

# Abel's Limit Theorem in Isabelle/HOL

Kangfeng Ye  
University of York, UK

kangfeng.ye@york.ac.uk

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## Abstract

This theory proves the Abel's limit theorem on power series of real numbers, and then an example is shown to use the theorem to cover the boundary cases of binomial series.

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## 1 Abel's limit theorem on real power series

**theory** *Abel-Limit-Theorem*

**imports** *HOL-Analysis.Generalised-Binomial-Theorem*

**begin**

Abel's theorem or Abel's limit theorem [3] provides a crucial link between the behavior of a power series inside its interval of convergence (such as  $(-1, 1)$ ) and its value at the boundary such as  $-1$  or  $1$ .

This section presents the proof of Abel's limit theorem, which relates a limit of a power series to the sum of its real coefficients, as shown below:

$$\lim_{x \rightarrow 1^-} f(x) = f(1) = \sum_{k=0}^{\infty} a_k \quad \text{where } f(x) = \sum_{k=0}^{\infty} a_k x^k$$

if the power series has its radius of convergence equal to 1 and  $\sum_{k=0}^{\infty} a_k$  converges, where  $a_k$  is the coefficient of the  $k$ -th term.

That is,  $f(x)$  is continuous from the left at 1.

The proof of continuity or the limit of  $f(x)$  is based on the  $\varepsilon$ - $\delta$  definition. This proof uses summation by parts or Abel transformation to express the power series  $f(x)$  as a power series whose coefficients are the partial sums  $(\sum_{k=0}^n a_k)$  of the coefficients of  $f(x)$ , instead of  $a_k$ . Then the new power series is split into two parts. The goal is to show that each part contributes to  $\varepsilon/2$  for any  $x$  satisfying  $(1 - x) < \delta$ .

Several references [3, 1, 2] are used to construct this proof.

**theorem** *Abel-limit-theorem:*

```

fixes a :: nat => real
defines f1 ≡ (λ(x::real) n. a n * x ^ n)
defines f ≡ (λ(x::real). ∑ n. f1 x n)
assumes summable-a: summable a and
          conv-radius-1: conv-radius a = 1
shows (f → (∑ n. a n)) (at-left 1)
⟨proof⟩

```

**lemma** *filterlim-at-right-at-left-eq:*

```

shows ((λx. f (-x)) → l) (at-right (-1)) ↔ ((λx. f (x)) → l) (at-left
(1::real))
⟨proof⟩

```

Abel's limit theorem is also suitable for continuous from the right at -1.

**corollary** *Abel-limit-theorem':*

```

fixes a :: nat => real
defines f1 ≡ (λ(x::real) n. a n * x ^ n)
defines f ≡ (λ(x::real). ∑ n. f1 x n)
assumes summable-a: summable a and
          conv-radius-1: conv-radius a = 1
shows ((λx. f (-x)) → (∑ n. a n)) (at-right (-1))
⟨proof⟩

```

**end**

## 2 Example application: boundary cases of binomial theorem

**theory** *Binomial-Sqrt-Series-Boundary*

```

imports
  Abel-Limit-Theorem
  Catalan-Numbers.Catalan-Numbers
  HOL-Real-Asymp.Real-Asymp

```

**begin**

Newton's generalized binomial theorem is applicable to  $|x| < 1$  as seen from this  $|?z| < 1 \implies (\lambda n. (1 / 2 gchoose n) * ?z^n) \text{ sums sqrt } (1 + ?z)$ .

However, it doesn't apply to the boundary cases where  $|x| = 1$  or  $|x| = -1$ . Here, Abel's limit theorem is applied to establish the binomial theorem for the boundary cases.

## 2.1 Binomial series

**lemma** *binomial-sqrt-series*:

**fixes**  $x :: \text{real}$

**assumes**  $|x| < 1$

**shows**  $\text{suminf } (\lambda n. ((1/2) \text{ gchoose } n) * x^n) = \text{sqrt } (1 + x)$

*<proof>*

The generalized binomial coefficient  $a \text{ gchoose } n$  where  $a = \frac{1}{2}$  can also be rewritten as an expression including a Catalan numbers. This is used to prove its summability using the property of Catalan numbers.

**lemma** *gbinomial-1-2-catalan*:  $((1/2) \text{ gchoose } (\text{Suc } n)) = ((-1)^n / (2^{2*n+1}))$

$* \text{real } (\text{catalan } n)$

*<proof>*

**lemma** *gbinomial-1-2-catalan'*:  $((1/2) \text{ gchoose } (\text{Suc } n)) = ((-1)^n / 2) * (1/4)^n$

$* \text{real } (\text{catalan } n)$

*<proof>*

Rewrite the generalized binomial coefficient  $a \text{ gchoose } n$  where  $a = \frac{1}{2}$  as a binomial coefficient.

**lemma** *gbinomial-1-2-simp*:

$((1/2) \text{ gchoose } (\text{Suc } n)) = ((-1)^n / \text{real } (2^{2*n+1}) * (\text{Suc } n)) * ((2*n) \text{ choose } n)$

*<proof>*

**lemma** *summable-real-powr-iff'*:  $\text{summable } (\lambda n. 1 / \text{of-nat } n \text{ powr } s :: \text{real}) \longleftrightarrow s > 1$

*<proof>*

**lemma** *summable-1-2-gchoose*:  $\text{summable } (\lambda n. ((1::\text{real})/2) \text{ gchoose } n)$

*<proof>*

**lemma** *gbinomial-1-2-gchoose-sum-sqrt-2*:

**shows**  $(\sum n. (((1::\text{real}) / (2::\text{real}) \text{ gchoose } n))) = \text{sqrt } 2$  (**is**  $(\sum n. ?f-1 n) = -$ )

*<proof>*

## 2.2 Alternating series

**lemma** *gbinomial-ratio-limit'*:

**fixes**  $a :: 'a :: \text{real-normed-field}$

**assumes**  $a \notin \mathbb{N}$

**shows**  $(\lambda n. ((a \text{ gchoose } n) * (-1) ^ n) / ((a \text{ gchoose } \text{Suc } n) * (-1) ^ (\text{Suc } n)))$   
 $\longrightarrow 1$   
 $\langle \text{proof} \rangle$

**lemma** *conv-radius-gchoose-alternating*:

**fixes**  $a :: 'a :: \{\text{real-normed-field}, \text{banach}\}$

**assumes**  $a \notin \mathbb{N}$

**shows** *conv-radius*  $(\lambda n::\text{nat}. (a \text{ gchoose } n) * (-1) ^ n) = (1::\text{ereal})$

$\langle \text{proof} \rangle$

**lemma** *summable-1-2-gchoose-alternating*:

*summable*  $(\lambda n::\text{nat}. (1 / 2 \text{ gchoose } n) * (-1) ^ n :: \text{real})$  (**is summable** ?f)

$\langle \text{proof} \rangle$

**lemma** *gbinomial-1-2-gchoose-alternating-sum-0*:

**shows**  $(\sum n. ((1/2 \text{ gchoose } n) * (-1::\text{real}) ^ n)) = 0$  (**is**  $(\sum n. ?f-1 n) = 0$ )

$\langle \text{proof} \rangle$

### 2.3 Binomial sqrt series with the boundary cases

This lemma incorporates the boundary values where  $x = 1$  and  $x = -1$ .

**theorem** *binomial-sqrt-series'*:

**assumes**  $|x| \leq (1 :: \text{real})$

**shows** *suminf*  $(\lambda n. ((1/2) \text{ gchoose } n) * x ^ n) = \text{sqrt } (1 + x)$

$\langle \text{proof} \rangle$

**end**

## References

- [1] Proof of Abel's limit theorem — planetmath.org. <https://planetmath.org/proofofabelslimittheorem>. [Accessed 11-11-2025].
- [2] F. Holland. Abel's limit theorem, its converse, and multiplication formulae for  $\Gamma(x)$ . *Irish Math. Soc. Bull.*, 0089:57–64, 2022.
- [3] Wikipedia contributors. Abel's theorem. [Accessed 11-11-2025]. URL: [https://en.wikipedia.org/wiki/Abel%27s\\_theorem](https://en.wikipedia.org/wiki/Abel%27s_theorem).