Interval Temporal Logic on Natural Numbers

David Trachtenherz

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Abstract

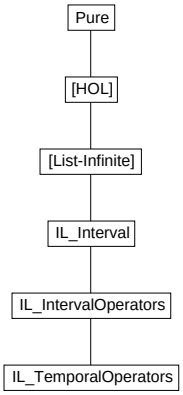
We introduce a theory of temporal logic operators using sets of natural numbers as time domain, formalized in a shallow embedding manner. The theory comprises special natural intervals (theory IL_Interval: open and closed intervals, continuous and modulo intervals, interval traversing results), operators for shifting intervals to left/right on the number axis as well as expanding/contracting intervals by constant factors (theory IL_IntervalOperators.thy), and ultimately definitions and results for unary and binary temporal operators on arbitrary natural sets (theory IL_TemporalOperators).

Contents

1 IL-Interval: Intervals and operations for temporal logic declarations **4**
- 1.1 Time intervals – definitions and basic lemmata 4
  - 1.1.1 Definitions 4
  - 1.1.2 Membership in an interval 5
  - 1.1.3 Interval conversions 7
  - 1.1.4 Finiteness and emptiness of intervals 9
  - 1.1.5 *Min* and *Max* element of an interval 10
- 1.2 Adding and subtracting constants to interval elements 11
- 1.3 Relations between intervals 14
  - 1.3.1 Auxiliary lemmata 14
  - 1.3.2 Subset relation between intervals 14
  - 1.3.3 Equality of intervals 21
  - 1.3.4 Inequality of intervals 22
- 1.4 Union and intersection of intervals 23
- 1.5 Cutting intervals 27
- 1.6 Cardinality of intervals 36
- 1.7 Functions *inext* and *iprev* with intervals 38
  - 1.7.1 Mirroring of intervals 42
  - 1.7.2 Functions *inext-nth* and *iprev-nth* on intervals 43
- 1.8 Induction with intervals 45

2 IL-IntervalOperators: Arithmetic operators on natural intervals 46
2.1 Arithmetic operations with intervals 46
2.1.1 Addition of and multiplication by constants 46
2.1.2 Some conversions between intervals using constant addition and multiplication 51
2.1.3 Subtraction of constants 51
2.1.4 Subtraction of intervals from constants 56
2.1.5 Division of intervals by constants 60
2.2 Interval cut operators with arithmetic interval operators . . . 69
2.3 *inext* and *iprev* with interval operators 72
2.4 Cardinality of intervals with interval operators 74
2.5 Results about sets of intervals 80
2.5.1 Set of intervals without and with emtpy interval . . . 80
2.5.2 Interval sets are closed under cutting 85
2.5.3 Interval sets are closed under addition and multiplication 86
2.5.4 Interval sets are closed with certain conditions under subtraction 86
2.5.5 Interval sets are not closed under division 87
2.5.6 Sets of intervals closed under division 88

3 IL-TemporalOperators: Temporal logic operators on natural intervals 92
3.1 Basic definitions 93
3.2 Basic lemmata for temporal operators 98
3.2.1 Intro/elim rules 98
3.2.2 Rewrite rules for trivial simplification 98
3.2.3 Empty sets and singletons 104
3.2.4 Conversions between temporal operators 105
3.2.5 Some implication results 108
3.2.6 Congruence rules for temporal operators' predicates . 109
3.2.7 Temporal operators with set unions/intersections and subsets 109
3.3 Further results for temporal operators 110
3.4 Temporal operators and arithmetic interval operators 113


Pure
[HOL]
[List-Infinite]
IL_Interval
IL_IntervalOperators
IL_TemporalOperators

1 IL-Interval: Intervals and operations for temporal logic declarations

theory *IL-Interval*
imports
Main Nat-Infinity
SetInterval2 InfiniteSet2 SetIntervalStep
Util-Nat Util-MinMax Util-Div
begin

1.1 Time intervals – definitions and basic lemmata

1.1.1 Definitions

types *Time* = *nat*

types *iT* = *Time set*
typ *iT*

Infinite interval starting at some natural *n*.

definition
iFROM :: *Time* ⇒ *iT* ([-...])
where
[*n*...] ≡ {*n*..}

Finite interval starting at *0*::'*a* and ending at some natural *n*.

definition
iTILL :: *Time* ⇒ *iT* ([...-])
where
[...*n*] ≡ {..*n*}

Finite bounded interval containing the naturals between *n* and *n* + *d*. *d* denotes the difference between left and right interval bound. The number of elements is *d* + (*1*::'*a*) so that an empty interval cannot be defined.

definition
iIN :: *Time* ⇒ *nat* ⇒ *iT* ([-...,-])
where
[*n*...,*d*] ≡ {*n*..*n*+*d*}

Infinite modulo interval containing all naturals having the same division remainder modulo *m* as *r*, and beginning at *n*.

definition
iMOD :: *Time* ⇒ *nat* ⇒ *iT* ([-, *mod* -])
where
[*r*, *mod m*] ≡ { *x*. *x mod m* = *r mod m* ∧ *r* ≤ *x*}

Finite bounded modulo interval containing all naturals having the same division remainder modulo *m* as *r*, beginning at *n*, and ending after *c* cycles

at $r + m * c$. The number of elements is $c + (1::'a)$ so that an empty interval cannot be defined.

definition
iMODb :: *Time* ⇒ *nat* ⇒ *nat* ⇒ *iT* ([-, *mod* -, -])
where
$[r, \mathit{mod}\ m, c] \equiv \{ x.\ x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x \wedge x \leq r + m * c\}$

1.1.2 Membership in an interval

lemmas *iT-defs* = *iFROM-def iTILL-def iIN-def iMOD-def iMODb-def*

lemma *iFROM-iff*: $x \in [n...] = (n \leq x)$
⟨*proof*⟩
lemma *iTILL-iff*: $x \in [...n] = (x \leq n)$
⟨*proof*⟩
lemma *iIN-iff*:$x \in [n...,d] = (n \leq x \wedge x \leq n + d)$
⟨*proof*⟩
lemma *iMOD-iff*: $x \in [r, \mathit{mod}\ m] = (x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x)$
⟨*proof*⟩
lemma *iMODb-iff*: $x \in [r, \mathit{mod}\ m, c] =$
$(x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x \wedge x \leq r + m * c)$
⟨*proof*⟩

lemma *iFROM-D*: $x \in [n...] \Longrightarrow (n \leq x)$
⟨*proof*⟩
lemma *iTILL-D*: $x \in [...n] \Longrightarrow (x \leq n)$
⟨*proof*⟩
corollary *iIN-geD*: $x \in [n...,d] \Longrightarrow n \leq x$
⟨*proof*⟩
corollary *iIN-leD*: $x \in [n...,d] \Longrightarrow x \leq n + d$
⟨*proof*⟩
corollary *iMOD-modD*: $x \in [r, \mathit{mod}\ m] \Longrightarrow x\ \mathit{mod}\ m = r\ \mathit{mod}\ m$
⟨*proof*⟩
corollary *iMOD-geD*: $x \in [r, \mathit{mod}\ m] \Longrightarrow r \leq x$
⟨*proof*⟩
corollary *iMODb-modD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow x\ \mathit{mod}\ m = r\ \mathit{mod}\ m$
⟨*proof*⟩
corollary *iMODb-geD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow r \leq x$
⟨*proof*⟩
corollary *iMODb-leD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow x \leq r + m * c$
⟨*proof*⟩

lemmas *iT-iff* = *iFROM-iff iTILL-iff iIN-iff iMOD-iff iMODb-iff*

lemmas *iT-drule* =
iFROM-D
iTILL-D
iIN-geD iIN-leD
iMOD-modD iMOD-geD

iMODb-modD iMODb-geD iMODb-leD
thm *iT-drule*

lemma
iFROM-I [*intro*]: $n \leq x \Longrightarrow x \in [n\ldots]$ **and**
iTILL-I [*intro*]: $x \leq n \Longrightarrow x \in [\ldots n]$ **and**
iIN-I [*intro*]: $n \leq x \Longrightarrow x \leq n + d \Longrightarrow x \in [n\ldots,d]$ **and**
iMOD-I [*intro*]: $x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \in [r,\ mod\ m]$ **and**
iMODb-I [*intro*]: $x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \leq r + m * c \Longrightarrow x \in [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma
iFROM-E [*elim*]: $x \in [n\ldots] \Longrightarrow (n \leq x \Longrightarrow P) \Longrightarrow P$ **and**
iTILL-E [*elim*]: $x \in [\ldots n] \Longrightarrow (x \leq n \Longrightarrow P) \Longrightarrow P$ **and**
iIN-E [*elim*]: $x \in [n\ldots,d] \Longrightarrow (n \leq x \Longrightarrow x \leq n + d \Longrightarrow P) \Longrightarrow P$ **and**
iMOD-E [*elim*]: $x \in [r,\ mod\ m] \Longrightarrow (x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow P) \Longrightarrow P$ **and**
iMODb-E [*elim*]: $x \in [r,\ mod\ m,\ c] \Longrightarrow (x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \leq r + m * c \Longrightarrow P) \Longrightarrow P$
⟨*proof*⟩

lemma *iIN-Suc-insert-conv*:
$insert\ (Suc\ (n + d))\ [n\ldots,d] = [n\ldots,Suc\ d]$
⟨*proof*⟩

lemma *iTILL-Suc-insert-conv*: $insert\ (Suc\ n)\ [\ldots n] = [\ldots Suc\ n]$
⟨*proof*⟩

lemma *iMODb-Suc-insert-conv*:
$insert\ (r + m * Suc\ c)\ [r,\ mod\ m,\ c] = [r,\ mod\ m,\ Suc\ c]$
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*
⟨*proof*⟩
thm *add-le-imp-le-right*
⟨*proof*⟩

lemma *iFROM-pred-insert-conv*: $insert\ (n - Suc\ 0)\ [n\ldots] = [n - Suc\ 0\ldots]$
⟨*proof*⟩

lemma *iIN-pred-insert-conv*:
$0 < n \Longrightarrow insert\ (n - Suc\ 0)\ [n\ldots,d] = [n - Suc\ 0\ldots,Suc\ d]$
⟨*proof*⟩

lemma *iMOD-pred-insert-conv*:

$m \leq r \Longrightarrow insert\ (r - m)\ [r,\ mod\ m] = [r - m,\ mod\ m]$
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*[*of* $r-m\ x\ m$]
⟨*proof*⟩

lemma *iMODb-pred-insert-conv*:
$m \leq r \Longrightarrow insert\ (r - m)\ [r,\ mod\ m,\ c] = [r - m,\ mod\ m,\ Suc\ c]$
⟨*proof*⟩

lemma *iFROM-Suc-pred-insert-conv*: $insert\ n\ [Suc\ n\ldots] = [n\ldots]$
⟨*proof*⟩
lemma *iIN-Suc-pred-insert-conv*: $insert\ n\ [Suc\ n\ldots,d] = [n\ldots,Suc\ d]$
⟨*proof*⟩
lemma *iMOD-Suc-pred-insert-conv*: $insert\ r\ [r + m,\ mod\ m] = [r,\ mod\ m]$
⟨*proof*⟩
lemma *iMODb-Suc-pred-insert-conv*: $insert\ r\ [r + m,\ mod\ m,\ c] = [r,\ mod\ m,\ Suc\ c]$
⟨*proof*⟩

lemmas *iT-Suc-insert* =
iIN-Suc-insert-conv
iTILL-Suc-insert-conv
iMODb-Suc-insert-conv
lemmas *iT-pred-insert* =
iFROM-pred-insert-conv
iIN-pred-insert-conv
iMOD-pred-insert-conv
iMODb-pred-insert-conv
lemmas *iT-Suc-pred-insert* =
iFROM-Suc-pred-insert-conv
iIN-Suc-pred-insert-conv
iMOD-Suc-pred-insert-conv
iMODb-Suc-pred-insert-conv

lemma *iMOD-mem-diff*: $[\![\ a \in [r,\ mod\ m];\ b \in [r,\ mod\ m]\ ]\!] \Longrightarrow (a - b)\ mod\ m = 0$
⟨*proof*⟩
lemma *iMODb-mem-diff*: $[\![\ a \in [r,\ mod\ m,\ c];\ b \in [r,\ mod\ m,\ c]\ ]\!] \Longrightarrow (a - b)\ mod\ m = 0$
⟨*proof*⟩

1.1.3 Interval conversions

lemma *iIN-0-iTILL-conv*:$[0\ldots,n] = [\ldots n]$
⟨*proof*⟩
lemma *iIN-iTILL-iTILL-conv*: $0 < n \Longrightarrow [n\ldots,d] = [\ldots n+d] - [\ldots n - Suc\ 0]$
⟨*proof*⟩
lemma *iIN-iFROM-iTILL-conv*: $[n\ldots,d] = [n\ldots] \cap [\ldots n+d]$
⟨*proof*⟩

lemma *iMODb-iMOD-iTILL-conv*: $[r,\ mod\ m,\ c] = [r,\ mod\ m] \cap [\ldots r+m*c]$
⟨*proof*⟩
lemma *iMODb-iMOD-iIN-conv*: $[r,\ mod\ m,\ c] = [r,\ mod\ m] \cap [r\ldots,m*c]$
⟨*proof*⟩

lemma *iFROM-iTILL-iIN-conv*: $n \leq n' \Longrightarrow [n\ldots] \cap [\ldots n'] = [n\ldots,n'-n]$
⟨*proof*⟩
lemma *iMOD-iTILL-iMODb-conv*:
$r \leq n \Longrightarrow [r,\ mod\ m] \cap [\ldots n] = [r,\ mod\ m,\ (n - r)\ div\ m]$
⟨*proof*⟩
thm *le-imp-sub-mod-le*
⟨*proof*⟩
lemma *iMOD-iIN-iMODb-conv*:
$[r,\ mod\ m] \cap [r\ldots,d] = [r,\ mod\ m,\ d\ div\ m]$
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*
⟨*proof*⟩
thm *subst*[*of* {} - $\lambda t.\ \forall x.(x - t) = x$]
thm *subst*[*of* {} - $\lambda t.\ \forall x.(x - t) = x$, *THEN spec*]
⟨*proof*⟩
thm *disjoint-iff-not-equal*
⟨*proof*⟩

thm *UNIV-def*
lemma *iFROM-0*: $[0\ldots] = UNIV$
⟨*proof*⟩
lemma *iTILL-0*: $[\ldots 0] = \{0\}$
⟨*proof*⟩
lemma *iIN-0*: $[n\ldots,0] = \{n\}$
⟨*proof*⟩
lemma *iMOD-0*: $[r,\ mod\ 0] = [r\ldots,0]$
⟨*proof*⟩
lemma *iMODb-mod-0*: $[r,\ mod\ 0,\ c] = [r\ldots,0]$
⟨*proof*⟩
lemma *iMODb-0*: $[r,\ mod\ m,\ 0] = [r\ldots,0]$
⟨*proof*⟩
lemmas *iT-0* =
iFROM-0
iTILL-0
iIN-0
iMOD-0
iMODb-mod-0
iMODb-0
thm *iT-0*

lemma *iMOD-1*: $[r,\ mod\ Suc\ 0] = [r\ldots]$

⟨*proof*⟩
lemma *iMODb-mod-1*: [*r*, *mod Suc 0*, *c*] = [*r*...,*c*]
⟨*proof*⟩

1.1.4 Finiteness and emptiness of intervals

lemma
iFROM-not-empty: [*n*...] ≠ {} **and**
iTILL-not-empty: [...*n*] ≠ {} **and**
iIN-not-empty: [*n*...,*d*] ≠ {} **and**
iMOD-not-empty: [*r*, *mod m*] ≠ {} **and**
iMODb-not-empty: [*r*, *mod m*, *c*] ≠ {}
⟨*proof*⟩

lemmas *iT-not-empty* =
iFROM-not-empty
iTILL-not-empty
iIN-not-empty
iMOD-not-empty
iMODb-not-empty
thm *iT-not-empty*

lemma
iTILL-finite: *finite* [...*n*] **and**
iIN-finite: *finite* [*n*...,*d*] **and**
iMODb-finite: *finite* [*r*, *mod m*, *c*] **and**
iMOD-0-finite: *finite* [*r*, *mod 0*]
⟨*proof*⟩
lemma *iFROM-infinite*: *infinite* [*n*...]
⟨*proof*⟩

lemma *iMOD-infinite*: *0* < *m* ⟹ *infinite* [*r*, *mod m*]
thm *infinite-nat-iff-asc-chain*
⟨*proof*⟩

lemmas *iT-finite* =
iTILL-finite
iIN-finite
iMODb-finite iMOD-0-finite
thm *iT-finite*

lemmas *iT-infinite* =
iFROM-infinite
iMOD-infinite
thm *iT-infinite*

thm
iMax-finite-conv
iMax-infinite-conv

1.1.5 *Min* and *Max* element of an interval

lemma
iTILL-Min: $iMin\ [\ldots n] = 0$ **and**
iFROM-Min: $iMin\ [n\ldots] = n$ **and**
iIN-Min: $iMin\ [n\ldots,d] = n$ **and**
iMOD-Min: $iMin\ [r,\ mod\ m] = r$ **and**
iMODb-Min: $iMin\ [r,\ mod\ m,\ c] = r$
thm *iMin-equality*
⟨*proof*⟩

lemmas *iT-Min* =
iIN-Min
iTILL-Min
iFROM-Min
iMOD-Min
iMODb-Min
thm *iT-Min*

lemma
iTILL-Max: $Max\ [\ldots n] = n$ **and**
iIN-Max: $Max\ [n\ldots,d] = n{+}d$ **and**
iMODb-Max: $Max\ [r,\ mod\ m,\ c] = r + m * c$ **and**
iMOD-0-Max: $Max\ [r,\ mod\ 0] = r$
⟨*proof*⟩

lemmas *iT-Max* =
iTILL-Max
iIN-Max
iMODb-Max
iMOD-0-Max
thm *iT-Max*

lemma
iTILL-iMax: $iMax\ [\ldots n] = Fin\ n$ **and**
iIN-iMax: $iMax\ [n\ldots,d] = Fin\ (n{+}d)$ **and**
iMODb-iMax: $iMax\ [r,\ mod\ m,\ c] = Fin\ (r + m * c)$ **and**
iMOD-0-iMax: $iMax\ [r,\ mod\ 0] = Fin\ r$ **and**
iFROM-iMax: $iMax\ [n\ldots] = \infty$ **and**
iMOD-iMax: $0 < m \Longrightarrow iMax\ [r,\ mod\ m] = \infty$
⟨*proof*⟩

lemmas *iT-iMax* =
iTILL-iMax
iIN-iMax

iMODb-iMax
iMOD-0-iMax
iFROM-iMax
iMOD-iMax
thm *iT-iMax*

1.2 Adding and subtracting constants to interval elements

lemma
iFROM-plus: $x \in [n\ldots] \Longrightarrow x + k \in [n\ldots]$ **and**
iFROM-Suc: $x \in [n\ldots] \Longrightarrow Suc\ x \in [n\ldots]$ **and**
iFROM-minus: $[\![\ x \in [n\ldots];\ k \leq x - n\ ]\!] \Longrightarrow x - k \in [n\ldots]$ **and**
iFROM-pred: $n < x \Longrightarrow x - Suc\ 0 \in [n\ldots]$
⟨*proof*⟩

lemma
iTILL-plus: $[\![\ x \in [\ldots n];\ k \leq n - x\ ]\!] \Longrightarrow x + k \in [\ldots n]$ **and**
iTILL-Suc: $x < n \Longrightarrow Suc\ x \in [\ldots n]$ **and**
iTILL-minus: $x \in [\ldots n] \Longrightarrow x - k \in [\ldots n]$ **and**
iTILL-pred: $x \in [\ldots n] \Longrightarrow x - Suc\ 0 \in [\ldots n]$
⟨*proof*⟩

lemma *iIN-plus*: $[\![\ x \in [n\ldots,d];\ k \leq n + d - x\ ]\!] \Longrightarrow x + k \in [n\ldots,d]$
⟨*proof*⟩
lemma *iIN-Suc*: $[\![\ x \in [n\ldots,d];\ x < n + d\ ]\!] \Longrightarrow Suc\ x \in [n\ldots,d]$
⟨*proof*⟩
lemma *iIN-minus*: $[\![\ x \in [n\ldots,d];\ k \leq x - n\ ]\!] \Longrightarrow x - k \in [n\ldots,d]$
⟨*proof*⟩
lemma *iIN-pred*: $[\![\ x \in [n\ldots,d];\ n < x\ ]\!] \Longrightarrow x - Suc\ 0 \in [n\ldots,d]$
⟨*proof*⟩

lemma *iMOD-plus-divisor-mult*: $x \in [r,\ mod\ m] \Longrightarrow x + k * m \in [r,\ mod\ m]$
⟨*proof*⟩
corollary *iMOD-plus-divisor*: $x \in [r,\ mod\ m] \Longrightarrow x + m \in [r,\ mod\ m]$
⟨*proof*⟩
lemma *iMOD-minus-divisor-mult*:
$[\![\ x \in [r,\ mod\ m];\ k * m \leq x - r\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
thm *mod-diff-mult-self1*
⟨*proof*⟩
corollary *iMOD-minus-divisor-mult2*:
$[\![\ x \in [r,\ mod\ m];\ k \leq (x - r)\ div\ m\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
⟨*proof*⟩
thm *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*]
⟨*proof*⟩
corollary *iMOD-minus-divisor*:
$[\![\ x \in [r,\ mod\ m];\ m + r \leq x\ ]\!] \Longrightarrow x - m \in [r,\ mod\ m]$
⟨*proof*⟩
thm *iMOD-minus-divisor-mult*[*of x r m 1*]
⟨*proof*⟩

lemma *iMOD-plus*:
$x \in [r,\ mod\ m] \Longrightarrow (x + k \in [r,\ mod\ m]) = (k\ mod\ m = 0)$
⟨*proof*⟩
thm *mod-add-eq-imp-mod-0*[*THEN iffD1*]
⟨*proof*⟩
corollary *iMOD-Suc*:
$x \in [r,\ mod\ m] \Longrightarrow (Suc\ x \in [r,\ mod\ m]) = (m = Suc\ 0)$
⟨*proof*⟩
lemma *iMOD-minus*:
$[\![\ x \in [r,\ mod\ m];\ k \leq x - r\ ]\!] \Longrightarrow (x - k \in [r,\ mod\ m]) = (k\ mod\ m = 0)$
⟨*proof*⟩
corollary *iMOD-pred*:
$[\![\ x \in [r,\ mod\ m];\ r < x\ ]\!] \Longrightarrow (x - Suc\ 0 \in [r,\ mod\ m]) = (m = Suc\ 0)$
⟨*proof*⟩
thm *iMOD-Suc*[*of* $x - Suc\ 0$, *THEN iffD1*]
⟨*proof*⟩

lemma *iMODb-plus-divisor-mult*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k * m \leq r + m * c - x\ ]\!] \Longrightarrow x + k * m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩
lemma *iMODb-plus-divisor-mult2*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k \leq c - (x - r)\ div\ m\ ]\!] \Longrightarrow$
$x + k * m \in [r,\ mod\ m,\ c]$
⟨*proof*⟩
lemma *iMODb-plus-divisor*:
$[\![\ x \in [r,\ mod\ m,\ c];\ x < r + m * c\ ]\!] \Longrightarrow x + m \in [r,\ mod\ m,\ c]$
thm *less-mod-eq-imp-add-divisor-le*
⟨*proof*⟩
lemma *iMODb-minus-divisor-mult*:
$[\![\ x \in [r,\ mod\ m,\ c];\ r + k * m \leq x\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m,\ c]$
thm *mod-diff-mult-self1*
⟨*proof*⟩

lemma *iMODb-plus*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k \leq r + m * c - x\ ]\!] \Longrightarrow$
$(x + k \in [r,\ mod\ m,\ c]) = (k\ mod\ m = 0)$
⟨*proof*⟩
thm *mod-add-eq-imp-mod-0*[*THEN iffD1*]
⟨*proof*⟩
corollary *iMODb-Suc*:
$[\![\ x \in [r,\ mod\ m,\ c];\ x < r + m * c\ ]\!] \Longrightarrow$
$(Suc\ x \in [r,\ mod\ m,\ c]) = (m = Suc\ 0)$
⟨*proof*⟩
lemma *iMODb-minus*:
$[\![\ x \in [r,\ mod\ m,\ c];\ k \leq x - r\ ]\!] \Longrightarrow$
$(x - k \in [r,\ mod\ m,\ c]) = (k\ mod\ m = 0)$
⟨*proof*⟩
corollary *iMODb-pred*:
$[\![\ x \in [r,\ mod\ m,\ c];\ r < x\ ]\!] \Longrightarrow$

$(x - Suc\ 0 \in [r,\ mod\ m,\ c]) = (m = Suc\ 0)$
⟨*proof*⟩
thm *iMOD-pred*[*THEN iffD1*, *of x r m*]
⟨*proof*⟩

lemmas *iFROM-plus-minus* =
iFROM-plus
iFROM-Suc
iFROM-minus
iFROM-pred
thm *iFROM-plus-minus*

lemmas *iTILL-plus-minus* =
iTILL-plus
iTILL-Suc
iTILL-minus
iTILL-pred
thm *iTILL-plus-minus*

lemmas *iIN-plus-minus* =
iIN-plus
iIN-Suc
iTILL-minus
iIN-pred
thm *iIN-plus-minus*

lemmas *iMOD-plus-minus-divisor* =
iMOD-plus-divisor-mult
iMOD-plus-divisor
iMOD-minus-divisor-mult
iMOD-minus-divisor-mult2
iMOD-minus-divisor
thm *iMOD-plus-minus-divisor*

lemmas *iMOD-plus-minus* =
iMOD-plus
iMOD-Suc
iMOD-minus
iMOD-pred
thm *iMOD-plus-minus*

lemmas *iMODb-plus-minus-divisor* =
iMODb-plus-divisor-mult
iMODb-plus-divisor-mult2
iMODb-plus-divisor
iMODb-minus-divisor-mult
thm *iMODb-plus-minus-divisor*

lemmas *iMODb-plus-minus* =

iMODb-plus
iMODb-Suc
iMODb-minus
iMODb-pred
thm *iMODb-plus-minus*

lemmas *iT-plus-minus* =
iFROM-plus-minus
iTILL-plus-minus
iIN-plus-minus
iMOD-plus-minus-divisor
iMOD-plus-minus
iMODb-plus-minus-divisor
iMODb-plus-minus
thm *iT-plus-minus*

1.3 Relations between intervals

1.3.1 Auxiliary lemmata

lemma *Suc-in-imp-not-subset-iMOD*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow \neg\ S \subseteq [r, mod\ m]$
thm *iMOD-Suc*[*THEN iffD1*]
⟨*proof*⟩

corollary *Suc-in-imp-neq-iMOD*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow S \neq [r, mod\ m]$
⟨*proof*⟩

lemma *Suc-in-imp-not-subset-iMODb*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow \neg\ S \subseteq [r, mod\ m, c]$
⟨*proof*⟩
thm *iMODb-Suc*[*THEN iffD1*]
⟨*proof*⟩
corollary *Suc-in-imp-neq-iMODb*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow S \neq [r, mod\ m, c]$
⟨*proof*⟩

1.3.2 Subset relation between intervals

lemma
iIN-iFROM-subset-same: $[n..,d] \subseteq [n..]$ **and**
iIN-iTILL-subset-same: $[n..,d] \subseteq [..n+d]$ **and**
iMOD-iFROM-subset-same: $[r, mod\ m] \subseteq [r..]$ **and**
iMODb-iTILL-subset-same: $[r, mod\ m, c] \subseteq [..r+m*c]$ **and**
iMODb-iIN-subset-same: $[r, mod\ m, c] \subseteq [r..,m*c]$ **and**
iMODb-iMOD-subset-same: $[r, mod\ m, c] \subseteq [r, mod\ m]$
⟨*proof*⟩

lemmas *iT-subset-same* =

iIN-iFROM-subset-same
iIN-iTILL-subset-same
iMOD-iFROM-subset-same
iMODb-iTILL-subset-same
iMODb-iIN-subset-same
iMODb-iTILL-subset-same
iMODb-iMOD-subset-same
thm *iT-subset-same*

lemma *iMODb-imp-iMOD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow x \in [r, \mathit{mod}\ m]$
⟨*proof*⟩
lemma *iMOD-imp-iMODb*:
$[\![\ x \in [r, \mathit{mod}\ m];\ x \leq r + m * c\ ]\!] \Longrightarrow x \in [r, \mathit{mod}\ m, c]$
⟨*proof*⟩

lemma *iMOD-singleton-subset-conv*: $([r, \mathit{mod}\ m] \subseteq \{a\}) = (r = a \wedge m = 0)$
⟨*proof*⟩
lemma *iMOD-singleton-eq-conv*: $([r, \mathit{mod}\ m] = \{a\}) = (r = a \wedge m = 0)$
⟨*proof*⟩

lemma *iMODb-singleton-subset-conv*:
$([r, \mathit{mod}\ m, c] \subseteq \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩
lemma *iMODb-singleton-eq-conv*:
$([r, \mathit{mod}\ m, c] = \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩

lemma *iMODb-subset-imp-divisor-mod-0*:
$[\![\ 0 < c';\ [r', \mathit{mod}\ m', c'] \subseteq [r, \mathit{mod}\ m, c]\ ]\!] \Longrightarrow m'\ \mathit{mod}\ m = 0$
⟨*proof*⟩
thm *mod-add-eq-imp-mod-0*[*of* r' m' m]
⟨*proof*⟩
lemma *iMOD-subset-imp-divisor-mod-0*:
$[r', \mathit{mod}\ m'] \subseteq [r, \mathit{mod}\ m] \Longrightarrow m'\ \mathit{mod}\ m = 0$
⟨*proof*⟩
thm *mod-add-eq-imp-mod-0*[*of* r' m' m]
⟨*proof*⟩

lemma *iMOD-subset-imp-iMODb-subset*:
$[\![\ [r', \mathit{mod}\ m'] \subseteq [r, \mathit{mod}\ m];\ r' + m' * c' \leq r + m * c\ ]\!] \Longrightarrow$
$[r', \mathit{mod}\ m', c'] \subseteq [r, \mathit{mod}\ m, c]$
⟨*proof*⟩

lemma *iMODb-subset-imp-iMOD-subset*:
⟦ [r′, mod m′, c′] ⊆ [r, mod m, c]; 0 < c′ ⟧ ⟹
[r′, mod m′] ⊆ [r, mod m]
thm *subsetD*
⟨*proof*⟩
thm *subsetI*
⟨*proof*⟩
thm *mod-eq-mod-0-imp-mod-eq*
⟨*proof*⟩
thm *iMODb-subset-imp-divisor-mod-0*
⟨*proof*⟩

lemma *iMODb-0-iMOD-subset-conv*:
([r′, mod m′, 0] ⊆ [r, mod m]) =
(r′ mod m = r mod m ∧ r ≤ r′)
⟨*proof*⟩

lemma *iFROM-subset-conv*: ([n′...] ⊆ [n...]) = (n ≤ n′)
⟨*proof*⟩

lemma *iFROM-iMOD-subset-conv*: ([n′...] ⊆ [r, mod m]) = (r ≤ n′ ∧ m = Suc 0)
⟨*proof*⟩
thm *iMin-subset*[*OF iFROM-not-empty*]
⟨*proof*⟩
thm *Suc-in-imp-not-subset-iMOD*
⟨*proof*⟩

lemma *iIN-subset-conv*: ([n′...,d′] ⊆ [n...,d]) = (n ≤ n′ ∧ n′+d′ ≤ n+d)
⟨*proof*⟩

lemma *iIN-iFROM-subset-conv*: ([n′...,d′] ⊆ [n...]) = (n ≤ n′)
⟨*proof*⟩

lemma *iIN-iTILL-subset-conv*: ([n′...,d′] ⊆ [...n]) = (n′ + d′ ≤ n)
⟨*proof*⟩

lemma *iIN-iMOD-subset-conv*:
0 < d′ ⟹ ([n′...,d′] ⊆ [r, mod m]) = (r ≤ n′ ∧ m = Suc 0)
⟨*proof*⟩
thm *Suc-in-imp-not-subset-iMOD*
⟨*proof*⟩
lemma *iIN-iMODb-subset-conv*:
0 < d′ ⟹
([n′...,d′] ⊆ [r, mod m, c]) =
(r ≤ n′ ∧ m = Suc 0 ∧ n′ + d′ ≤ r + m * c)
⟨*proof*⟩
thm *subset-trans*[*OF - iMODb-iMOD-subset-same*]

⟨*proof*⟩

lemma *iTILL-subset-conv*: $([\ldots n'] \subseteq [\ldots n]) = (n' \leq n)$
⟨*proof*⟩
lemma *iTILL-iFROM-subset-conv*: $([\ldots n'] \subseteq [n\ldots]) = (n = 0)$
⟨*proof*⟩
lemma *iTILL-iIN-subset-conv*: $([\ldots n'] \subseteq [n\ldots,d]) = (n = 0 \wedge n' \leq d)$
⟨*proof*⟩
lemma *iTILL-iMOD-subset-conv*:
$0 < n' \Longrightarrow ([\ldots n'] \subseteq [r, mod\ m]) = (r = 0 \wedge m = Suc\ 0)$
⟨*proof*⟩
lemma *iTILL-iMODb-subset-conv*:
$0 < n' \Longrightarrow ([\ldots n'] \subseteq [r, mod\ m, c]) = (r = 0 \wedge m = Suc\ 0 \wedge n' \leq r + m * c)$
⟨*proof*⟩

lemma *iMOD-iFROM-subset-conv*: $([r', mod\ m']) \subseteq [n\ldots] = (n \leq r')$
⟨*proof*⟩
lemma *iMODb-iFROM-subset-conv*: $([r', mod\ m', c'] \subseteq [n\ldots]) = (n \leq r')$
⟨*proof*⟩
lemma *iMODb-iIN-subset-conv*:
$([r', mod\ m', c'] \subseteq [n\ldots,d]) = (n \leq r' \wedge r' + m' * c' \leq n + d)$
⟨*proof*⟩
lemma *iMODb-iTILL-subset-conv*:
$([r', mod\ m', c'] \subseteq [\ldots n]) = (r' + m' * c' \leq n)$
⟨*proof*⟩

lemma *iMOD-0-subset-conv*: $([r', mod\ 0] \subseteq [r, mod\ m]) = (r'\ mod\ m = r\ mod\ m \wedge\ r \leq r')$
⟨*proof*⟩

lemma *iMOD-subset-conv*: $0 < m \Longrightarrow$
$([r', mod\ m'] \subseteq [r, mod\ m]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \leq r'\ \wedge\ m'\ mod\ m = 0)$
⟨*proof*⟩
thm *mod-eq-mod-0-imp-mod-eq*
⟨*proof*⟩

lemma *iMODb-subset-mod-0-conv*:
$([r', mod\ m', c'] \subseteq [r, mod\ 0, c\ ]) = (r'{=}r \wedge (m'{=}0 \vee c'{=}0))$
⟨*proof*⟩
lemma *iMODb-subset-0-conv*:
$([r', mod\ m', c'] \subseteq [r, mod\ m, 0\ ]) = (r'{=}r \wedge (m'{=}0 \vee c'{=}0))$
⟨*proof*⟩
lemma *iMODb-0-subset-conv*:
$([r', mod\ m', 0] \subseteq [r, mod\ m, c\ ]) = (r' \in [r, mod\ m, c])$
⟨*proof*⟩
lemma *iMODb-mod-0-subset-conv*:

$([r',\ mod\ 0,\ c'] \subseteq [r,\ mod\ m,\ c\ ]) = (r' \in [r,\ mod\ m,\ c])$
⟨*proof*⟩

lemma *iMODb-subset-conv'*: $[\![\ 0 < c;\ 0 < c'\ ]\!] \Longrightarrow$
$([r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \leq r'\ \wedge\ m'\ mod\ m = 0\ \wedge$
$r' + m' * c' \leq r + m * c)$
⟨*proof*⟩
thm *iMODb-subset-imp-iMOD-subset*
⟨*proof*⟩
thm *iMOD-subset-imp-iMODb-subset*
⟨*proof*⟩
lemma *iMODb-subset-conv*: $[\![\ 0 < m';\ 0 < c'\ ]\!] \Longrightarrow$
$([r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m,\ c]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \leq r'\ \wedge\ m'\ mod\ m = 0\ \wedge$
$r' + m' * c' \leq r + m * c)$
⟨*proof*⟩
thm *add-less-le-mono*[*of 0* $m' * c'$ *r r'*]
⟨*proof*⟩

lemma *iMODb-iMOD-subset-conv*: $0 < c' \Longrightarrow$
$([r',\ mod\ m',\ c'] \subseteq [r,\ mod\ m]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \leq r'\ \wedge\ m'\ mod\ m = 0)$
⟨*proof*⟩
thm *subsetD*[*OF - iMinI-ex2*[*OF iMODb-not-empty*]]
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv iMODb-subset-imp-divisor-mod-0*
⟨*proof*⟩
thm *subset-trans*[*OF iMODb-iMOD-subset-same*]
⟨*proof*⟩

lemmas *iT-subset-conv* =
iFROM-subset-conv
iFROM-iMOD-subset-conv
iTILL-subset-conv
iTILL-iFROM-subset-conv
iTILL-iIN-subset-conv
iTILL-iMOD-subset-conv
iTILL-iMODb-subset-conv
iIN-subset-conv
iIN-iFROM-subset-conv
iIN-iTILL-subset-conv
iIN-iMOD-subset-conv
iIN-iMODb-subset-conv
iMOD-subset-conv
iMOD-iFROM-subset-conv

iMODb-subset-conv'
iMODb-subset-conv
iMODb-iFROM-subset-conv
iMODb-iIN-subset-conv
iMODb-iTILL-subset-conv
iMODb-iMOD-subset-conv
thm *iT-subset-conv*

lemma *iFROM-subset*: $n \leq n' \Longrightarrow [n'\ldots] \subseteq [n\ldots]$
⟨*proof*⟩
lemma *not-iFROM-iIN-subset*: $\neg\ [n'\ldots] \subseteq [n\ldots,d]$
⟨*proof*⟩
lemma *not-iFROM-iTILL-subset*: $\neg\ [n'\ldots] \subseteq [\ldots n]$
⟨*proof*⟩
lemma *not-iFROM-iMOD-subset*: $m \neq Suc\ 0 \Longrightarrow \neg\ [n'\ldots] \subseteq [r,\ mod\ m]$
⟨*proof*⟩
lemma *not-iFROM-iMODb-subset*: $\neg\ [n'\ldots] \subseteq [r,\ mod\ m,\ c]$
thm *infinite-not-subset-finite*
⟨*proof*⟩

lemma *iIN-subset*: $[\![\ n \leq n';\ n' + d' \leq n + d\ ]\!] \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots,d]$
⟨*proof*⟩
lemma *iIN-iFROM-subset*: $n \leq n' \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots]$
⟨*proof*⟩
lemma *iIN-iTILL-subset*: $n' + d' \leq n \Longrightarrow [n'\ldots,d'] \subseteq [\ldots n]$
⟨*proof*⟩
lemma *not-iIN-iMODb-subset*: $[\![\ 0 < d';\ m \neq Suc\ 0\ ]\!] \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r,\ mod\ m,\ c]$
⟨*proof*⟩
lemma *not-iIN-iMOD-subset*: $[\![\ 0 < d';\ m \neq Suc\ 0\ ]\!] \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r,\ mod\ m]$
⟨*proof*⟩
thm *iIN-iTILL-subset*[*OF order-refl*]
thm *Int-greatest*[*OF - iIN-iTILL-subset*[*OF order-refl*]]
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv not-iIN-iMODb-subset*
⟨*proof*⟩
lemma *iTILL-subset*: $n' \leq n \Longrightarrow [\ldots n'] \subseteq [\ldots n]$
⟨*proof*⟩
lemma *iTILL-iFROM-subset*: $([\ldots n'] \subseteq [0\ldots])$
⟨*proof*⟩
lemma *iTILL-iIN-subset*: $n' \leq d \Longrightarrow ([\ldots n'] \subseteq [0\ldots,d])$
⟨*proof*⟩
thm *not-iIN-iMOD-subset*
lemma *not-iTILL-iMOD-subset*:
$[\![\ 0 < n';\ m \neq Suc\ 0\ ]\!] \Longrightarrow \neg\ [\ldots n'] \subseteq [r,\ mod\ m]$
⟨*proof*⟩

lemma *not-iTILL-iMODb-subset*:
⟦ $0 < n'$; $m \neq Suc\ 0$ ⟧ ⟹ ¬ $[\ldots n'] \subseteq [r, mod\ m, c]$
⟨*proof*⟩

lemma *iMOD-iFROM-subset*: $n \leq r' \Longrightarrow [r', mod\ m'] \subseteq [n\ldots]$
⟨*proof*⟩
lemma *not-iMOD-iIN-subset*: $0 < m' \Longrightarrow \neg\ [r', mod\ m'] \subseteq [n\ldots,d]$
⟨*proof*⟩
lemma *not-iMOD-iTILL-subset*: $0 < m' \Longrightarrow \neg\ [r', mod\ m'] \subseteq [\ldots n]$
⟨*proof*⟩
thm *iMOD-subset-conv*
lemma *iMOD-subset*:
⟦ $r \leq r'$; $r'\ mod\ m = r\ mod\ m$; $m'\ mod\ m = 0$ ⟧ ⟹ $[r', mod\ m'] \subseteq [r, mod\ m]$
⟨*proof*⟩
lemma *not-iMOD-iMODb-subset*: $0 < m' \Longrightarrow \neg\ [r', mod\ m'] \subseteq [r, mod\ m, c]$
⟨*proof*⟩

lemma *iMODb-iFROM-subset*: $n \leq r' \Longrightarrow [r', mod\ m', c'] \subseteq [n\ldots]$
thm *iMODb-iFROM-subset-conv*[*THEN iffD2*]
⟨*proof*⟩
lemma *iMODb-iTILL-subset*:
$r' + m' * c' \leq n \Longrightarrow [r', mod\ m', c'] \subseteq [\ldots n]$
⟨*proof*⟩
thm *iMODb-iIN-subset-conv*
lemma *iMODb-iIN-subset*:
⟦ $n \leq r'$; $r' + m' * c' \leq n + d$ ⟧ ⟹ $[r', mod\ m', c'] \subseteq [n\ldots,d]$
⟨*proof*⟩
thm *iMODb-iMOD-subset-conv*
lemma *iMODb-iMOD-subset*:
⟦ $r \leq r'$; $r'\ mod\ m = r\ mod\ m$; $m'\ mod\ m = 0$ ⟧ ⟹ $[r', mod\ m', c'] \subseteq [r, mod\ m]$
⟨*proof*⟩
thm *iMODb-iMOD-subset-conv*
⟨*proof*⟩

thm *iMODb-subset-conv*
lemma *iMODb-subset*:
⟦ $r \leq r'$; $r'\ mod\ m = r\ mod\ m$; $m'\ mod\ m = 0$; $r' + m' * c' \leq r + m * c$ ⟧ ⟹
$[r', mod\ m', c'] \subseteq [r, mod\ m, c]$
⟨*proof*⟩

lemma *iFROM-trans*: ⟦ $y \in [x\ldots]$; $z \in [y\ldots]$ ⟧ ⟹ $z \in [x\ldots]$
⟨*proof*⟩
lemma *iTILL-trans*: ⟦ $y \in [\ldots x]$; $z \in [\ldots y]$ ⟧ ⟹ $z \in [\ldots x]$

⟨*proof*⟩
thm *iIN-subset*
lemma *iIN-trans*:
⟦ $y \in [x\ldots,d]$; $z \in [y\ldots,d']$; $d' \leq x + d - y$ ⟧ ⟹ $z \in [x\ldots,d]$
⟨*proof*⟩
lemma *iMOD-trans*:
⟦ $y \in [x, \mathit{mod}\ m]$; $z \in [y, \mathit{mod}\ m]$ ⟧ ⟹ $z \in [x, \mathit{mod}\ m]$
⟨*proof*⟩
lemma *iMODb-trans*:
⟦ $y \in [x, \mathit{mod}\ m, c]$; $z \in [y, \mathit{mod}\ m, c']$; $m * c' \leq x + m * c - y$ ⟧ ⟹
$z \in [x, \mathit{mod}\ m, c]$
⟨*proof*⟩
lemma *iMODb-trans'*:
⟦ $y \in [x, \mathit{mod}\ m, c]$; $z \in [y, \mathit{mod}\ m, c']$; $c' \leq x\ \mathit{div}\ m + c - y\ \mathit{div}\ m$ ⟧ ⟹
$z \in [x, \mathit{mod}\ m, c]$
⟨*proof*⟩

1.3.3 Equality of intervals

lemma *iFROM-eq-conv*: $([n\ldots] = [n'\ldots]) = (n = n')$
⟨*proof*⟩

lemma *iIN-eq-conv*: $([n\ldots,d] = [n'\ldots,d']) = (n = n' \wedge d = d')$
⟨*proof*⟩

lemma *iTILL-eq-conv*: $([\ldots n] = [\ldots n']) = (n = n')$
thm *iIN-eq-conv*[*of 0 n 0 n'*]
⟨*proof*⟩

thm *iMOD-singleton-eq-conv*
lemma *iMOD-0-eq-conv*: $([r, \mathit{mod}\ 0] = [r', \mathit{mod}\ m']) = (r = r' \wedge m' = 0)$
⟨*proof*⟩
thm *iMOD-singleton-eq-conv*
⟨*proof*⟩
lemma *iMOD-eq-conv*: $0 < m \Longrightarrow ([r, \mathit{mod}\ m] = [r', \mathit{mod}\ m']) = (r = r' \wedge m = m')$
⟨*proof*⟩

thm *iMODb-singleton-eq-conv*
lemma *iMODb-mod-0-eq-conv*:
$([r, \mathit{mod}\ 0, c] = [r', \mathit{mod}\ m', c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
⟨*proof*⟩
lemma *iMODb-0-eq-conv*:
$([r, \mathit{mod}\ m, 0] = [r', \mathit{mod}\ m', c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
⟨*proof*⟩

lemma *iMODb-eq-conv*: ⟦ $0 < m$; $0 < c$ ⟧ ⟹
$([r, \mathit{mod}\ m, c] = [r', \mathit{mod}\ m', c']) = (r = r' \wedge m = m' \wedge c = c')$
⟨*proof*⟩

lemma *iMOD-iFROM-eq-conv*: $([n...] = [r, mod\ m]) = (n = r \wedge m = Suc\ 0)$
⟨*proof*⟩
thm *iMODb-singleton-eq-conv*
lemma *iMODb-iIN-0-eq-conv*:
$([n...,0] = [r, mod\ m, c]) = (n = r \wedge (m = 0 \vee c = 0))$
⟨*proof*⟩
lemma *iMODb-iIN-eq-conv*:
$0 < d \Longrightarrow ([n...,d] = [r, mod\ m, c]) = (n = r \wedge m = Suc\ 0 \wedge c = d)$
⟨*proof*⟩

1.3.4 Inequality of intervals

lemma *iFROM-iIN-neq*: $[n'...] \neq [n...,d]$
⟨*proof*⟩
corollary *iFROM-iTILL-neq*: $[n'...] \neq [...n]$
⟨*proof*⟩
corollary *iFROM-iMOD-neq*: $m \neq Suc\ 0 \Longrightarrow [n...] \neq [r, mod\ m]$
⟨*proof*⟩
corollary *iFROM-iMODb-neq*: $[n...] \neq [r, mod\ m, c]$
⟨*proof*⟩

corollary *iIN-iMOD-neq*: $0 < m \Longrightarrow [n...,d] \neq [r, mod\ m]$
⟨*proof*⟩
corollary *iIN-iMODb-neq2*: $[\![m \neq Suc\ 0; 0 < d]\!] \Longrightarrow [n...,d] \neq [r, mod\ m, c]$
⟨*proof*⟩
lemma *iIN-iMODb-neq*: $[\![2 \leq m; 0 < c]\!] \Longrightarrow [n...,d] \neq [r, mod\ m, c]$
⟨*proof*⟩
thm *iMODb-singleton-eq-conv*
⟨*proof*⟩

lemma *iTILL-iIN-neq*: $0 < n \Longrightarrow [...n'] \neq [n...,d]$
⟨*proof*⟩
corollary *iTILL-iMOD-neq*: $0 < m \Longrightarrow [...n] \neq [r, mod\ m]$
⟨*proof*⟩
corollary *iTILL-iMODb-neq*:
$[\![m \neq Suc\ 0; 0 < n]\!] \Longrightarrow [...n] \neq [r, mod\ m, c]$
⟨*proof*⟩

lemma *iMOD-iMODb-neq*: $0 < m \Longrightarrow [r, mod\ m] \neq [r', mod\ m', c']$
⟨*proof*⟩

lemmas *iT-neq* =
iFROM-iTILL-neq iFROM-iIN-neq iFROM-iMOD-neq iFROM-iMODb-neq
iTILL-iIN-neq iTILL-iMOD-neq iTILL-iMODb-neq
iIN-iMOD-neq iIN-iMODb-neq iIN-iMODb-neq2
iMOD-iMODb-neq
thm *iT-neq*

1.4 Union and intersection of intervals

lemma *iFROM-union'*: $[n\ldots] \cup [n'\ldots] = [min\ n\ n'\ldots]$
⟨*proof*⟩
corollary *iFROM-union*: $n \leq n' \Longrightarrow [n\ldots] \cup [n'\ldots] = [n\ldots]$
⟨*proof*⟩

lemma *iFROM-inter'*: $[n\ldots] \cap [n'\ldots] = [max\ n\ n'\ldots]$
⟨*proof*⟩
corollary *iFROM-inter*: $n' \leq n \Longrightarrow [n\ldots] \cap [n'\ldots] = [n\ldots]$
⟨*proof*⟩

lemma *iTILL-union'*: $[\ldots n] \cup [\ldots n'] = [\ldots max\ n\ n']$
⟨*proof*⟩
corollary *iTILL-union*: $n' \leq n \Longrightarrow [\ldots n] \cup [\ldots n'] = [\ldots n]$
⟨*proof*⟩

lemma *iTILL-iFROM-union*: $n \leq n' \Longrightarrow [\ldots n'] \cup [n\ldots] = UNIV$
⟨*proof*⟩

lemma *iTILL-inter'*: $[\ldots n] \cap [\ldots n'] = [\ldots min\ n\ n']$
⟨*proof*⟩
corollary *iTILL-inter*: $n \leq n' \Longrightarrow [\ldots n] \cap [\ldots n'] = [\ldots n]$
⟨*proof*⟩

Union and intersection for iIN only when there intersection is not empty and none of them is other's subset, for instance: .. n .. n+d .. n' .. n'+d'

lemma *iIN-union*:
$[\![\ n \leq n';\ n' \leq Suc\ (n + d);\ n + d \leq n' + d'\ ]\!] \Longrightarrow$
$[n\ldots,d] \cup [n'\ldots,d'] = [n\ldots,n' - n + d']$
⟨*proof*⟩

lemma *iIN-append-union*:
$[n\ldots,d] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
⟨*proof*⟩
lemma *iIN-append-union-Suc*:
$[n\ldots,d] \cup [Suc\ (n + d)\ldots,d'] = [n\ldots,Suc\ (d + d')]$
⟨*proof*⟩
lemma *iIN-append-union-pred*:
$0 < d \Longrightarrow [n\ldots,d - Suc\ 0] \cup [n + d\ldots,d'] = [n\ldots,d + d']$
⟨*proof*⟩

lemma *iIN-iFROM-union*:
$n' \leq Suc\ (n + d) \Longrightarrow [n\ldots,d] \cup [n'\ldots] = [min\ n\ n'\ldots]$
⟨*proof*⟩

lemma *iIN-iFROM-append-union*:
$[n\ldots,d] \cup [n + d\ldots] = [n\ldots]$
⟨*proof*⟩
lemma *iIN-iFROM-append-union-Suc*:

$[n...,d] \cup [Suc\ (n + d)...] = [n...]$
⟨*proof*⟩
lemma *iIN-iFROM-append-union-pred*:
$0 < d \Longrightarrow [n...,d - Suc\ 0] \cup [n + d...] = [n...]$
⟨*proof*⟩

lemma *iIN-inter*:
$[\![\ n \le n'; n' \le n + d; n + d \le n' + d'\ ]\!] \Longrightarrow$
$[n...,d] \cap [n'...,d'] = [n'...,n + d - n']$
⟨*proof*⟩

lemma *iMOD-union*:
$[\![\ r \le r'; r\ mod\ m = r'\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [r,\ mod\ m]$
⟨*proof*⟩
lemma *iMOD-union'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [min\ r\ r',\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-inter*:
$[\![\ r \le r'; r\ mod\ m = r'\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [r',\ mod\ m]$
⟨*proof*⟩
lemma *iMOD-inter'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [max\ r\ r',\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-union*:
$[\![\ r \le r'; r\ mod\ m = r'\ mod\ m; r' \le r + m * c; r + m * c \le r' + m * c'\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [r',\ mod\ m,\ c'] = [r,\ mod\ m,\ r'\ div\ m - r\ div\ m + c']$
⟨*proof*⟩
thm *iMODb-iMOD-subset-same*
thm *Un-absorb1*[*OF iMODb-iMOD-subset-same*]

lemma *iMODb-append-union*:
$[r,\ mod\ m,\ c] \cup [\ r + m * c,\ mod\ m,\ c'] = [r,\ mod\ m,\ c + c']$
thm *iMODb-union*[*of r r + m * c m c c'*]
⟨*proof*⟩

lemma *iMODb-iMOD-append-union'*:
$[\![\ r\ mod\ m = r'\ mod\ m; r' \le r + m * Suc\ c\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \cup [\ r',\ mod\ m\ ] = [min\ r\ r',\ mod\ m]$
⟨*proof*⟩

thm *add-le-imp-le-right*[*of* - *m*]
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*
⟨*proof*⟩
lemma *iMODb-iMOD-append-union*:
⟦ $r \leq r'$; r *mod* $m = r'$ *mod* m; $r' \leq r + m * Suc\ c$ ⟧ ⟹
$[r, mod\ m, c] \cup [\ r', mod\ m\ ] = [r, mod\ m]$
thm *iMODb-iMOD-append-union'*
⟨*proof*⟩

lemma *iMODb-append-union-Suc*:
$[r, mod\ m, c] \cup [\ r + m * Suc\ c, mod\ m, c'] = [r, mod\ m, Suc\ (c + c')]$
thm *insert-absorb*[*of* $r + m * c$ $[r, mod\ m, c] \cup [\ r + m * Suc\ c, mod\ m, c']$]
⟨*proof*⟩
thm *iMODb-append-union*[*of* r m c]
⟨*proof*⟩
lemma *iMODb-append-union-pred*:
$0 < c \Longrightarrow [r, mod\ m, c - Suc\ 0] \cup [\ r + m * c, mod\ m, c'] = [r, mod\ m, c + c']$
⟨*proof*⟩

lemma *iMODb-inter*:
⟦ $r \leq r'$; r *mod* $m = r'$ *mod* m; $r' \leq r + m * c$; $r + m * c \leq r' + m * c'$ ⟧ ⟹
$[r, mod\ m, c] \cap [r', mod\ m, c'] = [r', mod\ m, c - (r'-r)\ div\ m]$
⟨*proof*⟩
thm *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*]
⟨*proof*⟩

lemmas *iT-union'* =
iFROM-union'
iTILL-union'
iMOD-union'
iMODb-iMOD-append-union'
lemmas *iT-union* =
iFROM-union
iTILL-union
iTILL-iFROM-union
iIN-union
iIN-iFROM-union
iMOD-union
iMODb-union
lemmas *iT-union-append* =
iIN-append-union
iIN-append-union-Suc

iIN-append-union-pred
iIN-iFROM-append-union
iIN-iFROM-append-union-Suc
iIN-iFROM-append-union-pred
iMODb-append-union
iMODb-iMOD-append-union
iMODb-append-union-Suc
iMODb-append-union-pred

lemmas *iT-inter'* =
iFROM-inter'
iTILL-inter'
iMOD-inter'
lemmas *iT-inter* =
iFROM-inter
iTILL-inter
iIN-inter
iMOD-inter
iMODb-inter

thm *iT-union'*
thm *iT-union*
thm *iT-union-append*
thm *iT-inter'*
thm *iT-inter*

thm *partition-Union*
lemma *mod-partition-Union*:
$0 < m \Longrightarrow (\bigcup k.\ A \cap [k * m\ldots,m - Suc\ 0]) = A$
⟨*proof*⟩
thm *subst*[**where** $s{=}UNIV$ **and** $P{=}\lambda x.\ A \cap x = A$]
⟨*proof*⟩
thm *div-mult-cancel*
⟨*proof*⟩
thm *le-add-diff*
⟨*proof*⟩
thm *Suc-mod-le-divisor*
⟨*proof*⟩

thm *mod-partition-Union*
lemma *finite-mod-partition-Union*:
$[\![\ 0 < m;\ finite\ A\ ]\!] \Longrightarrow$
$(\bigcup k{\leq}Max\ A\ div\ m.\ A \cap [k{*}m\ldots,m - Suc\ 0]) = A$
thm *subst*[*OF mod-partition-Union*[*of m*], **where**
$P{=}\lambda x.\ (\bigcup k{\leq}Max\ A\ div\ m.\ A \cap [k{*}m\ldots,m - Suc\ 0]) = x$]
⟨*proof*⟩

thm *setsum-UN-disjoint*
lemma *mod-partition-is-disjoint*:
⟦ $0 < (m::nat)$; $k \neq k'$ ⟧ ⟹
$(A \cap [k * m \ldots, m - Suc\ 0]) \cap (A \cap [k' * m \ldots, m - Suc\ 0]) = \{\}$
⟨*proof*⟩
thm *le-less-imp-div*
⟨*proof*⟩

1.5 Cutting intervals

thm *i-cut-defs*
thm *i-cut-subset*
thm *i-cut-bound*

thm
cut-le-less-conv
cut-less-le-conv
thm
cut-ge-greater-conv
cut-greater-ge-conv

thm
cut-le-Min-empty
cut-less-Min-empty
cut-le-Min-not-empty
cut-less-Min-not-empty

thm
i-cut-min-empty
thm
i-cut-min-all
thm
i-cut-max-empty
thm
i-cut-max-all

lemma *iTILL-cut-le*: $[\ldots n] \downarrow\leq t = (\mathit{if}\ t \leq n\ \mathit{then}\ [\ldots t]\ \mathit{else}\ [\ldots n])$
⟨*proof*⟩
corollary *iTILL-cut-le1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\leq t = [\ldots t]$
⟨*proof*⟩
corollary *iTILL-cut-le2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\leq t = [\ldots n]$
⟨*proof*⟩

lemma *iFROM-cut-le*:
$[n\ldots] \downarrow\leq t =$
$(\mathit{if}\ t < n\ \mathit{then}\ \{\}\ \mathit{else}\ [n \ldots, t-n])$

⟨*proof*⟩
corollary *iFROM-cut-le1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\leq t = [n\ldots,t - n]$
⟨*proof*⟩

lemma *iIN-cut-le*:
$[n\ldots,d] \downarrow\leq t = ($
if $t < n$ *then* $\{\}$ *else*
if $t \leq n+d$ *then* $[n\ldots,t-n]$
else $[n\ldots,d])$
⟨*proof*⟩

corollary *iIN-cut-le1*:
$t \in [n\ldots,d] \Longrightarrow [n\ldots,d] \downarrow\leq t = [n\ldots,t - n]$
⟨*proof*⟩

lemma *iMOD-cut-le*:
$[r,\ mod\ m] \downarrow\leq t = ($
if $t < r$ *then* $\{\}$
else $[r,\ mod\ m,\ (t - r)\ div\ m])$
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*
⟨*proof*⟩
thm *mod-eq-imp-diff-mod-eq*[*of* - *m r t*, *rule-format*]
⟨*proof*⟩
thm *add-le-mono2*[*OF mod-le-divisor*]
thm *order-trans*[*OF add-le-mono2*[*OF mod-le-divisor*]]
⟨*proof*⟩
thm *le-imp-sub-mod-le*[*of* - *t*]
⟨*proof*⟩

lemma *iMOD-cut-le1*:
$t \in [r,\ mod\ m] \Longrightarrow$
$[r,\ mod\ m] \downarrow\leq t = [r,\ mod\ m,\ (t - r)\ div\ m]$
⟨*proof*⟩

lemma *iMODb-cut-le*:
$[r,\ mod\ m,\ c] \downarrow\leq t = ($
if $t < r$ *then* $\{\}$
else if $t < r + m * c$ *then* $[r,\ mod\ m,\ (t - r)\ div\ m]$
else $[r,\ mod\ m,\ c])$
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*[*of r r* + *m* ∗ *c m*]
⟨*proof*⟩
find-theorems *cut-le* (- ∩ -)
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*

⟨*proof*⟩

lemma *iMODb-cut-le1*:
$t \in [r,\ mod\ m,\ c] \Longrightarrow$
$[r,\ mod\ m,\ c]\ \downarrow\leq\ t = [r,\ mod\ m,\ (t - r)\ div\ m]$
⟨*proof*⟩

lemma *iTILL-cut-less*:
$[\ldots n]\ \downarrow<\ t =$ (
if $n < t$ *then* $[\ldots n]$ *else*
if $t = 0$ *then* $\{\}$
else $[\ldots t - Suc\ 0]$)
⟨*proof*⟩

lemma *iTILL-cut-less1*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow [\ldots n]\ \downarrow<\ t = [\ldots t - Suc\ 0]$
thm *iTILL-cut-less*
⟨*proof*⟩

lemma *iFROM-cut-less*:
$[n\ldots]\ \downarrow<\ t =$ (
if $t \leq n$ *then* $\{\}$
else $[n\ldots,t - Suc\ n]$)
⟨*proof*⟩

lemma *iFROM-cut-less1*:
$n < t \Longrightarrow [n\ldots]\ \downarrow<\ t = [n\ldots,t - Suc\ n]$
⟨*proof*⟩

lemma *iIN-cut-less*:
$[n\ldots,d]\ \downarrow<\ t =$ (
if $t \leq n$ *then* $\{\}$ *else*
if $t \leq n + d$ *then* $[n\ldots,\ t - Suc\ n]$
else $[n\ldots,d]$)
⟨*proof*⟩

lemma *iIN-cut-less1*:
$[\![\ t \in [n\ldots,d];\ n < t\ ]\!] \Longrightarrow [n\ldots,d]\ \downarrow<\ t = [n\ldots,\ t - Suc\ n]$
thm *iIN-cut-less*
⟨*proof*⟩

thm
cut-le-less-conv
cut-less-le-conv

lemma *iMOD-cut-less*:
[r, *mod* m] ↓< t = (
if t ≤ r *then* {}
else [r, *mod* m, (t − *Suc* r) *div* m])
thm *iMOD-cut-le*
⟨*proof*⟩

lemma *iMOD-cut-less1*:
⟦ t ∈ [r, *mod* m]; r < t ⟧ ⟹
[r, *mod* m] ↓< t = [r, *mod* m, (t − r) *div* m − *Suc* 0]
⟨*proof*⟩
thm *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0*
⟨*proof*⟩

lemma *iMODb-cut-less*:
[r, *mod* m, c] ↓< t = (
if t ≤ r *then* {} *else*
if r + m ∗ c < t *then* [r, *mod* m, c]
else [r, *mod* m, (t − *Suc* r) *div* m])
thm *iMODb-cut-le*
⟨*proof*⟩

lemma *iMODb-cut-less1*: ⟦ t ∈ [r, *mod* m, c]; r < t ⟧ ⟹
[r, *mod* m, c] ↓< t = [r, *mod* m, (t − r) *div* m − *Suc* 0]
⟨*proof*⟩
thm *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0*
⟨*proof*⟩

lemmas *iT-cut-le* =
iTILL-cut-le
iFROM-cut-le
iIN-cut-le
iMOD-cut-le
iMODb-cut-le
thm *iT-cut-le*
lemmas *iT-cut-le1* =
iTILL-cut-le1
iFROM-cut-le1
iIN-cut-le1
iMOD-cut-le1
iMODb-cut-le1
thm *iT-cut-le1*

lemmas *iT-cut-less* =
iTILL-cut-less
iFROM-cut-less
iIN-cut-less

iMOD-cut-less
iMODb-cut-less
thm *iT-cut-less*
lemmas *iT-cut-less1* =
iTILL-cut-less1
iFROM-cut-less1
iIN-cut-less1
iMOD-cut-less1
iMODb-cut-less1
thm *iT-cut-less1*

lemmas *iT-cut-le-less* =
iTILL-cut-le
iTILL-cut-less
iFROM-cut-le
iFROM-cut-less
iIN-cut-le
iIN-cut-less
iMOD-cut-le
iMOD-cut-less
iMODb-cut-le
iMODb-cut-less
lemmas *iT-cut-le-less1* =
iTILL-cut-le1
iTILL-cut-less1
iFROM-cut-le1
iFROM-cut-less1
iIN-cut-le1
iIN-cut-less1
iMOD-cut-le1
iMOD-cut-less1
iMODb-cut-le1
iMODb-cut-less1
thm *iT-cut-le-less*
thm *iT-cut-le-less1*

lemma *iTILL-cut-ge*:
$[\ldots n] \downarrow\geq t = (\mathit{if}\ n < t\ \mathit{then}\ \{\}\ \mathit{else}\ [t\ldots,n-t])$
⟨*proof*⟩
corollary *iTILL-cut-ge1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = [t\ldots,n-t]$
⟨*proof*⟩
corollary *iTILL-cut-ge2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = \{\}$
⟨*proof*⟩

lemma *iTILL-cut-greater*:
$[\ldots n] \downarrow> t = (\mathit{if}\ n \leq t\ \mathit{then}\ \{\}\ \mathit{else}\ [\mathit{Suc}\ t\ldots,n - \mathit{Suc}\ t])$
⟨*proof*⟩
corollary *iTILL-cut-greater1*:
$t \in [\ldots n] \Longrightarrow t < n \Longrightarrow [\ldots n] \downarrow> t = [\mathit{Suc}\ t\ldots,n - \mathit{Suc}\ t]$

⟨*proof*⟩
corollary *iTILL-cut-greater2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow> t = \{\}$
⟨*proof*⟩

lemma *iFROM-cut-ge*:
$[n\ldots] \downarrow\geq t = (\mathit{if}\ n \leq t\ \mathit{then}\ [t\ldots]\ \mathit{else}\ [n\ldots])$
⟨*proof*⟩
corollary *iFROM-cut-ge1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\geq t = [t\ldots]$
⟨*proof*⟩

lemma *iFROM-cut-greater*:
$[n\ldots] \downarrow> t = (\mathit{if}\ n \leq t\ \mathit{then}\ [\mathit{Suc}\ t\ldots]\ \mathit{else}\ [n\ldots])$
⟨*proof*⟩
corollary *iFROM-cut-greater1*:
$t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow> t = [\mathit{Suc}\ t\ldots]$
⟨*proof*⟩

lemma *iIN-cut-ge*:
$[n\ldots,d] \downarrow\geq t =$ (
if $t < n$ *then* $[n\ldots,d]$ *else*
if $t \leq n+d$ *then* $[t\ldots,n+d-t]$
else $\{\}$)
⟨*proof*⟩
corollary *iIN-cut-ge1*: $t \in [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = [t\ldots,n+d-t]$
⟨*proof*⟩
corollary *iIN-cut-ge2*: $t \notin [n\ldots,d] \Longrightarrow$
$[n\ldots,d] \downarrow\geq t = (\mathit{if}\ t < n\ \mathit{then}\ [n\ldots,d]\ \mathit{else}\ \{\})$
⟨*proof*⟩

lemma *iIN-cut-greater*:
$[n\ldots,d] \downarrow> t =$ (
if $t < n$ *then* $[n\ldots,d]$ *else*
if $t < n+d$ *then* $[\mathit{Suc}\ t\ldots,n + d - \mathit{Suc}\ t]$
else $\{\}$)
⟨*proof*⟩
corollary *iIN-cut-greater1*:
$[\![\ t \in [n\ldots,d];\ t < n + d\ ]\!] \Longrightarrow$
$[n\ldots,d] \downarrow> t = [\mathit{Suc}\ t\ldots,n + d - \mathit{Suc}\ t]$
⟨*proof*⟩

thm *sub-diff-mod-eq'*[*of r t 1 m*, *simplified*]
lemma *mod-cut-greater-aux-t-less*:
⟦ *0* < (*m*::*nat*); *r* ≤ *t* ⟧ ⟹
t < *t* + *m* − (*t* − *r*) *mod m*
thm *less-add-diff*
⟨*proof*⟩
lemma *mod-cut-greater-aux-le-x*:
⟦ (*r*::*nat*) ≤ *t*; *t* < *x*; *x mod m* = *r mod m*⟧ ⟹
t + *m* − (*t* − *r*) *mod m* ≤ *x*
thm *diff-mod-le*
⟨*proof*⟩
thm *diff-add-assoc2*[*of* (*t* − *r*) *mod m t m*]
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*[*of t* − (*t* − *r*) *mod m x m*]
⟨*proof*⟩
thm *sub-diff-mod-eq*
⟨*proof*⟩

lemma *iMOD-cut-greater*:
[*r*, *mod m*] ↓> *t* = (
if t < *r then* [*r*, *mod m*] *else*
if m = *0 then* {}
else [*t* + *m* − (*t* − *r*) *mod m*, *mod m*])
⟨*proof*⟩
thm *order-trans*[*OF mod-le-dividend*]
⟨*proof*⟩
thm *diff-add-assoc2*[*of* (*t* − *r*) *mod m t m*]
⟨*proof*⟩
thm *sub-diff-mod-eq*
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*[*of t* − (*t* − *r*) *mod m x m*]
⟨*proof*⟩

lemma *iMOD-cut-greater1*:
t ∈ [*r*, *mod m*] ⟹
[*r*, *mod m*] ↓> *t* = (
if m = *0 then* {}
else [*t* + *m*, *mod m*])
⟨*proof*⟩

lemma *iMODb-cut-greater-aux*:
⟦ *0* < *m*; *t* < *r* + *m* ∗ *c*; *r* ≤ *t*⟧ ⟹
(*r* + *m* ∗ *c* − (*t* + *m* − (*t* − *r*) *mod m*)) *div m* =
c − *Suc* ((*t* − *r*) *div m*)
thm *diff-diff-right*[*of r t*+*m*−(*t*−*r*)*mod m m*∗*c*, *symmetric*]
⟨*proof*⟩
thm *mod-cut-greater-aux-t-less*
⟨*proof*⟩

lemma *iMODb-cut-greater*:
$[r,\ mod\ m,\ c] \downarrow> t = (\ if\ t < r\ then\ [r,\ mod\ m,\ c]\ else\ if\ r + m * c \leq t\ then\ \{\}\ else\ [t + m - (t - r)\ mod\ m,\ mod\ m,\ c - Suc\ ((t-r)\ div\ m)])$
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*[*of r r* + *m* ∗ *c m*]
⟨*proof*⟩
thm *iMOD-cut-greater*
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*
⟨*proof*⟩
thm *mod-cut-greater-aux-le-x*
⟨*proof*⟩

lemma *iMODb-cut-greater1*:
$t \in [r,\ mod\ m,\ c] \Longrightarrow [r,\ mod\ m,\ c] \downarrow> t = (\ if\ r + m * c \leq t\ then\ \{\}\ else\ [t + m,\ mod\ m,\ c - Suc\ ((t-r)\ div\ m)])$
⟨*proof*⟩

lemma *iMOD-cut-ge*:
$[r,\ mod\ m] \downarrow\geq t = (\ if\ t \leq r\ then\ [r,\ mod\ m]\ else\ if\ m = 0\ then\ \{\}\ else\ [t + m - Suc\ ((t - Suc\ r)\ mod\ m),\ mod\ m])$
⟨*proof*⟩
thm *iMOD-cut-greater nat-cut-greater-ge-conv*[*symmetric*]
⟨*proof*⟩

lemma *iMOD-cut-ge1*:
$t \in [r,\ mod\ m] \Longrightarrow [r,\ mod\ m] \downarrow\geq t = [t,\ mod\ m]$
⟨*proof*⟩

lemma *iMODb-cut-ge*:
$[r,\ mod\ m,\ c] \downarrow\geq t = (\ if\ t \leq r\ then\ [r,\ mod\ m,\ c]\ else\ if\ r + m * c < t\ then\ \{\}\ else\ [t + m - Suc\ ((t - Suc\ r)\ mod\ m),\ mod\ m,\ c - (t + m - Suc\ r)\ div\ m])$
thm *iMOD-cut-ge*
⟨*proof*⟩
thm *diff-mod-pred*
⟨*proof*⟩

thm *mod-0-imp-diff-Suc-div-conv*
⟨*proof*⟩
thm *div-le-mono*
⟨*proof*⟩
thm *diff-mod-pred*
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*[*of r r + m * c m*]
⟨*proof*⟩
thm *iMOD-cut-ge*
⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*
⟨*proof*⟩
thm *mod-cut-greater-aux-le-x*
⟨*proof*⟩
thm *iMODb-cut-greater-aux*[*of m t r c*]
⟨*proof*⟩
thm *div-diff1-eq*
⟨*proof*⟩

thm *iMODb-cut-greater1*
lemma *iMODb-cut-ge1*:
t ∈ [*r*, *mod m*, *c*] ⟹
[*r*, *mod m*, *c*] ↓≥ *t* = (
*if r + m * c < t then* {}
else [*t*, *mod m*, *c* − (*t* − *r*) *div m*])
⟨*proof*⟩
thm *iMODb-cut-ge*
⟨*proof*⟩
thm *mod-eq-imp-diff-mod-eq-divisor*
⟨*proof*⟩
thm *div-add1-eq*
⟨*proof*⟩
lemma *iMOD-0-cut-greater*:
t ∈ [*r*, *mod 0*] ⟹ [*r*, *mod 0*] ↓> *t* = {}
⟨*proof*⟩
lemma *iMODb-0-cut-greater*: *t* ∈ [*r*, *mod 0*, *c*] ⟹
[*r*, *mod 0*, *c*] ↓> *t* = {}
⟨*proof*⟩

lemmas *iT-cut-ge* =
iTILL-cut-ge
iFROM-cut-ge
iIN-cut-ge
iMOD-cut-ge
iMODb-cut-ge
thm *iT-cut-ge*
lemmas *iT-cut-ge1* =
iTILL-cut-ge1
iFROM-cut-ge1

iIN-cut-ge1
iMOD-cut-ge1
iMODb-cut-ge1
thm *iT-cut-le1*

lemmas *iT-cut-greater* =
iTILL-cut-greater
iFROM-cut-greater
iIN-cut-greater
iMOD-cut-greater
iMODb-cut-greater
thm *iT-cut-greater*
lemmas *iT-cut-greater1* =
iTILL-cut-greater1
iFROM-cut-greater1
iIN-cut-greater1
iMOD-cut-greater1
iMODb-cut-greater1
thm *iT-cut-greater1*

lemmas *iT-cut-ge-greater* =
iTILL-cut-ge
iTILL-cut-greater
iFROM-cut-ge
iFROM-cut-greater
iIN-cut-ge
iIN-cut-greater
iMOD-cut-ge
iMOD-cut-greater
iMODb-cut-ge
iMODb-cut-greater
lemmas *iT-cut-ge-greater1* =
iTILL-cut-ge1
iTILL-cut-greater1
iFROM-cut-ge1
iFROM-cut-greater1
iIN-cut-ge1
iIN-cut-greater1
iMOD-cut-ge1
iMOD-cut-greater1
iMODb-cut-ge1
iMODb-cut-greater1
thm *iT-cut-ge-greater*
thm *iT-cut-ge-greater1*

1.6 Cardinality of intervals

lemma *iFROM-card*: *card* $[n\ldots] = 0$
⟨*proof*⟩

lemma *iTILL-card*: *card* [...*n*] = *Suc n*
⟨*proof*⟩
lemma *iIN-card*: *card* [*n*...,*d*] = *Suc d*
⟨*proof*⟩
lemma *iMOD-0-card*: *card* [*r*, *mod 0*] = *Suc 0*
⟨*proof*⟩
lemma *iMOD-card*: *0* < *m* ⟹ *card* [*r*, *mod m*] = *0*
⟨*proof*⟩
lemma *iMOD-card-if*: *card* [*r*, *mod m*] = (*if m* = *0 then Suc 0 else 0*)
⟨*proof*⟩
lemma *iMODb-mod-0-card*: *card* [*r*, *mod 0*, *c*] = *Suc 0*
⟨*proof*⟩
lemma *iMODb-card*: *0* < *m* ⟹ *card* [*r*, *mod m*, *c*] = *Suc c*
⟨*proof*⟩
thm *iMODb-Suc-insert-conv*
⟨*proof*⟩
lemma *iMODb-card-if*:
card [*r*, *mod m*, *c*] = (*if m* = *0 then Suc 0 else Suc c*)
⟨*proof*⟩

lemmas *iT-card* =
iFROM-card
iTILL-card
iIN-card
iMOD-card-if
iMODb-card-if

Cardinality with *icard*

lemma *iFROM-icard*: *icard* [*n*...] = ∞
⟨*proof*⟩
lemma *iTILL-icard*: *icard* [...*n*] = *Fin* (*Suc n*)
⟨*proof*⟩
lemma *iIN-icard*: *icard* [*n*...,*d*] = *Fin* (*Suc d*)
⟨*proof*⟩
lemma *iMOD-0-icard*: *icard* [*r*, *mod 0*] = *iSuc 0*
⟨*proof*⟩
lemma *iMOD-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*] = ∞
⟨*proof*⟩
lemma *iMOD-icard-if*: *icard* [*r*, *mod m*] = (*if m* = *0 then iSuc 0 else* ∞)
⟨*proof*⟩
lemma *iMODb-mod-0-icard*: *icard* [*r*, *mod 0*, *c*] = *iSuc 0*
⟨*proof*⟩
lemma *iMODb-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*, *c*] = *Fin* (*Suc c*)
⟨*proof*⟩
lemma *iMODb-icard-if*: *icard* [*r*, *mod m*, *c*] = *Fin* (*if m* = *0 then Suc 0 else Suc c*)
⟨*proof*⟩

lemmas *iT-icard* =

iFROM-icard
iTILL-icard
iIN-icard
iMOD-icard-if
iMODb-icard-if

1.7 Functions *inext* and *iprev* with intervals

thm
inext-def
iprev-def

thm
inext-Max
iprev-iMin
thm
inext-closed
iprev-closed

thm
iMin-subset
iMin-Un
thm
Max-Un
Min-Un

lemma
iFROM-inext: $t \in [n\ldots] \Longrightarrow \mathit{inext}\ t\ [n\ldots] = \mathit{Suc}\ t$ **and**
iTILL-inext: $t < n \Longrightarrow \mathit{inext}\ t\ [\ldots n] = \mathit{Suc}\ t$ **and**
iIN-inext: $[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow \mathit{inext}\ t\ [n\ldots,d] = \mathit{Suc}\ t$
⟨*proof*⟩

lemma
iFROM-iprev': $t \in [n\ldots] \Longrightarrow \mathit{iprev}\ (\mathit{Suc}\ t)\ [n\ldots] = t$ **and**
iFROM-iprev: $n < t \Longrightarrow \mathit{iprev}\ t\ [n\ldots] = t - \mathit{Suc}\ 0$ **and**
iTILL-iprev: $t \in [\ldots n] \Longrightarrow \mathit{iprev}\ t\ [\ldots n] = t - \mathit{Suc}\ 0$ **and**
iIN-iprev: $[\![\ n < t;\ t \leq n + d\ ]\!] \Longrightarrow \mathit{iprev}\ t\ [n\ldots,d] = t - \mathit{Suc}\ 0$ **and**
iIN-iprev': $[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow \mathit{iprev}\ (\mathit{Suc}\ t)\ [n\ldots,d] = t$
⟨*proof*⟩

lemma *iMOD-inext*: $t \in [r,\ \mathit{mod}\ m] \Longrightarrow \mathit{inext}\ t\ [r,\ \mathit{mod}\ m] = t + m$
⟨*proof*⟩
lemma *iMOD-iprev*: $[\![\ t \in [r,\ \mathit{mod}\ m];\ r < t\ ]\!] \Longrightarrow \mathit{iprev}\ t\ [r,\ \mathit{mod}\ m] = t - m$
⟨*proof*⟩
thm *mod-eq-imp-diff-mod-eq-divisor*

⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*
⟨*proof*⟩

lemma *iMOD-iprev'*: $t \in [r,\ mod\ m] \Longrightarrow iprev\ (t + m)\ [r,\ mod\ m] = t$
⟨*proof*⟩

lemma *iMODb-inext*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow$
$inext\ t\ [r,\ mod\ m,\ c] = t + m$
⟨*proof*⟩

lemma *iMODb-iprev*:
$[\![\ t \in [r,\ mod\ m,\ c];\ r < t\ ]\!] \Longrightarrow$
$iprev\ t\ [r,\ mod\ m,\ c] = t - m$
⟨*proof*⟩
thm *mod-eq-imp-diff-mod-eq-divisor*
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*
⟨*proof*⟩
lemma *iMODb-iprev'*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow$
$iprev\ (t + m)\ [r,\ mod\ m,\ c] = t$
⟨*proof*⟩
thm *less-mod-eq-imp-add-divisor-le*
⟨*proof*⟩

lemmas *iT-inext* =
iFROM-inext
iTILL-inext
iIN-inext
iMOD-inext
iMODb-inext
lemmas *iT-iprev* =
iFROM-iprev'
iFROM-iprev
iTILL-iprev
iIN-iprev
iIN-iprev'
iMOD-iprev
iMOD-iprev'
iMODb-iprev
iMODb-iprev'
thm *iT-inext*
thm *iT-iprev*

thm *iprev-iMin*
thm *iT-finite*[*THEN inext-Max*]
lemma *iFROM-inext-if*:
inext t [*n*...] = (*if t* ∈ [*n*...] *then Suc t else t*)
⟨*proof*⟩
lemma *iTILL-inext-if*:
inext t [...*n*] = (*if t* < *n then Suc t else t*)
⟨*proof*⟩
lemma *iIN-inext-if*:
inext t [*n*...,*d*] = (*if n* ≤ *t* ∧ *t* < *n* + *d then Suc t else t*)
⟨*proof*⟩
lemma *iMOD-inext-if*:
inext t [*r*, *mod m*] = (*if t* ∈ [*r*, *mod m*] *then t* + *m else t*)
⟨*proof*⟩
lemma *iMODb-inext-if*:
inext t [*r*, *mod m*, *c*] =
(*if t* ∈ [*r*, *mod m*, *c*] ∧ *t* < *r* + *m* ∗ *c then t* + *m else t*)
⟨*proof*⟩

lemmas *iT-inext-if* =
iFROM-inext-if
iTILL-inext-if
iIN-inext-if
iMOD-inext-if
iMODb-inext-if
thm *iT-inext-if*

lemma *iFROM-iprev-if*:
iprev t [*n*...] = (*if n* < *t then t* − *Suc 0 else t*)
⟨*proof*⟩
lemma *iTILL-iprev-if*:
iprev t [...*n*] = (*if t* ∈ [...*n*] *then t* − *Suc 0 else t*)
⟨*proof*⟩
lemma *iIN-iprev-if*:
iprev t [*n*...,*d*] = (*if n* < *t* ∧ *t* ≤ *n* + *d then t* − *Suc 0 else t*)
⟨*proof*⟩
lemma *iMOD-iprev-if*:
iprev t [*r*, *mod m*] =
(*if t* ∈ [*r*, *mod m*] ∧ *r* < *t then t* − *m else t*)
⟨*proof*⟩
lemma *iMODb-iprev-if*:
iprev t [*r*, *mod m*, *c*] =
(*if t* ∈ [*r*, *mod m*, *c*] ∧ *r* < *t then t* − *m else t*)
⟨*proof*⟩

lemmas *iT-iprev-if* =
iFROM-iprev-if
iTILL-iprev-if
iIN-iprev-if

iMOD-iprev-if
iMODb-iprev-if
thm *iT-iprev-if*

The difference between an element and the next/previous element is constant if the element is different from Min/Max of the interval

lemma *iFROM-inext-diff-const*:
$t \in [n\ldots] \Longrightarrow inext\ t\ [n\ldots] - t = Suc\ 0$
⟨*proof*⟩
lemma *iFROM-iprev-diff-const*:
$n < t \Longrightarrow t - iprev\ t\ [n\ldots] = Suc\ 0$
⟨*proof*⟩
lemma *iFROM-iprev-diff-const'*:
$t \in [n\ldots] \Longrightarrow Suc\ t - iprev\ (Suc\ t)\ [n\ldots] = Suc\ 0$
⟨*proof*⟩

lemma *iTILL-inext-diff-const*:
$t < n \Longrightarrow inext\ t\ [\ldots n] - t = Suc\ 0$
⟨*proof*⟩
lemma *iTILL-iprev-diff-const*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow t - iprev\ t\ [\ldots n] = Suc\ 0$
⟨*proof*⟩

thm *iIN-inext*
lemma *iIN-inext-diff-const*:
$[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow inext\ t\ [n\ldots,d] - t = Suc\ 0$
⟨*proof*⟩

thm *iIN-iprev*
lemma *iIN-iprev-diff-const*:
$[\![\ n < t;\ t \leq n + d\ ]\!] \Longrightarrow t - iprev\ t\ [n\ldots,d] = Suc\ 0$
⟨*proof*⟩
lemma *iIN-iprev-diff-const'*:
$[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow Suc\ t - iprev\ (Suc\ t)\ [n\ldots,d] = Suc\ 0$
⟨*proof*⟩

thm *iMOD-inext*
lemma *iMOD-inext-diff-const*:
$t \in [r,\ mod\ m] \Longrightarrow inext\ t\ [r,\ mod\ m] - t = m$
⟨*proof*⟩
lemma *iMOD-iprev-diff-const'*:
$t \in [r,\ mod\ m] \Longrightarrow (t + m) - iprev\ (t + m)\ [r,\ mod\ m] = m$
⟨*proof*⟩
thm *iMOD-iprev*
lemma *iMOD-iprev-diff-const*:
$[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r,\ mod\ m] = m$
⟨*proof*⟩

thm *iMODb-inext*

lemma *iMODb-inext-diff-const*:
⟦ $t \in [r,\ mod\ m,\ c]$; $t < r + m * c$ ⟧ ⟹ $inext\ t\ [r,\ mod\ m,\ c] - t = m$
⟨*proof*⟩
thm *iMODb-iprev'*
lemma *iMODb-iprev-diff-const'*:
⟦ $t \in [r,\ mod\ m,\ c]$; $t < r + m * c$ ⟧ ⟹ $(t + m) - iprev\ (t + m)\ [r,\ mod\ m,\ c] = m$
⟨*proof*⟩
thm *iMODb-iprev*
lemma *iMODb-iprev-diff-const*:
⟦ $t \in [r,\ mod\ m,\ c]$; $r < t$ ⟧ ⟹ $t - iprev\ t\ [r,\ mod\ m,\ c] = m$
⟨*proof*⟩

lemmas *iT-inext-diff-const* =
iFROM-inext-diff-const
iTILL-inext-diff-const
iIN-inext-diff-const
iMOD-inext-diff-const
iMODb-inext-diff-const
lemmas *iT-iprev-diff-const* =
iFROM-iprev-diff-const
iFROM-iprev-diff-const'
iTILL-iprev-diff-const
iIN-iprev-diff-const
iIN-iprev-diff-const'
iMOD-iprev-diff-const'
iMOD-iprev-diff-const
iMODb-iprev-diff-const'
iMODb-iprev-diff-const
thm *iT-inext-diff-const*
thm *iT-iprev-diff-const*

1.7.1 Mirroring of intervals

thm *mirror-elem-def*
lemma
iIN-mirror-elem: $mirror\text{-}elem\ x\ [n\ldots,d] = n + n + d - x$ **and**
iTILL-mirror-elem: $mirror\text{-}elem\ x\ [\ldots n] = n - x$ **and**
iMODb-mirror-elem: $mirror\text{-}elem\ x\ [r,\ mod\ m,\ c] = r + r + m * c - x$
⟨*proof*⟩

lemma *iMODb-imirror-bounds*:
$r' + m' * c' \leq l + r$ ⟹
$imirror\text{-}bounds\ [r',\ mod\ m',\ c']\ l\ r = [l + r - r' - m' * c',\ mod\ m',\ c']$
⟨*proof*⟩
thm *mod-diff-right-eq*
⟨*proof*⟩

thm *mod-diff-right-eq*
⟨*proof*⟩
thm *mod-sub-eq-mod-swap*
⟨*proof*⟩
thm *mod-diff-right-eq*
⟨*proof*⟩

thm *imirror-bounds-def*
lemma *iIN-imirror-bounds*:
$n + d \leq l + r \Longrightarrow$ *imirror-bounds* $[n..,d]\ l\ r = [l + r - n - d..,d]$
⟨*proof*⟩

lemma *iTILL-imirror-bounds*:
$n \leq l + r \Longrightarrow$ *imirror-bounds* $[..n]\ l\ r = [l + r - n..,n]$
⟨*proof*⟩

lemmas *iT-imirror-bounds* =
iTILL-imirror-bounds
iIN-imirror-bounds
iMODb-imirror-bounds
thm *iT-imirror-bounds*

lemma *iMODb-imirror-ident*: *imirror* $[r, mod\ m, c] = [r, mod\ m, c]$
⟨*proof*⟩
lemma *iIN-imirror-ident*: *imirror* $[n..,d] = [n..,d]$
⟨*proof*⟩
lemma *iTILL-imirror-ident*: *imirror* $[..n] = [..n]$
⟨*proof*⟩

lemmas *iT-imirror-ident* =
iTILL-imirror-ident
iIN-imirror-ident
iMODb-imirror-ident
thm *iT-imirror-ident*

1.7.2 Functions *inext-nth* and *iprev-nth* on intervals

lemma *iFROM-inext-nth* : $[n..] \rightarrow a = n + a$
⟨*proof*⟩
lemma *iIN-inext-nth* : $a \leq d \Longrightarrow [n..,d] \rightarrow a = n + a$
⟨*proof*⟩
lemma *iIN-iprev-nth*: $a \leq d \Longrightarrow [n..,d] \leftarrow a = n + d - a$
⟨*proof*⟩
lemma *iIN-inext-nth-if* :
$[n..,d] \rightarrow a = (if\ a \leq d\ then\ n + a\ else\ n + d)$
⟨*proof*⟩
lemma *iIN-iprev-nth-if*:
$[n..,d] \leftarrow a = (if\ a \leq d\ then\ n + d - a\ else\ n)$

⟨*proof*⟩

lemma *iTILL-inext-nth* : $a \leq n \Longrightarrow [\ldots n] \rightarrow a = a$
⟨*proof*⟩
lemma *iTILL-inext-nth-if* :
$[\ldots n] \rightarrow a = (\mathit{if}\ a \leq n\ \mathit{then}\ a\ \mathit{else}\ n)$
⟨*proof*⟩
lemma *iTILL-iprev-nth*: $a \leq n \Longrightarrow [\ldots n] \leftarrow a = n - a$
⟨*proof*⟩
lemma *iTILL-iprev-nth-if*:
$[\ldots n] \leftarrow a = (\mathit{if}\ a \leq n\ \mathit{then}\ n - a\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *iMOD-inext-nth*: $[r,\ mod\ m] \rightarrow a = r + m * a$
⟨*proof*⟩

lemma *iMODb-inext-nth*: $a \leq c \Longrightarrow [r,\ mod\ m,\ c] \rightarrow a = r + m * a$
⟨*proof*⟩
lemma *iMODb-inext-nth-if*:
$[r,\ mod\ m,\ c] \rightarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * a\ \mathit{else}\ r + m * c)$
⟨*proof*⟩

lemma *iMODb-iprev-nth*:
$a \leq c \Longrightarrow [r,\ mod\ m,\ c] \leftarrow a = r + m * (c - a)$
⟨*proof*⟩
lemma *iMODb-iprev-nth-if*:
$[r,\ mod\ m,\ c] \leftarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * (c - a)\ \mathit{else}\ r)$
⟨*proof*⟩

lemma *iIN-iFROM-inext-nth*:
$a \leq d \Longrightarrow [n\ldots,d] \rightarrow a = [n\ldots] \rightarrow a$
⟨*proof*⟩
lemma *iIN-iFROM-inext*:
$a < n + d \Longrightarrow \mathit{inext}\ a\ [n\ldots,d] = \mathit{inext}\ a\ [n\ldots]$
⟨*proof*⟩
lemma *iMOD-iMODb-inext-nth*:
$a \leq c \Longrightarrow [r,\ mod\ m,\ c] \rightarrow a = [r,\ mod\ m] \rightarrow a$
⟨*proof*⟩
lemma *iMOD-iMODb-inext*:
$a < r + m * c \Longrightarrow \mathit{inext}\ a\ [r,\ mod\ m,\ c] = \mathit{inext}\ a\ [r,\ mod\ m]$
⟨*proof*⟩

lemma *iMOD-inext-nth-Suc-diff*:
$([r,\ mod\ m] \rightarrow (Suc\ n)) - ([r,\ mod\ m] \rightarrow n) = m$
⟨*proof*⟩
lemma *iMOD-inext-nth-diff*:

$([r, mod\ m] \rightarrow a) - ([r, mod\ m] \rightarrow b) = (a - b) * m$
⟨*proof*⟩
lemma *iMODb-inext-nth-diff*: $[\![a \leq c; b \leq c]\!] \Longrightarrow$
$([r, mod\ m, c] \rightarrow a) - ([r, mod\ m, c] \rightarrow b) = (a - b) * m$
⟨*proof*⟩

1.8 Induction with intervals

thm
inext-induct
iFROM-inext
lemma *iFROM-induct*:
$[\![P\ n; \bigwedge k.\ [\![k \in [n...]; P\ k]\!] \Longrightarrow P\ (Suc\ k); a \in [n...]]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iIN-induct*:
$[\![P\ n; \bigwedge k.\ [\![k \in [n...,d]; k \neq n + d; P\ k]\!] \Longrightarrow P\ (Suc\ k); a \in [n...,d]]\!] \Longrightarrow$
$P\ a$
⟨*proof*⟩

lemma *iTILL-induct*:
$[\![P\ 0; \bigwedge k.\ [\![k \in [...n]; k \neq n; P\ k]\!] \Longrightarrow P\ (Suc\ k); a \in [...n]]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMOD-induct*:
$[\![P\ r; \bigwedge k.\ [\![k \in [r, mod\ m]; P\ k]\!] \Longrightarrow P\ (k + m); a \in [r, mod\ m]]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMODb-induct*:
$[\![P\ r; \bigwedge k.\ [\![k \in [r, mod\ m, c]; k \neq r + m * c; P\ k]\!] \Longrightarrow P\ (k + m); a \in [r,$
$mod\ m, c]]\!] \Longrightarrow P\ a$
⟨*proof*⟩

thm
iprev-induct
iIN-inext
lemma *iIN-rev-induct*:
$[\![P\ (n + d); \bigwedge k.\ [\![k \in [n...,d]; k \neq n; P\ k]\!] \Longrightarrow P\ (k - Suc\ 0); a \in [n...,d]$
$]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iTILL-rev-induct*:
$[\![P\ n; \bigwedge k.\ [\![k \in [...n]; 0 < k; P\ k]\!] \Longrightarrow P\ (k - Suc\ 0); a \in [...n]]\!] \Longrightarrow P\ a$
⟨*proof*⟩

lemma *iMODb-rev-induct*:
$[\![P\ (r + m * c); \bigwedge k.\ [\![k \in [r, mod\ m, c]; k \neq r; P\ k]\!] \Longrightarrow P\ (k - m); a \in [r,$
$mod\ m, c]]\!] \Longrightarrow P\ a$
⟨*proof*⟩

end

2 IL-IntervalOperators: Arithmetic operators on natural intervals

theory *IL-IntervalOperators*
imports *IL-Interval*
begin

2.1 Arithmetic operations with intervals

2.1.1 Addition of and multiplication by constants

definition *iT-Plus* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\oplus$ *55*) **where**
$I \oplus k \equiv (\lambda n.(n + k))\ `\ I$

definition *iT-Mult* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\otimes$ *55*) **where**
iT-Mult-def : $I \otimes k \equiv (\lambda n.(n * k))\ `\ I$

lemma *iT-Plus-image-conv*: $I \oplus k = (\lambda n.(n + k))\ `\ I$
⟨*proof*⟩
lemma *iT-Mult-image-conv*: $I \otimes k = (\lambda n.(n * k))\ `\ I$
⟨*proof*⟩

lemma *iT-Plus-empty*: $\{\} \oplus k = \{\}$
⟨*proof*⟩
lemma *iT-Mult-empty*: $\{\} \otimes k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-not-empty*: $I \neq \{\} \Longrightarrow I \oplus k \neq \{\}$
⟨*proof*⟩
lemma *iT-Mult-not-empty*: $I \neq \{\} \Longrightarrow I \otimes k \neq \{\}$
⟨*proof*⟩

lemma *iT-Plus-empty-iff*: $(I \oplus k = \{\}) = (I = \{\})$
⟨*proof*⟩
lemma *iT-Mult-empty-iff*: $(I \otimes k = \{\}) = (I = \{\})$
⟨*proof*⟩

lemma *iT-Plus-mono*: $A \subseteq B \Longrightarrow A \oplus k \subseteq B \oplus k$
⟨*proof*⟩
lemma *iT-Mult-mono*: $A \subseteq B \Longrightarrow A \otimes k \subseteq B \otimes k$
⟨*proof*⟩

lemma *iT-Mult-0*: $I \neq \{\} \Longrightarrow I \otimes 0 = [\ldots 0]$
⟨*proof*⟩
corollary *iT-Mult-0-if*: $I \otimes 0 = (\mathit{if}\ I = \{\}\ \mathit{then}\ \{\}\ \mathit{else}\ [\ldots 0])$
⟨*proof*⟩

lemma *iT-Plus-mem-iff*: $x \in (I \oplus k) = (k \leq x \wedge (x - k) \in I)$
⟨*proof*⟩
lemma *iT-Plus-mem-iff2*: $x + k \in (I \oplus k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Mult-mem-iff-0*: $x \in (I \otimes 0) = (I \neq \{\} \wedge x = 0)$
⟨*proof*⟩
lemma *iT-Mult-mem-iff*:
$0 < k \Longrightarrow x \in (I \otimes k) = (x\ \mathit{mod}\ k = 0 \wedge x\ \mathit{div}\ k \in I)$
⟨*proof*⟩
lemma *iT-Mult-mem-iff2*: $0 < k \Longrightarrow x * k \in (I \otimes k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Plus-singleton*: $\{a\} \oplus k = \{a + k\}$
⟨*proof*⟩
lemma *iT-Mult-singleton*: $\{a\} \otimes k = \{a * k\}$
⟨*proof*⟩

lemma *iT-Plus-Un*: $(A \cup B) \oplus k = (A \oplus k) \cup (B \oplus k)$
⟨*proof*⟩
lemma *iT-Mult-Un*: $(A \cup B) \otimes k = (A \otimes k) \cup (B \otimes k)$
⟨*proof*⟩

lemma *iT-Plus-Int*: $(A \cap B) \oplus k = (A \oplus k) \cap (B \oplus k)$
⟨*proof*⟩
lemma *iT-Mult-Int*: $0 < k \Longrightarrow (A \cap B) \otimes k = (A \otimes k) \cap (B \otimes k)$
⟨*proof*⟩

thm
iT-Plus-Un
iT-Mult-Un
iT-Plus-Int
iT-Mult-Int

thm *imageI*
lemma *iT-Plus-image*: $f\ `\ I \oplus n = (\lambda x.\ f\ x + n)\ `\ I$
⟨*proof*⟩
lemma *iT-Mult-image*: $f\ `\ I \otimes n = (\lambda x.\ f\ x * n)\ `\ I$
⟨*proof*⟩

lemma *iT-Plus-commute*: $I \oplus a \oplus b = I \oplus b \oplus a$
⟨*proof*⟩
lemma *iT-Mult-commute*: $I \otimes a \otimes b = I \otimes b \otimes a$
⟨*proof*⟩
lemma *iT-Plus-assoc*:$I \oplus a \oplus b = I \oplus (a + b)$
⟨*proof*⟩
lemma *iT-Mult-assoc*:$I \otimes a \otimes b = I \otimes (a * b)$
⟨*proof*⟩
lemma *iT-Plus-Mult-distrib*: $I \oplus n \otimes m = I \otimes m \oplus n * m$
⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩⟨*proof*⟩
lemma *iT-Plus-finite-iff*: *finite* $(I \oplus k) =$ *finite* I
⟨*proof*⟩
lemma *iT-Mult-0-finite*: *finite* $(I \otimes 0)$
⟨*proof*⟩
lemma *iT-Mult-finite-iff*: $0 < k \Longrightarrow$ *finite* $(I \otimes k) =$ *finite* I
⟨*proof*⟩

lemma *iT-Plus-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oplus k) = iMin\ I + k$
⟨*proof*⟩
lemma *iT-Mult-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \otimes k) = iMin\ I * k$
⟨*proof*⟩

thm *Max-mono2*
lemma *iT-Plus-Max*: $[\![$ *finite* I; $I \neq \{\}\ ]\!] \Longrightarrow Max\ (I \oplus k) = Max\ I + k$
⟨*proof*⟩
lemma *iT-Mult-Max*: $[\![$ *finite* I; $I \neq \{\}\ ]\!] \Longrightarrow Max\ (I \otimes k) = Max\ I * k$
⟨*proof*⟩

thm *iT-Mult-0*
corollary
iMOD-mult-0: $[r,\ mod\ m] \otimes 0 = [\ldots 0]$ **and**
iMODb-mult-0: $[r,\ mod\ m,\ c] \otimes 0 = [\ldots 0]$ **and**
iFROM-mult-0: $[n\ldots] \otimes 0 = [\ldots 0]$ **and**
iIN-mult-0: $[n\ldots,d] \otimes 0 = [\ldots 0]$ **and**
iTILL-mult-0: $[\ldots n] \otimes 0 = [\ldots 0]$
⟨*proof*⟩

lemmas *iT-mult-0* =
iTILL-mult-0
iFROM-mult-0
iIN-mult-0
iMOD-mult-0
iMODb-mult-0
thm *iT-mult-0*

lemma *iT-Plus-0*: $I \oplus 0 = I$
⟨*proof*⟩

lemma *iT-Mult-1*: $I \otimes Suc\ 0 = I$
⟨*proof*⟩

thm *iT-Plus-Min*
corollary
iFROM-add-Min: $iMin\ ([n\ldots] \oplus k) = n + k$ **and**
iIN-add-Min: $iMin\ ([n\ldots,d] \oplus k) = n + k$ **and**
iTILL-add-Min: $iMin\ ([\ldots n] \oplus k) = k$ **and**
iMOD-add-Min: $iMin\ ([r,\ mod\ m] \oplus k) = r + k$ **and**
iMODb-add-Min: $iMin\ ([r,\ mod\ m,\ c] \oplus k) = r + k$
⟨*proof*⟩

corollary
iFROM-mult-Min: $iMin\ ([n\ldots] \otimes k) = n * k$ **and**
iIN-mult-Min: $iMin\ ([n\ldots,d] \otimes k) = n * k$ **and**
iTILL-mult-Min: $iMin\ ([\ldots n] \otimes k) = 0$ **and**
iMOD-mult-Min: $iMin\ ([r,\ mod\ m] \otimes k) = r * k$ **and**
iMODb-mult-Min: $iMin\ ([r,\ mod\ m,\ c] \otimes k) = r * k$
⟨*proof*⟩

lemmas *iT-add-Min* =
iIN-add-Min
iTILL-add-Min
iFROM-add-Min
iMOD-add-Min
iMODb-add-Min
thm *iT-add-Min*

lemmas *iT-mult-Min* =
iIN-mult-Min
iTILL-mult-Min
iFROM-mult-Min
iMOD-mult-Min
iMODb-mult-Min
thm *iT-mult-Min*

lemma *iFROM-add*: $[n\ldots] \oplus k = [n{+}k\ldots]$
⟨*proof*⟩

lemma *iIN-add*: $[n\ldots,d] \oplus k = [n{+}k\ldots,d]$
⟨*proof*⟩

lemma *iTILL-add*: $[\ldots i] \oplus k = [k\ldots,i]$
⟨*proof*⟩

lemma *iMOD-add*: $[r,\ mod\ m] \oplus k = [r + k,\ mod\ m]$
⟨*proof*⟩
thm *mod-add*
⟨*proof*⟩
thm *mod-sub-add*[*of k x*]
⟨*proof*⟩

lemma *iMODb-add*: $[r,\ mod\ m,\ c] \oplus k = [r + k,\ mod\ m,\ c]$
⟨*proof*⟩

lemmas *iT-add* =
iMOD-add
iMODb-add
iFROM-add
iIN-add
iTILL-add
iT-Plus-singleton
thm *iT-add*

thm
mod-mult-distrib
mod-mult-distrib2
thm
mod-eq-mult-distrib
mod-factor-imp-mod-0
mod-factor-div
mod-factor-div-mod

lemma *iFROM-mult*: $[n...] \otimes k = [\ n * k,\ mod\ k\ ]$
⟨*proof*⟩

lemma *iIN-mult*: $[n...,d] \otimes k = [\ n * k,\ mod\ k,\ d\ ]$
⟨*proof*⟩

lemma *iTILL-mult*: $[...n] \otimes k = [\ 0,\ mod\ k,\ n\ ]$
⟨*proof*⟩

lemma *iMOD-mult*: $[r,\ mod\ m\ ] \otimes k = [\ r * k,\ mod\ m * k\ ]$
⟨*proof*⟩
thm *mod-mult2-eq*
⟨*proof*⟩

lemma *iMODb-mult*:
$[r,\ mod\ m,\ c\ ] \otimes k = [\ r * k,\ mod\ m * k,\ c\ ]$
⟨*proof*⟩
thm *iMODb-iMOD-iTILL-conv*
⟨*proof*⟩
thm *Int-absorb2*

⟨*proof*⟩
thm *iMOD-iTILL-iMODb-conv*
⟨*proof*⟩

lemmas *iT-mult* =
iTILL-mult
iFROM-mult
iIN-mult
iMOD-mult
iMODb-mult
iT-Mult-singleton
thm
iT-mult
iT-mult-0

2.1.2 Some conversions between intervals using constant addition and multiplication

lemma *iFROM-conv*: $[n...] = UNIV \oplus n$
⟨*proof*⟩
lemma *iIN-conv*: $[n...,d] = [...d] \oplus n$
⟨*proof*⟩
lemma *iMOD-conv*: $[r, mod\ m] = [0...] \otimes m \oplus r$
⟨*proof*⟩
lemma *iMODb-conv*: $[r, mod\ m, c] = [...c] \otimes m \oplus r$
⟨*proof*⟩

Some examples showing the utility of iMODb_conv

lemma $[12, mod\ 10, 4] = \{12, 22, 32, 42, 52\}$
⟨*proof*⟩
lemma $[12, mod\ 10, 4] = \{12, 22, 32, 42, 52\}$
⟨*proof*⟩
lemma $[12, mod\ 10, 4] = \{12, 22, 32, 42, 52\}$
⟨*proof*⟩

lemma $[r, mod\ m, 4] = \{r, r+m, r+2*m, r+3*m, r+4*m\}$
⟨*proof*⟩
thm *image-Collect*
⟨*proof*⟩

lemma $[2, mod\ 10, 4] = \{2, 12, 22, 32, 42\}$
⟨*proof*⟩

2.1.3 Subtraction of constants

definition *iT-Plus-neg* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\oplus-$ 55) **where**
$I \oplus- k \equiv \{x.\ x + k \in I\}$

lemma *iT-Plus-neg-mem-iff*: $(x \in I \oplus- k) = (x + k \in I)$

⟨*proof*⟩
thm *iT-Plus-mem-iff2*
lemma *iT-Plus-neg-mem-iff2*: $k \leq x \Longrightarrow (x - k \in I \oplus- k) = (x \in I)$
⟨*proof*⟩

lemma *iT-Plus-neg-image-conv*: $I \oplus- k = (\lambda n.(n - k))\ `\ (I \downarrow\geq k)$
⟨*proof*⟩
thm *image-eqI*
⟨*proof*⟩

lemma *iT-Plus-neg-cut-eq*: $t \leq k \Longrightarrow (I \downarrow\geq t) \oplus- k = I \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-mono*: $A \subseteq B \Longrightarrow A \oplus- k \subseteq B \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-empty*: $\{\} \oplus- k = \{\}$
⟨*proof*⟩
lemma *iT-Plus-neg-Max-less-empty*:
$[\![\ \mathit{finite}\ I;\ \mathit{Max}\ I < k\ ]\!] \Longrightarrow I \oplus- k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-neg-not-empty-iff*: $(I \oplus- k \neq \{\}) = (\exists\, x{\in}I.\ k \leq x)$
⟨*proof*⟩

lemma *iT-Plus-neg-empty-iff*:
$(I \oplus- k = \{\}) = (I = \{\} \vee (\mathit{finite}\ I \wedge \mathit{Max}\ I < k))$
⟨*proof*⟩

lemma *iT-Plus-neg-assoc*: $(I \oplus- a) \oplus- b = I \oplus- (a + b)$
⟨*proof*⟩

lemma *iT-Plus-neg-commute*: $I \oplus- a \oplus- b = I \oplus- b \oplus- a$
⟨*proof*⟩

lemma *iT-Plus-neg-0*: $I \oplus- 0 = I$
⟨*proof*⟩

thm *diff-add-assoc*
lemma *iT-Plus-Plus-neg-assoc*: $b \leq a \Longrightarrow I \oplus a \oplus- b = I \oplus (a - b)$
⟨*proof*⟩
lemma *iT-Plus-Plus-neg-assoc2*: $a \leq b \Longrightarrow I \oplus a \oplus- b = I \oplus- (b - a)$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-le-cut-eq*:
$a \leq b \Longrightarrow (I \oplus- a) \oplus b = (I \downarrow\geq a) \oplus (b - a)$
⟨*proof*⟩
corollary *iT-Plus-neg-Plus-le-Min-eq*:

$[\![\ a \leq b;\ a \leq iMin\ I\ ]\!] \Longrightarrow (I \oplus\!\!- a) \oplus b = I \oplus (b - a)$
⟨*proof*⟩

lemma *iT-Plus-neg-Plus-ge-cut-eq*:
$b \leq a \Longrightarrow (I \oplus\!\!- a) \oplus b = (I \downarrow\!\geq a) \oplus\!\!- (a - b)$
⟨*proof*⟩

corollary *iT-Plus-neg-Plus-ge-Min-eq*:
$[\![\ b \leq a;\ a \leq iMin\ I\ ]\!] \Longrightarrow (I \oplus\!\!- a) \oplus b = I \oplus\!\!- (a - b)$
⟨*proof*⟩

lemma *iT-Plus-neg-Mult-distrib*:
$0 < m \Longrightarrow I \oplus\!\!- n \otimes m = I \otimes m \oplus\!\!- n * m$
⟨*proof*⟩

thm *le-add-diff-inverse2*
lemma *iT-Plus-neg-Plus-le-inverse*: $k \leq iMin\ I \Longrightarrow I \oplus\!\!- k \oplus k = I$
⟨*proof*⟩
lemma *iT-Plus-neg-Plus-inverse*: $I \oplus\!\!- k \oplus k = I \downarrow\!\geq k$
⟨*proof*⟩

thm *diff-add-inverse2*
lemma *iT-Plus-Plus-neg-inverse*: $I \oplus k \oplus\!\!- k = I$
⟨*proof*⟩

lemma *iT-Plus-neg-Un*: $(A \cup B) \oplus\!\!- k = (A \oplus\!\!- k) \cup (B \oplus\!\!- k)$
⟨*proof*⟩

lemma *iT-Plus-neg-Int*: $(A \cap B) \oplus\!\!- k = (A \oplus\!\!- k) \cap (B \oplus\!\!- k)$
⟨*proof*⟩

lemma *iT-Plus-neg-Max-singleton*: $[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow I \oplus\!\!- Max\ I = \{0\}$
⟨*proof*⟩

lemma *iT-Plus-neg-singleton*: $\{a\} \oplus\!\!- k = (if\ k \leq a\ then\ \{a - k\}\ else\ \{\})$
⟨*proof*⟩
corollary *iT-Plus-neg-singleton1*: $k \leq a \Longrightarrow \{a\} \oplus\!\!- k = \{a-k\}$
⟨*proof*⟩
corollary *iT-Plus-neg-singleton2*: $a < k \Longrightarrow \{a\} \oplus\!\!- k = \{\}$
⟨*proof*⟩

lemma *iT-Plus-neg-finite-iff*: $finite\ (I \oplus\!\!- k) = finite\ I$
⟨*proof*⟩
thm *inj-on-finite-image-iff*
⟨*proof*⟩
thm *nat-cut-ge-finite-iff*
⟨*proof*⟩

lemma *iT-Plus-neg-Min*:
$I \oplus- k \neq \{\} \Longrightarrow iMin\ (I \oplus- k) = iMin\ (I \downarrow\geq k) - k$
⟨*proof*⟩

lemma *iT-Plus-neg-Max*:
$[\![\ finite\ I;\ I \oplus- k \neq \{\}\ ]\!] \Longrightarrow Max\ (I \oplus- k) = Max\ I - k$
⟨*proof*⟩

Subtractions of constants from intervals

lemma *iFROM-add-neg*: $[n\ldots] \oplus- k = [n - k\ldots]$
⟨*proof*⟩

lemma *iTILL-add-neg*: $[\ldots n] \oplus- k = (if\ k \leq n\ then\ [\ldots n - k]\ else\ \{\})$
⟨*proof*⟩
lemma *iTILL-add-neg1*: $k \leq n \Longrightarrow [\ldots n] \oplus- k = [\ldots n-k]$
⟨*proof*⟩
lemma *iTILL-add-neg2*: $n < k \Longrightarrow [\ldots n] \oplus- k = \{\}$
⟨*proof*⟩

lemma *iIN-add-neg*:
$[n\ldots,d] \oplus- k = ($
if $k \leq n$ *then* $[n - k\ldots,d]$
else if $k \leq n + d$ *then* $[\ldots n + d - k]$ *else* $\{\})$
⟨*proof*⟩
lemma *iIN-add-neg1*: $k \leq n \Longrightarrow [n\ldots,d] \oplus- k = [n - k\ldots,d]$
⟨*proof*⟩
lemma *iIN-add-neg2*: $[\![\ n \leq k;\ k \leq n + d\ ]\!] \Longrightarrow [n\ldots,d] \oplus- k = [\ldots n + d - k]$
⟨*proof*⟩
lemma *iIN-add-neg3*: $n + d < k \Longrightarrow [n\ldots,d] \oplus- k = \{\}$
⟨*proof*⟩

lemma *iMOD-0-add-neg*: $[r,\ mod\ 0] \oplus- k = \{r\} \oplus- k$
⟨*proof*⟩

lemma *iMOD-gr0-add-neg*:
$0 < m \Longrightarrow$
$[r,\ mod\ m] \oplus- k = ($
if $k \leq r$ *then* $[r - k,\ mod\ m]$
else $[(m + r\ mod\ m - k\ mod\ m)\ mod\ m,\ mod\ m])$
⟨*proof*⟩
thm *mod-sub-add*[*of k r m*] *le-diff-conv*
⟨*proof*⟩
thm *eq-commute*[*of r mod m*]
thm *mod-add-eq-mod-conv*[*of m x k r*]
⟨*proof*⟩
lemma *iMOD-add-neg*:

$[r,\ mod\ m] \oplus\!- k = ($
$\quad if\ k \le r\ then\ [r - k,\ mod\ m]$
$\quad else\ if\ 0 < m\ then\ [(m + r\ mod\ m - k\ mod\ m)\ mod\ m,\ mod\ m]\ else\ \{\})$
⟨*proof*⟩
corollary *iMOD-add-neg1*:
$k \le r \Longrightarrow [r,\ mod\ m] \oplus\!- k = [r - k,\ mod\ m]$
⟨*proof*⟩
lemma *iMOD-add-neg2*:
$[\![\ 0 < m;\ r < k\ ]\!] \Longrightarrow [r,\ mod\ m] \oplus\!- k = [(m + r\ mod\ m - k\ mod\ m)\ mod\ m,$
$mod\ m]$
⟨*proof*⟩

lemma *iMODb-mod-0-add-neg*: $[r,\ mod\ 0,\ c] \oplus\!- k = \{r\} \oplus\!- k$
⟨*proof*⟩

lemma *iMODb-add-neg*:
$[r,\ mod\ m,\ c] \oplus\!- k = ($
$\quad if\ k \le r\ then\ [r - k,\ mod\ m,\ c]$
$\quad else$
$\quad\quad if\ k \le r + m * c\ then$
$\quad\quad [(m + r\ mod\ m - k\ mod\ m)\ mod\ m,\ mod\ m,\ (r + m * c - k)\ div\ m]$
$\quad\quad else\ \{\})$
⟨*proof*⟩
thm *mod-diff1-eq*[*of k*]
⟨*proof*⟩
thm *sub-mod-div-eq-div*
⟨*proof*⟩

lemma *iMODb-add-neg′*:
$[r,\ mod\ m,\ c] \oplus\!- k = ($
$\quad if\ k \le r\ then\ [r - k,\ mod\ m,\ c]$
$\quad else\ if\ k \le r + m * c\ then$
$\quad\quad if\ k\ mod\ m \le r\ mod\ m$
$\quad\quad\quad then\ [\ r\ mod\ m - k\ mod\ m,\ mod\ m,\ c + r\ div\ m - k\ div\ m]$
$\quad\quad\quad else\ [\ m + r\ mod\ m - k\ mod\ m,\ mod\ m,\ c + r\ div\ m - Suc\ (k\ div\ m)\ ]$
$\quad\quad else\ \{\})$
⟨*proof*⟩
corollary *iMODb-add-neg1*:
$k \le r \Longrightarrow [r,\ mod\ m,\ c] \oplus\!- k = [r - k,\ mod\ m,\ c]$
⟨*proof*⟩
corollary *iMODb-add-neg2*:
$[\![\ r < k;\ k \le r + m * c\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oplus\!- k =$
$[(m + r\ mod\ m - k\ mod\ m)\ mod\ m,\ mod\ m,\ (r + m * c - k)\ div\ m]$
⟨*proof*⟩

corollary *iMODb-add-neg2-mod-le*:
$[\![\ r < k;\ k \leq r + m * c;\ k\ mod\ m \leq r\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oplus\!-\ k =$
$[\ r\ mod\ m - k\ mod\ m,\ mod\ m,\ c + r\ div\ m - k\ div\ m]$
⟨*proof*⟩
corollary *iMODb-add-neg2-mod-less*:
$[\![\ r < k;\ k \leq r + m * c;\ r\ mod\ m < k\ mod\ m]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oplus\!-\ k =$
$[\ m + r\ mod\ m - k\ mod\ m,\ mod\ m,\ c + r\ div\ m - Suc\ (k\ div\ m)\ ]$
⟨*proof*⟩
lemma *iMODb-add-neg3*: $r + m * c < k \Longrightarrow [r,\ mod\ m,\ c] \oplus\!-\ k = \{\}$
⟨*proof*⟩

lemmas *iT-add-neg* =
iFROM-add-neg
iIN-add-neg
iTILL-add-neg
iMOD-add-neg
iMODb-add-neg
iT-Plus-neg-singleton
thm *iT-add-neg*

2.1.4 Subtraction of intervals from constants

definition *iT-Minus* :: *Time* ⇒ *iT* ⇒ *iT* (**infixl** ⊖ 55) **where**
$k \ominus I \equiv \{x.\ x \leq k \wedge (k - x) \in I\}$

lemma *iT-Minus-mem-iff*: $(x \in k \ominus I) = (x \leq k \wedge k - x \in I)$
⟨*proof*⟩

lemma *iT-Minus-mono*: $A \subseteq B \Longrightarrow k \ominus A \subseteq k \ominus B$
⟨*proof*⟩

lemma *iT-Minus-image-conv*: $k \ominus I = (\lambda x.\ k - x)\ `\ (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-cut-eq*: $k \leq t \Longrightarrow k \ominus (I \downarrow\leq t) = k \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Minus-cut-eq*: $k \ominus (k \ominus (I \downarrow\leq k)) = I \downarrow\leq k$
⟨*proof*⟩

lemma *10* $\ominus$ [...*3*] = [*7*...,*3*]
⟨*proof*⟩

lemma *iT-Minus-empty*: $k \ominus \{\} = \{\}$
⟨*proof*⟩

lemma *iT-Minus-0*: $k \ominus \{0\} = \{k\}$

⟨*proof*⟩

lemma *iT-Minus-bound*: $x \in k \ominus I \Longrightarrow x \leq k$
⟨*proof*⟩

lemma *iT-Minus-finite*: *finite* $(k \ominus I)$
thm *finite-nat-iff-bounded-le2*
⟨*proof*⟩

lemma *iT-Minus-less-Min-empty*: $k < iMin\ I \Longrightarrow k \ominus I = \{\}$
⟨*proof*⟩

lemma *iT-Minus-Min-singleton*: $I \neq \{\} \Longrightarrow (iMin\ I) \ominus I = \{0\}$
⟨*proof*⟩

lemma *iT-Minus-empty-iff*: $(k \ominus I = \{\}) = (I = \{\} \vee k < iMin\ I)$
⟨*proof*⟩

lemma *iT-Minus-imirror-conv*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus\!- (iMin\ I + Max\ (I \downarrow\leq k))$
⟨*proof*⟩
thm *nat-add-right-cancel*[*THEN iffD2*, *of* - - *k*]
⟨*proof*⟩
thm *add-diff-assoc2*
⟨*proof*⟩
thm *iT-Minus-imirror-conv*
lemma *iT-Minus-imirror-conv'*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus\!- (iMin\ (I \downarrow\leq k) + Max\ (I \downarrow\leq k))$
⟨*proof*⟩
thm *cut-le-Min-eq*
⟨*proof*⟩

thm *iT-Minus-bound*
lemma *iT-Minus-Max*:
$[\![\ I \neq \{\};\ iMin\ I \leq k\ ]\!] \Longrightarrow Max\ (k \ominus I) = k - (iMin\ I)$
thm *Max-equality*
⟨*proof*⟩

lemma *iT-Minus-Min*:
$[\![\ I \neq \{\};\ iMin\ I \leq k\ ]\!] \Longrightarrow iMin\ (k \ominus I) = k - (Max\ (I \downarrow\leq k))$
⟨*proof*⟩
thm *subsetD*[*OF cut-le-subset*, *OF Max-in*]
⟨*proof*⟩

lemma *iT-Minus-Minus-eq*: $[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow k \ominus (k \ominus I) = I$
thm *iT-Minus-cut-eq*[*of k k I*, *symmetric*] *iT-Minus-Minus-cut-eq*
⟨*proof*⟩

lemma *iT-Minus-Minus-eq2*: $I \subseteq [\ldots k] \Longrightarrow k \ominus (k \ominus I) = I$
⟨*proof*⟩
thm *Max-subset*
⟨*proof*⟩

lemma *iT-Minus-Minus*: $a \ominus (b \ominus I) = (I \downarrow\leq b) \oplus a \oplus- b$
⟨*proof*⟩
lemma *iT-Minus-Plus-empty*: $k < n \Longrightarrow k \ominus (I \oplus n) = \{\}$
⟨*proof*⟩
lemma *iT-Minus-Plus-commute*: $n \leq k \Longrightarrow k \ominus (I \oplus n) = (k - n) \ominus I$
⟨*proof*⟩

lemma *iT-Minus-Plus-cut-assoc*: $(k \ominus I) \oplus n = (k + n) \ominus (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Minus-Plus-assoc*:
$[\![\textit{finite } I; \textit{Max } I \leq k]\!] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
⟨*proof*⟩
lemma *iT-Minus-Plus-assoc2*:
$I \subseteq [\ldots k] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
⟨*proof*⟩
thm *Max-subset*
⟨*proof*⟩

lemma *iT-Minus-Un*: $k \ominus (A \cup B) = (k \ominus A) \cup (k \ominus B)$
⟨*proof*⟩

lemma *iT-Minus-Int*: $k \ominus (A \cap B) = (k \ominus A) \cap (k \ominus B)$
⟨*proof*⟩

lemma *iT-Minus-singleton*: $k \ominus \{a\} = (\textit{if } a \leq k \textit{ then } \{k - a\} \textit{ else } \{\})$
⟨*proof*⟩
corollary *iT-Minus-singleton1*: $a \leq k \Longrightarrow k \ominus \{a\} = \{k-a\}$
⟨*proof*⟩
corollary *iT-Minus-singleton2*: $k < a \Longrightarrow k \ominus \{a\} = \{\}$
⟨*proof*⟩

lemma *iMOD-sub*:
$k \ominus [r, \textit{mod } m] =$
$(\textit{if } r \leq k \textit{ then } [(k - r) \textit{ mod } m, \textit{mod } m, (k - r) \textit{ div } m] \textit{ else } \{\})$
⟨*proof*⟩
thm *mod-sub-eq-mod-swap*[*of r k x m*, *symmetric*]
⟨*proof*⟩

corollary *iMOD-sub1*:
$r \leq k \Longrightarrow k \ominus [r, \mathit{mod}\ m] = [(k - r)\ \mathit{mod}\ m, \mathit{mod}\ m, (k - r)\ \mathit{div}\ m]$
⟨*proof*⟩
corollary *iMOD-sub2*: $k < r \Longrightarrow k \ominus [r, \mathit{mod}\ m] = \{\}$
thm *iT-Minus-less-Min-empty*
⟨*proof*⟩

lemma *iTILL-sub*: $k \ominus [\ldots n] = (\mathit{if}\ n \leq k\ \mathit{then}\ [k - n\ldots,n]\ \mathit{else}\ [\ldots k])$
⟨*proof*⟩
corollary *iTILL-sub1*: $n \leq k \Longrightarrow k \ominus [\ldots n] = [k - n\ldots,n]$
⟨*proof*⟩
corollary *iTILL-sub2*: $k \leq n \Longrightarrow k \ominus [\ldots n] = [\ldots k]$
⟨*proof*⟩

lemma *iMODb-sub*:
$k \ominus [r, \mathit{mod}\ m, c] = ($
$\mathit{if}\ r + m * c \leq k\ \mathit{then}\ [\ k - r - m * c, \mathit{mod}\ m, c]\ \mathit{else}$
$\mathit{if}\ r \leq k\ \mathit{then}\ [\ (k - r)\ \mathit{mod}\ m, \mathit{mod}\ m, (k - r)\ \mathit{div}\ m]\ \mathit{else}\ \{\})$
⟨*proof*⟩
thm *iMODb-iMOD-iTILL-conv*
⟨*proof*⟩
thm *iT-Minus-Int*
⟨*proof*⟩
thm *add-le-imp-le-diff2*
⟨*proof*⟩
thm *le-diff-conv2*[*OF mod-le-dividend*]
⟨*proof*⟩
thm *iMODb-iMOD-iIN-conv*
⟨*proof*⟩
thm *iIN-inter*
⟨*proof*⟩
corollary *iMODb-sub1*:
$[\![\ r \leq k; k \leq r + m * c\ ]\!] \Longrightarrow$
$k \ominus [r, \mathit{mod}\ m, c] = [(k - r)\ \mathit{mod}\ m, \mathit{mod}\ m, (k - r)\ \mathit{div}\ m]$
⟨*proof*⟩
corollary *iMODb-sub2*: $k < r \Longrightarrow k \ominus [r, \mathit{mod}\ m, c] = \{\}$
thm *iT-Minus-less-Min-empty*
⟨*proof*⟩
corollary *iMODb-sub3*:
$r + m * c \leq k \Longrightarrow k \ominus [r, \mathit{mod}\ m, c] = [\ k - r - m * c, \mathit{mod}\ m, c]$
⟨*proof*⟩

lemma *iFROM-sub*: $k \ominus [n\ldots] = (\mathit{if}\ n \leq k\ \mathit{then}\ [\ldots k - n]\ \mathit{else}\ \{\})$
⟨*proof*⟩
corollary *iFROM-sub1*: $n \leq k \Longrightarrow k \ominus [n\ldots] = [\ldots k{-}n]$
⟨*proof*⟩
corollary *iFROM-sub-empty*: $k < n \Longrightarrow k \ominus [n\ldots] = \{\}$

⟨*proof*⟩

lemma *iIN-sub*:
k ⊖ [*n*...,*d*] = (
if n + *d* ≤ *k then* [*k* − (*n* + *d*)...,*d*]
else if n ≤ *k then* [...*k* − *n*] *else* {})
⟨*proof*⟩
lemma *iIN-sub1*: *n* + *d* ≤ *k* ⟹ *k* ⊖ [*n*...,*d*] = [*k* − (*n* + *d*)...,*d*]
⟨*proof*⟩
lemma *iIN-sub2*: ⟦ *n* ≤ *k*; *k* ≤ *n* + *d* ⟧ ⟹ *k* ⊖ [*n*...,*d*] = [...*k* − *n*]
⟨*proof*⟩
lemma *iIN-sub3*: *k* < *n* ⟹ *k* ⊖ [*n*...,*d*] = {}
⟨*proof*⟩

lemmas *iT-sub* =
iFROM-sub
iIN-sub
iTILL-sub
iMOD-sub
iMODb-sub
iT-Minus-singleton
thm *iT-sub*

2.1.5 Division of intervals by constants

Monotonicity and injectivity of artithmetic operators

lemma *iMOD-div-right-strict-mono-on*:
⟦ 0 < *k*; *k* ≤ *m* ⟧ ⟹ *strict-mono-on* (λ*x*. *x div k*) [*r*, *mod m*]
⟨*proof*⟩
corollary *iMOD-div-right-inj-on*:
⟦ 0 < *k*; *k* ≤ *m* ⟧ ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*]
⟨*proof*⟩

lemma *iMOD-mult-div-right-inj-on*:
inj-on (λ*x*. *x div* (*k*::*nat*)) [*r*, *mod* (*k* ∗ *m*)]
⟨*proof*⟩
lemma *iMOD-mult-div-right-inj-on2*:
m mod k = 0 ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*]
⟨*proof*⟩

lemma *iMODb-div-right-strict-mono-on*:
⟦ 0 < *k*; *k* ≤ *m* ⟧ ⟹ *strict-mono-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
thm *strict-mono-on-subset*[*OF iMOD-div-right-strict-mono-on iMODb-iMOD-subset-same*]
⟨*proof*⟩
corollary *iMODb-div-right-inj-on*:
⟦ 0 < *k*; *k* ≤ *m* ⟧ ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]

⟨*proof*⟩
lemma *iMODb-mult-div-right-inj-on*:
inj-on (λ*x*. *x div* (*k*::*nat*)) [*r*, *mod* (*k* ∗ *m*), *c*]
thm *subset-inj-on*[*OF iMOD-div-right-inj-on iMODb-iMOD-subset-same*]
⟨*proof*⟩
corollary *iMODb-mult-div-right-inj-on2*:
m mod k = *0* ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
⟨*proof*⟩

definition *iT-Div* :: *iT* ⇒ *Time* ⇒ *iT* (**infixl** ⊘ *55*) **where**
I ⊘ *k* ≡ (λ*n*.(*n div k*)) ‘ *I*

lemma *iT-Div-image-conv*: *I* ⊘ *k* = (λ*n*.(*n div k*)) ‘ *I*
⟨*proof*⟩

lemma *iT-Div-mono*: *A* ⊆ *B* ⟹ *A* ⊘ *k* ⊆ *B* ⊘ *k*
⟨*proof*⟩

lemma *iT-Div-empty*: {} ⊘ *k* = {}
⟨*proof*⟩
lemma *iT-Div-not-empty*: *I* ≠ {} ⟹ *I* ⊘ *k* ≠ {}
⟨*proof*⟩

lemma *iT-Div-empty-iff*: (*I* ⊘ *k* = {}) = (*I* = {})
⟨*proof*⟩

lemma *iT-Div-0*: *I* ≠ {} ⟹ *I* ⊘ *0* = [. . .*0*]
⟨*proof*⟩
corollary *iT-Div-0-if*: *I* ⊘ *0* = (*if I* = {} *then* {} *else* [. . .*0*])
⟨*proof*⟩
corollary
iFROM-div-0: [*n*. . .] ⊘ *0* = [. . .*0*] **and**
iTILL-div-0: [. . .*n*] ⊘ *0* = [. . .*0*] **and**
iIN-div-0: [*n*. . .,*d*] ⊘ *0* = [. . .*0*] **and**
iMOD-div-0: [*r*, *mod m*] ⊘ *0* = [. . .*0*] **and**
iMODb-div-0: [*r*, *mod m*, *c*] ⊘ *0* = [. . .*0*]
⟨*proof*⟩

lemmas *iT-div-0* =
iTILL-div-0
iFROM-div-0
iIN-div-0

iMOD-div-0
iMODb-div-0
thm *iT-div-0*

lemma *iT-Div-1*: $I \oslash Suc\ 0 = I$
⟨*proof*⟩

lemma *iT-Div-mem-iff-0*: $x \in (I \oslash 0) = (I \neq \{\} \wedge x = 0)$
⟨*proof*⟩

lemma *iT-Div-mem-iff*:
$0 < k \Longrightarrow x \in (I \oslash k) = (\exists\, y \in I.\ y\ div\ k = x)$
⟨*proof*⟩

lemma *iT-Div-mem-iff2*:
$0 < k \Longrightarrow x\ div\ k \in (I \oslash k) = (\exists\, y \in I.\ y\ div\ k = x\ div\ k)$
⟨*proof*⟩

lemma *iT-Div-mem-iff-Int*:
$0 < k \Longrightarrow x \in (I \oslash k) = (I \cap [x * k \ldots, k - Suc\ 0] \neq \{\})$
thm *ex-in-conv*[*symmetric*]
⟨*proof*⟩
thm *le-less-div-conv*
⟨*proof*⟩
thm *less-eq-le-pred*
⟨*proof*⟩

lemma *iT-Div-imp-mem*:
$0 < k \Longrightarrow x \in I \Longrightarrow x\ div\ k \in (I \oslash k)$
⟨*proof*⟩

lemma *iT-Div-singleton*: $\{a\} \oslash k = \{a\ div\ k\}$
⟨*proof*⟩

lemma *iT-Div-Un*: $(A \cup B) \oslash k = (A \oslash k) \cup (B \oslash k)$
⟨*proof*⟩

lemma *iT-Div-insert*: $(insert\ n\ I) \oslash k = insert\ (n\ div\ k)\ (I \oslash k)$
⟨*proof*⟩

lemma *not-iT-Div-Int*: $\neg\ (\forall\ k\ A\ B.\ (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k))$
⟨*proof*⟩

thm *subset-image-Int*
lemma *subset-iT-Div-Int*: $A \subseteq B \Longrightarrow (A \cap B) \oslash k = (A \oslash k) \cap (B \oslash k)$
⟨*proof*⟩

lemma *iFROM-iT-Div-Int*:
$[\![\ 0 < k;\ n \leq iMin\ A\ ]\!] \Longrightarrow (A \cap [n\ldots]) \oslash k = (A \oslash k) \cap ([n\ldots] \oslash k)$
⟨*proof*⟩

lemma *iIN-iT-Div-Int*:
$[\![\ 0 < k;\ n \leq iMin\ A;\ \forall x{\in}A.\ x\ div\ k \leq (n + d)\ div\ k \longrightarrow x \leq n + d\ ]\!] \Longrightarrow$
$(A \cap [n\ldots,d]) \oslash k = (A \oslash k) \cap ([n\ldots,d] \oslash k)$
⟨*proof*⟩
corollary *iTILL-iT-Div-Int*:
$[\![\ 0 < k;\ \forall x{\in}A.\ x\ div\ k \leq n\ div\ k \longrightarrow x \leq n\ ]\!] \Longrightarrow$
$(A \cap [\ldots n]) \oslash k = (A \oslash k) \cap ([\ldots n] \oslash k)$
⟨*proof*⟩
lemma *iIN-iT-Div-Int-mod-0*:
$[\![\ 0 < k;\ n\ mod\ k = 0;\ \forall x{\in}A.\ x\ div\ k \leq (n + d)\ div\ k \longrightarrow x \leq n + d\ ]\!] \Longrightarrow$
$(A \cap [n\ldots,d]) \oslash k = (A \oslash k) \cap ([n\ldots,d] \oslash k)$
⟨*proof*⟩
thm *div-le-mono*
⟨*proof*⟩
thm *less-mod-0-imp-div-less*
⟨*proof*⟩

lemma *mod-partition-iT-Div-Int*:
$[\![\ 0 < k;\ 0 < d\ ]\!] \Longrightarrow$
$(A \cap [n * k\ldots,d * k - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n * k\ldots,d * k - Suc\ 0] \oslash k)$
thm *iIN-iT-Div-Int-mod-0*
⟨*proof*⟩⟨*proof*⟩
corollary *mod-partition-iT-Div-Int2*:
$[\![\ 0 < k;\ 0 < d;\ n\ mod\ k = 0;\ d\ mod\ k = 0\ ]\!] \Longrightarrow$
$(A \cap [n\ldots,d - Suc\ 0]) \oslash k =$
$(A \oslash k) \cap ([n\ldots,d - Suc\ 0] \oslash k)$
⟨*proof*⟩
thm *mod-partition-iT-Div-Int*
⟨*proof*⟩
corollary *mod-partition-iT-Div-Int-one-segment*:
$0 < k \Longrightarrow$
$(A \cap [n * k\ldots,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n * k\ldots,k - Suc\ 0] \oslash k)$
⟨*proof*⟩
corollary *mod-partition-iT-Div-Int-one-segment2*:
$[\![\ 0 < k;\ n\ mod\ k = 0\ ]\!] \Longrightarrow$
$(A \cap [n\ldots,k - Suc\ 0]) \oslash k = (A \oslash k) \cap ([n\ldots,k - Suc\ 0] \oslash k)$
⟨*proof*⟩

thm
iT-Div-Un
subset-iT-Div-Int
thm

mod-partition-iT-Div-Int
mod-partition-iT-Div-Int2

lemma *iT-Div-assoc*:$I \oslash a \oslash b = I \oslash (a * b)$
⟨*proof*⟩

lemma *iT-Div-commute*: $I \oslash a \oslash b = I \oslash b \oslash a$
⟨*proof*⟩

lemma *iT-Mult-Div-self*: $0 < k \Longrightarrow I \otimes k \oslash k = I$
⟨*proof*⟩
lemma *iT-Mult-Div*:
$[\![\ 0 < d;\ \ k\ mod\ d = 0\ ]\!] \Longrightarrow I \otimes k \oslash d = I \otimes (k\ div\ d)$
⟨*proof*⟩

lemma *iT-Div-Mult-self*:
$0 < k \Longrightarrow I \oslash k \otimes k = \{y.\ \exists x \in I.\ y = x - x\ mod\ k\}$
⟨*proof*⟩

thm *div-add1-eq*
lemma *iT-Plus-Div-distrib-mod-less*:
$\forall x{\in}I.\ x\ mod\ m + n\ mod\ m < m \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
⟨*proof*⟩
corollary *iT-Plus-Div-distrib-mod-0*:
$n\ mod\ m = 0 \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n\ div\ m$
⟨*proof*⟩

lemma *iT-Div-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oslash k) = iMin\ I\ div\ k$
⟨*proof*⟩

thm *iT-Div-Min*
corollary
iFROM-div-Min: $iMin\ ([n...] \oslash k) = n\ div\ k$ **and**
iIN-div-Min: $iMin\ ([n...,d] \oslash k) = n\ div\ k$ **and**
iTILL-div-Min: $iMin\ ([...n] \oslash k) = 0$ **and**
iMOD-div-Min: $iMin\ ([r,\ mod\ m] \oslash k) = r\ div\ k$ **and**
iMODb-div-Min: $iMin\ ([r,\ mod\ m,\ c] \oslash k) = r\ div\ k$
⟨*proof*⟩

lemmas *iT-div-Min* =
iFROM-div-Min
iIN-div-Min
iTILL-div-Min
iMOD-div-Min
iMODb-div-Min
thm *iT-div-Min*

lemma *iT-Div-Max*: $[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow Max\ (I \oslash k) = Max\ I\ div\ k$
⟨*proof*⟩

corollary
iIN-div-Max: $Max\ ([n...,d] \oslash k) = (n + d)\ div\ k$ **and**
iTILL-div-Max: $Max\ ([...n] \oslash k) = n\ div\ k$ **and**
iMODb-div-Max: $Max\ ([r, mod\ m, c] \oslash k) = (r + m * c)\ div\ k$
⟨*proof*⟩

lemma *iT-Div-0-finite*: $finite\ (I \oslash 0)$
⟨*proof*⟩

lemma *iT-Div-infinite-iff*: $0 < k \Longrightarrow infinite\ (I \oslash k) = infinite\ I$
⟨*proof*⟩
lemma *iT-Div-finite-iff*: $0 < k \Longrightarrow finite\ (I \oslash k) = finite\ I$
⟨*proof*⟩

lemma *iFROM-div*: $0 < k \Longrightarrow [n...] \oslash k = [n\ div\ k...]$
⟨*proof*⟩

thm *div-add1-eq*
lemma *iIN-div*:
$0 < k \Longrightarrow$
$[n...,d] \oslash k = [n\ div\ k..., d\ div\ k + (n\ mod\ k + d\ mod\ k)\ div\ k\ ]$
⟨*proof*⟩
corollary *iIN-div-if*:
$0 < k \Longrightarrow [n...,d] \oslash k =$
$[n\ div\ k..., d\ div\ k + (if\ n\ mod\ k + d\ mod\ k < k\ then\ 0\ else\ Suc\ 0)]$
⟨*proof*⟩
corollary *iIN-div-eq1*:
$[\![\ 0 < k; n\ mod\ k + d\ mod\ k < k\ ]\!] \Longrightarrow$
$[n...,d] \oslash k = [n\ div\ k...,d\ div\ k]$
⟨*proof*⟩
corollary *iIN-div-eq2*:
$[\![\ 0 < k; k \leq n\ mod\ k + d\ mod\ k\ ]\!] \Longrightarrow$
$[n...,d] \oslash k = [n\ div\ k..., Suc\ (d\ div\ k)]$
⟨*proof*⟩
corollary *iIN-div-mod-eq-0*:
$[\![\ 0 < k; n\ mod\ k = 0\ ]\!] \Longrightarrow [n...,d] \oslash k = [n\ div\ k...,d\ div\ k]$
⟨*proof*⟩

lemma *iTILL-div*:
$0 < k \Longrightarrow [...n] \oslash k = [...n\ div\ k]$
⟨*proof*⟩

lemma *iMOD-div-ge*:

$[\![\ 0 < m;\ m \leq k\ ]\!] \Longrightarrow [r,\ mod\ m] \oslash k = [r\ div\ k\ldots]$
⟨*proof*⟩
thm *div-mult-cancel*
⟨*proof*⟩
corollary *iMOD-div-self*:
$0 < m \Longrightarrow [r,\ mod\ m] \oslash m = [r\ div\ m\ldots]$
⟨*proof*⟩

lemma *iMOD-div*:
$[\![\ 0 < k;\ m\ mod\ k = 0\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \oslash k = [r\ div\ k,\ mod\ (m\ div\ k)\ ]$
⟨*proof*⟩
thm *iMOD-mult*[*of r div k q k*]
⟨*proof*⟩
thm *arg-cong*[**where** *f*=$\lambda x.\ x \oplus (r\ mod\ k)$]
⟨*proof*⟩
thm *iT-Plus-Div-distrib-mod-less*
⟨*proof*⟩
thm *mod-factor-imp-mod-0*[*of x q k*]
⟨*proof*⟩
thm *iT-Mult-Div*[*OF - mod-self*]
⟨*proof*⟩
thm
iMOD-div
iMOD-div-ge

lemma *iMODb-div-self*:
$0 < m \Longrightarrow [r,\ mod\ m,\ c] \oslash m = [r\ div\ m\ldots,c]$
thm *iMODb-iMOD-iTILL-conv*
⟨*proof*⟩
thm *iTILL-iT-Div-Int*
⟨*proof*⟩
thm *div-le-mod-le-imp-le*
⟨*proof*⟩

lemma *iMODb-div-ge*:
$[\![\ 0 < m;\ m \leq k\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k = [r\ div\ k\ldots,(r + m * c)\ div\ k - r\ div\ k]$
⟨*proof*⟩
thm *iMODb-append-union-Suc*[*of r m c 0, symmetric*]
⟨*proof*⟩
thm *div-add1-eq1*
⟨*proof*⟩
thm *div-add1-eq2*
⟨*proof*⟩
thm *iIN-Suc-insert-conv*
⟨*proof*⟩
thm *div-add1-eq-if*

corollary *iMODb-div-ge-if*:
$[\![\ 0 < m;\ m \leq k\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k =$
$[r\ div\ k \ldots,\ m * c\ div\ k + (if\ r\ mod\ k + m * c\ mod\ k < k\ then\ 0\ else\ Suc\ 0)]$
⟨*proof*⟩

thm *iMODb-div-ge*
lemma *iMODb-div*:
$[\![\ 0 < k;\ m\ mod\ k = 0\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k = [r\ div\ k,\ mod\ (m\ div\ k),\ c\ ]$
⟨*proof*⟩
thm *iTILL-iT-Div-Int*[*of k* [*r*, *mod m*] *m* * *c*]
⟨*proof*⟩
thm *mod-0-imp-mod-mult-right-0*
⟨*proof*⟩
thm *mod-eq-mod-0-imp-mod-eq*[*of x m r k*]
⟨*proof*⟩

lemmas *iT-div* =
iTILL-div
iFROM-div
iIN-div
iMOD-div
iMODb-div
iT-Div-singleton
thm
iT-div
iT-div-0

This lemma is valid for all $k \leq m$,i. e., also for k with $m\ mod\ k \neq (0::'a)$.

lemma *iMODb-div-unique*:
$[\![\ 0 < k;\ k \leq m;\ k \leq c;\ [r',\ mod\ m',\ c'] = [r,\ mod\ m,\ c] \oslash k\ ]\!] \Longrightarrow$
$r' = r\ div\ k \wedge m' = m\ div\ k \wedge c' = c$
⟨*proof*⟩
thm *iT-Div-Min*
⟨*proof*⟩
thm *div-add1-eq-if*[*of k r m* * *c*]
⟨*proof*⟩
thm *card-image*[*OF iMODb-div-right-inj-on*]
⟨*proof*⟩
thm *iMODb-div-right-strict-mono-on*
⟨*proof*⟩
thm *iMODb-inext-nth-diff*
⟨*proof*⟩
thm *inext-nth-image*[*OF iMODb-not-empty*]
⟨*proof*⟩

lemma *iMODb-div-mod-gr0-is-0-not-ex0*:
⟦ $0 < k$; $k < m$; $0 < m \bmod k$; $k \leq c$; $r \bmod k = 0$ ⟧ ⟹
¬(∃ r' m' c'. $[r', \bmod m', c'] = [r, \bmod m, c] \oslash k$)
⟨*proof*⟩
thm *iMODb-div-unique*
⟨*proof*⟩
thm *div-mult1-eq*[*of m c*]
⟨*proof*⟩

lemma *iMODb-div-mod-gr0-not-ex--arith-aux1*:
⟦ $(0::nat) < k$; $k < m$; $0 < x1$ ⟧ ⟹
$x1 * m + x2 - x \bmod k + x3 + x \bmod k = x1 * m + x2 + x3$
⟨*proof*⟩
lemma *iMODb-div-mod-gr0-not-ex*:
⟦ $0 < k$; $k < m$; $0 < m \bmod k$; $k \leq c$ ⟧ ⟹
¬(∃ r' m' c'. $[r', \bmod m', c'] = [r, \bmod m, c] \oslash k$)
⟨*proof*⟩
thm *iMODb-div-mod-gr0-is-0-not-ex0*
⟨*proof*⟩
thm *iMODb-div-unique*
⟨*proof*⟩
thm *div-add1-eq*
⟨*proof*⟩
thm *div-mult1-eq*[*of c m k*]
⟨*proof*⟩
thm *iMODb-div-mod-gr0-not-ex*

thm
cut-le-image
iMOD-div-right-strict-mono-on
iMOD-cut-le
thm *card-image*[*OF strict-mono-on-imp-inj-on*]

lemma *iMOD-div-eq-imp-iMODb-div-eq*:
⟦ $0 < k$; $k \leq m$; $[r', \bmod m'] = [r, \bmod m] \oslash k$ ⟧ ⟹
$[r', \bmod m', c] = [r, \bmod m, c] \oslash k$
⟨*proof*⟩
thm *iMODb-div-right-strict-mono-on*[*of k m r c*]
thm *card-image*[*OF strict-mono-on-imp-inj-on*]
thm *card-image*[*OF strict-mono-on-imp-inj-on*[*OF iMODb-div-right-strict-mono-on*[*of k m r c*]]]
⟨*proof*⟩
thm *iMOD-cut-le*[*of r m r + m * c*]
⟨*proof*⟩
thm *cut-le-image*[*OF - subset-refl*]
⟨*proof*⟩
thm *div-le-mono*
⟨*proof*⟩

lemma *iMOD-div-unique*:
⟦ $0 < k$; $k \leq m$; $[r', \mathit{mod}\ m'] = [r, \mathit{mod}\ m] \oslash k$ ⟧ ⟹
$r' = r\ \mathit{div}\ k \wedge m' = m\ \mathit{div}\ k$
thm *iMOD-div-eq-imp-iMODb-div-eq*
⟨*proof*⟩
thm *iMODb-div-unique*[*of k - k*]
⟨*proof*⟩

thm *iMODb-div-mod-gr0-not-ex*
lemma *iMOD-div-mod-gr0-not-ex*:
⟦ $0 < k$; $k < m$; $0 < m\ \mathit{mod}\ k$ ⟧ ⟹
$\neg\ (\exists\, r'\ m'.\ [r', \mathit{mod}\ m'] = [r, \mathit{mod}\ m] \oslash k)$
⟨*proof*⟩
thm *iMOD-div-eq-imp-iMODb-div-eq*[*OF - less-imp-le*]
⟨*proof*⟩
thm *iMODb-div-mod-gr0-not-ex*[*of k m k r*]
⟨*proof*⟩

2.2 Interval cut operators with arithmetic interval operators

lemma
iT-Plus-cut-le2: $(I \oplus k) \downarrow\leq (t + k) = (I \downarrow\leq t) \oplus k$ **and**
iT-Plus-cut-less2: $(I \oplus k) \downarrow< (t + k) = (I \downarrow< t) \oplus k$ **and**
iT-Plus-cut-ge2: $(I \oplus k) \downarrow\geq (t + k) = (I \downarrow\geq t) \oplus k$ **and**
iT-Plus-cut-greater2: $(I \oplus k) \downarrow> (t + k) = (I \downarrow> t) \oplus k$
⟨*proof*⟩

lemma *iT-Plus-cut-le*:
$(I \oplus k) \downarrow\leq t = (\mathit{if}\ t < k\ \mathit{then}\ \{\}\ \mathit{else}\ I \downarrow\leq (t - k) \oplus k)$
⟨*proof*⟩
thm *iT-Plus-cut-le2*[*of I k t - k*]
⟨*proof*⟩
lemma *iT-Plus-cut-less*: $(I \oplus k) \downarrow< t = I \downarrow< (t - k) \oplus k$
⟨*proof*⟩
lemma *iT-Plus-cut-ge*: $(I \oplus k) \downarrow\geq t = I \downarrow\geq (t - k) \oplus k$
⟨*proof*⟩
lemma *iT-Plus-cut-greater*:
$(I \oplus k) \downarrow> t = (\mathit{if}\ t < k\ \mathit{then}\ I \oplus k\ \mathit{else}\ I \downarrow> (t - k) \oplus k)$
⟨*proof*⟩

lemma
iT-Mult-cut-le2: $0 < k \Longrightarrow (I \otimes k) \downarrow\leq (t * k) = (I \downarrow\leq t) \otimes k$ **and**
iT-Mult-cut-less2: $0 < k \Longrightarrow (I \otimes k) \downarrow< (t * k) = (I \downarrow< t) \otimes k$ **and**
iT-Mult-cut-ge2: $0 < k \Longrightarrow (I \otimes k) \downarrow\geq (t * k) = (I \downarrow\geq t) \otimes k$ **and**
iT-Mult-cut-greater2: $0 < k \Longrightarrow (I \otimes k) \downarrow> (t * k) = (I \downarrow> t) \otimes k$
⟨*proof*⟩

lemma *iT-Mult-cut-le*:
$0 < k \Longrightarrow (I \otimes k) \downarrow\leq t = (I \downarrow\leq (t\ div\ k)) \otimes k$
⟨*proof*⟩
lemma *iT-Mult-cut-less*:
$0 < k \Longrightarrow (I \otimes k) \downarrow< t =$
$(if\ t\ mod\ k = 0\ then\ (I \downarrow< (t\ div\ k))\ else\ I \downarrow< Suc\ (t\ div\ k)) \otimes k$
⟨*proof*⟩
thm *less-mod-0-imp-div-less*[*of t x k*]
⟨*proof*⟩
lemma *iT-Mult-cut-greater*:
$0 < k \Longrightarrow (I \otimes k) \downarrow> t = (I \downarrow> (t\ div\ k)) \otimes k$
⟨*proof*⟩
lemma *iT-Mult-cut-ge*:
$0 < k \Longrightarrow (I \otimes k) \downarrow\geq t =$
$(if\ t\ mod\ k = 0\ then\ (I \downarrow\geq (t\ div\ k))\ else\ I \downarrow\geq Suc\ (t\ div\ k)) \otimes k$
⟨*proof*⟩
thm *div-le-mono*[*OF order-less-imp-le*]
⟨*proof*⟩

lemma *iT-Plus-neg-cut-le2*: $k \leq t \Longrightarrow (I \oplus- k) \downarrow\leq (t - k) = (I \downarrow\leq t) \oplus- k$
⟨*proof*⟩
thm *i-cut-commute-disj*[*of op* $\downarrow\leq$ *op* $\downarrow\geq$]
⟨*proof*⟩
lemma *iT-Plus-neg-cut-less2*: $(I \oplus- k) \downarrow< (t - k) = (I \downarrow< t) \oplus- k$
⟨*proof*⟩
lemma *iT-Plus-neg-cut-ge2*: $(I \oplus- k) \downarrow\geq (t - k) = (I \downarrow\geq t) \oplus- k$
⟨*proof*⟩
lemma *iT-Plus-neg-cut-greater2*: $k \leq t \Longrightarrow (I \oplus- k) \downarrow> (t - k) = (I \downarrow> t) \oplus- k$
⟨*proof*⟩

lemma *iT-Plus-neg-cut-le*: $(I \oplus- k) \downarrow\leq t = I \downarrow\leq (t + k) \oplus- k$
⟨*proof*⟩
lemma *iT-Plus-neg-cut-less*: $(I \oplus- k) \downarrow< t = I \downarrow< (t + k) \oplus- k$
⟨*proof*⟩
lemma *iT-Plus-neg-cut-ge*: $(I \oplus- k) \downarrow\geq t = I \downarrow\geq (t + k) \oplus- k$
⟨*proof*⟩
lemma *iT-Plus-neg-cut-greater*: $(I \oplus- k) \downarrow> t = I \downarrow> (t + k) \oplus- k$
⟨*proof*⟩

lemma *iT-Minus-cut-le2*: $t \leq k \Longrightarrow (k \ominus I) \downarrow\leq (k - t) = k \ominus (I \downarrow\geq t)$
⟨*proof*⟩
lemma *iT-Minus-cut-less2*: $(k \ominus I) \downarrow< (k - t) = k \ominus (I \downarrow> t)$
⟨*proof*⟩
lemma *iT-Minus-cut-ge2*: $(k \ominus I) \downarrow\geq (k - t) = k \ominus (I \downarrow\leq t)$

⟨*proof*⟩
lemma *iT-Minus-cut-greater2*: $t \leq k \Longrightarrow (k \ominus I) \downarrow> (k - t) = k \ominus (I \downarrow< t)$
⟨*proof*⟩

lemma *iT-Minus-cut-le*: $(k \ominus I) \downarrow\leq t = k \ominus (I \downarrow\geq (k - t))$
⟨*proof*⟩
lemma *iT-Minus-cut-less*:
$(k \ominus I) \downarrow< t = (\textit{if } t \leq k \textit{ then } k \ominus (I \downarrow> (k - t)) \textit{ else } k \ominus I)$
⟨*proof*⟩
lemma *iT-Minus-cut-ge*:
$(k \ominus I) \downarrow\geq t = (\textit{if } t \leq k \textit{ then } k \ominus (I \downarrow\leq (k - t)) \textit{ else } \{\})$
⟨*proof*⟩
lemma *iT-Minus-cut-greater*: $(k \ominus I) \downarrow> t = k \ominus (I \downarrow< (k - t))$
⟨*proof*⟩

thm *iT-Div-def*
thm *iT-Mult-cut-le2*
thm *iT-Div-mem-iff*
lemma *iT-Div-cut-le*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\leq t = I \downarrow\leq (t * k + (k - \textit{Suc } 0)) \oslash k$
⟨*proof*⟩
thm *div-le-conv*
⟨*proof*⟩
lemma *iT-Div-cut-less*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< t = I \downarrow< (t * k) \oslash k$
⟨*proof*⟩
lemma *iT-Div-cut-ge*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\geq t = I \downarrow\geq (t * k) \oslash k$
⟨*proof*⟩
thm *le-div-conv*
⟨*proof*⟩
lemma *iT-Div-cut-greater*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> t = I \downarrow> (t * k + (k - \textit{Suc } 0)) \oslash k$
⟨*proof*⟩

lemma *iT-Div-cut-le2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\leq (t \textit{ div } k) = I \downarrow\leq (t - t \textit{ mod } k + (k - \textit{Suc } 0)) \oslash k$
⟨*proof*⟩
lemma *iT-Div-cut-less2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< (t \textit{ div } k) = I \downarrow< (t - t \textit{ mod } k) \oslash k$
⟨*proof*⟩
lemma *iT-Div-cut-ge2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\geq (t \textit{ div } k) = I \downarrow\geq (t - t \textit{ mod } k) \oslash k$
⟨*proof*⟩
lemma *iT-Div-cut-greater2*:

$0 < k \Longrightarrow (I \oslash k) \downarrow> (t \ div \ k) = I \downarrow> (t - t \ mod \ k + (k - Suc\ 0)) \oslash k$
⟨*proof*⟩

2.3 *inext* and *iprev* with interval operators

lemma *iT-Plus-inext*: $inext\ (n + k)\ (I \oplus k) = (inext\ n\ I) + k$
⟨*proof*⟩

lemma *iT-Plus-iprev*: $iprev\ (n + k)\ (I \oplus k) = (iprev\ n\ I) + k$
⟨*proof*⟩

lemma *iT-Plus-inext2*: $k \leq n \Longrightarrow inext\ n\ (I \oplus k) = (inext\ (n - k)\ I) + k$
⟨*proof*⟩
lemma *iT-Plus-prev2*: $k \leq n \Longrightarrow iprev\ n\ (I \oplus k) = (iprev\ (n - k)\ I) + k$
⟨*proof*⟩

lemma *iT-Mult-inext*: $inext\ (n * k)\ (I \otimes k) = (inext\ n\ I) * k$
⟨*proof*⟩
lemma *iT-Mult-iprev*: $iprev\ (n * k)\ (I \otimes k) = (iprev\ n\ I) * k$
⟨*proof*⟩

lemma *iT-Mult-inext2-if*:
$inext\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (inext\ (n\ div\ k)\ I) * k\ else\ n)$
⟨*proof*⟩

lemma *iT-Mult-iprev2-if*:
$iprev\ n\ (I \otimes k) = (if\ n\ mod\ k = 0\ then\ (iprev\ (n\ div\ k)\ I) * k\ else\ n)$
⟨*proof*⟩

corollary *iT-Mult-inext2*:
$n\ mod\ k = 0 \Longrightarrow inext\ n\ (I \otimes k) = (inext\ (n\ div\ k)\ I) * k$
⟨*proof*⟩
corollary *iT-Mult-iprev2*:
$n\ mod\ k = 0 \Longrightarrow iprev\ n\ (I \otimes k) = (iprev\ (n\ div\ k)\ I) * k$
⟨*proof*⟩

lemma *iT-Plus-neg-inext*:
$k \leq n \Longrightarrow inext\ (n - k)\ (I \oplus - k) = inext\ n\ I - k$
⟨*proof*⟩
thm *subst*[*OF inext-cut-ge-conv*]
⟨*proof*⟩
thm *inext-image*
⟨*proof*⟩
thm *strict-mono-on-subset*[*OF - Int-lower2*]
⟨*proof*⟩
thm *sub-left-strict-mono-on*
⟨*proof*⟩

lemma *iT-Plus-neg-iprev*:
$iprev\ (n - k)\ (I \oplus- k) = iprev\ n\ (I \downarrow\geq k) - k$
⟨*proof*⟩
thm *iT-Plus-neg-mem-iff2*[*THEN iffD2*]
⟨*proof*⟩
thm *strict-mono-on-subset*[*OF - Int-lower2*]
⟨*proof*⟩
thm *sub-left-strict-mono-on*
⟨*proof*⟩

corollary *iT-Plus-neg-inext2*: $inext\ n\ (I \oplus- k) = inext\ (n + k)\ I - k$
⟨*proof*⟩
corollary *iT-Plus-neg-iprev2*: $iprev\ n\ (I \oplus- k) = iprev\ (n + k)\ (I \downarrow\geq k) - k$
⟨*proof*⟩

lemma *iT-Minus-inext*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow inext\ (k - n)\ (k \ominus I) = k - iprev\ n\ I$
⟨*proof*⟩
thm *iT-Minus-imirror-conv*
⟨*proof*⟩
thm *iT-Plus-inext*
⟨*proof*⟩
thm *iprev-mono*[*THEN order-trans*, *of n iMin* $(I \downarrow\leq k) + Max\ (I \downarrow\leq k)\ I$]
⟨*proof*⟩
corollary *iT-Minus-inext2*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow inext\ n\ (k \ominus I) = k - iprev\ (k - n)\ I$
⟨*proof*⟩

lemma *iT-Minus-iprev*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow iprev\ (k - n)\ (k \ominus I) = k - inext\ n\ (I \downarrow\leq k)$
⟨*proof*⟩
lemma *iT-Minus-iprev2*:
$[\![\ k \ominus I \neq \{\};\ n \leq k\ ]\!] \Longrightarrow iprev\ n\ (k \ominus I) = k - inext\ (k - n)\ (I \downarrow\leq k)$
⟨*proof*⟩

lemma *iT-Plus-inext-nth*: $I \neq \{\} \Longrightarrow (I \oplus k) \rightarrow n = (I \rightarrow n) + k$
⟨*proof*⟩
lemma *iT-Plus-iprev-nth*: $[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow (I \oplus k) \leftarrow n = (I \leftarrow n) + k$
⟨*proof*⟩

lemma *iT-Mult-inext-nth*: $I \neq \{\} \Longrightarrow (I \otimes k) \rightarrow n = (I \rightarrow n) * k$
⟨*proof*⟩

lemma *iT-Mult-iprev-nth*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow (I \otimes k) \leftarrow n = (I \leftarrow n) * k$
⟨*proof*⟩

lemma *iT-Plus-neg-inext-nth*:
$I \oplus- k \neq \{\} \Longrightarrow (I \oplus- k) \rightarrow n = (I \downarrow\geq k \rightarrow n) - k$
⟨*proof*⟩
thm *iT-Plus-neg-cut-eq*[*of k k I*]
⟨*proof*⟩
lemma *iT-Plus-neg-iprev-nth*:
$[\![$ *finite* I; $I \oplus- k \neq \{\}$ $]\!] \Longrightarrow (I \oplus- k) \leftarrow n = (I \downarrow\geq k \leftarrow n) - k$
⟨*proof*⟩

lemma *iT-Minus-inext-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \rightarrow n = k - ((I \downarrow\leq k) \leftarrow n)$
⟨*proof*⟩
lemma *iT-Minus-iprev-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \leftarrow n = k - ((I \downarrow\leq k) \rightarrow n)$
⟨*proof*⟩

lemma *iT-Div-ge-inext-nth*:
$[\![I \neq \{\}; \forall x{\in}I.\ \forall y{\in}I.\ x < y \longrightarrow x + k \leq y]\!] \Longrightarrow$
$(I \oslash k) \rightarrow n = (I \rightarrow n)\ div\ k$
⟨*proof*⟩
lemma *iT-Div-mod-inext-nth*:
$[\![I \neq \{\}; \forall x{\in}I.\ \forall y{\in}I.\ x\ mod\ k = y\ mod\ k]\!] \Longrightarrow$
$(I \oslash k) \rightarrow n = (I \rightarrow n)\ div\ k$
⟨*proof*⟩
lemma *iT-Div-ge-iprev-nth*:
$[\![$ *finite* I; $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x < y \longrightarrow x + k \leq y$ $]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)\ div\ k$
⟨*proof*⟩
lemma *iT-Div-mod-iprev-nth*:
$[\![$ *finite* I; $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x\ mod\ k = y\ mod\ k$ $]\!] \Longrightarrow$
$(I \oslash k) \leftarrow n = (I \leftarrow n)\ div\ k$
⟨*proof*⟩

2.4 Cardinality of intervals with interval operators

lemma *iT-Plus-card*: $card\ (I \oplus k) = card\ I$
⟨*proof*⟩
lemma *iT-Mult-card*: $0 < k \Longrightarrow card\ (I \otimes k) = card\ I$
⟨*proof*⟩

lemma *iT-Plus-neg-card*: $card\ (I \oplus- k) = card\ (I \downarrow\geq k)$
⟨*proof*⟩
thm *Int-lower2*
thm *subset-inj-on*[*OF - Int-lower2*]
⟨*proof*⟩
thm *sub-left-inj-on*

⟨*proof*⟩
lemma *iT-Plus-neg-card-le*: *card* $(I \oplus- k) \leq$ *card* I
⟨*proof*⟩
thm *nat-cut-ge-finite-iff card-infinite*
⟨*proof*⟩

lemma *iT-Minus-card*: *card* $(k \ominus I) =$ *card* $(I \downarrow\leq k)$
⟨*proof*⟩
thm *subset-inj-on*[*OF - Int-lower2*]
⟨*proof*⟩
lemma *iT-Minus-card-le*: *finite* $I \Longrightarrow$ *card* $(k \ominus I) \leq$ *card* I
⟨*proof*⟩

lemma *iT-Div-0-card-if*:
card $(I \oslash 0) =$ (*if* $I = \{\}$ *then* 0 *else Suc* 0)
⟨*proof*⟩

lemma *Int-empty-setsum*:
$(\sum k{\leq}(n::nat).$ *if* $\{\} \cap (I\ k) = \{\}$ *then* 0 *else Suc* 0$) = 0$
⟨*proof*⟩
lemma *iT-Div-mod-partition-card*:
card $(I \cap [n * d..,d - Suc\ 0] \oslash d) =$
(*if* $I \cap [n * d..,d - Suc\ 0] = \{\}$ *then* 0 *else Suc* 0)
⟨*proof*⟩
thm *iT-Div-mem-iff*
⟨*proof*⟩
thm *le-less-imp-div*
⟨*proof*⟩

thm *iT-Div-mem-iff-Int*

lemma *iT-Div-conv-count*:
$0 < d \Longrightarrow I \oslash d = \{k.\ I \cap [k * d..,d - Suc\ 0] \neq \{\}\}$
⟨*proof*⟩
thm *iT-Div-mem-iff-Int*
⟨*proof*⟩

lemma *iT-Div-conv-count2*:
$[\![\ 0 < d$; *finite* I; *Max* I *div* $d \leq n\ ]\!] \Longrightarrow$
$I \oslash d = \{k.\ k \leq n \wedge I \cap [k * d..,d - Suc\ 0] \neq \{\}\}$
⟨*proof*⟩
thm *div-less-conv*
⟨*proof*⟩

lemma *mod-partition-count-Suc*:
$\{k.\ k \leq Suc\ n \wedge I \cap [k * d..,d - Suc\ 0] \neq \{\}\} =$
$\{k.\ k \leq n \wedge I \cap [k * d..,d - Suc\ 0] \neq \{\}\} \cup$
(*if* $I \cap [Suc\ n * d..,d - Suc\ 0] \neq \{\}$ *then* $\{Suc\ n\}$ *else* $\{\}$)
⟨*proof*⟩

lemma *iT-Div-card*:
$\bigwedge I.[\![\ 0 < d;\ \mathit{finite}\ I;\ \mathit{Max}\ I\ \mathit{div}\ d \leq n]\!] \Longrightarrow$
$\mathit{card}\ (I \oslash d) = (\sum k{\leq}n.$
$\quad \mathit{if}\ I \cap [k * d \ldots, d - \mathit{Suc}\ 0] = \{\}\ \mathit{then}\ 0\ \mathit{else}\ \mathit{Suc}\ 0)$
⟨*proof*⟩
thm *iT-Div-conv-count2*
⟨*proof*⟩
thm *mod-partition-count-Suc*
⟨*proof*⟩
find-theorems - *div* - <= - *name*: *conv*
⟨*proof*⟩
thm *div-le-conv*
⟨*proof*⟩
thm *cut-le-Int-conv*
⟨*proof*⟩

thm *iT-Div-card*
thm *iT-Div-card*[*OF* - - *le-refl*, *of d i*]
corollary *iT-Div-card-Suc*:
$\bigwedge I.[\![\ 0 < d;\ \mathit{finite}\ I;\ \mathit{Max}\ I\ \mathit{div}\ d \leq n]\!] \Longrightarrow$
$\mathit{card}\ (I \oslash d) = (\sum k{<}\mathit{Suc}\ n.$
$\quad \mathit{if}\ I \cap [k * d \ldots, d - \mathit{Suc}\ 0] = \{\}\ \mathit{then}\ 0\ \mathit{else}\ \mathit{Suc}\ 0)$
⟨*proof*⟩
corollary *iT-Div-Max-card*: $[\![\ 0 < d;\ \mathit{finite}\ I\ ]\!] \Longrightarrow$
$\mathit{card}\ (I \oslash d) = (\sum k{\leq}\mathit{Max}\ I\ \mathit{div}\ d.$
$\quad \mathit{if}\ I \cap [k * d \ldots, d - \mathit{Suc}\ 0] = \{\}\ \mathit{then}\ 0\ \mathit{else}\ \mathit{Suc}\ 0)$
⟨*proof*⟩

lemma *iT-Div-card-le*: $0 < k \Longrightarrow \mathit{card}\ (I \oslash k) \leq \mathit{card}\ I$
⟨*proof*⟩

thm *iT-Div-def*
lemma *iT-Div-card-inj-on*:
$\mathit{inj\text{-}on}\ (\lambda n.\ n\ \mathit{div}\ k)\ I \Longrightarrow \mathit{card}\ (I \oslash k) = \mathit{card}\ I$
⟨*proof*⟩

thm *mod-Suc*
lemma *mod-Suc'*:
$0 < n \Longrightarrow \mathit{Suc}\ m\ \mathit{mod}\ n = (\mathit{if}\ m\ \mathit{mod}\ n < n - \mathit{Suc}\ 0\ \mathit{then}\ \mathit{Suc}\ (m\ \mathit{mod}\ n)\ \mathit{else}\ 0)$
⟨*proof*⟩

lemma *div-Suc*:
$0 < n \Longrightarrow \mathit{Suc}\ m\ \mathit{div}\ n = (\mathit{if}\ \mathit{Suc}\ (m\ \mathit{mod}\ n) = n\ \mathit{then}\ \mathit{Suc}\ (m\ \mathit{div}\ n)\ \mathit{else}\ m\ \mathit{div}$

n)
⟨*proof*⟩
thm *le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*]]
⟨*proof*⟩
lemma *div-Suc'*:
0 < n ⟹ Suc m div n = (if m mod n < n − Suc 0 then m div n else Suc (m div n))
⟨*proof*⟩

lemma *iT-Div-card-ge-aux*:
⋀I. ⟦ 0 < d; finite I; Max I div d ≤ n ⟧ ⟹
card I div d + (if card I mod d = 0 then 0 else Suc 0) ≤ card (I ⊘ d)
⟨*proof*⟩
thm *div-le-conv*[*THEN iffD1*]
⟨*proof*⟩
thm *nat-card-le-Max*
thm *order-trans*[*OF nat-card-le-Max*]
⟨*proof*⟩
thm *div-le-mono*[*of - - d*]
⟨*proof*⟩
thm *subst*[**where** *t=I* **and** *s=I ∩ [...n * d + d − Suc 0] ∪ I ∩ [Suc n * d...,d − Suc 0]*]
⟨*proof*⟩
thm *subset-atMost-Max-le-conv*
thm *le-less-div-conv*
⟨*proof*⟩
thm *card-Un-disjoint*
⟨*proof*⟩
thm *iT-Div-mod-partition-card*
⟨*proof*⟩
thm *Int-card2*[*OF iIN-finite*]
thm *order-trans*[*OF Int-card2*[*OF iIN-finite*]]
⟨*proof*⟩
thm *Int-card2*[*OF iIN-finite*, *rule-format*]
⟨*proof*⟩
thm *Max-Int-le2*[*OF - iTILL-finite*]
thm *order-trans*[*OF Max-Int-le2*[*OF - iTILL-finite*]]
⟨*proof*⟩
thm *card-Un-disjoint*
⟨*proof*⟩
thm *mod-partition-iT-Div-Int-one-segment*
⟨*proof*⟩
thm *mod-partition-iT-Div-Int-one-segment*
⟨*proof*⟩
thm *mod-partition-iT-Div-Int2*
⟨*proof*⟩
thm *disjoint-iff-in-not-in1*
⟨*proof*⟩
thm *iIN-div-mod-eq-0*

⟨*proof*⟩
thm *iT-Div-mod-partition-card*
⟨*proof*⟩
thm *add-le-divisor-imp-le-Suc-div*
⟨*proof*⟩
thm *Int-card2*[*OF iIN-finite*, *THEN le-trans*]
⟨*proof*⟩
thm *order-trans*[**where** *x*=*card I div d* + (*if card I mod d* ≠ *0 then Suc 0 else 0*)]
⟨*proof*⟩
thm *div-add1-eq1-mod-0-left*
⟨*proof*⟩
thm *add-le-divisor-imp-le-Suc-div*
⟨*proof*⟩

lemma *iT-Div-card-ge*:
card I div d + (*if card I mod d* = *0 then 0 else Suc 0*) ≤ *card* (*I* ⊘ *d*)
⟨*proof*⟩
thm *iT-Div-card-ge-aux*[*OF - - order-refl*]
⟨*proof*⟩
corollary *iT-Div-card-ge-div*: *card I div d* ≤ *card* (*I* ⊘ *d*)
⟨*proof*⟩

There is no better lower bound function *f* for *i* ⊘ *d* with *card i* and *d* as arguments.

lemma *iT-Div-card-ge--is-maximal-lower-bound*:
∀ *I d*. *card I div d* + (*if card I mod d* = *0 then 0 else Suc 0*) ≤ *f* (*card I*) *d* ∧
f (*card I*) *d* ≤ *card* (*I* ⊘ *d*) ⟹
f (*card* (*I*::*nat set*)) *d* = *card I div d* + (*if card I mod d* = *0 then 0 else Suc 0*)
⟨*proof*⟩
thm *iTILL-div iTILL-card*
⟨*proof*⟩
thm *div-diff1-eq1*[*of Suc 0 d card I*]
⟨*proof*⟩
thm *iT-Div-card-ge--is-maximal-lower-bound*[*rule-format*]

thm *iT-Plus-card*
lemma *iT-Plus-icard*: *icard* (*I* ⊕ *k*) = *icard I*
⟨*proof*⟩

thm *iT-Mult-card*
lemma *iT-Mult-icard*: *0* < *k* ⟹ *icard* (*I* ⊗ *k*) = *icard I*
⟨*proof*⟩

thm *iT-Plus-neg-card*
lemma *iT-Plus-neg-icard*: *icard* (*I* ⊕− *k*) = *icard* (*I* ↓≥ *k*)
⟨*proof*⟩
thm *iT-Plus-neg-finite-iff cut-ge-finite*

⟨*proof*⟩

thm *iT-Plus-neg-card-le*
lemma *iT-Plus-neg-icard-le*: *icard* (*I* ⊕− *k*) ≤ *icard I*
⟨*proof*⟩

thm *iT-Minus-card*
lemma *iT-Minus-icard*: *icard* (*k* ⊖ *I*) = *icard* (*I* ↓≤ *k*)
⟨*proof*⟩

thm *iT-Minus-card-le*
lemma *iT-Minus-icard-le*: *icard* (*k* ⊖ *I*) ≤ *icard I*
⟨*proof*⟩

thm *iT-Div-0-card-if*
lemma *iT-Div-0-icard-if*: *icard* (*I* ⊘ 0) = *Fin* (*if I* = {} *then 0 else Suc 0*)
thm *iT-Div-0-finite*
⟨*proof*⟩

thm *iT-Div-mod-partition-card*
lemma *iT-Div-mod-partition-icard*:
icard (*I* ∩ [*n* ∗ *d*...,*d* − *Suc 0*] ⊘ *d*) =
Fin (*if I* ∩ [*n* ∗ *d*...,*d* − *Suc 0*] = {} *then 0 else Suc 0*)
⟨*proof*⟩

thm *iT-Div-card*
lemma *iT-Div-icard*:
⟦ *0* < *d*; *finite I* ⟹ *Max I div d* ≤ *n*⟧ ⟹
icard (*I* ⊘ *d*) =
(*if finite I then Fin* (∑ *k*≤*n*. *if I* ∩ [*k* ∗ *d*...,*d* − *Suc 0*] = {} *then 0 else Suc 0*) *else* ∞)
⟨*proof*⟩

thm *iT-Div-Max-card*
corollary *iT-Div-Max-icard*: *0* < *d* ⟹
icard (*I* ⊘ *d*) = (*if finite I*
then Fin (∑ *k*≤*Max I div d*. *if I* ∩ [*k* ∗ *d*...,*d* − *Suc 0*] = {} *then 0 else Suc 0*) *else* ∞)
⟨*proof*⟩

thm *iT-Div-card-le*
lemma *iT-Div-icard-le*: *0* < *k* ⟹ *icard* (*I* ⊘ *k*) ≤ *icard I*
⟨*proof*⟩

thm *iT-Div-card-inj-on*
lemma *iT-Div-icard-inj-on*: *inj-on* (λ*n*. *n div k*) *I* ⟹ *icard* (*I* ⊘ *k*) = *icard I*
⟨*proof*⟩

thm *iT-Div-card-ge*

lemma *iT-Div-icard-ge*: *icard I div (Fin d) + Fin (if icard I mod (Fin d) = 0 then 0 else Suc 0) ≤ icard (I ⊘ d)*
⟨*proof*⟩

thm *iT-Div-card-ge-div*
corollary *iT-Div-icard-ge-div*: *icard I div (Fin d) ≤ icard (I ⊘ d)*
⟨*proof*⟩

thm *iT-Div-card-ge--is-maximal-lower-bound*
lemma *iT-Div-icard-ge--is-maximal-lower-bound*:
∀ I d. icard I div (Fin d) + Fin (if icard I mod (Fin d) = 0 then 0 else Suc 0) ≤ f (icard I) d ∧ f (icard I) d ≤ icard (I ⊘ d) ⟹ f (icard (I::nat set)) d = icard I div (Fin d) + Fin (if icard I mod (Fin d) = 0 then 0 else Suc 0)
⟨*proof*⟩
thm *iT-Div-card-ge--is-maximal-lower-bound*
⟨*proof*⟩
thm *iT-Div-card-ge--is-maximal-lower-bound*
⟨*proof*⟩

2.5 Results about sets of intervals

2.5.1 Set of intervals without and with emtpy interval

definition *iFROM-UN-set* :: *(nat set) set* **where**
iFROM-UN-set ≡ ⋃ n. {[n...]}
definition *iTILL-UN-set* :: *(nat set) set* **where**
iTILL-UN-set ≡ ⋃ n. {[...n]}
definition *iIN-UN-set* :: *(nat set) set* **where**
iIN-UN-set ≡ ⋃ n d. {[n...,d]}
definition *iMOD-UN-set* :: *(nat set) set* **where**
iMOD-UN-set ≡ ⋃ r m. {[r, mod m]}
definition *iMODb-UN-set* :: *(nat set) set* **where**
iMODb-UN-set ≡ ⋃ r m c. {[r, mod m, c]}

definition *iFROM-set* :: *(nat set) set* **where**
iFROM-set ≡ {[n...] |n. True}
definition *iTILL-set* :: *(nat set) set* **where**
iTILL-set ≡ {[...n] |n. True}
definition *iIN-set* :: *(nat set) set* **where**
iIN-set ≡ {[n...,d] |n d. True}
definition *iMOD-set* :: *(nat set) set* **where**
iMOD-set ≡ {[r, mod m] |r m. True}
definition *iMODb-set* :: *(nat set) set* **where**
iMODb-set ≡ {[r, mod m, c] |r m c. True}

definition *iMOD2-set* :: *(nat set) set* **where**
iMOD2-set ≡ {[r, mod m] |r m. 2 ≤ m}

definition *iMODb2-set* :: (*nat set*) *set* **where**
iMODb2-set ≡ {[*r*, *mod m*, *c*] |*r m c*. *2* ≤ *m* ∧ *1* ≤ *c*}

definition *iMOD2-UN-set* :: (*nat set*) *set* **where**
iMOD2-UN-set ≡ ⋃*r*. ⋃*m*∈{*2*..}. {[*r*, *mod m*]}
definition *iMODb2-UN-set* :: (*nat set*) *set* **where**
iMODb2-UN-set ≡ ⋃*r*. ⋃*m*∈{*2*..}. ⋃*c*∈{*1*..}. {[*r*, *mod m*, *c*]}

lemmas *i-set-defs* =
iFROM-set-def iTILL-set-def iIN-set-def
iMOD-set-def iMODb-set-def
iMOD2-set-def iMODb2-set-def
lemmas *i-UN-set-defs* =
iFROM-UN-set-def iTILL-UN-set-def iIN-UN-set-def
iMOD-UN-set-def iMODb-UN-set-def
iMOD2-UN-set-def iMODb2-UN-set-def

lemma *iFROM-set-UN-set-eq*: *iFROM-set* = *iFROM-UN-set*
⟨*proof*⟩
lemma
iTILL-set-UN-set-eq: *iTILL-set* = *iTILL-UN-set* **and**
iIN-set-UN-set-eq: *iIN-set* = *iIN-UN-set* **and**
iMOD-set-UN-set-eq: *iMOD-set* = *iMOD-UN-set* **and**
iMODb-set-UN-set-eq: *iMODb-set* = *iMODb-UN-set*
⟨*proof*⟩

lemma *iMOD2-set-UN-set-eq*: *iMOD2-set* = *iMOD2-UN-set*
⟨*proof*⟩
lemma *iMODb2-set-UN-set-eq*: *iMODb2-set* = *iMODb2-UN-set*
⟨*proof*⟩

lemmas *i-set-i-UN-set-sets-eq* =
iFROM-set-UN-set-eq
iTILL-set-UN-set-eq
iIN-set-UN-set-eq
iMOD-set-UN-set-eq
iMODb-set-UN-set-eq
iMOD2-set-UN-set-eq
iMODb2-set-UN-set-eq
thm *i-set-i-UN-set-sets-eq*

lemma *iMOD2-set-iMOD-set-subset*: *iMOD2-set* ⊆ *iMOD-set*
⟨*proof*⟩
lemma *iMODb2-set-iMODb-set-subset*: *iMODb2-set* ⊆ *iMODb-set*
⟨*proof*⟩

definition *i-set* :: (*nat set*) *set* **where**
i-set ≡ *iFROM-set* ∪ *iTILL-set* ∪ *iIN-set* ∪ *iMOD-set* ∪ *iMODb-set*
definition *i-UN-set* :: (*nat set*) *set* **where**

i-UN-set ≡ *iFROM-UN-set* ∪ *iTILL-UN-set* ∪ *iIN-UN-set* ∪ *iMOD-UN-set* ∪ *iMODb-UN-set*

Minimal definitions for *i-set* and *i-set*

definition *i-set-min* :: (*nat set*) *set* **where**
i-set-min ≡ *iFROM-set* ∪ *iIN-set* ∪ *iMOD2-set* ∪ *iMODb2-set*
definition *i-UN-set-min* :: (*nat set*) *set* **where**
i-UN-set-min ≡ *iFROM-UN-set* ∪ *iIN-UN-set* ∪ *iMOD2-UN-set* ∪ *iMODb2-UN-set*

definition *i-set0* :: (*nat set*) *set* **where**
i-set0 ≡ *insert* {} *i-set*
thm *i-set0-def*

thm *i-set-i-UN-set-sets-eq*
lemma *i-set-i-UN-set-eq*: *i-set* = *i-UN-set*
⟨*proof*⟩
lemma *i-set-min-i-UN-set-min-eq*: *i-set-min* = *i-UN-set-min*
⟨*proof*⟩

lemma *i-set-min-eq*: *i-set* = *i-set-min*
⟨*proof*⟩
thm *Un-mono*
⟨*proof*⟩
corollary *i-UN-set-i-UN-min-set-eq*: *i-UN-set* = *i-UN-set-min*
⟨*proof*⟩

thm
i-set-def i-set-min-def
i-set-min-eq

lemma *i-set-min-is-minimal-let*:
let s1 = *iFROM-set*; *s2*= *iIN-set*; *s3*= *iMOD2-set*; *s4*= *iMODb2-set in*
s1 ∩ *s2* = {} ∧ *s1* ∩ *s3* = {} ∧ *s1* ∩ *s4* = {} ∧
s2 ∩ *s3* = {} ∧ *s2* ∩ *s4* = {} ∧ *s3* ∩ *s4* = {}
⟨*proof*⟩
lemmas *i-set-min-is-minimal* = *i-set-min-is-minimal-let*[*simplified*]
thm *i-set-min-is-minimal*
thm *conjunct1*
thm
i-set-min-is-minimal[*THEN conjunct1*]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*]]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*[*OF conjunct2*]]]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*]]]]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*]]]]]

i-set-min-is-minimal[*THEN conjunct2*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*]]]]]

thm
i-set-def
i-set-defs
inductive-set
i-set-ind:: (*nat set*) *set*
where
i-set-ind-FROM[*intro*!]: $[n\ldots] \in$ *i-set-ind*
| *i-set-ind-TILL*[*intro*!]: $[\ldots n] \in$ *i-set-ind*
| *i-set-ind-IN*[*intro*!]: $[n\ldots,d] \in$ *i-set-ind*
| *i-set-ind-MOD*[*intro*!]: $[r,\ mod\ m] \in$ *i-set-ind*
| *i-set-ind-MODb*[*intro*!]: $[r,\ mod\ m,\ c] \in$ *i-set-ind*

inductive-set
i-set0-ind :: (*nat set*) *set*
where
i-set0-ind-empty[*intro*!] : $\{\} \in$ *i-set0-ind*
| *i-set0-ind-i-set*[*intro*]: $I \in$ *i-set-ind* $\Longrightarrow I \in$ *i-set0-ind*

The introduction rule *i-set0-ind-i-set* is not declared a safe introduction rule, because it would disturb the correct usage of the *safe* method.

lemma *i-set-ind-subset-i-set0-ind*: *i-set-ind* $\subseteq$ *i-set0-ind*
⟨*proof*⟩
thm
i-set-ind-FROM
i-set-ind-TILL
i-set-ind-IN
i-set-ind-MOD
i-set-ind-MODb
lemma
i-set0-ind-FROM[*intro*!] : $[n\ldots] \in$ *i-set0-ind* **and**
i-set0-ind-TILL[*intro*!] : $[\ldots n] \in$ *i-set0-ind* **and**
i-set0-ind-IN[*intro*!] : $[n\ldots,d] \in$ *i-set0-ind* **and**
i-set0-ind-MOD[*intro*!] : $[r,\ mod\ m] \in$ *i-set0-ind* **and**
i-set0-ind-MODb[*intro*!] : $[r,\ mod\ m,\ c] \in$ *i-set0-ind*
⟨*proof*⟩
lemmas *i-set0-ind-intros2* =
i-set0-ind-empty
i-set0-ind-FROM
i-set0-ind-TILL
i-set0-ind-IN
i-set0-ind-MOD
i-set0-ind-MODb
thm *i-set0-ind-intros2*

lemma *i-set-i-set-ind-eq*: $i\text{-}set = i\text{-}set\text{-}ind$
⟨*proof*⟩

lemma *i-set0-i-set0-ind-eq*: $i\text{-}set0 = i\text{-}set0\text{-}ind$
⟨*proof*⟩
thm *i-set-i-set-ind-eq*
⟨*proof*⟩

lemma *i-set-imp-not-empty*: $I \in i\text{-}set \Longrightarrow I \neq \{\}$
⟨*proof*⟩

lemma *i-set0-i-set-mem-conv*: $(I \in i\text{-}set0) = (I \in i\text{-}set \vee I = \{\})$
⟨*proof*⟩

lemma *i-set-i-set0-mem-conv*: $(I \in i\text{-}set) = (I \in i\text{-}set0 \wedge I \neq \{\})$
⟨*proof*⟩

lemma *i-set0-i-set-conv*: $i\text{-}set0 - \{\{\}\} = i\text{-}set$
⟨*proof*⟩
corollary *i-set-subset-i-set0*: $i\text{-}set \subseteq i\text{-}set0$
⟨*proof*⟩
lemma *i-set-singleton*: $\{a\} \in i\text{-}set$
⟨*proof*⟩
lemma *i-set0-singleton*: $\{a\} \in i\text{-}set0$
⟨*proof*⟩

thm *i-set-ind.intros*
corollary
$i\text{-}set\text{-}FROM[intro!] : [n...] \in i\text{-}set$ **and**
$i\text{-}set\text{-}TILL[intro!] : [...n] \in i\text{-}set$ **and**
$i\text{-}set\text{-}IN[intro!] : [n...,d] \in i\text{-}set$ **and**
$i\text{-}set\text{-}MOD[intro!] : [r,\ mod\ m] \in i\text{-}set$ **and**
$i\text{-}set\text{-}MODb[intro!] : [r,\ mod\ m,\ c] \in i\text{-}set$
thm *i-set-i-set-ind-eq*
⟨*proof*⟩
lemmas *i-set-intros* =
i-set-FROM
i-set-TILL
i-set-IN
i-set-MOD
i-set-MODb
thm *i-set0-ind-intros2*
lemma
$i\text{-}set0\text{-}empty[intro!]: \{\} \in i\text{-}set0$ **and**
$i\text{-}set0\text{-}FROM[intro!] : [n...] \in i\text{-}set0$ **and**
$i\text{-}set0\text{-}TILL[intro!] : [...n] \in i\text{-}set0$ **and**
$i\text{-}set0\text{-}IN[intro!] : [n...,d] \in i\text{-}set0$ **and**
$i\text{-}set0\text{-}MOD[intro!] : [r,\ mod\ m] \in i\text{-}set0$ **and**

i-set0-MODb[*intro*!] : $[r,\ mod\ m,\ c] \in$ *i-set0*
⟨*proof*⟩
lemmas *i-set0-intros* =
i-set0-empty
i-set0-FROM
i-set0-TILL
i-set0-IN
i-set0-MOD
i-set0-MODb
thm *i-set0-intros*

lemma *i-set-infinite-as-iMOD*:
$[\![\ I \in$ *i-set*; *infinite* $I\ ]\!] \Longrightarrow \exists\, r\ m.\ I = [r,\ mod\ m]$
⟨*proof*⟩

lemma *i-set-finite-as-iMODb*:
$[\![\ I \in$ *i-set*; *finite* $I\ ]\!] \Longrightarrow \exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c]$
⟨*proof*⟩

lemma *i-set-as-iMOD-iMODb*:
$I \in$ *i-set* $\Longrightarrow (\exists\, r\ m.\ I = [r,\ mod\ m]) \vee (\exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c])$
⟨*proof*⟩

2.5.2 Interval sets are closed under cutting

lemma *i-set-cut-le-ge-closed-disj*:
$[\![\ I \in$ *i-set*; $t \in I$; *cut-op* $=$ *op* $\downarrow\leq \vee$ *cut-op* $=$ *op* $\downarrow\geq\ ]\!] \Longrightarrow$
cut-op $I\ t \in$ *i-set*
⟨*proof*⟩

thm *i-set-cut-le-ge-closed-disj*[*of - - op* $\downarrow\geq$, *simplified*]
corollary
i-set-cut-le-closed: $[\![\ I \in$ *i-set*; $t \in I\ ]\!] \Longrightarrow I \downarrow\leq t \in$ *i-set* **and**
i-set-cut-ge-closed: $[\![\ I \in$ *i-set*; $t \in I\ ]\!] \Longrightarrow I \downarrow\geq t \in$ *i-set*
⟨*proof*⟩

lemmas *i-set-cut-le-ge-closed* = *i-set-cut-le-closed i-set-cut-ge-closed*
thm *i-set-cut-le-ge-closed*

lemma *i-set0-cut-closed-disj*:
$[\![\ I \in$ *i-set0*;
cut-op $=$ *op* $\downarrow\leq \vee$ *cut-op* $=$ *op* $\downarrow\geq \vee$
cut-op $=$ *op* $\downarrow< \vee$ *cut-op* $=$ *op* $\downarrow>\ ]\!] \Longrightarrow$
cut-op $I\ t \in$ *i-set0*
⟨*proof*⟩
thm *i-set0-ind-empty*

thm *ssubst*[*OF set-restriction-empty*, **where** $P{=}\lambda x.\ x \in$ *i-set0-ind*]
⟨*proof*⟩
thm *i-cut-set-restriction-disj*
⟨*proof*⟩

thm *i-set0-cut-closed-disj*[*of - op* $\downarrow>$, *simplified*]
corollary
i-set0-cut-le-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow\leq t \in$ *i-set0* **and**
i-set0-cut-less-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow< t \in$ *i-set0* **and**
i-set0-cut-ge-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow\geq t \in$ *i-set0* **and**
i-set0-cut-greater-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow> t \in$ *i-set0*
⟨*proof*⟩

thm *i-set0-ind.intros*
thm
i-set0-FROM[*THEN i-set0-cut-closed-disj*[*of - op* $\downarrow<$, *simplified*]]
i-set0-TILL[*THEN i-set0-cut-closed-disj*[*of - op* $\downarrow\geq$, *simplified*]]
i-set0-IN[*THEN i-set0-cut-closed-disj*[*of - op* $\downarrow\leq$, *simplified*]]
i-set0-MOD[*THEN i-set0-cut-closed-disj*[*of - op* $\downarrow>$, *simplified*]]
i-set0-MODb[*THEN i-set0-cut-closed-disj*[*of - op* $\downarrow>$, *simplified*]]
lemmas *i-set0-cut-closed* =
i-set0-cut-le-closed
i-set0-cut-less-closed
i-set0-cut-ge-closed
i-set0-cut-greater-closed
thm *i-set0-cut-closed*

2.5.3 Interval sets are closed under addition and multiplication

lemma *i-set-Plus-closed*: $I \in$ *i-set* $\Longrightarrow I \oplus k \in$ *i-set*
⟨*proof*⟩

lemma *i-set-Mult-closed*: $I \in$ *i-set* $\Longrightarrow I \otimes k \in$ *i-set*
⟨*proof*⟩

lemma *i-set0-Plus-closed*: $I \in$ *i-set0* $\Longrightarrow I \oplus k \in$ *i-set0*
⟨*proof*⟩

lemma *i-set0-Mult-closed*: $I \in$ *i-set0* $\Longrightarrow I \otimes k \in$ *i-set0*
⟨*proof*⟩

2.5.4 Interval sets are closed with certain conditions under subtraction

lemma *i-set-Plus-neg-closed*:
$[\![\ I \in$ *i-set*; $\exists\, x{\in}I.\ k \leq x\ ]\!] \Longrightarrow I \oplus{-}\ k \in$ *i-set*
⟨*proof*⟩

lemma *i-set-Minus-closed*:
⟦ $I \in$ *i-set*; *iMin* $I \leq k$ ⟧ ⟹ $k \ominus I \in$ *i-set*
⟨*proof*⟩

lemma *i-set0-Plus-neg-closed*: $I \in$ *i-set0* ⟹ $I \oplus- k \in$ *i-set0*
⟨*proof*⟩

lemma *i-set0-Minus-closed*: $I \in$ *i-set0* ⟹ $k \ominus I \in$ *i-set0*
⟨*proof*⟩

lemmas *i-set-IntOp-closed* =
i-set-Plus-closed
i-set-Mult-closed
i-set-Plus-neg-closed
i-set-Minus-closed
lemmas *i-set0-IntOp-closed* =
i-set0-Plus-closed
i-set0-Mult-closed
i-set0-Plus-neg-closed
i-set0-Minus-closed
thm *i-set-IntOp-closed*
thm *i-set0-IntOp-closed*

2.5.5 Interval sets are not closed under division

thm *i-set-as-iMOD-iMODb*
thm *iMOD-div-mod-gr0-not-ex*
lemma *iMOD-div-mod-gr0-not-in-i-set*:
⟦ $0 < k$; $k < m$; $0 < m \bmod k$ ⟧ ⟹ $[r, \bmod m] \oslash k \notin$ *i-set*
⟨*proof*⟩
thm *i-set-infinite-as-iMOD*
⟨*proof*⟩
lemma *iMODb-div-mod-gr0-not-in-i-set*:
⟦ $0 < k$; $k < m$; $0 < m \bmod k$; $k \leq c$ ⟧ ⟹ $[r, \bmod m, c] \oslash k \notin$ *i-set*
⟨*proof*⟩

lemma $[0, \bmod 3] \oslash 2 \notin$ *i-set*
⟨*proof*⟩

lemma *i-set-Div-not-closed*: *Suc* $0 < k \Longrightarrow \exists I{\in}$*i-set*. $I \oslash k \notin$ *i-set*
⟨*proof*⟩
lemma *i-set0-Div-not-closed*: *Suc* $0 < k \Longrightarrow \exists I{\in}$*i-set0*. $I \oslash k \notin$ *i-set0*
⟨*proof*⟩
thm
i-set-Div-not-closed
i-set0-Div-not-closed

2.5.6 Sets of intervals closed under division

inductive-set
NatMultiples :: *nat set* ⇒ *nat set*
for *F* :: *nat set*
where
NatMultiples-Factor:
$k \in F \Longrightarrow k \in NatMultiples\ F$
| *NatMultiples-Product*:
$[\![\ k \in F;\ m \in NatMultiples\ F\ ]\!] \Longrightarrow k * m \in NatMultiples\ F$
thm *NatMultiples.induct*
lemma *NatMultiples-ex-divisor*: $m \in NatMultiples\ F \Longrightarrow \exists\, k{\in}F.\ m\ mod\ k = 0$
⟨*proof*⟩
lemma *NatMultiples-product-factor*: $[\![\ a \in F;\ b \in F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩
lemma *NatMultiples-product-factor-multiple*:
$[\![\ a \in F;\ b \in NatMultiples\ F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩
lemma *NatMultiples-product-multiple-factor*:
$[\![\ a \in NatMultiples\ F;\ b \in F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩
lemma *NatMultiples-product-multiple*:
$[\![\ a \in NatMultiples\ F;\ b \in NatMultiples\ F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
⟨*proof*⟩

Subset of *i-set* containing only continuous intervals, i. e., without *iMOD* and *iMODb*.

inductive-set *i-set-cont* :: (*nat set*) *set*
where
i-set-cont-FROM[*intro*]: $[n\ldots] \in$ *i-set-cont*
| *i-set-cont-TILL*[*intro*]: $[\ldots n] \in$ *i-set-cont*
| *i-set-cont-IN*[*intro*]: $[n\ldots,d] \in$ *i-set-cont*

definition *i-set0-cont* :: (*nat set*) *set* **where**
i-set0-cont ≡ *insert* {} *i-set-cont*

lemma *i-set-cont-subset-i-set*: *i-set-cont* ⊆ *i-set*
⟨*proof*⟩

lemma *i-set0-cont-subset-i-set0*: *i-set0-cont* ⊆ *i-set0*
⟨*proof*⟩

Minimal definition of *i-set-cont*

inductive-set *i-set-cont-min* :: (*nat set*) *set*
where
i-set-cont-FROM[*intro*]: $[n\ldots] \in$ *i-set-cont-min*
| *i-set-cont-IN*[*intro*]: $[n\ldots,d] \in$ *i-set-cont-min*

definition *i-set0-cont-min* :: (*nat set*) *set* **where**
i-set0-cont-min ≡ *insert* {} *i-set-cont-min*

lemma *i-set-cont-min-eq*: *i-set-cont* = *i-set-cont-min*
⟨*proof*⟩

inext and *iprev* with continuous intervals

lemma *i-set-cont-inext*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *finite I* ⟹ *n* < *Max I* ⟧ ⟹ *inext n I* = *Suc n*
⟨*proof*⟩
lemma *i-set-cont-iprev*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *iMin I* < *n* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
⟨*proof*⟩

lemma *i-set-cont-inext--less*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n* < *n0*; *n0* ∈ *I* ⟧ ⟹ *inext n I* = *Suc n*
⟨*proof*⟩
thm *order-less-le-trans*[*OF - Max-ge*]
⟨*proof*⟩
lemma *i-set-cont-iprev--greater*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n0* < *n*; *n0* ∈ *I* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
⟨*proof*⟩
thm *order-le-less-trans*[*OF iMin-le*, *of n0*]
⟨*proof*⟩

Sets of modulo intervals

inductive-set *i-set-mult* :: *nat* ⇒ (*nat set*) *set*
for *k* :: *nat*
where
i-set-mult-FROM[*intro*!]: [*n*...] ∈ *i-set-mult k*
| *i-set-mult-TILL*[*intro*!]: [...*n*] ∈ *i-set-mult k*
| *i-set-mult-IN*[*intro*!]: [*n*...,*d*] ∈ *i-set-mult k*
| *i-set-mult-MOD*[*intro*!]: [*r*, *mod m* ∗ *k*] ∈ *i-set-mult k*
| *i-set-mult-MODb*[*intro*!]: [*r*, *mod m* ∗ *k*, *c*] ∈ *i-set-mult k*

definition *i-set0-mult* :: *nat* ⇒ (*nat set*) *set* **where**
i-set0-mult k ≡ *insert* {} (*i-set-mult k*)

lemma
i-set0-mult-empty[*intro*!]: {} ∈ *i-set0-mult k* **and**
i-set0-mult-FROM[*intro*!] : [*n*...] ∈ *i-set0-mult k* **and**
i-set0-mult-TILL[*intro*!] : [...*n*] ∈ *i-set0-mult k* **and**
i-set0-mult-IN[*intro*!] : [*n*...,*d*] ∈ *i-set0-mult k* **and**
i-set0-mult-MOD[*intro*!] : [*r*, *mod m* ∗ *k*] ∈ *i-set0-mult k* **and**
i-set0-mult-MODb[*intro*!] : [*r*, *mod m* ∗ *k*, *c*] ∈ *i-set0-mult k*
⟨*proof*⟩
lemmas *i-set0-mult-intros* =
i-set0-mult-empty
i-set0-mult-FROM

i-set0-mult-TILL
i-set0-mult-IN
i-set0-mult-MOD
i-set0-mult-MODb
thm *i-set0-mult-intros*

lemma *i-set-mult-subset-i-set0-mult*: *i-set-mult k* $\subseteq$ *i-set0-mult k*
⟨*proof*⟩

lemma *i-set-mult-subset-i-set*: *i-set-mult k* $\subseteq$ *i-set*
⟨*proof*⟩

thm *subsetD*[*OF i-set-mult-subset-i-set*]

lemma *i-set0-mult-subset-i-set0*: *i-set0-mult k* $\subseteq$ *i-set0*
⟨*proof*⟩
thm *order-trans*[*OF - i-set-subset-i-set0*, *OF i-set-mult-subset-i-set*]
⟨*proof*⟩

lemma *i-set-mult-0-eq-i-set-cont*: *i-set-mult 0* = *i-set-cont*
⟨*proof*⟩

lemma *i-set0-mult-0-eq-i-set0-cont*: *i-set0-mult 0* = *i-set0-cont*
⟨*proof*⟩

lemma *i-set-mult-1-eq-i-set*: *i-set-mult* (*Suc 0*) = *i-set*
⟨*proof*⟩

lemma *i-set0-mult-1-eq-i-set0*: *i-set0-mult* (*Suc 0*) = *i-set0*
⟨*proof*⟩

lemma *i-set-mult-imp-not-empty*: *I* $\in$ *i-set-mult k* $\Longrightarrow$ *I* $\neq$ {}
⟨*proof*⟩

lemma *iMOD-in-i-set-mult-imp-divisor-mod-0*:
⟦ *m* $\neq$ *Suc 0*; [*r*, *mod m*] $\in$ *i-set-mult k* ⟧ $\Longrightarrow$ *m mod k* = *0*
⟨*proof*⟩

lemma
divisor-mod-0-imp-iMOD-in-i-set-mult: *m mod k* = *0* $\Longrightarrow$ [*r*, *mod m*] $\in$ *i-set-mult k* **and**
divisor-mod-0-imp-iMODb-in-i-set-mult: *m mod k* = *0* $\Longrightarrow$ [*r*, *mod m*, *c*] $\in$ *i-set-mult k*
⟨*proof*⟩

lemma *iMOD-in-i-set-mult--divisor-mod-0-conv*:
m $\neq$ *Suc 0* $\Longrightarrow$ ([*r*, *mod m*] $\in$ *i-set-mult k*) = (*m mod k* = *0*)
⟨*proof*⟩

lemma *i-set-mult-neq1-subset-i-set*: $k \neq Suc\ 0 \Longrightarrow i\text{-}set\text{-}mult\ k \subset i\text{-}set$
⟨*proof*⟩

lemma *mod-0-imp-i-set-mult-subset*:
$a\ mod\ b = 0 \Longrightarrow i\text{-}set\text{-}mult\ a \subseteq i\text{-}set\text{-}mult\ b$
⟨*proof*⟩
thm *i-set-mult.cases*
⟨*proof*⟩
lemma *i-set-mult-subset-imp-mod-0*:
$[\![\ a \neq Suc\ 0;\ (i\text{-}set\text{-}mult\ a \subseteq i\text{-}set\text{-}mult\ b)\ ]\!] \Longrightarrow a\ mod\ b = 0$
⟨*proof*⟩
thm *iMOD-in-i-set-mult-imp-divisor-mod-0*[*of - 0 b*]
⟨*proof*⟩
lemma *i-set-mult-subset-conv*:
$a \neq Suc\ 0 \Longrightarrow (i\text{-}set\text{-}mult\ a \subseteq i\text{-}set\text{-}mult\ b) = (a\ mod\ b = 0)$
⟨*proof*⟩

lemma *i-set-mult-mod-0-div*:
$[\![\ I \in i\text{-}set\text{-}mult\ k;\ k\ mod\ d = 0\ ]\!] \Longrightarrow I \oslash d \in i\text{-}set\text{-}mult\ (k\ div\ d)$
⟨*proof*⟩
thm *iT-Div-0*[*OF i-set-mult-imp-not-empty*]
⟨*proof*⟩
thm *mod-0-imp-mod-mult-left-0*
thm *mod-0-imp-div-mult-right-eq*
thm *iMOD-div iMODb-div*
⟨*proof*⟩

Intervals from *i-set-mult k* remain in *i-set* after division by d a divisor of k.

corollary *i-set-mult-mod-0-div-i-set*:
$[\![\ I \in i\text{-}set\text{-}mult\ k;\ k\ mod\ d = 0\ ]\!] \Longrightarrow I \oslash d \in i\text{-}set$
thm *subsetD*[*OF i-set-mult-subset-i-set*[*of k div d*]]
⟨*proof*⟩
corollary *i-set-mult-div-self-i-set*:
$I \in i\text{-}set\text{-}mult\ k \Longrightarrow I \oslash k \in i\text{-}set$
⟨*proof*⟩

lemma *i-set-mod-0-mult-in-i-set-mult*:
$[\![\ I \in i\text{-}set;\ m\ mod\ k = 0\ ]\!] \Longrightarrow I \otimes m \in i\text{-}set\text{-}mult\ k$
⟨*proof*⟩
lemma *i-set-self-mult-in-i-set-mult*:
$I \in i\text{-}set \Longrightarrow I \otimes k \in i\text{-}set\text{-}mult\ k$
⟨*proof*⟩

lemma *i-set-mult-mod-gr0-div-not-in-i-set*:
$[\![\ 0 < k;\ 0 < d;\ 0 < k\ mod\ d\ ]\!] \Longrightarrow \exists I{\in}i\text{-}set\text{-}mult\ k.\ I \oslash d \notin i\text{-}set$
⟨*proof*⟩
thm *iMOD-div-mod-gr0-not-ex*[*of d Suc d * k 0*]

⟨*proof*⟩
thm *i-set-infinite-as-iMOD*
⟨*proof*⟩

lemma *i-set0-mult-mod-0-div*:
⟦ $I \in$ *i-set0-mult* k; k *mod* $d = 0$ ⟧ $\Longrightarrow I \oslash d \in$ *i-set0-mult* $(k$ *div* $d)$
⟨*proof*⟩
corollary *i-set0-mult-mod-0-div-i-set0*:
⟦ $I \in$ *i-set0-mult* k; k *mod* $d = 0$ ⟧ $\Longrightarrow I \oslash d \in$ *i-set0*
⟨*proof*⟩
corollary *i-set0-mult-div-self-i-set0*:
$I \in$ *i-set0-mult* $k \Longrightarrow I \oslash k \in$ *i-set0*
⟨*proof*⟩

lemma *i-set0-mod-0-mult-in-i-set0-mult*:
⟦ $I \in$ *i-set0*; m *mod* $k = 0$ ⟧ $\Longrightarrow I \otimes m \in$ *i-set0-mult* k
⟨*proof*⟩

lemma *i-set0-self-mult-in-i-set0-mult*:
$I \in$ *i-set0* $\Longrightarrow I \otimes k \in$ *i-set0-mult* k
⟨*proof*⟩

lemma *i-set0-mult-mod-gr0-div-not-in-i-set0*:
⟦ $0 < k$; $0 < d$; $0 <$ k *mod* d ⟧ $\Longrightarrow \exists I \in$*i-set0-mult* k. $I \oslash d \notin$ *i-set0*
⟨*proof*⟩

end

3 IL-TemporalOperators: Temporal logic operators on natural intervals

theory *IL-TemporalOperators*
imports *IL-IntervalOperators*
begin

Bool : some additional properties

instantiation *bool* :: {*ord*, *zero*, *one*, *plus*, *times*, *order*}
begin

definition *Zero-bool-def* [*simp*]: $0 \equiv$ *False*
definition *One-bool-def* [*simp*]: $1 \equiv$ *True*
definition *add-bool-def*: $a + b \equiv a \vee b$
definition *mult-bool-def*: $a * b \equiv a \wedge b$

instance ⟨*proof*⟩
end

value *False* $<$ *True*

value *True* < *True*
value *True* ≤ *True*
thm *le-bool-def*

lemmas *bool-op-rel-defs* =
add-bool-def
mult-bool-def
less-bool-def
le-bool-def
thm *bool-op-rel-defs*

instance *bool* :: *linorder*
⟨*proof*⟩

instance *bool* :: *wellorder*
⟨*proof*⟩
thm *wf-def*
⟨*proof*⟩

instance *bool* :: *comm-semiring-1*
⟨*proof*⟩

3.1 Basic definitions

typ *iT*

thm *Ex-def*
thm *All-def*

term *All*
term *Ball*
lemma *UNIV-nat*: ℕ = (*UNIV*::*nat set*)
⟨*proof*⟩

Universal temporal operator: Always/Globally

definition
iAll :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Always
where
iAll I P ≡ ∀ *t*∈*I*. *P t*

Existential temporal operator: Eventually/Finally

definition
iEx :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Eventually
where
iEx-def : *iEx I P* ≡ ∃ *t*∈*I*. *P t*

syntax (*xsymbols*)
-iAll :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*□ - -./ -) [*0*, *0*, *10*] *10*)
-iEx :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◇ - -./ -) [*0*, *0*, *10*] *10*)
syntax (*HTML* **output**)
-iAll :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*□ - -./ -) [*0*, *0*, *10*] *10*)
-iEx :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◇ - -./ -) [*0*, *0*, *10*] *10*)
translations
□ *t I*. *P* ⇌ *CONST iAll I* (λ*t*. *P*)
◇ *t I*. *P* ⇌ *CONST iEx I* (λ*t*. *P*)

Future temporal operator: Next

definition
iNext :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Next
where
iNext t0 I P ≡ *P* (*inext t0 I*)

Past temporal operator: Last/Previous

definition
iLast :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Last
where
iLast t0 I P ≡ *P* (*iprev t0 I*)

syntax (*xsymbols*)
-iNext :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯ - - -./ -) [*0*, *0*, *10*] *10*)
-iLast :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖ - - -./ -) [*0*, *0*, *10*] *10*)
syntax (*HTML* **output**)
-iNext :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯ - - -./ -) [*0*, *0*, *10*] *10*)
-iLast :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖ - - -./ -) [*0*, *0*, *10*] *10*)
translations
◯ *t t0 I*. *P* ⇌ *CONST iNext t0 I* (λ*t*. *P*)
⊖ *t t0 I*. *P* ⇌ *CONST iLast t0 I* (λ*t*. *P*)
lemma ◯ *t 10* [*0*...]. (*t* + *10* > *10*)
⟨*proof*⟩

The following versions of Next and Last operator differ in the cases where no next/previous element exists or specified time point is not in interval: the weak versions return *True* and the strong versions return *False*.

definition
iNextWeak :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Next
where
iNextWeak t0 I P ≡ (□ *t* {*inext t0 I*} ↓> *t0*. *P t*)
definition
iNextStrong :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Next

where
iNextStrong t0 I P ≡ (◊ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition
iLastWeak :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Last
where
iLastWeak t0 I P ≡ (□ *t* {*iprev t0 I*} ↓< *t0*. *P t*)
definition
iLastStrong :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Last
where
iLastStrong t0 I P ≡ (◊ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

syntax (*xsymbols*)
-iNextWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
-iNextStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
-iLastWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
-iLastStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
syntax (*HTML* **output**)
-iNextWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
-iNextStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
-iLastWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
-iLastStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
translations
◯$_W$ *t t0 I*. *P* ⇌ *CONST iNextWeak t0 I* (λ*t*. *P*)
◯$_S$ *t t0 I*. *P* ⇌ *CONST iNextStrong t0 I* (λ*t*. *P*)
⊖$_W$ *t t0 I*. *P* ⇌ *CONST iLastWeak t0 I* (λ*t*. *P*)
⊖$_S$ *t t0 I*. *P* ⇌ *CONST iLastStrong t0 I* (λ*t*. *P*)

Some examples for Next and Last operator

lemma ◯ *t 5* [*0*...,*10*]. ([*0*::*int*,*10*,*20*,*30*,*40*,*50*,*60*,*70*,*80*,*90*] ! *t* < *80*)
⟨*proof*⟩
lemma ⊖ *t 5* [*0*...,*10*]. ([*0*::*int*,*10*,*20*,*30*,*40*,*50*,*60*,*70*,*80*,*90*] ! *t* < *80*)
⟨*proof*⟩

Temporal Until operator

definition
iUntil :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Until
where
iUntil I P Q ≡ ◊ *t I*. *Q t* ∧ (□ *t'* (*I* ↓< *t*). *P t'*)

Temporal Since operator (past operator corresponding to Until)

definition
iSince :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Since
where
$iSince\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t')$

syntax (*xsymbols*)
-iUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{U}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
-iSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{S}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
translations
$P.\ t\ \mathcal{U}\ t'\ I.\ Q \rightleftharpoons CONST\ iUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{S}\ t'\ I.\ Q \rightleftharpoons CONST\ iSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition
iWeakUntil :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Until/Wating-for/Unless
where
$iWeakUntil\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow< t).\ P\ t'))$

definition
iWeakSince :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Since/Back-to
where
$iWeakSince\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I \downarrow> t).\ P\ t'))$

syntax (*xsymbols*)
-iWeakUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{W}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
-iWeakSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{B}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
translations
$P.\ t\ \mathcal{W}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{B}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition
iRelease :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Release
where
iRelease-def : $iRelease\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \vee (\Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow\leq t).\ Q\ t'))$

definition
iTrigger :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Trigger
where
iTrigger-def : $iTrigger\ I\ P\ Q \equiv (\Box\ t\ I.\ Q\ t) \vee (\Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow\geq t).\ Q$

t'))

syntax (*xsymbols*)
  -*iRelease* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ (*Time* ⇒ *bool*) ⇒ *bool*
    ((-./ - (3$\mathcal{R}$ - -)./ -) [*10*, *0*, *0*, *0*, *10*] *10*)
  -*iTrigger* :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ (*Time* ⇒ *bool*) ⇒ *bool*
    ((-./ - (3$\mathcal{T}$ - -)./ -) [*10*, *0*, *0*, *0*, *10*] *10*)
translations
  P. *t* $\mathcal{R}$ *t'* *I*. *Q* ⇌ *CONST iRelease I* (λ*t*. *P*) (λ*t'*. *Q*)
  P. *t* $\mathcal{T}$ *t'* *I*. *Q* ⇌ *CONST iTrigger I* (λ*t*. *P*) (λ*t'*. *Q*)

lemmas *iTL-Next-defs* =
  iNext-def iLast-def
  iNextWeak-def iLastWeak-def
  iNextStrong-def iLastStrong-def
lemmas *iTL-defs* =
  iAll-def iEx-def
  iUntil-def iSince-def
  iWeakUntil-def iWeakSince-def
  iRelease-def iTrigger-def
  iTL-Next-defs

⟨*ML*⟩

term □ *t I*. *P t*
term ◇ *t I*. *P t*

term *P1 t1*. *t1* $\mathcal{U}$ *t2 I*. *P2 t2*
term *P1 t1*. *t1* $\mathcal{S}$ *t2 I*. *P2 t2*
⟨*ML*⟩

term *P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2*
term *P t1*. *t1* $\mathcal{S}$ *t2 I*. *Q t2*

⟨*ML*⟩

term *P t1*. *t1* $\mathcal{W}$ *t2 I*. *Q t2*
term *P t1*. *t1* $\mathcal{B}$ *t2 I*. *Q t2*

⟨*ML*⟩

term $P\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2$
term $P\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2$

3.2 Basic lemmata for temporal operators

3.2.1 Intro/elim rules

thm *bexI rev-bexI*
lemma
iexI[*intro*]: $[\![\ P\ t;\ t \in I\ ]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
rev-iexI[*intro?*]: $[\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
iexE[*elim*!]: $[\![\ \Diamond\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
⟨*proof*⟩
lemma
iallI[*intro*!]: $(\bigwedge t.\ t \in I \Longrightarrow P\ t) \Longrightarrow \Box\ t\ I.\ P\ t$ **and**
ispec[*dest?*]: $[\![\ \Box\ t\ I.\ P\ t;\ t \in I\ ]\!] \Longrightarrow P\ t$ **and**
iallE[*elim*]: $[\![\ \Box\ t\ I.\ P\ t;\ P\ t \Longrightarrow Q;\ t \notin I \Longrightarrow Q\ ]\!] \Longrightarrow Q$
⟨*proof*⟩

lemma
iuntilI[*intro*]:
$[\![\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t');\ t \in I\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$ **and**
rev-iuntilI[*intro?*]:
$[\![\ t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t')\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
⟨*proof*⟩
lemma
iuntilE[*elim*]:
$[\![\ P'\ t'.\ t'\ \mathcal{U}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
⟨*proof*⟩

thm *iSince-def*
lemma
isinceI[*intro*]:
$[\![\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t');\ t \in I\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$ **and**
rev-isinceI[*intro?*]:
$[\![\ t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t')\ ]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
⟨*proof*⟩
lemma
isinceE[*elim*]:
$[\![\ P'\ t'.\ t'\ \mathcal{S}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![\ t \in I;\ P\ t\ ]\!] \Longrightarrow Q\ ]\!] \Longrightarrow Q$
⟨*proof*⟩

3.2.2 Rewrite rules for trivial simplification

thm *ball-triv*
lemma *iall-triv*[*simp*]: $(\Box\ t\ I.\ P) = ((\exists\, t.\ t \in I) \longrightarrow P)$
⟨*proof*⟩
thm *bex-triv*
lemma *iex-triv*[*simp*]: $(\Diamond\ t\ I.\ P) = ((\exists\, t.\ t \in I) \wedge P)$

⟨*proof*⟩

lemma *iex-conjL1*:
(◇ *t1 I1*. (*P1 t1* ∧ (◇ *t2 I2*. *P2 t1 t2*))) =
(◇ *t1 I1*. ◇ *t2 I2*. *P1 t1* ∧ *P2 t1 t2*)
⟨*proof*⟩
lemma *iex-conjR1*:
(◇ *t1 I1*. ((◇ *t2 I2*. *P2 t1 t2*) ∧ *P1 t1*)) =
(◇ *t1 I1*. ◇ *t2 I2*. *P2 t1 t2* ∧ *P1 t1*)
⟨*proof*⟩
lemma *iex-conjL2*:
(◇ *t1 I1*. (*P1 t1* ∧ (◇ *t2* (*I2 t1*). *P2 t1 t2*))) =
(◇ *t1 I1*. ◇ *t2* (*I2 t1*). *P1 t1* ∧ *P2 t1 t2*)
⟨*proof*⟩
lemma *iex-conjR2*:
(◇ *t1 I1*. ((◇ *t2* (*I2 t1*). *P2 t1 t2*) ∧ *P1 t1*)) =
(◇ *t1 I1*. ◇ *t2* (*I2 t1*). *P2 t1 t2* ∧ *P1 t1*)
⟨*proof*⟩
lemma *iex-commute*:
(◇ *t1 I1*. ◇ *t2 I2*. *P t1 t2*) =
(◇ *t2 I2*. ◇ *t1 I1*. *P t1 t2*)
⟨*proof*⟩

lemma *iall-conjL1*:
I2 ≠ {} ⟹
(□ *t1 I1*. (*P1 t1* ∧ (□ *t2 I2*. *P2 t1 t2*))) =
(□ *t1 I1*. □ *t2 I2*. *P1 t1* ∧ *P2 t1 t2*)
⟨*proof*⟩
lemma *iall-conjR1*:
I2 ≠ {} ⟹
(□ *t1 I1*. ((□ *t2 I2*. *P2 t1 t2*) ∧ *P1 t1*)) =
(□ *t1 I1*. □ *t2 I2*. *P2 t1 t2* ∧ *P1 t1*)
⟨*proof*⟩
lemma *iall-conjL2*:
□ *t1 I1*. *I2 t1* ≠ {} ⟹
(□ *t1 I1*. (*P1 t1* ∧ (□ *t2* (*I2 t1*). *P2 t1 t2*))) =
(□ *t1 I1*. □ *t2* (*I2 t1*). *P1 t1* ∧ *P2 t1 t2*)
⟨*proof*⟩
lemma *iall-conjR2*:
□ *t1 I1*. *I2 t1* ≠ {} ⟹
(□ *t1 I1*. ((□ *t2* (*I2 t1*). *P2 t1 t2*) ∧ *P1 t1*)) =
(□ *t1 I1*. □ *t2* (*I2 t1*). *P2 t1 t2* ∧ *P1 t1*)
⟨*proof*⟩
lemma *iall-commute*:
(□ *t1 I1*. □ *t2 I2*. *P t1 t2*) =
(□ *t2 I2*. □ *t1 I1*. *P t1 t2*)
⟨*proof*⟩

lemma *iall-conj-distrib*:
$(\Box\ t\ I.\ P\ t \wedge Q\ t) = ((\Box\ t\ I.\ P\ t) \wedge (\Box\ t\ I.\ Q\ t))$
⟨*proof*⟩
lemma *iex-disj-distrib*:
$(\Diamond\ t\ I.\ P\ t \vee Q\ t) = ((\Diamond\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩

lemma *iall-conj-distrib2*:
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2 \wedge Q\ t1\ t2) =$
$((\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P\ t1\ t2) \wedge (\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ Q\ t1\ t2))$
⟨*proof*⟩
lemma *iex-disj-distrib2*:
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P\ t1\ t2 \vee Q\ t1\ t2) =$
$((\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P\ t1\ t2) \vee (\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ Q\ t1\ t2))$
⟨*proof*⟩

lemma *iUntil-disj-distrib*:
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ (Q1\ t2 \vee Q2\ t2)) = ((P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q1\ t2) \vee (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q2\ t2))$
⟨*proof*⟩
lemma *iSince-disj-distrib*:
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ (Q1\ t2 \vee Q2\ t2)) = ((P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q1\ t2) \vee (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q2\ t2))$
⟨*proof*⟩

thm *iNextWeak-def*
lemma
iNext-iff : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Box\ t\ [\ldots 0] \oplus (inext\ t0\ I).\ P\ t)$ **and**
iLast-iff : $(\ominus\ t\ t0\ I.\ P\ t) = (\Box\ t\ [\ldots 0] \oplus (iprev\ t0\ I).\ P\ t)$
⟨*proof*⟩
lemma
iNext-iEx-iff : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ [\ldots 0] \oplus (inext\ t0\ I).\ P\ t)$ **and**
iLast-iEx-iff : $(\ominus\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ [\ldots 0] \oplus (iprev\ t0\ I).\ P\ t)$
⟨*proof*⟩

lemma *inext-singleton-cut-greater-not-empty-iff*:
$(\{inext\ t0\ I\} \downarrow> t0 \neq \{\}) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩
thm *inext-all-le-fix*
⟨*proof*⟩
lemma *iprev-singleton-cut-less-not-empty-iff*:
$(\{iprev\ t0\ I\} \downarrow< t0 \neq \{\}) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩
thm *iprev-all-ge-fix*
⟨*proof*⟩
lemma *inext-singleton-cut-greater-empty-iff*:
$(\{inext\ t0\ I\} \downarrow> t0 = \{\}) = (I \downarrow> t0 = \{\} \vee t0 \notin I)$

⟨*proof*⟩
thm *inext-singleton-cut-greater-not-empty-iff*
⟨*proof*⟩
lemma *iprev-singleton-cut-less-empty-iff*:
(⦃*iprev t0 I*⦄ ↓< *t0* = {}) = (*I* ↓< *t0* = {} ∨ *t0* ∉ *I*)
⟨*proof*⟩
thm *iprev-singleton-cut-less-not-empty-iff*
⟨*proof*⟩

lemma *iNextWeak-iff* : (◯$_W$ *t t0 I*. *P t*) = ((◯ *t t0 I*. *P t*) ∨ (*I* ↓> *t0* = {}) ∨ *t0* ∉ *I*)
⟨*proof*⟩
thm *inext-singleton-cut-greater-empty-iff*[*symmetric*]
⟨*proof*⟩
lemma *iNextStrong-iff* : (◯$_S$ *t t0 I*. *P t*) = ((◯ *t t0 I*. *P t*) ∧ (*I* ↓> *t0* ≠ {}) ∧ *t0* ∈ *I*)
⟨*proof*⟩
thm *inext-singleton-cut-greater-not-empty-iff*[*symmetric*]
⟨*proof*⟩
lemma *iLastWeak-iff* : (⊖$_W$ *t t0 I*. *P t*) = ((⊖ *t t0 I*. *P t*) ∨ (*I* ↓< *t0* = {}) ∨ *t0* ∉ *I*)
⟨*proof*⟩
thm *iprev-singleton-cut-less-empty-iff*[*symmetric*]
⟨*proof*⟩
lemma *iLastStrong-iff* : (⊖$_S$ *t t0 I*. *P t*) = ((⊖ *t t0 I*. *P t*) ∧ (*I* ↓< *t0* ≠ {}) ∧ *t0* ∈ *I*)
⟨*proof*⟩
thm *iprev-singleton-cut-less-not-empty-iff*[*symmetric*]
⟨*proof*⟩

lemmas *iTL-Next-iff* =
iNext-iff iLast-iff
iNextWeak-iff iNextStrong-iff
iLastWeak-iff iLastStrong-iff

lemma
iNext-iff-singleton : (◯ *t t0 I*. *P t*) = (□ *t* ⦃*inext t0 I*⦄. *P t*) **and**
iLast-iff-singleton : (⊖ *t t0 I*. *P t*) = (□ *t* ⦃*iprev t0 I*⦄. *P t*)
⟨*proof*⟩
lemmas *iNextLast-iff-singleton* = *iNext-iff-singleton iLast-iff-singleton*

lemma
iNext-iEx-iff-singleton : (◯ *t t0 I*. *P t*) = (◇ *t* ⦃*inext t0 I*⦄. *P t*) **and**
iLast-iEx-iff-singleton : (⊖ *t t0 I*. *P t*) = (◇ *t* ⦃*iprev t0 I*⦄. *P t*)
⟨*proof*⟩

lemma
iNextWeak-iAll-conv: $(\bigcirc_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iNextStrong-iEx-conv: $(\bigcirc_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iLastWeak-iAll-conv: $(\ominus_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$ **and**
iLastStrong-iEx-conv: $(\ominus_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$
⟨*proof*⟩

lemma
iAll-True[*simp*]: $\Box\ t\ I.\ True$ **and**
iAll-False[*simp*]: $(\Box\ t\ I.\ False) = (I = \{\})$ **and**
iEx-True[*simp*]: $(\Diamond\ t\ I.\ True) = (I \neq \{\})$ **and**
iEx-False[*simp*]: $\neg\ (\Diamond\ t\ I.\ False)$
⟨*proof*⟩

lemma *empty-iff-iAll-False*: $(I = \{\}) = (\Box\ t\ I.\ False)$ ⟨*proof*⟩
lemma *not-empty-iff-iEx-True*: $(I \neq \{\}) = (\Diamond\ t\ I.\ True)$ ⟨*proof*⟩

lemma
iNext-True: $\bigcirc\ t\ t0\ I.\ True$ **and**
iNextWeak-True: $(\bigcirc_W\ t\ t0\ I.\ True)$ **and**
iNext-False: $\neg\ (\bigcirc\ t\ t0\ I.\ False)$ **and**
iNextStrong-False: $\neg\ (\bigcirc_S\ t\ t0\ I.\ False)$
⟨*proof*⟩
lemma
iNextStrong-True: $(\bigcirc_S\ t\ t0\ I.\ True) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$ **and**
iNextWeak-False: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ False)) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma
iLast-True: $\ominus\ t\ t0\ I.\ True$ **and**
iLastWeak-True: $(\ominus_W\ t\ t0\ I.\ True)$ **and**
iLast-False: $\neg\ (\ominus\ t\ t0\ I.\ False)$ **and**
iLastStrong-False: $\neg\ (\ominus_S\ t\ t0\ I.\ False)$
⟨*proof*⟩
lemma
iLastStrong-True: $(\ominus_S\ t\ t0\ I.\ True) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$ **and**
iLastWeak-False: $(\neg\ (\ominus_W\ t\ t0\ I.\ False)) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$
⟨*proof*⟩

lemma *iUntil-True-left*[*simp*]: $(True.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (\Diamond\ t\ I.\ Q\ t)$
⟨*proof*⟩
lemma *iUntil-True*[*simp*]: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ True) = (I \neq \{\})$
⟨*proof*⟩
lemma *iUntil-False-left*[*simp*]: $(False.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (I \neq \{\} \wedge Q\ (iMin\ I))$
⟨*proof*⟩
lemma *iUntil-False*[*simp*]: $\neg\ (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ False)$

⟨*proof*⟩

lemma *iSince-True-left*[*simp*]: (*True*. t' $\mathcal{S}$ t I. Q t) = ($\Diamond$ t I. Q t)
⟨*proof*⟩
lemma *iSince-True-if*:
(*P* t'. t' $\mathcal{S}$ t I. *True*) =
(*if finite I then* $I \neq \{\}$ *else* $\Diamond$ *t1* I. $\Box$ *t2* (I $\downarrow>$ *t1*). *P t2*)
⟨*proof*⟩
thm *cut-greater-Max-empty*
⟨*proof*⟩
corollary *iSince-True-finite*[*simp*]: *finite* $I \Longrightarrow$ (*P* t'. t' $\mathcal{S}$ t I. *True*) = ($I \neq \{\}$)
⟨*proof*⟩
lemma *iSince-False-left*[*simp*]: (*False*. t' $\mathcal{S}$ t I. Q t) = (*finite* I $\wedge$ $I \neq \{\}$ $\wedge$ Q (*Max* I))
⟨*proof*⟩
lemma *iSince-False*[*simp*]: $\neg$ (*P* t'. t' $\mathcal{S}$ t I. *False*)
⟨*proof*⟩

lemma *iWeakUntil-True-left*[*simp*]: *True*. t' $\mathcal{W}$ t I. Q t
⟨*proof*⟩
lemma *iWeakUntil-True*[*simp*]: *P* t'. t' $\mathcal{W}$ t I. *True*
⟨*proof*⟩
lemma *iWeakUntil-False-left*[*simp*]: (*False*. t' $\mathcal{W}$ t I. Q t) = ($I = \{\}$ $\vee$ Q (*iMin* I))
⟨*proof*⟩
lemma *iWeakUntil-False*[*simp*]: (*P* t'. t' $\mathcal{W}$ t I. *False*) = ($\Box$ t I. P t)
⟨*proof*⟩

lemma *iWeakSince-True-left*[*simp*]: *True*. t' $\mathcal{B}$ t I. Q t
⟨*proof*⟩
lemma *iWeakSince-True-disj*:
(*P* t'. t' $\mathcal{B}$ t I. *True*) =
($I = \{\}$ $\vee$ ($\Diamond$ *t1* I. $\Box$ *t2* (I $\downarrow>$ *t1*). *P t2*))
⟨*proof*⟩
lemma *iWeakSince-True-finite*[*simp*]: *finite* $I \Longrightarrow$ (*P* t'. t' $\mathcal{B}$ t I. *True*)
⟨*proof*⟩
lemma *iWeakSince-False-left*[*simp*]: (*False*. t' $\mathcal{B}$ t I. Q t) = ($I = \{\}$ $\vee$ (*finite* I $\wedge$ Q (*Max* I)))
⟨*proof*⟩
lemma *iWeakSince-False*[*simp*]: (*P* t'. t' $\mathcal{B}$ t I. *False*) = ($\Box$ t I. P t)
⟨*proof*⟩

lemma *iRelease-True-left*[*simp*]: (*True*. t' $\mathcal{R}$ t I. Q t) = ($I = \{\}$ $\vee$ Q (*iMin* I))
⟨*proof*⟩
lemma *iRelease-True*[*simp*]: *P* t'. t' $\mathcal{R}$ t I. *True*
⟨*proof*⟩
lemma *iRelease-False-left*[*simp*]: (*False*. t' $\mathcal{R}$ t I. Q t) = ($\Box$ t I. Q t)
⟨*proof*⟩
lemma *iRelease-False*[*simp*]: (*P* t'. t' $\mathcal{R}$ t I. *False*) = ($I = \{\}$)

⟨*proof*⟩

lemma *iTrigger-True-left*[*simp*]: (*True*. t' $\mathcal{T}$ t I. Q t) = (I = {} ∨ (◇ *t1* I. □ *t2* (I ↓≥ *t1*). Q *t2*))
⟨*proof*⟩
lemma *iTrigger-True*[*simp*]: P t'. t' $\mathcal{T}$ t I. *True*
⟨*proof*⟩
lemma *iTrigger-False-left*[*simp*]: (*False*. t' $\mathcal{T}$ t I. Q t) = (□ t I. Q t)
⟨*proof*⟩
lemma *iTrigger-False*[*simp*]: (P t'. t' $\mathcal{T}$ t I. *False*) = (I = {})
⟨*proof*⟩

lemma
iUntil-TrueTrue[*simp*]: (*True*. t' $\mathcal{U}$ t I. *True*) = (I ≠ {}) **and**
iSince-TrueTrue[*simp*]: (*True*. t' $\mathcal{S}$ t I. *True*) = (I ≠ {}) **and**
iWeakUntil-TrueTrue[*simp*]: *True*. t' $\mathcal{W}$ t I. *True* **and**
iWeakSince-TrueTrue[*simp*]: *True*. t' $\mathcal{B}$ t I. *True* **and**
iRelease-TrueTrue[*simp*]: *True*. t' $\mathcal{R}$ t I. *True* **and**
iTrigger-TrueTrue[*simp*]: *True*. t' $\mathcal{T}$ t I. *True*
⟨*proof*⟩

3.2.3 Empty sets and singletons

lemma *iAll-empty*[*simp*]: □ t {}. P t ⟨*proof*⟩
lemma *iEx-empty*[*simp*]: ¬ (◇ t {}. P t) ⟨*proof*⟩

thm *iAll-empty iEx-empty*
lemma *iUntil-empty*[*simp*]: ¬ (P *t0*. *t0* $\mathcal{U}$ *t1* {}. Q *t1*) ⟨*proof*⟩
lemma *iSince-empty*[*simp*]: ¬ (P *t0*. *t0* $\mathcal{S}$ *t1* {}. Q *t1*) ⟨*proof*⟩
lemma *iWeakUntil-empty*[*simp*]: P *t0*. *t0* $\mathcal{W}$ *t1* {}. Q *t1* ⟨*proof*⟩
lemma *iWeakSince-empty*[*simp*]: P *t0*. *t0* $\mathcal{B}$ *t1* {}. Q *t1* ⟨*proof*⟩

lemma *iRelease-empty*[*simp*]: P *t0*. *t0* $\mathcal{R}$ *t1* {}. Q *t1* ⟨*proof*⟩
lemma *iTrigger-empty*[*simp*]: P *t0*. *t0* $\mathcal{T}$ *t1* {}. Q *t1* ⟨*proof*⟩

lemmas *iTL-empty* =
iAll-empty iEx-empty
iUntil-empty iSince-empty
iWeakUntil-empty iWeakSince-empty
iRelease-empty iTrigger-empty

lemma *iAll-singleton*[*simp*]: (□ t' {t}. P t') = P t ⟨*proof*⟩
lemma *iEx-singleton*[*simp*]: (◇ t' {t}. P t') = P t ⟨*proof*⟩

lemma *iUntil-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{U}$ *t1* {t}. Q *t1*) = Q t
⟨*proof*⟩
lemma *iSince-singleton*[*simp*]: (P *t0*. *t0* $\mathcal{S}$ *t1* {t}. Q *t1*) = Q t
⟨*proof*⟩

lemma *iWeakUntil-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{W}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
⟨*proof*⟩
lemma *iWeakSince-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{B}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
⟨*proof*⟩

lemma *iRelease-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{R}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
⟨*proof*⟩
lemma *iTrigger-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{T}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
⟨*proof*⟩

lemmas *iTL-singleton* =
iAll-singleton iEx-singleton
iUntil-singleton iSince-singleton
iWeakUntil-singleton iWeakSince-singleton
iRelease-singleton iTrigger-singleton
thm *iTL-singleton*

3.2.4 Conversions between temporal operators

lemma *iAll-iEx-conv*: $(\Box\ t\ I.\ P\ t) = (\neg\ (\Diamond\ t\ I.\ \neg\ P\ t))$ ⟨*proof*⟩
lemma *iEx-iAll-conv*: $(\Diamond\ t\ I.\ P\ t) = (\neg\ (\Box\ t\ I.\ \neg\ P\ t))$ ⟨*proof*⟩

lemma *not-iAll*[*simp*]: $(\neg\ (\Box\ t\ I.\ P\ t)) = (\Diamond\ t\ I.\ \neg\ P\ t)$ ⟨*proof*⟩
lemma *not-iEx*[*simp*]: $(\neg\ (\Diamond\ t\ I.\ P\ t)) = (\Box\ t\ I.\ \neg\ P\ t)$ ⟨*proof*⟩

lemma *iUntil-iEx-conv*: $(True.\ t'\ \mathcal{U}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ ⟨*proof*⟩
lemma *iSince-iEx-conv*: $(True.\ t'\ \mathcal{S}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ ⟨*proof*⟩

lemma *iRelease-iAll-conv*: $(False.\ t'\ \mathcal{R}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩
lemma *iTrigger-iAll-conv*: $(False.\ t'\ \mathcal{T}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
⟨*proof*⟩

lemma *iWeakUntil-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
⟨*proof*⟩
lemma *iWeakSince-iSince-conv*: $(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
⟨*proof*⟩

lemma *iUntil-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩
lemma *iSince-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
⟨*proof*⟩

thm *iRelease-def iWeakUntil-def*

lemma *iRelease-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{W}\ t\ I.\ (Q\ t \wedge P\ t))$
⟨*proof*⟩
lemma *iRelease-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t'\ \mathcal{U}\ t\ I.\ (Q\ t \wedge P\ t)))$
⟨*proof*⟩

lemma *iTrigger-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{B}\ t\ I.\ (Q\ t \wedge P\ t))$
⟨*proof*⟩
lemma *iTrigger-iSince-conv*: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t'\ \mathcal{S}\ t\ I.\ (Q\ t \wedge P\ t)))$
⟨*proof*⟩

lemma *iRelease-not-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩
lemma *iUntil-not-iRelease-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{R}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

The Trigger operator $\mathcal{T}$is a past operator, so that it is used for time intervals, that are bounded by a current time point, and thus are finite. For an infinite interval the stated relation to the Since operator $\mathcal{S}$would not be fulfilled.

lemma *iTrigger-not-iSince-conv*: *finite* $I \Longrightarrow (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩
thm *not-greater-Max*[*rotated 1*]
⟨*proof*⟩
lemma *iSince-not-iTrigger-conv*: *finite* $I \Longrightarrow (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{T}\ t\ I.\ \neg\ Q\ t))$
⟨*proof*⟩

lemma *not-iUntil*:
$(\neg\ (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)) =$
$(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ \downarrow<\ t).\ \neg\ P\ t')))$
⟨*proof*⟩
lemma *not-iSince*:
$(\neg\ (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)) =$
$(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ \downarrow>\ t).\ \neg\ P\ t')))$
⟨*proof*⟩

lemma *iWeakUntil-conj-iUntil-conv*:
$(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ (P\ t2 \wedge Q\ t2)) = (\neg\ (\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ \neg\ P\ t2))$
⟨*proof*⟩
lemma *iUntil-disj-iUntil-conv*:
$(P\ t1 \vee Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) =$

$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩
thm *subsetD*[*OF - iMinI*]
⟨*proof*⟩
lemma *iWeakUntil-disj-iWeakUntil-conv*:
$(P\ t1 \vee Q\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) =$
$(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩
lemma *iWeakUntil-iUntil-conj-conv*:
$(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) =$
$(\neg\ (\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ (\neg\ P\ t2 \wedge \neg\ Q\ t2)))$
⟨*proof*⟩

Negation and temporal operators

thm *iNext-iff*
lemma
not-iNext[*simp*]: $(\neg\ (\bigcirc\ t\ t0\ I.\ P\ t)) = (\bigcirc\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextWeak[*simp*]: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ P\ t)) = (\bigcirc_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextStrong[*simp*]: $(\neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)) = (\bigcirc_W\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLast[*simp*]: $(\neg\ (\ominus\ t\ t0\ I.\ P\ t)) = (\ominus\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastWeak[*simp*]: $(\neg\ (\ominus_W\ t\ t0\ I.\ P\ t)) = (\ominus_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastStrong[*simp*]: $(\neg\ (\ominus_S\ t\ t0\ I.\ P\ t)) = (\ominus_W\ t\ t0\ I.\ \neg\ P\ t)$
⟨*proof*⟩

thm *not-iUntil*
thm *not-iSince*

lemma *not-iWeakUntil*:
$(\neg\ (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ {\downarrow}{<}\ t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
⟨*proof*⟩
lemma *not-iWeakSince*:
$(\neg\ (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ {\downarrow}{>}\ t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
⟨*proof*⟩

lemma *not-iRelease*:
$(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ P\ t \longrightarrow (\Diamond\ t\ I\ {\downarrow}{\leq}\ t.\ \neg\ Q\ t)))$
⟨*proof*⟩
lemma *not-iRelease-iUntil-conv*:
$(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t)$
⟨*proof*⟩
lemma *not-iTrigger*:
$(\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ \neg\ P\ t \vee (\Diamond\ t\ I\ {\downarrow}{\geq}\ t.\ \neg\ Q\ t)))$
⟨*proof*⟩
lemma *not-iTrigger-iSince-conv*:
finite $I \Longrightarrow (\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t)$

⟨*proof*⟩

3.2.5 Some implication results

lemma *all-imp-iall*: $\forall x.\ P\ x \Longrightarrow \Box\ t\ I.\ P\ t$ ⟨*proof*⟩
lemma *bex-imp-lex*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \exists x.\ P\ x$ ⟨*proof*⟩

lemma *iAll-imp-iEx*: $I \neq \{\} \Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$ ⟨*proof*⟩
lemma *i-set-iAll-imp-iEx*: $I \in$ *i-set* $\Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$
⟨*proof*⟩
lemmas *iT-iAll-imp-iEx* = *iT-not-empty*[*THEN iAll-imp-iEx*]
thm *iT-iAll-imp-iEx*

lemma *iUntil-imp-iEx*: $P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$
⟨*proof*⟩
lemma *iSince-imp-iEx*: $P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$
⟨*proof*⟩

thm *ball-subset-imp-ball*
lemma *iall-subset-imp-iall*: $[\![\ \Box\ t\ B.\ P\ t;\ A \subseteq B\ ]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
⟨*proof*⟩
thm *bex-subset-imp-bex*
lemma *iex-subset-imp-iex*: $[\![\ \Diamond\ t\ A.\ P\ t;\ A \subseteq B\ ]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$
⟨*proof*⟩

lemma *iall-mp*: $[\![\ \Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Box\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ I.\ Q\ t$ ⟨*proof*⟩
lemma *iex-mp*: $[\![\ \Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Diamond\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ I.\ Q\ t$ ⟨*proof*⟩

lemma *iUntil-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2$
⟨*proof*⟩
lemma *iSince-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2$
⟨*proof*⟩
lemma *iWeakUntil-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2$
⟨*proof*⟩
lemma *iWeakSince-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2$
⟨*proof*⟩
lemma *iRelease-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2$
⟨*proof*⟩
lemma *iTrigger-imp*:
$[\![\ P1\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2;\ \Box\ t\ I.\ P1\ t \longrightarrow P2\ t\ ]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2$
⟨*proof*⟩

lemma *iMin-imp-iUntil*:
$[\![I \neq \{\}; Q\ (iMin\ I)]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
⟨*proof*⟩
lemma *Max-imp-iSince*:
$[\![finite\ I; I \neq \{\}; Q\ (Max\ I)]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
⟨*proof*⟩

3.2.6 Congruence rules for temporal operators' predicates

thm *arg-cong*
lemma *iAll-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Box\ t\ I.\ P\ (f\ t)\ t) = (\Box\ t\ I.\ P\ (g\ t)\ t)$
⟨*proof*⟩
lemma *iEx-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Diamond\ t\ I.\ P\ (f\ t)\ t) = (\Diamond\ t\ I.\ P\ (g\ t)\ t)$
⟨*proof*⟩
lemma *iUntil-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩
lemma *iUntil-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (g\ t2)\ t2)$
⟨*proof*⟩
thm *subst*
lemma *iSince-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩
lemma *iSince-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (g\ t2)\ t2)$
⟨*proof*⟩
thm *iAll-cong*

lemma *bex-subst*:
$\forall x{\in}A.\ P\ x \longrightarrow (Q\ x = Q'\ x) \Longrightarrow$
$(\exists x{\in}A.\ P\ x \wedge Q\ x) = (\exists x{\in}A.\ P\ x \wedge Q'\ x)$
⟨*proof*⟩
lemma *iEx-subst*:
$\Box\ t\ I.\ P\ t \longrightarrow (Q\ t = Q'\ t) \Longrightarrow$
$(\Diamond\ t\ I.\ P\ t \wedge Q\ t) = (\Diamond\ t\ I.\ P\ t \wedge Q'\ t)$
⟨*proof*⟩

3.2.7 Temporal operators with set unions/intersections and subsets

lemma *iAll-subset*: $[\![A \subseteq B; \Box\ t\ B.\ P\ t]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
⟨*proof*⟩
lemma *iEx-subset*: $[\![A \subseteq B; \Diamond\ t\ A.\ P\ t]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$
⟨*proof*⟩

lemma *iUntil-append*:
$[\![$ *finite A*; *Max A* $\leq$ *iMin B* $]\!] \Longrightarrow$
P t1. *t1* $\mathcal{U}$ *t2 A*. *Q t2* $\Longrightarrow$ *P t1*. *t1* $\mathcal{U}$ *t2* $(A \cup B)$. *Q t2*
⟨*proof*⟩
thm *iEx-subset*[*OF Un-upper1*]
⟨*proof*⟩
thm *subst*[*OF iEx-cong*, *rule-format*]
⟨*proof*⟩
lemma *iSince-prepend*:
$[\![$ *finite A*; *Max A* $\leq$ *iMin B* $]\!] \Longrightarrow$
P t1. *t1* $\mathcal{S}$ *t2 B*. *Q t2* $\Longrightarrow$ *P t1*. *t1* $\mathcal{S}$ *t2* $(A \cup B)$. *Q t2*
⟨*proof*⟩

lemma
iAll-union: $[\![\Box\ t\ A.\ P\ t; \Box\ t\ B.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ (A \cup B).\ P\ t$ **and**
iAll-union-conv: $(\Box\ t\ A \cup B.\ P\ t) = ((\Box\ t\ A.\ P\ t) \wedge (\Box\ t\ B.\ P\ t))$
⟨*proof*⟩
lemma
iEx-union: $(\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t) \Longrightarrow \Diamond\ t\ (A \cup B).\ P\ t$ **and**
iEx-union-conv: $(\Diamond\ t\ (A \cup B).\ P\ t) = ((\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t))$
⟨*proof*⟩
lemma *iAll-inter*: $(\Box\ t\ A.\ P\ t) \vee (\Box\ t\ B.\ P\ t) \Longrightarrow \Box\ t\ (A \cap B).\ P\ t$ ⟨*proof*⟩
lemma *not-iEx-inter*:
$\exists A\ B\ P.\ (\Diamond\ t\ A.\ P\ t) \wedge (\Diamond\ t\ B.\ P\ t) \wedge \neg\ (\Diamond\ t\ (A \cap B).\ P\ t)$
⟨*proof*⟩

lemma
iAll-insert: $[\![\ P\ t; \Box\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Box\ t'\ (\mathit{insert}\ t\ I).\ P\ t'$ **and**
iAll-insert-conv: $(\Box\ t'\ (\mathit{insert}\ t\ I).\ P\ t') = (P\ t \wedge (\Box\ t'\ I.\ P\ t'))$
⟨*proof*⟩
lemma
iEx-insert: $[\![\ P\ t \vee (\Diamond\ t\ I.\ P\ t)\ ]\!] \Longrightarrow \Diamond\ t'\ (\mathit{insert}\ t\ I).\ P\ t'$ **and**
iEx-insert-conv: $(\Diamond\ t'\ (\mathit{insert}\ t\ I).\ P\ t') = (P\ t \vee (\Diamond\ t'\ I.\ P\ t'))$
⟨*proof*⟩

3.3 Further results for temporal operators

thm *Collect-minI-ex*
thm *Collect-minI-ex-cut*
lemma *Collect-minI-iEx*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow< t).\ \neg\ P\ t')$
⟨*proof*⟩

thm *iUntil-def*
lemma *iUntil-disj-conv1*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (Q\ (\mathit{iMin}\ I) \vee (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t \wedge \mathit{iMin}\ I < t))$
⟨*proof*⟩
lemma *iSince-disj-conv1*:

$[\![\mathit{finite}\ I;\ I \neq \{\}\ ]\!] \Longrightarrow$
$(P\ t'.\ t' \mathcal{S}\ t\ I.\ Q\ t) = (Q\ (\mathit{Max}\ I) \vee (P\ t'.\ t' \mathcal{S}\ t\ I.\ Q\ t \wedge t < \mathit{Max}\ I))$
⟨*proof*⟩

lemma *iUntil-next*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t' \mathcal{U}\ t\ I.\ Q\ t) =$
$(Q\ (\mathit{iMin}\ I) \vee (P\ (\mathit{iMin}\ I) \wedge (P\ t'.\ t' \mathcal{U}\ t\ (I \downarrow> (\mathit{iMin}\ I)).\ Q\ t)))$
⟨*proof*⟩

lemma *iSince-prev*: $[\![\mathit{finite}\ I;\ I \neq \{\}\ ]\!] \Longrightarrow$
$(P\ t'.\ t' \mathcal{S}\ t\ I.\ Q\ t) =$
$(Q\ (\mathit{Max}\ I) \vee (P\ (\mathit{Max}\ I) \wedge (P\ t'.\ t' \mathcal{S}\ t\ (I \downarrow< \mathit{Max}\ I).\ Q\ t)))$
⟨*proof*⟩

lemma *iNext-induct-rule*:
$[\![P\ (\mathit{iMin}\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t')); t \in I\ ]\!] \Longrightarrow P\ t$
⟨*proof*⟩
lemma *iNext-induct*:
$[\![P\ (\mathit{iMin}\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t'))\ ]\!] \Longrightarrow \Box\ t\ I.\ P\ t$
⟨*proof*⟩

lemma *iLast-induct-rule*:
$[\![P\ (\mathit{Max}\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t')); \mathit{finite}\ I; t \in I\ ]\!] \Longrightarrow P\ t$
⟨*proof*⟩
lemma *iLast-induct*:
$[\![P\ (\mathit{Max}\ I); \Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t')); \mathit{finite}\ I\ ]\!] \Longrightarrow \Box\ t\ I.\ P\ t$
⟨*proof*⟩

lemma *iUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩
thm *iMin-le*
⟨*proof*⟩
thm *subsetD*[*OF - iMinI*]
⟨*proof*⟩

lemma *iWeakUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩

lemma *iSince-conj-not*: *finite* $I \Longrightarrow$
$((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
⟨*proof*⟩
thm *MaxI2*
⟨*proof*⟩
thm *not-greater-Max*

⟨*proof*⟩
thm *subsetD*[*OF - MaxI*]
⟨*proof*⟩
lemma *iWeakSince-conj-not*: *finite I* ⟹
((*P t1* ∧ ¬ *Q t1*). *t1* $\mathcal{B}$ *t2 I*. *Q t2*) = (*P t1*. *t1* $\mathcal{B}$ *t2 I*. *Q t2*)
⟨*proof*⟩

lemma *iNextStrong-imp-iNextWeak*: ($\bigcirc_S$ *t t0 I*. *P t*) ⟶ ($\bigcirc_W$ *t t0 I*. *P t*)
⟨*proof*⟩
lemma *iLastStrong-imp-iLastWeak*: ($\ominus_S$ *t t0 I*. *P t*) ⟶ ($\ominus_W$ *t t0 I*. *P t*)
⟨*proof*⟩

lemma *infin-imp-iNextWeak-iNextStrong-eq-iNext*:
⟦ *infinite I*; *t0* ∈ *I* ⟧ ⟹
(($\bigcirc_W$ *t t0 I*. *P t*) = ($\bigcirc$ *t t0 I*. *P t*)) ∧ (($\bigcirc_S$ *t t0 I*. *P t*) = ($\bigcirc$ *t t0 I*. *P t*))
⟨*proof*⟩

lemma *infin-imp-iNextWeak-eq-iNext*: ⟦ *infinite I*; *t0* ∈ *I* ⟧ ⟹ ($\bigcirc_W$ *t t0 I*. *P t*)
= ($\bigcirc$ *t t0 I*. *P t*)
⟨*proof*⟩
lemma *infin-imp-iNextStrong-eq-iNext*: ⟦ *infinite I*; *t0* ∈ *I* ⟧ ⟹ ($\bigcirc_S$ *t t0 I*. *P t*)
= ($\bigcirc$ *t t0 I*. *P t*)
⟨*proof*⟩
lemma *infin-imp-iNextStrong-eq-iNextWeak*: ⟦ *infinite I*; *t0* ∈ *I* ⟧ ⟹ ($\bigcirc_S$ *t t0*
I. *P t*) = ($\bigcirc_W$ *t t0 I*. *P t*)
⟨*proof*⟩

lemma
not-in-iNext-eq: *t0* ∉ *I* ⟹ ($\bigcirc$ *t t0 I*. *P t*) = (*P t0*) **and**
not-in-iLast-eq: *t0* ∉ *I* ⟹ ($\ominus$ *t t0 I*. *P t*) = (*P t0*)
⟨*proof*⟩
lemma
not-in-iNextWeak-eq: *t0* ∉ *I* ⟹ ($\bigcirc_W$ *t t0 I*. *P t*) **and**
not-in-iLastWeak-eq: *t0* ∉ *I* ⟹ ($\ominus_W$ *t t0 I*. *P t*)
⟨*proof*⟩
lemma
not-in-iNextStrong-eq: *t0* ∉ *I* ⟹ ¬ ($\bigcirc_S$ *t t0 I*. *P t*) **and**
not-in-iLastStrong-eq: *t0* ∉ *I* ⟹ ¬ ($\ominus_S$ *t t0 I*. *P t*)
⟨*proof*⟩

lemma
iNext-UNIV: ($\bigcirc$ *t t0 UNIV*. *P t*) = *P* (*Suc t0*) **and**
iNextWeak-UNIV: ($\bigcirc_W$ *t t0 UNIV*. *P t*) = *P* (*Suc t0*) **and**
iNextStrong-UNIV: ($\bigcirc_S$ *t t0 UNIV*. *P t*) = *P* (*Suc t0*)
⟨*proof*⟩
lemma
iLast-UNIV: ($\ominus$ *t t0 UNIV*. *P t*) = *P* (*t0* − *Suc 0*) **and**

iLastWeak-UNIV: $(\ominus_W\ t\ t0\ UNIV.\ P\ t) = (if\ 0 < t0\ then\ P\ (t0 - Suc\ 0)\ else\ True)$ **and**
iLastStrong-UNIV: $(\ominus_S\ t\ t0\ UNIV.\ P\ t) = (if\ 0 < t0\ then\ P\ (t0 - Suc\ 0)\ else\ False)$
⟨*proof*⟩

lemmas *iTL-Next-UNIV* =
iNext-UNIV iNextWeak-UNIV iNextStrong-UNIV
iLast-UNIV iLastWeak-UNIV iLastStrong-UNIV

lemma *inext-nth-iNext-Suc*: $(\bigcirc\ t\ (I \rightarrow n)\ I.\ P\ t) = P\ (I \rightarrow Suc\ n)$
⟨*proof*⟩
lemma *iprev-nth-iLast-Suc*: $(\ominus\ t\ (I \leftarrow n)\ I.\ P\ t) = P\ (I \leftarrow Suc\ n)$
⟨*proof*⟩

3.4 Temporal operators and arithmetic interval operators

Shifting intervals through addition and subtraction of constants. Mirroring intervals through subtraction of intervals from constants. Expanding and compressing intervals through multiplication and division by constants.

Always operator

lemma *iT-Plus-iAll-conv*: $(\Box\ t\ I \oplus k.\ P\ t) = (\Box\ t\ I.\ P\ (t + k))$
⟨*proof*⟩
lemma *iT-Mult-iAll-conv*: $(\Box\ t\ I \otimes k.\ P\ t) = (\Box\ t\ I.\ P\ (t * k))$
⟨*proof*⟩
lemma *iT-Plus-neg-iAll-conv*: $(\Box\ t\ I \oplus\!- k.\ P\ t) = (\Box\ t\ (I \downarrow\geq k).\ P\ (t - k))$
⟨*proof*⟩
lemma *iT-Minus-iAll-conv*: $(\Box\ t\ k \ominus I.\ P\ t) = (\Box\ t\ (I \downarrow\leq k).\ P\ (k - t))$
⟨*proof*⟩
lemma *iT-Div-iAll-conv*: $(\Box\ t\ I \oslash k.\ P\ t) = (\Box\ t\ I.\ P\ (t\ div\ k))$
⟨*proof*⟩

lemmas *iT-arith-iAll-conv* =
iT-Plus-iAll-conv
iT-Mult-iAll-conv
iT-Plus-neg-iAll-conv
iT-Minus-iAll-conv
iT-Div-iAll-conv

Eventually operator

lemma
iT-Plus-iEx-conv: $(\Diamond\ t\ I \oplus k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t + k))$ **and**
iT-Mult-iEx-conv: $(\Diamond\ t\ I \otimes k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t * k))$ **and**
iT-Plus-neg-iEx-conv: $(\Diamond\ t\ I \oplus\!- k.\ P\ t) = (\Diamond\ t\ (I \downarrow\geq k).\ P\ (t - k))$ **and**
iT-Minus-iEx-conv: $(\Diamond\ t\ k \ominus I.\ P\ t) = (\Diamond\ t\ (I \downarrow\leq k).\ P\ (k - t))$ **and**
iT-Div-iEx-conv: $(\Diamond\ t\ I \oslash k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t\ div\ k))$
⟨*proof*⟩

Until and Since operators

lemma *iT-Plus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩
lemma *iT-Mult-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩
lemma *iT-Plus-neg-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{U}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
thm *i-cut-commute-disj*[*of op* $\downarrow<$ *op* $\downarrow\geq$, *simplified*]
⟨*proof*⟩
lemma *iT-Minus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩
lemma *iT-Div-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩
thm *cut-less-cut-ge-ident*[*OF order-refl*]
⟨*proof*⟩
find-theorems - $<$ *iMin* - - $\notin$ -
⟨*proof*⟩
thm *subsetD*[*OF* - *iMinI-ex2*]
⟨*proof*⟩

Until and Since operators can be converted into each other through substraction of intervals from constants

thm *iT-Minus-iUntil-conv*
lemma *iUntil-iSince-conv*:
$[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (k \ominus I).\ Q\ (k - t2))$
⟨*proof*⟩
thm *iT-Minus-iUntil-conv*
⟨*proof*⟩
thm *iT-Minus-iUntil-conv*
⟨*proof*⟩
lemma *iSince-iUntil-conv*:
$[\![\ finite\ I;\ Max\ I \leq k\ ]\!] \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ (k - t2))$
⟨*proof*⟩
thm *cut-less-Max-all*
⟨*proof*⟩

lemma *iT-Plus-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩
lemma *iT-Mult-iSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1$

$* k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩
lemma *iT-Plus-neg-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{S}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
lemma *iT-Minus-iSince-conv*:
$(P\ t1.\ t1\ \mathcal{S}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{U}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩
thm *iT-Minus-cut-eq*[*OF order-refl*]
⟨*proof*⟩
thm *iSince-iUntil-conv*
⟨*proof*⟩
lemma *iT-Div-iSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩
thm *le-add-diff Suc-mod-le-divisor*
⟨*proof*⟩

Weak Until and Weak Since operators

lemma *iT-Plus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩
lemma *iT-Mult-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩
lemma *iT-Plus-neg-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{W}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
lemma *iT-Minus-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{B}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩
lemma *iT-Div-iWeakUntil-conv*: $(P\ t1.\ t1\ \mathcal{W}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{W}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

lemma *iT-Plus-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩
lemma *iT-Mult-iWeakSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩
lemma *iT-Plus-neg-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{B}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
lemma *iT-Minus-iWeakSince-conv*:
$(P\ t1.\ t1\ \mathcal{B}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{W}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
thm *iT-Minus-iWeakUntil-conv*[*symmetric*]

⟨*proof*⟩
lemma *iT-Div-iWeakSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

Release and Trigger operators

lemma *iT-Plus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩
lemma *iT-Mult-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩
lemma *iT-Plus-neg-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{R}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
lemma *iT-Minus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{T}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩
lemma *iT-Div-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

lemma *iT-Plus-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 + k))$
⟨*proof*⟩
lemma *iT-Mult-iTrigger-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{T}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 * k))$
⟨*proof*⟩
lemma *iT-Plus-neg-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{T}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
⟨*proof*⟩
lemma *iT-Minus-iTrigger-conv*:
$(P\ t1.\ t1\ \mathcal{T}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{R}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
⟨*proof*⟩
lemma *iT-Div-iTrigger-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2\ div\ k))$
⟨*proof*⟩

end