary *x-ex-const-iff-l-const*: x + $e‘∘-x = y + $e‘∘-y ⟷ $l-x = $l-y
if *total-finite* x ≤ y y < $ψ **for** x y :: *real*
⟨*proof*⟩

end

6.3 Piecewise Differentiable Life Table

locale *smooth-life-table* = *life-table* +
assumes *l-piecewise-differentiable*[*simp*]: *l piecewise-differentiable-on UNIV*
begin

lemma *smooth-survival-function-MM-X*: *smooth-survival-function* 𝔐 *X*
⟨*proof*⟩

end

sublocale *smooth-life-table* $\subseteq$ *smooth-survival-function* $\mathfrak{M}$ X
⟨*proof*⟩

context *smooth-life-table*
begin

notation *force-mortal* (‹$\mu'--› [101] 200)

lemma *l-continuous*[*simp*]: *continuous-on UNIV l*
⟨*proof*⟩

lemma *l-nondifferentiable-finite-set*[*simp*]: *finite* {*x*. ¬ *l differentiable at x*}
⟨*proof*⟩

lemma *l-differentiable-borel-set*[*measurable*, *simp*]: {*x*. *l differentiable at x*} ∈ *sets borel*
⟨*proof*⟩

lemma *l-differentiable-AE*: *AE x in lborel*. *l differentiable at x*
⟨*proof*⟩

lemma *deriv-l-measurable*[*measurable*]: *deriv l* ∈ *borel-measurable borel*
⟨*proof*⟩

lemma *pdfX-l-normal*:
pdfX x = (*if l differentiable at x then* − *deriv l x* / $*l-0 else 0*) **for** *x*::*real*
⟨*proof*⟩

lemma *mu-deriv-l*: $*μ-x* = − *deriv l x* / $*l-x* **if** *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *mu-nonneg-differentiable-l*: $*μ-x* ≥ *0* **if** *l differentiable at x* **for** *x*::*real*
⟨*proof*⟩

lemma *mu-deriv-ln-l*:
$*μ-x* = − *deriv* (λ*x*. *ln* ($*l-x*)) *x* **if** *l differentiable at x x* < $*ψ* **for** *x*::*real*
⟨*proof*⟩

lemma *deriv-l-shift*: *deriv l* (*x*+*t*) = *deriv* (λ*t*. $*l-*(*x*+*t*)) *t*
if *l differentiable at* (*x*+*t*) **for** *x t* :: *real*
⟨*proof*⟩

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: *x* < $*ψ*
begin

lemma *p-mu-l*: $*p-*{*t*&*x*} ∗ $*μ-*(*x*+*t*) = − *deriv l* (*x*+*t*) / $*l-x*

if *l differentiable at* (*x*+*t*) *t* > *0 x*+*t* < $*ψ* **for** *t*::*real*
⟨*proof*⟩

lemma *p-mu-l-AE*: *AE s in lborel. 0* < *s* ∧ *x*+*s* < $*ψ* ⟶ $*p*-{*s*&*x*} ∗ $*μ*-(*x*+*s*) = − *deriv l* (*x*+*s*) / $*l*-*x*
⟨*proof*⟩

lemma *LBINT-l-mu-q*: (*LBINT s*:{*f*<..*f*+*t*}. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*)) / $*l*-*x* = $*q*-{*f*|*t*&*x*}
if *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *set-integrable-l-mu*: *set-integrable lborel* {*f*<..*f*+*t*} (λ*s*. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*))
if *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *l-mu-has-integral-q-defer*:
((λ*s*. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*) / $*l*-*x*) *has-integral* $*q*-{*f*|*t*&*x*}) {*f*..*f*+*t*}
if *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

corollary *l-mu-has-integral-q*:
((λ*s*. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*) / $*l*-*x*) *has-integral* $*q*-{*t*&*x*}) {*0*..*t*} **if** *t* ≥ *0* **for** *t*::*real*
⟨*proof*⟩

lemma *l-mu-has-integral-d*:
((λ*s*. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*)) *has-integral* $*d*-{*t* & *x*+*f*}) {*f*..*f*+*t*}
if *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

corollary *l-mu-has-integral-d-1*:
((λ*s*. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*)) *has-integral* $*d*-(*x*+*f*)) {*f*..*f*+*1*} **if** *t* ≥ *0 f* ≥ *0* **for** *t f* :: *real*
⟨*proof*⟩

lemma *e-LBINT-l*: $*e*'∘-*x* = (*LBINT s*:{*0*..}. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*) ∗ *s*) / $*l*-*x*
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

lemma *e-integral-l*: $*e*'∘-*x* = *integral* {*0*..} (λ*s*. $*l*-(*x*+*s*) ∗ $*μ*-(*x*+*s*) ∗ *s*) / $*l*-*x*
— Note that *0* = *0* holds when the life expectation diverges.
⟨*proof*⟩

lemma *m-LBINT-p-mu*: $*m*-{*n*&*x*} = (*LBINT t*:{*0*<..*n*}. $*p*-{*t*&*x*} ∗ $*μ*-(*x*+*t*)) / (*LBINT t*:{*0*..*n*}. $*p*-{*t*&*x*})
if *n* ≥ *0* **for** *n*::*real*
⟨*proof*⟩

lemma *m-integral-p-mu*:
$*m*-{*n*&*x*} = *integral* {*0*..*n*} (λ*t*. $*p*-{*t*&*x*} ∗ $*μ*-(*x*+*t*)) / *integral* {*0*..*n*} (λ*t*.

$p-{t&x})
if $n \geq 0$ **for** n::*real*
⟨*proof*⟩

end

lemma *deriv-x-p-mu-l*: *deriv* (λy. $p-{t&y}) x = $p-{t&x} * ($\mu$-x − μ-(x+t))
if *l differentiable at x l differentiable at* (x+t) $t \geq 0$ $x <$ ψ **for** *x t* :: *real*
⟨*proof*⟩

lemma *e-has-derivative-mu-e-l*: ((λx. $e`o-x) *has-real-derivative* ($\mu$-x * $e`o-x − 1)) (*at x*)
if *total-finite l differentiable at x* x∈{a<..<b} $b \leq$ ψ **for** *a b x* :: *real*
⟨*proof*⟩

corollary *e-has-derivative-mu-e-l'*: ((λx. $e`o-x) *has-real-derivative* ($\mu$-x * $e`o-x − 1)) (*at x*)
if *total-finite l differentiable at x* x∈{a<..<b} $b \leq$ ψ **for** *a b x* :: *real*
⟨*proof*⟩

context
fixes *x*::*real*
assumes *x-lt-psi*[*simp*]: $x <$ ψ
begin

lemma *curt-e-sum-l-smooth*: $e-x = ($\sum$ k. $l-(x+k+1)) / $l-x **if** *total-finite*
⟨*proof*⟩

lemma *curt-e-sum-l-finite-smooth*: $e-x = ($\sum$ k<n. $l-(x+k+1)) / $l-x **if** x+n+1 > ψ **for** *n*::*nat*
⟨*proof*⟩

lemma *temp-curt-e-sum-l-smooth*: $e-{x:n} = ($\sum$ k<n. $l-(x+k+1)) / $l-x **for** *n*::*nat*
⟨*proof*⟩

end

end

6.4 Interpolations

context *life-table*
begin

definition *linear-interpolation* ≡
∀(x::*nat*)(t::*real*). $0 \leq t \wedge t \leq 1 \longrightarrow$ $l-(x+t) = (1-t)*$l-x + t*$l-(x+1)

lemma *linear-l*: $l-(x+t) = (1-t)*$l-x + t*$l-(x+1)

if *linear-interpolation 0* $\leq$ *t t* $\leq$ *1* **for** *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

lemma *linear-l-d*: $*l*-(*x*+*t*) = $*l*-*x* − *t*∗$*d*-*x*
if *linear-interpolation 0* $\leq$ *t t* $\leq$ *1* **for** *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

lemma *linear-p-q*: $*p*-{*t*&*x*} = *1* − *t*∗$*q*-*x*
if *linear-interpolation 0* $\leq$ *t t* $\leq$ *1 x* < $*ψ* **for** *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

lemma *linear-q*: $*q*-{*t*&*x*} = *t*∗$*q*-*x*
if *linear-interpolation 0* $\leq$ *t t* $\leq$ *1 x* < $*ψ* **for** *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

lemma *linear-L-l-d*: $*L*-*x* = $*l*-*x* − $*d*-*x* / *2* **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-L-l-d′*: $*L*-*x* = $*l*-(*x*+*1*) + $*d*-*x* / *2* **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-continuous*: *continuous-on UNIV l* **if** *linear-interpolation*
⟨*proof*⟩

lemma *linear-l-sums-T-l*: (*λk*. $*l*-(*x* + *Suc k*)) *sums* ($*T*-*x* − $*l*-*x* / *2*)
if *linear-interpolation total-finite* **for** *x*::*nat*
⟨*proof*⟩

corollary *linear-T-suminf-l*: $*T*-*x* = (∑ *k*. $*l*-(*x*+*k*+*1*)) + $*l*-*x* / *2*
if *linear-interpolation total-finite* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-mx-q*: $*m*-*x* = $*q*-*x* / (*1* − $*q*-*x* / *2*) **if** *linear-interpolation x* < $*ψ*
for *x*::*nat*
⟨*proof*⟩

lemma *linear-e-curt-e*: $*e*‘∘-*x* = $*e*-*x* + *1*/*2*
if *linear-interpolation total-finite x* < $*ψ* **for** *x*::*nat*
⟨*proof*⟩

end

context *smooth-life-table*
begin

lemma *linear-l-has-derivative-at-frac*:
((*λs*. $*l*-(*x*+*s*)) *has-real-derivative* − $*d*-*x*) (*at t*)
if *linear-interpolation 0* < *t t* < *1* **for** *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

lemma *linear-l-has-derivative-at-frac'*:
(*l has-real-derivative* − $*d-x*) (*at y*)
if *linear-interpolation x* < *y y* < *x*+*1* **for** *x*::*nat* **and** *y*::*real*
⟨*proof*⟩

lemma *linear-l-differentiable-on-frac*:
l differentiable-on {*x*<..<*x*+*1*} **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-has-right-derivative-at-nat*:
(*l has-real-derivative* − $*d-x*) (*at-right x*) **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-has-left-derivative-at-nat*:
(*l has-real-derivative* − $*d-*(*real x* − *1*)) (*at-left x*) **if** *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-has-derivative-at-nat-iff-d*:
(*l has-real-derivative* − $*d-x*) (*at x*) ⟷ $*d-x* = $*d-*(*real x* − *1*)
if *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-differentiable-at-nat-iff-d*:
l differentiable at x ⟷ $*d-x* = $*d-*(*real x* − *1*)
if *linear-interpolation* **for** *x*::*nat*
⟨*proof*⟩

lemma *linear-l-limited*: $*ψ* < ∞ **if** *linear-interpolation*
⟨*proof*⟩

lemma *linear-mu-q*: $*μ-*(*x*+*t*) = $*q-x* / (*1* − *t*∗$*q-x*)
if *linear-interpolation l differentiable at* (*x*+*t*) *0* < *t t* < *1 x*+*t* < $*ψ*
for *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

definition *exponential-interpolation* ≡
∀(*x*::*nat*)(*t*::*real*). *x*+*1* < $*ψ* ⟶ *0* ≤ *t* ∧ *t* < *1* ⟶ $*μ-*(*x*+*t*) = $*μ-x*
— Without *x*+*1* < $*ψ*, the smooth life table could not be limited.

lemma *exponential-mu*: $*μ-*(*x*+*t*) = $*μ-x*
if *exponential-interpolation x*+*1* < $*ψ* *0* ≤ *t t* < *1* **for** *x*::*nat* **and** *t*::*real*
⟨*proof*⟩

corollary *exponential-mu'*: $*μ-y* = $*μ-x*
if *exponential-interpolation x* ≤ *y y* < *x*+*1 x*+*1* < $*ψ* **for** *x*::*nat* **and** *y*::*real*
⟨*proof*⟩

lemma *exponential-integral-mu*: *integral* $\{x..<x+t\}$ $(\lambda y.\ \$\mu\text{-}y) = \$\mu\text{-}x * t$
if *exponential-interpolation* $x+1 < \$\psi$ $0 \leq t$ $t \leq 1$ **for** x::*nat* **and** t::*real*
⟨*proof*⟩

lemma *exponential-p-mu*: $\$p\text{-}x = exp\ (-\$\mu\text{-}x)$ **if** *exponential-interpolation* $x+1 < \$\psi$ **for** x::*nat*
⟨*proof*⟩

corollary *exponential-mu-p*: $\$\mu\text{-}x = -\ ln\ (\$p\text{-}x)$ **if** *exponential-interpolation* $x+1 < \$\psi$ **for** x::*nat*
⟨*proof*⟩

corollary *exponential-mu-xt-p*: $\$\mu\text{-}(x+t) = -\ ln\ (\$p\text{-}x)$
if *exponential-interpolation* $x+1 < \$\psi$ $0 \leq t$ $t < 1$ **for** x::*nat* **and** t::*real*
⟨*proof*⟩

corollary *exponential-q-mu*: $\$q\text{-}x = 1 - exp\ (-\$\mu\text{-}x)$
if *exponential-interpolation* $x+1 < \$\psi$ **for** x::*nat*
⟨*proof*⟩

lemma *exponential-p*: $\$p\text{-}\{t\&x\} = (\$p\text{-}x).\hat{}t$
if *exponential-interpolation* $x+1 < \$\psi$ $0 \leq t$ $t \leq 1$ **for** x::*nat* **and** t::*real*
⟨*proof*⟩

lemma *exponential-q*: $\$q\text{-}\{t\&x\} = 1 - (1 - \$q\text{-}x).\hat{}t$
if *exponential-interpolation* $x+1 < \$\psi$ $0 \leq t$ $t \leq 1$ **for** x::*nat* **and** t::*real*
⟨*proof*⟩

lemma *exponential-l-p*: $\$l\text{-}(x+t) = \$l\text{-}x * (\$p\text{-}x).\hat{}t$
if *exponential-interpolation* $x+1 < \$\psi$ $0 \leq t$ $t \leq 1$ **for** x::*nat* **and** t::*real*
⟨*proof*⟩

lemma *exponential-l-has-derivative-at-frac*:
$((\lambda s.\ \$l\text{-}(x+s))$ *has-real-derivative* $(-\ \$l\text{-}x * \$\mu\text{-}x * (\$p\text{-}x).\hat{}t))$ $(at\ t)$
if *exponential-interpolation* $x+1 < \$\psi$ $0 < t$ $t < 1$ **for** x::*nat* **and** t::*real*
⟨*proof*⟩

lemma *exponential-l-has-derivative-at-frac'*:
$(l$ *has-real-derivative* $(-\ \$l\text{-}x * \$\mu\text{-}x * (\$p\text{-}x).\hat{}(y-x)))$ $(at\ y)$
if *exponential-interpolation* $x+1 < \$\psi$ $x < y$ $y < x+1$ **for** x::*nat* **and** y::*real*
⟨*proof*⟩

lemma *exponential-l-differentiable-on-frac*:
l *differentiable-on* $\{x<..<x+1\}$ **if** *exponential-interpolation* $x+1 < \$\psi$ **for** x::*nat*
⟨*proof*⟩

lemma *exponential-l-has-right-derivative-at-nat*:
$(l$ *has-real-derivative* $(-\ \$l\text{-}x * \$\mu\text{-}x))$ $(at\text{-}right\ x)$
if *exponential-interpolation* $x+1 < \$\psi$ **for** x::*nat*

⟨*proof*⟩

lemma *exponential-l-has-left-derivative-at-nat*:
(*l has-real-derivative* (− $*l-x* ∗ $*μ*-(*real x* − *1*))) (*at-left x*)
if *exponential-interpolation x* < $*ψ* **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-l-has-derivative-at-nat-iff-mu*:
(*l has-real-derivative* (− $*l-x* ∗ $*μ-x*)) (*at x*) ⟷ $*μ-x* = $*μ*-(*real x* − *1*)
if *exponential-interpolation x+1* < $*ψ* **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-l-differentiable-at-nat-iff-mu*:
l differentiable at x ⟷ $*μ-x* = $*μ*-(*real x* − *1*)
if *exponential-interpolation x+1* < $*ψ* **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-L-d-mu*: $*L-x* = $*d-x* / $*μ-x*
if *exponential-interpolation* $*μ-x* ≠ *0 x+1* < $*ψ* **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-mx-mu*: $*m-x* = $*μ-x* **if** *exponential-interpolation x+1* < $*ψ*
for *x*::*nat*
⟨*proof*⟩

lemma *exponential-d-mu-sums-T*: (*λk*. $*d*-(*x*+*k*) / $*μ*-(*x*+*k*)) *sums* $*T-x*
if *exponential-interpolation total-finite* ⋀*k*::*nat*. $*μ*-(*x*+*k*) ≠ *0* **for** *x*::*nat*
⟨*proof*⟩

lemma *exponential-e-d-l-mu*: (*λk*. $*d*-(*x*+*k*) / ($*l-x* ∗ $*μ*-(*x*+*k*))) *sums* $*e*‘∘-*x*
if *exponential-interpolation total-finite* ⋀*k*::*nat*. $*μ*-(*x*+*k*) ≠ *0* **for** *x*::*nat*
⟨*proof*⟩

end

6.5 Limited Life Table

locale *limited-life-table* = *life-table* +
assumes *l-limited*: ∃ *x*::*real*. $*l-x* = *0*
begin

lemma *limited-survival-function-MM-X*: *limited-survival-function* 𝔐 *X*
⟨*proof*⟩

end

sublocale *limited-life-table* ⊆ *limited-survival-function* 𝔐 *X*
⟨*proof*⟩

context *limited-life-table*
begin

notation *ult-age* (‹$ω›)

lemma *l-omega-0*: $*l*-$ω = *0*
⟨*proof*⟩

lemma *l-0-equiv-nat*: $*l*-*x* = *0* ⟷ *x* ≥ $ω **for** *x*::*nat*
⟨*proof*⟩

lemma *d-l-equiv-nat*: $*d*-{*t*&*x*} = $*l*-*x* ⟷ *x*+*t* ≥ $ω **if** *t* ≥ *0* **for** *x t* :: *nat*
⟨*proof*⟩

corollary *d-1-omega-l*: $*d*-($ω−*1*) = $*l*-($ω−*1*)
⟨*proof*⟩

lemma *limited-life-table-imp-total-finite*: *total-finite*
⟨*proof*⟩

context
fixes *x*::*nat*
assumes *x-lt-omega*[*simp*]: *x* < $ω
begin

lemma *curt-e-sum-l-finite-nat*: $*e*-*x* = (∑ *k*<*n*. $*l*-(*x*+*k*+*1*)) / $*l*-*x*
if ⋀*k*::*nat*. *k* < *n* ⟹ *isCont l* (*x*+*k*+*1*) *x*+*n* ≥ $ω **for** *n*::*nat*
⟨*proof*⟩

end

end

end
theory *Examples*
imports *Life-Table*
begin

7 Examples

The following lemma is a verification of the solution to the multiple choice question No. 3 of Exam LTAM Spring 2022 by Society of Actuaries.

context *smooth-survival-function*
begin

lemma *SoA-LTAM-2022-Spring-MCQ-No3*:
assumes ⋀*x*::*real*. *0* ≤ *x* ⟹ *x* ≤ *100* ⟹ *ccdf* (*distr* 𝔐 *borel X*) *x* = (*1* − *0.01*∗*x*).^*0.5*

shows $|1000*\$\mu\text{-}25 - 6.7| < 0.05$
⟨*proof*⟩

end

The following lemma is a verification of the solution to the problem No. 2. (1)-1 of Life Insurance Mathematics 2016 by the Institute of Actuaries of Japan, slightly modified; see the remark below.

context *smooth-life-table*
begin

lemma *IAJ-Life-Insurance-Math-2016-2-1-1*:
fixes $a\ b :: real$
assumes $-1 < a\ a < 0\ 0 < b\ -b/a \leq \$\psi$ **and**
total-finite **and**
$\bigwedge x.\ 0 < x \Longrightarrow x < -b/a \Longrightarrow l$ *differentiable at* x **and**
$\bigwedge x.\ 0 \leq x \Longrightarrow x < -b/a \Longrightarrow \$e\text{`}\circ\text{-}x = a*x + b$
shows $\bigwedge x.\ 0 \leq x \Longrightarrow x < -b/a \Longrightarrow \$l\text{-}x = \$l\text{-}0 * (b\ /\ (a*x + b)).\hat{}((a+1)/a)$
⟨*proof*⟩

REMARK. The original problem lacks the following hypotheses: (i) $0 < b$, (ii) $-b/a \leq \$\psi$, (iii) $\forall x.\ 0 < x < -b/a \longrightarrow l$ *differentiable at* x, (iv) $\forall x.\ 0 \leq x < -b/a \longrightarrow l$ *integrable-on* $\{x..\}$. Moreover, the hypothesis $\forall x.\ 0 \leq x < -b/a$ is originally $\forall x.\ 0 \leq x \leq -b/a$.

end

end