Interval Temporal Logic on Natural Numbers

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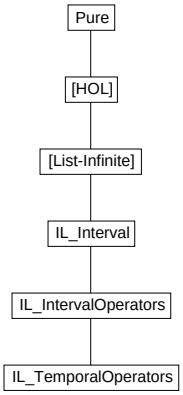
Abstract

We introduce a theory of temporal logic operators using sets of natural numbers as time domain, formalized in a shallow embedding manner. The theory comprises special natural intervals (theory IL_Interval: open and closed intervals, continuous and modulo intervals, interval traversing results), operators for shifting intervals to left/right on the number axis as well as expanding/contracting intervals by constant factors (theory IL_IntervalOperators.thy), and ultimately definitions and results for unary and binary temporal operators on arbitrary natural sets (theory IL_TemporalOperators).

Contents


Pure
[HOL]
[List-Infinite]
IL_Interval
IL_IntervalOperators
IL_TemporalOperators

1 IL-Interval: Intervals and operations for temporal logic declarations

theory *IL-Interval*
imports
Main Nat-Infinity
SetInterval2 InfiniteSet2 SetIntervalStep
Util-Nat Util-MinMax Util-Div
begin

1.1 Time intervals – definitions and basic lemmata

1.1.1 Definitions

types *Time* = *nat*

types *iT* = *Time set*
typ *iT*

Infinite interval starting at some natural *n*.

definition
iFROM :: *Time* ⇒ *iT* ([-...])
where
[*n*...] ≡ {*n*..}

Finite interval starting at *0*::'*a* and ending at some natural *n*.

definition
iTILL :: *Time* ⇒ *iT* ([...-])
where
[...*n*] ≡ {..*n*}

Finite bounded interval containing the naturals between *n* and *n* + *d*. *d* denotes the difference between left and right interval bound. The number of elements is *d* + (*1*::'*a*) so that an empty interval cannot be defined.

definition
iIN :: *Time* ⇒ *nat* ⇒ *iT* ([-...,-])
where
[*n*...,*d*] ≡ {*n*..*n*+*d*}

Infinite modulo interval containing all naturals having the same division remainder modulo *m* as *r*, and beginning at *n*.

definition
iMOD :: *Time* ⇒ *nat* ⇒ *iT* ([-, *mod* -])
where
[*r*, *mod m*] ≡ { *x*. *x mod m* = *r mod m* ∧ *r* ≤ *x*}

Finite bounded modulo interval containing all naturals having the same division remainder modulo *m* as *r*, beginning at *n*, and ending after *c* cycles

at $r + m * c$. The number of elements is $c + (1::'a)$ so that an empty interval cannot be defined.

definition
iMODb :: *Time* ⇒ *nat* ⇒ *nat* ⇒ *iT* ([-, *mod* -, -])
where
$[r, \mathit{mod}\ m, c] \equiv \{ x.\ x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x \wedge x \leq r + m * c\}$

1.1.2 Membership in an interval

lemmas *iT-defs* = *iFROM-def iTILL-def iIN-def iMOD-def iMODb-def*

lemma *iFROM-iff*: $x \in [n...] = (n \leq x)$
by (*simp add*: *iFROM-def*)
lemma *iTILL-iff*: $x \in [...n] = (x \leq n)$
by (*simp add*: *iTILL-def*)
lemma *iIN-iff*:$x \in [n...,d] = (n \leq x \wedge x \leq n + d)$
by (*simp add*: *iIN-def*)
lemma *iMOD-iff*: $x \in [r, \mathit{mod}\ m] = (x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x)$
by (*simp add*: *iMOD-def*)
lemma *iMODb-iff*: $x \in [r, \mathit{mod}\ m, c] =$
$(x\ \mathit{mod}\ m = r\ \mathit{mod}\ m \wedge r \leq x \wedge x \leq r + m * c)$
by (*simp add*: *iMODb-def*)

lemma *iFROM-D*: $x \in [n...] \Longrightarrow (n \leq x)$
by (*rule iFROM-iff*[*THEN iffD1*])
lemma *iTILL-D*: $x \in [...n] \Longrightarrow (x \leq n)$
by (*rule iTILL-iff*[*THEN iffD1*])
corollary *iIN-geD*: $x \in [n...,d] \Longrightarrow n \leq x$
by (*simp add*: *iIN-iff*)
corollary *iIN-leD*: $x \in [n...,d] \Longrightarrow x \leq n + d$
by (*simp add*: *iIN-iff*)
corollary *iMOD-modD*: $x \in [r, \mathit{mod}\ m] \Longrightarrow x\ \mathit{mod}\ m = r\ \mathit{mod}\ m$
by (*simp add*: *iMOD-iff*)
corollary *iMOD-geD*: $x \in [r, \mathit{mod}\ m] \Longrightarrow r \leq x$
by (*simp add*: *iMOD-iff*)
corollary *iMODb-modD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow x\ \mathit{mod}\ m = r\ \mathit{mod}\ m$
by (*simp add*: *iMODb-iff*)
corollary *iMODb-geD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow r \leq x$
by (*simp add*: *iMODb-iff*)
corollary *iMODb-leD*: $x \in [r, \mathit{mod}\ m, c] \Longrightarrow x \leq r + m * c$
by (*simp add*: *iMODb-iff*)

lemmas *iT-iff* = *iFROM-iff iTILL-iff iIN-iff iMOD-iff iMODb-iff*

lemmas *iT-drule* =
iFROM-D
iTILL-D
iIN-geD iIN-leD
iMOD-modD iMOD-geD

iMODb-modD iMODb-geD iMODb-leD
thm *iT-drule*

lemma
iFROM-I [*intro*]: $n \leq x \Longrightarrow x \in [n\ldots]$ **and**
iTILL-I [*intro*]: $x \leq n \Longrightarrow x \in [\ldots n]$ **and**
iIN-I [*intro*]: $n \leq x \Longrightarrow x \leq n + d \Longrightarrow x \in [n\ldots,d]$ **and**
iMOD-I [*intro*]: $x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \in [r,\ mod\ m]$ **and**
iMODb-I [*intro*]: $x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \leq r + m * c \Longrightarrow x \in [r,\ mod\ m,\ c]$
by (*simp add*: *iT-iff*)+

lemma
iFROM-E [*elim*]: $x \in [n\ldots] \Longrightarrow (n \leq x \Longrightarrow P) \Longrightarrow P$ **and**
iTILL-E [*elim*]: $x \in [\ldots n] \Longrightarrow (x \leq n \Longrightarrow P) \Longrightarrow P$ **and**
iIN-E [*elim*]: $x \in [n\ldots,d] \Longrightarrow (n \leq x \Longrightarrow x \leq n + d \Longrightarrow P) \Longrightarrow P$ **and**
iMOD-E [*elim*]: $x \in [r,\ mod\ m] \Longrightarrow (x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow P) \Longrightarrow P$ **and**
iMODb-E [*elim*]: $x \in [r,\ mod\ m,\ c] \Longrightarrow (x\ mod\ m = r\ mod\ m \Longrightarrow r \leq x \Longrightarrow x \leq r + m * c \Longrightarrow P) \Longrightarrow P$
by (*simp add*: *iT-iff*)+

lemma *iIN-Suc-insert-conv*:
$insert\ (Suc\ (n + d))\ [n\ldots,d] = [n\ldots,Suc\ d]$
by (*fastsimp simp*: *iIN-iff*)

lemma *iTILL-Suc-insert-conv*: $insert\ (Suc\ n)\ [\ldots n] = [\ldots Suc\ n]$
by (*fastsimp simp*: *iIN-Suc-insert-conv*[*of 0 n*])

lemma *iMODb-Suc-insert-conv*:
$insert\ (r + m * Suc\ c)\ [r,\ mod\ m,\ c] = [r,\ mod\ m,\ Suc\ c]$
apply (*rule set-eqI*)
apply (*simp add*: *iMODb-iff add-commute*[*of - r*])
apply (*simp add*: *add-commute*[*of m*])
apply (*simp add*: *add-assoc*[*symmetric*])
apply (*rule iffI*)
apply *fastsimp*
apply (*elim conjE*)
apply (*drule-tac x*=*x* **in** *order-le-less*[*THEN iffD1*, *rule-format*])
apply (*erule disjE*)
thm *less-mod-eq-imp-add-divisor-le*
apply (*frule less-mod-eq-imp-add-divisor-le*[**where** *m*=*m*], *simp*)
thm *add-le-imp-le-right*
apply (*drule add-le-imp-le-right*)
apply *simp*

apply *simp*
done

lemma *iFROM-pred-insert-conv*: *insert* $(n - Suc\ 0)$ $[n\ldots] = [n - Suc\ 0\ldots]$
by (*fastsimp simp*: *iFROM-iff*)

lemma *iIN-pred-insert-conv*:
$0 < n \Longrightarrow insert\ (n - Suc\ 0)\ [n\ldots,d] = [n - Suc\ 0\ldots,Suc\ d]$
by (*fastsimp simp*: *iIN-iff*)

lemma *iMOD-pred-insert-conv*:
$m \leq r \Longrightarrow insert\ (r - m)\ [r,\ mod\ m] = [r - m,\ mod\ m]$
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMOD-iff insert-absorb*)
apply *simp*
apply (*rule set-eqI*)
apply (*simp add*: *iMOD-iff mod-diff-self2*)
apply (*rule iffI*)
apply (*erule disjE*)
apply (*simp add*: *mod-diff-self2*)
apply (*simp add*: *le-imp-diff-le*)
apply (*erule conjE*)
apply (*drule order-le-less*[*THEN iffD1*, *of* $r-m$], *erule disjE*)
prefer *2*
apply *simp*
apply (*frule order-less-le-trans*[*of - m r*], *assumption*)
thm *less-mod-eq-imp-add-divisor-le*[*of* $r-m$ *x m*]
apply (*drule less-mod-eq-imp-add-divisor-le*[*of* $r-m$ *- m*])
apply (*simp add*: *mod-diff-self2*)
apply *simp*
done

lemma *iMODb-pred-insert-conv*:
$m \leq r \Longrightarrow insert\ (r - m)\ [r,\ mod\ m,\ c] = [r - m,\ mod\ m,\ Suc\ c]$
apply (*rule set-eqI*)
apply (*frule iMOD-pred-insert-conv*)
apply (*drule-tac* $f=\lambda s.\ x \in s$ **in** *arg-cong*)
apply (*force simp*: *iMOD-iff iMODb-iff*)
done

lemma *iFROM-Suc-pred-insert-conv*: *insert n* $[Suc\ n\ldots] = [n\ldots]$
by (*insert iFROM-pred-insert-conv*[*of Suc n*], *simp*)
lemma *iIN-Suc-pred-insert-conv*: *insert n* $[Suc\ n\ldots,d] = [n\ldots,Suc\ d]$
by (*insert iIN-pred-insert-conv*[*of Suc n*], *simp*)
lemma *iMOD-Suc-pred-insert-conv*: *insert r* $[r + m,\ mod\ m] = [r,\ mod\ m]$
by (*insert iMOD-pred-insert-conv*[*of m r* $+$ *m*], *simp*)
lemma *iMODb-Suc-pred-insert-conv*: *insert r* $[r + m,\ mod\ m,\ c] = [r,\ mod\ m,$
Suc c]
by (*insert iMODb-pred-insert-conv*[*of m r* $+$ *m*], *simp*)

lemmas *iT-Suc-insert* =
iIN-Suc-insert-conv
iTILL-Suc-insert-conv
iMODb-Suc-insert-conv
lemmas *iT-pred-insert* =
iFROM-pred-insert-conv
iIN-pred-insert-conv
iMOD-pred-insert-conv
iMODb-pred-insert-conv
lemmas *iT-Suc-pred-insert* =
iFROM-Suc-pred-insert-conv
iIN-Suc-pred-insert-conv
iMOD-Suc-pred-insert-conv
iMODb-Suc-pred-insert-conv

lemma *iMOD-mem-diff*: $[\![\ a \in [r,\ mod\ m];\ b \in [r,\ mod\ m]\ ]\!] \Longrightarrow (a - b)\ mod\ m = 0$
by (*simp add*: *iMOD-iff mod-eq-imp-diff-mod-0*)
lemma *iMODb-mem-diff*: $[\![\ a \in [r,\ mod\ m,\ c];\ b \in [r,\ mod\ m,\ c]\ ]\!] \Longrightarrow (a - b)\ mod\ m = 0$
by (*simp add*: *iMODb-iff mod-eq-imp-diff-mod-0*)

1.1.3 Interval conversions

lemma *iIN-0-iTILL-conv*:$[0\ldots,n] = [\ldots n]$
by (*simp add*: *iTILL-def iIN-def atMost-atLeastAtMost-0-conv*)
lemma *iIN-iTILL-iTILL-conv*: $0 < n \Longrightarrow [n\ldots,d] = [\ldots n{+}d] - [\ldots n - Suc\ 0]$
by (*fastsimp simp*: *iTILL-iff iIN-iff*)
lemma *iIN-iFROM-iTILL-conv*: $[n\ldots,d] = [n\ldots] \cap [\ldots n{+}d]$
by (*simp add*: *iT-defs atLeastAtMost-def*)
lemma *iMODb-iMOD-iTILL-conv*: $[r,\ mod\ m,\ c] = [r,\ mod\ m] \cap [\ldots r{+}m{*}c]$
by (*force simp*: *iT-defs set-interval-defs*)
lemma *iMODb-iMOD-iIN-conv*: $[r,\ mod\ m,\ c] = [r,\ mod\ m] \cap [r\ldots,m{*}c]$
by (*force simp*: *iT-defs set-interval-defs*)

lemma *iFROM-iTILL-iIN-conv*: $n \leq n' \Longrightarrow [n\ldots] \cap [\ldots n'] = [n\ldots,n'{-}n]$
by (*simp add*: *iT-defs atLeastAtMost-def*)
lemma *iMOD-iTILL-iMODb-conv*:
$r \leq n \Longrightarrow [r,\ mod\ m] \cap [\ldots n] = [r,\ mod\ m,\ (n - r)\ div\ m]$
apply (*rule set-eqI*)
apply (*simp add*: *iT-iff mult-div-cancel*)
apply (*rule iffI*)
apply *clarify*
thm *le-imp-sub-mod-le*
apply (*frule-tac x=x* **and** *y=n* **and** *m=m* **in** *le-imp-sub-mod-le*)
apply (*simp add*: *mod-diff-right-eq*)
apply *fastsimp*
done

lemma *iMOD-iIN-iMODb-conv*:
  $[r,\ mod\ m] \cap [r..,d] = [r,\ mod\ m,\ d\ div\ m]$
apply (*case-tac* $r = 0$)
 thm *iMOD-iTILL-iMODb-conv*
 apply (*simp add*: *iIN-0-iTILL-conv iMOD-iTILL-iMODb-conv*)
apply (*simp add*: *iIN-iTILL-iTILL-conv Diff-Int-distrib iMOD-iTILL-iMODb-conv diff-add-inverse*)
thm *subst*[*of* $\{\}$ - $\lambda t.\ \forall x.(x - t) = x$]
thm *subst*[*of* $\{\}$ - $\lambda t.\ \forall x.(x - t) = x$, *THEN spec*]
apply (*rule subst*[*of* $\{\}$ - $\lambda t.\ \forall x.(x - t) = x$, *THEN spec*])
 prefer *2*
 apply *simp*
apply (*rule sym*)
thm *disjoint-iff-not-equal*
apply (*fastsimp simp*: *disjoint-iff-not-equal iMOD-iff iTILL-iff*)
done

thm *UNIV-def*
lemma *iFROM-0*: $[0..] = UNIV$
by (*simp add*: *iFROM-def*)
lemma *iTILL-0*: $[..0] = \{0\}$
by (*simp add*: *iTILL-def*)
lemma *iIN-0*: $[n..,0] = \{n\}$
by (*simp add*: *iIN-def*)
lemma *iMOD-0*: $[r,\ mod\ 0] = [r..,0]$
by (*fastsimp simp*: *iIN-0 iMOD-def*)
lemma *iMODb-mod-0*: $[r,\ mod\ 0,\ c] = [r..,0]$
by (*fastsimp simp*: *iMODb-def iIN-0*)
lemma *iMODb-0*: $[r,\ mod\ m,\ 0] = [r..,0]$
by (*fastsimp simp*: *iMODb-def iIN-0 set-eq-iff*)
lemmas *iT-0* =
  iFROM-0
  iTILL-0
  iIN-0
  iMOD-0
  iMODb-mod-0
  iMODb-0
thm *iT-0*

lemma *iMOD-1*: $[r,\ mod\ Suc\ 0] = [r..]$
by (*fastsimp simp*: *iFROM-iff*)
lemma *iMODb-mod-1*: $[r,\ mod\ Suc\ 0,\ c] = [r..,c]$
by (*fastsimp simp*: *iT-iff*)

1.1.4 Finiteness and emptiness of intervals

lemma
iFROM-not-empty: $[n...] \neq \{\}$ **and**
iTILL-not-empty: $[...n] \neq \{\}$ **and**
iIN-not-empty: $[n...,d] \neq \{\}$ **and**
iMOD-not-empty: $[r,\ mod\ m] \neq \{\}$ **and**
iMODb-not-empty: $[r,\ mod\ m,\ c] \neq \{\}$
by (*fastsimp simp*: *iT-iff*)+

lemmas *iT-not-empty* =
iFROM-not-empty
iTILL-not-empty
iIN-not-empty
iMOD-not-empty
iMODb-not-empty
thm *iT-not-empty*

lemma
iTILL-finite: *finite* $[...n]$ **and**
iIN-finite: *finite* $[n...,d]$ **and**
iMODb-finite: *finite* $[r,\ mod\ m,\ c]$ **and**
iMOD-0-finite: *finite* $[r,\ mod\ 0]$
by (*simp add*: *iT-defs*)+
lemma *iFROM-infinite*: *infinite* $[n...]$
by (*simp add*: *iT-defs infinite-atLeast*)

lemma *iMOD-infinite*: $0 < m \Longrightarrow$ *infinite* $[r,\ mod\ m]$
thm *infinite-nat-iff-asc-chain*
apply (*rule infinite-nat-iff-asc-chain*[*THEN iffD2*])
apply (*rule iT-not-empty*)
apply (*rule ballI*, *rename-tac n*)
apply (*rule-tac x*=*n*+*m* **in** *bexI*, *simp*)
apply (*simp add*: *iMOD-iff*)
done

lemmas *iT-finite* =
iTILL-finite
iIN-finite
iMODb-finite iMOD-0-finite
thm *iT-finite*

lemmas *iT-infinite* =
iFROM-infinite
iMOD-infinite
thm *iT-infinite*

thm
iMax-finite-conv
iMax-infinite-conv

1.1.5 *Min* and *Max* element of an interval

lemma
iTILL-Min: $iMin\ [\ldots n] = 0$ **and**
iFROM-Min: $iMin\ [n\ldots] = n$ **and**
iIN-Min: $iMin\ [n\ldots,d] = n$ **and**
iMOD-Min: $iMin\ [r, mod\ m] = r$ **and**
iMODb-Min: $iMin\ [r, mod\ m, c] = r$
thm *iMin-equality*
by (*rule iMin-equality*, (*simp add*: *iT-iff*)+)+

lemmas *iT-Min* =
iIN-Min
iTILL-Min
iFROM-Min
iMOD-Min
iMODb-Min
thm *iT-Min*

lemma
iTILL-Max: $Max\ [\ldots n] = n$ **and**
iIN-Max: $Max\ [n\ldots,d] = n+d$ **and**
iMODb-Max: $Max\ [r, mod\ m, c] = r + m * c$ **and**
iMOD-0-Max: $Max\ [r, mod\ 0] = r$
by (*rule Max-equality*, (*simp add*: *iT-iff iT-finite*)+)+

lemmas *iT-Max* =
iTILL-Max
iIN-Max
iMODb-Max
iMOD-0-Max
thm *iT-Max*

lemma
iTILL-iMax: $iMax\ [\ldots n] = Fin\ n$ **and**
iIN-iMax: $iMax\ [n\ldots,d] = Fin\ (n+d)$ **and**
iMODb-iMax: $iMax\ [r, mod\ m, c] = Fin\ (r + m * c)$ **and**
iMOD-0-iMax: $iMax\ [r, mod\ 0] = Fin\ r$ **and**
iFROM-iMax: $iMax\ [n\ldots] = \infty$ **and**
iMOD-iMax: $0 < m \Longrightarrow iMax\ [r, mod\ m] = \infty$
by (*simp add*: *iMax-def iT-finite iT-infinite iT-Max*)+

lemmas *iT-iMax* =
iTILL-iMax
iIN-iMax

iMODb-iMax
iMOD-0-iMax
iFROM-iMax
iMOD-iMax
thm *iT-iMax*

1.2 Adding and subtracting constants to interval elements

lemma
iFROM-plus: $x \in [n\ldots] \Longrightarrow x + k \in [n\ldots]$ **and**
iFROM-Suc: $x \in [n\ldots] \Longrightarrow Suc\ x \in [n\ldots]$ **and**
iFROM-minus: $[\![\ x \in [n\ldots];\ k \leq x - n\ ]\!] \Longrightarrow x - k \in [n\ldots]$ **and**
iFROM-pred: $n < x \Longrightarrow x - Suc\ 0 \in [n\ldots]$
by (*simp add*: *iFROM-iff*)+

lemma
iTILL-plus: $[\![\ x \in [\ldots n];\ k \leq n - x\ ]\!] \Longrightarrow x + k \in [\ldots n]$ **and**
iTILL-Suc: $x < n \Longrightarrow Suc\ x \in [\ldots n]$ **and**
iTILL-minus: $x \in [\ldots n] \Longrightarrow x - k \in [\ldots n]$ **and**
iTILL-pred: $x \in [\ldots n] \Longrightarrow x - Suc\ 0 \in [\ldots n]$
by (*simp add*: *iTILL-iff*)+

lemma *iIN-plus*: $[\![\ x \in [n\ldots,d];\ k \leq n + d - x\ ]\!] \Longrightarrow x + k \in [n\ldots,d]$
by (*fastsimp simp*: *iIN-iff*)
lemma *iIN-Suc*: $[\![\ x \in [n\ldots,d];\ x < n + d\ ]\!] \Longrightarrow Suc\ x \in [n\ldots,d]$
by (*simp add*: *iIN-iff*)
lemma *iIN-minus*: $[\![\ x \in [n\ldots,d];\ k \leq x - n\ ]\!] \Longrightarrow x - k \in [n\ldots,d]$
by (*fastsimp simp*: *iIN-iff*)
lemma *iIN-pred*: $[\![\ x \in [n\ldots,d];\ n < x\ ]\!] \Longrightarrow x - Suc\ 0 \in [n\ldots,d]$
by (*fastsimp simp*: *iIN-iff*)

lemma *iMOD-plus-divisor-mult*: $x \in [r,\ mod\ m] \Longrightarrow x + k * m \in [r,\ mod\ m]$
by (*simp add*: *iMOD-def*)
corollary *iMOD-plus-divisor*: $x \in [r,\ mod\ m] \Longrightarrow x + m \in [r,\ mod\ m]$
by (*simp add*: *iMOD-def*)
lemma *iMOD-minus-divisor-mult*:
$[\![\ x \in [r,\ mod\ m];\ k * m \leq x - r\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
thm *mod-diff-mult-self1*
by (*fastsimp simp*: *iMOD-def mod-diff-mult-self1*)
corollary *iMOD-minus-divisor-mult2*:
$[\![\ x \in [r,\ mod\ m];\ k \leq (x - r)\ div\ m\ ]\!] \Longrightarrow x - k * m \in [r,\ mod\ m]$
apply (*rule iMOD-minus-divisor-mult*, *assumption*)
apply (*clarsimp simp*: *iMOD-iff*)
apply (*drule mult-le-mono1*[*of* - - *m*])
thm *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*]
apply (*simp add*: *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*])
done
corollary *iMOD-minus-divisor*:
$[\![\ x \in [r,\ mod\ m];\ m + r \leq x\ ]\!] \Longrightarrow x - m \in [r,\ mod\ m]$

apply (*frule iMOD-geD*)
thm *iMOD-minus-divisor-mult*[*of x r m 1*]
apply (*insert iMOD-minus-divisor-mult*[*of x r m 1*])
apply *simp*
done
lemma *iMOD-plus*:
  $x \in [r, mod\ m] \Longrightarrow (x + k \in [r, mod\ m]) = (k\ mod\ m = 0)$
apply *safe*
 apply (*drule iMOD-modD*)+
 thm *mod-add-eq-imp-mod-0*[*THEN iffD1*]
 apply (*rule mod-add-eq-imp-mod-0*[*of x, THEN iffD1*])
 apply *simp*
apply (*simp add*: *mult-commute iMOD-plus-divisor-mult*)
done
corollary *iMOD-Suc*:
  $x \in [r, mod\ m] \Longrightarrow (Suc\ x \in [r, mod\ m]) = (m = Suc\ 0)$
apply (*simp add*: *iMOD-iff*, *safe*)
apply (*simp add*: *mod-Suc*, *split split-if-asm*)
apply *simp*+
done
lemma *iMOD-minus*:
  $[\![\ x \in [r, mod\ m]; k \leq x - r\ ]\!] \Longrightarrow (x - k \in [r, mod\ m]) = (k\ mod\ m = 0)$
apply *safe*
 apply (*clarsimp simp*: *iMOD-iff*)
 apply (*rule mod-add-eq-imp-mod-0*[*of x − k k, THEN iffD1*])
 apply *simp*
apply (*simp add*: *mult-commute iMOD-minus-divisor-mult*)
done
corollary *iMOD-pred*:
  $[\![\ x \in [r, mod\ m]; r < x\ ]\!] \Longrightarrow (x - Suc\ 0 \in [r, mod\ m]) = (m = Suc\ 0)$
apply *safe*
 thm *iMOD-Suc*[*of x − Suc 0, THEN iffD1*]
 apply (*simp add*: *iMOD-Suc*[*of x − Suc 0 r, THEN iffD1*])
apply (*simp add*: *iMOD-iff*)
done

lemma *iMODb-plus-divisor-mult*:
  $[\![\ x \in [r, mod\ m, c]; k * m \leq r + m * c - x\ ]\!] \Longrightarrow x + k * m \in [r, mod\ m, c]$
by (*fastsimp simp*: *iMODb-def*)
lemma *iMODb-plus-divisor-mult2*:
  $[\![\ x \in [r, mod\ m, c]; k \leq c - (x - r)\ div\ m\ ]\!] \Longrightarrow$
  $x + k * m \in [r, mod\ m, c]$
apply (*rule iMODb-plus-divisor-mult*, *assumption*)
apply (*clarsimp simp*: *iMODb-iff*)
apply (*drule mult-le-mono1*[*of - - m*])
apply (*simp add*: *diff-mult-distrib*
  mod-0-div-mult-cancel[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*]
  add-commute[*of r*] *mult-commute*[*of c*])
done

lemma *iMODb-plus-divisor*:
⟦ $x \in [r, \mathit{mod}\ m, c]$; $x < r + m * c$ ⟧ ⟹ $x + m \in [r, \mathit{mod}\ m, c]$
thm *less-mod-eq-imp-add-divisor-le*
by (*simp add*: *iMODb-iff less-mod-eq-imp-add-divisor-le*)
lemma *iMODb-minus-divisor-mult*:
⟦ $x \in [r, \mathit{mod}\ m, c]$; $r + k * m \leq x$ ⟧ ⟹ $x - k * m \in [r, \mathit{mod}\ m, c]$
thm *mod-diff-mult-self1*
by (*fastsimp simp*: *iMODb-def mod-diff-mult-self1*)

lemma *iMODb-plus*:
⟦ $x \in [r, \mathit{mod}\ m, c]$; $k \leq r + m * c - x$ ⟧ ⟹
$(x + k \in [r, \mathit{mod}\ m, c]) = (k\ \mathit{mod}\ m = 0)$
apply *safe*
thm *mod-add-eq-imp-mod-0*[*THEN iffD1*]
apply (*rule mod-add-eq-imp-mod-0*[*of x*, *THEN iffD1*])
apply (*simp add*: *iT-iff*)
apply *fastsimp*
done
corollary *iMODb-Suc*:
⟦ $x \in [r, \mathit{mod}\ m, c]$; $x < r + m * c$ ⟧ ⟹
$(\mathit{Suc}\ x \in [r, \mathit{mod}\ m, c]) = (m = \mathit{Suc}\ 0)$
apply (*rule iffI*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-Suc*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-1 iFROM-Suc iTILL-Suc*)
done
lemma *iMODb-minus*:
⟦ $x \in [r, \mathit{mod}\ m, c]$; $k \leq x - r$ ⟧ ⟹
$(x - k \in [r, \mathit{mod}\ m, c]) = (k\ \mathit{mod}\ m = 0)$
apply (*rule iffI*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-minus*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv iMOD-minus iTILL-minus*)
done
corollary *iMODb-pred*:
⟦ $x \in [r, \mathit{mod}\ m, c]$; $r < x$ ⟧ ⟹
$(x - \mathit{Suc}\ 0 \in [r, \mathit{mod}\ m, c]) = (m = \mathit{Suc}\ 0)$
apply (*rule iffI*)
thm *iMOD-pred*[*THEN iffD1*, *of x r m*]
apply (*subgoal-tac* $x \in [r, \mathit{mod}\ m] \wedge x - \mathit{Suc}\ 0 \in [r, \mathit{mod}\ m]$)
prefer *2*
apply (*simp add*: *iT-iff*)
apply (*clarsimp simp*: *iMOD-pred*)
apply (*fastsimp simp add*: *iMODb-iff*)
done

lemmas *iFROM-plus-minus* =
iFROM-plus
iFROM-Suc
iFROM-minus
iFROM-pred

thm *iFROM-plus-minus*

lemmas *iTILL-plus-minus* =
  iTILL-plus
  iTILL-Suc
  iTILL-minus
  iTILL-pred
thm *iTILL-plus-minus*

lemmas *iIN-plus-minus* =
  iIN-plus
  iIN-Suc
  iTILL-minus
  iIN-pred
thm *iIN-plus-minus*

lemmas *iMOD-plus-minus-divisor* =
  iMOD-plus-divisor-mult
  iMOD-plus-divisor
  iMOD-minus-divisor-mult
  iMOD-minus-divisor-mult2
  iMOD-minus-divisor
thm *iMOD-plus-minus-divisor*

lemmas *iMOD-plus-minus* =
  iMOD-plus
  iMOD-Suc
  iMOD-minus
  iMOD-pred
thm *iMOD-plus-minus*

lemmas *iMODb-plus-minus-divisor* =
  iMODb-plus-divisor-mult
  iMODb-plus-divisor-mult2
  iMODb-plus-divisor
  iMODb-minus-divisor-mult
thm *iMODb-plus-minus-divisor*

lemmas *iMODb-plus-minus* =
  iMODb-plus
  iMODb-Suc
  iMODb-minus
  iMODb-pred
thm *iMODb-plus-minus*

lemmas *iT-plus-minus* =
  iFROM-plus-minus
  iTILL-plus-minus
  iIN-plus-minus

iMOD-plus-minus-divisor
iMOD-plus-minus
iMODb-plus-minus-divisor
iMODb-plus-minus
thm *iT-plus-minus*

1.3 Relations between intervals

1.3.1 Auxiliary lemmata

lemma *Suc-in-imp-not-subset-iMOD*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow \neg\ S \subseteq [r, mod\ m]$
thm *iMOD-Suc*[*THEN iffD1*]
by (*blast intro*: *iMOD-Suc*[*THEN iffD1*])

corollary *Suc-in-imp-neq-iMOD*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow S \neq [r, mod\ m]$
by (*blast dest*: *Suc-in-imp-not-subset-iMOD*)

lemma *Suc-in-imp-not-subset-iMODb*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow \neg\ S \subseteq [r, mod\ m, c]$
apply (*rule ccontr*, *simp*)
apply (*frule subsetD*[*of - - n*], *assumption*)
apply (*drule subsetD*[*of - - Suc n*], *assumption*)
thm *iMODb-Suc*[*THEN iffD1*]
apply (*frule iMODb-Suc*[*THEN iffD1*])
apply (*drule iMODb-leD*[*of Suc n*])
apply *simp*
apply *blast*+
done
corollary *Suc-in-imp-neq-iMODb*:
$[\![n \in S; Suc\ n \in S; m \neq Suc\ 0]\!] \Longrightarrow S \neq [r, mod\ m, c]$
by (*blast dest*: *Suc-in-imp-not-subset-iMODb*)

1.3.2 Subset relation between intervals

lemma
iIN-iFROM-subset-same: $[n..,d] \subseteq [n..]$ **and**
iIN-iTILL-subset-same: $[n..,d] \subseteq [..n+d]$ **and**
iMOD-iFROM-subset-same: $[r, mod\ m] \subseteq [r..]$ **and**
iMODb-iTILL-subset-same: $[r, mod\ m, c] \subseteq [..r+m*c]$ **and**
iMODb-iIN-subset-same: $[r, mod\ m, c] \subseteq [r..,m*c]$ **and**
iMODb-iMOD-subset-same: $[r, mod\ m, c] \subseteq [r, mod\ m]$
by (*simp add*: *subset-iff iT-iff*)+

lemmas *iT-subset-same* =
iIN-iFROM-subset-same
iIN-iTILL-subset-same
iMOD-iFROM-subset-same
iMODb-iTILL-subset-same

iMODb-iIN-subset-same
iMODb-iTILL-subset-same
iMODb-iMOD-subset-same
thm *iT-subset-same*

lemma *iMODb-imp-iMOD*: $x \in [r,\ mod\ m,\ c] \Longrightarrow x \in [r,\ mod\ m]$
by (*blast intro*: *iMODb-iMOD-subset-same*)
lemma *iMOD-imp-iMODb*:
$[\![\ x \in [r,\ mod\ m];\ x \leq r + m * c\ ]\!] \Longrightarrow x \in [r,\ mod\ m,\ c]$
by (*simp add*: *iT-iff*)

lemma *iMOD-singleton-subset-conv*: $([r,\ mod\ m] \subseteq \{a\}) = (r = a \wedge m = 0)$
apply (*rule iffI*)
apply (*simp add*: *subset-singleton-conv iT-not-empty*)
apply (*simp add*: *set-eq-iff iT-iff*)
apply (*frule-tac x=r* **in** *spec*, *drule-tac x=r+m* **in** *spec*)
apply *simp*
apply (*simp add*: *iMOD-0 iIN-0*)
done
lemma *iMOD-singleton-eq-conv*: $([r,\ mod\ m] = \{a\}) = (r = a \wedge m = 0)$
apply (*rule-tac t=*$[r,\ mod\ m] = \{a\}$ **and** *s=*$[r,\ mod\ m] \subseteq \{a\}$ **in** *subst*)
apply (*simp add*: *subset-singleton-conv iMOD-not-empty*)
apply (*simp add*: *iMOD-singleton-subset-conv*)
done

lemma *iMODb-singleton-subset-conv*:
$([r,\ mod\ m,\ c] \subseteq \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
apply (*rule iffI*)
apply (*simp add*: *subset-singleton-conv iT-not-empty*)
apply (*simp add*: *set-eq-iff iT-iff*)
apply (*frule-tac x=r* **in** *spec*, *drule-tac x=r+m* **in** *spec*)
apply *clarsimp*
apply (*fastsimp simp*: *iMODb-0 iMODb-mod-0 iIN-0*)
done
lemma *iMODb-singleton-eq-conv*:
$([r,\ mod\ m,\ c] = \{a\}) = (r = a \wedge (m = 0 \vee c = 0))$
apply (*rule-tac t=*$[r,\ mod\ m,\ c] = \{a\}$ **and** *s=*$[r,\ mod\ m,\ c] \subseteq \{a\}$ **in** *subst*)
apply (*simp add*: *subset-singleton-conv iMODb-not-empty*)
apply (*simp add*: *iMODb-singleton-subset-conv*)
done

lemma *iMODb-subset-imp-divisor-mod-0*:

⟦ *0 < c'*; [*r'*, *mod m'*, *c'*] ⊆ [*r*, *mod m*, *c*] ⟧ ⟹ *m' mod m = 0*
apply (*simp add*: *subset-iff iMODb-iff*)
apply (*drule gr0-imp-self-le-mult1*[*of* - *m'*])
thm *mod-add-eq-imp-mod-0*[*of r' m' m*]
apply (*rule mod-add-eq-imp-mod-0*[*of r' m' m*, *THEN iffD1*])
apply (*frule-tac x=r'* **in** *spec*, *drule-tac x=r'+m'* **in** *spec*)
apply *simp*
done
lemma *iMOD-subset-imp-divisor-mod-0*:
[*r'*, *mod m'*] ⊆ [*r*, *mod m*] ⟹ *m' mod m = 0*
apply (*simp add*: *subset-iff iMOD-iff*)
thm *mod-add-eq-imp-mod-0*[*of r' m' m*]
apply (*rule mod-add-eq-imp-mod-0*[*of r' m' m*, *THEN iffD1*])
apply *simp*
done

lemma *iMOD-subset-imp-iMODb-subset*:
⟦ [*r'*, *mod m'*] ⊆ [*r*, *mod m*]; *r' + m' * c' ≤ r + m * c* ⟧ ⟹
[*r'*, *mod m'*, *c'*] ⊆ [*r*, *mod m*, *c*]
by (*simp add*: *subset-iff iT-iff*)

lemma *iMODb-subset-imp-iMOD-subset*:
⟦ [*r'*, *mod m'*, *c'*] ⊆ [*r*, *mod m*, *c*]; *0 < c'* ⟧ ⟹
[*r'*, *mod m'*] ⊆ [*r*, *mod m*]
thm *subsetD*
apply (*frule subsetD*[*of* - - *r'*])
apply (*simp add*: *iMODb-iff*)
thm *subsetI*
apply (*rule subsetI*)
apply (*simp add*: *iMOD-iff iMODb-iff*, *clarify*)
thm *mod-eq-mod-0-imp-mod-eq*
apply (*drule mod-eq-mod-0-imp-mod-eq*[**where** *m=m* **and** *m'=m'*])
thm *iMODb-subset-imp-divisor-mod-0*
apply (*simp add*: *iMODb-subset-imp-divisor-mod-0*)
apply *simp*
done

lemma *iMODb-0-iMOD-subset-conv*:
([*r'*, *mod m'*, *0*] ⊆ [*r*, *mod m*]) =
(*r' mod m = r mod m ∧ r ≤ r'*)
by (*simp add*: *iMODb-0 iIN-0 singleton-subset-conv iMOD-iff*)

lemma *iFROM-subset-conv*: ([*n'*...] ⊆ [*n*...]) = (*n ≤ n'*)
by (*simp add*: *iFROM-def*)

lemma *iFROM-iMOD-subset-conv*: ([*n'*...] ⊆ [*r*, *mod m*]) = (*r ≤ n' ∧ m = Suc 0*)
apply (*rule iffI*)
apply (*rule conjI*)

thm *iMin-subset*[*OF iFROM-not-empty*]
apply (*drule iMin-subset*[*OF iFROM-not-empty*])
apply (*simp add*: *iT-Min*)
apply (*rule ccontr*)
thm *Suc-in-imp-not-subset-iMOD*
apply (*cut-tac Suc-in-imp-not-subset-iMOD*[*of* n' $[n'...]$ m r])
apply (*simp add*: *iT-iff*)+
apply (*simp add*: *subset-iff iT-iff*)
done

lemma *iIN-subset-conv*: $([n'...,d'] \subseteq [n...,d]) = (n \leq n' \wedge n'+d' \leq n+d)$
apply (*rule iffI*)
apply (*frule iMin-subset*[*OF iIN-not-empty*])
apply (*drule Max-subset*[*OF iIN-not-empty - iIN-finite*])
apply (*simp add*: *iIN-Min iIN-Max*)
apply (*simp add*: *subset-iff iIN-iff*)
done

lemma *iIN-iFROM-subset-conv*: $([n'...,d'] \subseteq [n...]) = (n \leq n')$
by (*fastsimp simp*: *subset-iff iFROM-iff iIN-iff*)

lemma *iIN-iTILL-subset-conv*: $([n'...,d'] \subseteq [...n]) = (n' + d' \leq n)$
by (*fastsimp simp*: *subset-iff iT-iff*)

lemma *iIN-iMOD-subset-conv*:
$0 < d' \Longrightarrow ([n'...,d'] \subseteq [r, mod\ m]) = (r \leq n' \wedge m = Suc\ 0)$
apply (*rule iffI*)
apply (*frule iMin-subset*[*OF iIN-not-empty*])
apply (*simp add*: *iT-Min*)
apply (*subgoal-tac* $n' \in [n'...,d']$)
prefer *2*
apply (*simp add*: *iIN-iff*)
apply (*rule ccontr*)
thm *Suc-in-imp-not-subset-iMOD*
apply (*frule Suc-in-imp-not-subset-iMOD*[**where** r=r **and** m=m])
apply (*simp add*: *iIN-Suc*)+
apply (*simp add*: *iMOD-1 iIN-iFROM-subset-conv*)
done
lemma *iIN-iMODb-subset-conv*:
$0 < d' \Longrightarrow$
$([n'...,d'] \subseteq [r, mod\ m, c]) =$
$(r \leq n' \wedge m = Suc\ 0 \wedge n' + d' \leq r + m * c)$
apply (*rule iffI*)
thm *subset-trans*[*OF - iMODb-iMOD-subset-same*]
apply (*frule subset-trans*[*OF - iMODb-iMOD-subset-same*])
apply (*simp add*: *iIN-iMOD-subset-conv iMODb-mod-1 iIN-subset-conv*)
apply (*clarsimp simp*: *iMODb-mod-1 iIN-subset-conv*)
done

lemma *iTILL-subset-conv*: $([...n'] \subseteq [...n]) = (n' \leq n)$
by (*simp add*: *iTILL-def*)
lemma *iTILL-iFROM-subset-conv*: $([...n'] \subseteq [n...]) = (n = 0)$
apply (*rule iffI*)
apply (*drule subsetD*[*of* - - *0*])
apply (*simp add*: *iT-iff*)+
apply (*simp add*: *iFROM-0*)
done
lemma *iTILL-iIN-subset-conv*: $([...n'] \subseteq [n...,d]) = (n = 0 \wedge n' \leq d)$
apply (*rule iffI*)
apply (*frule iMin-subset*[*OF iTILL-not-empty*])
apply (*drule Max-subset*[*OF iTILL-not-empty - iIN-finite*])
apply (*simp add*: *iT-Min iT-Max*)
apply (*simp add*: *iIN-0-iTILL-conv iTILL-subset-conv*)
done
lemma *iTILL-iMOD-subset-conv*:
$0 < n' \Longrightarrow ([...n'] \subseteq [r, mod\ m]) = (r = 0 \wedge m = Suc\ 0)$
apply (*drule iIN-iMOD-subset-conv*[*of n' 0 r m*])
apply (*simp add*: *iIN-0-iTILL-conv*)
done
lemma *iTILL-iMODb-subset-conv*:
$0 < n' \Longrightarrow ([...n'] \subseteq [r, mod\ m, c]) = (r = 0 \wedge m = Suc\ 0 \wedge n' \leq r + m * c)$
apply (*drule iIN-iMODb-subset-conv*[*of n' 0 r m c*])
apply (*simp add*: *iIN-0-iTILL-conv*)
done

lemma *iMOD-iFROM-subset-conv*: $([r', mod\ m']) \subseteq [n...] = (n \leq r')$
by (*fastsimp simp*: *subset-iff iT-iff*)
lemma *iMODb-iFROM-subset-conv*: $([r', mod\ m', c'] \subseteq [n...]) = (n \leq r')$
by (*fastsimp simp*: *subset-iff iT-iff*)
lemma *iMODb-iIN-subset-conv*:
$([r', mod\ m', c'] \subseteq [n...,d]) = (n \leq r' \wedge r' + m' * c' \leq n + d)$
by (*fastsimp simp*: *subset-iff iT-iff*)
lemma *iMODb-iTILL-subset-conv*:
$([r', mod\ m', c'] \subseteq [...n]) = (r' + m' * c' \leq n)$
by (*fastsimp simp*: *subset-iff iT-iff*)

lemma *iMOD-0-subset-conv*: $([r', mod\ 0] \subseteq [r, mod\ m]) = (r'\ mod\ m = r\ mod\ m \wedge\ r \leq r')$
by (*fastsimp simp*: *iMOD-0 iIN-0 singleton-subset-conv iMOD-iff*)

lemma *iMOD-subset-conv*: $0 < m \Longrightarrow$
$([r', mod\ m'] \subseteq [r, mod\ m]) =$
$(r'\ mod\ m = r\ mod\ m\ \wedge\ r \leq r'\ \wedge\ m'\ mod\ m = 0)$
apply (*rule iffI*)
apply (*frule subsetD*[*of* - - *r'*])
apply (*simp add*: *iMOD-iff*)
apply (*drule iMOD-subset-imp-divisor-mod-0*)

apply (*simp add*: *iMOD-iff*)
apply (*rule subsetI*)
apply (*simp add*: *iMOD-iff*, *elim conjE*)
thm *mod-eq-mod-0-imp-mod-eq*
apply (*drule mod-eq-mod-0-imp-mod-eq*[**where** m'=m' **and** m=m])
apply *simp*+
done

lemma *iMODb-subset-mod-0-conv*:
$([r', \mathit{mod}\ m', c'] \subseteq [r, \mathit{mod}\ 0, c\ ]) = (r'=r \wedge (m'=0 \vee c'=0))$
by (*simp add*: *iMODb-mod-0 iIN-0 iMODb-singleton-subset-conv*)
lemma *iMODb-subset-0-conv*:
$([r', \mathit{mod}\ m', c'] \subseteq [r, \mathit{mod}\ m, 0\ ]) = (r'=r \wedge (m'=0 \vee c'=0))$
by (*simp add*: *iMODb-0 iIN-0 iMODb-singleton-subset-conv*)
lemma *iMODb-0-subset-conv*:
$([r', \mathit{mod}\ m', 0] \subseteq [r, \mathit{mod}\ m, c\ ]) = (r' \in [r, \mathit{mod}\ m, c])$
by (*simp add*: *iMODb-0 iIN-0*)
lemma *iMODb-mod-0-subset-conv*:
$([r', \mathit{mod}\ 0, c'] \subseteq [r, \mathit{mod}\ m, c\ ]) = (r' \in [r, \mathit{mod}\ m, c])$
by (*simp add*: *iMODb-mod-0 iIN-0*)

lemma *iMODb-subset-conv'*: $[\![\ 0 < c;\ 0 < c'\ ]\!] \Longrightarrow$
$([r', \mathit{mod}\ m', c'] \subseteq [r, \mathit{mod}\ m, c]) =$
$(r'\ \mathit{mod}\ m = r\ \mathit{mod}\ m\ \wedge\ r \le r'\ \wedge\ m'\ \mathit{mod}\ m = 0\ \wedge$
$r' + m' * c' \le r + m * c)$
apply (*rule iffI*)
thm *iMODb-subset-imp-iMOD-subset*
apply (*frule iMODb-subset-imp-iMOD-subset*, *assumption*)
apply (*drule iMOD-subset-imp-divisor-mod-0*)
apply (*frule subsetD*[*OF - iMinI-ex2*[*OF iMODb-not-empty*]])
apply (*drule Max-subset*[*OF iMODb-not-empty - iMODb-finite*])
apply (*simp add*: *iMODb-iff iMODb-Min iMODb-Max*)
apply (*elim conjE*)
thm *iMOD-subset-imp-iMODb-subset*
apply (*case-tac* $m = 0$, *simp add*: *iMODb-mod-0*)
apply (*simp add*: *iMOD-subset-imp-iMODb-subset iMOD-subset-conv*)
done
lemma *iMODb-subset-conv*: $[\![\ 0 < m';\ 0 < c'\ ]\!] \Longrightarrow$
$([r', \mathit{mod}\ m', c'] \subseteq [r, \mathit{mod}\ m, c]) =$
$(r'\ \mathit{mod}\ m = r\ \mathit{mod}\ m\ \wedge\ r \le r'\ \wedge\ m'\ \mathit{mod}\ m = 0\ \wedge$
$r' + m' * c' \le r + m * c)$
apply (*case-tac* $c = 0$)
apply (*simp add*: *iMODb-0 iIN-0 iMODb-singleton-subset-conv linorder-not-le*, *intro impI*)
apply (*case-tac* $r' < r$, *simp*)
apply (*simp add*: *linorder-not-less*)
thm *add-less-le-mono*[*of* 0 $m' * c'$ r r']

apply (*insert add-less-le-mono*[*of* 0 $m' * c'$ r r'])
apply *simp*
apply (*simp add*: *iMODb-subset-conv'*)
done

lemma *iMODb-iMOD-subset-conv*: $0 < c' \Longrightarrow$
$([r', \ mod\ m', \ c'] \subseteq [r, \ mod\ m]) =$
$(r' \ mod\ m = r \ mod\ m \ \wedge \ r \leq r' \ \wedge \ m' \ mod\ m = 0)$
apply (*rule iffI*)
thm *subsetD*[*OF - iMinI-ex2*[*OF iMODb-not-empty*]]
apply (*frule subsetD*[*OF - iMinI-ex2*[*OF iMODb-not-empty*]])
apply (*simp add*: *iMODb-Min iMOD-iff*, *elim conjE*)
apply (*simp add*: *iMODb-iMOD-iTILL-conv*)
apply (*subgoal-tac* [r', *mod* m', c'] $\subseteq$ [r, *mod* m] $\cap$ [$...r' + m' * c'$])
prefer *2*
apply (*simp add*: *iMODb-iMOD-iTILL-conv*)
thm *iMOD-iTILL-iMODb-conv iMODb-subset-imp-divisor-mod-0*
apply (*simp add*: *iMOD-iTILL-iMODb-conv iMODb-subset-imp-divisor-mod-0*)
thm *subset-trans*[*OF iMODb-iMOD-subset-same*]
apply (*rule subset-trans*[*OF iMODb-iMOD-subset-same*])
apply (*case-tac* $m = 0$, *simp*)
apply (*simp add*: *iMOD-subset-conv*)
done

lemmas *iT-subset-conv* =
iFROM-subset-conv
iFROM-iMOD-subset-conv
iTILL-subset-conv
iTILL-iFROM-subset-conv
iTILL-iIN-subset-conv
iTILL-iMOD-subset-conv
iTILL-iMODb-subset-conv
iIN-subset-conv
iIN-iFROM-subset-conv
iIN-iTILL-subset-conv
iIN-iMOD-subset-conv
iIN-iMODb-subset-conv
iMOD-subset-conv
iMOD-iFROM-subset-conv
iMODb-subset-conv'
iMODb-subset-conv
iMODb-iFROM-subset-conv
iMODb-iIN-subset-conv
iMODb-iTILL-subset-conv
iMODb-iMOD-subset-conv
thm *iT-subset-conv*

lemma *iFROM-subset*: $n \leq n' \Longrightarrow [n'\ldots] \subseteq [n\ldots]$
by (*simp add*: *iFROM-subset-conv*)
lemma *not-iFROM-iIN-subset*: $\neg\ [n'\ldots] \subseteq [n\ldots,d]$
apply (*rule ccontr*, *simp*)
apply (*drule subsetD*[*of* - - *max* n' (*Suc* (n + d))])
apply (*simp add*: *iFROM-iff*)
apply (*simp add*: *iIN-iff*)
done
lemma *not-iFROM-iTILL-subset*: $\neg\ [n'\ldots] \subseteq [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv* [*symmetric*] *not-iFROM-iIN-subset*)
lemma *not-iFROM-iMOD-subset*: $m \neq Suc\ 0 \Longrightarrow \neg\ [n'\ldots] \subseteq [r, mod\ m]$
apply (*rule Suc-in-imp-not-subset-iMOD*[*of* n'])
apply (*simp add*: *iT-iff*)+
done
lemma *not-iFROM-iMODb-subset*: $\neg\ [n'\ldots] \subseteq [r, mod\ m, c]$
thm *infinite-not-subset-finite*
by (*rule infinite-not-subset-finite*[*OF iFROM-infinite iMODb-finite*])

lemma *iIN-subset*: $[\![\ n \leq n';\ n' + d' \leq n + d\ ]\!] \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots,d]$
by (*simp add*: *iIN-subset-conv*)
lemma *iIN-iFROM-subset*: $n \leq n' \Longrightarrow [n'\ldots,d'] \subseteq [n\ldots]$
by (*simp add*: *subset-iff iT-iff*)
lemma *iIN-iTILL-subset*: $n' + d' \leq n \Longrightarrow [n'\ldots,d'] \subseteq [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-subset*)
lemma *not-iIN-iMODb-subset*: $[\![\ 0 < d';\ m \neq Suc\ 0\ ]\!] \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r, mod\ m, c]$
apply (*rule Suc-in-imp-not-subset-iMODb*[*of* n'])
apply (*simp add*: *iIN-iff*)+
done
lemma *not-iIN-iMOD-subset*: $[\![\ 0 < d';\ m \neq Suc\ 0\ ]\!] \Longrightarrow \neg\ [n'\ldots,d'] \subseteq [r, mod\ m]$
apply (*rule ccontr*, *simp*)
apply (*case-tac* $r \leq n' + d'$)
thm *iIN-iTILL-subset*[*OF order-refl*]
thm *Int-greatest*[*OF* - *iIN-iTILL-subset*[*OF order-refl*]]
apply (*drule Int-greatest*[*OF* - *iIN-iTILL-subset*[*OF order-refl*]])
thm *iMOD-iTILL-iMODb-conv not-iIN-iMODb-subset*
apply (*simp add*: *iMOD-iTILL-iMODb-conv not-iIN-iMODb-subset*)
apply (*drule subsetD*[*of* - - n'+d'])
apply (*simp add*: *iT-iff*)+
done
lemma *iTILL-subset*: $n' \leq n \Longrightarrow [\ldots n'] \subseteq [\ldots n]$
by (*rule iTILL-subset-conv*[*THEN iffD2*])
lemma *iTILL-iFROM-subset*: $([\ldots n'] \subseteq [0\ldots])$
by (*simp add*: *iFROM-0*)
lemma *iTILL-iIN-subset*: $n' \leq d \Longrightarrow ([\ldots n'] \subseteq [0\ldots,d])$
by (*simp add*: *iIN-0-iTILL-conv iTILL-subset*)

thm *not-iIN-iMOD-subset*
lemma *not-iTILL-iMOD-subset*:
⟦ $0 < n'$; $m \neq Suc\ 0$ ⟧ ⟹ ¬ $[\ldots n']$ ⊆ $[r, mod\ m]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *not-iIN-iMOD-subset*)
lemma *not-iTILL-iMODb-subset*:
⟦ $0 < n'$; $m \neq Suc\ 0$ ⟧ ⟹ ¬ $[\ldots n']$ ⊆ $[r, mod\ m, c]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *not-iIN-iMODb-subset*)

lemma *iMOD-iFROM-subset*: $n \leq r' \Longrightarrow [r', mod\ m'] \subseteq [n\ldots]$
by (*rule iMOD-iFROM-subset-conv*[*THEN iffD2*])
lemma *not-iMOD-iIN-subset*: $0 < m' \Longrightarrow$ ¬ $[r', mod\ m'] \subseteq [n\ldots,d]$
by (*rule infinite-not-subset-finite*[*OF iMOD-infinite iIN-finite*])
lemma *not-iMOD-iTILL-subset*: $0 < m' \Longrightarrow$ ¬ $[r', mod\ m'] \subseteq [\ldots n]$
by (*rule infinite-not-subset-finite*[*OF iMOD-infinite iTILL-finite*])
thm *iMOD-subset-conv*
lemma *iMOD-subset*:
⟦ $r \leq r'$; $r'\ mod\ m = r\ mod\ m$; $m'\ mod\ m = 0$ ⟧ ⟹ $[r', mod\ m'] \subseteq [r, mod\ m]$
apply (*case-tac* $m = 0$, *simp*)
apply (*simp add*: *iMOD-subset-conv*)
done
lemma *not-iMOD-iMODb-subset*: $0 < m' \Longrightarrow$ ¬ $[r', mod\ m'] \subseteq [r, mod\ m, c]$
by (*rule infinite-not-subset-finite*[*OF iMOD-infinite iMODb-finite*])

lemma *iMODb-iFROM-subset*: $n \leq r' \Longrightarrow [r', mod\ m', c'] \subseteq [n\ldots]$
thm *iMODb-iFROM-subset-conv*[*THEN iffD2*]
by (*rule iMODb-iFROM-subset-conv*[*THEN iffD2*])
lemma *iMODb-iTILL-subset*:
$r' + m' * c' \leq n \Longrightarrow [r', mod\ m', c'] \subseteq [\ldots n]$
by (*rule iMODb-iTILL-subset-conv*[*THEN iffD2*])
thm *iMODb-iIN-subset-conv*
lemma *iMODb-iIN-subset*:
⟦ $n \leq r'$; $r' + m' * c' \leq n + d$ ⟧ ⟹ $[r', mod\ m', c'] \subseteq [n\ldots,d]$
by (*simp add*: *iMODb-iIN-subset-conv*)
thm *iMODb-iMOD-subset-conv*
lemma *iMODb-iMOD-subset*:
⟦ $r \leq r'$; $r'\ mod\ m = r\ mod\ m$; $m'\ mod\ m = 0$ ⟧ ⟹ $[r', mod\ m', c'] \subseteq [r, mod\ m]$
apply (*case-tac* $c' = 0$)
apply (*simp add*: *iMODb-0 iIN-0 iMOD-iff*)
thm *iMODb-iMOD-subset-conv*
apply (*simp add*: *iMODb-iMOD-subset-conv*)
done

thm *iMODb-subset-conv*
lemma *iMODb-subset*:
⟦ $r \leq r'$; $r'\ mod\ m = r\ mod\ m$; $m'\ mod\ m = 0$; $r' + m' * c' \leq r + m * c$ ⟧ ⟹

$[r', mod\ m', c'] \subseteq [r, mod\ m, c]$
apply (*case-tac* $m' = 0$)
apply (*simp add*: *iMODb-mod-0 iIN-0 iMODb-iff*)
apply (*case-tac* $c' = 0$)
apply (*simp add*: *iMODb-0 iIN-0 iMODb-iff*)
apply (*simp add*: *iMODb-subset-conv*)
done

lemma *iFROM-trans*: $[\![y \in [x...]; z \in [y...]]\!] \Longrightarrow z \in [x...]$
by (*rule subsetD*[*OF iFROM-subset*[*OF iFROM-D*]])
lemma *iTILL-trans*: $[\![y \in [...x]; z \in [...y]]\!] \Longrightarrow z \in [...x]$
by (*rule subsetD*[*OF iTILL-subset*[*OF iTILL-D*]])
thm *iIN-subset*
lemma *iIN-trans*:
$[\![y \in [x...,d]; z \in [y...,d']; d' \leq x + d - y]\!] \Longrightarrow z \in [x...,d]$
by *fastsimp*
lemma *iMOD-trans*:
$[\![y \in [x, mod\ m]; z \in [y, mod\ m]]\!] \Longrightarrow z \in [x, mod\ m]$
by (*rule subsetD*[*OF iMOD-subset*[*OF iMOD-geD iMOD-modD mod-self*]])
lemma *iMODb-trans*:
$[\![y \in [x, mod\ m, c]; z \in [y, mod\ m, c']; m * c' \leq x + m * c - y]\!] \Longrightarrow$
$z \in [x, mod\ m, c]$
by *fastsimp*
lemma *iMODb-trans'*:
$[\![y \in [x, mod\ m, c]; z \in [y, mod\ m, c']; c' \leq x\ div\ m + c - y\ div\ m]\!] \Longrightarrow$
$z \in [x, mod\ m, c]$
apply (*rule iMODb-trans*[**where** c'=c'], *assumption*+)
apply (*frule iMODb-geD*, *frule div-le-mono*[*of x y m*])
apply (*simp add*: *add-commute*[*of - c*] *add-commute*[*of - m*c*])
apply (*drule mult-le-mono*[*OF le-refl*, *of - - m*])
apply (*simp add*: *add-mult-distrib2 diff-mult-distrib2 mult-div-cancel*)
apply (*simp add*: *iMODb-iff*)
done

1.3.3 Equality of intervals

lemma *iFROM-eq-conv*: $([n...] = [n'...]) = (n = n')$
apply (*rule iffI*)
apply (*drule set-eq-subset*[*THEN iffD1*])
apply (*simp add*: *iFROM-subset-conv*)
apply *simp*
done

lemma *iIN-eq-conv*: $([n...,d] = [n'...,d']) = (n = n' \wedge d = d')$
apply (*rule iffI*)
apply (*drule set-eq-subset*[*THEN iffD1*])
apply (*simp add*: *iIN-subset-conv*)

apply *simp*
done

lemma *iTILL-eq-conv*: $([\ldots n] = [\ldots n']) = (n = n')$
thm *iIN-eq-conv*[*of 0 n 0 n'*]
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-eq-conv*)

thm *iMOD-singleton-eq-conv*
lemma *iMOD-0-eq-conv*: $([r, mod\ 0] = [r', mod\ m']) = (r = r' \wedge m' = 0)$
apply (*simp add*: *iMOD-0 iIN-0*)
thm *iMOD-singleton-eq-conv*
apply (*simp add*: *iMOD-singleton-eq-conv eq-sym-conv*[*of* {*r*}] *eq-sym-conv*[*of r*])
done
lemma *iMOD-eq-conv*: $0 < m \Longrightarrow ([r, mod\ m] = [r', mod\ m']) = (r = r' \wedge m = m')$
apply (*case-tac* $m' = 0$)
apply (*simp add*: *eq-sym-conv*[*of* $[r, mod\ m]$] *iMOD-0-eq-conv*)
apply (*rule iffI*)
apply (*fastsimp simp add*: *set-eq-subset iMOD-subset-conv*)
apply *simp*
done

thm *iMODb-singleton-eq-conv*
lemma *iMODb-mod-0-eq-conv*:
$([r, mod\ 0, c] = [r', mod\ m', c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
apply (*simp add*: *iMODb-mod-0 iIN-0*)
apply (*fastsimp simp*: *iMODb-singleton-eq-conv eq-sym-conv*[*of* {*r*}])
done
lemma *iMODb-0-eq-conv*:
$([r, mod\ m, 0] = [r', mod\ m', c']) = (r = r' \wedge (m' = 0 \vee c' = 0))$
apply (*simp add*: *iMODb-0 iIN-0*)
apply (*fastsimp simp*: *iMODb-singleton-eq-conv eq-sym-conv*[*of* {*r*}])
done

lemma *iMODb-eq-conv*: $[\![\ 0 < m;\ 0 < c\ ]\!] \Longrightarrow$
$([r, mod\ m, c] = [r', mod\ m', c']) = (r = r' \wedge m = m' \wedge c = c')$
apply (*case-tac* $c' = 0$)
apply (*simp add*: *iMODb-0 iIN-0 iMODb-singleton-eq-conv*)
apply (*rule iffI*)
apply (*fastsimp simp*: *set-eq-subset iMODb-subset-conv'*)
apply *simp*
done

lemma *iMOD-iFROM-eq-conv*: $([n\ldots] = [r, mod\ m]) = (n = r \wedge m = Suc\ 0)$
by (*fastsimp simp*: *iMOD-1*[*symmetric*] *iMOD-eq-conv*)
thm *iMODb-singleton-eq-conv*
lemma *iMODb-iIN-0-eq-conv*:
$([n\ldots,0] = [r, mod\ m, c]) = (n = r \wedge (m = 0 \vee c = 0))$
by (*simp add*: *iIN-0 eq-commute*[*of* {*n*}] *eq-commute*[*of n*] *iMODb-singleton-eq-conv*)

lemma *iMODb-iIN-eq-conv*:
$0 < d \Longrightarrow ([n\ldots,d] = [r,\ mod\ m,\ c]) = (n = r \wedge m = Suc\ 0 \wedge c = d)$
by (*fastsimp simp*: *iMODb-mod-1*[*symmetric*] *iMODb-eq-conv*)

1.3.4 Inequality of intervals

lemma *iFROM-iIN-neq*: $[n'\ldots] \neq [n\ldots,d]$
apply (*rule ccontr*)
apply (*insert iFROM-infinite*[*of n'*], *insert iIN-finite*[*of n d*])
apply *simp*
done
corollary *iFROM-iTILL-neq*: $[n'\ldots] \neq [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iFROM-iIN-neq*)
corollary *iFROM-iMOD-neq*: $m \neq Suc\ 0 \Longrightarrow [n\ldots] \neq [r,\ mod\ m]$
apply (*subgoal-tac* $n \in [n\ldots]$)
prefer *2*
apply (*simp add*: *iFROM-iff*)
apply (*simp add*: *Suc-in-imp-neq-iMOD iFROM-Suc*)
done
corollary *iFROM-iMODb-neq*: $[n\ldots] \neq [r,\ mod\ m,\ c]$
apply (*rule ccontr*)
apply (*insert iMODb-finite*[*of r m c*], *insert iFROM-infinite*[*of n*])
apply *simp*
done

corollary *iIN-iMOD-neq*: $0 < m \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m]$
apply (*rule ccontr*)
apply (*insert iMOD-infinite*[*of m r*], *insert iIN-finite*[*of n d*])
apply *simp*
done
corollary *iIN-iMODb-neq2*: $[\![\ m \neq Suc\ 0;\ 0 < d\ ]\!] \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m,\ c]$
apply (*subgoal-tac* $n \in [n\ldots,d]$)
prefer *2*
apply (*simp add*: *iIN-iff*)
apply (*simp add*: *Suc-in-imp-neq-iMODb iIN-Suc*)
done
lemma *iIN-iMODb-neq*: $[\![\ 2 \leq m;\ 0 < c\ ]\!] \Longrightarrow [n\ldots,d] \neq [r,\ mod\ m,\ c]$
apply (*simp add*: *nat-ge2-conv*, *elim conjE*)
apply (*case-tac d=0*)
thm *iMODb-singleton-eq-conv*
apply (*rule not-sym*)
apply (*simp add*: *iIN-0 iMODb-singleton-eq-conv*)
apply (*simp add*: *iIN-iMODb-neq2*)
done

lemma *iTILL-iIN-neq*: $0 < n \Longrightarrow [\ldots n'] \neq [n\ldots,d]$
by (*fastsimp simp*: *set-eq-iff iT-iff*)
corollary *iTILL-iMOD-neq*: $0 < m \Longrightarrow [\ldots n] \neq [r,\ mod\ m]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-iMOD-neq*)

corollary *iTILL-iMODb-neq*:
⟦ $m \neq Suc\ 0$; $0 < n$ ⟧ ⟹ $[\ldots n] \neq [r, mod\ m, c]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-iMODb-neq2*)

lemma *iMOD-iMODb-neq*: $0 < m \Longrightarrow [r, mod\ m] \neq [r', mod\ m', c']$
apply (*rule ccontr*)
apply (*insert iMODb-finite*[*of r' m' c'*], *insert iMOD-infinite*[*of m r*])
apply *simp*
done

lemmas *iT-neq* =
iFROM-iTILL-neq iFROM-iIN-neq iFROM-iMOD-neq iFROM-iMODb-neq
iTILL-iIN-neq iTILL-iMOD-neq iTILL-iMODb-neq
iIN-iMOD-neq iIN-iMODb-neq iIN-iMODb-neq2
iMOD-iMODb-neq
thm *iT-neq*

1.4 Union and intersection of intervals

lemma *iFROM-union'*: $[n\ldots] \cup [n'\ldots] = [min\ n\ n'\ldots]$
by (*fastsimp simp*: *iFROM-iff*)
corollary *iFROM-union*: $n \leq n' \Longrightarrow [n\ldots] \cup [n'\ldots] = [n\ldots]$
by (*simp add*: *iFROM-union' min-eqL*)

lemma *iFROM-inter'*: $[n\ldots] \cap [n'\ldots] = [max\ n\ n'\ldots]$
by (*fastsimp simp*: *iFROM-iff*)
corollary *iFROM-inter*: $n' \leq n \Longrightarrow [n\ldots] \cap [n'\ldots] = [n\ldots]$
by (*simp add*: *iFROM-inter' max-eqL*)

lemma *iTILL-union'*: $[\ldots n] \cup [\ldots n'] = [\ldots max\ n\ n']$
by (*fastsimp simp*: *iTILL-iff*)
corollary *iTILL-union*: $n' \leq n \Longrightarrow [\ldots n] \cup [\ldots n'] = [\ldots n]$
by (*simp add*: *iTILL-union' max-eqL*)

lemma *iTILL-iFROM-union*: $n \leq n' \Longrightarrow [\ldots n'] \cup [n\ldots] = UNIV$
by (*fastsimp simp*: *iT-iff*)

lemma *iTILL-inter'*: $[\ldots n] \cap [\ldots n'] = [\ldots min\ n\ n']$
by (*fastsimp simp*: *iT-iff*)
corollary *iTILL-inter*: $n \leq n' \Longrightarrow [\ldots n] \cap [\ldots n'] = [\ldots n]$
by (*simp add*: *iTILL-inter' min-eqL*)

Union and intersection for iIN only when there intersection is not empty and none of them is other's subset, for instance: .. n .. n+d .. n' .. n'+d'

lemma *iIN-union*:
⟦ $n \leq n'$; $n' \leq Suc\ (n + d)$; $n + d \leq n' + d'$ ⟧ ⟹
$[n\ldots,d] \cup [n'\ldots,d'] = [n\ldots,n' - n + d']$
by (*fastsimp simp add*: *iIN-iff*)

lemma *iIN-append-union*:
$[n...,d] \cup [n + d...,d'] = [n...,d + d']$
by (*simp add*: *iIN-union*)
lemma *iIN-append-union-Suc*:
$[n...,d] \cup [Suc\ (n + d)...,d'] = [n...,Suc\ (d + d')]$
by (*simp add*: *iIN-union*)
lemma *iIN-append-union-pred*:
$0 < d \Longrightarrow [n...,d - Suc\ 0] \cup [n + d...,d'] = [n...,d + d']$
by (*simp add*: *iIN-union*)

lemma *iIN-iFROM-union*:
$n' \leq Suc\ (n + d) \Longrightarrow [n...,d] \cup [n'...] = [min\ n\ n'...]$
by (*fastsimp simp*: *iIN-iff*)

lemma *iIN-iFROM-append-union*:
$[n...,d] \cup [n + d...] = [n...]$
by (*simp add*: *iIN-iFROM-union min-eqL*)
lemma *iIN-iFROM-append-union-Suc*:
$[n...,d] \cup [Suc\ (n + d)...] = [n...]$
by (*simp add*: *iIN-iFROM-union min-eqL*)
lemma *iIN-iFROM-append-union-pred*:
$0 < d \Longrightarrow [n...,d - Suc\ 0] \cup [n + d...] = [n...]$
by (*simp add*: *iIN-iFROM-union min-eqL*)

lemma *iIN-inter*:
$[\![\ n \leq n';\ n' \leq n + d;\ n + d \leq n' + d'\ ]\!] \Longrightarrow$
$[n...,d] \cap [n'...,d'] = [n'...,n + d - n']$
by (*fastsimp simp*: *iIN-iff*)

lemma *iMOD-union*:
$[\![\ r \leq r';\ r\ mod\ m = r'\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [r,\ mod\ m]$
by (*fastsimp simp*: *iT-iff*)
lemma *iMOD-union'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cup [r',\ mod\ m] = [min\ r\ r',\ mod\ m]$
apply (*case-tac* $r \leq r'$)
apply (*fastsimp simp*: *iMOD-union min-eq*)+
done

lemma *iMOD-inter*:
$[\![\ r \leq r';\ r\ mod\ m = r'\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [r',\ mod\ m]$
by (*fastsimp simp*: *iT-iff*)
lemma *iMOD-inter'*:
$r\ mod\ m = r'\ mod\ m \Longrightarrow$
$[r,\ mod\ m] \cap [r',\ mod\ m] = [max\ r\ r',\ mod\ m]$

apply (*case-tac* $r \leq r'$)
apply (*fastsimp simp*: *iMOD-inter max-eq*)+
done

lemma *iMODb-union*:
$[\![r \leq r'; r \bmod m = r' \bmod m; r' \leq r + m * c; r + m * c \leq r' + m * c']\!] \Longrightarrow$
$[r, \bmod m, c] \cup [r', \bmod m, c'] = [r, \bmod m, r' \operatorname{div} m - r \operatorname{div} m + c']$
apply (*rule set-eqI*)
apply (*simp add*: *iMODb-iff*)
apply (*drule sym*[*of r mod m*], *simp*)
apply (*fastsimp simp*: *add-mult-distrib2 diff-mult-distrib2 mult-div-cancel*)
done
thm *iMODb-iMOD-subset-same*
thm *Un-absorb1*[*OF iMODb-iMOD-subset-same*]

lemma *iMODb-append-union*:
$[r, \bmod m, c] \cup [\ r + m * c, \bmod m, c'] = [r, \bmod m, c + c']$
thm *iMODb-union*[*of r r + m * c m c c'*]
apply (*insert iMODb-union*[*of r r + m * c m c c'*])
apply (*case-tac m = 0*)
apply (*simp add*: *iMODb-mod-0*)
apply *simp*
done

lemma *iMODb-iMOD-append-union'*:
$[\![r \bmod m = r' \bmod m; r' \leq r + m * Suc\ c]\!] \Longrightarrow$
$[r, \bmod m, c] \cup [\ r', \bmod m\ ] = [min\ r\ r', \bmod m]$
apply (*subgoal-tac* (*min r r'*) *mod m = r' mod m*)
prefer *2*
apply (*simp add*: *min-def*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-iff*)
apply (*drule sym*[*of r mod m*], *simp*)
apply (*rule iffI*)
apply *fastsimp*
apply (*clarsimp simp*: *linorder-not-le*)
apply (*case-tac* $r \leq r'$)
apply (*simp add*: *min-eqL*)
thm *add-le-imp-le-right*[*of - m*]
apply (*rule add-le-imp-le-right*[*of - m*])
thm *less-mod-eq-imp-add-divisor-le*
apply (*rule less-mod-eq-imp-add-divisor-le*)
apply *simp*+
done
lemma *iMODb-iMOD-append-union*:
$[\![r \leq r'; r \bmod m = r' \bmod m; r' \leq r + m * Suc\ c]\!] \Longrightarrow$
$[r, \bmod m, c] \cup [\ r', \bmod m\ ] = [r, \bmod m]$

thm *iMODb-iMOD-append-union'*
by (*simp add*: *iMODb-iMOD-append-union' min-eqL*)

lemma *iMODb-append-union-Suc*:
[*r*, *mod m*, *c*] ∪ [*r* + *m* ∗ *Suc c*, *mod m*, *c'*] = [*r*, *mod m*, *Suc* (*c* + *c'*)]
thm *insert-absorb*[*of r* + *m* ∗ *c* [*r*, *mod m*, *c*] ∪ [*r* + *m* ∗ *Suc c*, *mod m*, *c'*]]
apply (*subst insert-absorb*[*of r* + *m* ∗ *c* [*r*, *mod m*, *c*] ∪ [*r* + *m* ∗ *Suc c*, *mod m*, *c'*], *symmetric*])
apply (*simp add*: *iT-iff*)
apply (*simp del*: *Un-insert-right add*: *Un-insert-right*[*symmetric*] *add-commute*[*of m*] *add-assoc*[*symmetric*] *iMODb-Suc-pred-insert-conv*)
thm *iMODb-append-union*[*of r m c*]
apply (*simp add*: *iMODb-append-union*)
done
lemma *iMODb-append-union-pred*:
0 < *c* ⟹ [*r*, *mod m*, *c* − *Suc 0*] ∪ [*r* + *m* ∗ *c*, *mod m*, *c'*] = [*r*, *mod m*, *c* + *c'*]
by (*insert iMODb-append-union-Suc*[*of r m c* − *Suc 0 c'*], *simp*)

lemma *iMODb-inter*:
⟦ *r* ≤ *r'*; *r mod m* = *r' mod m*; *r'* ≤ *r* + *m* ∗ *c*; *r* + *m* ∗ *c* ≤ *r'* + *m* ∗ *c'* ⟧ ⟹
[*r*, *mod m*, *c*] ∩ [*r'*, *mod m*, *c'*] = [*r'*, *mod m*, *c* − (*r'*−*r*) *div m*]
apply (*rule set-eqI*)
apply (*simp add*: *iMODb-iff*)
apply (*simp add*: *diff-mult-distrib2*)
apply (*simp add*: *mult-commute*[*of* - (*r'* − *r*) *div m*])
thm *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*]
apply (*simp add*: *mod-0-div-mult-cancel*[*THEN iffD1*, *OF mod-eq-imp-diff-mod-0*])
apply (*simp add*: *add-commute*[*of* - *r*])
apply *fastsimp*
done

lemmas *iT-union'* =
iFROM-union'
iTILL-union'
iMOD-union'
iMODb-iMOD-append-union'
lemmas *iT-union* =
iFROM-union
iTILL-union
iTILL-iFROM-union
iIN-union
iIN-iFROM-union
iMOD-union

iMODb-union
lemmas *iT-union-append* =
iIN-append-union
iIN-append-union-Suc
iIN-append-union-pred
iIN-iFROM-append-union
iIN-iFROM-append-union-Suc
iIN-iFROM-append-union-pred
iMODb-append-union
iMODb-iMOD-append-union
iMODb-append-union-Suc
iMODb-append-union-pred

lemmas *iT-inter'* =
iFROM-inter'
iTILL-inter'
iMOD-inter'
lemmas *iT-inter* =
iFROM-inter
iTILL-inter
iIN-inter
iMOD-inter
iMODb-inter

thm *iT-union'*
thm *iT-union*
thm *iT-union-append*
thm *iT-inter'*
thm *iT-inter*

thm *partition-Union*
lemma *mod-partition-Union*:
$0 < m \Longrightarrow (\bigcup k.\ A \cap [k * m \ldots, m - Suc\ 0]) = A$
apply *simp*
thm *subst*[**where** s=*UNIV* **and** P=$\lambda x.\ A \cap x = A$]
apply (*rule subst*[**where** s=*UNIV* **and** P=$\lambda x.\ A \cap x = A$])
apply (*rule set-eqI*)
apply (*simp add*: *iT-iff*)
apply (*rule-tac* x=x *div* m **in** *exI*)
thm *div-mult-cancel*
apply (*simp add*: *div-mult-cancel*)
thm *le-add-diff*
apply (*subst add-commute*)
apply (*rule le-add-diff*)
thm *Suc-mod-le-divisor*
apply (*simp add*: *Suc-mod-le-divisor*)

apply *simp*
done

thm *mod-partition-Union*
lemma *finite-mod-partition-Union*:
⟦ *0* < *m*; *finite A* ⟧ ⟹
(⋃ *k*≤*Max A div m*. *A* ∩ [*k*∗*m*...,*m* − *Suc 0*]) = *A*
thm *subst*[*OF mod-partition-Union*[*of m*], **where**
P=λ*x*. (⋃ *k*≤*Max A div m*. *A* ∩ [*k*∗*m*...,*m* − *Suc 0*]) = *x*]
apply (*rule subst*[*OF mod-partition-Union*[*of m*], **where**
P=λ*x*. (⋃ *k*≤*Max A div m*. *A* ∩ [*k*∗*m*...,*m* − *Suc 0*]) = *x*])
apply *assumption*
apply (*rule set-eqI*)
apply (*simp add*: *Bex-def iIN-iff*)
apply (*rule iffI*, *blast*)
apply *clarsimp*
apply (*rename-tac x x1*)
apply (*rule-tac x*=*x div m* **in** *exI*)
apply (*frule in-imp-not-empty*[**where** *A*=*A*])
apply (*frule-tac Max-ge*, *assumption*)
apply (*cut-tac n*=*x* **and** *k*=*x div m* **and** *m*=*m* **in** *div-imp-le-less*)
apply *clarsimp*+
apply (*drule-tac m*=*x* **in** *less-imp-le-pred*)
apply (*simp add*: *add-commute*[*of m*])
apply (*simp add*: *div-le-mono*)
done
thm *setsum-UN-disjoint*
lemma *mod-partition-is-disjoint*:
⟦ *0* < (*m*::*nat*); *k* ≠ *k*′ ⟧ ⟹
(*A* ∩ [*k* ∗ *m*...,*m* − *Suc 0*]) ∩ (*A* ∩ [*k*′ ∗ *m*...,*m* − *Suc 0*]) = {}
apply (*clarsimp simp add*: *all-not-in-conv*[*symmetric*] *iT-iff*)
apply (*subgoal-tac*
⋀*k*. ⟦ *k* ∗ *m* ≤ *x*; *x* ≤ *k* ∗ *m* + *m* − *Suc 0* ⟧ ⟹ *x div m* = *k*, *blast*)
thm *le-less-imp-div*
apply (*rule le-less-imp-div*, *assumption*)
apply *simp*
done

1.5 Cutting intervals

thm *i-cut-defs*
thm *i-cut-subset*
thm *i-cut-bound*

thm
cut-le-less-conv
cut-less-le-conv
thm
cut-ge-greater-conv

cut-greater-ge-conv

thm
cut-le-Min-empty
cut-less-Min-empty
cut-le-Min-not-empty
cut-less-Min-not-empty

thm
i-cut-min-empty
thm
i-cut-min-all
thm
i-cut-max-empty
thm
i-cut-max-all

lemma *iTILL-cut-le*: $[\ldots n] \downarrow\leq t = (\mathit{if}\ t \leq n\ \mathit{then}\ [\ldots t]\ \mathit{else}\ [\ldots n])$
unfolding *i-cut-defs iT-defs atMost-def*
by *force*
corollary *iTILL-cut-le1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\leq t = [\ldots t]$
by (*simp add*: *iTILL-cut-le iT-iff*)
corollary *iTILL-cut-le2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\leq t = [\ldots n]$
by (*simp add*: *iTILL-cut-le iT-iff*)

lemma *iFROM-cut-le*:
$[n\ldots] \downarrow\leq t =$
$(\mathit{if}\ t < n\ \mathit{then}\ \{\}\ \mathit{else}\ [n\ldots,t-n])$
by (*simp add*: *set-eq-iff i-cut-mem-iff iT-iff*)
corollary *iFROM-cut-le1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\leq t = [n\ldots,t - n]$
by (*simp add*: *iFROM-cut-le iT-iff*)

lemma *iIN-cut-le*:
$[n\ldots,d] \downarrow\leq t = ($
if $t < n$ *then* $\{\}$ *else*
if $t \leq n+d$ *then* $[n\ldots,t-n]$
else $[n\ldots,d])$
by (*force simp*: *set-eq-iff i-cut-mem-iff iT-iff*)

corollary *iIN-cut-le1*:
$t \in [n\ldots,d] \Longrightarrow [n\ldots,d] \downarrow\leq t = [n\ldots,t - n]$
by (*simp add*: *iIN-cut-le iT-iff*)

lemma *iMOD-cut-le*:

```
  [r, mod m] ↓≤ t = (
  if t < r then {}
  else [r, mod m, (t − r) div m])
apply (case-tac m = 0)
 apply (simp add: iMOD-0 iMODb-0 iIN-0 i-cut-empty i-cut-singleton)
apply (case-tac t < r)
 apply (simp add: cut-le-Min-empty iMOD-Min)
apply (clarsimp simp: linorder-not-less set-eq-iff i-cut-mem-iff iT-iff)
apply (rule conj-cong, simp)+
apply (clarsimp simp: mult-div-cancel)
apply (drule-tac x=r and y=x in le-imp-less-or-eq, erule disjE)
 prefer 2
 apply simp
thm less-mod-eq-imp-add-divisor-le
apply (drule-tac x=r and y=x and m=m in less-mod-eq-imp-add-divisor-le, simp)
apply (rule iffI)
 thm mod-eq-imp-diff-mod-eq[of - m r t, rule-format]
 apply (rule-tac x=x in subst[OF mod-eq-imp-diff-mod-eq[of - m r t], rule-format],
simp+)
 apply (subgoal-tac r + (t − x) mod m ≤ t)
  prefer 2
  thm add-le-mono2[OF mod-le-divisor]
  thm order-trans[OF add-le-mono2[OF mod-le-divisor]]
  apply (simp add: order-trans[OF add-le-mono2[OF mod-le-divisor]])
 apply simp
 thm le-imp-sub-mod-le[of - t]
 apply (simp add: le-imp-sub-mod-le)
apply (subgoal-tac r + (t − r) mod m ≤ t)
 prefer 2
 apply (rule ccontr)
 apply simp
apply simp
done


lemma iMOD-cut-le1:
  t ∈ [r, mod m] ⟹
  [r, mod m] ↓≤ t = [r, mod m, (t − r) div m]
by (simp add: iMOD-cut-le iT-iff)

lemma iMODb-cut-le:
  [r, mod m, c] ↓≤ t = (
    if t < r then {}
    else if t < r + m ∗ c then [r, mod m, (t − r) div m]
    else [r, mod m, c])
apply (case-tac m = 0)
 apply (simp add: iMODb-mod-0 iIN-0 cut-le-singleton)
apply (case-tac t < r)
 apply (simp add: cut-le-Min-empty iT-Min)
```

apply (*case-tac* $r + m * c \leq t$)
apply (*simp add*: *cut-le-Max-all iT-Max iT-finite*)
apply (*simp add*: *linorder-not-le linorder-not-less*)
thm *iMOD-iTILL-iMODb-conv*[*of r r* + *m* ∗ *c m*]
apply (*rule-tac t*=*c* **and** *s*=(*r* + *m* ∗ *c* − *r*) *div m* **in** *subst*, *simp*)
apply (*subst iMOD-iTILL-iMODb-conv*[*symmetric*], *simp*)
find-theorems *cut-le* (- ∩ -)
apply (*simp add*: *cut-le-Int-right iTILL-cut-le*)
thm *iMOD-iTILL-iMODb-conv*
apply (*simp add*: *iMOD-iTILL-iMODb-conv*)
done

lemma *iMODb-cut-le1*:
$t \in [r, \mathit{mod}\ m, c] \Longrightarrow$
$[r, \mathit{mod}\ m, c] \downarrow\leq t = [r, \mathit{mod}\ m, (t - r)\ \mathit{div}\ m]$
by (*clarsimp simp*: *iMODb-cut-le iT-iff iMODb-mod-0*)

lemma *iTILL-cut-less*:
$[\ldots n] \downarrow< t =$ (
if $n < t$ *then* $[\ldots n]$ *else*
if $t = 0$ *then* $\{\}$
else $[\ldots t - \mathit{Suc}\ 0]$)
apply (*case-tac* $n < t$)
apply (*simp add*: *cut-less-Max-all iT-Max iT-finite*)
apply (*case-tac* $t = 0$)
apply (*simp add*: *cut-less-0-empty*)
apply (*fastsimp simp*: *nat-cut-less-le-conv iTILL-cut-le*)
done

lemma *iTILL-cut-less1*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow [\ldots n] \downarrow< t = [\ldots t - \mathit{Suc}\ 0]$
thm *iTILL-cut-less*
by (*simp add*: *iTILL-cut-less iT-iff*)

lemma *iFROM-cut-less*:
$[n\ldots] \downarrow< t =$ (
if $t \leq n$ *then* $\{\}$
else $[n\ldots,t - \mathit{Suc}\ n]$)
apply (*case-tac* $t \leq n$)
apply (*simp add*: *cut-less-Min-empty iT-Min*)
apply (*fastsimp simp*: *nat-cut-less-le-conv iFROM-cut-le*)
done

lemma *iFROM-cut-less1*:
$n < t \Longrightarrow [n\ldots] \downarrow< t = [n\ldots,t - \mathit{Suc}\ n]$
by (*simp add*: *iFROM-cut-less*)

lemma *iIN-cut-less*:
  $[n...,d] \downarrow< t = ($
    if $t \leq n$ *then* $\{\}$ *else*
    if $t \leq n + d$ *then* $[n..., t - Suc\ n]$
    else $[n...,d])$
apply (*case-tac* $t \leq n$)
 apply (*simp add*: *cut-less-Min-empty iT-Min*)
apply (*case-tac* $n + d < t$)
 apply (*simp add*: *cut-less-Max-all iT-Max iT-finite*)
apply (*fastsimp simp*: *nat-cut-less-le-conv iIN-cut-le*)
done

lemma *iIN-cut-less1*:
  $[\![\ t \in [n...,d];\ n < t\ ]\!] \Longrightarrow [n...,d] \downarrow< t = [n..., t - Suc\ n]$
thm *iIN-cut-less*
by (*simp add*: *iIN-cut-less iT-iff*)

thm
  cut-le-less-conv
  cut-less-le-conv

lemma *iMOD-cut-less*:
  $[r,\ mod\ m] \downarrow< t = ($
    if $t \leq r$ *then* $\{\}$
    else $[r,\ mod\ m,\ (t - Suc\ r)\ div\ m])$
thm *iMOD-cut-le*
apply (*case-tac* $t = 0$)
 apply (*simp add*: *cut-less-0-empty*)
apply (*simp add*: *nat-cut-less-le-conv iMOD-cut-le*)
apply *fastsimp*
done

lemma *iMOD-cut-less1*:
  $[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow$
  $[r,\ mod\ m] \downarrow< t = [r,\ mod\ m,\ (t - r)\ div\ m - Suc\ 0]$
apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMOD-0 iMODb-mod-0 iIN-0*)
apply (*simp add*: *iMOD-cut-less*)
thm *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0*
apply (*simp add*: *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0 iT-iff*)
done

lemma *iMODb-cut-less*:
  $[r,\ mod\ m,\ c] \downarrow< t = ($
    if $t \leq r$ *then* $\{\}$ *else*
    if $r + m * c < t$ *then* $[r,\ mod\ m,\ c]$

else [*r*, *mod m*, (*t* − *Suc r*) *div m*])
thm *iMODb-cut-le*
apply (*case-tac t* = *0*)
apply (*simp add*: *cut-less-0-empty*)
apply (*simp add*: *nat-cut-less-le-conv iMODb-cut-le*)
apply *fastsimp*
done

lemma *iMODb-cut-less1*: ⟦ *t* ∈ [*r*, *mod m*, *c*]; *r* < *t* ⟧ ⟹
[*r*, *mod m*, *c*] ↓< *t* = [*r*, *mod m*, (*t* − *r*) *div m* − *Suc 0*]
apply (*case-tac m* = *0*)
apply (*simp add*: *iMODb-mod-0 iIN-0*)
apply (*simp add*: *iMODb-cut-less*)
thm *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0*
apply (*simp add*: *mod-0-imp-diff-Suc-div-conv mod-eq-imp-diff-mod-0 iT-iff*)
done

lemmas *iT-cut-le* =
iTILL-cut-le
iFROM-cut-le
iIN-cut-le
iMOD-cut-le
iMODb-cut-le
thm *iT-cut-le*
lemmas *iT-cut-le1* =
iTILL-cut-le1
iFROM-cut-le1
iIN-cut-le1
iMOD-cut-le1
iMODb-cut-le1
thm *iT-cut-le1*

lemmas *iT-cut-less* =
iTILL-cut-less
iFROM-cut-less
iIN-cut-less
iMOD-cut-less
iMODb-cut-less
thm *iT-cut-less*
lemmas *iT-cut-less1* =
iTILL-cut-less1
iFROM-cut-less1
iIN-cut-less1
iMOD-cut-less1
iMODb-cut-less1
thm *iT-cut-less1*

lemmas *iT-cut-le-less* =

iTILL-cut-le
iTILL-cut-less
iFROM-cut-le
iFROM-cut-less
iIN-cut-le
iIN-cut-less
iMOD-cut-le
iMOD-cut-less
iMODb-cut-le
iMODb-cut-less
lemmas *iT-cut-le-less1* =
iTILL-cut-le1
iTILL-cut-less1
iFROM-cut-le1
iFROM-cut-less1
iIN-cut-le1
iIN-cut-less1
iMOD-cut-le1
iMOD-cut-less1
iMODb-cut-le1
iMODb-cut-less1
thm *iT-cut-le-less*
thm *iT-cut-le-less1*

lemma *iTILL-cut-ge*:
$[\ldots n] \downarrow\geq t = (\textit{if } n < t \textit{ then } \{\} \textit{ else } [t\ldots,n-t])$
by (*force simp*: *i-cut-mem-iff iT-iff*)
corollary *iTILL-cut-ge1*: $t \in [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = [t\ldots,n-t]$
by (*simp add*: *iTILL-cut-ge iT-iff*)
corollary *iTILL-cut-ge2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow\geq t = \{\}$
by (*simp add*: *iTILL-cut-ge iT-iff*)

lemma *iTILL-cut-greater*:
$[\ldots n] \downarrow> t = (\textit{if } n \leq t \textit{ then } \{\} \textit{ else } [\textit{Suc } t\ldots,n - \textit{Suc } t])$
by (*force simp*: *i-cut-mem-iff iT-iff*)
corollary *iTILL-cut-greater1*:
$t \in [\ldots n] \Longrightarrow t < n \Longrightarrow [\ldots n] \downarrow> t = [\textit{Suc } t\ldots,n - \textit{Suc } t]$
by (*simp add*: *iTILL-cut-greater iT-iff*)
corollary *iTILL-cut-greater2*: $t \notin [\ldots n] \Longrightarrow [\ldots n] \downarrow> t = \{\}$
by (*simp add*: *iTILL-cut-greater iT-iff*)

lemma *iFROM-cut-ge*:
$[n\ldots] \downarrow\geq t = (\textit{if } n \leq t \textit{ then } [t\ldots] \textit{ else } [n\ldots])$
by (*force simp*: *i-cut-mem-iff iT-iff*)
corollary *iFROM-cut-ge1*: $t \in [n\ldots] \Longrightarrow [n\ldots] \downarrow\geq t = [t\ldots]$
by (*simp add*: *iFROM-cut-ge iT-iff*)

lemma *iFROM-cut-greater*:

$[n...] \downarrow> t = (\mathit{if}\ n \leq t\ \mathit{then}\ [\mathit{Suc}\ t...]\ \mathit{else}\ [n...])$
by (*force simp*: *i-cut-mem-iff iT-iff*)
corollary *iFROM-cut-greater1*:
$t \in [n...] \Longrightarrow [n...] \downarrow> t = [\mathit{Suc}\ t...]$
by (*simp add*: *iFROM-cut-greater iT-iff*)

lemma *iIN-cut-ge*:
$[n...,d] \downarrow\geq t = ($
$\mathit{if}\ t < n\ \mathit{then}\ [n...,d]\ \mathit{else}$
$\mathit{if}\ t \leq n+d\ \mathit{then}\ [t...,n+d-t]$
$\mathit{else}\ \{\})$
by (*force simp*: *i-cut-mem-iff iT-iff*)
corollary *iIN-cut-ge1*: $t \in [n...,d] \Longrightarrow$
$[n...,d] \downarrow\geq t = [t...,n+d-t]$
by (*simp add*: *iIN-cut-ge iT-iff*)
corollary *iIN-cut-ge2*: $t \notin [n...,d] \Longrightarrow$
$[n...,d] \downarrow\geq t = (\mathit{if}\ t < n\ \mathit{then}\ [n...,d]\ \mathit{else}\ \{\})$
by (*simp add*: *iIN-cut-ge iT-iff*)

lemma *iIN-cut-greater*:
$[n...,d] \downarrow> t = ($
$\mathit{if}\ t < n\ \mathit{then}\ [n...,d]\ \mathit{else}$
$\mathit{if}\ t < n+d\ \mathit{then}\ [\mathit{Suc}\ t...,n\ +\ d\ -\ \mathit{Suc}\ t]$
$\mathit{else}\ \{\})$
by (*force simp*: *i-cut-mem-iff iT-iff*)
corollary *iIN-cut-greater1*:
$[\![\ t \in [n...,d];\ t < n\ +\ d\ ]\!] \Longrightarrow$
$[n...,d] \downarrow> t = [\mathit{Suc}\ t...,n\ +\ d\ -\ \mathit{Suc}\ t]$
by (*simp add*: *iIN-cut-greater iT-iff*)

thm *sub-diff-mod-eq'*[*of r t 1 m*, *simplified*]
lemma *mod-cut-greater-aux-t-less*:
$[\![\ 0 < (m::nat);\ r \leq t\ ]\!] \Longrightarrow$
$t < t\ +\ m\ -\ (t\ -\ r)\ \mathit{mod}\ m$
thm *less-add-diff*
by (*simp add*: *less-add-diff add-commute*)
lemma *mod-cut-greater-aux-le-x*:
$[\![\ (r::nat) \leq t;\ t < x;\ x\ \mathit{mod}\ m = r\ \mathit{mod}\ m]\!] \Longrightarrow$
$t\ +\ m\ -\ (t\ -\ r)\ \mathit{mod}\ m \leq x$
thm *diff-mod-le*
apply (*insert diff-mod-le*[*of t r m*])
thm *diff-add-assoc2*[*of* $(t\ -\ r)\ \mathit{mod}\ m\ t\ m$]

apply (*subst diff-add-assoc2*, *simp*)
thm *less-mod-eq-imp-add-divisor-le*[*of* $t - (t - r) \bmod m\ x\ m$]
apply (*rule less-mod-eq-imp-add-divisor-le*, *simp*)
thm *sub-diff-mod-eq*
apply (*simp add*: *sub-diff-mod-eq*)
done

lemma *iMOD-cut-greater*:
$[r, \bmod m] \downarrow> t =$ (
if $t < r$ *then* $[r, \bmod m]$ *else*
if $m = 0$ *then* $\{\}$
else $[t + m - (t - r) \bmod m, \bmod m]$)
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMOD-0 iIN-0 i-cut-singleton*)
apply (*case-tac* $t < r$)
apply (*simp add*: *iT-Min cut-greater-Min-all*)
apply (*simp add*: *linorder-not-less*)
apply (*simp add*: *set-eq-iff i-cut-mem-iff iT-iff*, *clarify*)
apply (*subgoal-tac* $(t - r) \bmod m \leq t$)
prefer *2*
thm *order-trans*[*OF mod-le-dividend*]
apply (*rule order-trans*[*OF mod-le-dividend*], *simp*)
thm *diff-add-assoc2*[*of* $(t - r) \bmod m\ t\ m$]
apply (*simp add*: *diff-add-assoc2 del*: *add-diff-assoc2*)
thm *sub-diff-mod-eq*
apply (*simp add*: *sub-diff-mod-eq del*: *add-diff-assoc2*)
apply (*rule conj-cong*, *simp*)
apply (*rule iffI*)
apply *clarify*
thm *less-mod-eq-imp-add-divisor-le*[*of* $t - (t - r) \bmod m\ x\ m$]
apply (*rule less-mod-eq-imp-add-divisor-le*)
apply *simp*
apply (*simp add*: *sub-diff-mod-eq*)
apply (*subgoal-tac* $t < x$)
prefer *2*
apply (*rule-tac* $y{=}t - (t - r) \bmod m + m$ **in** *order-less-le-trans*)
apply (*simp add*: *mod-cut-greater-aux-t-less*)
apply *simp+*
done

lemma *iMOD-cut-greater1*:
$t \in [r, \bmod m] \Longrightarrow$
$[r, \bmod m] \downarrow> t =$ (
if $m = 0$ *then* $\{\}$
else $[t + m, \bmod m]$)
by (*simp add*: *iMOD-cut-greater iT-iff mod-eq-imp-diff-mod-0*)

lemma *iMODb-cut-greater-aux*:
$[\![\ 0 < m;\ t < r + m * c;\ r \leq t\ ]\!] \Longrightarrow$

$(r + m * c - (t + m - (t - r) \ mod \ m)) \ div \ m =$
$c - Suc \ ((t - r) \ div \ m)$
thm *diff-diff-right*[*of* $r \ t+m-(t-r) mod \ m \ m*c$, *symmetric*]
apply (*subgoal-tac* $r \leq t + m - (t - r) \ mod \ m$)
prefer *2*
apply (*rule order-trans*[*of* - t], *simp*)
thm *mod-cut-greater-aux-t-less*
apply (*simp add*: *mod-cut-greater-aux-t-less less-imp-le*)
apply (*rule-tac* $t=(r + m * c - (t + m - (t - r) \ mod \ m))$ **and** $s=m * (c - Suc \ ((t - r) \ div \ m))$ **in** *subst*)
apply (*simp add*: *diff-mult-distrib2 mult-div-cancel del*: *diff-diff-left*)
apply *simp*
done

lemma *iMODb-cut-greater*:
$[r, \ mod \ m, \ c] \downarrow> t = ($
if $t < r$ *then* $[r, \ mod \ m, \ c]$ *else*
if $r + m * c \leq t$ *then* $\{\}$
else $[t + m - (t - r) \ mod \ m, \ mod \ m, \ c - Suc \ ((t-r) \ div \ m)])$
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMODb-mod-0 iIN-0 i-cut-singleton*)
apply (*case-tac* $r + m * c \leq t$)
apply (*simp add*: *cut-greater-Max-empty iT-Max iT-finite*)
apply (*case-tac* $t < r$)
apply (*simp add*: *cut-greater-Min-all iT-Min*)
apply (*simp add*: *linorder-not-less linorder-not-le*)
thm *iMOD-iTILL-iMODb-conv*[*of* $r \ r + m * c \ m$]
apply (*rule-tac* $t=[\ r, \ mod \ m, \ c \ ]$ **and** $s=[\ r, \ mod \ m \ ] \cap [...r + m * c]$ **in** *subst*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv*)
thm *iMOD-cut-greater*
apply (*simp add*: *i-cut-Int-left iMOD-cut-greater*)
thm *iMOD-iTILL-iMODb-conv*
apply (*subst iMOD-iTILL-iMODb-conv*)
thm *mod-cut-greater-aux-le-x*
apply (*rule mod-cut-greater-aux-le-x*, *simp+*)
apply (*simp add*: *iMODb-cut-greater-aux*)
done

lemma *iMODb-cut-greater1*:
$t \in [r, \ mod \ m, \ c] \Longrightarrow$
$[r, \ mod \ m, \ c] \downarrow> t = ($
if $r + m * c \leq t$ *then* $\{\}$
else $[t + m, \ mod \ m, \ c - Suc \ ((t-r) \ div \ m)])$
by (*simp add*: *iMODb-cut-greater iT-iff mod-eq-imp-diff-mod-0*)

lemma *iMOD-cut-ge*:
$[r, \ mod \ m] \downarrow\geq t = ($

```
    if t ≤ r then [r, mod m] else
    if m = 0 then {}
    else [t + m − Suc ((t − Suc r) mod m), mod m])
apply (case-tac t = 0)
 apply (simp add: cut-ge-0-all)
thm iMOD-cut-greater nat-cut-greater-ge-conv[symmetric]
apply (force simp: nat-cut-greater-ge-conv[symmetric] iMOD-cut-greater)
done

lemma iMOD-cut-ge1:
  t ∈ [r, mod m] ⟹
  [r, mod m] ↓≥ t = [t, mod m]
by (fastsimp simp: iMOD-cut-ge)



lemma iMODb-cut-ge:
  [r, mod m, c] ↓≥ t = (
    if t ≤ r then [r, mod m, c] else
    if r + m ∗ c < t then {}
    else [t + m − Suc ((t − Suc r) mod m), mod m, c − (t + m − Suc r) div m])
thm iMOD-cut-ge
apply (case-tac m = 0)
 apply (simp add: iMODb-mod-0 iIN-0 i-cut-singleton)
apply (case-tac r + m ∗ c < t)
 apply (simp add: cut-ge-Max-empty iT-Max iT-finite)
apply (case-tac t ≤ r)
 apply (simp add: cut-ge-Min-all iT-Min)
apply (simp add: linorder-not-less linorder-not-le)
apply (case-tac r mod m = t mod m)
 thm diff-mod-pred
 apply (simp add: diff-mod-pred)
 thm mod-0-imp-diff-Suc-div-conv
 apply (simp add: mod-0-imp-diff-Suc-div-conv mod-eq-diff-mod-0-conv diff-add-assoc2
del: add-diff-assoc2)
 apply (subgoal-tac 0 < (t − r) div m)
  prefer 2
  apply (frule-tac x=r in less-mod-eq-imp-add-divisor-le)
   apply (simp add: mod-eq-diff-mod-0-conv)
  apply (drule add-le-imp-le-diff2)
  thm div-le-mono
  apply (drule-tac m=m and k=m in div-le-mono)
  apply simp
 apply (simp add: set-eq-iff i-cut-mem-iff iT-iff , intro allI)
 apply (simp add: mod-eq-diff-mod-0-conv[symmetric])
 apply (rule conj-cong, simp)
 apply (case-tac t ≤ x)
  prefer 2
  apply simp
```

apply (*simp add*: *diff-mult-distrib2 mult-div-cancel mod-eq-diff-mod-0-conv add-commute*[*of r*])
apply (*subgoal-tac Suc* ((*t* − *Suc r*) *mod m*) = (*t* − *r*) *mod m*)
prefer *2*
thm *diff-mod-pred*
apply (*clarsimp simp add*: *diff-mod-pred mod-eq-diff-mod-0-conv*)
thm *iMOD-iTILL-iMODb-conv*[*of r r* + *m* ∗ *c m*]
apply (*rule-tac t*=[*r*, *mod m*, *c*] **and** *s*=[*r*, *mod m*] ∩ [. . .*r* + *m* ∗ *c*] **in** *subst*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv*)
thm *iMOD-cut-ge*
apply (*simp add*: *i-cut-Int-left iMOD-cut-ge*)
thm *iMOD-iTILL-iMODb-conv*
apply (*subst iMOD-iTILL-iMODb-conv*)
apply (*drule-tac x*=*t* **in** *le-imp-less-or-eq*, *erule disjE*)
thm *mod-cut-greater-aux-le-x*
apply (*rule mod-cut-greater-aux-le-x*, *simp*+)
apply (*rule arg-cong*)
apply (*drule-tac x*=*t* **in** *le-imp-less-or-eq*, *erule disjE*)
prefer *2*
apply *simp*
thm *iMODb-cut-greater-aux*[*of m t r c*]
apply (*simp add*: *iMODb-cut-greater-aux*)
apply (*rule arg-cong*[**where** *f*=*op* − *c*])
apply (*simp add*: *diff-add-assoc2 del*: *add-diff-assoc2*)
apply (*rule-tac t*=*t* − *Suc r* **and** *s*=*t* − *r* − *Suc 0* **in** *subst*, *simp*)
thm *div-diff1-eq*
apply (*subst div-diff1-eq*[*of* - *Suc 0*])
apply (*case-tac m* = *Suc 0*, *simp*)
apply *simp*
done

thm *iMODb-cut-greater1*
lemma *iMODb-cut-ge1*:
t ∈ [*r*, *mod m*, *c*] ⟹
[*r*, *mod m*, *c*] ↓≥ *t* = (
if r + *m* ∗ *c* < *t then* {}
else [*t*, *mod m*, *c* − (*t* − *r*) *div m*])
apply (*case-tac m* = *0*)
apply (*simp add*: *iMODb-mod-0 iT-iff iIN-0 i-cut-singleton*)
thm *iMODb-cut-ge*
apply (*clarsimp simp*: *iMODb-cut-ge iT-iff*)
thm *mod-eq-imp-diff-mod-eq-divisor*
apply (*simp add*: *mod-eq-imp-diff-mod-eq-divisor*)
apply (*rule-tac t*=*t* + *m* − *Suc r* **and** *s*=*t* − *r* + (*m* − *Suc 0*) **in** *subst*, *simp*)
thm *div-add1-eq*
apply (*subst div-add1-eq*)
apply (*simp add*: *mod-eq-imp-diff-mod-0*)
done
lemma *iMOD-0-cut-greater*:

$t \in [r,\ mod\ 0] \Longrightarrow [r,\ mod\ 0] \downarrow> t = \{\}$
by (*simp add*: *iT-iff iMOD-0 iIN-0 i-cut-singleton*)
lemma *iMODb-0-cut-greater*: $t \in [r,\ mod\ 0,\ c] \Longrightarrow$
$[r,\ mod\ 0,\ c] \downarrow> t = \{\}$
by (*simp add*: *iT-iff iMODb-mod-0 iIN-0 i-cut-singleton*)

lemmas *iT-cut-ge* =
iTILL-cut-ge
iFROM-cut-ge
iIN-cut-ge
iMOD-cut-ge
iMODb-cut-ge
thm *iT-cut-ge*
lemmas *iT-cut-ge1* =
iTILL-cut-ge1
iFROM-cut-ge1
iIN-cut-ge1
iMOD-cut-ge1
iMODb-cut-ge1
thm *iT-cut-le1*

lemmas *iT-cut-greater* =
iTILL-cut-greater
iFROM-cut-greater
iIN-cut-greater
iMOD-cut-greater
iMODb-cut-greater
thm *iT-cut-greater*
lemmas *iT-cut-greater1* =
iTILL-cut-greater1
iFROM-cut-greater1
iIN-cut-greater1
iMOD-cut-greater1
iMODb-cut-greater1
thm *iT-cut-greater1*

lemmas *iT-cut-ge-greater* =
iTILL-cut-ge
iTILL-cut-greater
iFROM-cut-ge
iFROM-cut-greater
iIN-cut-ge
iIN-cut-greater
iMOD-cut-ge
iMOD-cut-greater
iMODb-cut-ge
iMODb-cut-greater
lemmas *iT-cut-ge-greater1* =
iTILL-cut-ge1

iTILL-cut-greater1
iFROM-cut-ge1
iFROM-cut-greater1
iIN-cut-ge1
iIN-cut-greater1
iMOD-cut-ge1
iMOD-cut-greater1
iMODb-cut-ge1
iMODb-cut-greater1
thm *iT-cut-ge-greater*
thm *iT-cut-ge-greater1*

1.6 Cardinality of intervals

lemma *iFROM-card*: *card* [*n*...] = *0*
by (*simp add*: *iFROM-infinite*)
lemma *iTILL-card*: *card* [...*n*] = *Suc n*
by (*simp add*: *iTILL-def*)
lemma *iIN-card*: *card* [*n*...,*d*] = *Suc d*
by (*simp add*: *iIN-def*)
lemma *iMOD-0-card*: *card* [*r*, *mod 0*] = *Suc 0*
by (*simp add*: *iMOD-0 iIN-card*)
lemma *iMOD-card*: *0* < *m* ⟹ *card* [*r*, *mod m*] = *0*
by (*simp add*: *iMOD-infinite*)
lemma *iMOD-card-if*: *card* [*r*, *mod m*] = (*if m* = *0 then Suc 0 else 0*)
by (*simp add*: *iMOD-0-card iMOD-card*)
lemma *iMODb-mod-0-card*: *card* [*r*, *mod 0*, *c*] = *Suc 0*
by (*simp add*: *iMODb-mod-0 iIN-card*)
lemma *iMODb-card*: *0* < *m* ⟹ *card* [*r*, *mod m*, *c*] = *Suc c*
apply (*induct c*)
apply (*simp add*: *iMODb-0 iIN-card*)
thm *iMODb-Suc-insert-conv*
apply (*subst iMODb-Suc-insert-conv*[*symmetric*])
apply (*subst card-insert-disjoint*)
apply (*simp add*: *iT-finite iT-iff*)+
done
lemma *iMODb-card-if*:
card [*r*, *mod m*, *c*] = (*if m* = *0 then Suc 0 else Suc c*)
by (*simp add*: *iMODb-mod-0-card iMODb-card*)

lemmas *iT-card* =
iFROM-card
iTILL-card
iIN-card
iMOD-card-if
iMODb-card-if

Cardinality with *icard*

lemma *iFROM-icard*: *icard* [*n*...] = ∞

by (*simp add*: *iFROM-infinite*)
lemma *iTILL-icard*: *icard* [...*n*] = *Fin* (*Suc n*)
by (*simp add*: *icard-finite iT-finite iT-card*)
lemma *iIN-icard*: *icard* [*n*...,*d*] = *Fin* (*Suc d*)
by (*simp add*: *icard-finite iT-finite iT-card*)
lemma *iMOD-0-icard*: *icard* [*r*, *mod 0*] = *iSuc 0*
by (*simp add*: *icard-finite iT-finite iT-card iSuc-Fin*)
lemma *iMOD-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*] = ∞
by (*simp add*: *iMOD-infinite*)
lemma *iMOD-icard-if*: *icard* [*r*, *mod m*] = (*if m = 0 then iSuc 0 else* ∞)
by (*simp add*: *icard-finite iT-finite iT-infinite iSuc-Fin iT-card*)
lemma *iMODb-mod-0-icard*: *icard* [*r*, *mod 0*, *c*] = *iSuc 0*
by (*simp add*: *icard-finite iT-finite iSuc-Fin iT-card*)
lemma *iMODb-icard*: *0* < *m* ⟹ *icard* [*r*, *mod m*, *c*] = *Fin* (*Suc c*)
by (*simp add*: *icard-finite iT-finite iMODb-card*)
lemma *iMODb-icard-if*: *icard* [*r*, *mod m*, *c*] = *Fin* (*if m = 0 then Suc 0 else Suc c*)
by (*simp add*: *icard-finite iT-finite iMODb-card-if*)

lemmas *iT-icard* =
iFROM-icard
iTILL-icard
iIN-icard
iMOD-icard-if
iMODb-icard-if

1.7 Functions *inext* and *iprev* with intervals

thm
inext-def
iprev-def

thm
inext-Max
iprev-iMin
thm
inext-closed
iprev-closed

thm
iMin-subset
iMin-Un
thm
Max-Un
Min-Un

lemma
iFROM-inext: $t \in [n\ldots] \Longrightarrow inext\ t\ [n\ldots] = Suc\ t$ **and**
iTILL-inext: $t < n \Longrightarrow inext\ t\ [\ldots n] = Suc\ t$ **and**
iIN-inext: $[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow inext\ t\ [n\ldots,d] = Suc\ t$
by (*simp add*: *iT-defs inext-atLeast inext-atMost inext-atLeastAtMost*)+

lemma
iFROM-iprev': $t \in [n\ldots] \Longrightarrow iprev\ (Suc\ t)\ [n\ldots] = t$ **and**
iFROM-iprev: $n < t \Longrightarrow iprev\ t\ [n\ldots] = t - Suc\ 0$ **and**
iTILL-iprev: $t \in [\ldots n] \Longrightarrow iprev\ t\ [\ldots n] = t - Suc\ 0$ **and**
iIN-iprev: $[\![\ n < t;\ t \leq n + d\ ]\!] \Longrightarrow iprev\ t\ [n\ldots,d] = t - Suc\ 0$ **and**
iIN-iprev': $[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow iprev\ (Suc\ t)\ [n\ldots,d] = t$
by (*simp add*: *iT-defs iprev-atLeast iprev-atMost iprev-atLeastAtMost*)+

lemma *iMOD-inext*: $t \in [r,\ mod\ m] \Longrightarrow inext\ t\ [r,\ mod\ m] = t + m$
by (*clarsimp simp add*: *inext-def iMOD-cut-greater iT-iff iT-Min iT-not-empty mod-eq-imp-diff-mod-0*)
lemma *iMOD-iprev*: $[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow iprev\ t\ [r,\ mod\ m] = t - m$
apply (*case-tac* $m = 0$, *simp add*: *iMOD-iff*)
apply (*clarsimp simp add*: *iprev-def iMOD-cut-less iT-iff iT-Max iT-not-empty mult-div-cancel*)
thm *mod-eq-imp-diff-mod-eq-divisor*
apply (*simp del*: *add-Suc-right add*: *add-Suc-right*[*symmetric*] *mod-eq-imp-diff-mod-eq-divisor*)
thm *less-mod-eq-imp-add-divisor-le*
apply (*simp add*: *less-mod-eq-imp-add-divisor-le*)
done

lemma *iMOD-iprev'*: $t \in [r,\ mod\ m] \Longrightarrow iprev\ (t + m)\ [r,\ mod\ m] = t$
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMOD-0 iIN-0 iprev-singleton*)
apply (*simp add*: *iMOD-iprev iT-iff*)
done

lemma *iMODb-inext*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow$
$inext\ t\ [r,\ mod\ m,\ c] = t + m$
by (*clarsimp simp add*: *inext-def iMODb-cut-greater iT-iff iT-Min iT-not-empty mod-eq-imp-diff-mod-0*)

lemma *iMODb-iprev*:
$[\![\ t \in [r,\ mod\ m,\ c];\ r < t\ ]\!] \Longrightarrow$
$iprev\ t\ [r,\ mod\ m,\ c] = t - m$
apply (*case-tac* $m = 0$, *simp add*: *iMODb-iff*)
apply (*clarsimp simp add*: *iprev-def iMODb-cut-less iT-iff iT-Max iT-not-empty mult-div-cancel*)
thm *mod-eq-imp-diff-mod-eq-divisor*
apply (*simp del*: *add-Suc-right add*: *add-Suc-right*[*symmetric*] *mod-eq-imp-diff-mod-eq-divisor*)
thm *less-mod-eq-imp-add-divisor-le*

apply (*simp add*: *less-mod-eq-imp-add-divisor-le*)
done
lemma *iMODb-iprev'*:
⟦ $t \in [r, \mathit{mod}\ m, c]$; $t < r + m * c$ ⟧ ⟹
iprev $(t + m)$ $[r, \mathit{mod}\ m, c] = t$
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMODb-mod-0 iIN-0 iprev-singleton*)
thm *less-mod-eq-imp-add-divisor-le*
apply (*simp add*: *iMODb-iprev iT-iff less-mod-eq-imp-add-divisor-le*)
done

lemmas *iT-inext* =
iFROM-inext
iTILL-inext
iIN-inext
iMOD-inext
iMODb-inext
lemmas *iT-iprev* =
iFROM-iprev'
iFROM-iprev
iTILL-iprev
iIN-iprev
iIN-iprev'
iMOD-iprev
iMOD-iprev'
iMODb-iprev
iMODb-iprev'
thm *iT-inext*
thm *iT-iprev*

thm *iprev-iMin*
thm *iT-finite*[*THEN inext-Max*]
lemma *iFROM-inext-if*:
inext t $[n\ldots]$ = (*if* $t \in [n\ldots]$ *then Suc t else t*)
by (*simp add*: *iFROM-inext not-in-inext-fix*)
lemma *iTILL-inext-if*:
inext t $[\ldots n]$ = (*if* $t < n$ *then Suc t else t*)
by (*simp add*: *iTILL-inext iT-finite iT-Max inext-ge-Max*)
lemma *iIN-inext-if*:
inext t $[n\ldots,d]$ = (*if* $n \leq t \wedge t < n + d$ *then Suc t else t*)
by (*fastsimp simp*: *iIN-inext iT-iff not-in-inext-fix iT-finite iT-Max inext-ge-Max*)
lemma *iMOD-inext-if*:
inext t $[r, \mathit{mod}\ m]$ = (*if* $t \in [r, \mathit{mod}\ m]$ *then* $t + m$ *else t*)
by (*simp add*: *iMOD-inext not-in-inext-fix*)
lemma *iMODb-inext-if*:
inext t $[r, \mathit{mod}\ m, c]$ =

(*if* $t \in [r, \mathit{mod}\ m, c] \wedge t < r + m * c$ *then* $t + m$ *else* t)
by (*fastsimp simp*: *iMODb-inext iT-iff not-in-inext-fix iT-finite iT-Max inext-ge-Max*)

lemmas *iT-inext-if* =
iFROM-inext-if
iTILL-inext-if
iIN-inext-if
iMOD-inext-if
iMODb-inext-if
thm *iT-inext-if*

lemma *iFROM-iprev-if*:
iprev t $[n\ldots] =$ (*if* $n < t$ *then* $t -$ *Suc 0 else* t)
by (*simp add*: *iFROM-iprev iT-Min iprev-le-iMin*)
lemma *iTILL-iprev-if*:
iprev t $[\ldots n] =$ (*if* $t \in [\ldots n]$ *then* $t -$ *Suc 0 else* t)
by (*simp add*: *iTILL-iprev not-in-iprev-fix*)
lemma *iIN-iprev-if*:
iprev t $[n\ldots,d] =$ (*if* $n < t \wedge t \leq n + d$ *then* $t -$ *Suc 0 else* t)
by (*fastsimp simp*: *iIN-iprev iT-iff not-in-iprev-fix iT-Min iprev-le-iMin*)
lemma *iMOD-iprev-if*:
iprev t $[r, \mathit{mod}\ m] =$
(*if* $t \in [r, \mathit{mod}\ m] \wedge r < t$ *then* $t - m$ *else* t)
by (*fastsimp simp add*: *iMOD-iprev iT-iff not-in-iprev-fix iT-Min iprev-le-iMin*)
lemma *iMODb-iprev-if*:
iprev t $[r, \mathit{mod}\ m, c] =$
(*if* $t \in [r, \mathit{mod}\ m, c] \wedge r < t$ *then* $t - m$ *else* t)
by (*fastsimp simp add*: *iMODb-iprev iT-iff not-in-iprev-fix iT-Min iprev-le-iMin*)

lemmas *iT-iprev-if* =
iFROM-iprev-if
iTILL-iprev-if
iIN-iprev-if
iMOD-iprev-if
iMODb-iprev-if
thm *iT-iprev-if*

The difference between an element and the next/previous element is constant if the element is different from Min/Max of the interval

lemma *iFROM-inext-diff-const*:
$t \in [n\ldots] \Longrightarrow$ *inext t* $[n\ldots] - t =$ *Suc 0*
by (*simp add*: *iFROM-inext*)
lemma *iFROM-iprev-diff-const*:
$n < t \Longrightarrow t -$ *iprev t* $[n\ldots] =$ *Suc 0*
by (*simp add*: *iFROM-iprev*)
lemma *iFROM-iprev-diff-const′*:
$t \in [n\ldots] \Longrightarrow$ *Suc t* $-$ *iprev* (*Suc t*) $[n\ldots] =$ *Suc 0*
by (*simp add*: *iFROM-iprev′*)

lemma *iTILL-inext-diff-const*:
$t < n \Longrightarrow inext\ t\ [\ldots n] - t = Suc\ 0$
by (*simp add*: *iTILL-inext*)
lemma *iTILL-iprev-diff-const*:
$[\![\ t \in [\ldots n];\ 0 < t\ ]\!] \Longrightarrow t - iprev\ t\ [\ldots n] = Suc\ 0$
by (*simp add*: *iTILL-iprev*)

thm *iIN-inext*
lemma *iIN-inext-diff-const*:
$[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow inext\ t\ [n\ldots,d] - t = Suc\ 0$
by (*simp add*: *iIN-inext*)

thm *iIN-iprev*
lemma *iIN-iprev-diff-const*:
$[\![\ n < t;\ t \leq n + d\ ]\!] \Longrightarrow t - iprev\ t\ [n\ldots,d] = Suc\ 0$
by (*simp add*: *iIN-iprev*)
lemma *iIN-iprev-diff-const'*:
$[\![\ n \leq t;\ t < n + d\ ]\!] \Longrightarrow Suc\ t - iprev\ (Suc\ t)\ [n\ldots,d] = Suc\ 0$
by (*simp add*: *iIN-iprev*)

thm *iMOD-inext*
lemma *iMOD-inext-diff-const*:
$t \in [r,\ mod\ m] \Longrightarrow inext\ t\ [r,\ mod\ m] - t = m$
by (*simp add*: *iMOD-inext*)
lemma *iMOD-iprev-diff-const'*:
$t \in [r,\ mod\ m] \Longrightarrow (t + m) - iprev\ (t + m)\ [r,\ mod\ m] = m$
by (*simp add*: *iMOD-iprev'*)
thm *iMOD-iprev*
lemma *iMOD-iprev-diff-const*:
$[\![\ t \in [r,\ mod\ m];\ r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r,\ mod\ m] = m$
apply (*simp add*: *iMOD-iprev iT-iff*)
apply (*drule less-mod-eq-imp-add-divisor-le*[**where** m=m], *simp*+)
done

thm *iMODb-inext*
lemma *iMODb-inext-diff-const*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow inext\ t\ [r,\ mod\ m,\ c] - t = m$
by (*simp add*: *iMODb-inext*)
thm *iMODb-iprev'*
lemma *iMODb-iprev-diff-const'*:
$[\![\ t \in [r,\ mod\ m,\ c];\ t < r + m * c\ ]\!] \Longrightarrow (t + m) - iprev\ (t + m)\ [r,\ mod\ m,$
$c] = m$
by (*simp add*: *iMODb-iprev'*)
thm *iMODb-iprev*
lemma *iMODb-iprev-diff-const*:
$[\![\ t \in [r,\ mod\ m,\ c];\ r < t\ ]\!] \Longrightarrow t - iprev\ t\ [r,\ mod\ m,\ c] = m$
apply (*simp add*: *iMODb-iprev iT-iff*)
apply (*drule less-mod-eq-imp-add-divisor-le*[**where** m=m], *simp*+)
done

lemmas *iT-inext-diff-const* =
iFROM-inext-diff-const
iTILL-inext-diff-const
iIN-inext-diff-const
iMOD-inext-diff-const
iMODb-inext-diff-const
lemmas *iT-iprev-diff-const* =
iFROM-iprev-diff-const
iFROM-iprev-diff-const'
iTILL-iprev-diff-const
iIN-iprev-diff-const
iIN-iprev-diff-const'
iMOD-iprev-diff-const'
iMOD-iprev-diff-const
iMODb-iprev-diff-const'
iMODb-iprev-diff-const
thm *iT-inext-diff-const*
thm *iT-iprev-diff-const*

1.7.1 Mirroring of intervals

thm *mirror-elem-def*
lemma
iIN-mirror-elem: *mirror-elem x* [*n*...,*d*] = *n* + *n* + *d* − *x* **and**
iTILL-mirror-elem: *mirror-elem x* [...*n*] = *n* − *x* **and**
iMODb-mirror-elem: *mirror-elem x* [*r*, *mod m*, *c*] = *r* + *r* + *m* ∗ *c* − *x*
by (*simp add*: *mirror-elem-def nat-mirror-def iT-Min iT-Max*)+

lemma *iMODb-imirror-bounds*:
r' + *m'* ∗ *c'* ≤ *l* + *r* ⟹
imirror-bounds [*r'*, *mod m'*, *c'*] *l r* = [*l* + *r* − *r'* − *m'* ∗ *c'*, *mod m'*, *c'*]
apply (*clarsimp simp*: *set-eq-iff Bex-def imirror-bounds-iff iT-iff*)
apply (*frule diff-le-mono*[*of - - r'*], *simp*)
thm *mod-diff-right-eq*
apply (*simp add*: *mod-diff-right-eq*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac x'*)
thm *mod-diff-right-eq*
apply (*rule-tac a=x'* **in** *ssubst*[*OF mod-diff-right-eq*, *rule-format*], *simp*+)
apply (*simp add*: *diff-le-mono2*)
apply *clarsimp*
apply (*rule-tac x=l+r−x* **in** *exI*)
apply (*simp add*: *le-diff-swap*)
apply (*simp add*: *le-diff-conv2*)
thm *mod-sub-eq-mod-swap*
apply (*subst mod-sub-eq-mod-swap*, *simp*+)

thm *mod-diff-right-eq*
apply (*simp add*: *mod-diff-right-eq*[*symmetric*])
done

thm *imirror-bounds-def*
lemma *iIN-imirror-bounds*:
  $n + d \leq l + r \Longrightarrow$ *imirror-bounds* $[n\ldots,d]$ l $r = [l + r - n - d\ldots,d]$
apply (*insert iMODb-imirror-bounds*[*of n Suc 0 d l r*])
apply (*simp add*: *iMODb-mod-1*)
done

lemma *iTILL-imirror-bounds*:
  $n \leq l + r \Longrightarrow$ *imirror-bounds* $[\ldots n]$ l $r = [l + r - n\ldots,n]$
apply (*insert iIN-imirror-bounds*[*of 0 n l r*])
apply (*simp add*: *iIN-0-iTILL-conv*)
done

lemmas *iT-imirror-bounds* =
  iTILL-imirror-bounds
  iIN-imirror-bounds
  iMODb-imirror-bounds
thm *iT-imirror-bounds*

lemma *iMODb-imirror-ident*: *imirror* $[r,\ mod\ m,\ c] = [r,\ mod\ m,\ c]$
by (*simp add*: *imirror-eq-imirror-bounds iMODb-Min iMODb-Max iMODb-imirror-bounds*)
lemma *iIN-imirror-ident*: *imirror* $[n\ldots,d] = [n\ldots,d]$
by (*simp add*: *iMODb-mod-1*[*symmetric*] *iMODb-imirror-ident*)
lemma *iTILL-imirror-ident*: *imirror* $[\ldots n] = [\ldots n]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-imirror-ident*)

lemmas *iT-imirror-ident* =
  iTILL-imirror-ident
  iIN-imirror-ident
  iMODb-imirror-ident
thm *iT-imirror-ident*

1.7.2 Functions *inext-nth* and *iprev-nth* on intervals

lemma *iFROM-inext-nth* : $[n\ldots] \rightarrow a = n + a$
by (*simp add*: *iT-defs inext-nth-atLeast*)
lemma *iIN-inext-nth* : $a \leq d \Longrightarrow [n\ldots,d] \rightarrow a = n + a$
by (*simp add*: *iT-defs inext-nth-atLeastAtMost*)
lemma *iIN-iprev-nth*: $a \leq d \Longrightarrow [n\ldots,d] \leftarrow a = n + d - a$
by (*simp add*: *iT-defs iprev-nth-atLeastAtMost*)
lemma *iIN-inext-nth-if* :
  $[n\ldots,d] \rightarrow a =$ (*if* $a \leq d$ *then* $n + a$ *else* $n + d$)
by (*simp add*: *iIN-inext-nth inext-nth-card-Max iT-finite iT-not-empty iT-Max iT-card*)

lemma *iIN-iprev-nth-if*:
$[n..,d] \leftarrow a = (\mathit{if}\ a \leq d\ \mathit{then}\ n + d - a\ \mathit{else}\ n)$
by (*simp add*: *iIN-iprev-nth iprev-nth-card-iMin iT-finite iT-not-empty iT-Min iT-card*)

lemma *iTILL-inext-nth* : $a \leq n \Longrightarrow [..n] \rightarrow a = a$
by (*simp add*: *iTILL-def inext-nth-atMost*)
lemma *iTILL-inext-nth-if* :
$[..n] \rightarrow a = (\mathit{if}\ a \leq n\ \mathit{then}\ a\ \mathit{else}\ n)$
by (*insert iIN-inext-nth-if*[*of 0 n a*], *simp add*: *iIN-0-iTILL-conv*)
lemma *iTILL-iprev-nth*: $a \leq n \Longrightarrow [..n] \leftarrow a = n - a$
by (*simp add*: *iTILL-def iprev-nth-atMost*)
lemma *iTILL-iprev-nth-if*:
$[..n] \leftarrow a = (\mathit{if}\ a \leq n\ \mathit{then}\ n - a\ \mathit{else}\ 0)$
by (*insert iIN-iprev-nth-if*[*of 0 n a*], *simp add*: *iIN-0-iTILL-conv*)

lemma *iMOD-inext-nth*: $[r, \mathit{mod}\ m] \rightarrow a = r + m * a$
apply (*induct a*)
apply (*simp add*: *iT-Min*)
apply (*simp add*: *iMOD-inext-if iT-iff*)
done

lemma *iMODb-inext-nth*: $a \leq c \Longrightarrow [r, \mathit{mod}\ m, c] \rightarrow a = r + m * a$
apply (*case-tac m = 0*)
apply (*simp add*: *iMODb-mod-0 iIN-0 inext-nth-singleton*)
apply (*induct a*)
apply (*simp add*: *iMODb-Min*)
apply (*simp add*: *iMODb-inext-if iT-iff*)
done
lemma *iMODb-inext-nth-if*:
$[r, \mathit{mod}\ m, c] \rightarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * a\ \mathit{else}\ r + m * c)$
by (*simp add*: *iMODb-inext-nth inext-nth-card-Max iT-finite iT-not-empty iT-Max iT-card*)

lemma *iMODb-iprev-nth*:
$a \leq c \Longrightarrow [r, \mathit{mod}\ m, c] \leftarrow a = r + m * (c - a)$
apply (*case-tac m = 0*)
apply (*simp add*: *iMODb-mod-0 iIN-0 iprev-nth-singleton*)
apply (*induct a*)
apply (*simp add*: *iMODb-Max*)
apply (*simp add*: *iMODb-iprev-if iT-iff*)
apply (*frule mult-left-mono*[*of - - m*], *simp*)
apply (*simp add*: *diff-mult-distrib2*)
done
lemma *iMODb-iprev-nth-if*:
$[r, \mathit{mod}\ m, c] \leftarrow a = (\mathit{if}\ a \leq c\ \mathit{then}\ r + m * (c - a)\ \mathit{else}\ r)$
by (*simp add*: *iMODb-iprev-nth iprev-nth-card-iMin iT-finite iT-not-empty iT-Min iT-card*)

lemma *iIN-iFROM-inext-nth*:
$a \leq d \Longrightarrow [n..,d] \rightarrow a = [n..] \rightarrow a$
by (*simp add*: *iIN-inext-nth iFROM-inext-nth*)
lemma *iIN-iFROM-inext*:
$a < n + d \Longrightarrow inext\ a\ [n..,d] = inext\ a\ [n..]$
by (*simp add*: *iT-inext-if iT-iff*)
lemma *iMOD-iMODb-inext-nth*:
$a \leq c \Longrightarrow [r,\ mod\ m,\ c] \rightarrow a = [r,\ mod\ m] \rightarrow a$
by (*simp add*: *iMOD-inext-nth iMODb-inext-nth*)
lemma *iMOD-iMODb-inext*:
$a < r + m * c \Longrightarrow inext\ a\ [r,\ mod\ m,\ c] = inext\ a\ [r,\ mod\ m]$
by (*simp add*: *iT-inext-if iT-iff*)

lemma *iMOD-inext-nth-Suc-diff*:
$([r,\ mod\ m] \rightarrow (Suc\ n)) - ([r,\ mod\ m] \rightarrow n) = m$
by (*simp add*: *iMOD-inext-nth del*: *inext-nth.simps*)
lemma *iMOD-inext-nth-diff*:
$([r,\ mod\ m] \rightarrow a) - ([r,\ mod\ m] \rightarrow b) = (a - b) * m$
by (*simp add*: *iMOD-inext-nth diff-mult-distrib mult-commute*[*of m*])
lemma *iMODb-inext-nth-diff*: $[\![\ a \leq c;\ b \leq c\ ]\!] \Longrightarrow$
$([r,\ mod\ m,\ c] \rightarrow a) - ([r,\ mod\ m,\ c] \rightarrow b) = (a - b) * m$
by (*simp add*: *iMODb-inext-nth diff-mult-distrib mult-commute*[*of m*])

1.8 Induction with intervals

thm
inext-induct
iFROM-inext
lemma *iFROM-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n..];\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n..]\ ]\!] \Longrightarrow P\ a$
apply (*rule inext-induct*[*of* - [*n*..]])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iIN-induct*:
$[\![\ P\ n;\ \bigwedge k.\ [\![\ k \in [n..,d];\ k \neq n + d;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [n..,d]\ ]\!] \Longrightarrow$
$P\ a$
apply (*rule inext-induct*[*of* - [*n*..,*d*]])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iTILL-induct*:
$[\![\ P\ 0;\ \bigwedge k.\ [\![\ k \in [..n];\ k \neq n;\ P\ k\ ]\!] \Longrightarrow P\ (Suc\ k);\ a \in [..n]\ ]\!] \Longrightarrow P\ a$
apply (*rule inext-induct*[*of* - [..*n*]])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iMOD-induct*:
⟦ *P r*; ⋀*k*. ⟦ *k* ∈ [*r*, *mod m*]; *P k* ⟧ ⟹ *P* (*k* + *m*); *a* ∈ [*r*, *mod m*] ⟧ ⟹ *P a*
apply (*rule inext-induct*[*of* - [*r*, *mod m*]])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

lemma *iMODb-induct*:
⟦ *P r*; ⋀*k*. ⟦ *k* ∈ [*r*, *mod m*, *c*]; *k* ≠ *r* + *m* ∗ *c*; *P k* ⟧ ⟹ *P* (*k* + *m*); *a* ∈ [*r*, *mod m*, *c*] ⟧ ⟹ *P a*
apply (*rule inext-induct*[*of* - [*r*, *mod m*, *c*]])
apply (*simp add*: *iT-Min iT-inext-if*)+
done

thm
iprev-induct
iIN-inext
lemma *iIN-rev-induct*:
⟦ *P* (*n* + *d*); ⋀*k*. ⟦ *k* ∈ [*n*...,*d*]; *k* ≠ *n*; *P k* ⟧ ⟹ *P* (*k* − *Suc 0*); *a* ∈ [*n*...,*d*] ⟧ ⟹ *P a*
apply (*rule iprev-induct*[*of* - [*n*...,*d*]])
apply (*simp add*: *iT-Max iT-finite iT-iprev-if*)+
done

lemma *iTILL-rev-induct*:
⟦ *P n*; ⋀*k*. ⟦ *k* ∈ [...*n*]; *0* < *k*; *P k* ⟧ ⟹ *P* (*k* − *Suc 0*); *a* ∈ [...*n*] ⟧ ⟹ *P a*
apply (*rule iprev-induct*[*of* - [...*n*]])
apply (*fastsimp simp*: *iT-Max iT-finite iT-iprev-if*)+
done

lemma *iMODb-rev-induct*:
⟦ *P* (*r* + *m* ∗ *c*); ⋀*k*. ⟦ *k* ∈ [*r*, *mod m*, *c*]; *k* ≠ *r*; *P k* ⟧ ⟹ *P* (*k* − *m*); *a* ∈ [*r*, *mod m*, *c*] ⟧ ⟹ *P a*
apply (*rule iprev-induct*[*of* - [*r*, *mod m*, *c*]])
apply (*simp add*: *iT-Max iT-finite iT-iprev-if*)+
done

end

2 IL-IntervalOperators: Arithmetic operators on natural intervals

theory *IL-IntervalOperators*
imports *IL-Interval*
begin

2.1 Arithmetic operations with intervals

2.1.1 Addition of and multiplication by constants

definition *iT-Plus* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\oplus$ *55*) **where**
$I \oplus k \equiv (\lambda n.(n + k))$ ' I

definition *iT-Mult* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\otimes$ *55*) **where**
iT-Mult-def : $I \otimes k \equiv (\lambda n.(n * k))$ ' I

lemma *iT-Plus-image-conv*: $I \oplus k = (\lambda n.(n + k))$ ' I
by (*simp add*: *iT-Plus-def*)
lemma *iT-Mult-image-conv*: $I \otimes k = (\lambda n.(n * k))$ ' I
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-empty*: $\{\} \oplus k = \{\}$
by (*simp add*: *iT-Plus-def*)
lemma *iT-Mult-empty*: $\{\} \otimes k = \{\}$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-not-empty*: $I \neq \{\} \Longrightarrow I \oplus k \neq \{\}$
by (*simp add*: *iT-Plus-def*)
lemma *iT-Mult-not-empty*: $I \neq \{\} \Longrightarrow I \otimes k \neq \{\}$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-empty-iff*: $(I \oplus k = \{\}) = (I = \{\})$
by (*simp add*: *iT-Plus-def*)
lemma *iT-Mult-empty-iff*: $(I \otimes k = \{\}) = (I = \{\})$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-mono*: $A \subseteq B \Longrightarrow A \oplus k \subseteq B \oplus k$
by (*simp add*: *iT-Plus-def image-mono*)
lemma *iT-Mult-mono*: $A \subseteq B \Longrightarrow A \otimes k \subseteq B \otimes k$
by (*simp add*: *iT-Mult-def image-mono*)

lemma *iT-Mult-0*: $I \neq \{\} \Longrightarrow I \otimes 0 = [\ldots 0]$
by (*fastsimp simp add*: *iTILL-def iT-Mult-def*)
corollary *iT-Mult-0-if*: $I \otimes 0 = (\textit{if } I = \{\} \textit{ then } \{\} \textit{ else } [\ldots 0])$
by (*simp add*: *iT-Mult-empty iT-Mult-0*)

lemma *iT-Plus-mem-iff*: $x \in (I \oplus k) = (k \leq x \wedge (x - k) \in I)$
apply (*simp add*: *iT-Plus-def image-iff*)
apply (*rule iffI*)
apply *fastsimp*
apply (*rule-tac x*=$x - k$ **in** *bexI*, *simp*+)
done
lemma *iT-Plus-mem-iff2*: $x + k \in (I \oplus k) = (x \in I)$

by (*simp add*: *iT-Plus-def image-iff*)

lemma *iT-Mult-mem-iff-0*: $x \in (I \otimes 0) = (I \neq \{\} \wedge x = 0)$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Mult-empty*)
apply (*simp add*: *iT-Mult-0 iT-iff*)
done
lemma *iT-Mult-mem-iff*:
$0 < k \Longrightarrow x \in (I \otimes k) = (x \ mod \ k = 0 \wedge x \ div \ k \in I)$
by (*fastsimp simp*: *iT-Mult-def image-iff*)
lemma *iT-Mult-mem-iff2*: $0 < k \Longrightarrow x * k \in (I \otimes k) = (x \in I)$
by (*simp add*: *iT-Mult-def image-iff*)

lemma *iT-Plus-singleton*: $\{a\} \oplus k = \{a + k\}$
by (*simp add*: *iT-Plus-def*)
lemma *iT-Mult-singleton*: $\{a\} \otimes k = \{a * k\}$
by (*simp add*: *iT-Mult-def*)

lemma *iT-Plus-Un*: $(A \cup B) \oplus k = (A \oplus k) \cup (B \oplus k)$
by (*simp add*: *iT-Plus-def image-Un*)
lemma *iT-Mult-Un*: $(A \cup B) \otimes k = (A \otimes k) \cup (B \otimes k)$
by (*simp add*: *iT-Mult-def image-Un*)

lemma *iT-Plus-Int*: $(A \cap B) \oplus k = (A \oplus k) \cap (B \oplus k)$
by (*simp add*: *iT-Plus-def image-Int*)
lemma *iT-Mult-Int*: $0 < k \Longrightarrow (A \cap B) \otimes k = (A \otimes k) \cap (B \otimes k)$
by (*simp add*: *iT-Mult-def image-Int mult-right-inj*)

thm
iT-Plus-Un
iT-Mult-Un
iT-Plus-Int
iT-Mult-Int

thm *imageI*
lemma *iT-Plus-image*: $f \ ` \ I \oplus n = (\lambda x.\ f\, x + n) \ ` \ I$
by (*fastsimp simp*: *iT-Plus-def*)
lemma *iT-Mult-image*: $f \ ` \ I \otimes n = (\lambda x.\ f\, x * n) \ ` \ I$
by (*fastsimp simp*: *iT-Mult-def*)

lemma *iT-Plus-commute*: $I \oplus a \oplus b = I \oplus b \oplus a$
by (*fastsimp simp*: *iT-Plus-def*)
lemma *iT-Mult-commute*: $I \otimes a \otimes b = I \otimes b \otimes a$
by (*fastsimp simp*: *iT-Mult-def*)
lemma *iT-Plus-assoc*:$I \oplus a \oplus b = I \oplus (a + b)$
by (*fastsimp simp*: *iT-Plus-def*)
lemma *iT-Mult-assoc*:$I \otimes a \otimes b = I \otimes (a * b)$

by (*fastsimp simp*: *iT-Mult-def*)
lemma *iT-Plus-Mult-distrib*: $I \oplus n \otimes m = I \otimes m \oplus n * m$
by (*simp add*: *iT-Plus-def iT-Mult-def image-image add-mult-distrib*)

lemma *iT-Plus-finite-iff*: *finite* $(I \oplus k)$ = *finite I*
by (*simp add*: *iT-Plus-def inj-on-finite-image-iff*)
lemma *iT-Mult-0-finite*: *finite* $(I \otimes 0)$
by (*simp add*: *iT-Mult-0-if iTILL-0*)
lemma *iT-Mult-finite-iff*: $0 < k \Longrightarrow$ *finite* $(I \otimes k)$ = *finite I*
by (*simp add*: *iT-Mult-def inj-on-finite-image-iff* [*OF inj-imp-inj-on*] *mult-right-inj*)

lemma *iT-Plus-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oplus k) = iMin\ I + k$
by (*simp add*: *iT-Plus-def iMin-mono2 mono-def*)
lemma *iT-Mult-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \otimes k) = iMin\ I * k$
by (*simp add*: *iT-Mult-def iMin-mono2 mono-def*)

thm *Max-mono2*
lemma *iT-Plus-Max*: $[\![$ *finite I*; $I \neq \{\}$ $]\!] \Longrightarrow Max\ (I \oplus k) = Max\ I + k$
by (*simp add*: *iT-Plus-def Max-mono2 mono-def*)
lemma *iT-Mult-Max*: $[\![$ *finite I*; $I \neq \{\}$ $]\!] \Longrightarrow Max\ (I \otimes k) = Max\ I * k$
by (*simp add*: *iT-Mult-def Max-mono2 mono-def*)

thm *iT-Mult-0*
corollary
iMOD-mult-0: $[r, mod\ m] \otimes 0 = [\ldots 0]$ **and**
iMODb-mult-0: $[r, mod\ m, c] \otimes 0 = [\ldots 0]$ **and**
iFROM-mult-0: $[n\ldots] \otimes 0 = [\ldots 0]$ **and**
iIN-mult-0: $[n\ldots,d] \otimes 0 = [\ldots 0]$ **and**
iTILL-mult-0: $[\ldots n] \otimes 0 = [\ldots 0]$
by (*simp add*: *iT-not-empty iT-Mult-0*)+

lemmas *iT-mult-0* =
iTILL-mult-0
iFROM-mult-0
iIN-mult-0
iMOD-mult-0
iMODb-mult-0
thm *iT-mult-0*

lemma *iT-Plus-0*: $I \oplus 0 = I$
by (*simp add*: *iT-Plus-def*)

lemma *iT-Mult-1*: $I \otimes Suc\ 0 = I$
by (*simp add*: *iT-Mult-def*)

thm *iT-Plus-Min*

corollary
iFROM-add-Min: *iMin* $([n...] \oplus k) = n + k$ **and**
iIN-add-Min: *iMin* $([n...,d] \oplus k) = n + k$ **and**
iTILL-add-Min: *iMin* $([...n] \oplus k) = k$ **and**
iMOD-add-Min: *iMin* $([r, mod\ m] \oplus k) = r + k$ **and**
iMODb-add-Min: *iMin* $([r, mod\ m, c] \oplus k) = r + k$
by (*simp add*: *iT-not-empty iT-Plus-Min iT-Min*)+

corollary
iFROM-mult-Min: *iMin* $([n...] \otimes k) = n * k$ **and**
iIN-mult-Min: *iMin* $([n...,d] \otimes k) = n * k$ **and**
iTILL-mult-Min: *iMin* $([...n] \otimes k) = 0$ **and**
iMOD-mult-Min: *iMin* $([r, mod\ m] \otimes k) = r * k$ **and**
iMODb-mult-Min: *iMin* $([r, mod\ m, c] \otimes k) = r * k$
by (*simp add*: *iT-not-empty iT-Mult-Min iT-Min*)+

lemmas *iT-add-Min* =
iIN-add-Min
iTILL-add-Min
iFROM-add-Min
iMOD-add-Min
iMODb-add-Min
thm *iT-add-Min*

lemmas *iT-mult-Min* =
iIN-mult-Min
iTILL-mult-Min
iFROM-mult-Min
iMOD-mult-Min
iMODb-mult-Min
thm *iT-mult-Min*

lemma *iFROM-add*: $[n...] \oplus k = [n+k...]$
by (*simp add*: *iFROM-def iT-Plus-def image-add-atLeast*)

lemma *iIN-add*: $[n...,d] \oplus k = [n+k...,d]$
by (*fastsimp simp add*: *iIN-def iT-Plus-def image-add-atLeastAtMost*)

lemma *iTILL-add*: $[...i] \oplus k = [k...,i]$
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-add*)

lemma *iMOD-add*: $[r, mod\ m] \oplus k = [r + k, mod\ m]$
apply (*clarsimp simp*: *set-eq-iff iMOD-def iT-Plus-def image-iff*)
apply (*rule iffI*)
thm *mod-add*

apply (*clarsimp simp*: *mod-add*)
apply (*rule-tac x*=*x* − *k* **in** *exI*)
apply *clarsimp*
thm *mod-sub-add*[*of k x*]
apply (*simp add*: *mod-sub-add*)
done

lemma *iMODb-add*: [*r*, *mod m*, *c*] ⊕ *k* = [*r* + *k*, *mod m*, *c*]
by (*simp add*: *iMODb-iMOD-iIN-conv iT-Plus-Int iMOD-add iIN-add*)

lemmas *iT-add* =
iMOD-add
iMODb-add
iFROM-add
iIN-add
iTILL-add
iT-Plus-singleton
thm *iT-add*

thm
mod-mult-distrib
mod-mult-distrib2
thm
mod-eq-mult-distrib
mod-factor-imp-mod-0
mod-factor-div
mod-factor-div-mod

lemma *iFROM-mult*: [*n*...] ⊗ *k* = [*n* ∗ *k*, *mod k*]
apply (*case-tac k* = *0*)
apply (*simp add*: *iMOD-0 iT-mult-0 iIN-0 iTILL-0*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff iT-iff*)
apply (*rule conj-cong*, *simp*)
apply (*rule iffI*)
apply (*drule mult-le-mono1*[*of* - - *k*])
apply (*rule order-trans*, *assumption*)
apply (*simp add*: *div-mult-cancel*)
apply (*drule div-le-mono*[*of* - - *k*])
apply *simp*
done

lemma *iIN-mult*: [*n*...,*d*] ⊗ *k* = [*n* ∗ *k*, *mod k*, *d*]
apply (*case-tac k* = *0*)
apply (*simp add*: *iMODb-mod-0 iT-mult-0 iIN-0 iTILL-0*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff iT-iff*)
apply (*rule conj-cong*, *simp*)
apply (*rule iffI*)
apply (*elim conjE*)
apply (*drule mult-le-mono1*[*of* - - *k*], *drule mult-le-mono1*[*of* - - *k*])

apply (*rule conjI*)
apply (*rule order-trans*, *assumption*)
apply (*simp add*: *div-mult-cancel*)
apply (*simp add*: *div-mult-cancel add-mult-distrib mult-commute*[*of k*])
apply (*erule conjE*)
apply (*drule div-le-mono*[*of - - k*], *drule div-le-mono*[*of - - k*])
apply *simp*
done

lemma *iTILL-mult*: [*...n*] ⊗ *k* = [*0*, *mod k*, *n*]
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-mult*)

lemma *iMOD-mult*: [*r*, *mod m*] ⊗ *k* = [*r* ∗ *k*, *mod m* ∗ *k*]
apply (*case-tac k = 0*)
apply (*simp add*: *iMOD-0 iT-mult-0 iIN-0 iTILL-0*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff iT-iff*)
apply (*subst mult-commute*[*of m k*])
thm *mod-mult2-eq*
apply (*simp add*: *mod-mult2-eq*)
apply (*rule iffI*)
apply (*elim conjE*)
apply (*drule mult-le-mono1*[*of - - k*])
apply (*simp add*: *div-mult-cancel*)
apply (*elim conjE*)
apply (*subgoal-tac x mod k = 0*)
prefer *2*
apply (*drule-tac arg-cong*[**where** *f*=λ*x*. *x mod k*])
apply (*simp add*: *mult-commute*[*of k*])
apply (*drule div-le-mono*[*of - - k*])
apply *simp*
done

lemma *iMODb-mult*:
[*r*, *mod m*, *c*] ⊗ *k* = [*r* ∗ *k*, *mod m* ∗ *k*, *c*]
apply (*case-tac k = 0*)
apply (*simp add*: *iMODb-mod-0 iT-mult-0 iIN-0 iTILL-0*)
thm *iMODb-iMOD-iTILL-conv*
apply (*subst iMODb-iMOD-iTILL-conv*)
apply (*simp add*: *iT-Mult-Int iMOD-mult iTILL-mult iMODb-iMOD-iTILL-conv*)
apply (*subst Int-assoc*[*symmetric*])
thm *Int-absorb2*
apply (*subst Int-absorb2*)
apply (*simp add*: *iMOD-subset*)
thm *iMOD-iTILL-iMODb-conv*
apply (*simp add*: *iMOD-iTILL-iMODb-conv add-mult-distrib2*)
done

lemmas *iT-mult* =

iTILL-mult
iFROM-mult
iIN-mult
iMOD-mult
iMODb-mult
iT-Mult-singleton
thm
iT-mult
iT-mult-0

2.1.2 Some conversions between intervals using constant addition and multiplication

lemma *iFROM-conv*: $[n\ldots] = UNIV \oplus n$
by (*simp add*: *iFROM-0*[*symmetric*] *iFROM-add*)
lemma *iIN-conv*: $[n\ldots,d] = [\ldots d] \oplus n$
by (*simp add*: *iTILL-add*)
lemma *iMOD-conv*: $[r,\ mod\ m] = [0\ldots] \otimes m \oplus r$
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMOD-0 iT-mult-0 iTILL-add*)
by (*simp add*: *iFROM-mult iMOD-add*)
lemma *iMODb-conv*: $[r,\ mod\ m,\ c] = [\ldots c] \otimes m \oplus r$
apply (*case-tac* $m = 0$)
apply (*simp add*: *iMODb-mod-0 iT-mult-0 iTILL-add*)
apply (*simp add*: *iTILL-mult iMODb-add*)
done

Some examples showing the utility of iMODb_conv

lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
apply (*simp add*: *iT-defs*)
apply *safe*
defer *1*
apply *simp+*

The direct proof without iMODb_conv fails

oops
lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Mult-def iT-Plus-def*)
apply *safe*
apply *simp+*
done
lemma $[12,\ mod\ 10,\ 4] = \{12,\ 22,\ 32,\ 42,\ 52\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Mult-def iT-Plus-def*)
apply (*simp add*: *atMost-def*)
apply *safe*
apply *simp+*
done

lemma $[r,\ mod\ m,\ 4] = \{r,\ r+m,\ r+2*m,\ r+3*m,\ r+4*m\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Mult-def iT-Plus-def atMost-def*)
thm *image-Collect*
apply (*simp add*: *image-Collect*)
apply *safe*
apply *fastsimp+*
done

lemma $[2,\ mod\ 10,\ 4] = \{2,\ 12,\ 22,\ 32,\ 42\}$
apply (*simp only*: *iMODb-conv*)
apply (*simp add*: *iT-defs iT-Plus-def iT-Mult-def*)
apply *fastsimp*
done

2.1.3 Subtraction of constants

definition *iT-Plus-neg* :: $iT \Rightarrow Time \Rightarrow iT$ (**infixl** $\oplus-$ *55*) **where**
$I \oplus- k \equiv \{x.\ x + k \in I\}$

lemma *iT-Plus-neg-mem-iff*: $(x \in I \oplus- k) = (x + k \in I)$
by (*simp add*: *iT-Plus-neg-def*)
thm *iT-Plus-mem-iff2*
lemma *iT-Plus-neg-mem-iff2*: $k \leq x \Longrightarrow (x - k \in I \oplus- k) = (x \in I)$
by (*simp add*: *iT-Plus-neg-def*)

lemma *iT-Plus-neg-image-conv*: $I \oplus- k = (\lambda n.(n - k))\ `\ (I \downarrow\geq k)$
apply (*simp add*: *iT-Plus-neg-def cut-ge-def*, *safe*)
thm *image-eqI*
apply (*rule-tac x=x+k* **in** *image-eqI*)
apply *simp+*
done

lemma *iT-Plus-neg-cut-eq*: $t \leq k \Longrightarrow (I \downarrow\geq t) \oplus- k = I \oplus- k$
by (*simp add*: *set-eq-iff iT-Plus-neg-mem-iff cut-ge-mem-iff*)

lemma *iT-Plus-neg-mono*: $A \subseteq B \Longrightarrow A \oplus- k \subseteq B \oplus- k$
by (*simp add*: *iT-Plus-neg-def subset-iff*)

lemma *iT-Plus-neg-empty*: $\{\} \oplus- k = \{\}$
by (*simp add*: *iT-Plus-neg-def*)
lemma *iT-Plus-neg-Max-less-empty*:
$[\![\ finite\ I;\ Max\ I < k\ ]\!] \Longrightarrow I \oplus- k = \{\}$
by (*simp add*: *iT-Plus-neg-image-conv cut-ge-Max-empty*)

lemma *iT-Plus-neg-not-empty-iff*: $(I \oplus- k \neq \{\}) = (\exists\, x{\in}I.\ k \leq x)$
by (*simp add*: *iT-Plus-neg-image-conv cut-ge-not-empty-iff*)

lemma *iT-Plus-neg-empty-iff*:
$(I \oplus\!- k = \{\}) = (I = \{\} \vee (\mathit{finite}\ I \wedge \mathit{Max}\ I < k))$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Plus-neg-empty*)
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*case-tac infinite I*)
apply (*simp add*: *nat-cut-ge-infinite-not-empty*)
apply (*simp add*: *cut-ge-empty-iff*)
done

lemma *iT-Plus-neg-assoc*: $(I \oplus\!- a) \oplus\!- b = I \oplus\!- (a + b)$
apply (*simp add*: *iT-Plus-neg-def*)
apply (*simp add*: *add-assoc add-commute*[*of b*])
done

lemma *iT-Plus-neg-commute*: $I \oplus\!- a \oplus\!- b = I \oplus\!- b \oplus\!- a$
by (*simp add*: *iT-Plus-neg-assoc add-commute*[*of b*])

lemma *iT-Plus-neg-0*: $I \oplus\!- 0 = I$
by (*simp add*: *iT-Plus-neg-image-conv cut-ge-0-all*)

thm *diff-add-assoc*
lemma *iT-Plus-Plus-neg-assoc*: $b \leq a \Longrightarrow I \oplus a \oplus\!- b = I \oplus (a - b)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*clarsimp simp add*: *set-eq-iff image-iff Bex-def cut-ge-mem-iff iT-Plus-mem-iff*)
apply (*rule iffI*)
apply *fastsimp*
apply (*rule-tac* $x{=}x + b$ **in** *exI*)
apply (*simp add*: *le-diff-conv*)
done
lemma *iT-Plus-Plus-neg-assoc2*: $a \leq b \Longrightarrow I \oplus a \oplus\!- b = I \oplus\!- (b - a)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*clarsimp simp add*: *set-eq-iff image-iff Bex-def cut-ge-mem-iff iT-Plus-mem-iff*)
apply (*rule iffI*)
apply *fastsimp*
apply (*clarify*, *rename-tac* x')
apply (*rule-tac* $x{=}x' + a$ **in** *exI*)
apply *simp*
done

lemma *iT-Plus-neg-Plus-le-cut-eq*:
$a \leq b \Longrightarrow (I \oplus\!- a) \oplus b = (I \downarrow\!\geq a) \oplus (b - a)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*clarsimp simp add*: *set-eq-iff image-iff Bex-def cut-ge-mem-iff iT-Plus-mem-iff*)
apply (*rule iffI*)
apply (*clarify*, *rename-tac* x')
apply (*subgoal-tac* $x' = x + a - b$)
prefer *2*

apply *simp*
apply (*simp add*: *le-imp-diff-le le-add-diff*)
apply *fastsimp*
done
corollary *iT-Plus-neg-Plus-le-Min-eq*:
$[\![a \leq b; a \leq iMin\ I]\!] \Longrightarrow (I \oplus- a) \oplus b = I \oplus (b - a)$
by (*simp add*: *iT-Plus-neg-Plus-le-cut-eq cut-ge-Min-all*)

lemma *iT-Plus-neg-Plus-ge-cut-eq*:
$b \leq a \Longrightarrow (I \oplus- a) \oplus b = (I \downarrow\geq a) \oplus- (a - b)$
apply (*simp add*: *iT-Plus-neg-image-conv iT-Plus-def cut-cut-ge max-eqL*)
apply (*subst image-compose*[*symmetric*])
apply (*rule image-cong*, *simp*)
apply (*simp add*: *cut-ge-mem-iff*)
done

corollary *iT-Plus-neg-Plus-ge-Min-eq*:
$[\![b \leq a; a \leq iMin\ I]\!] \Longrightarrow (I \oplus- a) \oplus b = I \oplus- (a - b)$
by (*simp add*: *iT-Plus-neg-Plus-ge-cut-eq cut-ge-Min-all*)

lemma *iT-Plus-neg-Mult-distrib*:
$0 < m \Longrightarrow I \oplus- n \otimes m = I \otimes m \oplus- n * m$
apply (*clarsimp simp*: *set-eq-iff iT-Plus-neg-image-conv image-iff iT-Plus-def iT-Mult-def Bex-def cut-ge-mem-iff*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac x'*)
apply (*rule-tac x=x' * m* **in** *exI*)
apply (*simp add*: *diff-mult-distrib*)
apply (*clarsimp*, *rename-tac x'*)
apply (*rule-tac x=x' − n* **in** *exI*)
apply (*simp add*: *diff-mult-distrib*)
apply *fastsimp*
done

thm *le-add-diff-inverse2*
lemma *iT-Plus-neg-Plus-le-inverse*: $k \leq iMin\ I \Longrightarrow I \oplus- k \oplus k = I$
by (*simp add*: *iT-Plus-neg-Plus-le-Min-eq iT-Plus-0*)
lemma *iT-Plus-neg-Plus-inverse*: $I \oplus- k \oplus k = I \downarrow\geq k$
by (*simp add*: *iT-Plus-neg-Plus-ge-cut-eq iT-Plus-neg-0*)

thm *diff-add-inverse2*
lemma *iT-Plus-Plus-neg-inverse*: $I \oplus k \oplus- k = I$
by (*simp add*: *iT-Plus-Plus-neg-assoc iT-Plus-0*)

lemma *iT-Plus-neg-Un*: $(A \cup B) \oplus- k = (A \oplus- k) \cup (B \oplus- k)$
by (*fastsimp simp*: *iT-Plus-neg-def*)

lemma *iT-Plus-neg-Int*: $(A \cap B) \oplus\!- k = (A \oplus\!- k) \cap (B \oplus\!- k)$
by (*fastsimp simp*: *iT-Plus-neg-def*)

lemma *iT-Plus-neg-Max-singleton*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow I \oplus\!- Max\ I = \{0\}$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Plus-neg-def*)
apply (*case-tac* $x = 0$)
apply *simp*
apply *fastsimp*
done

lemma *iT-Plus-neg-singleton*: $\{a\} \oplus\!- k = (\textit{if } k \leq a \textit{ then } \{a - k\} \textit{ else } \{\})$
by (*force simp add*: *set-eq-iff iT-Plus-neg-mem-iff singleton-iff*)
corollary *iT-Plus-neg-singleton1*: $k \leq a \Longrightarrow \{a\} \oplus\!- k = \{a-k\}$
by (*simp add*: *iT-Plus-neg-singleton*)
corollary *iT-Plus-neg-singleton2*: $a < k \Longrightarrow \{a\} \oplus\!- k = \{\}$
by (*simp add*: *iT-Plus-neg-singleton*)

lemma *iT-Plus-neg-finite-iff*: *finite* $(I \oplus\!- k) =$ *finite* I
apply (*simp add*: *iT-Plus-neg-image-conv*)
thm *inj-on-finite-image-iff*
apply (*simp add*: *inj-on-finite-image-iff inj-on-diff-nat cut-ge-mem-iff*)
thm *nat-cut-ge-finite-iff*
apply (*simp add*: *nat-cut-ge-finite-iff*)
done

lemma *iT-Plus-neg-Min*:
$I \oplus\!- k \neq \{\} \Longrightarrow iMin\ (I \oplus\!- k) = iMin\ (I \downarrow\geq k) - k$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*simp add*: *iMin-mono2 monoI*)
done

lemma *iT-Plus-neg-Max*:
$[\![$ *finite* I; $I \oplus\!- k \neq \{\}$ $]\!] \Longrightarrow Max\ (I \oplus\!- k) = Max\ I - k$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*simp add*: *Max-mono2 monoI cut-ge-finite cut-ge-Max-eq*)
done

Subtractions of constants from intervals

lemma *iFROM-add-neg*: $[n\ldots] \oplus\!- k = [n - k\ldots]$
by (*fastsimp simp*: *set-eq-iff iT-Plus-neg-mem-iff*)

lemma *iTILL-add-neg*: $[\ldots n] \oplus\!- k = (\textit{if } k \leq n \textit{ then } [\ldots n - k] \textit{ else } \{\})$
by (*force simp add*: *set-eq-iff iT-Plus-neg-mem-iff iT-iff*)
lemma *iTILL-add-neg1*: $k \leq n \Longrightarrow [\ldots n] \oplus\!- k = [\ldots n-k]$
by (*simp add*: *iTILL-add-neg*)
lemma *iTILL-add-neg2*: $n < k \Longrightarrow [\ldots n] \oplus\!- k = \{\}$
by (*simp add*: *iTILL-add-neg*)

lemma *iIN-add-neg*:
[*n*...,*d*] ⊕− *k* = (
if *k* ≤ *n* *then* [*n* − *k*...,*d*]
else *if* *k* ≤ *n* + *d* *then* [...*n* + *d* − *k*] *else* {})
by (*simp add*: *iIN-iFROM-iTILL-conv* *iT-Plus-neg-Int* *iFROM-add-neg* *iTILL-add-neg* *iFROM-0*)
lemma *iIN-add-neg1*: *k* ≤ *n* ⟹ [*n*...,*d*] ⊕− *k* = [*n* − *k*...,*d*]
by (*simp add*: *iIN-add-neg*)
lemma *iIN-add-neg2*: ⟦ *n* ≤ *k*; *k* ≤ *n* + *d* ⟧ ⟹ [*n*...,*d*] ⊕− *k* = [...*n* + *d* − *k*]
by (*simp add*: *iIN-add-neg* *iIN-0-iTILL-conv*)
lemma *iIN-add-neg3*: *n* + *d* < *k* ⟹ [*n*...,*d*] ⊕− *k* = {}
by (*simp add*: *iT-Plus-neg-Max-less-empty* *iT-finite* *iT-Max*)

lemma *iMOD-0-add-neg*: [*r*, *mod* *0*] ⊕− *k* = {*r*} ⊕− *k*
by (*simp add*: *iMOD-0* *iIN-0*)

lemma *iMOD-gr0-add-neg*:
0 < *m* ⟹
[*r*, *mod* *m*] ⊕− *k* = (
if *k* ≤ *r* *then* [*r* − *k*, *mod* *m*]
else [(*m* + *r* *mod* *m* − *k* *mod* *m*) *mod* *m*, *mod* *m*])
apply (*rule* *set-eqI*)
apply (*simp add*: *iMOD-def* *iT-Plus-neg-def*)
apply (*simp add*: *eq-sym-conv*[*of* - *r* *mod* *m*])
apply (*intro* *conjI* *impI*)
thm *mod-sub-add*[*of* *k* *r* *m*] *le-diff-conv*
apply (*simp add*: *eq-sym-conv*[*of* - (*r* − *k*) *mod* *m*] *mod-sub-add* *le-diff-conv*)
thm *eq-commute*[*of* *r* *mod* *m*]
thm *mod-add-eq-mod-conv*[*of* *m* *x* *k* *r*]
apply (*simp add*: *eq-commute*[*of* *r* *mod* *m*] *mod-add-eq-mod-conv*)
apply *safe*
apply (*drule* *sym*)
apply *simp*
done
lemma *iMOD-add-neg*:
[*r*, *mod* *m*] ⊕− *k* = (
if *k* ≤ *r* *then* [*r* − *k*, *mod* *m*]
else *if* *0* < *m* *then* [(*m* + *r* *mod* *m* − *k* *mod* *m*) *mod* *m*, *mod* *m*] *else* {})
apply (*case-tac* *0* < *m*)
apply (*simp add*: *iMOD-gr0-add-neg*)
apply (*simp add*: *iMOD-0* *iIN-0* *iT-Plus-neg-singleton*)
done
corollary *iMOD-add-neg1*:
k ≤ *r* ⟹ [*r*, *mod* *m*] ⊕− *k* = [*r* − *k*, *mod* *m*]
by (*simp add*: *iMOD-add-neg*)
lemma *iMOD-add-neg2*:
⟦ *0* < *m*; *r* < *k* ⟧ ⟹ [*r*, *mod* *m*] ⊕− *k* = [(*m* + *r* *mod* *m* − *k* *mod* *m*) *mod* *m*,

mod m]
by (*simp add*: *iMOD-add-neg*)

lemma *iMODb-mod-0-add-neg*: [*r*, *mod 0*, *c*] ⊕− *k* = {*r*} ⊕− *k*
by (*simp add*: *iMODb-mod-0 iIN-0*)

lemma *iMODb-add-neg*:
[*r*, *mod m*, *c*] ⊕− *k* = (
if k ≤ *r then* [*r* − *k*, *mod m*, *c*]
else
if k ≤ *r* + *m* ∗ *c then*
[(*m* + *r mod m* − *k mod m*) *mod m*, *mod m*, (*r* + *m* ∗ *c* − *k*) *div m*]
else {})
apply (*clarsimp simp add*: *iMODb-iMOD-iIN-conv iT-Plus-neg-Int iMOD-add-neg iIN-add-neg*)
apply (*simp add*: *iMOD-iIN-iMODb-conv*)
apply (*rule-tac t*=(*m* + *r mod m* − *k mod m*) *mod m* **and** *s*=(*r* + *m* ∗ *c* − *k*) *mod m* **in** *subst*)
thm *mod-diff1-eq*[*of k*]
apply (*simp add*: *mod-diff1-eq*[*of k*])
apply (*subst iMOD-iTILL-iMODb-conv*, *simp*)
thm *sub-mod-div-eq-div*
apply (*subst sub-mod-div-eq-div*, *simp*)
done

lemma *iMODb-add-neg'*:
[*r*, *mod m*, *c*] ⊕− *k* = (
if k ≤ *r then* [*r* − *k*, *mod m*, *c*]
else if k ≤ *r* + *m* ∗ *c then*
if k mod m ≤ *r mod m*
then [*r mod m* − *k mod m*, *mod m*, *c* + *r div m* − *k div m*]
else [*m* + *r mod m* − *k mod m*, *mod m*, *c* + *r div m* − *Suc* (*k div m*)]
else {})
apply (*clarsimp simp add*: *iMODb-add-neg*)
apply (*case-tac m* = *0*, *simp*+)
apply (*case-tac k mod m* ≤ *r mod m*)
apply (*clarsimp simp*: *linorder-not-le*)
apply (*simp add*: *divisor-add-diff-mod-if*)
apply (*simp add*: *div-diff1-eq-if*)
apply (*clarsimp simp*: *linorder-not-le*)
apply (*simp add*: *div-diff1-eq-if*)
done
corollary *iMODb-add-neg1*:
k ≤ *r* ⟹ [*r*, *mod m*, *c*] ⊕− *k* = [*r* − *k*, *mod m*, *c*]
by (*simp add*: *iMODb-add-neg*)

corollary *iMODb-add-neg2*:
$[\![\ r < k;\ k \leq r + m * c\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oplus- k =$
$[(m + r\ mod\ m - k\ mod\ m)\ mod\ m,\ mod\ m,\ (r + m * c - k)\ div\ m]$
by (*simp add*: *iMODb-add-neg*)
corollary *iMODb-add-neg2-mod-le*:
$[\![\ r < k;\ k \leq r + m * c;\ k\ mod\ m \leq r\ mod\ m\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oplus- k =$
$[\ r\ mod\ m - k\ mod\ m,\ mod\ m,\ c + r\ div\ m - k\ div\ m]$
by (*simp add*: *iMODb-add-neg'*)
corollary *iMODb-add-neg2-mod-less*:
$[\![\ r < k;\ k \leq r + m * c;\ r\ mod\ m < k\ mod\ m]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oplus- k =$
$[\ m + r\ mod\ m - k\ mod\ m,\ mod\ m,\ c + r\ div\ m - Suc\ (k\ div\ m)\ ]$
by (*simp add*: *iMODb-add-neg'*)
lemma *iMODb-add-neg3*: $r + m * c < k \Longrightarrow [r,\ mod\ m,\ c] \oplus- k = \{\}$
by (*simp add*: *iMODb-add-neg*)

lemmas *iT-add-neg* =
iFROM-add-neg
iIN-add-neg
iTILL-add-neg
iMOD-add-neg
iMODb-add-neg
iT-Plus-neg-singleton
thm *iT-add-neg*

2.1.4 Subtraction of intervals from constants

definition *iT-Minus* :: *Time* ⇒ *iT* ⇒ *iT* (**infixl** ⊖ *55*) **where**
$k \ominus I \equiv \{x.\ x \leq k \wedge (k - x) \in I\}$

lemma *iT-Minus-mem-iff*: $(x \in k \ominus I) = (x \leq k \wedge k - x \in I)$
by (*simp add*: *iT-Minus-def*)

lemma *iT-Minus-mono*: $A \subseteq B \Longrightarrow k \ominus A \subseteq k \ominus B$
by (*simp add*: *subset-iff iT-Minus-mem-iff*)

lemma *iT-Minus-image-conv*: $k \ominus I = (\lambda x.\ k - x)\ `\ (I \downarrow\leq k)$
by (*fastsimp simp*: *iT-Minus-def cut-le-def image-iff*)

lemma *iT-Minus-cut-eq*: $k \leq t \Longrightarrow k \ominus (I \downarrow\leq t) = k \ominus I$
by (*fastsimp simp*: *set-eq-iff iT-Minus-mem-iff*)

lemma *iT-Minus-Minus-cut-eq*: $k \ominus (k \ominus (I \downarrow\leq k)) = I \downarrow\leq k$
by (*fastsimp simp*: *iT-Minus-def*)

lemma *10* $\ominus$ [...*3*] = [*7*...,*3*]
by (*fastsimp simp*: *iT-Minus-def*)

lemma *iT-Minus-empty*: $k \ominus \{\} = \{\}$
by (*simp add*: *iT-Minus-def*)

lemma *iT-Minus-0*: $k \ominus \{0\} = \{k\}$
by (*simp add*: *iT-Minus-image-conv cut-le-def image-Collect*)

lemma *iT-Minus-bound*: $x \in k \ominus I \Longrightarrow x \leq k$
by (*simp add*: *iT-Minus-def*)

lemma *iT-Minus-finite*: *finite* $(k \ominus I)$
thm *finite-nat-iff-bounded-le2*
apply (*rule finite-nat-iff-bounded-le2*[*THEN iffD2*])
apply (*rule-tac* $x{=}k$ **in** *exI*)
apply (*simp add*: *iT-Minus-bound*)
done

lemma *iT-Minus-less-Min-empty*: $k < iMin\ I \Longrightarrow k \ominus I = \{\}$
by (*simp add*: *iT-Minus-image-conv cut-le-Min-empty*)

lemma *iT-Minus-Min-singleton*: $I \neq \{\} \Longrightarrow (iMin\ I) \ominus I = \{0\}$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-mem-iff*)
apply (*fastsimp intro*: *iMinI-ex2*)
done

lemma *iT-Minus-empty-iff*: $(k \ominus I = \{\}) = (I = \{\} \vee k < iMin\ I)$
apply (*case-tac* $I = \{\}$, *simp add*: *iT-Minus-empty*)
apply (*simp add*: *iT-Minus-image-conv cut-le-empty-iff iMin-gr-iff*)
done

lemma *iT-Minus-imirror-conv*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus- (iMin\ I + Max\ (I \downarrow\leq k))$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Minus-empty cut-le-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
apply (*case-tac* $k < iMin\ I$)
apply (*simp add*: *iT-Minus-less-Min-empty cut-le-Min-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
apply (*simp add*: *linorder-not-less*)
apply (*frule cut-le-Min-not-empty*[*of* - k], *assumption*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-image-conv iT-Plus-neg-image-conv iT-Plus-neg-mem-iff iT-Plus-mem-iff imirror-iff image-iff Bex-def i-cut-mem-iff cut-le-Min-eq*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac* x')
apply (*rule-tac* $x{=}k - x' + iMin\ I + Max\ (I \downarrow\leq k)$ **in** *exI*, *simp*)
apply (*simp add*: *add-assoc le-add-diff*)

apply (*simp add*: *add-commute*[*of k*] *le-add-diff nat-cut-le-finite cut-leI trans-le-add2*)
apply (*rule-tac x=x′* **in** *exI*, *simp*)
apply (*clarsimp*, *rename-tac x1 x2*)
apply (*rule-tac x=x2* **in** *exI*)
apply *simp*
thm *nat-add-right-cancel*[*THEN iffD2*, *of - - k*]
apply (*drule nat-add-right-cancel*[*THEN iffD2*, *of - - k*], *simp*)
thm *add-diff-assoc2*
apply (*simp add*: *trans-le-add2 nat-cut-le-finite cut-le-mem-iff*)
done
thm *iT-Minus-imirror-conv*
lemma *iT-Minus-imirror-conv′*:
$k \ominus I = imirror\ (I \downarrow\leq k) \oplus k \oplus- (iMin\ (I \downarrow\leq k) + Max\ (I \downarrow\leq k))$
apply (*case-tac I* = {})
apply (*simp add*: *iT-Minus-empty cut-le-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
apply (*case-tac k* < *iMin I*)
apply (*simp add*: *iT-Minus-less-Min-empty cut-le-Min-empty imirror-empty iT-Plus-empty iT-Plus-neg-empty*)
thm *cut-le-Min-eq*
apply (*simp add*: *cut-le-Min-not-empty cut-le-Min-eq iT-Minus-imirror-conv*)
done

thm *iT-Minus-bound*
lemma *iT-Minus-Max*:
$[\![\ I \neq \{\};\ iMin\ I \leq k\ ]\!] \Longrightarrow Max\ (k \ominus I) = k - (iMin\ I)$
thm *Max-equality*
apply (*rule Max-equality*)
apply (*simp add*: *iT-Minus-mem-iff iMinI-ex2*)
apply (*simp add*: *iT-Minus-finite*)
apply (*fastsimp simp*: *iT-Minus-def*)
done

lemma *iT-Minus-Min*:
$[\![\ I \neq \{\};\ iMin\ I \leq k\ ]\!] \Longrightarrow iMin\ (k \ominus I) = k - (Max\ (I \downarrow\leq k))$
apply (*insert nat-cut-le-finite*[*of I k*])
apply (*frule cut-le-Min-not-empty*[*of - k*], *assumption*)
apply (*rule iMin-equality*)
apply (*simp add*: *iT-Minus-mem-iff nat-cut-le-Max-le del*: *Max-le-iff*)
thm *subsetD*[*OF cut-le-subset*, *OF Max-in*]
apply (*simp add*: *subsetD*[*OF cut-le-subset*, *OF Max-in*])
apply (*clarsimp simp add*: *iT-Minus-image-conv image-iff*, *rename-tac x′*)
apply (*rule diff-le-mono2*)
apply (*simp add*: *Max-ge-iff cut-le-mem-iff*)
done

lemma *iT-Minus-Minus-eq*: $[\![$ *finite I*; *Max I* $\leq k\ ]\!] \Longrightarrow k \ominus (k \ominus I) = I$
thm *iT-Minus-cut-eq*[*of k k I*, *symmetric*] *iT-Minus-Minus-cut-eq*
apply (*simp add*: *iT-Minus-cut-eq*[*of k k I*, *symmetric*] *iT-Minus-Minus-cut-eq*)
apply (*simp add*: *cut-le-Max-all*)

done
lemma *iT-Minus-Minus-eq2*: $I \subseteq [\ldots k] \Longrightarrow k \ominus (k \ominus I) = I$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Minus-empty*)
apply (*rule iT-Minus-Minus-eq*)
apply (*simp add*: *finite-subset iTILL-finite*)
thm *Max-subset*
apply (*frule Max-subset*)
apply (*simp add*: *iTILL-finite iTILL-Max*)+
done

lemma *iT-Minus-Minus*: $a \ominus (b \ominus I) = (I \downarrow\leq b) \oplus a \oplus\!- b$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-image-conv iT-Plus-image-conv iT-Plus-neg-image-conv image-iff Bex-def i-cut-mem-iff*)
apply *fastsimp*
done
lemma *iT-Minus-Plus-empty*: $k < n \Longrightarrow k \ominus (I \oplus n) = \{\}$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Plus-empty iT-Minus-empty*)
apply (*simp add*: *iT-Minus-empty-iff iT-Plus-empty-iff iT-Plus-Min*)
done
lemma *iT-Minus-Plus-commute*: $n \leq k \Longrightarrow k \ominus (I \oplus n) = (k - n) \ominus I$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-image-conv iT-Plus-image-conv image-iff Bex-def i-cut-mem-iff*)
apply *fastsimp*
done

lemma *iT-Minus-Plus-cut-assoc*: $(k \ominus I) \oplus n = (k + n) \ominus (I \downarrow\leq k)$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Plus-mem-iff iT-Minus-mem-iff cut-le-mem-iff*)
apply *fastsimp*
done

lemma *iT-Minus-Plus-assoc*:
$[\![\mathit{finite}\ I;\ \mathit{Max}\ I \leq k\ ]\!] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
by (*insert iT-Minus-Plus-cut-assoc*[*of k I n*], *simp add*: *cut-le-Max-all*)
lemma *iT-Minus-Plus-assoc2*:
$I \subseteq [\ldots k] \Longrightarrow (k \ominus I) \oplus n = (k + n) \ominus I$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Minus-empty iT-Plus-empty*)
apply (*rule iT-Minus-Plus-assoc*)
apply (*simp add*: *finite-subset iTILL-finite*)
thm *Max-subset*
apply (*frule Max-subset*)
apply (*simp add*: *iTILL-finite iTILL-Max*)+
done

lemma *iT-Minus-Un*: $k \ominus (A \cup B) = (k \ominus A) \cup (k \ominus B)$
by (*fastsimp simp*: *iT-Minus-def*)

lemma *iT-Minus-Int*: $k \ominus (A \cap B) = (k \ominus A) \cap (k \ominus B)$
by (*fastsimp simp*: *set-eq-iff iT-Minus-mem-iff*)

lemma *iT-Minus-singleton*: $k \ominus \{a\} = (\mathit{if}\ a \leq k\ \mathit{then}\ \{k - a\}\ \mathit{else}\ \{\})$
by (*simp add*: *iT-Minus-image-conv cut-le-singleton*)
corollary *iT-Minus-singleton1*: $a \leq k \Longrightarrow k \ominus \{a\} = \{k-a\}$
by (*simp add*: *iT-Minus-singleton*)
corollary *iT-Minus-singleton2*: $k < a \Longrightarrow k \ominus \{a\} = \{\}$
by (*simp add*: *iT-Minus-singleton*)

lemma *iMOD-sub*:
$k \ominus [r, \mathit{mod}\ m] =$
$(\mathit{if}\ r \leq k\ \mathit{then}\ [(k - r)\ \mathit{mod}\ m, \mathit{mod}\ m, (k - r)\ \mathit{div}\ m]\ \mathit{else}\ \{\})$
apply (*rule set-eqI*)
apply (*simp add*: *iT-Minus-mem-iff iT-iff*)
thm *mod-sub-eq-mod-swap*[*of r k x m*, *symmetric*]
apply (*fastsimp simp add*: *mod-sub-eq-mod-swap*[*of r*, *symmetric*])
done

corollary *iMOD-sub1*:
$r \leq k \Longrightarrow k \ominus [r, \mathit{mod}\ m] = [(k - r)\ \mathit{mod}\ m, \mathit{mod}\ m, (k - r)\ \mathit{div}\ m]$
by (*simp add*: *iMOD-sub*)
corollary *iMOD-sub2*: $k < r \Longrightarrow k \ominus [r, \mathit{mod}\ m] = \{\}$
thm *iT-Minus-less-Min-empty*
apply (*rule iT-Minus-less-Min-empty*)
apply (*simp add*: *iMOD-Min*)
done

lemma *iTILL-sub*: $k \ominus [\ldots n] = (\mathit{if}\ n \leq k\ \mathit{then}\ [k - n\ldots,n]\ \mathit{else}\ [\ldots k])$
by (*force simp add*: *set-eq-iff iT-Minus-mem-iff iT-iff*)
corollary *iTILL-sub1*: $n \leq k \Longrightarrow k \ominus [\ldots n] = [k - n\ldots,n]$
by (*simp add*: *iTILL-sub*)
corollary *iTILL-sub2*: $k \leq n \Longrightarrow k \ominus [\ldots n] = [\ldots k]$
by (*simp add*: *iTILL-sub iIN-0-iTILL-conv*)

lemma *iMODb-sub*:
$k \ominus [r, \mathit{mod}\ m, c] = ($
$\mathit{if}\ r + m * c \leq k\ \mathit{then}\ [\ k - r - m * c, \mathit{mod}\ m, c]\ \mathit{else}$
$\mathit{if}\ r \leq k\ \mathit{then}\ [\ (k - r)\ \mathit{mod}\ m, \mathit{mod}\ m, (k - r)\ \mathit{div}\ m]\ \mathit{else}\ \{\})$
apply (*case-tac m = 0*)

apply (*simp add*: *iMODb-mod-0 iIN-0 iT-Minus-singleton*)
thm *iMODb-iMOD-iTILL-conv*
apply (*subst iMODb-iMOD-iTILL-conv*)
thm *iT-Minus-Int*
apply (*subst iT-Minus-Int*)
apply (*simp add*: *iMOD-sub iTILL-sub*)
apply (*intro conjI impI*)
apply *simp*
apply (*subgoal-tac* $(k - r)$ *mod* $m \leq k - (r + m * c)$)
prefer *2*
apply (*subgoal-tac* $m * c \leq k - r - (k - r)$ *mod* m)
prefer *2*
thm *add-le-imp-le-diff2*
apply (*drule add-le-imp-le-diff2*)
apply (*drule div-le-mono*[*of* - - m], *simp*)
apply (*drule mult-le-mono2*[*of* - - m])
apply (*simp add*: *mult-div-cancel*)
thm *le-diff-conv2*[*OF mod-le-dividend*]
apply (*simp add*: *le-diff-conv2*[*OF mod-le-dividend*] *del*: *diff-diff-left*)
thm *iMODb-iMOD-iIN-conv*
apply (*subst iMODb-iMOD-iIN-conv*)
apply (*simp add*: *Int-assoc mult-div-cancel*)
thm *iIN-inter*
apply (*subst iIN-inter*, *simp*+)
apply (*rule set-eqI*)
apply (*fastsimp simp add*: *iT-iff mod-diff-mult-self2 diff-diff-left*[*symmetric*] *simp del*: *diff-diff-left*)
apply (*simp add*: *Int-absorb2 iMODb-iTILL-subset*)
done
corollary *iMODb-sub1*:
$[\![\ r \leq k;\ k \leq r + m * c\ ]\!] \Longrightarrow$
$k \ominus [r,\ mod\ m,\ c] = [(k - r)\ mod\ m,\ mod\ m,\ (k - r)\ div\ m]$
by (*clarsimp simp*: *iMODb-sub iMODb-mod-0*)
corollary *iMODb-sub2*: $k < r \Longrightarrow k \ominus [r,\ mod\ m,\ c] = \{\}$
thm *iT-Minus-less-Min-empty*
apply (*rule iT-Minus-less-Min-empty*)
apply (*simp add*: *iMODb-Min*)
done
corollary *iMODb-sub3*:
$r + m * c \leq k \Longrightarrow k \ominus [r,\ mod\ m,\ c] = [\ k - r - m * c,\ mod\ m,\ c]$
by (*simp add*: *iMODb-sub*)

lemma *iFROM-sub*: $k \ominus [n\ldots] =$ (*if* $n \leq k$ *then* $[\ldots k - n]$ *else* $\{\}$)
by (*simp add*: *iMOD-1*[*symmetric*] *iMOD-sub iMODb-mod-1 iIN-0-iTILL-conv*)
corollary *iFROM-sub1*: $n \leq k \Longrightarrow k \ominus [n\ldots] = [\ldots k-n]$
by (*simp add*: *iFROM-sub*)
corollary *iFROM-sub-empty*: $k < n \Longrightarrow k \ominus [n\ldots] = \{\}$
by (*simp add*: *iFROM-sub*)

lemma *iIN-sub*:
k ⊖ [*n*...,*d*] = (
if n + *d* ≤ *k then* [*k* − (*n* + *d*)...,*d*]
else if n ≤ *k then* [...*k* − *n*] *else* {})
apply (*simp add*: *iMODb-mod-1*[*symmetric*] *iMODb-sub*)
apply (*simp add*: *iMODb-mod-1 iIN-0-iTILL-conv*)
done
lemma *iIN-sub1*: *n* + *d* ≤ *k* ⟹ *k* ⊖ [*n*...,*d*] = [*k* − (*n* + *d*)...,*d*]
by (*simp add*: *iIN-sub*)
lemma *iIN-sub2*: ⟦ *n* ≤ *k*; *k* ≤ *n* + *d* ⟧ ⟹ *k* ⊖ [*n*...,*d*] = [...*k* − *n*]
by (*clarsimp simp*: *iIN-sub iIN-0-iTILL-conv*)
lemma *iIN-sub3*: *k* < *n* ⟹ *k* ⊖ [*n*...,*d*] = {}
by (*simp add*: *iIN-sub*)

lemmas *iT-sub* =
iFROM-sub
iIN-sub
iTILL-sub
iMOD-sub
iMODb-sub
iT-Minus-singleton
thm *iT-sub*

2.1.5 Division of intervals by constants

Monotonicity and injectivity of artithmetic operators

lemma *iMOD-div-right-strict-mono-on*:
⟦ 0 < *k*; *k* ≤ *m* ⟧ ⟹ *strict-mono-on* (λ*x*. *x div k*) [*r*, *mod m*]
apply (*rule div-right-strict-mono-on*, *assumption*)
apply (*clarsimp simp*: *iT-iff*)
apply (*drule-tac s*=*y mod m* **in** *sym*, *simp*)
apply (*rule-tac y*=*x* + *m* **in** *order-trans*, *simp*)
apply (*simp add*: *less-mod-eq-imp-add-divisor-le*)
done
corollary *iMOD-div-right-inj-on*:
⟦ 0 < *k*; *k* ≤ *m* ⟧ ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*]
by (*rule strict-mono-on-imp-inj-on*[*OF iMOD-div-right-strict-mono-on*])

lemma *iMOD-mult-div-right-inj-on*:
inj-on (λ*x*. *x div* (*k*::*nat*)) [*r*, *mod* (*k* ∗ *m*)]
apply (*case-tac k* ∗ *m* = 0)
apply (*simp del*: *mult-is-0 add*: *iMOD-0 iIN-0*)
apply (*simp add*: *iMOD-div-right-inj-on*)
done
lemma *iMOD-mult-div-right-inj-on2*:
m mod k = 0 ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*]

by (*clarsimp simp add*: *iMOD-mult-div-right-inj-on*)

lemma *iMODb-div-right-strict-mono-on*:
⟦ *0* < *k*; *k* ≤ *m* ⟧ ⟹ *strict-mono-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
thm *strict-mono-on-subset*[*OF iMOD-div-right-strict-mono-on iMODb-iMOD-subset-same*]
by (*rule strict-mono-on-subset*[*OF iMOD-div-right-strict-mono-on iMODb-iMOD-subset-same*])
corollary *iMODb-div-right-inj-on*:
⟦ *0* < *k*; *k* ≤ *m* ⟧ ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
by (*rule strict-mono-on-imp-inj-on*[*OF iMODb-div-right-strict-mono-on*])
lemma *iMODb-mult-div-right-inj-on*:
inj-on (λ*x*. *x div* (*k*::*nat*)) [*r*, *mod* (*k* ∗ *m*), *c*]
thm *subset-inj-on*[*OF iMOD-div-right-inj-on iMODb-iMOD-subset-same*]
by (*rule subset-inj-on*[*OF iMOD-mult-div-right-inj-on iMODb-iMOD-subset-same*])
corollary *iMODb-mult-div-right-inj-on2*:
m mod k = *0* ⟹ *inj-on* (λ*x*. *x div k*) [*r*, *mod m*, *c*]
by (*clarsimp simp*: *iMODb-mult-div-right-inj-on*)

definition *iT-Div* :: *iT* ⇒ *Time* ⇒ *iT* (**infixl** ⊘ *55*) **where**
I ⊘ *k* ≡ (λ*n*.(*n div k*)) ‘ *I*

lemma *iT-Div-image-conv*: *I* ⊘ *k* = (λ*n*.(*n div k*)) ‘ *I*
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-mono*: *A* ⊆ *B* ⟹ *A* ⊘ *k* ⊆ *B* ⊘ *k*
by (*simp add*: *iT-Div-def image-mono*)

lemma *iT-Div-empty*: {} ⊘ *k* = {}
by (*simp add*: *iT-Div-def*)
lemma *iT-Div-not-empty*: *I* ≠ {} ⟹ *I* ⊘ *k* ≠ {}
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-empty-iff*: (*I* ⊘ *k* = {}) = (*I* = {})
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-0*: *I* ≠ {} ⟹ *I* ⊘ *0* = [. . .*0*]
by (*force simp*: *iT-Div-def*)
corollary *iT-Div-0-if*: *I* ⊘ *0* = (*if I* = {} *then* {} *else* [. . .*0*])
by (*force simp*: *iT-Div-def*)
corollary
iFROM-div-0: [*n*. . .] ⊘ *0* = [. . .*0*] **and**
iTILL-div-0: [. . .*n*] ⊘ *0* = [. . .*0*] **and**
iIN-div-0: [*n*. . .,*d*] ⊘ *0* = [. . .*0*] **and**

iMOD-div-0: $[r,\ mod\ m] \oslash 0 = [\ldots 0]$ **and**
iMODb-div-0: $[r,\ mod\ m,\ c] \oslash 0 = [\ldots 0]$
by (*simp add*: *iT-Div-0 iT-not-empty*)+

lemmas *iT-div-0* =
iTILL-div-0
iFROM-div-0
iIN-div-0
iMOD-div-0
iMODb-div-0
thm *iT-div-0*

lemma *iT-Div-1*: $I \oslash Suc\ 0 = I$
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-mem-iff-0*: $x \in (I \oslash 0) = (I \neq \{\} \wedge x = 0)$
by (*force simp*: *iT-Div-0-if*)

lemma *iT-Div-mem-iff*:
$0 < k \Longrightarrow x \in (I \oslash k) = (\exists\, y \in I.\ y\ div\ k = x)$
by (*force simp*: *iT-Div-def*)

lemma *iT-Div-mem-iff2*:
$0 < k \Longrightarrow x\ div\ k \in (I \oslash k) = (\exists\, y \in I.\ y\ div\ k = x\ div\ k)$
by (*rule iT-Div-mem-iff*)

lemma *iT-Div-mem-iff-Int*:
$0 < k \Longrightarrow x \in (I \oslash k) = (I \cap [x * k\ldots,k - Suc\ 0] \neq \{\})$
thm *ex-in-conv*[*symmetric*]
apply (*simp add*: *ex-in-conv*[*symmetric*] *iT-Div-mem-iff iT-iff*)
thm *le-less-div-conv*
apply (*simp add*: *le-less-div-conv*[*symmetric*] *add-commute*[*of k*])
thm *less-eq-le-pred*
apply (*subst less-eq-le-pred*, *simp*)
apply *blast*
done

lemma *iT-Div-imp-mem*:
$0 < k \Longrightarrow x \in I \Longrightarrow x\ div\ k \in (I \oslash k)$
by (*force simp*: *iT-Div-mem-iff2*)

lemma *iT-Div-singleton*: $\{a\} \oslash k = \{a\ div\ k\}$
by (*simp add*: *iT-Div-def*)

lemma *iT-Div-Un*: $(A \cup B) \oslash k = (A \oslash k) \cup (B \oslash k)$
by (*fastsimp simp*: *iT-Div-def*)

lemma *iT-Div-insert*: $(insert\ n\ I) \oslash k = insert\ (n\ div\ k)\ (I \oslash k)$

by (*fastsimp simp*: *iT-Div-def*)

lemma *not-iT-Div-Int*: ¬ (∀ *k A B*. (*A* ∩ *B*) ⊘ *k* = (*A* ⊘ *k*) ∩ (*B* ⊘ *k*))
apply *simp*
apply (
  rule-tac x=3 **in** *exI*,
  rule-tac x={0} **in** *exI*,
  rule-tac x={1} **in** *exI*)
by (*simp add*: *iT-Div-def*)

thm *subset-image-Int*
lemma *subset-iT-Div-Int*: *A* ⊆ *B* ⟹ (*A* ∩ *B*) ⊘ *k* = (*A* ⊘ *k*) ∩ (*B* ⊘ *k*)
by (*simp add*: *iT-Div-def subset-image-Int*)

lemma *iFROM-iT-Div-Int*:
  ⟦ *0* < *k*; *n* ≤ *iMin A* ⟧ ⟹ (*A* ∩ [*n*...]) ⊘ *k* = (*A* ⊘ *k*) ∩ ([*n*...] ⊘ *k*)
apply (*rule subset-iT-Div-Int*)
apply (*blast intro*: *order-trans iMin-le*)
done

lemma *iIN-iT-Div-Int*:
  ⟦ *0* < *k*; *n* ≤ *iMin A*; ∀ *x*∈*A*. *x div k* ≤ (*n* + *d*) *div k* ⟶ *x* ≤ *n* + *d* ⟧ ⟹
  (*A* ∩ [*n*...,*d*]) ⊘ *k* = (*A* ⊘ *k*) ∩ ([*n*...,*d*] ⊘ *k*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff Bex-def iIN-iff*)
apply (*rule iffI*)
 apply *blast*
apply (*clarsimp*, *rename-tac x1 x2*)
apply (*frule iMin-le*)
apply (*rule-tac x=x1* **in** *exI*, *simp*)
apply (*drule-tac x=x1* **in** *bspec*, *simp*)
apply (*drule div-le-mono*[*of* - *n* + *d k*])
apply *simp*
done
corollary *iTILL-iT-Div-Int*:
  ⟦ *0* < *k*; ∀ *x*∈*A*. *x div k* ≤ *n div k* ⟶ *x* ≤ *n* ⟧ ⟹
  (*A* ∩ [*n*...]) ⊘ *k* = (*A* ⊘ *k*) ∩ ([*n*...] ⊘ *k*)
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-iT-Div-Int*)
lemma *iIN-iT-Div-Int-mod-0*:
  ⟦ *0* < *k*; *n mod k* = *0*; ∀ *x*∈*A*. *x div k* ≤ (*n* + *d*) *div k* ⟶ *x* ≤ *n* + *d* ⟧ ⟹
  (*A* ∩ [*n*...,*d*]) ⊘ *k* = (*A* ⊘ *k*) ∩ ([*n*...,*d*] ⊘ *k*)
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff Bex-def iIN-iff*)
apply (*rule iffI*)
 apply *blast*
apply (*elim conjE exE*, *rename-tac x1 x2*)
apply (*rule-tac x=x1* **in** *exI*, *simp*)
apply (*rule conjI*)

apply (*rule ccontr*, *simp add*: *linorder-not-le*)
thm *div-le-mono*
apply (*drule-tac m=n* **and** *n=x2* **and** *k=k* **in** *div-le-mono*)
thm *less-mod-0-imp-div-less*
apply (*drule-tac a=x1* **and** *m=k* **in** *less-mod-0-imp-div-less*)
apply *simp+*
apply (*drule-tac x=x1* **in** *bspec*, *simp*)
apply (*drule div-le-mono*[*of - n + d k*])
apply *simp*
done

lemma *mod-partition-iT-Div-Int*:
⟦ *0 < k*; *0 < d* ⟧ ⟹
(*A* ∩ [*n* ∗ *k*...,*d* ∗ *k* − *Suc 0*]) ⊘ *k* =
(*A* ⊘ *k*) ∩ ([*n* ∗ *k*...,*d* ∗ *k* − *Suc 0*] ⊘ *k*)
thm *iIN-iT-Div-Int-mod-0*
apply (*rule iIN-iT-Div-Int-mod-0*, *simp+*)
apply (*clarify*, *rename-tac x*)
apply (*simp add*: *mod-0-imp-sub-1-div-conv*)
apply (*rule ccontr*, *simp add*: *linorder-not-le pred-less-eq-le*)
apply (*drule-tac n=x* **and** *k=k* **in** *div-le-mono*)
apply *simp*
done

corollary *mod-partition-iT-Div-Int2*:
⟦ *0 < k*; *0 < d*; *n mod k = 0*; *d mod k = 0* ⟧ ⟹
(*A* ∩ [*n*...,*d* − *Suc 0*]) ⊘ *k* =
(*A* ⊘ *k*) ∩ ([*n*...,*d* − *Suc 0*] ⊘ *k*)
apply (*clarsimp simp*: *mult-commute*[*of k*])
thm *mod-partition-iT-Div-Int*
apply (*simp add*: *mod-partition-iT-Div-Int*)
done
corollary *mod-partition-iT-Div-Int-one-segment*:
0 < k ⟹
(*A* ∩ [*n* ∗ *k*...,*k* − *Suc 0*]) ⊘ *k* = (*A* ⊘ *k*) ∩ ([*n* ∗ *k*...,*k* − *Suc 0*] ⊘ *k*)
by (*insert mod-partition-iT-Div-Int*[**where** *d=1*], *simp*)
corollary *mod-partition-iT-Div-Int-one-segment2*:
⟦ *0 < k*; *n mod k = 0* ⟧ ⟹
(*A* ∩ [*n*...,*k* − *Suc 0*]) ⊘ *k* = (*A* ⊘ *k*) ∩ ([*n*...,*k* − *Suc 0*] ⊘ *k*)
by (*insert mod-partition-iT-Div-Int2*[**where** *k=k* **and** *d=k* **and** *n=n*], *simp*)

thm
iT-Div-Un
subset-iT-Div-Int
thm
mod-partition-iT-Div-Int
mod-partition-iT-Div-Int2

lemma *iT-Div-assoc*:$I \oslash a \oslash b = I \oslash (a * b)$
by (*simp add*: *iT-Div-def image-image div-mult2-eq*)

lemma *iT-Div-commute*: $I \oslash a \oslash b = I \oslash b \oslash a$
by (*simp add*: *iT-Div-assoc mult-commute*[*of a*])

lemma *iT-Mult-Div-self*: $0 < k \Longrightarrow I \otimes k \oslash k = I$
by (*simp add*: *iT-Mult-def iT-Div-def image-image*)
lemma *iT-Mult-Div*:
$[\![\ 0 < d;\ \ k \bmod d = 0\ ]\!] \Longrightarrow I \otimes k \oslash d = I \otimes (k \operatorname{div} d)$
apply (*clarsimp simp*: *mult-commute*[*of d*])
apply (*simp add*: *iT-Mult-assoc*[*symmetric*] *iT-Mult-Div-self*)
done

lemma *iT-Div-Mult-self*:
$0 < k \Longrightarrow I \oslash k \otimes k = \{y.\ \exists x \in I.\ y = x - x \bmod k\}$
by (*simp add*: *set-eq-iff iT-Mult-def iT-Div-def image-image image-iff div-mult-cancel*)

thm *div-add1-eq*
lemma *iT-Plus-Div-distrib-mod-less*:
$\forall x{\in}I.\ x \bmod m + n \bmod m < m \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n \operatorname{div} m$
by (*simp add*: *set-eq-iff iT-Div-def iT-Plus-def image-image image-iff div-add1-eq1*)
corollary *iT-Plus-Div-distrib-mod-0*:
$n \bmod m = 0 \Longrightarrow I \oplus n \oslash m = I \oslash m \oplus n \operatorname{div} m$
apply (*case-tac* $m = 0$, *simp add*: *iT-Plus-0 iT-Div-0*)
apply (*simp add*: *iT-Plus-Div-distrib-mod-less*)
done

lemma *iT-Div-Min*: $I \neq \{\} \Longrightarrow iMin\ (I \oslash k) = iMin\ I \operatorname{div} k$
by (*simp add*: *iT-Div-def iMin-mono2 mono-def div-le-mono*)

thm *iT-Div-Min*
corollary
iFROM-div-Min: $iMin\ ([n\ldots] \oslash k) = n \operatorname{div} k$ **and**
iIN-div-Min: $iMin\ ([n\ldots,d] \oslash k) = n \operatorname{div} k$ **and**
iTILL-div-Min: $iMin\ ([\ldots n] \oslash k) = 0$ **and**
iMOD-div-Min: $iMin\ ([r, \bmod m] \oslash k) = r \operatorname{div} k$ **and**
iMODb-div-Min: $iMin\ ([r, \bmod m, c] \oslash k) = r \operatorname{div} k$
by (*simp add*: *iT-not-empty iT-Div-Min iT-Min*)+

lemmas *iT-div-Min* =
iFROM-div-Min
iIN-div-Min
iTILL-div-Min
iMOD-div-Min
iMODb-div-Min
thm *iT-div-Min*

lemma *iT-Div-Max*: $[\![$ *finite* I; $I \neq \{\}$ $]\!] \Longrightarrow$ *Max* $(I \oslash k)$ = *Max* I *div* k
by (*simp add*: *iT-Div-def Max-mono2 mono-def div-le-mono*)

corollary
iIN-div-Max: *Max* $([n\ldots,d] \oslash k)$ = $(n + d)$ *div* k **and**
iTILL-div-Max: *Max* $([\ldots n] \oslash k)$ = n *div* k **and**
iMODb-div-Max: *Max* $([r, mod\ m, c] \oslash k)$ = $(r + m * c)$ *div* k
by (*simp add*: *iT-not-empty iT-finite iT-Div-Max iT-Max*)+

lemma *iT-Div-0-finite*: *finite* $(I \oslash 0)$
by (*simp add*: *iT-Div-0-if iTILL-0*)

lemma *iT-Div-infinite-iff*: $0 < k \Longrightarrow$ *infinite* $(I \oslash k)$ = *infinite* I
apply (*unfold iT-Div-def*)
apply (*rule iffI*)
apply (*rule infinite-image-imp-infinite*, *assumption*)
apply (*clarsimp simp*: *infinite-nat-iff-unbounded-le image-iff*, *rename-tac x1*)
apply (*drule-tac x=x1 * k* **in** *spec*, *clarsimp*, *rename-tac x2*)
apply (*drule div-le-mono*[*of - - k*], *simp*)
apply (*rule-tac x=x2 div k* **in** *exI*)
apply *fastsimp*
done
lemma *iT-Div-finite-iff*: $0 < k \Longrightarrow$ *finite* $(I \oslash k)$ = *finite* I
by (*insert iT-Div-infinite-iff*, *simp*)

lemma *iFROM-div*: $0 < k \Longrightarrow [n\ldots] \oslash k = [n\ div\ k\ldots]$
apply (*clarsimp simp*: *set-eq-iff iT-Div-def image-iff Bex-def iFROM-iff*, *rename-tac x*)
apply (*rule iffI*)
apply (*clarsimp simp*: *div-le-mono*)
apply (*rule-tac x=n mod k + k * x* **in** *exI*)
apply *simp*
apply (*subst add-commute*, *subst le-diff-conv*[*symmetric*])
apply (*subst mult-div-cancel*[*symmetric*])
apply *simp*
done

thm *div-add1-eq*
lemma *iIN-div*:
$0 < k \Longrightarrow$
$[n\ldots,d] \oslash k = [n\ div\ k\ldots,\ d\ div\ k + (n\ mod\ k + d\ mod\ k)\ div\ k\ ]$
apply (*clarsimp simp*: *set-eq-iff iT-Div-def image-iff Bex-def iIN-iff*, *rename-tac x*)
apply (*rule iffI*)
apply *clarify*

apply (*drule div-le-mono*[*of n - k*])
apply (*drule div-le-mono*[*of - n + d k*])
apply (*simp add*: *div-add1-eq*[*of n d*])
apply (*clarify*, *rename-tac x*)
apply (*simp add*: *add-assoc*[*symmetric*] *div-add1-eq*[*symmetric*])
apply (*frule mult-le-mono1*[*of n div k - k*])
apply (*frule mult-le-mono1*[*of - (n + d) div k k*])
apply (*simp add*: *mult-commute*[*of - k*] *mult-div-cancel*)
apply (*simp add*: *le-diff-conv le-diff-conv2*[*OF mod-le-dividend*])
apply (*drule order-le-less*[*of - (n + d) div k*, *THEN iffD1*], *erule disjE*)
apply (*rule-tac x=k * x + n mod k* **in** *exI*)
apply (*simp add*: *add-commute*[*of - n mod k*])
apply (*case-tac n mod k ≤ (n + d) mod k*, *simp*)
apply (*simp add*: *linorder-not-le*)
apply (*drule-tac m=x* **in** *less-imp-le-pred*)
apply (*drule-tac i=x* **and** *k=k* **in** *mult-le-mono2*)
apply (*simp add*: *diff-mult-distrib2 mult-div-cancel*)
apply (*subst add-commute*[*of n mod k*])
apply (*subst le-diff-conv2*[*symmetric*])
apply (*simp add*: *trans-le-add1*)
apply (*rule order-trans*, *assumption*)
apply (*rule diff-le-mono2*)
apply (*simp add*: *trans-le-add2*)
apply (*rule-tac x=n + d* **in** *exI*, *simp*)
done
corollary *iIN-div-if*:
0 < k ⟹ [n...,d] ⊘ k =
[n div k..., d div k + (if n mod k + d mod k < k then 0 else Suc 0)]
apply (*simp add*: *iIN-div*)
apply (*simp add*: *iIN-def add-assoc*[*symmetric*] *div-add1-eq*[*symmetric*] *div-add1-eq2*[**where** *a=n*])
done
corollary *iIN-div-eq1*:
⟦ 0 < k; n mod k + d mod k < k ⟧ ⟹
[n...,d] ⊘ k = [n div k...,d div k]
by (*simp add*: *iIN-div-if*)
corollary *iIN-div-eq2*:
⟦ 0 < k; k ≤ n mod k + d mod k ⟧ ⟹
[n...,d] ⊘ k = [n div k..., Suc (d div k)]
by (*simp add*: *iIN-div-if*)
corollary *iIN-div-mod-eq-0*:
⟦ 0 < k; n mod k = 0 ⟧ ⟹ [n...,d] ⊘ k = [n div k...,d div k]
by (*simp add*: *iIN-div-eq1*)

lemma *iTILL-div*:
0 < k ⟹ [...n] ⊘ k = [...n div k]
by (*simp add*: *iIN-0-iTILL-conv*[*symmetric*] *iIN-div-if*)

lemma *iMOD-div-ge*:
⟦ $0 < m$; $m \leq k$ ⟧ ⟹ $[r, \mathit{mod}\ m] \oslash k = [r\ \mathit{div}\ k \ldots]$
apply (*frule less-le-trans*[*of* - - *k*], *assumption*)
apply (*clarsimp simp*: *set-eq-iff iT-Div-mem-iff Bex-def iT-iff*, *rename-tac x*)
apply (*rule iffI*)
apply (*fastsimp simp*: *div-le-mono*)
apply (*rule-tac x*=
if $x * k < r$ *then r else*
$((\mathit{if}\ x * k\ \mathit{mod}\ m \leq r\ \mathit{mod}\ m\ \mathit{then}\ 0\ \mathit{else}\ m) + r\ \mathit{mod}\ m + (x * k - x * k\ \mathit{mod}\ m))$
in *exI*)
apply (*case-tac* $x * k < r$)
apply *simp*
apply (*drule less-imp-le*[*of* - *r*], *drule div-le-mono*[*of* - *r k*], *simp*)
apply (*simp add*: *linorder-not-less linorder-not-le*)
apply (*simp add*: *div-le-conv add-commute*[*of k*])
apply (*subst diff-add-assoc*, *simp*)+
thm *div-mult-cancel*
apply (*simp add*: *div-mult-cancel*[*symmetric*] *del*: *add-diff-assoc*)
apply (*case-tac* $x * k\ \mathit{mod}\ m = 0$)
apply *clarsimp*
apply (*drule sym*)
apply (*simp add*: *mult-commute*[*of m*])
apply (*blast intro*: *div-less order-less-le-trans mod-less-divisor*)
apply *simp*
apply (*intro conjI impI*)
apply (*simp add*: *div-mult-cancel*)
apply (*simp add*: *div-mult-cancel*)
apply (*subst add-commute*, *subst diff-add-assoc*, *simp*)
apply (*subst add-commute*, *subst div-mult-self1*, *simp*)
apply (*subst div-less*)
apply (*rule order-less-le-trans*[*of* - *m*], *simp add*: *less-imp-diff-less*)
apply *simp*
apply *simp*
apply (*rule-tac y*=$x * k$ **in** *order-trans*, *assumption*)
apply (*simp add*: *div-mult-cancel*)
apply (*rule le-add-diff*)
apply (*simp add*: *trans-le-add1*)
apply (*simp add*: *div-mult-cancel*)
apply (*subst diff-add-assoc2*, *simp add*: *trans-le-add1*)
apply *simp*
done
corollary *iMOD-div-self*:
$0 < m \Longrightarrow [r, \mathit{mod}\ m] \oslash m = [r\ \mathit{div}\ m \ldots]$
by (*simp add*: *iMOD-div-ge*)

lemma *iMOD-div*:
⟦ $0 < k$; $m\ \mathit{mod}\ k = 0$ ⟧ ⟹
$[r, \mathit{mod}\ m] \oslash k = [r\ \mathit{div}\ k, \mathit{mod}\ (m\ \mathit{div}\ k)]$

apply (*case-tac* $m = 0$)
 apply (*simp add*: *iMOD-0 iIN-0 iT-Div-singleton*)
apply (*clarsimp*, *rename-tac q*)
thm *iMOD-mult*[*of r div k q k*]
apply (*cut-tac* $r{=}r$ *div* k **and** $k{=}k$ **and** $m{=}q$ **in** *iMOD-mult*)
thm *arg-cong*[**where** $f{=}\lambda x.\ x \oplus (r\ mod\ k)$]
apply (*drule arg-cong*[**where** $f{=}\lambda x.\ x \oplus (r\ mod\ k)$])
apply (*drule sym*)
apply (*simp add*: *iMOD-add mult-commute*[*of k*])
thm *iT-Plus-Div-distrib-mod-less*
apply (*cut-tac* $I{=}[r\ div\ k, mod\ q] \otimes k$ **and** $m{=}k$ **and** $n{=}r\ mod\ k$ **in** *iT-Plus-Div-distrib-mod-less*)
 apply (*rule ballI*)
 apply (*simp only*: *iMOD-mult iMOD-iff*, *elim conjE*)
 thm *mod-factor-imp-mod-0*[*of x q k*]
 apply (*drule mod-factor-imp-mod-0*)
 apply *simp*
apply (*simp add*: *iT-Plus-0*)
thm *iT-Mult-Div*[*OF - mod-self*]
apply (*simp add*: *iT-Mult-Div*[*OF - mod-self*] *iT-Mult-1*)
done
thm
  iMOD-div
  iMOD-div-ge

lemma *iMODb-div-self*:
  $0 < m \Longrightarrow [r, mod\ m, c] \oslash m = [r\ div\ m\ldots,c]$
thm *iMODb-iMOD-iTILL-conv*
apply (*subst iMODb-iMOD-iTILL-conv*)
thm *iTILL-iT-Div-Int*
apply (*subst iTILL-iT-Div-Int*)
  apply *simp*
 apply (*clarsimp simp*: *iT-iff simp del*: *div-mult-self1 div-mult-self2*, *rename-tac x*)
 thm *div-le-mod-le-imp-le*
 apply (*drule div-le-mod-le-imp-le*)
 apply *simp+*
apply (*simp add*: *iMOD-div-self iTILL-div iFROM-iTILL-iIN-conv*)
done

lemma *iMODb-div-ge*:
  $[\![\ 0 < m; m \le k\ ]\!] \Longrightarrow$
  $[r, mod\ m, c] \oslash k = [r\ div\ k\ldots,(r + m * c)\ div\ k - r\ div\ k]$
apply (*case-tac* $m = k$)
 apply (*simp add*: *iMODb-div-self*)
apply (*drule le-neq-trans*, *simp+*)
apply (*induct c*)
 apply (*simp add*: *iMODb-0 iIN-0 iT-Div-singleton*)
apply (*rule-tac* $t{=}[\ r, mod\ m, Suc\ c\ ]$ **and** $s{=}[\ r, mod\ m, c\ ] \cup \{r + m * c +$

$m\}$ **in** *subst*)
thm *iMODb-append-union-Suc*[*of r m c 0*, *symmetric*]
apply (*cut-tac c=c* **and** *c'=0* **and** *r=r* **and** *m=m* **in** *iMODb-append-union-Suc*[*symmetric*])
apply (*simp add*: *iMODb-0 iIN-0 add-commute*[*of m*] *add-assoc*)
apply (*subst iT-Div-Un*)
apply (*simp add*: *iT-Div-singleton*)
apply (*simp add*: *add-commute*[*of m*] *add-assoc*[*symmetric*])
apply (*case-tac* $(r + m * c)$ *mod k* $+$ *m mod k* $< k$)
thm *div-add1-eq1*
apply (*simp add*: *div-add1-eq1*)
apply (*rule insert-absorb*)
apply (*simp add*: *iIN-iff div-le-mono*)
apply (*simp add*: *linorder-not-less*)
thm *div-add1-eq2*
apply (*simp add*: *div-add1-eq2*)
apply (*rule-tac t=Suc* $((r + m * c)$ *div k*) **and** *s=Suc* $(r$ *div k* $+ ((r + m * c)$
div k $-$ *r div k*)) **in** *subst*)
apply (*simp add*: *div-le-mono*)
thm *iIN-Suc-insert-conv*
apply (*simp add*: *iIN-Suc-insert-conv*)
done
thm *div-add1-eq-if*
corollary *iMODb-div-ge-if*:
$[\![\ 0 < m;\ m \leq k\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k =$
$[r\ div\ k \ldots,\ m * c\ div\ k + (if\ r\ mod\ k + m * c\ mod\ k < k\ then\ 0\ else\ Suc\ 0)]$
by (*simp add*: *iMODb-div-ge div-add1-eq-if*[*of - r*])

thm *iMODb-div-ge*
lemma *iMODb-div*:
$[\![\ 0 < k;\ m\ mod\ k = 0\ ]\!] \Longrightarrow$
$[r,\ mod\ m,\ c] \oslash k = [r\ div\ k,\ mod\ (m\ div\ k),\ c\ ]$
apply (*subst iMODb-iMOD-iTILL-conv*)
thm *iTILL-iT-Div-Int*[*of k* $[r,\ mod\ m]$ $m * c$]
apply (*subst iTILL-iT-Div-Int*)
apply *simp*
apply (*simp add*: *Ball-def iMOD-iff*, *intro allI impI*, *elim conjE*, *rename-tac x*)
apply (*drule div-le-mod-le-imp-le*)
apply (*subst mod-add1-eq-if*)
thm *mod-0-imp-mod-mult-right-0*
apply (*simp add*: *mod-0-imp-mod-mult-right-0*)
thm *mod-eq-mod-0-imp-mod-eq*[*of x m r k*]
apply (*drule mod-eq-mod-0-imp-mod-eq*, *simp+*)
apply (*simp add*: *iMOD-div iTILL-div*)
apply (*simp add*: *iMOD-iTILL-iMODb-conv div-le-mono*)
apply (*clarsimp simp*: *mult-assoc iMODb-mod-0 iMOD-0*)
done

lemmas *iT-div* $=$

iTILL-div
iFROM-div
iIN-div
iMOD-div
iMODb-div
iT-Div-singleton
thm
iT-div
iT-div-0

This lemma is valid for all $k \leq m$,i. e., also for k with $m\ mod\ k \neq (0::'a)$.

lemma *iMODb-div-unique*:
$[\![\ 0 < k;\ k \leq m;\ k \leq c;\ [r',\ mod\ m',\ c'] = [r,\ mod\ m,\ c] \oslash k\ ]\!] \Longrightarrow$
$r' = r\ div\ k \wedge m' = m\ div\ k \wedge c' = c$
apply (*case-tac* $r' \neq r\ div\ k$)
thm *iT-Div-Min*
apply (*drule arg-cong*[**where** f=*iMin*])
apply (*simp add*: *iT-Min iT-not-empty iT-Div-Min*)
apply *simp*
apply (*case-tac* $m' = 0 \vee c' = 0$)
apply (*subgoal-tac* [$r\ div\ k$, $mod\ m'$, c'] = $\{r\ div\ k\}$)
prefer *2*
apply (*rule iMODb-singleton-eq-conv*[*THEN iffD2*], *simp*)
apply *simp*
apply (*drule arg-cong*[**where** f=*Max*])
apply (*simp add*: *iMODb-mod-0 iIN-0 iT-Max iT-Div-Max iT-Div-finite-iff iT-Div-not-empty iT-finite iT-not-empty*)
apply (*subgoal-tac* $r\ div\ k < (r + m * c)\ div\ k$, *simp*)
thm *div-add1-eq-if*[*of k r m* $*$ *c*]
apply (*subst div-add1-eq-if*, *simp*)
apply *clarsimp*
apply (*rule order-less-le-trans*[*of* - $k * k\ div\ k$], *simp*)
apply (*rule div-le-mono*)
apply (*simp add*: *mult-mono*)
apply (*subgoal-tac* $c' = c$)
prefer *2*
apply (*drule arg-cong*[**where** f=$\lambda A.\ card\ A$])
thm *card-image*[*OF iMODb-div-right-inj-on*]
apply (*simp add*: *iT-Div-def card-image*[*OF iMODb-div-right-inj-on*] *iMODb-card*)
apply *clarsimp*
thm *iMODb-div-right-strict-mono-on*
apply (*frule iMODb-div-right-strict-mono-on*[*of k m r c*], *assumption*)
thm *iMODb-inext-nth-diff*
apply (*frule-tac* a=k **and** b=*0* **and** m=m' **and** r=$r\ div\ k$ **and** c=c **in** *iMODb-inext-nth-diff*, *simp*)
thm *inext-nth-image*[*OF iMODb-not-empty*]
apply (*simp add*: *iT-Div-Min iT-not-empty iT-Min*)
apply (*simp add*: *iT-Div-def inext-nth-image*[*OF iMODb-not-empty*])
apply (*simp add*: *iMODb-inext-nth*)

done

lemma *iMODb-div-mod-gr0-is-0-not-ex0*:
⟦ $0 < k$; $k < m$; $0 < m \bmod k$; $k \leq c$; $r \bmod k = 0$ ⟧ ⟹
¬(∃ r' m' c'. $[r', \bmod m', c'] = [r, \bmod m, c] \oslash k$)
apply (*rule ccontr*, *simp*, *elim exE conjE*)
thm *iMODb-div-unique*
apply (*frule-tac* r'=r' **and** m'=m' **and** c'=c' **and** r=r **and** k=k **and** m=m **and** c=c
in *iMODb-div-unique*[*OF - less-imp-le*], *simp*+)
apply (*drule arg-cong*[**where** f=*Max*])
apply (*simp add*: *iT-Max iT-Div-Max iT-Div-finite-iff iT-Div-not-empty iT-finite iT-not-empty*)
apply (*simp add*: *div-add1-eq1*)
apply (*simp add*: *mult-commute*[*of m*])
thm *div-mult1-eq*[*of m c*]
apply (*simp add*: *div-mult1-eq*[*of c m*] *div-eq-0-conv*)
apply (*subgoal-tac* $c \leq c * (m \bmod k)$)
apply *simp*+
done

lemma *iMODb-div-mod-gr0-not-ex--arith-aux1*:
⟦ (*0*::*nat*) $< k$; $k < m$; $0 < x1$ ⟧ ⟹
$x1 * m + x2 - x \bmod k + x3 + x \bmod k = x1 * m + x2 + x3$
apply (*drule Suc-leI*[*of - x1*])
apply (*drule mult-le-mono1*[*of Suc 0 - m*])
apply (*subgoal-tac* $x \bmod k \leq x1 * m$)
prefer *2*
apply (*rule order-trans*[*OF mod-le-divisor*], *assumption*)
apply (*rule order-less-imp-le*)
apply (*rule order-less-le-trans*)
apply *simp*+
done
lemma *iMODb-div-mod-gr0-not-ex*:
⟦ $0 < k$; $k < m$; $0 < m \bmod k$; $k \leq c$ ⟧ ⟹
¬(∃ r' m' c'. $[r', \bmod m', c'] = [r, \bmod m, c] \oslash k$)
apply (*case-tac* $r \bmod k = 0$)
thm *iMODb-div-mod-gr0-is-0-not-ex0*
apply (*simp add*: *iMODb-div-mod-gr0-is-0-not-ex0*)
apply (*rule ccontr*, *simp*, *elim exE conjE*)
thm *iMODb-div-unique*
apply (*frule-tac* r'=r' **and** m'=m' **and** c'=c' **and** r=r **and** k=k **and** m=m **and** c=c
in *iMODb-div-unique*[*OF - less-imp-le*], *simp*+)
apply *clarsimp*
apply (*drule arg-cong*[**where** f=*Max*])
apply (*simp add*: *iT-Max iT-Div-Max iT-Div-finite-iff iT-Div-not-empty iT-finite*

iT-not-empty)
thm *div-add1-eq*
apply (*simp add*: *div-add1-eq*[*of r m* ∗ *c*])
apply (*simp add*: *mult-commute*[*of* - *c*])
thm *div-mult1-eq*[*of c m k*]
apply (*clarsimp simp add*: *div-mult1-eq*[*of c m k*])
apply (*subgoal-tac Suc 0* ≤ *c* ∗ (*m mod k*) *div k*, *simp*)
apply (*thin-tac ?x* = *0*)+
apply (*drule div-le-mono*[*of k c k*], *simp*)
apply (*rule order-trans*[*of* - *c div k*], *simp*)
apply (*rule div-le-mono*, *simp*)
done
thm *iMODb-div-mod-gr0-not-ex*

thm
cut-le-image
iMOD-div-right-strict-mono-on
iMOD-cut-le
thm *card-image*[*OF strict-mono-on-imp-inj-on*]

lemma *iMOD-div-eq-imp-iMODb-div-eq*:
⟦ *0* < *k*; *k* ≤ *m*; [*r'*, *mod m'*] = [*r*, *mod m*] ⊘ *k* ⟧ ⟹
[*r'*, *mod m'*, *c*] = [*r*, *mod m*, *c*] ⊘ *k*
apply (*subgoal-tac r'* = *r div k*)
prefer *2*
apply (*drule arg-cong*[**where** *f*=*iMin*])
apply (*simp add*: *iT-Div-Min iMOD-not-empty iMOD-Min*)
apply *clarsimp*
apply (*frule iMOD-div-right-strict-mono-on*[*of* - *m r*], *assumption*)
thm *iMODb-div-right-strict-mono-on*[*of k m r c*]
thm *card-image*[*OF strict-mono-on-imp-inj-on*]
thm *card-image*[*OF strict-mono-on-imp-inj-on*[*OF iMODb-div-right-strict-mono-on*[*of k m r c*]]]
apply (*frule card-image*[*OF strict-mono-on-imp-inj-on*[*OF iMODb-div-right-strict-mono-on*[*of k m r c*]]], *assumption*)
apply (*simp add*: *iMODb-card*)
apply (*subgoal-tac r* + *m* ∗ *c* ∈ [*r*, *mod m*])
prefer *2*
apply (*simp add*: *iMOD-iff*)
thm *iMOD-cut-le*[*of r m r* + *m* ∗ *c*]
apply (*subgoal-tac* [*r*, *mod m*, *c*] = [*r*, *mod m*] ↓≤ (*r* + *m* ∗ *c*))
prefer *2*
apply (*simp add*: *iMOD-cut-le1*)
apply (*simp add*: *iT-Div-def*)
thm *cut-le-image*[*OF* - *subset-refl*]
apply (*simp add*: *cut-le-image*[*symmetric*])
apply (*drule sym*)
apply (*simp add*: *iMOD-cut-le*)

apply (*simp add*: *linorder-not-le*[*of r div k*, *symmetric*])
thm *div-le-mono*
apply (*simp add*: *div-le-mono*)
apply (*case-tac* $m' = 0$)
apply (*simp add*: *iMODb-mod-0-card*)
apply (*rule arg-cong*[**where** $f{=}\lambda c.\ [r\ div\ k,\ mod\ m',\ c]$])
apply (*simp add*: *iMODb-card*)
done

lemma *iMOD-div-unique*:
$[\![\ 0 < k;\ k \le m;\ [r',\ mod\ m'] = [r,\ mod\ m] \oslash k\ ]\!] \Longrightarrow$
$r' = r\ div\ k \wedge m' = m\ div\ k$
thm *iMOD-div-eq-imp-iMODb-div-eq*
apply (*frule iMOD-div-eq-imp-iMODb-div-eq*[*of k m r' m' r k*], *assumption*+)
thm *iMODb-div-unique*[*of k - k*]
apply (*simp add*: *iMODb-div-unique*[*of k - k*])
done

thm *iMODb-div-mod-gr0-not-ex*
lemma *iMOD-div-mod-gr0-not-ex*:
$[\![\ 0 < k;\ k < m;\ 0 < m\ mod\ k\ ]\!] \Longrightarrow$
$\neg\ (\exists\, r'\ m'.\ [r',\ mod\ m'] = [r,\ mod\ m] \oslash k)$
apply (*rule ccontr*, *clarsimp*)
thm *iMOD-div-eq-imp-iMODb-div-eq*[*OF - less-imp-le*]
apply (*frule-tac* $k{=}k$ **and** $m{=}m$ **and** $r'{=}r'$ **and** $m'{=}m'$ **and** $c{=}k$
in *iMOD-div-eq-imp-iMODb-div-eq*[*OF - less-imp-le*], *assumption*+)
thm *iMODb-div-mod-gr0-not-ex*[*of k m k r*]
apply (*frule iMODb-div-mod-gr0-not-ex*[*of k m k r*], *simp*+)
done

2.2 Interval cut operators with arithmetic interval operators

lemma
iT-Plus-cut-le2: $(I \oplus k) \downarrow\le (t + k) = (I \downarrow\le t) \oplus k$ **and**
iT-Plus-cut-less2: $(I \oplus k) \downarrow< (t + k) = (I \downarrow< t) \oplus k$ **and**
iT-Plus-cut-ge2: $(I \oplus k) \downarrow\ge (t + k) = (I \downarrow\ge t) \oplus k$ **and**
iT-Plus-cut-greater2: $(I \oplus k) \downarrow> (t + k) = (I \downarrow> t) \oplus k$
unfolding *iT-Plus-def* **by** *fastsimp*+

lemma *iT-Plus-cut-le*:
$(I \oplus k) \downarrow\le t = (\mathit{if}\ t < k\ \mathit{then}\ \{\}\ \mathit{else}\ I \downarrow\le (t - k) \oplus k)$
apply (*case-tac* $t < k$)
apply (*simp add*: *cut-le-empty-iff iT-Plus-mem-iff*)
thm *iT-Plus-cut-le2*[*of I k t − k*]
apply (*insert iT-Plus-cut-le2*[*of I k t − k*], *simp*)
done
lemma *iT-Plus-cut-less*: $(I \oplus k) \downarrow< t = I \downarrow< (t - k) \oplus k$
apply (*case-tac* $t < k$)

apply (*simp add*: *cut-less-0-empty iT-Plus-empty cut-less-empty-iff iT-Plus-mem-iff*)
apply (*insert iT-Plus-cut-less2*[*of I k t − k*], *simp*)
done
lemma *iT-Plus-cut-ge*: $(I \oplus k) \downarrow\geq t = I \downarrow\geq (t - k) \oplus k$
apply (*case-tac* $t < k$)
apply (*simp add*: *cut-ge-0-all cut-ge-all-iff iT-Plus-mem-iff*)
apply (*insert iT-Plus-cut-ge2*[*of I k t − k*], *simp*)
done
lemma *iT-Plus-cut-greater*:
$(I \oplus k) \downarrow> t = (\textit{if } t < k \textit{ then } I \oplus k \textit{ else } I \downarrow> (t - k) \oplus k)$
apply (*case-tac* $t < k$)
apply (*simp add*: *cut-greater-all-iff iT-Plus-mem-iff*)
apply (*insert iT-Plus-cut-greater2*[*of I k t − k*], *simp*)
done

lemma
iT-Mult-cut-le2: $0 < k \Longrightarrow (I \otimes k) \downarrow\leq (t * k) = (I \downarrow\leq t) \otimes k$ **and**
iT-Mult-cut-less2: $0 < k \Longrightarrow (I \otimes k) \downarrow< (t * k) = (I \downarrow< t) \otimes k$ **and**
iT-Mult-cut-ge2: $0 < k \Longrightarrow (I \otimes k) \downarrow\geq (t * k) = (I \downarrow\geq t) \otimes k$ **and**
iT-Mult-cut-greater2: $0 < k \Longrightarrow (I \otimes k) \downarrow> (t * k) = (I \downarrow> t) \otimes k$
unfolding *iT-Mult-def* **by** *fastsimp*+

lemma *iT-Mult-cut-le*:
$0 < k \Longrightarrow (I \otimes k) \downarrow\leq t = (I \downarrow\leq (t \textit{ div } k)) \otimes k$
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-le-mem-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*rule iffI*)
apply (*simp add*: *div-le-mono*)
apply (*rule div-le-mod-le-imp-le*, *simp*+)
done
lemma *iT-Mult-cut-less*:
$0 < k \Longrightarrow (I \otimes k) \downarrow< t =$
$(\textit{if } t \textit{ mod } k = 0 \textit{ then } (I \downarrow< (t \textit{ div } k)) \textit{ else } I \downarrow< \textit{Suc } (t \textit{ div } k)) \otimes k$
apply (*case-tac* $t \textit{ mod } k = 0$)
apply (*clarsimp simp add*: *mult-commute*[*of k*] *iT-Mult-cut-less2*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-less-mem-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*subst less-Suc-eq-le*)
apply (*rule iffI*)
apply (*rule div-le-mono*, *simp*)
apply (*rule ccontr*, *simp add*: *linorder-not-less*)
apply (*drule le-imp-less-or-eq*[*of t*], *erule disjE*)
thm *less-mod-0-imp-div-less*[*of t x k*]
apply (*fastsimp dest*: *less-mod-0-imp-div-less*[*of t - k*])
apply *simp*
done
lemma *iT-Mult-cut-greater*:
$0 < k \Longrightarrow (I \otimes k) \downarrow> t = (I \downarrow> (t \textit{ div } k)) \otimes k$

apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-greater-mem-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*rule iffI*)
apply (*simp add*: *less-mod-ge-imp-div-less*)
apply (*rule ccontr*, *simp add*: *linorder-not-less*)
apply (*fastsimp dest*: *div-le-mono*[*of* - - *k*])
done
lemma *iT-Mult-cut-ge*:
$0 < k \Longrightarrow (I \otimes k) \downarrow\geq t =$
(*if t mod k* = *0 then* $(I \downarrow\geq (t \text{ div } k))$ *else* $I \downarrow\geq Suc\ (t \text{ div } k)) \otimes k$
apply (*case-tac t mod k* = *0*)
apply (*clarsimp simp add*: *mult-commute*[*of k*] *iT-Mult-cut-ge2*)
apply (*clarsimp simp*: *set-eq-iff iT-Mult-mem-iff cut-ge-mem-iff*)
apply (*rule conj-cong*, *simp*)+
apply (*rule iffI*)
apply (*rule Suc-leI*)
apply (*simp add*: *le-mod-greater-imp-div-less*)
apply (*rule ccontr*)
apply (*drule Suc-le-lessD*)
apply (*simp add*: *linorder-not-le*)
thm *div-le-mono*[*OF order-less-imp-le*]
apply (*fastsimp dest*: *div-le-mono*[*OF order-less-imp-le*, *of* - *t k*])
done

lemma *iT-Plus-neg-cut-le2*: $k \leq t \Longrightarrow (I \oplus- k) \downarrow\leq (t - k) = (I \downarrow\leq t) \oplus- k$
apply (*simp add*: *iT-Plus-neg-image-conv*)
thm *i-cut-commute-disj*[*of op* $\downarrow\leq$ *op* $\downarrow\geq$]
apply (*simp add*: *i-cut-commute-disj*[*of op* $\downarrow\leq$ *op* $\downarrow\geq$])
apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done
lemma *iT-Plus-neg-cut-less2*: $(I \oplus- k) \downarrow< (t - k) = (I \downarrow< t) \oplus- k$
apply (*case-tac* $t \leq k$)
apply (*simp add*: *cut-less-0-empty*)
apply (*case-tac* $I \downarrow< t = \{\}$)
apply (*simp add*: *iT-Plus-neg-empty*)
apply (*rule sym*, *rule iT-Plus-neg-Max-less-empty*[*OF nat-cut-less-finite*])
apply (*rule order-less-le-trans*[*OF cut-less-Max-less*[*OF nat-cut-less-finite*]], *assumption*+)
apply (*simp add*: *linorder-not-le iT-Plus-neg-image-conv*)
apply (*simp add*: *i-cut-commute-disj*[*of op* $\downarrow<$ *op* $\downarrow\geq$])
apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done
lemma *iT-Plus-neg-cut-ge2*: $(I \oplus- k) \downarrow\geq (t - k) = (I \downarrow\geq t) \oplus- k$
apply (*case-tac* $t \leq k$)
apply (*simp add*: *cut-ge-0-all iT-Plus-neg-cut-eq*)
apply (*simp add*: *linorder-not-le iT-Plus-neg-image-conv*)
apply (*simp add*: *i-cut-commute-disj*[*of op* $\downarrow\geq$ *op* $\downarrow\geq$])

apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done
lemma *iT-Plus-neg-cut-greater2*: $k \leq t \Longrightarrow (I \oplus\!- k) \downarrow\!> (t - k) = (I \downarrow\!> t) \oplus\!- k$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*simp add*: *i-cut-commute-disj*[*of op* $\downarrow\!>$ *op* $\downarrow\!\geq$])
apply (*rule i-cut-image*[*OF sub-left-strict-mono-on*])
apply (*simp add*: *cut-ge-Int-conv*)+
done

lemma *iT-Plus-neg-cut-le*: $(I \oplus\!- k) \downarrow\!\leq t = I \downarrow\!\leq (t + k) \oplus\!- k$
by (*insert iT-Plus-neg-cut-le2*[*of k t* + *k I*, *OF le-add2*], *simp*)
lemma *iT-Plus-neg-cut-less*: $(I \oplus\!- k) \downarrow\!< t = I \downarrow\!< (t + k) \oplus\!- k$
by (*insert iT-Plus-neg-cut-less2*[*of I k t* + *k*], *simp*)
lemma *iT-Plus-neg-cut-ge*: $(I \oplus\!- k) \downarrow\!\geq t = I \downarrow\!\geq (t + k) \oplus\!- k$
by (*insert iT-Plus-neg-cut-ge2*[*of I k t* + *k*], *simp*)
lemma *iT-Plus-neg-cut-greater*: $(I \oplus\!- k) \downarrow\!> t = I \downarrow\!> (t + k) \oplus\!- k$
by (*insert iT-Plus-neg-cut-greater2*[*of k t* + *k I*], *simp*)

lemma *iT-Minus-cut-le2*: $t \leq k \Longrightarrow (k \ominus I) \downarrow\!\leq (k - t) = k \ominus (I \downarrow\!\geq t)$
by (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
lemma *iT-Minus-cut-less2*: $(k \ominus I) \downarrow\!< (k - t) = k \ominus (I \downarrow\!> t)$
by (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
lemma *iT-Minus-cut-ge2*: $(k \ominus I) \downarrow\!\geq (k - t) = k \ominus (I \downarrow\!\leq t)$
by (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
lemma *iT-Minus-cut-greater2*: $t \leq k \Longrightarrow (k \ominus I) \downarrow\!> (k - t) = k \ominus (I \downarrow\!< t)$
by (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)

lemma *iT-Minus-cut-le*: $(k \ominus I) \downarrow\!\leq t = k \ominus (I \downarrow\!\geq (k - t))$
by (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
lemma *iT-Minus-cut-less*:
$(k \ominus I) \downarrow\!< t = (\textit{if } t \leq k \textit{ then } k \ominus (I \downarrow\!> (k - t)) \textit{ else } k \ominus I)$
apply (*case-tac* $t \leq k$)
apply (*cut-tac iT-Minus-cut-less2*[*of k I k* − *t*], *simp*)
apply (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
done
lemma *iT-Minus-cut-ge*:
$(k \ominus I) \downarrow\!\geq t = (\textit{if } t \leq k \textit{ then } k \ominus (I \downarrow\!\leq (k - t)) \textit{ else } \{\})$
apply (*case-tac* $t \leq k$)
apply (*cut-tac iT-Minus-cut-ge2*[*of k I k* − *t*], *simp*)
apply (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
done
lemma *iT-Minus-cut-greater*: $(k \ominus I) \downarrow\!> t = k \ominus (I \downarrow\!< (k - t))$
apply (*case-tac* $t \leq k$)
apply (*cut-tac iT-Minus-cut-greater2*[*of k* − *t k I*], *simp*+)

apply (*fastsimp simp*: *i-cut-mem-iff iT-Minus-mem-iff*)
done

thm *iT-Div-def*
thm *iT-Mult-cut-le2*
thm *iT-Div-mem-iff*
lemma *iT-Div-cut-le*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\leq t = I \downarrow\leq (t * k + (k - Suc\ 0)) \oslash k$
apply (*simp add*: *set-eq-iff i-cut-mem-iff iT-Div-mem-iff Bex-def*)
thm *div-le-conv*
apply (*fastsimp simp*: *div-le-conv*)
done
lemma *iT-Div-cut-less*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< t = I \downarrow< (t * k) \oslash k$
apply (*case-tac t = 0*)
apply (*simp add*: *cut-less-0-empty iT-Div-empty*)
apply (*simp add*: *nat-cut-less-le-conv iT-Div-cut-le diff-mult-distrib*)
done
lemma *iT-Div-cut-ge*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\geq t = I \downarrow\geq (t * k) \oslash k$
apply (*simp add*: *set-eq-iff i-cut-mem-iff iT-Div-mem-iff Bex-def*)
thm *le-div-conv*
apply (*fastsimp simp*: *le-div-conv*)
done
lemma *iT-Div-cut-greater*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> t = I \downarrow> (t * k + (k - Suc\ 0)) \oslash k$
by (*simp add*: *nat-cut-ge-greater-conv*[*symmetric*] *iT-Div-cut-ge add-commute*[*of k*])

lemma *iT-Div-cut-le2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\leq (t\ div\ k) = I \downarrow\leq (t - t\ mod\ k + (k - Suc\ 0)) \oslash k$
by (*frule iT-Div-cut-le*[*of k I t div k*], *simp add*: *div-mult-cancel*)
lemma *iT-Div-cut-less2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow< (t\ div\ k) = I \downarrow< (t - t\ mod\ k) \oslash k$
by (*frule iT-Div-cut-less*[*of k I t div k*], *simp add*: *div-mult-cancel*)
lemma *iT-Div-cut-ge2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow\geq (t\ div\ k) = I \downarrow\geq (t - t\ mod\ k) \oslash k$
by (*frule iT-Div-cut-ge*[*of k I t div k*], *simp add*: *div-mult-cancel*)
lemma *iT-Div-cut-greater2*:
$0 < k \Longrightarrow (I \oslash k) \downarrow> (t\ div\ k) = I \downarrow> (t - t\ mod\ k + (k - Suc\ 0)) \oslash k$
by (*frule iT-Div-cut-greater*[*of k I t div k*], *simp add*: *div-mult-cancel*)

2.3 *inext* and *iprev* with interval operators

lemma *iT-Plus-inext*: $inext\ (n + k)\ (I \oplus k) = (inext\ n\ I) + k$

by (*unfold iT-Plus-def*, *rule inext-image2*[*OF add-right-strict-mono*])

lemma *iT-Plus-iprev*: *iprev* (*n* + *k*) (*I* ⊕ *k*) = (*iprev n I*) + *k*
by (*unfold iT-Plus-def*, *rule iprev-image2*[*OF add-right-strict-mono*])

lemma *iT-Plus-inext2*: *k* ≤ *n* ⟹ *inext n* (*I* ⊕ *k*) = (*inext* (*n* − *k*) *I*) + *k*
by (*insert iT-Plus-inext*[*of n* − *k k I*], *simp*)
lemma *iT-Plus-prev2*: *k* ≤ *n* ⟹ *iprev n* (*I* ⊕ *k*) = (*iprev* (*n* − *k*) *I*) + *k*
by (*insert iT-Plus-iprev*[*of n* − *k k I*], *simp*)

lemma *iT-Mult-inext*: *inext* (*n* ∗ *k*) (*I* ⊗ *k*) = (*inext n I*) ∗ *k*
apply (*case-tac I* = {})
apply (*simp add*: *iT-Mult-empty inext-empty*)
apply (*case-tac k* = *0*)
apply (*simp add*: *iT-Mult-0 iTILL-0 inext-singleton*)
apply (*simp add*: *iT-Mult-def inext-image2*[*OF mult-right-strict-mono*])
done
lemma *iT-Mult-iprev*: *iprev* (*n* ∗ *k*) (*I* ⊗ *k*) = (*iprev n I*) ∗ *k*
apply (*case-tac I* = {})
apply (*simp add*: *iT-Mult-empty iprev-empty*)
apply (*case-tac k* = *0*)
apply (*simp add*: *iT-Mult-0 iTILL-0 iprev-singleton*)
apply (*simp add*: *iT-Mult-def iprev-image2*[*OF mult-right-strict-mono*])
done

lemma *iT-Mult-inext2-if*:
inext n (*I* ⊗ *k*) = (*if n mod k* = *0 then* (*inext* (*n div k*) *I*) ∗ *k else n*)
apply (*case-tac I* = {})
apply (*simp add*: *iT-Mult-empty inext-empty div-mult-cancel*)
apply (*case-tac k* = *0*)
apply (*simp add*: *iT-Mult-0 iTILL-0 inext-singleton*)
apply (*case-tac n mod k* = *0*)
apply (*clarsimp simp*: *mult-commute*[*of k*] *iT-Mult-inext*)
apply (*simp add*: *not-in-inext-fix iT-Mult-mem-iff*)
done

lemma *iT-Mult-iprev2-if*:
iprev n (*I* ⊗ *k*) = (*if n mod k* = *0 then* (*iprev* (*n div k*) *I*) ∗ *k else n*)
apply (*case-tac I* = {})
apply (*simp add*: *iT-Mult-empty iprev-empty div-mult-cancel*)
apply (*case-tac k* = *0*)
apply (*simp add*: *iT-Mult-0 iTILL-0 iprev-singleton*)
apply (*case-tac n mod k* = *0*)
apply (*clarsimp simp*: *mult-commute*[*of k*] *iT-Mult-iprev*)
apply (*simp add*: *not-in-iprev-fix iT-Mult-mem-iff*)
done

corollary *iT-Mult-inext2*:
  $n \ mod \ k = 0 \Longrightarrow inext \ n \ (I \otimes k) = (inext \ (n \ div \ k) \ I) * k$
by (*simp add*: *iT-Mult-inext2-if*)
corollary *iT-Mult-iprev2*:
  $n \ mod \ k = 0 \Longrightarrow iprev \ n \ (I \otimes k) = (iprev \ (n \ div \ k) \ I) * k$
by (*simp add*: *iT-Mult-iprev2-if*)

lemma *iT-Plus-neg-inext*:
  $k \leq n \Longrightarrow inext \ (n - k) \ (I \oplus- \ k) = inext \ n \ I - k$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Plus-neg-empty inext-empty*)
apply (*case-tac* $n \in I$)
 apply (*simp add*: *iT-Plus-neg-image-conv*)
 thm *subst*[*OF inext-cut-ge-conv*]
 apply (*rule subst*[*OF inext-cut-ge-conv*, *of k*], *simp*)
 thm *inext-image*
 apply (*rule inext-image*)
  apply (*simp add*: *cut-ge-mem-iff*)
 apply (*subst cut-ge-Int-conv*)
 thm *strict-mono-on-subset*[*OF - Int-lower2*]
 apply (*rule strict-mono-on-subset*[*OF - Int-lower2*])
 thm *sub-left-strict-mono-on*
 apply (*rule sub-left-strict-mono-on*)
apply (*subgoal-tac* $n - k \notin I \oplus- \ k$)
 prefer *2*
 apply (*simp add*: *iT-Plus-neg-mem-iff*)
apply (*simp add*: *not-in-inext-fix*)
done

lemma *iT-Plus-neg-iprev*:
  $iprev \ (n - k) \ (I \oplus- \ k) = iprev \ n \ (I \downarrow\geq k) - k$
apply (*case-tac* $I = \{\}$)
 apply (*simp add*: *iT-Plus-neg-empty i-cut-empty iprev-empty*)
apply (*case-tac* $n < k$)
 apply (*simp add*: *iprev-le-iMin*)
 apply (*simp add*: *order-trans*[*OF iprev-mono*])
apply (*simp add*: *linorder-not-less*)
apply (*case-tac* $n \in I$)
 thm *iT-Plus-neg-mem-iff2*[*THEN iffD2*]
 apply (*frule iT-Plus-neg-mem-iff2*[*THEN iffD2*, *of - - I*], *assumption*)
 apply (*simp add*: *iT-Plus-neg-image-conv*)
 apply (*rule iprev-image*)
  apply (*simp add*: *cut-ge-mem-iff*)
 apply (*subst cut-ge-Int-conv*)
 thm *strict-mono-on-subset*[*OF - Int-lower2*]
 apply (*rule strict-mono-on-subset*[*OF - Int-lower2*])
 thm *sub-left-strict-mono-on*
 apply (*rule sub-left-strict-mono-on*)
apply (*frule cut-ge-not-in-imp*[*of - - k*])

apply (*subgoal-tac* $n - k \notin I \oplus- k$)
prefer *2*
apply (*simp add*: *iT-Plus-neg-mem-iff*)
apply (*simp add*: *not-in-iprev-fix*)
done

corollary *iT-Plus-neg-inext2*: $inext\ n\ (I \oplus- k) = inext\ (n + k)\ I - k$
by (*insert iT-Plus-neg-inext*[*of k n + k I*, *OF le-add2*], *simp*)
corollary *iT-Plus-neg-iprev2*: $iprev\ n\ (I \oplus- k) = iprev\ (n + k)\ (I \downarrow\geq k) - k$
by (*insert iT-Plus-neg-iprev*[*of n + k k I*], *simp*)

lemma *iT-Minus-inext*:
$[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow inext\ (k - n)\ (k \ominus I) = k - iprev\ n\ I$
apply (*subgoal-tac iMin I* $\leq k$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff*)
apply (*subgoal-tac* $I \downarrow\leq k \neq \{\}$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*case-tac* $n \in I$)
thm *iT-Minus-imirror-conv*
apply (*simp add*: *iT-Minus-imirror-conv*)
apply (*simp add*: *iT-Plus-neg-inext2*)
apply (*subgoal-tac* $n \leq iMin\ I + Max\ (I \downarrow\leq k)$)
prefer *2*
apply (*rule trans-le-add2*)
apply (*rule Max-ge*[*OF nat-cut-le-finite*])
apply (*simp add*: *cut-le-mem-iff*)
thm *iT-Plus-inext*
apply (*simp add*: *diff-add-assoc del*: *add-diff-assoc*)
apply (*subst add-commute*[*of k*], *subst iT-Plus-inext*)
apply (*simp add*: *cut-le-Min-eq*[*of I*, *symmetric*])
apply (*fold nat-mirror-def mirror-elem-def*)
apply (*simp add*: *inext-imirror-iprev-conv*[*OF nat-cut-le-finite*])
apply (*simp add*: *iprev-cut-le-conv*)
apply (*simp add*: *mirror-elem-def nat-mirror-def*)
thm *iprev-mono*[*THEN order-trans*, *of n iMin* $(I \downarrow\leq k) + Max\ (I \downarrow\leq k)\ I$]
apply (*frule iprev-mono*[*THEN order-trans*, *of n iMin* $(I \downarrow\leq k) + Max\ (I \downarrow\leq k)$ *I*])
apply *simp*
apply (*subgoal-tac* $k - n \notin k \ominus I$)
prefer *2*
apply (*simp add*: *iT-Minus-mem-iff*)
apply (*simp add*: *not-in-inext-fix not-in-iprev-fix*)
done
corollary *iT-Minus-inext2*:
$[\![k \ominus I \neq \{\}; n \leq k]\!] \Longrightarrow inext\ n\ (k \ominus I) = k - iprev\ (k - n)\ I$

by (*insert iT-Minus-inext*[*of k I k − n*], *simp*)

lemma *iT-Minus-iprev*:
⟦ $k \ominus I \neq \{\}$; $n \leq k$ ⟧ ⟹ $iprev\ (k - n)\ (k \ominus I) = k - inext\ n\ (I \downarrow\leq k)$
apply (*subgoal-tac I* $\downarrow\leq k \neq \{\}$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*subst iT-Minus-cut-eq*[*OF le-refl*, *of - I*, *symmetric*])
apply (*insert iT-Minus-inext2*[*of k k* $\ominus$ ($I \downarrow\leq k$) *n*])
apply (*simp add*: *iT-Minus-Minus-cut-eq*)
apply (*rule diff-diff-cancel*[*symmetric*])
apply (*rule order-trans*[*OF iprev-mono*])
apply *simp*
done
lemma *iT-Minus-iprev2*:
⟦ $k \ominus I \neq \{\}$; $n \leq k$ ⟧ ⟹ $iprev\ n\ (k \ominus I) = k - inext\ (k - n)\ (I \downarrow\leq k)$
by (*insert iT-Minus-iprev*[*of k I k − n*], *simp*)

lemma *iT-Plus-inext-nth*: $I \neq \{\} \Longrightarrow (I \oplus k) \rightarrow n = (I \rightarrow n) + k$
apply (*induct n*)
apply (*simp add*: *iT-Plus-Min*)
apply (*simp add*: *iT-Plus-inext*)
done
lemma *iT-Plus-iprev-nth*: ⟦ *finite I*; $I \neq \{\}$ ⟧ ⟹ $(I \oplus k) \leftarrow n = (I \leftarrow n) + k$
apply (*induct n*)
apply (*simp add*: *iT-Plus-Max*)
apply (*simp add*: *iT-Plus-iprev*)
done

lemma *iT-Mult-inext-nth*: $I \neq \{\} \Longrightarrow (I \otimes k) \rightarrow n = (I \rightarrow n) * k$
apply (*induct n*)
apply (*simp add*: *iT-Mult-Min*)
apply (*simp add*: *iT-Mult-inext*)
done
lemma *iT-Mult-iprev-nth*: ⟦ *finite I*; $I \neq \{\}$ ⟧ ⟹ $(I \otimes k) \leftarrow n = (I \leftarrow n) * k$
apply (*induct n*)
apply (*simp add*: *iT-Mult-Max*)
apply (*simp add*: *iT-Mult-iprev*)
done

lemma *iT-Plus-neg-inext-nth*:
$I \oplus- k \neq \{\} \Longrightarrow (I \oplus- k) \rightarrow n = (I \downarrow\geq k \rightarrow n) - k$
apply (*subgoal-tac I* $\downarrow\geq k \neq \{\}$)
prefer *2*
apply (*simp add*: *cut-ge-not-empty-iff iT-Plus-neg-not-empty-iff*)

apply (*induct n*)
apply (*simp add*: *iT-Plus-neg-Min*)
thm *iT-Plus-neg-cut-eq*[*of k k I*]
apply (*simp add*: *iT-Plus-neg-cut-eq*[*of k k I*, *symmetric*])
apply (*rule iT-Plus-neg-inext*)
apply (*rule cut-ge-bound*[*of - I*])
apply (*simp add*: *inext-nth-closed*)
done
lemma *iT-Plus-neg-iprev-nth*:
⟦ *finite I*; $I \oplus- k \neq \{\}$ ⟧ ⟹ $(I \oplus- k) \leftarrow n = (I \downarrow\geq k \leftarrow n) - k$
apply (*subgoal-tac* $I \downarrow\geq k \neq \{\}$)
prefer *2*
apply (*simp add*: *cut-ge-not-empty-iff iT-Plus-neg-not-empty-iff*)
apply (*induct n*)
apply (*simp add*: *iT-Plus-neg-Max cut-ge-Max-eq*)
apply (*simp add*: *iT-Plus-neg-iprev*)
done

lemma *iT-Minus-inext-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \rightarrow n = k - ((I \downarrow\leq k) \leftarrow n)$
apply (*subgoal-tac* $I \downarrow\leq k \neq \{\} \wedge I \neq \{\} \wedge iMin\ I \leq k$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*elim conjE*)
apply (*induct n*)
apply(*simp add*: *iT-Minus-Min*)
apply (*simp add*: *iT-Minus-cut-eq*[*OF order-refl*, *of - I*, *symmetric*])
apply (*rule iT-Minus-inext*)
apply *simp*
by (*rule cut-le-bound*, *rule iprev-nth-closed*[*OF nat-cut-le-finite*])
lemma *iT-Minus-iprev-nth*:
$k \ominus I \neq \{\} \Longrightarrow (k \ominus I) \leftarrow n = k - ((I \downarrow\leq k) \rightarrow n)$
apply (*subgoal-tac* $I \downarrow\leq k \neq \{\} \wedge I \neq \{\} \wedge iMin\ I \leq k$)
prefer *2*
apply (*simp add*: *iT-Minus-empty-iff cut-le-Min-not-empty*)
apply (*elim conjE*)
apply (*induct n*)
apply(*simp add*: *iT-Minus-Max cut-le-Min-eq*)
apply *simp*
apply (*rule iT-Minus-iprev*)
apply *simp*
by (*rule cut-le-bound*, *rule inext-nth-closed*)

lemma *iT-Div-ge-inext-nth*:
⟦ $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x < y \longrightarrow x + k \leq y$ ⟧ ⟹
$(I \oslash k) \rightarrow n = (I \rightarrow n)\ div\ k$
apply (*case-tac* $k = 0$)
apply (*simp add*: *iT-Div-0 iTILL-0 inext-nth-singleton*)
apply (*simp add*: *iT-Div-def*)

by (*rule inext-nth-image*[*OF - div-right-strict-mono-on*])
lemma *iT-Div-mod-inext-nth*:
⟦ $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x \bmod k = y \bmod k$ ⟧ ⟹
$(I \oslash k) \rightarrow n = (I \rightarrow n)\ div\ k$
apply (*case-tac* $k = 0$)
apply (*simp add*: *iT-Div-0 iTILL-0 inext-nth-singleton*)
apply (*simp add*: *iT-Div-def*)
by (*rule inext-nth-image*[*OF - mod-eq-div-right-strict-mono-on*])
lemma *iT-Div-ge-iprev-nth*:
⟦ *finite* I; $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x < y \longrightarrow x + k \leq y$ ⟧ ⟹
$(I \oslash k) \leftarrow n = (I \leftarrow n)\ div\ k$
apply (*case-tac* $k = 0$)
apply (*simp add*: *iT-Div-0 iTILL-0 iprev-nth-singleton*)
apply (*simp add*: *iT-Div-def*)
by (*rule iprev-nth-image*[*OF - - div-right-strict-mono-on*])
lemma *iT-Div-mod-iprev-nth*:
⟦ *finite* I; $I \neq \{\}$; $\forall x{\in}I.\ \forall y{\in}I.\ x \bmod k = y \bmod k$ ⟧ ⟹
$(I \oslash k) \leftarrow n = (I \leftarrow n)\ div\ k$
apply (*case-tac* $k = 0$)
apply (*simp add*: *iT-Div-0 iTILL-0 iprev-nth-singleton*)
apply (*simp add*: *iT-Div-def*)
by (*rule iprev-nth-image*[*OF - - mod-eq-div-right-strict-mono-on*])

2.4 Cardinality of intervals with interval operators

lemma *iT-Plus-card*: *card* $(I \oplus k) =$ *card* I
apply (*unfold iT-Plus-def*)
apply (*rule card-image*)
apply (*rule inj-imp-inj-on*)
apply (*rule add-right-inj*)
done
lemma *iT-Mult-card*: $0 < k \Longrightarrow$ *card* $(I \otimes k) =$ *card* I
apply (*unfold iT-Mult-def*)
apply (*rule card-image*)
apply (*rule inj-imp-inj-on*)
apply (*rule mult-right-inj*)
apply *assumption*
done

lemma *iT-Plus-neg-card*: *card* $(I \oplus\!- k) =$ *card* $(I \downarrow\!\geq k)$
apply (*simp add*: *iT-Plus-neg-image-conv*)
apply (*rule card-image*)
apply (*subst cut-ge-Int-conv*)
thm *Int-lower2*
thm *subset-inj-on*[*OF - Int-lower2*]
apply (*rule subset-inj-on*[*OF - Int-lower2*])
thm *sub-left-inj-on*
apply (*rule sub-left-inj-on*)
done

lemma *iT-Plus-neg-card-le*: *card* $(I \oplus- k) \leq$ *card* I
apply (*simp add*: *iT-Plus-neg-card*)
apply (*case-tac finite* I)
apply (*rule cut-ge-card*, *assumption*)
thm *nat-cut-ge-finite-iff card-infinite*
apply (*simp add*: *nat-cut-ge-finite-iff*)
done

lemma *iT-Minus-card*: *card* $(k \ominus I) =$ *card* $(I \downarrow\leq k)$
apply (*simp add*: *iT-Minus-image-conv*)
apply (*rule card-image*)
apply (*subst cut-le-Int-conv*)
thm *subset-inj-on*[*OF - Int-lower2*]
apply (*rule subset-inj-on*[*OF - Int-lower2*])
apply (*rule sub-right-inj-on*)
done
lemma *iT-Minus-card-le*: *finite* $I \Longrightarrow$ *card* $(k \ominus I) \leq$ *card* I
by (*subst iT-Minus-card*, *rule cut-le-card*)

lemma *iT-Div-0-card-if*:
card $(I \oslash 0) =$ (*if* $I = \{\}$ *then 0 else Suc 0*)
by (*fastsimp simp*: *iT-Div-empty iT-Div-0 iTILL-0*)

lemma *Int-empty-setsum*:
$(\sum k{\leq}(n::nat).$ *if* $\{\} \cap (I\ k) = \{\}$ *then 0 else Suc 0*$) = 0$
apply (*rule setsum-eq-0-iff*[*THEN iffD2*])
apply (*rule finite-atMost*)
apply *simp*
done
lemma *iT-Div-mod-partition-card*:
card $(I \cap [n * d\ldots,d - Suc\ 0] \oslash d) =$
(*if* $I \cap [n * d\ldots,d - Suc\ 0] = \{\}$ *then 0 else Suc 0*)
apply (*case-tac* $d = 0$)
apply (*simp add*: *iIN-0 iTILL-0 iT-Div-0-if*)
apply (*split split-if*, *rule conjI*)
apply (*simp add*: *iT-Div-empty*)
apply *clarsimp*
apply (*subgoal-tac* $I \cap [n * d\ldots,d - Suc\ 0] \oslash d = \{n\}$, *simp*)
apply (*rule set-eqI*)
thm *iT-Div-mem-iff*
apply (*simp add*: *iT-Div-mem-iff Bex-def iIN-iff*)
apply (*rule iffI*)
thm *le-less-imp-div*
apply (*clarsimp simp*: *le-less-imp-div*)
apply (*drule ex-in-conv*[*THEN iffD2*], *clarsimp simp*: *iIN-iff*, *rename-tac* x')
apply (*rule-tac* $x{=}x'$ **in** *exI*)
apply (*simp add*: *le-less-imp-div*)
done

thm *iT-Div-mem-iff-Int*

lemma *iT-Div-conv-count*:
*0 < d ⟹ I ⊘ d = {k. I ∩ [k * d...,d − Suc 0] ≠ {}}*
apply (*case-tac I = {}*)
apply (*simp add*: *iT-Div-empty*)
apply (*rule set-eqI*)
thm *iT-Div-mem-iff-Int*
apply (*simp add*: *iT-Div-mem-iff-Int*)
done

lemma *iT-Div-conv-count2*:
⟦ 0 < d; finite I; Max I div d ≤ n ⟧ ⟹
*I ⊘ d = {k. k ≤ n ∧ I ∩ [k * d...,d − Suc 0] ≠ {}}*
apply (*simp add*: *iT-Div-conv-count*)
apply (*rule set-eqI*, *simp*)
apply (*rule iffI*)
apply *simp*
apply (*rule ccontr*)
apply (*drule ex-in-conv*[*THEN iffD2*], *clarify*, *rename-tac x′*)
apply (*clarsimp simp*: *linorder-not-le iIN-iff*)
apply (*drule order-le-less-trans*, *simp*)
thm *div-less-conv*
apply (*drule div-less-conv*[*THEN iffD1*, *of - Max I*], *simp*)
apply (*drule-tac x=x′* **in** *Max-ge*, *simp*)
apply *simp+*
done

lemma *mod-partition-count-Suc*:
*{k. k ≤ Suc n ∧ I ∩ [k * d...,d − Suc 0] ≠ {}} =*
*{k. k ≤ n ∧ I ∩ [k * d...,d − Suc 0] ≠ {}} ∪*
*(if I ∩ [Suc n * d...,d − Suc 0] ≠ {} then {Suc n} else {})*
apply (*rule set-eqI*, *rename-tac x*)
apply (*simp add*: *le-less*[*of - Suc n*] *less-Suc-eq-le*)
apply (*simp add*: *conj-disj-distribR*)
apply (*intro conjI impI*)
apply *fastsimp*
apply (*rule iffI*, *clarsimp+*)
done

lemma *iT-Div-card*:
⋀I.⟦ 0 < d; finite I; Max I div d ≤ n⟧ ⟹
card (I ⊘ d) = (∑ k≤n.
*if I ∩ [k * d...,d − Suc 0] = {} then 0 else Suc 0)*
apply (*case-tac I = {}*)
apply (*simp add*: *iT-Div-empty*)
thm *iT-Div-conv-count2*
apply (*simp add*: *iT-Div-conv-count2*)
apply (*induct n*)

apply (*simp add*: *div-eq-0-conv iIN-0-iTILL-conv*)
apply (*subgoal-tac* $I \cap [\ldots d - Suc\ 0] \neq \{\}$)
prefer *2*
apply (*simp add*: *ex-in-conv*[*symmetric*], *fastsimp*)
apply (*simp add*: *card-1-singleton-conv*)
apply (*rule-tac x=0* **in** *exI*)
apply (*rule set-eqI*)
apply (*simp add*: *ex-in-conv*[*symmetric*], *fastsimp*)
apply *simp*
thm *mod-partition-count-Suc*
apply (*simp add*: *mod-partition-count-Suc*)
apply (*drule-tac x=I* $\cap [\ldots n * d + d - Suc\ 0]$ **in** *meta-spec*)
apply *simp*
apply (*case-tac* $I \cap [\ldots n * d + d - Suc\ 0] = \{\}$)
apply *simp*
apply (*subgoal-tac* $\{k.\ k \leq n \wedge I \cap [k * d \ldots, d - Suc\ 0] \neq \{\}\} = \{\}$, *simp*)
apply (*clarsimp*, *rename-tac x*)
apply (*subgoal-tac* $I \cap [x * d \ldots, d - Suc\ 0] \subseteq I \cap [\ldots n * d + d - Suc\ 0]$, *simp*)
apply (*rule Int-mono*[*OF order-refl*])
apply (*simp add*: *iIN-iTILL-subset-conv*)
apply (*simp add*: *diff-le-mono*)
find-theorems - *div* - <= - *name*: *conv*
apply (*subgoal-tac Max* $(I \cap [\ldots n * d + d - Suc\ 0])$ *div* $d \leq n$)
prefer *2*
thm *div-le-conv*
apply (*simp add*: *div-le-conv add-commute*[*of d*] *iTILL-iff*)
apply (*subgoal-tac* $\bigwedge k.\ k \leq n \Longrightarrow [\ldots n * d + d - Suc\ 0] \cap [k * d \ldots, d - Suc\ 0]$
$= [k * d \ldots, d - Suc\ 0]$)
prefer *2*
apply (*subst Int-commute*)
thm *cut-le-Int-conv*
apply (*simp add*: *iTILL-def cut-le-Int-conv*[*symmetric*])
apply (*rule cut-le-Max-all*[*OF iIN-finite*])
apply (*simp add*: *iIN-Max diff-le-mono*)
apply (*simp add*: *Int-assoc*)
apply (*subgoal-tac*
$\{k.\ k \leq n \wedge I \cap ([\ldots n * d + d - Suc\ 0] \cap [k * d \ldots, d - Suc\ 0]) \neq \{\}\} =$
$\{k.\ k \leq n \wedge I \cap [k * d \ldots, d - Suc\ 0] \neq \{\}\}$)
prefer *2*
apply (*rule set-eqI*, *rename-tac x*)
apply *simp*
apply (*rule conj-cong*, *simp*)
apply *simp*
apply *simp*
done

thm *iT-Div-card*
thm *iT-Div-card*[*OF - - le-refl*, *of d i*]
corollary *iT-Div-card-Suc*:

$\bigwedge I.[\![\ 0 < d;\ \mathit{finite}\ I;\ \mathit{Max}\ I\ \mathit{div}\ d \leq n\ ]\!] \Longrightarrow$
$\mathit{card}\ (I \oslash d) = (\sum k{<}\mathit{Suc}\ n.$
$\quad \mathit{if}\ I \cap [k * d \ldots, d - \mathit{Suc}\ 0] = \{\}\ \mathit{then}\ 0\ \mathit{else}\ \mathit{Suc}\ 0)$
by (*simp add*: *lessThan-Suc-atMost iT-Div-card*)
corollary *iT-Div-Max-card*: $[\![\ 0 < d;\ \mathit{finite}\ I\ ]\!] \Longrightarrow$
$\mathit{card}\ (I \oslash d) = (\sum k{\leq}\mathit{Max}\ I\ \mathit{div}\ d.$
$\quad \mathit{if}\ I \cap [k * d \ldots, d - \mathit{Suc}\ 0] = \{\}\ \mathit{then}\ 0\ \mathit{else}\ \mathit{Suc}\ 0)$
by (*simp add*: *iT-Div-card*)

lemma *iT-Div-card-le*: $0 < k \Longrightarrow \mathit{card}\ (I \oslash k) \leq \mathit{card}\ I$
apply (*case-tac finite I*)
apply (*simp add*: *iT-Div-def card-image-le*)
apply (*simp add*: *iT-Div-finite-iff*)
done

thm *iT-Div-def*
lemma *iT-Div-card-inj-on*:
$\mathit{inj\text{-}on}\ (\lambda n.\ n\ \mathit{div}\ k)\ I \Longrightarrow \mathit{card}\ (I \oslash k) = \mathit{card}\ I$
by (*unfold iT-Div-def*, *rule card-image*)

thm *mod-Suc*
lemma *mod-Suc'*:
$0 < n \Longrightarrow \mathit{Suc}\ m\ \mathit{mod}\ n = (\mathit{if}\ m\ \mathit{mod}\ n < n - \mathit{Suc}\ 0\ \mathit{then}\ \mathit{Suc}\ (m\ \mathit{mod}\ n)\ \mathit{else}\ 0)$
apply (*simp add*: *mod-Suc*)
apply (*intro conjI impI*)
apply *simp*
apply (*insert le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*, *of n m*]], *simp*)
done

lemma *div-Suc*:
$0 < n \Longrightarrow \mathit{Suc}\ m\ \mathit{div}\ n = (\mathit{if}\ \mathit{Suc}\ (m\ \mathit{mod}\ n) = n\ \mathit{then}\ \mathit{Suc}\ (m\ \mathit{div}\ n)\ \mathit{else}\ m\ \mathit{div}\ n)$
apply (*drule Suc-leI*, *drule le-imp-less-or-eq*)
apply (*case-tac n = Suc 0*, *simp*)
apply (*split split-if*, *intro conjI impI*)
apply (*rule-tac t=Suc m* **and** *s=m + 1* **in** *subst*, *simp*)
apply (*subst div-add1-eq2*, *simp+*)
thm *le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*]]
apply (*insert le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*, *of n m*]], *simp*)
apply (*rule-tac t=Suc m* **and** *s=m + 1* **in** *subst*, *simp*)
apply (*subst div-add1-eq1*, *simp+*)
done
lemma *div-Suc'*:
$0 < n \Longrightarrow \mathit{Suc}\ m\ \mathit{div}\ n = (\mathit{if}\ m\ \mathit{mod}\ n < n - \mathit{Suc}\ 0\ \mathit{then}\ m\ \mathit{div}\ n\ \mathit{else}\ \mathit{Suc}\ (m$

div n))
apply (*simp add*: *div-Suc*)
apply (*intro conjI impI*)
apply *simp*
apply (*insert le-neq-trans*[*OF mod-less-divisor*[*THEN Suc-leI*, *of n m*]], *simp*)
done

lemma *iT-Div-card-ge-aux*:
⋀*I*. ⟦ *0* < *d*; *finite I*; *Max I div d* ≤ *n* ⟧ ⟹
card I div d + (*if card I mod d* = *0 then 0 else Suc 0*) ≤ *card* (*I* ⊘ *d*)
apply (*induct n*)
apply (*case-tac I* = {}, *simp*)
thm *div-le-conv*[*THEN iffD1*]
apply (*frule-tac m*=*d* **and** *n*=*Max I* **and** *k*=*0* **in** *div-le-conv*[*THEN iffD1*, *rule-format*], *assumption*)
apply (*simp del*: *Max-le-iff*)
apply (*subgoal-tac I* ⊘ *d* = {*0*})
prefer *2*
apply (*rule set-eqI*)
apply (*simp add*: *iT-Div-mem-iff*)
apply (*rule iffI*)
apply (*fastsimp simp*: *div-eq-0-conv'*)
apply *fastsimp*
apply (*simp add*: *iT-Div-singleton card-singleton del*: *Max-le-iff*)
apply (*drule Suc-le-mono*[*THEN iffD2*, *of - d* − *Suc 0*])
thm *nat-card-le-Max*
thm *order-trans*[*OF nat-card-le-Max*]
apply (*drule order-trans*[*OF nat-card-le-Max*])
apply (*simp*, *intro conjI impI*)
thm *div-le-mono*[*of - - d*]
apply (*drule div-le-mono*[*of - d d*])
apply *simp*
apply (*subgoal-tac card I* ≠ *d*, *simp*)
apply *clarsimp*
apply (*drule order-le-less*[*THEN iffD1*], *erule disjE*)
apply *simp*
thm *subst*[**where** *t*=*I* **and** *s*=*I* ∩ [*...n* ∗ *d* + *d* − *Suc 0*] ∪ *I* ∩ [*Suc n* ∗ *d...*,*d* − *Suc 0*]]
apply (*rule-tac t*=*I* **and** *s*=*I* ∩ [*...n* ∗ *d* + *d* − *Suc 0*] ∪ *I* ∩ [*Suc n* ∗ *d...*,*d* − *Suc 0*] **in** *subst*)
apply (*simp add*: *Int-Un-distrib*[*symmetric*] *add-commute*[*of d*])
apply (*subst iIN-0-iTILL-conv*[*symmetric*])
apply (*simp add*: *iIN-union*)
apply (*rule Int-absorb2*)
apply (*simp add*: *iIN-0-iTILL-conv iTILL-def*)
apply (*case-tac I* = {}, *simp*)
thm *subset-atMost-Max-le-conv*
thm *le-less-div-conv*
apply (*simp add*: *subset-atMost-Max-le-conv le-less-div-conv*[*symmetric*] *less-eq-le-pred*[*symmetric*]

add-commute[*of d*])
thm *card-Un-disjoint*
apply (*cut-tac A*=*I* ∩ [*...n* ∗ *d* + *d* − *Suc 0*] **and** *B*=*I* ∩ [*Suc n* ∗ *d...*,*d* − *Suc 0*] **in** *card-Un-disjoint*)
apply *simp*
apply *simp*
apply (*clarsimp simp*: *disjoint-iff-in-not-in1 iT-iff*)
apply (*case-tac I* ∩ [*...n* ∗ *d* + *d* − *Suc 0*] = {})
thm *iT-Div-mod-partition-card*
apply (*simp add*: *iT-Div-mod-partition-card del*: *mult-Suc*)
apply (*intro conjI impI*)
apply (*rule div-le-conv*[*THEN iffD2*], *assumption*)
apply *simp*
thm *Int-card2*[*OF iIN-finite*]
thm *order-trans*[*OF Int-card2*[*OF iIN-finite*]]
apply (*rule order-trans*[*OF Int-card2*[*OF iIN-finite*]])
apply (*simp add*: *iIN-card*)
thm *Int-card2*[*OF iIN-finite*, *rule-format*]
apply (*cut-tac A*=*I* **and** *n*=*Suc n* ∗ *d* **and** *d*=*d* − *Suc 0* **in** *Int-card2*[*OF iIN-finite*, *rule-format*])
apply (*simp add*: *iIN-card*)
apply (*drule order-le-less*[*THEN iffD1*], *erule disjE*)
apply *simp*
apply *simp*
apply (*subgoal-tac Max* (*I* ∩ [*...n* ∗ *d* + *d* − *Suc 0*]) *div d* ≤ *n*)
prefer *2*
apply (*rule div-le-conv*[*THEN iffD2*], *assumption*)
thm *Max-Int-le2*[*OF - iTILL-finite*]
thm *order-trans*[*OF Max-Int-le2*[*OF - iTILL-finite*]]
apply (*rule order-trans*[*OF Max-Int-le2*[*OF - iTILL-finite*]], *assumption*)
apply (*simp add*: *iTILL-Max*)
apply (*simp only*: *iT-Div-Un*)
thm *card-Un-disjoint*
apply (*cut-tac A*=*I* ∩ [*...n* ∗ *d* + *d* − *Suc 0*] ⊘ *d* **and** *B*=*I* ∩ [*Suc n* ∗ *d...*,*d* − *Suc 0*] ⊘ *d* **in** *card-Un-disjoint*)
apply (*simp add*: *iT-Div-finite-iff*)
apply (*simp add*: *iT-Div-finite-iff*)
thm *mod-partition-iT-Div-Int-one-segment*
apply (*subst iIN-0-iTILL-conv*[*symmetric*])
thm *mod-partition-iT-Div-Int-one-segment*
apply (*subst mod-partition-iT-Div-Int-one-segment*, *simp*)
thm *mod-partition-iT-Div-Int2*
apply (*cut-tac n*=*0* **and** *d*=*n* ∗ *d*+*d* **and** *k*=*d* **and** *A*=*I* **in** *mod-partition-iT-Div-Int2*, *simp*+)
thm *disjoint-iff-in-not-in1*
apply (*rule disjoint-iff-in-not-in1*[*THEN iffD2*])
apply *clarsimp*
thm *iIN-div-mod-eq-0*
apply (*simp add*: *iIN-div-mod-eq-0*)

apply (*simp add*: *mod-0-imp-sub-1-div-conv iIN-0-iTILL-conv iIN-0 iTILL-iff*)
thm *iT-Div-mod-partition-card*
apply (*simp only*: *iT-Div-mod-partition-card*)
apply (*subgoal-tac finite* $(I \cap [\ldots n * d + d - Suc\ 0])$)
prefer *2*
apply *simp*
apply *simp*
apply (*intro conjI impI*)
thm *add-le-divisor-imp-le-Suc-div*
apply (*rule add-le-divisor-imp-le-Suc-div*)
apply (*rule add-leD1*, *blast*)
thm *Int-card2*[*OF iIN-finite*, *THEN le-trans*]
apply (*rule Int-card2*[*OF iIN-finite*, *THEN le-trans*])
apply (*simp add*: *iIN-card*)
apply (*cut-tac* $A=I$ **and** $n=Suc\ n * d$ **and** $d=d - Suc\ 0$ **in** *Int-card2*[*OF iIN-finite*, *rule-format*])
apply (*simp add*: *iIN-card*)
thm *order-trans*[**where** $x=card\ I\ div\ d + (if\ card\ I\ mod\ d \neq 0\ then\ Suc\ 0\ else\ 0)$]
apply (*rule-tac* $y=let\ I=I \cap [\ldots n * d + d - Suc\ 0]\ in$
$card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0)$ **in** *order-trans*)
prefer *2*
apply (*simp add*: *Let-def*)
apply (*unfold Let-def*)
apply (*split split-if*, *intro conjI impI*)
apply (*subgoal-tac card* $(I \cap [Suc\ n * d\ldots,d - Suc\ 0]) \neq d$)
prefer *2*
apply (*rule ccontr*, *simp*)
thm *div-add1-eq1-mod-0-left*
apply (*simp add*: *div-add1-eq1-mod-0-left*)
thm *add-le-divisor-imp-le-Suc-div*
apply (*simp add*: *add-le-divisor-imp-le-Suc-div*)
done

lemma *iT-Div-card-ge*:
$card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \leq card\ (I \oslash d)$
apply (*case-tac* $I = \{\}$, *simp*)
apply (*case-tac finite I*)
prefer *2*
apply *simp*
apply (*case-tac* $d = 0$)
apply (*simp add*: *iT-Div-0 iTILL-0*)
thm *iT-Div-card-ge-aux*[*OF - - order-refl*]
apply (*simp add*: *iT-Div-card-ge-aux*[*OF - - order-refl*])
done
corollary *iT-Div-card-ge-div*: $card\ I\ div\ d \leq card\ (I \oslash d)$
by (*rule iT-Div-card-ge*[*THEN add-leD1*])

There is no better lower bound function f for $i \oslash d$ with *card* i and d as arguments.

lemma *iT-Div-card-ge--is-maximal-lower-bound*:
 $\forall I\ d.\ card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0) \leq f\ (card\ I)\ d \wedge$
 $f\ (card\ I)\ d \leq card\ (I \oslash d) \Longrightarrow$
 $f\ (card\ (I::nat\ set))\ d = card\ I\ div\ d + (if\ card\ I\ mod\ d = 0\ then\ 0\ else\ Suc\ 0)$
apply (*case-tac* $I = \{\}$)
 apply (*erule-tac* $x{=}I$ **in** *allE*, *erule-tac* $x{=}d$ **in** *allE*)
 apply (*simp add*: *iT-Div-empty*)
apply (*case-tac* $d = 0$)
 apply (*frule-tac* $x{=}\{\}$ **in** *spec*, *erule-tac* $x{=}I$ **in** *allE*)
 apply (*erule-tac* $x{=}d$ **in** *allE*, *erule-tac* $x{=}d$ **in** *allE*)
 apply (*clarsimp simp*: *iT-Div-0 iTILL-card iT-Div-empty*)
apply (*rule order-antisym*)
 prefer *2*
 apply *simp*
apply (*case-tac finite I*)
 prefer *2*
 apply (*erule-tac* $x{=}I$ **in** *allE*, *erule-tac* $x{=}d$ **in** *allE*)
 apply (*simp add*: *iT-Div-finite-iff*)
apply (*erule-tac* $x{=}[\ldots card\ I - Suc\ 0]$ **in** *allE*, *erule-tac* $x{=}d$ **in** *allE*, *elim conjE*)
apply (*frule not-empty-card-gr0-conv*[*THEN iffD1*], *assumption*)
thm *iTILL-div iTILL-card*
apply (*simp add*: *iTILL-card iTILL-div*)
apply (*intro conjI impI*)
 apply (*simp add*: *mod-0-imp-sub-1-div-conv*)
 apply (*subgoal-tac* $d \leq card\ I$)
 prefer *2*
  apply *clarsimp*
 apply (*drule div-le-mono*[*of d - d*])
 apply *simp*
apply (*case-tac* $d = Suc\ 0$, *simp*)
thm *div-diff1-eq1*[*of Suc 0 d card I*]
apply (*simp add*: *div-diff1-eq1*)
done
thm *iT-Div-card-ge--is-maximal-lower-bound*[*rule-format*]

thm *iT-Plus-card*
lemma *iT-Plus-icard*: $icard\ (I \oplus k) = icard\ I$
by (*simp add*: *iT-Plus-def icard-image*)

thm *iT-Mult-card*
lemma *iT-Mult-icard*: $0 < k \Longrightarrow icard\ (I \otimes k) = icard\ I$
apply (*unfold iT-Mult-def*)
apply (*rule icard-image*)
apply (*rule inj-imp-inj-on*)
apply (*simp add*: *mult-right-inj*)
done

thm *iT-Plus-neg-card*
lemma *iT-Plus-neg-icard*: *icard* (*I* ⊕− *k*) = *icard* (*I* ↓≥ *k*)
apply (*case-tac finite I*)
thm *iT-Plus-neg-finite-iff cut-ge-finite*
apply (*simp add*: *iT-Plus-neg-finite-iff cut-ge-finite icard-finite iT-Plus-neg-card*)
apply (*simp add*: *iT-Plus-neg-finite-iff nat-cut-ge-finite-iff*)
done

thm *iT-Plus-neg-card-le*
lemma *iT-Plus-neg-icard-le*: *icard* (*I* ⊕− *k*) ≤ *icard I*
apply (*case-tac finite I*)
apply (*simp add*: *iT-Plus-neg-finite-iff icard-finite iT-Plus-neg-card-le*)
apply *simp*
done

thm *iT-Minus-card*
lemma *iT-Minus-icard*: *icard* (*k* ⊖ *I*) = *icard* (*I* ↓≤ *k*)
by (*simp add*: *icard-finite iT-Minus-finite nat-cut-le-finite iT-Minus-card*)

thm *iT-Minus-card-le*
lemma *iT-Minus-icard-le*: *icard* (*k* ⊖ *I*) ≤ *icard I*
apply (*case-tac finite I*)
apply (*simp add*: *icard-finite iT-Minus-finite iT-Minus-card-le*)
apply *simp*
done

thm *iT-Div-0-card-if*
lemma *iT-Div-0-icard-if*: *icard* (*I* ⊘ *0*) = *Fin* (*if I* = {} *then 0 else Suc 0*)
thm *iT-Div-0-finite*
by (*simp add*: *icard-finite iT-Div-0-finite iT-Div-0-card-if*)

thm *iT-Div-mod-partition-card*
lemma *iT-Div-mod-partition-icard*:
icard (*I* ∩ [*n* ∗ *d*...,*d* − *Suc 0*] ⊘ *d*) =
Fin (*if I* ∩ [*n* ∗ *d*...,*d* − *Suc 0*] = {} *then 0 else Suc 0*)
apply (*subgoal-tac finite* (*I* ∩ [*n* ∗ *d*...,*d* − *Suc 0*] ⊘ *d*))
prefer *2*
apply (*case-tac d* = *0*, *simp add*: *iT-Div-0-finite*)
apply (*simp add*: *iT-Div-finite-iff iIN-finite*)
apply (*simp add*: *icard-finite iT-Div-mod-partition-card*)
done

thm *iT-Div-card*
lemma *iT-Div-icard*:
⟦ *0* < *d*; *finite I* ⟹ *Max I div d* ≤ *n*⟧ ⟹
icard (*I* ⊘ *d*) =
(*if finite I then Fin* (∑ *k*≤*n*. *if I* ∩ [*k* ∗ *d*...,*d* − *Suc 0*] = {} *then 0 else Suc 0*) *else* ∞)
by (*simp add*: *icard-finite iT-Div-finite-iff iT-Div-card*)

thm *iT-Div-Max-card*
corollary *iT-Div-Max-icard*: $0 < d \Longrightarrow$
$icard\ (I \oslash d) = (if\ finite\ I$
$then\ Fin\ (\sum k \leq Max\ I\ div\ d.\ if\ I \cap [k * d \ldots, d - Suc\ 0] = \{\}\ then\ 0\ else\ Suc\ 0)\ else\ \infty)$
by (*simp add*: *iT-Div-icard*)

thm *iT-Div-card-le*
lemma *iT-Div-icard-le*: $0 < k \Longrightarrow icard\ (I \oslash k) \leq icard\ I$
apply (*case-tac finite I*)
apply (*simp add*: *iT-Div-finite-iff icard-finite iT-Div-card-le*)
apply *simp*
done

thm *iT-Div-card-inj-on*
lemma *iT-Div-icard-inj-on*: $inj\text{-}on\ (\lambda n.\ n\ div\ k)\ I \Longrightarrow icard\ (I \oslash k) = icard\ I$
by (*simp add*: *iT-Div-def icard-image*)

thm *iT-Div-card-ge*
lemma *iT-Div-icard-ge*: $icard\ I\ div\ (Fin\ d) + Fin\ (if\ icard\ I\ mod\ (Fin\ d) = 0\ then\ 0\ else\ Suc\ 0) \leq icard\ (I \oslash d)$
apply (*case-tac* $d = 0$)
apply (*simp add*: *icard-finite iT-Div-0-finite*)
apply (*case-tac icard I*)
apply (*fastsimp simp*: *iT-Div-0-card-if*)
apply (*simp add*: *iT-Div-0-card-if icard-infinite-conv infinite-imp-nonempty*)
apply (*case-tac finite I*)
apply (*simp add*: *iT-Div-finite-iff icard-finite iT-Div-card-ge*)
apply (*simp add*: *iT-Div-finite-iff*)
done

thm *iT-Div-card-ge-div*
corollary *iT-Div-icard-ge-div*: $icard\ I\ div\ (Fin\ d) \leq icard\ (I \oslash d)$
by (*rule iT-Div-icard-ge*[*THEN iadd-ileD1*])

thm *iT-Div-card-ge--is-maximal-lower-bound*
lemma *iT-Div-icard-ge--is-maximal-lower-bound*:
$\forall I\ d.\ icard\ I\ div\ (Fin\ d) + Fin\ (if\ icard\ I\ mod\ (Fin\ d) = 0\ then\ 0\ else\ Suc\ 0) \leq f\ (icard\ I)\ d \wedge f\ (icard\ I)\ d \leq icard\ (I \oslash d) \Longrightarrow$
$f\ (icard\ (I{::}nat\ set))\ d =$
$icard\ I\ div\ (Fin\ d) + Fin\ (if\ icard\ I\ mod\ (Fin\ d) = 0\ then\ 0\ else\ Suc\ 0)$
apply (*case-tac* $d = 0$)
apply (*drule-tac* $x{=}I$ **in** *spec*, *drule-tac* $x{=}d$ **in** *spec*, *erule conjE*)
apply (*simp add*: *iT-Div-0-icard-if icard-0-eq*[*unfolded zero-inat-def*])
thm *iT-Div-card-ge--is-maximal-lower-bound*
apply (*case-tac finite I*)

prefer *2*
apply (*drule-tac x=I* **in** *spec*, *drule-tac x=d* **in** *spec*)
apply *simp*
apply *simp*
apply (*frule-tac iT-Div-finite-iff*[*THEN iffD2*], *assumption*)
thm *iT-Div-card-ge--is-maximal-lower-bound*
apply (*cut-tac f=λc d. the-Fin* (*f* (*Fin c*) *d*) **and** *I=I* **and** *d=d* **in** *iT-Div-card-ge--is-maximal-lower-bound*)
apply (*intro allI*, *rename-tac I′ d′*)
apply (*subgoal-tac* ⋀*k. f 0 k = 0*)
prefer *2*
apply (*drule-tac x={}* **in** *spec*, *drule-tac x=k* **in** *spec*, *erule conjE*)
apply (*simp add*: *iT-Div-empty*)
apply (*drule-tac x=I′* **in** *spec*, *drule-tac x=d′* **in** *spec*, *erule conjE*)
apply (*case-tac d′ = 0*)
apply (*simp add*: *idiv-by-0 imod-by-0 iT-Div-0-card-if iT-Div-0-icard-if*)
apply (*case-tac I′ = {}*, *simp*)
apply (*case-tac finite I′*)
apply (*simp add*: *icard-finite*)
apply *simp*
apply *simp*
apply (*case-tac finite I′*)
apply (*frule-tac I=I′* **and** *k=d′* **in** *iT-Div-finite-iff*[*THEN iffD2*, *rule-format*], *assumption*)
apply (*simp add*: *icard-finite*)
apply (*subgoal-tac* ∃*n. f* (*Fin* (*card I′*)) *d′ = Fin n*)
prefer *2*
apply (*rule Fin-ile*, *assumption*)
apply *clarsimp*
apply (*subgoal-tac infinite* (*I′* ⊘ *d′*))
prefer *2*
apply (*simp add*: *iT-Div-finite-iff*)
apply *simp*
apply (*drule-tac x=I* **in** *spec*, *drule-tac x=d* **in** *spec*, *erule conjE*)
apply (*simp add*: *icard-finite*)
apply (*subgoal-tac* ∃*n. f* (*Fin* (*card I*)) *d = Fin n*)
prefer *2*
apply (*rule Fin-ile*, *assumption*)
apply *clarsimp*
done

2.5 Results about sets of intervals

2.5.1 Set of intervals without and with emtpy interval

definition *iFROM-UN-set* :: (*nat set*) *set* **where**
iFROM-UN-set ≡ ⋃*n*. {[*n*...]}
definition *iTILL-UN-set* :: (*nat set*) *set* **where**
iTILL-UN-set ≡ ⋃*n*. {[...*n*]}
definition *iIN-UN-set* :: (*nat set*) *set* **where**
iIN-UN-set ≡ ⋃*n d*. {[*n*...,*d*]}

definition *iMOD-UN-set* :: (*nat set*) *set* **where**
iMOD-UN-set ≡ ⋃ *r m*. {[*r*, *mod m*]}
definition *iMODb-UN-set* :: (*nat set*) *set* **where**
iMODb-UN-set ≡ ⋃ *r m c*. {[*r*, *mod m*, *c*]}

definition *iFROM-set* :: (*nat set*) *set* **where**
iFROM-set ≡ {[*n*...] |*n*. *True*}
definition *iTILL-set* :: (*nat set*) *set* **where**
iTILL-set ≡ {[...*n*] |*n*. *True*}
definition *iIN-set* :: (*nat set*) *set* **where**
iIN-set ≡ {[*n*...,*d*] |*n d*. *True*}
definition *iMOD-set* :: (*nat set*) *set* **where**
iMOD-set ≡ {[*r*, *mod m*] |*r m*. *True*}
definition *iMODb-set* :: (*nat set*) *set* **where**
iMODb-set ≡ {[*r*, *mod m*, *c*] |*r m c*. *True*}

definition *iMOD2-set* :: (*nat set*) *set* **where**
iMOD2-set ≡ {[*r*, *mod m*] |*r m*. *2* ≤ *m*}
definition *iMODb2-set* :: (*nat set*) *set* **where**
iMODb2-set ≡ {[*r*, *mod m*, *c*] |*r m c*. *2* ≤ *m* ∧ *1* ≤ *c*}

definition *iMOD2-UN-set* :: (*nat set*) *set* **where**
iMOD2-UN-set ≡ ⋃ *r*. ⋃ *m*∈{*2*..}. {[*r*, *mod m*]}
definition *iMODb2-UN-set* :: (*nat set*) *set* **where**
iMODb2-UN-set ≡ ⋃ *r*. ⋃ *m*∈{*2*..}. ⋃ *c*∈{*1*..}. {[*r*, *mod m*, *c*]}

lemmas *i-set-defs* =
iFROM-set-def iTILL-set-def iIN-set-def
iMOD-set-def iMODb-set-def
iMOD2-set-def iMODb2-set-def
lemmas *i-UN-set-defs* =
iFROM-UN-set-def iTILL-UN-set-def iIN-UN-set-def
iMOD-UN-set-def iMODb-UN-set-def
iMOD2-UN-set-def iMODb2-UN-set-def

lemma *iFROM-set-UN-set-eq*: *iFROM-set* = *iFROM-UN-set*
by (*fastsimp simp*: *iFROM-set-def iFROM-UN-set-def*)
lemma
iTILL-set-UN-set-eq: *iTILL-set* = *iTILL-UN-set* **and**
iIN-set-UN-set-eq: *iIN-set* = *iIN-UN-set* **and**
iMOD-set-UN-set-eq: *iMOD-set* = *iMOD-UN-set* **and**
iMODb-set-UN-set-eq: *iMODb-set* = *iMODb-UN-set*
by (*fastsimp simp*: *i-set-defs i-UN-set-defs*)+

lemma *iMOD2-set-UN-set-eq*: *iMOD2-set* = *iMOD2-UN-set*
by (*fastsimp simp*: *i-set-defs i-UN-set-defs*)
lemma *iMODb2-set-UN-set-eq*: *iMODb2-set* = *iMODb2-UN-set*
by (*fastsimp simp*: *i-set-defs i-UN-set-defs*)

lemmas *i-set-i-UN-set-sets-eq* =
iFROM-set-UN-set-eq
iTILL-set-UN-set-eq
iIN-set-UN-set-eq
iMOD-set-UN-set-eq
iMODb-set-UN-set-eq
iMOD2-set-UN-set-eq
iMODb2-set-UN-set-eq
thm *i-set-i-UN-set-sets-eq*

lemma *iMOD2-set-iMOD-set-subset*: *iMOD2-set* ⊆ *iMOD-set*
by (*fastsimp simp*: *i-set-defs*)
lemma *iMODb2-set-iMODb-set-subset*: *iMODb2-set* ⊆ *iMODb-set*
by (*fastsimp simp*: *i-set-defs*)

definition *i-set* :: (*nat set*) *set* **where**
i-set ≡ *iFROM-set* ∪ *iTILL-set* ∪ *iIN-set* ∪ *iMOD-set* ∪ *iMODb-set*
definition *i-UN-set* :: (*nat set*) *set* **where**
i-UN-set ≡ *iFROM-UN-set* ∪ *iTILL-UN-set* ∪ *iIN-UN-set* ∪ *iMOD-UN-set* ∪ *iMODb-UN-set*

Minimal definitions for *i-set* and *i-set*

definition *i-set-min* :: (*nat set*) *set* **where**
i-set-min ≡ *iFROM-set* ∪ *iIN-set* ∪ *iMOD2-set* ∪ *iMODb2-set*
definition *i-UN-set-min* :: (*nat set*) *set* **where**
i-UN-set-min ≡ *iFROM-UN-set* ∪ *iIN-UN-set* ∪ *iMOD2-UN-set* ∪ *iMODb2-UN-set*

definition *i-set0* :: (*nat set*) *set* **where**
i-set0 ≡ *insert* {} *i-set*
thm *i-set0-def*

thm *i-set-i-UN-set-sets-eq*
lemma *i-set-i-UN-set-eq*: *i-set* = *i-UN-set*
by (*simp add*: *i-set-def i-UN-set-def i-set-i-UN-set-sets-eq*)
lemma *i-set-min-i-UN-set-min-eq*: *i-set-min* = *i-UN-set-min*
by (*simp add*: *i-set-min-def i-UN-set-min-def i-set-i-UN-set-sets-eq*)

lemma *i-set-min-eq*: *i-set* = *i-set-min*
apply (*unfold i-set-def i-set-min-def*)
apply (*rule equalityI*)
apply (*rule subsetI*)
apply (*simp add*: *i-set-defs*)
apply (*elim disjE*)
apply *blast*
apply (*fastsimp simp*: *iIN-0-iTILL-conv*[*symmetric*])

apply *blast*
apply (*elim exE*)
apply (*case-tac* $2 \leq m$, *blast*)
apply (*simp add*: *nat-ge2-conv*)
apply (*fastsimp simp*: *iMOD-0 iMOD-1*)
apply (*elim exE*)
apply (*case-tac* $1 \leq c$)
apply (*case-tac* $2 \leq m$, *fastsimp*)
apply (*simp add*: *nat-ge2-conv*)
apply (*fastsimp simp*: *iMODb-mod-0 iMODb-mod-1*)
apply (*fastsimp simp*: *linorder-not-le less-Suc-eq-le iMODb-0*)
thm *Un-mono*
apply (*rule Un-mono*)+
apply (*simp-all add*: *iMOD2-set-iMOD-set-subset iMODb2-set-iMODb-set-subset*)
done
corollary *i-UN-set-i-UN-min-set-eq*: *i-UN-set* = *i-UN-set-min*
by (*simp add*: *i-set-min-eq i-set-i-UN-set-eq*[*symmetric*] *i-set-min-i-UN-set-min-eq*[*symmetric*])

thm
i-set-def i-set-min-def
i-set-min-eq

lemma *i-set-min-is-minimal-let*:
let s1 = *iFROM-set*; *s2*= *iIN-set*; *s3*= *iMOD2-set*; *s4*= *iMODb2-set in*
$s1 \cap s2 = \{\} \wedge s1 \cap s3 = \{\} \wedge s1 \cap s4 = \{\} \wedge$
$s2 \cap s3 = \{\} \wedge s2 \cap s4 = \{\} \wedge s3 \cap s4 = \{\}$
apply (*unfold Let-def i-set-defs*, *intro conjI*)
apply (*rule disjoint-iff-in-not-in1*[*THEN iffD2*], *clarsimp simp*: *iT-neq*)+
done
lemmas *i-set-min-is-minimal* = *i-set-min-is-minimal-let*[*simplified*]
thm *i-set-min-is-minimal*
thm *conjunct1*
thm
i-set-min-is-minimal[*THEN conjunct1*]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*]]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*[*OF conjunct2*]]]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*]]]]
i-set-min-is-minimal[*THEN conjunct1*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*]]]]]
i-set-min-is-minimal[*THEN conjunct2*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*[*OF conjunct2*]]]]]

thm
i-set-def
i-set-defs
inductive-set
i-set-ind:: (*nat set*) *set*
where
i-set-ind-FROM[*intro*!]: $[n\ldots] \in$ *i-set-ind*

| *i-set-ind-TILL*[*intro*!]: $[\ldots n] \in$ *i-set-ind*
| *i-set-ind-IN*[*intro*!]: $[n\ldots,d] \in$ *i-set-ind*
| *i-set-ind-MOD*[*intro*!]: $[r,\ mod\ m] \in$ *i-set-ind*
| *i-set-ind-MODb*[*intro*!]: $[r,\ mod\ m,\ c] \in$ *i-set-ind*

inductive-set
i-set0-ind :: (*nat set*) *set*
where
i-set0-ind-empty[*intro*!] : $\{\} \in$ *i-set0-ind*
| *i-set0-ind-i-set*[*intro*]: $I \in$ *i-set-ind* $\Longrightarrow I \in$ *i-set0-ind*

The introduction rule *i-set0-ind-i-set* is not declared a safe introduction rule, because it would disturb the correct usage of the *safe* method.

lemma *i-set-ind-subset-i-set0-ind*: *i-set-ind* $\subseteq$ *i-set0-ind*
by (*rule subsetI*, *rule i-set0-ind-i-set*)
thm
i-set-ind-FROM
i-set-ind-TILL
i-set-ind-IN
i-set-ind-MOD
i-set-ind-MODb
lemma
i-set0-ind-FROM[*intro*!] : $[n\ldots] \in$ *i-set0-ind* **and**
i-set0-ind-TILL[*intro*!] : $[\ldots n] \in$ *i-set0-ind* **and**
i-set0-ind-IN[*intro*!] : $[n\ldots,d] \in$ *i-set0-ind* **and**
i-set0-ind-MOD[*intro*!] : $[r,\ mod\ m] \in$ *i-set0-ind* **and**
i-set0-ind-MODb[*intro*!] : $[r,\ mod\ m,\ c] \in$ *i-set0-ind*
by (*rule subsetD*[*OF i-set-ind-subset-i-set0-ind*], *rule i-set-ind.intros*)+
lemmas *i-set0-ind-intros2* =
i-set0-ind-empty
i-set0-ind-FROM
i-set0-ind-TILL
i-set0-ind-IN
i-set0-ind-MOD
i-set0-ind-MODb
thm *i-set0-ind-intros2*

lemma *i-set-i-set-ind-eq*: *i-set* = *i-set-ind*
apply (*rule set-eqI*, *unfold i-set-def i-set-defs*)
apply (*rule iffI*, *blast*)
apply (*induct-tac x rule*: *i-set-ind.induct*)
apply *blast*+
done

lemma *i-set0-i-set0-ind-eq*: *i-set0* = *i-set0-ind*
apply (*rule set-eqI*, *unfold i-set0-def*)
thm *i-set-i-set-ind-eq*

apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*rule iffI*)
apply *blast*
apply (*rule-tac a=x* **in** *i-set0-ind.cases*)
apply *blast+*
done

lemma *i-set-imp-not-empty*: $I \in$ *i-set* $\Longrightarrow I \neq \{\}$
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct I rule*: *i-set-ind.induct*)
apply (*rule iT-not-empty*)*+*
done

lemma *i-set0-i-set-mem-conv*: $(I \in$ *i-set0*$) = (I \in$ *i-set* $\vee I = \{\})$
apply (*simp add*: *i-set-i-set-ind-eq i-set0-i-set0-ind-eq*)
apply (*rule iffI*)
apply (*rule i-set0-ind.cases*[*of I*])
apply *blast+*
done

lemma *i-set-i-set0-mem-conv*: $(I \in$ *i-set*$) = (I \in$ *i-set0* $\wedge I \neq \{\})$
apply (*insert i-set-imp-not-empty*[*of I*])
apply (*fastsimp simp*: *i-set0-i-set-mem-conv*)
done

lemma *i-set0-i-set-conv*: *i-set0* $- \{\{\}\} =$ *i-set*
by (*fastsimp simp*: *i-set-i-set0-mem-conv*)
corollary *i-set-subset-i-set0*: *i-set* $\subseteq$ *i-set0*
by (*simp add*: *i-set0-i-set-conv*[*symmetric*] *Diff-subset*)
lemma *i-set-singleton*: $\{a\} \in$ *i-set*
by (*fastsimp simp*: *i-set-def iIN-set-def iIN-0*[*symmetric*])
lemma *i-set0-singleton*: $\{a\} \in$ *i-set0*
apply (*rule subsetD*[*OF i-set-subset-i-set0*])
apply (*simp add*: *iIN-0*[*symmetric*] *i-set-i-set-ind-eq i-set-ind.intros*)
done

thm *i-set-ind.intros*
corollary
  i-set-FROM[*intro*!] : $[n\ldots] \in$ *i-set* **and**
  i-set-TILL[*intro*!] : $[\ldots n] \in$ *i-set* **and**
  i-set-IN[*intro*!] : $[n\ldots,d] \in$ *i-set* **and**
  i-set-MOD[*intro*!] : $[r,$ *mod* $m] \in$ *i-set* **and**
  i-set-MODb[*intro*!] : $[r,$ *mod* $m,\ c] \in$ *i-set*
thm *i-set-i-set-ind-eq*
by (*rule ssubst*[*OF i-set-i-set-ind-eq*], *rule i-set-ind.intros*)*+*
lemmas *i-set-intros* =
  i-set-FROM
  i-set-TILL

i-set-IN
i-set-MOD
i-set-MODb
thm *i-set0-ind-intros2*
lemma
i-set0-empty[*intro*!]: {} ∈ *i-set0* **and**
i-set0-FROM[*intro*!] : [*n*...] ∈ *i-set0* **and**
i-set0-TILL[*intro*!] : [...*n*] ∈ *i-set0* **and**
i-set0-IN[*intro*!] : [*n*...,*d*] ∈ *i-set0* **and**
i-set0-MOD[*intro*!] : [*r*, *mod m*] ∈ *i-set0* **and**
i-set0-MODb[*intro*!] : [*r*, *mod m*, *c*] ∈ *i-set0*
by (*rule ssubst*[*OF i-set0-i-set0-ind-eq*], *rule i-set0-ind-intros2*)+
lemmas *i-set0-intros* =
i-set0-empty
i-set0-FROM
i-set0-TILL
i-set0-IN
i-set0-MOD
i-set0-MODb
thm *i-set0-intros*

lemma *i-set-infinite-as-iMOD*:
⟦ *I* ∈ *i-set*; *infinite I* ⟧ ⟹ ∃ *r m*. *I* = [*r*, *mod m*]
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct I rule*: *i-set-ind.induct*)
apply (*simp-all add*: *iT-finite*)
apply (*rule-tac x*=*n* **in** *exI*, *rule-tac x*=*Suc 0* **in** *exI*, *simp add*: *iMOD-1*)
apply *blast*
done

lemma *i-set-finite-as-iMODb*:
⟦ *I* ∈ *i-set*; *finite I* ⟧ ⟹ ∃ *r m c*. *I* = [*r*, *mod m*, *c*]
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct I rule*: *i-set-ind.induct*)
apply (*simp add*: *iT-infinite*)
apply (*rule-tac x*=*0* **in** *exI*, *rule-tac x*=*Suc 0* **in** *exI*, *rule-tac x*=*n* **in** *exI*)
apply (*simp add*: *iMODb-mod-1 iIN-0-iTILL-conv*)
apply (*rule-tac x*=*n* **in** *exI*, *rule-tac x*=*Suc 0* **in** *exI*, *rule-tac x*=*d* **in** *exI*)
apply (*simp add*: *iMODb-mod-1*)
apply (*case-tac m* = *0*)
apply (*rule-tac x*=*r* **in** *exI*, *rule-tac x*=*Suc 0* **in** *exI*, *rule-tac x*=*0* **in** *exI*)
apply (*simp add*: *iMOD-0 iIN-0 iMODb-0*)
apply (*simp add*: *iT-infinite*)
apply *blast*
done

lemma *i-set-as-iMOD-iMODb*:
  $I \in$ *i-set* $\Longrightarrow (\exists\, r\ m.\ I = [r,\ mod\ m]) \vee (\exists\, r\ m\ c.\ I = [r,\ mod\ m,\ c])$
by (*blast intro*: *i-set-finite-as-iMODb i-set-infinite-as-iMOD*)

2.5.2 Interval sets are closed under cutting

lemma *i-set-cut-le-ge-closed-disj*:
  $[\![\ I \in$ *i-set*; $t \in I$; *cut-op* $=$ *op* $\downarrow\leq \vee$ *cut-op* $=$ *op* $\downarrow\geq\ ]\!] \Longrightarrow$
  cut-op I t $\in$ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply *safe*
apply (*simp-all add*: *iT-cut-le1 iT-cut-ge1 i-set-ind.intros iMODb-iff*)
done

thm *i-set-cut-le-ge-closed-disj*[*of - - op* $\downarrow\geq$, *simplified*]
corollary
  i-set-cut-le-closed: $[\![\ I \in$ *i-set*; $t \in I\ ]\!] \Longrightarrow I \downarrow\leq t \in$ *i-set* **and**
  i-set-cut-ge-closed: $[\![\ I \in$ *i-set*; $t \in I\ ]\!] \Longrightarrow I \downarrow\geq t \in$ *i-set*
by (*simp-all add*: *i-set-cut-le-ge-closed-disj*)

lemmas *i-set-cut-le-ge-closed* = *i-set-cut-le-closed i-set-cut-ge-closed*
thm *i-set-cut-le-ge-closed*

lemma *i-set0-cut-closed-disj*:
  $[\![\ I \in$ *i-set0*;
    cut-op $=$ *op* $\downarrow\leq \vee$ *cut-op* $=$ *op* $\downarrow\geq \vee$
    cut-op $=$ *op* $\downarrow< \vee$ *cut-op* $=$ *op* $\downarrow>\ ]\!] \Longrightarrow$
  cut-op I t $\in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct rule*: *i-set0-ind.induct*)
 thm *i-set0-ind-empty*
 thm *ssubst*[*OF set-restriction-empty*, **where** $P{=}\lambda x.\ x \in$ *i-set0-ind*]
 apply (*rule ssubst*[*OF set-restriction-empty*, **where** $P{=}\lambda x.\ x \in$ *i-set0-ind*])
  thm *i-cut-set-restriction-disj*
  apply (*rule i-cut-set-restriction-disj*[*of cut-op*], *blast*)
  apply *blast*
 apply *blast*
apply (*induct-tac I rule*: *i-set-ind.induct*)
apply *safe*
apply (*simp-all add*: *iT-cut-le iT-cut-ge iT-cut-less iT-cut-greater i-set0-ind-intros2*)
done

thm *i-set0-cut-closed-disj*[*of - op* $\downarrow>$, *simplified*]
corollary
  i-set0-cut-le-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow\leq t \in$ *i-set0* **and**
  i-set0-cut-less-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow< t \in$ *i-set0* **and**
  i-set0-cut-ge-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow\geq t \in$ *i-set0* **and**
  i-set0-cut-greater-closed: $I \in$ *i-set0* $\Longrightarrow I \downarrow> t \in$ *i-set0*

by (*simp-all add*: *i-set0-cut-closed-disj*)

thm *i-set0-ind.intros*
thm
i-set0-FROM[*THEN i-set0-cut-closed-disj*[*of* - *op* ↓<, *simplified*]]
i-set0-TILL[*THEN i-set0-cut-closed-disj*[*of* - *op* ↓≥, *simplified*]]
i-set0-IN[*THEN i-set0-cut-closed-disj*[*of* - *op* ↓≤, *simplified*]]
i-set0-MOD[*THEN i-set0-cut-closed-disj*[*of* - *op* ↓>, *simplified*]]
i-set0-MODb[*THEN i-set0-cut-closed-disj*[*of* - *op* ↓>, *simplified*]]
lemmas *i-set0-cut-closed* =
i-set0-cut-le-closed
i-set0-cut-less-closed
i-set0-cut-ge-closed
i-set0-cut-greater-closed
thm *i-set0-cut-closed*

2.5.3 Interval sets are closed under addition and multiplication

lemma *i-set-Plus-closed*: $I \in$ *i-set* $\Longrightarrow I \oplus k \in$ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*simp-all add*: *iT-add i-set-ind.intros*)
done

lemma *i-set-Mult-closed*: $I \in$ *i-set* $\Longrightarrow I \otimes k \in$ *i-set*
apply (*case-tac* $k = 0$)
apply (*simp add*: *i-set-imp-not-empty iT-Mult-0-if i-set-intros*)
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*simp-all add*: *iT-mult i-set-ind.intros*)
done

lemma *i-set0-Plus-closed*: $I \in$ *i-set0* $\Longrightarrow I \oplus k \in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct I rule*: *i-set0-ind.induct*)
apply (*simp add*: *iT-Plus-empty i-set0-ind-empty*)
apply (*rule subsetD*[*OF i-set-ind-subset-i-set0-ind*])
apply (*simp add*: *i-set-i-set-ind-eq*[*symmetric*] *i-set-Plus-closed*)
done

lemma *i-set0-Mult-closed*: $I \in$ *i-set0* $\Longrightarrow I \otimes k \in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct I rule*: *i-set0-ind.induct*)
apply (*simp add*: *iT-Mult-empty i-set0-ind-empty*)
apply (*rule subsetD*[*OF i-set-ind-subset-i-set0-ind*])
apply (*simp add*: *i-set-i-set-ind-eq*[*symmetric*] *i-set-Mult-closed*)
done

2.5.4 Interval sets are closed with certain conditions under subtraction

lemma *i-set-Plus-neg-closed*:
⟦ $I \in$ *i-set*; $\exists x{\in}I.\ k \leq x$ ⟧ $\Longrightarrow$ $I \oplus{-}\ k \in$ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*fastsimp simp*: *iT-iff iT-add-neg*)+
done

lemma *i-set-Minus-closed*:
⟦ $I \in$ *i-set*; *iMin* $I \leq k$ ⟧ $\Longrightarrow$ $k \ominus I \in$ *i-set*
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*induct rule*: *i-set-ind.induct*)
apply (*fastsimp simp*: *iT-Min iT-sub*)+
done

lemma *i-set0-Plus-neg-closed*: $I \in$ *i-set0* $\Longrightarrow$ $I \oplus{-}\ k \in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct rule*: *i-set0-ind.induct*)
apply (*fastsimp simp*: *iT-Plus-neg-empty*)
apply (*induct-tac I rule*: *i-set-ind.induct*)
apply (*fastsimp simp*: *iT-add-neg*)+
done

lemma *i-set0-Minus-closed*: $I \in$ *i-set0* $\Longrightarrow$ $k \ominus I \in$ *i-set0*
apply (*simp add*: *i-set0-i-set0-ind-eq*)
apply (*induct rule*: *i-set0-ind.induct*)
apply (*simp add*: *iT-Minus-empty i-set0-ind-empty*)
apply (*induct-tac I rule*: *i-set-ind.induct*)
apply (*fastsimp simp*: *iT-sub*)+
done

lemmas *i-set-IntOp-closed* =
i-set-Plus-closed
i-set-Mult-closed
i-set-Plus-neg-closed
i-set-Minus-closed
lemmas *i-set0-IntOp-closed* =
i-set0-Plus-closed
i-set0-Mult-closed
i-set0-Plus-neg-closed
i-set0-Minus-closed
thm *i-set-IntOp-closed*
thm *i-set0-IntOp-closed*

2.5.5 Interval sets are not closed under division

thm *i-set-as-iMOD-iMODb*

thm *iMOD-div-mod-gr0-not-ex*
lemma *iMOD-div-mod-gr0-not-in-i-set*:
⟦ *0 < k*; *k < m*; *0 < m mod k* ⟧ ⟹ [*r*, *mod m*] ⊘ *k* ∉ *i-set*
apply (*rule ccontr*, *simp*)
thm *i-set-infinite-as-iMOD*
apply (*drule i-set-infinite-as-iMOD*)
apply (*simp add*: *iT-Div-finite-iff iMOD-infinite*)
apply (*elim exE*, *rename-tac r' m'*)
apply (*frule iMOD-div-mod-gr0-not-ex*[*of k m r*], *assumption*+)
apply *fastsimp*
done
lemma *iMODb-div-mod-gr0-not-in-i-set*:
⟦ *0 < k*; *k < m*; *0 < m mod k*; *k ≤ c* ⟧ ⟹ [*r*, *mod m*, *c*] ⊘ *k* ∉ *i-set*
apply (*rule ccontr*, *simp*)
apply (*drule i-set-finite-as-iMODb*)
apply (*simp add*: *iT-Div-finite-iff iMODb-finite*)
apply (*elim exE*, *rename-tac r' m' c'*)
apply (*frule iMODb-div-mod-gr0-not-ex*[*of k m c r*], *assumption*+)
apply *fastsimp*
done

lemma [*0*, *mod 3*] ⊘ *2* ∉ *i-set*
by (*rule iMOD-div-mod-gr0-not-in-i-set*, *simp*+)

lemma *i-set-Div-not-closed*: *Suc 0 < k* ⟹ ∃ *I*∈*i-set*. *I* ⊘ *k* ∉ *i-set*
apply (*rule-tac x*=[*0*, *mod* (*Suc k*)] **in** *bexI*)
apply (*rule iMOD-div-mod-gr0-not-in-i-set*)
apply (*simp-all add*: *mod-Suc i-set-MOD*)
done
lemma *i-set0-Div-not-closed*: *Suc 0 < k* ⟹ ∃ *I*∈*i-set0*. *I* ⊘ *k* ∉ *i-set0*
apply (*frule i-set-Div-not-closed*, *erule bexE*)
apply (*rule-tac x*=*I* **in** *bexI*)
apply (*simp add*: *i-set0-def iT-Div-not-empty i-set-imp-not-empty*)
apply (*rule subsetD*[*OF i-set-subset-i-set0*], *assumption*)
done
thm
i-set-Div-not-closed
i-set0-Div-not-closed

2.5.6 Sets of intervals closed under division

inductive-set
NatMultiples :: *nat set* ⇒ *nat set*
for *F* :: *nat set*
where
NatMultiples-Factor:
k ∈ *F* ⟹ *k* ∈ *NatMultiples F*
| *NatMultiples-Product*:
⟦ *k* ∈ *F*; *m* ∈ *NatMultiples F* ⟧ ⟹ *k* ∗ *m* ∈ *NatMultiples F*

thm *NatMultiples.induct*
lemma *NatMultiples-ex-divisor*: $m \in NatMultiples\ F \Longrightarrow \exists\, k{\in}F.\ m\ mod\ k = 0$
apply (*induct m rule*: *NatMultiples.induct*)
apply (*rule-tac x*=*k* **in** *bexI*, *simp*+)+
done
lemma *NatMultiples-product-factor*: $[\![\ a \in F;\ b \in F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
by (*drule NatMultiples-Factor*[*of b*], *rule NatMultiples-Product*)
lemma *NatMultiples-product-factor-multiple*:
$[\![\ a \in F;\ b \in NatMultiples\ F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
by (*rule NatMultiples-Product*)
lemma *NatMultiples-product-multiple-factor*:
$[\![\ a \in NatMultiples\ F;\ b \in F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
by (*simp add*: *mult-commute*[*of a*] *NatMultiples-Product*)
lemma *NatMultiples-product-multiple*:
$[\![\ a \in NatMultiples\ F;\ b \in NatMultiples\ F\ ]\!] \Longrightarrow a * b \in NatMultiples\ F$
apply (*induct a rule*: *NatMultiples.induct*)
apply (*simp add*: *NatMultiples-Product*)
apply (*simp add*: *mult-assoc*[*of - - b*] *NatMultiples-Product*)
done

Subset of *i-set* containing only continuous intervals, i. e., without *iMOD* and *iMODb*.

inductive-set *i-set-cont* :: (*nat set*) *set*
where
i-set-cont-FROM[*intro*]: $[n\ldots] \in$ *i-set-cont*
| *i-set-cont-TILL*[*intro*]: $[\ldots n] \in$ *i-set-cont*
| *i-set-cont-IN*[*intro*]: $[n\ldots,d] \in$ *i-set-cont*

definition *i-set0-cont* :: (*nat set*) *set* **where**
i-set0-cont $\equiv$ *insert* {} *i-set-cont*

lemma *i-set-cont-subset-i-set*: *i-set-cont* $\subseteq$ *i-set*
apply (*unfold subset-eq*, *rule ballI*, *rename-tac x*)
apply (*rule-tac a*=*x* **in** *i-set-cont.cases*)
apply *blast*+
done

lemma *i-set0-cont-subset-i-set0*: *i-set0-cont* $\subseteq$ *i-set0*
apply (*unfold i-set0-cont-def i-set0-def*)
apply (*rule insert-mono*)
apply (*rule i-set-cont-subset-i-set*)
done

Minimal definition of *i-set-cont*

inductive-set *i-set-cont-min* :: (*nat set*) *set*
where
i-set-cont-FROM[*intro*]: $[n\ldots] \in$ *i-set-cont-min*
| *i-set-cont-IN*[*intro*]: $[n\ldots,d] \in$ *i-set-cont-min*

definition *i-set0-cont-min* :: (*nat set*) *set* **where**
i-set0-cont-min ≡ *insert* {} *i-set-cont-min*

lemma *i-set-cont-min-eq*: *i-set-cont* = *i-set-cont-min*
apply (*rule set-eqI*, *rule iffI*)
apply (*rename-tac x*, *rule-tac a=x* **in** *i-set-cont.cases*)
apply (*fastsimp simp*: *iIN-0-iTILL-conv*[*symmetric*])+
apply (*rename-tac x*, *rule-tac a=x* **in** *i-set-cont-min.cases*)
apply *blast*+
done

inext and *iprev* with continuous intervals

lemma *i-set-cont-inext*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *finite I* ⟹ *n* < *Max I* ⟧ ⟹ *inext n I* = *Suc n*
apply (*simp add*: *i-set-cont-min-eq*)
apply (*rule i-set-cont-min.cases*, *assumption*)
apply (*simp-all add*: *iT-finite iT-Max iT-inext-if iT-iff*)
done
lemma *i-set-cont-iprev*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *iMin I* < *n* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
apply (*simp add*: *i-set-cont-min-eq*)
apply (*rule i-set-cont-min.cases*, *assumption*)
apply (*simp-all add*: *iT-iprev-if iT-Min iT-iff*)
done

lemma *i-set-cont-inext--less*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n* < *n0*; *n0* ∈ *I* ⟧ ⟹ *inext n I* = *Suc n*
apply (*case-tac finite I*)
apply (*rule i-set-cont-inext*, *assumption*+)
thm *order-less-le-trans*[*OF - Max-ge*]
apply (*rule order-less-le-trans*[*OF - Max-ge*], *assumption*+)
apply (*rule i-set-cont.cases*, *assumption*)
apply (*simp-all add*: *iT-finite iT-inext*)
done
lemma *i-set-cont-iprev--greater*:
⟦ *I* ∈ *i-set-cont*; *n* ∈ *I*; *n0* < *n*; *n0* ∈ *I* ⟧ ⟹ *iprev n I* = *n* − *Suc 0*
apply (*rule i-set-cont-iprev*, *assumption*+)
thm *order-le-less-trans*[*OF iMin-le*, *of n0*]
apply (*rule order-le-less-trans*[*OF iMin-le*, *of n0*], *assumption*+)
done

Sets of modulo intervals

inductive-set *i-set-mult* :: *nat* ⇒ (*nat set*) *set*
for *k* :: *nat*
where
i-set-mult-FROM[*intro*!]: [*n*...] ∈ *i-set-mult k*
| *i-set-mult-TILL*[*intro*!]: [...*n*] ∈ *i-set-mult k*
| *i-set-mult-IN*[*intro*!]: [*n*...,*d*] ∈ *i-set-mult k*

| *i-set-mult-MOD*[*intro*!]: $[r,\ mod\ m * k] \in$ *i-set-mult k*
| *i-set-mult-MODb*[*intro*!]: $[r,\ mod\ m * k,\ c] \in$ *i-set-mult k*

definition *i-set0-mult* :: *nat* ⇒ (*nat set*) *set* **where**
i-set0-mult k ≡ *insert* {} (*i-set-mult k*)

lemma
i-set0-mult-empty[*intro*!]: {} ∈ *i-set0-mult k* **and**
i-set0-mult-FROM[*intro*!] : [*n*...] ∈ *i-set0-mult k* **and**
i-set0-mult-TILL[*intro*!] : [...*n*] ∈ *i-set0-mult k* **and**
i-set0-mult-IN[*intro*!] : [*n*...,*d*] ∈ *i-set0-mult k* **and**
i-set0-mult-MOD[*intro*!] : $[r,\ mod\ m * k] \in$ *i-set0-mult k* **and**
i-set0-mult-MODb[*intro*!] : $[r,\ mod\ m * k,\ c] \in$ *i-set0-mult k*
by (*simp-all add*: *i-set0-mult-def i-set-mult.intros*)
lemmas *i-set0-mult-intros* =
i-set0-mult-empty
i-set0-mult-FROM
i-set0-mult-TILL
i-set0-mult-IN
i-set0-mult-MOD
i-set0-mult-MODb
thm *i-set0-mult-intros*

lemma *i-set-mult-subset-i-set0-mult*: *i-set-mult k* ⊆ *i-set0-mult k*
by (*unfold i-set0-mult-def*, *rule subset-insertI*)

lemma *i-set-mult-subset-i-set*: *i-set-mult k* ⊆ *i-set*
apply (*clarsimp simp*: *subset-iff*)
apply (*rule-tac a*=*t* **in** *i-set-mult.cases*[*of* - *k*])
apply (*simp-all add*: *i-set-intros*)
done

thm *subsetD*[*OF i-set-mult-subset-i-set*]

lemma *i-set0-mult-subset-i-set0*: *i-set0-mult k* ⊆ *i-set0*
apply (*simp add*: *i-set0-mult-def i-set0-empty*)
thm *order-trans*[*OF* - *i-set-subset-i-set0*, *OF i-set-mult-subset-i-set*]
apply (*rule order-trans*[*OF* - *i-set-subset-i-set0*, *OF i-set-mult-subset-i-set*])
done

lemma *i-set-mult-0-eq-i-set-cont*: *i-set-mult 0* = *i-set-cont*
apply (*rule set-eqI*, *rule iffI*)
apply (*rename-tac x*, *rule-tac a*=*x* **in** *i-set-mult.cases*[*of* - *0*])
apply (*simp-all add*: *i-set-cont.intros iMOD-0 iMODb-mod-0*)
apply (*rename-tac x*, *rule-tac a*=*x* **in** *i-set-cont.cases*)
apply (*simp-all add*: *i-set-mult.intros*)
done

lemma *i-set0-mult-0-eq-i-set0-cont*: *i-set0-mult 0* = *i-set0-cont*

by (*simp add*: *i-set0-mult-def i-set0-cont-def i-set-mult-0-eq-i-set-cont*)

lemma *i-set-mult-1-eq-i-set*: *i-set-mult* (*Suc 0*) = *i-set*
apply (*rule set-eqI*, *rule iffI*)
apply (*rename-tac x*, *induct-tac x rule*: *i-set-mult.induct*[*of - 1*])
apply (*simp-all add*: *i-set-intros*)
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*rename-tac x*, *induct-tac x rule*: *i-set-ind.induct*)
apply (*simp-all add*: *i-set-mult.intros*)
apply (*rule-tac t*=*m* **and** *s*=*m* ∗ *Suc 0* **in** *subst*, *simp*, *rule i-set-mult.intros*)+
done

lemma *i-set0-mult-1-eq-i-set0*: *i-set0-mult* (*Suc 0*) = *i-set0*
by (*simp add*: *i-set0-mult-def i-set0-def i-set-mult-1-eq-i-set*)

lemma *i-set-mult-imp-not-empty*: *I* ∈ *i-set-mult k* ⟹ *I* ≠ {}
by (*induct I rule*: *i-set-mult.induct*, *simp-all add*: *iT-not-empty*)

lemma *iMOD-in-i-set-mult-imp-divisor-mod-0*:
⟦ *m* ≠ *Suc 0*; [*r*, *mod m*] ∈ *i-set-mult k* ⟧ ⟹ *m mod k* = *0*
apply (*case-tac m* = *0*, *simp*)
apply (*rule i-set-mult.cases*[*of* [*r*, *mod m*] *k*], *assumption*)
apply (*blast dest*: *iFROM-iMOD-neq*)
apply (*blast dest*: *iTILL-iMOD-neq*)
apply (*blast dest*: *iIN-iMOD-neq*)
apply (*simp add*: *iMOD-eq-conv*)
apply (*blast dest*: *iMOD-iMODb-neq*)
done

lemma
divisor-mod-0-imp-iMOD-in-i-set-mult: *m mod k* = *0* ⟹ [*r*, *mod m*] ∈ *i-set-mult k* **and**
divisor-mod-0-imp-iMODb-in-i-set-mult: *m mod k* = *0* ⟹ [*r*, *mod m*, *c*] ∈ *i-set-mult k*
by (*clarsimp simp*: *mult-commute*[*of k*])+

lemma *iMOD-in-i-set-mult--divisor-mod-0-conv*:
m ≠ *Suc 0* ⟹ ([*r*, *mod m*] ∈ *i-set-mult k*) = (*m mod k* = *0*)
apply (*rule iffI*)
apply (*simp add*: *iMOD-in-i-set-mult-imp-divisor-mod-0*)
apply (*simp add*: *divisor-mod-0-imp-iMOD-in-i-set-mult*)
done

lemma *i-set-mult-neq1-subset-i-set*: *k* ≠ *Suc 0* ⟹ *i-set-mult k* ⊂ *i-set*
apply (*rule psubsetI*, *rule i-set-mult-subset-i-set*)
apply (*simp add*: *set-eq-iff*)
apply (*drule neq-iff*[*THEN iffD1*], *erule disjE*)
apply (*simp add*: *i-set-mult-0-eq-i-set-cont*)
apply (*thin-tac k* = *0*)

apply (*rule-tac x=*[*0*, *mod 2*] **in** *exI*)
apply (*rule ccontr*)
apply (*simp add*: *i-set-intros*)
apply (*drule i-set-cont.cases*[**where** *P=False*])
apply (*drule sym*, *simp add*: *iT-neq*)+
apply *simp*
apply (*rule-tac x=*[*0*, *mod Suc k*] **in** *exI*)
apply (*rule ccontr*)
apply (*simp add*: *i-set-intros*)
apply (*insert iMOD-in-i-set-mult-imp-divisor-mod-0*[*of Suc k 0 k*])
apply (*simp add*: *mod-Suc*)
done

lemma *mod-0-imp-i-set-mult-subset*:
a mod b = 0 ⟹ i-set-mult a ⊆ i-set-mult b
apply (*clarsimp simp*: *mult-commute*[*of b*], *rename-tac q*)
thm *i-set-mult.cases*
apply (*rule-tac a=x* **and** *k=q * b* **in** *i-set-mult.cases*)
apply (*simp-all add*: *i-set-mult.intros mult-assoc*[*symmetric*])
done
lemma *i-set-mult-subset-imp-mod-0*:
⟦ *a ≠ Suc 0*; (*i-set-mult a ⊆ i-set-mult b*) ⟧ *⟹ a mod b = 0*
apply (*simp add*: *subset-iff*)
apply (*erule-tac x=*[*0*, *mod a*] **in** *allE*)
apply (*simp add*: *divisor-mod-0-imp-iMOD-in-i-set-mult*)
thm *iMOD-in-i-set-mult-imp-divisor-mod-0*[*of - 0 b*]
apply (*simp add*: *iMOD-in-i-set-mult-imp-divisor-mod-0*[*of - 0 b*])
done
lemma *i-set-mult-subset-conv*:
a ≠ Suc 0 ⟹ (*i-set-mult a ⊆ i-set-mult b*) = (*a mod b = 0*)
apply (*rule iffI*)
apply (*simp add*: *i-set-mult-subset-imp-mod-0*)
apply (*simp add*: *mod-0-imp-i-set-mult-subset*)
done

lemma *i-set-mult-mod-0-div*:
⟦ *I ∈ i-set-mult k*; *k mod d = 0* ⟧ *⟹ I ⊘ d ∈ i-set-mult* (*k div d*)
apply (*case-tac d = 0*)
thm *iT-Div-0*[*OF i-set-mult-imp-not-empty*]
apply (*simp add*: *iT-Div-0*[*OF i-set-mult-imp-not-empty*] *i-set-mult.intros*)
apply (*induct I rule*: *i-set-mult.induct*)
apply (*simp-all add*: *iT-div i-set-mult.intros*)
thm *mod-0-imp-mod-mult-left-0*
thm *mod-0-imp-div-mult-right-eq*
thm *iMOD-div iMODb-div*
apply (*simp-all add*: *iMOD-div iMODb-div mod-0-imp-mod-mult-left-0 mod-0-imp-div-mult-left-eq i-set-mult.intros*)
done

Intervals from *i-set-mult k* remain in *i-set* after division by d a divisor of k.

corollary *i-set-mult-mod-0-div-i-set*:
$[\![I \in \textit{i-set-mult}\ k;\ k\ \textit{mod}\ d = 0]\!] \Longrightarrow I \oslash d \in \textit{i-set}$
thm *subsetD*[*OF i-set-mult-subset-i-set*[*of k div d*]]
by (*rule subsetD*[*OF i-set-mult-subset-i-set*[*of k div d*]], *rule i-set-mult-mod-0-div*)
corollary *i-set-mult-div-self-i-set*:
$I \in \textit{i-set-mult}\ k \Longrightarrow I \oslash k \in \textit{i-set}$
by (*simp add*: *i-set-mult-mod-0-div-i-set*)

lemma *i-set-mod-0-mult-in-i-set-mult*:
$[\![I \in \textit{i-set};\ m\ \textit{mod}\ k = 0]\!] \Longrightarrow I \otimes m \in \textit{i-set-mult}\ k$
apply (*case-tac m = 0*)
apply (*simp add*: *iT-Mult-0 i-set-imp-not-empty i-set-mult.intros*)
apply (*clarsimp simp*: *mult-commute*[*of k*])
apply (*simp add*: *i-set-i-set-ind-eq*)
apply (*rule-tac a=I* **in** *i-set-ind.cases*)
apply (*simp-all add*: *iT-mult mult-assoc*[*symmetric*] *i-set-mult.intros*)
done
lemma *i-set-self-mult-in-i-set-mult*:
$I \in \textit{i-set} \Longrightarrow I \otimes k \in \textit{i-set-mult}\ k$
by (*rule i-set-mod-0-mult-in-i-set-mult*[*OF - mod-self*])

lemma *i-set-mult-mod-gr0-div-not-in-i-set*:
$[\![0 < k;\ 0 < d;\ 0 < k\ \textit{mod}\ d]\!] \Longrightarrow \exists I \in \textit{i-set-mult}\ k.\ I \oslash d \notin \textit{i-set}$
apply (*case-tac d = Suc 0*, *simp*)
thm *iMOD-div-mod-gr0-not-ex*[*of d Suc d * k 0*]
apply (*frule iMOD-div-mod-gr0-not-ex*[*of d Suc d * k 0*])
apply (*rule Suc-le-lessD*, *rule gr0-imp-self-le-mult1*, *assumption*)
apply *simp*
apply (*rule-tac x=*[*0*, *mod Suc d * k*] **in** *bexI*)
apply (*rule ccontr*, *simp*)
thm *i-set-infinite-as-iMOD*
apply (*frule i-set-infinite-as-iMOD*)
apply (*simp add*: *iT-Div-finite-iff iMOD-infinite*)
apply *fastsimp*
apply (*simp add*: *i-set-mult.intros del*: *mult-Suc*)
done

lemma *i-set0-mult-mod-0-div*:
$[\![I \in \textit{i-set0-mult}\ k;\ k\ \textit{mod}\ d = 0]\!] \Longrightarrow I \oslash d \in \textit{i-set0-mult}\ (k\ \textit{div}\ d)$
apply (*simp add*: *i-set0-mult-def*)
apply (*elim disjE*)
apply (*simp add*: *iT-Div-empty*)
apply (*simp add*: *i-set-mult-mod-0-div*)
done
corollary *i-set0-mult-mod-0-div-i-set0*:
$[\![I \in \textit{i-set0-mult}\ k;\ k\ \textit{mod}\ d = 0]\!] \Longrightarrow I \oslash d \in \textit{i-set0}$
by (*rule subsetD*[*OF i-set0-mult-subset-i-set0*[*of k div d*]], *rule i-set0-mult-mod-0-div*)

corollary *i-set0-mult-div-self-i-set0*:
$I \in$ *i-set0-mult* $k \Longrightarrow I \oslash k \in$ *i-set0*
by (*simp add*: *i-set0-mult-mod-0-div-i-set0*)

lemma *i-set0-mod-0-mult-in-i-set0-mult*:
$[\![I \in$ *i-set0*; *m mod k* $= 0]\!] \Longrightarrow I \otimes m \in$ *i-set0-mult k*
apply (*simp add*: *i-set0-def*)
apply (*erule disjE*)
apply (*simp add*: *iT-Mult-empty i-set0-mult-empty*)
apply (*rule subsetD*[*OF i-set-mult-subset-i-set0-mult*])
apply (*simp add*: *i-set-mod-0-mult-in-i-set-mult*)
done

lemma *i-set0-self-mult-in-i-set0-mult*:
$I \in$ *i-set0* $\Longrightarrow I \otimes k \in$ *i-set0-mult k*
by (*simp add*: *i-set0-mod-0-mult-in-i-set0-mult*)

lemma *i-set0-mult-mod-gr0-div-not-in-i-set0*:
$[\![0 < k; 0 < d; 0 <$ *k mod d* $]\!] \Longrightarrow \exists I \in$*i-set0-mult k*. $I \oslash d \notin$ *i-set0*
apply (*frule i-set-mult-mod-gr0-div-not-in-i-set*[*of k d*], *assumption*+)
apply (*erule bexE*, *rename-tac I*, *rule-tac x*=*I* **in** *bexI*)
apply (*simp add*: *i-set0-def iT-Div-not-empty i-set-mult-imp-not-empty*)
apply (*simp add*: *subsetD*[*OF i-set-mult-subset-i-set0-mult*])
done

end

3 IL-TemporalOperators: Temporal logic operators on natural intervals

theory *IL-TemporalOperators*
imports *IL-IntervalOperators*
begin

Bool : some additional properties

instantiation *bool* :: {*ord*, *zero*, *one*, *plus*, *times*, *order*}
begin

definition *Zero-bool-def* [*simp*]: $0 \equiv$ *False*
definition *One-bool-def* [*simp*]: $1 \equiv$ *True*
definition *add-bool-def*: $a + b \equiv a \vee b$
definition *mult-bool-def*: $a * b \equiv a \wedge b$

instance ..
end

value *False* $<$ *True*
value *True* $<$ *True*

value *True* ≤ *True*
thm *le-bool-def*

lemmas *bool-op-rel-defs* =
add-bool-def
mult-bool-def
less-bool-def
le-bool-def
thm *bool-op-rel-defs*

instance *bool* :: *linorder*
apply *intro-classes*
apply (*unfold le-bool-def less-bool-def*)
apply *blast+*
done

instance *bool* :: *wellorder*
apply (*rule wf-wellorderI*)
apply (*rule-tac t*={(*x*, *y*). *x* < *y*} **and** *s*={(*False*, *True*)} **in** *subst*)
apply (*fastsimp simp add*: *less-bool-def*)
thm *wf-def*
apply (*unfold wf-def*, *blast*)
apply *intro-classes*
done

instance *bool* :: *comm-semiring-1*
apply *intro-classes*
apply (*unfold bool-op-rel-defs Zero-bool-def One-bool-def*)
apply *blast+*
done

3.1 Basic definitions

typ *iT*

thm *Ex-def*
thm *All-def*

term *All*
term *Ball*
lemma *UNIV-nat*: ℕ = (*UNIV*::*nat set*)
by (*simp add*: *Nats-def*)

Universal temporal operator: Always/Globally

definition
iAll :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Always

where
iAll I P ≡ ∀ *t*∈*I*. *P t*

Existential temporal operator: Eventually/Finally

definition
iEx :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Eventually
where
iEx-def : *iEx I P* ≡ ∃ *t*∈*I*. *P t*

syntax (*xsymbols*)
-iAll :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*□ - -./ -) [*0*, *0*, *10*] *10*)
-iEx :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◊ - -./ -) [*0*, *0*, *10*] *10*)
syntax (*HTML* **output**)
-iAll :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*□ - -./ -) [*0*, *0*, *10*] *10*)
-iEx :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◊ - -./ -) [*0*, *0*, *10*] *10*)
translations
□ *t I*. *P* ⇌ *CONST iAll I* (λ*t*. *P*)
◊ *t I*. *P* ⇌ *CONST iEx I* (λ*t*. *P*)

Future temporal operator: Next

definition
iNext :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Next
where
iNext t0 I P ≡ *P* (*inext t0 I*)

Past temporal operator: Last/Previous

definition
iLast :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Last
where
iLast t0 I P ≡ *P* (*iprev t0 I*)

syntax (*xsymbols*)
-iNext :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯ - - -./ -) [*0*, *0*, *10*] *10*)
-iLast :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖ - - -./ -) [*0*, *0*, *10*] *10*)
syntax (*HTML* **output**)
-iNext :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯ - - -./ -) [*0*, *0*, *10*] *10*)
-iLast :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖ - - -./ -) [*0*, *0*, *10*] *10*)
translations
◯ *t t0 I*. *P* ⇌ *CONST iNext t0 I* (λ*t*. *P*)
⊖ *t t0 I*. *P* ⇌ *CONST iLast t0 I* (λ*t*. *P*)
lemma ◯ *t 10* [*0*...]. (*t* + *10* > *10*)
by (*simp add*: *iNext-def iT-inext-if*)

The following versions of Next and Last operator differ in the cases where

no next/previous element exists or specified time point is not in interval: the weak versions return *True* and the strong versions return *False*.

definition
  iNextWeak :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Next
where
  iNextWeak t0 I P ≡ (□ *t* {*inext t0 I*} ↓> *t0*. *P t*)
definition
  iNextStrong :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Next
where
  iNextStrong t0 I P ≡ (◊ *t* {*inext t0 I*} ↓> *t0*. *P t*)

definition
  iLastWeak :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Weak Last
where
  iLastWeak t0 I P ≡ (□ *t* {*iprev t0 I*} ↓< *t0*. *P t*)
definition
  iLastStrong :: *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Strong Last
where
  iLastStrong t0 I P ≡ (◊ *t* {*iprev t0 I*} ↓< *t0*. *P t*)

syntax (*xsymbols*)
  -iNextWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
  -iNextStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
  -iLastWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
  -iLastStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
syntax (*HTML* **output**)
  -iNextWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
  -iNextStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*◯$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
  -iLastWeak :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_W$ - - -./ -) [*0*, *0*, *10*] *10*)
  -iLastStrong :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ *bool* ((*3*⊖$_S$ - - -./ -) [*0*, *0*, *10*] *10*)
translations
  ◯$_W$ *t t0 I*. *P* ⇌ *CONST iNextWeak t0 I* (λ*t*. *P*)
  ◯$_S$ *t t0 I*. *P* ⇌ *CONST iNextStrong t0 I* (λ*t*. *P*)
  ⊖$_W$ *t t0 I*. *P* ⇌ *CONST iLastWeak t0 I* (λ*t*. *P*)
  ⊖$_S$ *t t0 I*. *P* ⇌ *CONST iLastStrong t0 I* (λ*t*. *P*)

Some examples for Next and Last operator

lemma ◯ *t 5* [*0...,10*]. ([*0*::*int*,*10*,*20*,*30*,*40*,*50*,*60*,*70*,*80*,*90*] ! *t* < *80*)
by (*simp add*: *iNext-def iIN-inext*)
lemma ⊖ *t 5* [*0...,10*]. ([*0*::*int*,*10*,*20*,*30*,*40*,*50*,*60*,*70*,*80*,*90*] ! *t* < *80*)
by (*simp add*: *iLast-def iIN-iprev*)

Temporal Until operator

definition
iUntil :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Until
where
$iUntil\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I\ {\downarrow}{<}\ t).\ P\ t')$

Temporal Since operator (past operator corresponding to Until)

definition
iSince :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Since
where
$iSince\ I\ P\ Q \equiv \Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I\ {\downarrow}{>}\ t).\ P\ t')$

syntax (*xsymbols*)
-iUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{U}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
-iSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{S}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
translations
$P.\ t\ \mathcal{U}\ t'\ I.\ Q \rightleftharpoons CONST\ iUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{S}\ t'\ I.\ Q \rightleftharpoons CONST\ iSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition
iWeakUntil :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Until/Wating-for/Unless
where
$iWeakUntil\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I\ {\downarrow}{<}\ t).\ P\ t'))$

definition
iWeakSince :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Weak Since/Back-to
where
$iWeakSince\ I\ P\ Q \equiv (\Box\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t \wedge (\Box\ t'\ (I\ {\downarrow}{>}\ t).\ P\ t'))$

syntax (*xsymbols*)
-iWeakUntil :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{W}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
-iWeakSince :: $Time \Rightarrow Time \Rightarrow iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$
$((-./\ -\ (3\mathcal{B}\ -\ -)./\ -)\ [10,\ 0,\ 0,\ 0,\ 10]\ 10)$
translations
$P.\ t\ \mathcal{W}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakUntil\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$
$P.\ t\ \mathcal{B}\ t'\ I.\ Q \rightleftharpoons CONST\ iWeakSince\ I\ (\lambda t.\ P)\ (\lambda t'.\ Q)$

definition
iRelease :: $iT \Rightarrow (Time \Rightarrow bool) \Rightarrow (Time \Rightarrow bool) \Rightarrow bool$ — Release
where

iRelease-def : *iRelease I P Q* ≡ (□ *t I*. *Q t*) ∨ (◇ *t I*. *P t* ∧ (□ *t'* (*I* ↓≤ *t*). *Q t'*))

definition
iTrigger :: *iT* ⇒ (*Time* ⇒ *bool*) ⇒ (*Time* ⇒ *bool*) ⇒ *bool* — Trigger
where
iTrigger-def : *iTrigger I P Q* ≡ (□ *t I*. *Q t*) ∨ (◇ *t I*. *P t* ∧ (□ *t'* (*I* ↓≥ *t*). *Q t'*))

syntax (*xsymbols*)
-iRelease :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ (*Time* ⇒ *bool*) ⇒ *bool*
((-./ - (3ℛ - -)./ -) [10, 0, 0, 0, 10] 10)
-iTrigger :: *Time* ⇒ *Time* ⇒ *iT* ⇒ (*Time* ⇒ *bool*) ⇒ (*Time* ⇒ *bool*) ⇒ *bool*
((-./ - (3𝒯 - -)./ -) [10, 0, 0, 0, 10] 10)
translations
P. *t* ℛ *t'* *I*. *Q* ⇌ *CONST iRelease I* (λ*t*. *P*) (λ*t'*. *Q*)
P. *t* 𝒯 *t'* *I*. *Q* ⇌ *CONST iTrigger I* (λ*t*. *P*) (λ*t'*. *Q*)

lemmas *iTL-Next-defs* =
iNext-def iLast-def
iNextWeak-def iLastWeak-def
iNextStrong-def iLastStrong-def
lemmas *iTL-defs* =
iAll-def iEx-def
iUntil-def iSince-def
iWeakUntil-def iWeakSince-def
iRelease-def iTrigger-def
iTL-Next-defs

print-translation ⟪
[*Syntax.preserve-binder-abs2-tr'* @{*const-syntax iAll*} @{*syntax-const -iAll*},
Syntax.preserve-binder-abs2-tr' @{*const-syntax iEx*} @{*syntax-const -iEx*}]
⟫

term □ *t I*. *P t*
term ◇ *t I*. *P t*

term *P1 t1*. *t1* 𝒰 *t2 I*. *P2 t2*
term *P1 t1*. *t1* 𝒮 *t2 I*. *P2 t2*
print-translation ⟪
let
fun btr' syn [*i*,*Abs abs*,*Abs abs'*] =
let

```
      val (t,P) = atomic-abs-tr' abs;
      val (t',Q) = atomic-abs-tr' abs'
    in Syntax.const syn $ P $ t $ t' $ i $ Q end
in
[(@{const-syntax iUntil}, btr' -iUntil), (@{const-syntax iSince}, btr' -iSince)]
end
⟩⟩
```

term *P t1. t1 𝒰 t2 I. Q t2*
term *P t1. t1 𝒮 t2 I. Q t2*

```
print-translation ⟨⟨
let
  fun btr' syn [i,Abs abs,Abs abs'] =
    let
      val (t,P) = atomic-abs-tr' abs;
      val (t',Q) = atomic-abs-tr' abs'
    in Syntax.const syn $ P $ t $ t' $ i $ Q end
in
[(@{const-syntax iWeakUntil}, btr' -iWeakUntil), (@{const-syntax iWeakSince},
btr' -iWeakSince)]
end
⟩⟩
```

term *P t1. t1 𝒲 t2 I. Q t2*
term *P t1. t1 ℬ t2 I. Q t2*

```
print-translation ⟨⟨
let
  fun btr' syn [i,Abs abs,Abs abs'] =
    let
      val (t,P) = atomic-abs-tr' abs;
      val (t',Q) = atomic-abs-tr' abs'
    in Syntax.const syn $ P $ t $ t' $ i $ Q end
in
[(@{const-syntax iRelease}, btr' -iRelease), (@{const-syntax iTrigger}, btr' -iTrigger)]
end
⟩⟩
```

term *P t1. t1 ℛ t2 I. Q t2*
term *P t1. t1 𝒯 t2 I. Q t2*

3.2 Basic lemmata for temporal operators

3.2.1 Intro/elim rules

thm *bexI rev-bexI*

lemma
iexI[*intro*]: $[\![P\ t;\ t \in I]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
rev-iexI[*intro?*]: $[\![t \in I;\ P\ t]\!] \Longrightarrow \Diamond\ t\ I.\ P\ t$ **and**
iexE[*elim*!]: $[\![\Diamond\ t\ I.\ P\ t;\ \bigwedge t.\ [\![t \in I;\ P\ t]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
by (*unfold iEx-def*, *blast*+)
lemma
iallI[*intro*!]: $(\bigwedge t.\ t \in I \Longrightarrow P\ t) \Longrightarrow \Box\ t\ I.\ P\ t$ **and**
ispec[*dest?*]: $[\![\Box\ t\ I.\ P\ t;\ t \in I]\!] \Longrightarrow P\ t$ **and**
iallE[*elim*]: $[\![\Box\ t\ I.\ P\ t;\ P\ t \Longrightarrow Q;\ t \notin I \Longrightarrow Q]\!] \Longrightarrow Q$
by (*unfold iAll-def*, *blast*+)

lemma
iuntilI[*intro*]:
$[\![Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t');\ t \in I]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$ **and**
rev-iuntilI[*intro?*]:
$[\![t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow< t \Longrightarrow P\ t')]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
by (*unfold iUntil-def*, *blast*+)
lemma
iuntilE[*elim*]:
$[\![P'\ t'.\ t'\ \mathcal{U}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![t \in I;\ P\ t]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
by (*unfold iUntil-def*, *blast*)

thm *iSince-def*
lemma
isinceI[*intro*]:
$[\![Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t');\ t \in I]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$ **and**
rev-isinceI[*intro?*]:
$[\![t \in I;\ Q\ t;\ (\bigwedge t'.\ t' \in I \downarrow> t \Longrightarrow P\ t')]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
by (*unfold iSince-def*, *blast*+)
lemma
isinceE[*elim*]:
$[\![P'\ t'.\ t'\ \mathcal{S}\ t\ I.\ P\ t;\ \bigwedge t.\ [\![t \in I;\ P\ t]\!] \Longrightarrow Q]\!] \Longrightarrow Q$
by (*unfold iSince-def*, *blast*)

3.2.2 Rewrite rules for trivial simplification

thm *ball-triv*
lemma *iall-triv*[*simp*]: $(\Box\ t\ I.\ P) = ((\exists\, t.\ t \in I) \longrightarrow P)$
by (*simp add*: *iAll-def*)
thm *bex-triv*
lemma *iex-triv*[*simp*]: $(\Diamond\ t\ I.\ P) = ((\exists\, t.\ t \in I) \wedge P)$
by (*simp add*: *iEx-def*)

lemma *iex-conjL1*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*
lemma *iex-conjR1*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$

$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
by *blast*
lemma *iex-conjL2*:
$(\Diamond\ t1\ I1.\ (P1\ t1 \wedge (\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*
lemma *iex-conjR2*:
$(\Diamond\ t1\ I1.\ ((\Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$
by *blast*
lemma *iex-commute*:
$(\Diamond\ t1\ I1.\ \Diamond\ t2\ I2.\ P\ t1\ t2) =$
$(\Diamond\ t2\ I2.\ \Diamond\ t1\ I1.\ P\ t1\ t2)$
by *blast*

lemma *iall-conjL1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ I2.\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*
lemma *iall-conjR1*:
$I2 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ I2.\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P2\ t1\ t2 \wedge P1\ t1)$
by *blast*
lemma *iall-conjL2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ (P1\ t1 \wedge (\Box\ t2\ (I2\ t1).\ P2\ t1\ t2))) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P1\ t1 \wedge P2\ t1\ t2)$
by *blast*
lemma *iall-conjR2*:
$\Box\ t1\ I1.\ I2\ t1 \neq \{\} \Longrightarrow$
$(\Box\ t1\ I1.\ ((\Box\ t2\ (I2\ t1).\ P2\ t1\ t2) \wedge P1\ t1)) =$
$(\Box\ t1\ I1.\ \Box\ t2\ (I2\ t1).\ P2\ t1\ t2 \wedge P1\ t1)$
by *blast*
lemma *iall-commute*:
$(\Box\ t1\ I1.\ \Box\ t2\ I2.\ P\ t1\ t2) =$
$(\Box\ t2\ I2.\ \Box\ t1\ I1.\ P\ t1\ t2)$
by *blast*

lemma *iall-conj-distrib*:
$(\Box\ t\ I.\ P\ t \wedge Q\ t) = ((\Box\ t\ I.\ P\ t) \wedge (\Box\ t\ I.\ Q\ t))$
by *blast*
lemma *iex-disj-distrib*:
$(\Diamond\ t\ I.\ P\ t \vee Q\ t) = ((\Diamond\ t\ I.\ P\ t) \vee (\Diamond\ t\ I.\ Q\ t))$
by *blast*

lemma *iall-conj-distrib2*:

(□ *t1 I1*. □ *t2* (*I2 t1*). *P t1 t2* ∧ *Q t1 t2*) =
(□ *t1 I1*. □ *t2* (*I2 t1*). *P t1 t2*) ∧ (□ *t1 I1*. □ *t2* (*I2 t1*). *Q t1 t2*))
by *blast*
lemma *iex-disj-distrib2*:
(◇ *t1 I1*. ◇ *t2* (*I2 t1*). *P t1 t2* ∨ *Q t1 t2*) =
(◇ *t1 I1*. ◇ *t2* (*I2 t1*). *P t1 t2*) ∨ (◇ *t1 I1*. ◇ *t2* (*I2 t1*). *Q t1 t2*))
by *blast*

lemma *iUntil-disj-distrib*:
(*P t1*. *t1* 𝒰 *t2 I*. (*Q1 t2* ∨ *Q2 t2*)) = ((*P t1*. *t1* 𝒰 *t2 I*. *Q1 t2*) ∨ (*P t1*. *t1* 𝒰 *t2 I*. *Q2 t2*))
unfolding *iUntil-def* **by** *blast*
lemma *iSince-disj-distrib*:
(*P t1*. *t1* 𝒮 *t2 I*. (*Q1 t2* ∨ *Q2 t2*)) = ((*P t1*. *t1* 𝒮 *t2 I*. *Q1 t2*) ∨ (*P t1*. *t1* 𝒮 *t2 I*. *Q2 t2*))
unfolding *iSince-def* **by** *blast*

thm *iNextWeak-def*
lemma
iNext-iff : (◯ *t t0 I*. *P t*) = (□ *t* [*...0*] ⊕ (*inext t0 I*). *P t*) **and**
iLast-iff : (⊖ *t t0 I*. *P t*) = (□ *t* [*...0*] ⊕ (*iprev t0 I*). *P t*)
by (*fastsimp simp*: *iTL-Next-defs iT-add iIN-0*)+
lemma
iNext-iEx-iff : (◯ *t t0 I*. *P t*) = (◇ *t* [*...0*] ⊕ (*inext t0 I*). *P t*) **and**
iLast-iEx-iff : (⊖ *t t0 I*. *P t*) = (◇ *t* [*...0*] ⊕ (*iprev t0 I*). *P t*)
by (*fastsimp simp*: *iTL-Next-defs iT-add iIN-0*)+

lemma *inext-singleton-cut-greater-not-empty-iff*:
({*inext t0 I*} ↓> *t0* ≠ {}) = (*I* ↓> *t0* ≠ {} ∧ *t0* ∈ *I*)
apply (*simp add*: *cut-greater-singleton*)
apply (*case-tac t0* ∈ *I*)
prefer *2*
apply (*simp add*: *not-in-inext-fix*)
apply *simp*
apply (*case-tac I* ↓> *t0* = {})
thm *inext-all-le-fix*
apply (*simp add*: *cut-greater-empty-iff inext-all-le-fix*)
apply (*simp add*: *cut-greater-not-empty-iff inext-mono2*)
done
lemma *iprev-singleton-cut-less-not-empty-iff*:
({*iprev t0 I*} ↓< *t0* ≠ {}) = (*I* ↓< *t0* ≠ {} ∧ *t0* ∈ *I*)
apply (*simp add*: *cut-less-singleton*)
apply (*case-tac t0* ∈ *I*)
prefer *2*
apply (*simp add*: *not-in-iprev-fix*)
apply *simp*
apply (*case-tac I* ↓< *t0* = {})

thm *iprev-all-ge-fix*
apply (*simp add*: *cut-less-empty-iff iprev-all-ge-fix*)
apply (*simp add*: *cut-less-not-empty-iff iprev-mono2*)
done
lemma *inext-singleton-cut-greater-empty-iff*:
(\{*inext t0 I*\} $\downarrow>$ *t0* = \{\}) = (*I* $\downarrow>$ *t0* = \{\} $\vee$ *t0* $\notin$ *I*)
apply (*subst Not-eq-iff*[*symmetric*])
thm *inext-singleton-cut-greater-not-empty-iff*
apply (*simp add*: *inext-singleton-cut-greater-not-empty-iff*)
done
lemma *iprev-singleton-cut-less-empty-iff*:
(\{*iprev t0 I*\} $\downarrow<$ *t0* = \{\}) = (*I* $\downarrow<$ *t0* = \{\} $\vee$ *t0* $\notin$ *I*)
apply (*subst Not-eq-iff*[*symmetric*])
thm *iprev-singleton-cut-less-not-empty-iff*
apply (*simp add*: *iprev-singleton-cut-less-not-empty-iff*)
done

lemma *iNextWeak-iff* : ($\bigcirc_W$ *t t0 I*. *P t*) = (($\bigcirc$ *t t0 I*. *P t*) $\vee$ (*I* $\downarrow>$ *t0* = \{\}) $\vee$ *t0* $\notin$ *I*)
apply (*unfold iTL-defs*)
thm *inext-singleton-cut-greater-empty-iff*[*symmetric*]
apply (*simp add*: *inext-singleton-cut-greater-empty-iff*[*symmetric*] *cut-greater-singleton*)
done
lemma *iNextStrong-iff* : ($\bigcirc_S$ *t t0 I*. *P t*) = (($\bigcirc$ *t t0 I*. *P t*) $\wedge$ (*I* $\downarrow>$ *t0* $\neq$ \{\}) $\wedge$ *t0* $\in$ *I*)
apply (*unfold iTL-defs*)
thm *inext-singleton-cut-greater-not-empty-iff*[*symmetric*]
apply (*simp add*: *inext-singleton-cut-greater-not-empty-iff*[*symmetric*] *cut-greater-singleton*)
done
lemma *iLastWeak-iff* : ($\ominus_W$ *t t0 I*. *P t*) = (($\ominus$ *t t0 I*. *P t*) $\vee$ (*I* $\downarrow<$ *t0* = \{\}) $\vee$ *t0* $\notin$ *I*)
apply (*unfold iTL-defs*)
thm *iprev-singleton-cut-less-empty-iff*[*symmetric*]
apply (*simp add*: *iprev-singleton-cut-less-empty-iff*[*symmetric*] *cut-less-singleton*)
done
lemma *iLastStrong-iff* : ($\ominus_S$ *t t0 I*. *P t*) = (($\ominus$ *t t0 I*. *P t*) $\wedge$ (*I* $\downarrow<$ *t0* $\neq$ \{\}) $\wedge$ *t0* $\in$ *I*)
apply (*unfold iTL-defs*)
thm *iprev-singleton-cut-less-not-empty-iff*[*symmetric*]
apply (*simp add*: *iprev-singleton-cut-less-not-empty-iff*[*symmetric*] *cut-less-singleton*)
done

lemmas *iTL-Next-iff* =
iNext-iff iLast-iff
iNextWeak-iff iNextStrong-iff
iLastWeak-iff iLastStrong-iff

lemma
iNext-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
iLast-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Box\ t\ \{iprev\ t0\ I\}.\ P\ t)$
by (*fastsimp simp*: *iTL-Next-defs iT-add iIN-0*)+
lemmas *iNextLast-iff-singleton = iNext-iff-singleton iLast-iff-singleton*

lemma
iNext-iEx-iff-singleton : $(\bigcirc\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{inext\ t0\ I\}.\ P\ t)$ **and**
iLast-iEx-iff-singleton : $(\ominus\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ \{iprev\ t0\ I\}.\ P\ t)$
by (*fastsimp simp*: *iTL-Next-defs iT-add iIN-0*)+

lemma
iNextWeak-iAll-conv: $(\bigcirc_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iNextStrong-iEx-conv: $(\bigcirc_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{inext\ t0\ I\} \downarrow> t0).\ P\ t)$ **and**
iLastWeak-iAll-conv: $(\ominus_W\ t\ t0\ I.\ P\ t) = (\Box\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$ **and**
iLastStrong-iEx-conv: $(\ominus_S\ t\ t0\ I.\ P\ t) = (\Diamond\ t\ (\{iprev\ t0\ I\} \downarrow< t0).\ P\ t)$
by (*simp-all add*: *iTL-Next-defs*)

lemma
iAll-True[*simp*]: $\Box\ t\ I.\ True$ **and**
iAll-False[*simp*]: $(\Box\ t\ I.\ False) = (I = \{\})$ **and**
iEx-True[*simp*]: $(\Diamond\ t\ I.\ True) = (I \neq \{\})$ **and**
iEx-False[*simp*]: $\neg\ (\Diamond\ t\ I.\ False)$
by *blast*+

lemma *empty-iff-iAll-False*: $(I = \{\}) = (\Box\ t\ I.\ False)$ **by** *blast*
lemma *not-empty-iff-iEx-True*: $(I \neq \{\}) = (\Diamond\ t\ I.\ True)$ **by** *blast*

lemma
iNext-True: $\bigcirc\ t\ t0\ I.\ True$ **and**
iNextWeak-True: $(\bigcirc_W\ t\ t0\ I.\ True)$ **and**
iNext-False: $\neg\ (\bigcirc\ t\ t0\ I.\ False)$ **and**
iNextStrong-False: $\neg\ (\bigcirc_S\ t\ t0\ I.\ False)$
by (*simp-all add*: *iTL-defs*)
lemma
iNextStrong-True: $(\bigcirc_S\ t\ t0\ I.\ True) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$ **and**
iNextWeak-False: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ False)) = (I \downarrow> t0 \neq \{\} \wedge t0 \in I)$
by (*simp-all add*: *iTL-defs ex-in-conv inext-singleton-cut-greater-not-empty-iff*)

lemma
iLast-True: $\ominus\ t\ t0\ I.\ True$ **and**
iLastWeak-True: $(\ominus_W\ t\ t0\ I.\ True)$ **and**
iLast-False: $\neg\ (\ominus\ t\ t0\ I.\ False)$ **and**
iLastStrong-False: $\neg\ (\ominus_S\ t\ t0\ I.\ False)$
by (*simp-all add*: *iTL-defs*)

lemma
iLastStrong-True: $(\ominus_S\ t\ t0\ I.\ True) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$ **and**
iLastWeak-False: $(\neg\ (\ominus_W\ t\ t0\ I.\ False)) = (I \downarrow< t0 \neq \{\} \wedge t0 \in I)$
by (*simp-all add*: *iTL-defs ex-in-conv iprev-singleton-cut-less-not-empty-iff*)

lemma *iUntil-True-left*[*simp*]: $(True.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (\Diamond\ t\ I.\ Q\ t)$
by *blast*
lemma *iUntil-True*[*simp*]: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ True) = (I \neq \{\})$
apply (*unfold iTL-defs*)
apply (*rule iffI*)
apply *blast*
apply (*rule-tac x=iMin I* **in** *bexI*)
apply (*simp add*: *cut-less-Min-empty iMinI-ex2*)+
done
lemma *iUntil-False-left*[*simp*]: $(False.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (I \neq \{\} \wedge Q\ (iMin\ I))$
apply (*case-tac* $I = \{\}$, *blast*)
apply (*simp add*: *iTL-defs*)
apply (*rule iffI*)
apply *clarsimp*
apply (*drule iMin-equality*)
apply (*simp add*: *cut-less-empty-iff*)
apply *simp*
apply (*rule-tac x=iMin I* **in** *bexI*)
apply (*simp add*: *cut-less-Min-empty*)
apply (*simp add*: *iMinI-ex2*)
done
lemma *iUntil-False*[*simp*]: $\neg\ (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ False)$
by *blast*

lemma *iSince-True-left*[*simp*]: $(True.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (\Diamond\ t\ I.\ Q\ t)$
by *blast*
lemma *iSince-True-if*:
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ True) =$
$(if\ finite\ I\ then\ I \neq \{\}\ else\ \Diamond\ t1\ I.\ \Box\ t2\ (I \downarrow> t1).\ P\ t2)$
apply (*clarsimp simp*: *iTL-defs*)
apply (*rule iffI*)
apply *clarsimp*
apply (*rule-tac x=Max I* **in** *bexI*)
thm *cut-greater-Max-empty*
apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done
corollary *iSince-True-finite*[*simp*]: $finite\ I \Longrightarrow (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ True) = (I \neq \{\})$
by (*simp add*: *iSince-True-if*)
lemma *iSince-False-left*[*simp*]: $(False.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (finite\ I \wedge I \neq \{\} \wedge Q\ (Max\ I))$
apply (*simp add*: *iTL-defs*)
apply (*case-tac* $I = \{\}$, *simp*)
apply (*case-tac infinite I*)

apply (*simp add*: *nat-cut-greater-infinite-not-empty*)
apply (*rule iffI*)
apply *clarsimp*
apply (*drule Max-equality*)
apply *simp*
apply (*simp add*: *cut-greater-empty-iff*)
apply *simp*
apply (*rule-tac x*=*Max I* **in** *bexI*)
apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done
lemma *iSince-False*[*simp*]: ¬ (*P t'*. *t'* $\mathcal{S}$ *t I*. *False*)
by *blast*

lemma *iWeakUntil-True-left*[*simp*]: *True*. *t'* $\mathcal{W}$ *t I*. *Q t*
by (*simp add*: *iWeakUntil-def*)
lemma *iWeakUntil-True*[*simp*]: *P t'*. *t'* $\mathcal{W}$ *t I*. *True*
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*rule disjI2*)
apply (*rule-tac x*=*iMin I* **in** *bexI*)
apply (*simp add*: *cut-less-Min-empty*)
apply (*simp add*: *iMinI-ex2*)
done
lemma *iWeakUntil-False-left*[*simp*]: (*False*. *t'* $\mathcal{W}$ *t I*. *Q t*) = (*I* = {} ∨ *Q* (*iMin I*))
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*rule iffI*)
apply (*clarsimp simp*: *cut-less-empty-iff*)
apply (*frule iMin-equality*)
apply *simp*+
apply (*rule-tac x*=*iMin I* **in** *bexI*)
apply (*simp add*: *cut-less-Min-empty*)
apply (*simp add*: *iMinI-ex2*)
done
lemma *iWeakUntil-False*[*simp*]: (*P t'*. *t'* $\mathcal{W}$ *t I*. *False*) = (□ *t I*. *P t*)
by (*simp add*: *iWeakUntil-def*)

lemma *iWeakSince-True-left*[*simp*]: *True*. *t'* $\mathcal{B}$ *t I*. *Q t*
by (*simp add*: *iTL-defs*)
lemma *iWeakSince-True-disj*:
(*P t'*. *t'* $\mathcal{B}$ *t I*. *True*) =
(*I* = {} ∨ (◊ *t1 I*. □ *t2* (*I* ↓> *t1*). *P t2*))
unfolding *iTL-defs* **by** *blast*
lemma *iWeakSince-True-finite*[*simp*]: *finite I* ⟹ (*P t'*. *t'* $\mathcal{B}$ *t I*. *True*)
apply (*simp add*: *iTL-defs*)
apply (*case-tac I* = {}, *simp*)
apply (*rule disjI2*)

apply (*rule-tac x=Max I* **in** *bexI*)
apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done
lemma *iWeakSince-False-left*[*simp*]: (*False*. t' $\mathcal{B}$ t I. Q t) = (I = {} ∨ (*finite* I ∧ Q (*Max* I)))
apply (*simp add*: *iTL-defs*)
apply (*case-tac* I = {}, *simp*)
apply (*case-tac infinite* I)
apply (*simp add*: *nat-cut-greater-infinite-not-empty*)
apply (*rule iffI*)
apply *clarsimp*
apply (*drule Max-equality*)
apply *simp*
apply (*simp add*: *cut-greater-empty-iff*)
apply *simp*
apply *simp*
apply (*rule-tac x=Max I* **in** *bexI*)
apply (*simp add*: *cut-greater-Max-empty*)
apply *simp*
done
lemma *iWeakSince-False*[*simp*]: (P t'. t' $\mathcal{B}$ t I. *False*) = (□ t I. P t)
by (*simp add*: *iWeakSince-def*)

lemma *iRelease-True-left*[*simp*]: (*True*. t' $\mathcal{R}$ t I. Q t) = (I = {} ∨ Q (*iMin* I))
apply (*simp add*: *iTL-defs*)
apply (*case-tac* I = {}, *simp*)
apply (*rule iffI*)
apply (*erule disjE*)
apply (*blast intro*: *iMinI2-ex2*)
apply *clarsimp*
apply (*drule-tac x=iMin I* **in** *bspec*)
apply (*blast intro*: *iMinI-ex2*)
apply *simp*
apply (*rule disjI2*)
apply (*rule-tac x=iMin I* **in** *bexI*)
apply *fastsimp*
apply (*simp add*: *iMinI-ex2*)
done
lemma *iRelease-True*[*simp*]: P t'. t' $\mathcal{R}$ t I. *True*
by (*simp add*: *iTL-defs*)
lemma *iRelease-False-left*[*simp*]: (*False*. t' $\mathcal{R}$ t I. Q t) = (□ t I. Q t)
by (*simp add*: *iTL-defs*)
lemma *iRelease-False*[*simp*]: (P t'. t' $\mathcal{R}$ t I. *False*) = (I = {})
unfolding *iTL-defs* **by** *blast*

lemma *iTrigger-True-left*[*simp*]: (*True*. t' $\mathcal{T}$ t I. Q t) = (I = {} ∨ (◇ *t1* I. □ *t2* (I ↓≥ *t1*). Q *t2*))
unfolding *iTL-defs* **by** *blast*

lemma *iTrigger-True*[*simp*]: $P\ t'.\ t'\ \mathcal{T}\ t\ I.\ True$
by (*simp add*: *iTL-defs*)
lemma *iTrigger-False-left*[*simp*]: $(False.\ t'\ \mathcal{T}\ t\ I.\ Q\ t) = (\Box\ t\ I.\ Q\ t)$
by (*simp add*: *iTL-defs*)
lemma *iTrigger-False*[*simp*]: $(P\ t'.\ t'\ \mathcal{T}\ t\ I.\ False) = (I = \{\})$
unfolding *iTL-defs* **by** *blast*

lemma
iUntil-TrueTrue[*simp*]: $(True.\ t'\ \mathcal{U}\ t\ I.\ True) = (I \neq \{\})$ **and**
iSince-TrueTrue[*simp*]: $(True.\ t'\ \mathcal{S}\ t\ I.\ True) = (I \neq \{\})$ **and**
iWeakUntil-TrueTrue[*simp*]: $True.\ t'\ \mathcal{W}\ t\ I.\ True$ **and**
iWeakSince-TrueTrue[*simp*]: $True.\ t'\ \mathcal{B}\ t\ I.\ True$ **and**
iRelease-TrueTrue[*simp*]: $True.\ t'\ \mathcal{R}\ t\ I.\ True$ **and**
iTrigger-TrueTrue[*simp*]: $True.\ t'\ \mathcal{T}\ t\ I.\ True$
by (*simp-all add*: *iTL-defs ex-in-conv*)

3.2.3 Empty sets and singletons

lemma *iAll-empty*[*simp*]: $\Box\ t\ \{\}.\ P\ t$ **by** *blast*
lemma *iEx-empty*[*simp*]: $\neg\ (\Diamond\ t\ \{\}.\ P\ t)$ **by** *blast*

thm *iAll-empty iEx-empty*
lemma *iUntil-empty*[*simp*]: $\neg\ (P\ t0.\ t0\ \mathcal{U}\ t1\ \{\}.\ Q\ t1)$ **by** *blast*
lemma *iSince-empty*[*simp*]: $\neg\ (P\ t0.\ t0\ \mathcal{S}\ t1\ \{\}.\ Q\ t1)$ **by** *blast*
lemma *iWeakUntil-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{W}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iWeakUntil-def*)
lemma *iWeakSince-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{B}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iWeakSince-def*)

lemma *iRelease-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{R}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iRelease-def*)
lemma *iTrigger-empty*[*simp*]: $P\ t0.\ t0\ \mathcal{T}\ t1\ \{\}.\ Q\ t1$ **by** (*simp add*: *iTrigger-def*)

lemmas *iTL-empty* =
iAll-empty iEx-empty
iUntil-empty iSince-empty
iWeakUntil-empty iWeakSince-empty
iRelease-empty iTrigger-empty

lemma *iAll-singleton*[*simp*]: $(\Box\ t'\ \{t\}.\ P\ t') = P\ t$ **by** *blast*
lemma *iEx-singleton*[*simp*]: $(\Diamond\ t'\ \{t\}.\ P\ t') = P\ t$ **by** *blast*

lemma *iUntil-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{U}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
by (*simp add*: *iUntil-def cut-less-singleton*)
lemma *iSince-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{S}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
by (*simp add*: *iSince-def cut-greater-singleton*)

lemma *iWeakUntil-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{W}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
by (*simp add*: *iWeakUntil-def cut-less-singleton*)
lemma *iWeakSince-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{B}\ t1\ \{t\}.\ Q\ t1) = (P\ t \vee Q\ t)$
by (*simp add*: *iWeakSince-def cut-greater-singleton*)

lemma *iRelease-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{R}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
unfolding *iRelease-def* **by** *blast*
lemma *iTrigger-singleton*[*simp*]: $(P\ t0.\ t0\ \mathcal{T}\ t1\ \{t\}.\ Q\ t1) = Q\ t$
unfolding *iTrigger-def* **by** *blast*

lemmas *iTL-singleton* =
iAll-singleton iEx-singleton
iUntil-singleton iSince-singleton
iWeakUntil-singleton iWeakSince-singleton
iRelease-singleton iTrigger-singleton
thm *iTL-singleton*

3.2.4 Conversions between temporal operators

lemma *iAll-iEx-conv*: $(\Box\ t\ I.\ P\ t) = (\neg\ (\Diamond\ t\ I.\ \neg\ P\ t))$ **by** *blast*
lemma *iEx-iAll-conv*: $(\Diamond\ t\ I.\ P\ t) = (\neg\ (\Box\ t\ I.\ \neg\ P\ t))$ **by** *blast*

lemma *not-iAll*[*simp*]: $(\neg\ (\Box\ t\ I.\ P\ t)) = (\Diamond\ t\ I.\ \neg\ P\ t)$ **by** *blast*
lemma *not-iEx*[*simp*]: $(\neg\ (\Diamond\ t\ I.\ P\ t)) = (\Box\ t\ I.\ \neg\ P\ t)$ **by** *blast*

lemma *iUntil-iEx-conv*: $(True.\ t'\ \mathcal{U}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ **by** *blast*
lemma *iSince-iEx-conv*: $(True.\ t'\ \mathcal{S}\ t\ I.\ P\ t) = (\Diamond\ t\ I.\ P\ t)$ **by** *blast*

lemma *iRelease-iAll-conv*: $(False.\ t'\ \mathcal{R}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
by (*simp add*: *iRelease-def*)
lemma *iTrigger-iAll-conv*: $(False.\ t'\ \mathcal{T}\ t\ I.\ P\ t) = (\Box\ t\ I.\ P\ t)$
by (*simp add*: *iTrigger-def*)

lemma *iWeakUntil-iUntil-conv*: $(P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
unfolding *iWeakUntil-def iUntil-def* **by** *blast*
lemma *iWeakSince-iSince-conv*: $(P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) \vee (\Box\ t\ I.\ P\ t))$
unfolding *iWeakSince-def iSince-def* **by** *blast*

lemma *iUntil-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{W}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
by (*subst iWeakUntil-iUntil-conv*, *blast*)
lemma *iSince-iWeakSince-conv*: $(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = ((P\ t'.\ t'\ \mathcal{B}\ t\ I.\ Q\ t) \wedge (\Diamond\ t\ I.\ Q\ t))$
by (*subst iWeakSince-iSince-conv*, *blast*)

thm *iRelease-def iWeakUntil-def*
lemma *iRelease-iWeakUntil-conv*: $(P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t) = (Q\ t'.\ t'\ \mathcal{W}\ t\ I.\ (Q\ t \wedge P\ t))$
apply (*unfold iRelease-def iWeakUntil-def*)
apply (*simp add*: *cut-le-less-conv-if*)
apply *blast*

done
lemma *iRelease-iUntil-conv*: $(P\ t'.\ t' \mathcal{R}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t' \mathcal{U}\ t\ I.\ (Q\ t \wedge P\ t)))$
by (*fastsimp simp*: *iRelease-iWeakUntil-conv iWeakUntil-iUntil-conv*)

lemma *iTrigger-iWeakSince-conv*: $(P\ t'.\ t' \mathcal{T}\ t\ I.\ Q\ t) = (Q\ t'.\ t' \mathcal{B}\ t\ I.\ (Q\ t \wedge P\ t))$
apply (*unfold iTrigger-def iWeakSince-def*)
apply (*simp add*: *cut-ge-greater-conv-if*)
apply *blast*
done
lemma *iTrigger-iSince-conv*: $(P\ t'.\ t' \mathcal{T}\ t\ I.\ Q\ t) = ((\Box\ t\ I.\ Q\ t) \vee (Q\ t'.\ t' \mathcal{S}\ t\ I.\ (Q\ t \wedge P\ t)))$
by (*fastsimp simp*: *iTrigger-iWeakSince-conv iWeakSince-iSince-conv*)

lemma *iRelease-not-iUntil-conv*: $(P\ t'.\ t' \mathcal{R}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t' \mathcal{U}\ t\ I.\ \neg\ Q\ t))$
apply (*simp only*: *iUntil-def iRelease-def not-iAll not-iEx de-Morgan-conj not-not*)
apply (*case-tac* $\Box\ t\ I.\ Q\ t$, *blast*)
apply (*simp* (*no-asm-simp*))
apply *clarsimp*
apply (*rule iffI*)
apply (*elim iexE*, *intro iallI*, *rename-tac t1 t2*)
apply (*case-tac t2* $\leq$ *t1*, *blast*)
apply (*simp add*: *linorder-not-le*, *blast*)
apply (*frule-tac t=t* **in** *ispec*, *assumption*)
apply *clarsimp*
apply (*rule-tac t=iMin* $\{t \in I.\ P\ t\}$ **in** *iexI*)
prefer *2*
apply (*blast intro*: *subsetD*[*OF - iMinI-ex*])
apply (*rule conjI*)
apply (*blast intro*: *iMinI2*)
apply (*clarsimp simp*: *cut-le-mem-iff*, *rename-tac t1 t2*)
apply (*drule-tac t=t2* **in** *ispec*, *assumption*)
apply (*clarsimp simp*: *cut-less-mem-iff*)
apply (*frule-tac x=t'* **in** *order-less-le-trans*, *assumption*)
apply (*drule not-less-iMin*)
apply *simp*
done
lemma *iUntil-not-iRelease-conv*: $(P\ t'.\ t' \mathcal{U}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t' \mathcal{R}\ t\ I.\ \neg\ Q\ t))$
by (*simp add*: *iRelease-not-iUntil-conv*)

The Trigger operator $\mathcal{T}$is a past operator, so that it is used for time intervals, that are bounded by a current time point, and thus are finite. For an infinite interval the stated relation to the Since operator $\mathcal{S}$would not be fulfilled.

lemma *iTrigger-not-iSince-conv*: *finite I* $\Longrightarrow (P\ t'.\ t' \mathcal{T}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t' \mathcal{S}\ t\ I.\ \neg\ Q\ t))$
apply (*unfold iTrigger-def iSince-def*)

apply (*case-tac* $\Box\ t\ I.\ Q\ t$, *blast*)
apply (*simp* (*no-asm-simp*))
apply *clarsimp*
apply (*rule iffI*)
 apply (*elim iexE conjE*, *rule iallI*, *rename-tac t1 t2*)
 apply (*case-tac t2* $\geq$ *t1*, *blast*)
 apply (*simp add*: *linorder-not-le*, *blast*)
apply (*frule-tac t=t* **in** *ispec*, *assumption*)
 apply (*erule disjE*, *blast*)
apply (*erule iexE*)
apply (*subgoal-tac finite* $\{t \in I.\ P\ t\}$)
 prefer *2*
 apply (*blast intro*: *subset-finite-imp-finite*)
apply (*rule-tac t=Max* $\{t \in I.\ P\ t\}$ **in** *iexI*)
 prefer *2*
 apply (*blast intro*: *subsetD*[*OF - MaxI*])
apply (*rule conjI*)
 apply (*blast intro*: *MaxI2*)
apply (*clarsimp simp*: *cut-ge-mem-iff*, *rename-tac t1 t2*)
apply (*drule-tac t=t2* **in** *ispec*, *assumption*)
apply (*clarsimp simp*: *cut-greater-mem-iff*, *rename-tac t'*)
apply (*frule-tac z=t'* **in** *order-le-less-trans*, *assumption*)
thm *not-greater-Max*[*rotated 1*]
apply (*drule-tac A=*$\{t \in I.\ P\ t\}$ **in** *not-greater-Max*[*rotated 1*])
apply *simp+*
done
lemma *iSince-not-iTrigger-conv*: *finite I* $\Longrightarrow$ $(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (\neg\ (\neg\ P\ t'.\ t'\ \mathcal{T}\ t\ I.\ \neg\ Q\ t))$
by (*simp add*: *iTrigger-not-iSince-conv*)

lemma *not-iUntil*:
  $(\neg\ (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)) =$
  $(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ {\downarrow}{<}\ t).\ \neg\ P\ t')))$
unfolding *iTL-defs* **by** *blast*
lemma *not-iSince*:
  $(\neg\ (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)) =$
  $(\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\ {\downarrow}{>}\ t).\ \neg\ P\ t')))$
unfolding *iTL-defs* **by** *blast*

lemma *iWeakUntil-conj-iUntil-conv*:
  $(P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ (P\ t2 \wedge Q\ t2)) = (\neg\ (\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ \neg\ P\ t2))$
by (*simp add*: *iRelease-not-iUntil-conv*[*symmetric*] *iRelease-iWeakUntil-conv*)
lemma *iUntil-disj-iUntil-conv*:
  $(P\ t1 \vee Q\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) =$
  $(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
apply (*unfold iUntil-def*)
apply (*rule iffI*)

prefer *2*
apply *blast*
apply (*clarsimp*, *rename-tac t1*)
apply (*rule-tac t=iMin* {$t \in I.\ Q\ t$} **in** *iexI*)
apply (*subgoal-tac Q* (*iMin* {$t \in I.\ Q\ t$}))
prefer *2*
apply (*blast intro*: *iMinI2*)
apply (*clarsimp*, *rename-tac t2*)
apply (*frule Collect-not-less-iMin*, *simp*)
apply (*subgoal-tac t2* $<$ *t1*)
prefer *2*
apply (*rule order-less-le-trans*, *assumption*)
apply (*simp add*: *Collect-iMin-le*)
apply *blast*
thm *subsetD*[*OF - iMinI*]
apply (*rule subsetD*[*OF - iMinI*])
apply *blast+*
done
lemma *iWeakUntil-disj-iWeakUntil-conv*:
($P\ t1 \vee Q\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2$) =
($P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2$)
apply (*simp only*: *iWeakUntil-iUntil-conv iUntil-disj-iUntil-conv*)
apply (*case-tac P t1. t1* $\mathcal{U}$ *t2 I. Q t2*, *simp+*)
apply (*case-tac* $\Box$ *t I. P t*, *blast*)
apply (*simp add*: *not-iUntil*)
apply (*clarsimp*, *rename-tac t1*)
apply (*case-tac* $\neg$ *Q t1*, *blast*)
apply (*subgoal-tac iMin* {$t \in I.\ Q\ t$} $\in I$)
prefer *2*
apply (*blast intro*: *subsetD*[*OF - iMinI*])
apply (*frule-tac t=iMin* {$t \in I.\ Q\ t$} **in** *ispec*, *assumption*)
apply (*drule mp*)
apply (*blast intro*: *iMinI2*)
apply (*clarsimp*, *rename-tac t2*)
apply (*subgoal-tac* $\neg$ *Q t2*)
prefer *2*
apply (*drule Collect-not-less-iMin*)
apply (*simp add*: *cut-less-mem-iff*)
apply *blast*
done
lemma *iWeakUntil-iUntil-conv-conv*:
($P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2$) =
($\neg$ ($\neg\ Q\ t1.\ t1\ \mathcal{U}\ t2\ I.$ ($\neg\ P\ t2 \wedge \neg\ Q\ t2$)))
apply (*subst iWeakUntil-disj-iWeakUntil-conv*[*symmetric*])
apply (*subst de-Morgan-disj*[*symmetric*])
apply (*subst iWeakUntil-conj-iUntil-conv*[*symmetric*])
apply (*simp add*: *conj-disj-distribR conj-disj-absorb*)
done

Negation and temporal operators

thm *iNext-iff*
lemma
not-iNext[*simp*]: $(\neg\ (\bigcirc\ t\ t0\ I.\ P\ t)) = (\bigcirc\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextWeak[*simp*]: $(\neg\ (\bigcirc_W\ t\ t0\ I.\ P\ t)) = (\bigcirc_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iNextStrong[*simp*]: $(\neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)) = (\bigcirc_W\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLast[*simp*]: $(\neg\ (\ominus\ t\ t0\ I.\ P\ t)) = (\ominus\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastWeak[*simp*]: $(\neg\ (\ominus_W\ t\ t0\ I.\ P\ t)) = (\ominus_S\ t\ t0\ I.\ \neg\ P\ t)$ **and**
not-iLastStrong[*simp*]: $(\neg\ (\ominus_S\ t\ t0\ I.\ P\ t)) = (\ominus_W\ t\ t0\ I.\ \neg\ P\ t)$
by (*simp-all add*: *iTL-Next-defs*)

thm *not-iUntil*
thm *not-iSince*

lemma *not-iWeakUntil*:
$(\neg\ (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\downarrow< t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
by (*simp add*: *iWeakUntil-iUntil-conv not-iUntil*)
lemma *not-iWeakSince*:
$(\neg\ (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)) =$
$((\Box\ t\ I.\ (Q\ t \longrightarrow (\Diamond\ t'\ (I\downarrow> t).\ \neg\ P\ t'))) \wedge (\Diamond\ t\ I.\ \neg\ P\ t))$
by (*simp add*: *iWeakSince-iSince-conv not-iSince*)

lemma *not-iRelease*:
$(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ P\ t \longrightarrow (\Diamond\ t\ I\downarrow\leq t.\ \neg\ Q\ t)))$
by (*simp add*: *iRelease-def*)
lemma *not-iRelease-iUntil-conv*:
$(\neg\ (P\ t'.\ t'\ \mathcal{R}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{U}\ t\ I.\ \neg\ Q\ t)$
by (*simp add*: *iUntil-not-iRelease-conv*)
lemma *not-iTrigger*:
$(\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) =$
$((\Diamond\ t\ I.\ \neg\ Q\ t) \wedge (\Box\ t\ I.\ \neg\ P\ t \vee (\Diamond\ t\ I\downarrow\geq t.\ \neg\ Q\ t)))$
by (*simp add*: *iTrigger-def*)
lemma *not-iTrigger-iSince-conv*:
finite $I \Longrightarrow (\neg\ (P\ t'.\ t'\ \mathcal{T}\ t\ I.\ Q\ t)) = (\neg\ P\ t'.\ t'\ \mathcal{S}\ t\ I.\ \neg\ Q\ t)$
by (*simp add*: *iSince-not-iTrigger-conv*)

3.2.5 Some implication results

lemma *all-imp-iall*: $\forall x.\ P\ x \Longrightarrow \Box\ t\ I.\ P\ t$ **by** *blast*
lemma *bex-imp-lex*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \exists x.\ P\ x$ **by** *blast*

lemma *iAll-imp-iEx*: $I \neq \{\} \Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$ **by** *blast*
lemma *i-set-iAll-imp-iEx*: $I \in$ *i-set* $\Longrightarrow \Box\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t$
by (*rule iAll-imp-iEx*, *rule i-set-imp-not-empty*)
lemmas *iT-iAll-imp-iEx* = *iT-not-empty*[*THEN iAll-imp-iEx*]
thm *iT-iAll-imp-iEx*

lemma *iUntil-imp-iEx*: $P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$

unfolding *iTL-defs* **by** *blast*
lemma *iSince-imp-iEx*: $P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2 \Longrightarrow \Diamond\ t\ I.\ Q\ t$
unfolding *iTL-defs* **by** *blast*

thm *ball-subset-imp-ball*
lemma *iall-subset-imp-iall*: $[\![\Box\ t\ B.\ P\ t;\ A \subseteq B]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
by *blast*
thm *bex-subset-imp-bex*
lemma *iex-subset-imp-iex*: $[\![\Diamond\ t\ A.\ P\ t;\ A \subseteq B]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$
by *blast*

lemma *iall-mp*: $[\![\Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Box\ t\ I.\ P\ t]\!] \Longrightarrow \Box\ t\ I.\ Q\ t$ **by** *blast*
lemma *iex-mp*: $[\![\Box\ t\ I.\ P\ t \longrightarrow Q\ t;\ \Diamond\ t\ I.\ P\ t]\!] \Longrightarrow \Diamond\ t\ I.\ Q\ t$ **by** *blast*

lemma *iUntil-imp*:
$[\![P1\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2; \Box\ t\ I.\ P1\ t \longrightarrow P2\ t]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*
lemma *iSince-imp*:
$[\![P1\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2; \Box\ t\ I.\ P1\ t \longrightarrow P2\ t]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*
lemma *iWeakUntil-imp*:
$[\![P1\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2; \Box\ t\ I.\ P1\ t \longrightarrow P2\ t]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*
lemma *iWeakSince-imp*:
$[\![P1\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2; \Box\ t\ I.\ P1\ t \longrightarrow P2\ t]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*
lemma *iRelease-imp*:
$[\![P1\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2; \Box\ t\ I.\ P1\ t \longrightarrow P2\ t]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{R}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*
lemma *iTrigger-imp*:
$[\![P1\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2; \Box\ t\ I.\ P1\ t \longrightarrow P2\ t]\!] \Longrightarrow P2\ t1.\ t1\ \mathcal{T}\ t2\ I.\ Q\ t2$
unfolding *iTL-defs* **by** *blast*

lemma *iMin-imp-iUntil*:
$[\![I \neq \{\};\ Q\ (iMin\ I)]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t$
apply (*unfold iUntil-def*)
apply (*rule-tac t=iMin I* **in** *iexI*)
apply (*simp add*: *cut-less-Min-empty*)
apply (*blast intro*: *iMinI-ex2*)
done
lemma *Max-imp-iSince*:
$[\![finite\ I;\ I \neq \{\};\ Q\ (Max\ I)]\!] \Longrightarrow P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t$
apply (*unfold iSince-def*)
apply (*rule-tac t=Max I* **in** *iexI*)
apply (*simp add*: *cut-greater-Max-empty*)
apply (*blast intro*: *Max-in*)
done

3.2.6 Congruence rules for temporal operators' predicates

thm *arg-cong*
lemma *iAll-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Box\ t\ I.\ P\ (f\ t)\ t) = (\Box\ t\ I.\ P\ (g\ t)\ t)$
unfolding *iTL-defs* **by** *simp*
lemma *iEx-cong*: $\Box\ t\ I.\ f\ t = g\ t \Longrightarrow (\Diamond\ t\ I.\ P\ (f\ t)\ t) = (\Diamond\ t\ I.\ P\ (g\ t)\ t)$
unfolding *iTL-defs* **by** *simp*
lemma *iUntil-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
apply (*unfold iUntil-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*)
apply (*rule-tac* $f{=}\lambda x.\ (Q\ t \wedge x)$ **in** *arg-cong*, *rename-tac t*)
apply (*rule iAll-cong*[*OF iall-subset-imp-iall*[*OF - cut-less-subset*]])
apply (*rule iallI*, *rename-tac* t')
apply (*drule-tac* $t{=}t'$ **in** *ispec*)
apply *simp+*
done
lemma *iUntil-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ (g\ t2)\ t2)$
apply (*unfold iUntil-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*, *rename-tac t*)
apply (*drule-tac t=t* **in** *ispec*)
apply *simp+*
done
thm *subst*
lemma *iSince-cong1*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ (f\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ (g\ t1)\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
apply (*unfold iSince-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*, *rename-tac t*)
apply (*rule-tac* $f{=}\lambda x.\ (Q\ t \wedge x)$ **in** *arg-cong*)
apply (*rule iAll-cong*[*OF iall-subset-imp-iall*[*OF - cut-greater-subset*]])
apply (*rule iallI*, *rename-tac* t')
apply (*drule-tac* $t{=}t'$ **in** *ispec*)
apply *simp+*
done
lemma *iSince-cong2*:
$\Box\ t\ I.\ f\ t = g\ t \Longrightarrow$
$(P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (f\ t2)\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ (g\ t2)\ t2)$
apply (*unfold iSince-def*)
apply (*rule iEx-cong*)
apply (*rule iallI*, *rename-tac t*)
apply (*drule-tac t=t* **in** *ispec*)
apply *simp+*
done

thm *iAll-cong*

lemma *bex-subst*:
$\forall x{\in}A.\ P\ x \longrightarrow (Q\ x = Q'\ x) \Longrightarrow$
$(\exists x{\in}A.\ P\ x \wedge Q\ x) = (\exists x{\in}A.\ P\ x \wedge Q'\ x)$
by *blast*
lemma *iEx-subst*:
$\Box\ t\ I.\ P\ t \longrightarrow (Q\ t = Q'\ t) \Longrightarrow$
$(\Diamond\ t\ I.\ P\ t \wedge Q\ t) = (\Diamond\ t\ I.\ P\ t \wedge Q'\ t)$
by *blast*

3.2.7 Temporal operators with set unions/intersections and subsets

lemma *iAll-subset*: $[\![\ A \subseteq B;\ \Box\ t\ B.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ A.\ P\ t$
by (*rule iall-subset-imp-iall*)
lemma *iEx-subset*: $[\![\ A \subseteq B;\ \Diamond\ t\ A.\ P\ t\ ]\!] \Longrightarrow \Diamond\ t\ B.\ P\ t$
by (*rule iex-subset-imp-iex*)

lemma *iUntil-append*:
$[\![$ *finite A*; *Max A* $\leq$ *iMin B* $]\!] \Longrightarrow$
P t1. t1 $\mathcal{U}$ *t2 A. Q t2* $\Longrightarrow$ *P t1. t1* $\mathcal{U}$ *t2* $(A \cup B)$*. Q t2*
apply (*case-tac A* $= \{\}$, *simp*)
apply (*unfold iUntil-def*)
thm *iEx-subset*[*OF Un-upper1*]
apply (*rule iEx-subset*[*OF Un-upper1*])
thm *subst*[*OF iEx-cong*, *rule-format*]
apply (*rule-tac f*=$\lambda t.\ A \downarrow< t$ **and** *g*=$\lambda t.\ (A \cup B) \downarrow< t$ **in** *subst*[*OF iEx-cong*, *rule-format*])
apply (*clarsimp simp*: *cut-less-Un*, *rename-tac t t'*)
apply (*cut-tac t*=*t* **and** *I*=*B* **in** *cut-less-Min-empty*)
apply *simp*+
done
lemma *iSince-prepend*:
$[\![$ *finite A*; *Max A* $\leq$ *iMin B* $]\!] \Longrightarrow$
P t1. t1 $\mathcal{S}$ *t2 B. Q t2* $\Longrightarrow$ *P t1. t1* $\mathcal{S}$ *t2* $(A \cup B)$*. Q t2*
apply (*case-tac B* $= \{\}$, *simp*)
apply (*unfold iSince-def*)
apply (*rule iEx-subset*[*OF Un-upper2*])
apply (*rule-tac f*=$\lambda t.\ B \downarrow> t$ **and** *g*=$\lambda t.\ (A \cup B) \downarrow> t$ **in** *subst*[*OF iEx-cong*, *rule-format*])
apply (*clarsimp simp*: *cut-greater-Un*, *rename-tac t t'*)
apply (*cut-tac t*=*t* **and** *I*=*A* **in** *cut-greater-Max-empty*)
apply (*simp add*: *iMin-ge-iff*)+
done

lemma
iAll-union: $[\![\ \Box\ t\ A.\ P\ t; \Box\ t\ B.\ P\ t\ ]\!] \Longrightarrow \Box\ t\ (A \cup B).\ P\ t$ **and**

iAll-union-conv: $(\Box\ t\ A \cup B.\ P\ t) = ((\Box\ t\ A.\ P\ t) \wedge (\Box\ t\ B.\ P\ t))$
by *blast+*
lemma
iEx-union: $(\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t) \Longrightarrow \Diamond\ t\ (A \cup B).\ P\ t$ **and**
iEx-union-conv: $(\Diamond\ t\ (A \cup B).\ P\ t) = ((\Diamond\ t\ A.\ P\ t) \vee (\Diamond\ t\ B.\ P\ t))$
by *blast+*
lemma *iAll-inter*: $(\Box\ t\ A.\ P\ t) \vee (\Box\ t\ B.\ P\ t) \Longrightarrow \Box\ t\ (A \cap B).\ P\ t$ **by** *blast*
lemma *not-iEx-inter*:
$\exists A\ B\ P.\ (\Diamond\ t\ A.\ P\ t) \wedge (\Diamond\ t\ B.\ P\ t) \wedge \neg\ (\Diamond\ t\ (A \cap B).\ P\ t)$
by (*rule-tac* $x=\{0\}$ **in** *exI*, *rule-tac* $x=\{Suc\ 0\}$ **in** *exI*, *blast*)

lemma
iAll-insert: $[\![\ P\ t; \Box\ t\ I.\ P\ t\ ]\!] \Longrightarrow \Box\ t'\ (insert\ t\ I).\ P\ t'$ **and**
iAll-insert-conv: $(\Box\ t'\ (insert\ t\ I).\ P\ t') = (P\ t \wedge (\Box\ t'\ I.\ P\ t'))$
by *blast+*
lemma
iEx-insert: $[\![\ P\ t \vee (\Diamond\ t\ I.\ P\ t)\ ]\!] \Longrightarrow \Diamond\ t'\ (insert\ t\ I).\ P\ t'$ **and**
iEx-insert-conv: $(\Diamond\ t'\ (insert\ t\ I).\ P\ t') = (P\ t \vee (\Diamond\ t'\ I.\ P\ t'))$
by *blast+*

3.3 Further results for temporal operators

thm *Collect-minI-ex*
thm *Collect-minI-ex-cut*
lemma *Collect-minI-iEx*: $\Diamond\ t\ I.\ P\ t \Longrightarrow \Diamond\ t\ I.\ P\ t \wedge (\Box\ t'\ (I \downarrow< t).\ \neg\ P\ t')$
by (*unfold iAll-def iEx-def*, *rule Collect-minI-ex-cut*)

thm *iUntil-def*
lemma *iUntil-disj-conv1*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) = (Q\ (iMin\ I) \vee (P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t \wedge iMin\ I < t))$
apply (*case-tac* $Q\ (iMin\ I)$)
apply (*simp add*: *iMin-imp-iUntil*)
apply (*unfold iUntil-def*, *blast*)
done
lemma *iSince-disj-conv1*:
$[\![\ finite\ I;\ I \neq \{\}\ ]\!] \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) = (Q\ (Max\ I) \vee (P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t \wedge t < Max\ I))$
apply (*case-tac* $Q\ (Max\ I)$)
apply (*simp add*: *Max-imp-iSince*)
apply (*unfold iSince-def*, *blast*)
done

lemma *iUntil-next*:
$I \neq \{\} \Longrightarrow$
$(P\ t'.\ t'\ \mathcal{U}\ t\ I.\ Q\ t) =$
$(Q\ (iMin\ I) \vee (P\ (iMin\ I) \wedge (P\ t'.\ t'\ \mathcal{U}\ t\ (I \downarrow> (iMin\ I)).\ Q\ t)))$
apply (*case-tac* $Q\ (iMin\ I)$)
apply (*simp add*: *iMin-imp-iUntil*)

apply (*simp add*: *iUntil-def*)
apply (*frule iMinI-ex2*)
apply *blast*
done

lemma *iSince-prev*: ⟦ *finite I*; $I \neq \{\}$ ⟧ ⟹
$(P\ t'.\ t'\ \mathcal{S}\ t\ I.\ Q\ t) =$
$(Q\ (Max\ I) \vee (P\ (Max\ I) \wedge (P\ t'.\ t'\ \mathcal{S}\ t\ (I \downarrow< Max\ I).\ Q\ t)))$
apply (*case-tac Q* (*Max I*))
apply (*simp add*: *Max-imp-iSince*)
apply (*simp add*: *iSince-def*)
apply (*frule Max-in*, *assumption*)
apply *blast*
done

lemma *iNext-induct-rule*:
⟦ $P\ (iMin\ I)$; $\Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t'))$; $t \in I$ ⟧ ⟹ $P\ t$
apply (*rule inext-induct*[*of - I*])
apply *simp*
apply (*drule-tac t=n* **in** *ispec*, *assumption*)
apply (*simp add*: *iNext-def*)
apply *assumption*
done
lemma *iNext-induct*:
⟦ $P\ (iMin\ I)$; $\Box\ t\ I.\ (P\ t \longrightarrow (\bigcirc\ t'\ t\ I.\ P\ t'))$ ⟧ ⟹ $\Box\ t\ I.\ P\ t$
by (*rule iallI*, *rule iNext-induct-rule*)

lemma *iLast-induct-rule*:
⟦ $P\ (Max\ I)$; $\Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t'))$; *finite I*; $t \in I$ ⟧ ⟹ $P\ t$
apply (*rule iprev-induct*[*of - I*])
apply *assumption*
apply (*drule-tac t=n* **in** *ispec*, *assumption*)
apply (*simp add*: *iLast-def*)
apply *assumption*+
done
lemma *iLast-induct*:
⟦ $P\ (Max\ I)$; $\Box\ t\ I.\ (P\ t \longrightarrow (\ominus\ t'\ t\ I.\ P\ t'))$; *finite I* ⟧ ⟹ $\Box\ t\ I.\ P\ t$
by (*rule iallI*, *rule iLast-induct-rule*)

lemma *iUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{U}\ t2\ I.\ Q\ t2)$
apply (*unfold iUntil-def*)
apply (*rule iffI*)
apply *blast*
apply (*clarsimp*, *rename-tac t*)
apply (*rule-tac t=iMin* $\{x \in I.\ Q\ x\}$ **in** *iexI*)
apply (*rule conjI*)

apply (*blast intro*: *iMinI2*)
apply (*clarsimp simp*: *cut-less-mem-iff*, *rename-tac t1*)
apply (*subgoal-tac iMin* $\{x \in I.\ Q\ x\} \leq t$)
prefer *2*
apply (*simp add*: *iMin-le*)
apply (*frule order-less-le-trans*, *assumption*)
apply (*drule-tac t=t1* **in** *ispec*, *simp add*: *cut-less-mem-iff*)
apply (*rule ccontr*, *simp*)
apply (*subgoal-tac t1* $\in \{x \in I.\ Q\ x\}$)
prefer *2*
apply *blast*
thm *iMin-le*
apply (*drule-tac k=t1* **and** $I=\{x \in I.\ Q\ x\}$ **in** *iMin-le*)
apply *simp*
thm *subsetD*[*OF - iMinI*]
apply (*blast intro*: *subsetD*[*OF - iMinI*])
done

lemma *iWeakUntil-conj-not*: $((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{W}\ t2\ I.\ Q\ t2)$
by (*simp only*: *iWeakUntil-iUntil-conv iUntil-conj-not*, *blast*)

lemma *iSince-conj-not*: *finite I* $\Longrightarrow$
$((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{S}\ t2\ I.\ Q\ t2)$
apply (*simp only*: *iSince-def*)
apply (*case-tac* $I = \{\}$, *simp*)
apply (*rule iffI*)
apply *blast*
apply (*clarsimp*, *rename-tac t*)
apply (*subgoal-tac finite* $\{x \in I.\ Q\ x\}$)
prefer *2*
apply *fastsimp*
apply (*rule-tac t=Max* $\{x \in I.\ Q\ x\}$ **in** *iexI*)
apply (*rule conjI*)
thm *MaxI2*
apply (*blast intro*: *MaxI2*)
apply (*clarsimp simp*: *cut-greater-mem-iff*, *rename-tac t1*)
apply (*subgoal-tac* $t \leq Max\ \{x \in I.\ Q\ x\}$)
prefer *2*
apply *simp*
apply (*frule order-le-less-trans*, *assumption*)
apply (*drule-tac t=t1* **in** *ispec*, *simp add*: *cut-greater-mem-iff*)
apply (*rule ccontr*, *simp*)
apply (*subgoal-tac t1* $\in \{x \in I.\ Q\ x\}$)
prefer *2*
apply *blast*
thm *not-greater-Max*
apply (*drule not-greater-Max*[*rotated 1*], *simp+*)
thm *subsetD*[*OF - MaxI*]

apply (*rule subsetD*[*OF - MaxI*], *fastsimp*+)
done
lemma *iWeakSince-conj-not*: *finite I* $\Longrightarrow$
$((P\ t1 \wedge \neg\ Q\ t1).\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2) = (P\ t1.\ t1\ \mathcal{B}\ t2\ I.\ Q\ t2)$
by (*simp only*: *iWeakSince-iSince-conv iSince-conj-not*, *blast*)

lemma *iNextStrong-imp-iNextWeak*: $(\bigcirc_S\ t\ t0\ I.\ P\ t) \longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$
unfolding *iTL-Next-defs* **by** *blast*
lemma *iLastStrong-imp-iLastWeak*: $(\ominus_S\ t\ t0\ I.\ P\ t) \longrightarrow (\ominus_W\ t\ t0\ I.\ P\ t)$
unfolding *iTL-Next-defs* **by** *blast*

lemma *infin-imp-iNextWeak-iNextStrong-eq-iNext*:
$[\![$ *infinite I*; *t0* $\in$ *I* $]\!] \Longrightarrow$
$((\bigcirc_W\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t)) \wedge ((\bigcirc_S\ t\ t0\ I.\ P\ t) = (\bigcirc\ t\ t0\ I.\ P\ t))$
by (*simp add*: *iTL-Next-iff nat-cut-greater-infinite-not-empty*)

lemma *infin-imp-iNextWeak-eq-iNext*: $[\![$ *infinite I*; *t0* $\in$ *I* $]\!] \Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$
$= (\bigcirc\ t\ t0\ I.\ P\ t)$
by (*simp add*: *infin-imp-iNextWeak-iNextStrong-eq-iNext*)
lemma *infin-imp-iNextStrong-eq-iNext*: $[\![$ *infinite I*; *t0* $\in$ *I* $]\!] \Longrightarrow (\bigcirc_S\ t\ t0\ I.\ P\ t)$
$= (\bigcirc\ t\ t0\ I.\ P\ t)$
by (*simp add*: *infin-imp-iNextWeak-iNextStrong-eq-iNext*)
lemma *infin-imp-iNextStrong-eq-iNextWeak*: $[\![$ *infinite I*; *t0* $\in$ *I* $]\!] \Longrightarrow (\bigcirc_S\ t\ t0$
$I.\ P\ t) = (\bigcirc_W\ t\ t0\ I.\ P\ t)$
by (*simp add*: *infin-imp-iNextWeak-eq-iNext infin-imp-iNextStrong-eq-iNext*)

lemma
not-in-iNext-eq: *t0* $\notin$ *I* $\Longrightarrow (\bigcirc\ t\ t0\ I.\ P\ t) = (P\ t0)$ **and**
not-in-iLast-eq: *t0* $\notin$ *I* $\Longrightarrow (\ominus\ t\ t0\ I.\ P\ t) = (P\ t0)$
by (*simp-all add*: *iTL-defs not-in-inext-fix not-in-iprev-fix*)
lemma
not-in-iNextWeak-eq: *t0* $\notin$ *I* $\Longrightarrow (\bigcirc_W\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastWeak-eq: *t0* $\notin$ *I* $\Longrightarrow (\ominus_W\ t\ t0\ I.\ P\ t)$
by (*simp-all add*: *iNextWeak-iff iLastWeak-iff*)
lemma
not-in-iNextStrong-eq: *t0* $\notin$ *I* $\Longrightarrow \neg\ (\bigcirc_S\ t\ t0\ I.\ P\ t)$ **and**
not-in-iLastStrong-eq: *t0* $\notin$ *I* $\Longrightarrow \neg\ (\ominus_S\ t\ t0\ I.\ P\ t)$
by (*simp-all add*: *iNextStrong-iff iLastStrong-iff*)

lemma
iNext-UNIV: $(\bigcirc\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextWeak-UNIV: $(\bigcirc_W\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$ **and**
iNextStrong-UNIV: $(\bigcirc_S\ t\ t0\ UNIV.\ P\ t) = P\ (Suc\ t0)$
by (*simp-all add*: *iTL-Next-defs inext-UNIV cut-greater-singleton*)
lemma
iLast-UNIV: $(\ominus\ t\ t0\ UNIV.\ P\ t) = P\ (t0 - Suc\ 0)$ **and**
iLastWeak-UNIV: $(\ominus_W\ t\ t0\ UNIV.\ P\ t) = (if\ 0 < t0\ then\ P\ (t0 - Suc\ 0)\ else$

True) **and**
iLastStrong-UNIV: $(\ominus_S\ t\ t0\ UNIV.\ P\ t) = (if\ 0 < t0\ then\ P\ (t0 - Suc\ 0)\ else\ False)$
by (*simp-all add*: *iTL-Next-defs iprev-UNIV cut-less-singleton*)

lemmas *iTL-Next-UNIV* =
iNext-UNIV iNextWeak-UNIV iNextStrong-UNIV
iLast-UNIV iLastWeak-UNIV iLastStrong-UNIV

lemma *inext-nth-iNext-Suc*: $(\bigcirc\ t\ (I \rightarrow n)\ I.\ P\ t) = P\ (I \rightarrow Suc\ n)$
by (*simp add*: *iNext-def*)
lemma *iprev-nth-iLast-Suc*: $(\ominus\ t\ (I \leftarrow n)\ I.\ P\ t) = P\ (I \leftarrow Suc\ n)$
by (*simp add*: *iLast-def*)

3.4 Temporal operators and arithmetic interval operators

Shifting intervals through addition and subtraction of constants. Mirroring intervals through subtraction of intervals from constants. Expanding and compressing intervals through multiplication and division by constants.

Always operator

lemma *iT-Plus-iAll-conv*: $(\Box\ t\ I \oplus k.\ P\ t) = (\Box\ t\ I.\ P\ (t + k))$
apply (*unfold iAll-def Ball-def*)
apply (*rule iffI*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x + k* **in** *spec*)
apply (*simp add*: *iT-Plus-mem-iff2*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x − k* **in** *spec*)
apply (*simp add*: *iT-Plus-mem-iff*)
done
lemma *iT-Mult-iAll-conv*: $(\Box\ t\ I \otimes k.\ P\ t) = (\Box\ t\ I.\ P\ (t * k))$
apply (*unfold iAll-def Ball-def*)
apply (*case-tac I = {}*)
apply (*simp add*: *iT-Mult-empty*)
apply (*case-tac k = 0*)
apply (*force simp*: *iT-Mult-0 iTILL-0*)
apply (*rule iffI*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x * k* **in** *spec*)
apply (*simp add*: *iT-Mult-mem-iff2*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x div k* **in** *spec*)
apply (*simp add*: *iT-Mult-mem-iff mod-0-div-mult-cancel*)
done
lemma *iT-Plus-neg-iAll-conv*: $(\Box\ t\ I \oplus\!-\ k.\ P\ t) = (\Box\ t\ (I \downarrow\geq k).\ P\ (t - k))$
apply (*unfold iAll-def Ball-def*)
apply (*rule iffI*)
apply (*clarify*, *rename-tac x*)

apply (*drule-tac x=x − k* **in** *spec*)
apply (*simp add*: *iT-Plus-neg-mem-iff2*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x + k* **in** *spec*)
apply (*simp add*: *iT-Plus-neg-mem-iff cut-ge-mem-iff*)
done
lemma *iT-Minus-iAll-conv*: $(\Box\ t\ k \ominus I.\ P\ t) = (\Box\ t\ (I \downarrow\leq k).\ P\ (k - t))$
apply (*unfold iAll-def Ball-def*)
apply (*rule iffI*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=k − x* **in** *spec*)
apply (*simp add*: *iT-Minus-mem-iff*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=k − x* **in** *spec*)
apply (*simp add*: *iT-Minus-mem-iff cut-le-mem-iff*)
done
lemma *iT-Div-iAll-conv*: $(\Box\ t\ I \oslash k.\ P\ t) = (\Box\ t\ I.\ P\ (t\ div\ k))$
apply (*case-tac I = {}*)
apply (*simp add*: *iT-Div-empty*)
apply (*case-tac k = 0*)
apply (*force simp*: *iT-Div-0 iTILL-0*)
apply (*unfold iAll-def Ball-def*)
apply (*rule iffI*)
apply (*clarify*, *rename-tac x*)
apply (*drule-tac x=x div k* **in** *spec*)
apply (*simp add*: *iT-Div-imp-mem*)
apply (*blast dest*: *iT-Div-mem-iff*[*THEN iffD1*])
done

lemmas *iT-arith-iAll-conv* =
iT-Plus-iAll-conv
iT-Mult-iAll-conv
iT-Plus-neg-iAll-conv
iT-Minus-iAll-conv
iT-Div-iAll-conv

Eventually operator

lemma
iT-Plus-iEx-conv: $(\Diamond\ t\ I \oplus k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t + k))$ **and**
iT-Mult-iEx-conv: $(\Diamond\ t\ I \otimes k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t * k))$ **and**
iT-Plus-neg-iEx-conv: $(\Diamond\ t\ I \oplus- k.\ P\ t) = (\Diamond\ t\ (I \downarrow\geq k).\ P\ (t - k))$ **and**
iT-Minus-iEx-conv: $(\Diamond\ t\ k \ominus I.\ P\ t) = (\Diamond\ t\ (I \downarrow\leq k).\ P\ (k - t))$ **and**
iT-Div-iEx-conv: $(\Diamond\ t\ I \oslash k.\ P\ t) = (\Diamond\ t\ I.\ P\ (t\ div\ k))$
by (*simp-all only*: *iEx-iAll-conv iT-arith-iAll-conv*)

Until and Since operators

lemma *iT-Plus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iUntil-def iT-Plus-iAll-conv iT-Plus-iEx-conv iT-Plus-cut-less2*)

lemma *iT-Mult-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2 * k))$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Mult-empty*)
apply (*case-tac* $k = 0$)
apply (*force simp add*: *iT-Mult-0 iTILL-0*)
apply (*simp add*: *iUntil-def iT-Mult-iAll-conv iT-Mult-iEx-conv iT-Mult-cut-less2*)
done
lemma *iT-Plus-neg-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{U}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
apply (*simp add*: *iUntil-def iT-Plus-neg-iAll-conv iT-Plus-neg-iEx-conv iT-Plus-neg-cut-less2*)
thm *i-cut-commute-disj*[*of op* $\downarrow<$ *op* $\downarrow\geq$, *simplified*]
apply (*simp add*: *i-cut-commute-disj*)
done
lemma *iT-Minus-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{S}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
apply (*simp add*: *iUntil-def iSince-def iT-Minus-iAll-conv iT-Minus-iEx-conv iT-Minus-cut-less2*)
apply (*simp add*: *i-cut-commute-disj*)
done
lemma *iT-Div-iUntil-conv*: $(P\ t1.\ t1\ \mathcal{U}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{U}\ t2\ I.\ Q\ (t2\ div\ k))$
apply (*case-tac* $I = \{\}$)
apply (*simp add*: *iT-Div-empty*)
apply (*case-tac* $k = 0$)
apply (*force simp add*: *iT-Div-0 iTILL-0*)
apply (*simp add*: *iUntil-def iT-Div-iAll-conv iT-Div-iEx-conv iT-Div-cut-less2*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac t*)
apply (*subgoal-tac* $I \downarrow\geq (t - t\ mod\ k) \neq \{\}$)
prefer *2*
apply (*simp add*: *cut-ge-not-empty-iff*)
apply (*rule-tac x*=*t* **in** *bexI*)
apply *simp*+
apply (*case-tac t mod k* = *0*)
apply (*rule-tac t*=*t* **in** *iexI*)
apply *simp*+
apply (*rule-tac t*=*iMin* $(I \downarrow\geq (t - t\ mod\ k))$ **in** *iexI*)
apply (*subgoal-tac*
$t - t\ mod\ k \leq iMin\ (I \downarrow\geq (t - t\ mod\ k)) \wedge$
$iMin\ (I \downarrow\geq (t - t\ mod\ k)) \leq t$)
prefer *2*
apply (*rule conjI*)
apply (*blast intro*: *cut-ge-Min-greater*)
apply (*simp add*: *iMin-le cut-ge-mem-iff*)
apply *clarify*
apply (*rule-tac t*=*iMin* $(I \downarrow\geq (t - t\ mod\ k))$ *div k* **and** *s*=*t div k* **in** *subst*)
apply (*rule order-antisym*)
apply (*drule-tac m*=$t - t\ mod\ k$ **and** *k*=*k* **in** *div-le-mono*)
apply (*simp add*: *sub-mod-div-eq-div*)

apply (*rule div-le-mono*, *assumption*)
apply (*clarsimp*, *rename-tac t1*)
thm *cut-less-cut-ge-ident*[*OF order-refl*]
apply (*subgoal-tac t1* $\in I \downarrow< (t - t \bmod k) \cup I \downarrow\geq (t - t \bmod k)$)
prefer *2*
apply (*simp add*: *cut-less-cut-ge-ident*)
find-theorems - $<$ *iMin* - - $\notin$ -
apply (*subgoal-tac t1* $\notin I \downarrow\geq (t - t \bmod k)$)
prefer *2*
apply (*blast dest*: *not-less-iMin*)
apply *blast*
thm *subsetD*[*OF - iMinI-ex2*]
apply (*blast intro*: *subsetD*[*OF - iMinI-ex2*])
apply (*clarsimp*, *rename-tac t*)
apply (*rule-tac t=t* **in** *iexI*)
apply *simp*
apply (*rule-tac B=I* $\downarrow<$ *t* **in** *iAll-subset*)
apply (*simp add*: *cut-less-mono*)
apply *simp+*
done

Until and Since operators can be converted into each other through substraction of intervals from constants

thm *iT-Minus-iUntil-conv*
lemma *iUntil-iSince-conv*:
$[\![$ *finite I*; *Max I* $\leq k$ $]\!] \Longrightarrow$
(*P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2*) = (*P* $(k - t1)$. *t1* $\mathcal{S}$ *t2* $(k \ominus I)$. *Q* $(k - t2)$)
apply (*case-tac I* = {})
apply (*simp add*: *iT-Minus-empty*)
apply (*frule le-trans*[*OF iMin-le-Max*], *assumption+*)
apply (*subgoal-tac Max* $(k \ominus I) \leq k$)
prefer *2*
apply (*simp add*: *iT-Minus-Max*)
apply (*subgoal-tac iMin* $(k \ominus I) \leq k$)
prefer *2*
apply (*rule order-trans*[*OF iMin-le-Max*])
apply (*simp add*: *iT-Minus-finite iT-Minus-empty-iff del*: *Max-le-iff*)+
thm *iT-Minus-iUntil-conv*
apply (*rule-tac t=P t1*. *t1* $\mathcal{U}$ *t2 I*. *Q t2* **and** *s=P t1*. *t1* $\mathcal{U}$ *t2* $(k \ominus (k \ominus I))$. *Q t2* **in** *subst*)
apply (*simp add*: *iT-Minus-Minus-eq*)
thm *iT-Minus-iUntil-conv*
apply (*simp add*: *iT-Minus-iUntil-conv cut-le-Max-all iT-Minus-finite*)
done
lemma *iSince-iUntil-conv*:
$[\![$ *finite I*; *Max I* $\leq k$ $]\!] \Longrightarrow$
(*P t1*. *t1* $\mathcal{S}$ *t2 I*. *Q t2*) = (*P* $(k - t1)$. *t1* $\mathcal{U}$ *t2* $(k \ominus I)$. *Q* $(k - t2)$)
apply (*case-tac I* = {})
apply (*simp add*: *iT-Minus-empty*)

apply (*simp* (*no-asm-simp*) *add*: *iT-Minus-iUntil-conv*)
thm *cut-less-Max-all*
apply (*simp* (*no-asm-simp*) *add*: *cut-le-Max-all*)
apply (*unfold iSince-def*)
apply (*rule iffI*)
 apply (*clarsimp*, *rename-tac t*)
 apply (*rule-tac t=t* **in** *iexI*)
  apply (*frule-tac x=t* **in** *bspec*, *assumption*)
  apply (*clarsimp*, *rename-tac t1*)
  apply (*drule-tac t=t1* **in** *ispec*)
   apply (*simp add*: *cut-greater-mem-iff*)
  apply *simp+*
apply (*clarsimp*, *rename-tac t*)
apply (*rule-tac t=t* **in** *iexI*)
 apply (*clarsimp*, *rename-tac t'*)
 apply (*drule-tac t=t'* **in** *ispec*)
  apply (*simp add*: *cut-greater-mem-iff*)
 apply *simp+*
done

lemma *iT-Plus-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iSince-def iT-Plus-iAll-conv iT-Plus-iEx-conv iT-Plus-cut-greater2*)
lemma *iT-Mult-iSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{S}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{S}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iSince-def iT-Mult-iAll-conv iT-Mult-iEx-conv iT-Mult-cut-greater2*)
lemma *iT-Plus-neg-iSince-conv*: $(P\ t1.\ t1\ \mathcal{S}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{S}\ t2\ (I \downarrow\!\geq k).\ Q\ (t2 - k))$
apply (*simp add*: *iSince-def iT-Plus-neg-iAll-conv iT-Plus-neg-iEx-conv*)
apply (*rule iffI*)
 apply (*clarsimp*, *rename-tac t*)
 apply (*simp add*: *iT-Plus-neg-cut-greater2*)
 apply (*rule-tac t=t* **in** *iexI*)
  apply (*clarsimp*, *rename-tac t'*)
  apply (*drule-tac t=t' − k* **in** *ispec*)
   apply (*simp add*: *iT-Plus-neg-mem-iff2 cut-greater-mem-iff*)
  apply *simp*
 apply *blast*
apply (*clarsimp*, *rename-tac t*)
apply (*rule-tac t=t* **in** *iexI*)
 apply (*clarsimp*, *rename-tac t'*)
 apply (*drule-tac t=t' + k* **in** *ispec*)
  apply (*simp add*: *iT-Plus-neg-mem-iff i-cut-mem-iff*)
 apply *simp*
apply *blast*
done
lemma *iT-Minus-iSince-conv*:

(*P t1*. *t1* $\mathcal{S}$ *t2* (*k* $\ominus$ *I*). *Q t2*) = (*P* (*k* − *t1*). *t1* $\mathcal{U}$ *t2* (*I* $\downarrow\leq$ *k*). *Q* (*k* − *t2*))
apply (*case-tac I* = {})
apply (*simp add*: *iT-Minus-empty cut-le-empty*)
apply (*case-tac I* $\downarrow\leq$ *k* = {})
apply (*simp add*: *iT-Minus-image-conv*)
thm *iT-Minus-cut-eq*[*OF order-refl*]
apply (*subst iT-Minus-cut-eq*[*OF order-refl*, *symmetric*])
thm *iSince-iUntil-conv*
apply (*subst iSince-iUntil-conv*[**where** *k*=*k*])
apply (*rule iT-Minus-finite*)
apply (*subst iT-Minus-Max*)
apply *simp*
apply (*rule cut-le-bound*, *rule iMinI-ex2*, *simp*)
apply *simp*
apply (*simp add*: *iT-Minus-Minus-cut-eq*)
done
lemma *iT-Div-iSince-conv*:
0 < *k* ⟹ (*P t1*. *t1* $\mathcal{S}$ *t2* (*I* $\oslash$ *k*). *Q t2*) = (*P* (*t1 div k*). *t1* $\mathcal{S}$ *t2 I*. *Q* (*t2 div k*))
apply (*case-tac I* = {})
apply (*simp add*: *iT-Div-empty*)
apply (*simp add*: *iSince-def iT-Div-iAll-conv iT-Div-iEx-conv*)
apply (*simp add*: *iT-Div-cut-greater*)
apply (*subgoal-tac* ∀ *t*. *t* ≤ *t div k* ∗ *k* + (*k* − *Suc 0*))
prefer *2*
apply *clarsimp*
apply (*simp add*: *div-mult-cancel add-commute*[*of - k*])
thm *le-add-diff Suc-mod-le-divisor*
apply (*simp add*: *le-add-diff Suc-mod-le-divisor*)
apply (*rule iffI*)
apply (*clarsimp*, *rename-tac t*)
apply (*drule-tac x*=*t* **in** *spec*)
apply (*subgoal-tac I* $\downarrow\leq$ (*t div k* ∗ *k* + (*k* − *Suc 0*)) ≠ {})
prefer *2*
apply (*simp add*: *cut-le-not-empty-iff*)
apply (*rule-tac x*=*t* **in** *bexI*, *assumption*+)
apply (*subgoal-tac t* ≤ *Max* (*I* $\downarrow\leq$ (*t div k* ∗ *k* + (*k* − *Suc 0*))))
prefer *2*
apply (*simp add*: *nat-cut-le-finite cut-le-mem-iff*)
apply (*subgoal-tac Max* (*I* $\downarrow\leq$ (*t div k* ∗ *k* + (*k* − *Suc 0*))) ≤ *t div k* ∗ *k* + (*k* − *Suc 0*))
prefer *2*
apply (*simp add*: *nat-cut-le-finite cut-le-mem-iff*)
apply (*subgoal-tac Max* (*I* $\downarrow\leq$ (*t div k* ∗ *k* + (*k* − *Suc 0*))) *div k* = *t div k*)
prefer *2*
apply (*rule order-antisym*)
apply (*rule-tac t*=*t div k* **and** *s*=(*t div k* ∗ *k* + (*k* − *Suc 0*)) *div k* **in** *subst*)
apply (*simp only*: *div-add1-eq1-mod-0-left*[*OF mod-mult-self2-is-0*])
apply *simp*

apply (*rule div-le-mono*)
apply (*simp only*: *div-add1-eq1-mod-0-left*[*OF mod-mult-self2-is-0*])
apply *simp*
apply (*rule div-le-mono*, *assumption*)
apply (*rule-tac t=Max* (*I* ↓≤ (*t div k* ∗ *k* + (*k* − *Suc 0*))) **in** *iexI*)
apply (*clarsimp*, *rename-tac t1*)
apply (*subgoal-tac t1* ∈ *I*)
prefer *2*
apply *assumption*
apply (*subgoal-tac t div k* ∗ *k* + (*k* − *Suc 0*) < *t1*)
prefer *2*
apply (*rule ccontr*)
apply (*drule not-greater-Max*[*OF nat-cut-le-finite*])
apply (*simp add*: *i-cut-mem-iff*)
apply (*drule-tac t=t1 div k* **in** *ispec*)
apply (*simp add*: *iT-Div-imp-mem cut-greater-mem-iff*)
apply *assumption*
apply (*blast intro*: *subsetD*[*OF - Max-in*[*OF nat-cut-le-finite*]])
apply (*clarsimp*, *rename-tac t*)
apply (*drule-tac x=t* **in** *spec*)
apply (*rule-tac t=t* **in** *iexI*)
apply (*clarsimp simp*: *iT-Div-mem-iff*, *rename-tac t1 t2*)
apply (*drule-tac t=t2* **in** *ispec*)
apply (*simp add*: *cut-greater-mem-iff*)
apply *simp*+
done

Weak Until and Weak Since operators

lemma *iT-Plus-iWeakUntil-conv*: (*P t1*. *t1* 𝒲 *t2* (*I* ⊕ *k*). *Q t2*) = (*P* (*t1* + *k*). *t1* 𝒲 *t2 I*. *Q* (*t2* + *k*))
by (*simp add*: *iWeakUntil-iUntil-conv iT-Plus-iUntil-conv iT-Plus-iAll-conv*)
lemma *iT-Mult-iWeakUntil-conv*: (*P t1*. *t1* 𝒲 *t2* (*I* ⊗ *k*). *Q t2*) = (*P* (*t1* ∗ *k*). *t1* 𝒲 *t2 I*. *Q* (*t2* ∗ *k*))
by (*simp add*: *iWeakUntil-iUntil-conv iT-Mult-iUntil-conv iT-Mult-iAll-conv*)
lemma *iT-Plus-neg-iWeakUntil-conv*: (*P t1*. *t1* 𝒲 *t2* (*I* ⊕− *k*). *Q t2*) = (*P* (*t1* − *k*). *t1* 𝒲 *t2* (*I* ↓≥ *k*). *Q* (*t2* − *k*))
by (*simp add*: *iWeakUntil-iUntil-conv iT-Plus-neg-iUntil-conv iT-Plus-neg-iAll-conv*)
lemma *iT-Minus-iWeakUntil-conv*: (*P t1*. *t1* 𝒲 *t2* (*k* ⊖ *I*). *Q t2*) = (*P* (*k* − *t1*). *t1* ℬ *t2* (*I* ↓≤ *k*). *Q* (*k* − *t2*))
by (*simp add*: *iWeakUntil-iUntil-conv iWeakSince-iSince-conv iT-Minus-iUntil-conv iT-Minus-iAll-conv*)
lemma *iT-Div-iWeakUntil-conv*: (*P t1*. *t1* 𝒲 *t2* (*I* ⊘ *k*). *Q t2*) = (*P* (*t1 div k*). *t1* 𝒲 *t2 I*. *Q* (*t2 div k*))
by (*simp add*: *iWeakUntil-iUntil-conv iT-Div-iUntil-conv iT-Div-iAll-conv*)

lemma *iT-Plus-iWeakSince-conv*: (*P t1*. *t1* ℬ *t2* (*I* ⊕ *k*). *Q t2*) = (*P* (*t1* + *k*). *t1* ℬ *t2 I*. *Q* (*t2* + *k*))
by (*simp add*: *iWeakSince-iSince-conv iT-Plus-iSince-conv iT-Plus-iAll-conv*)

lemma *iT-Mult-iWeakSince-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iWeakSince-iSince-conv iT-Mult-iSince-conv iT-Mult-iAll-conv*)
lemma *iT-Plus-neg-iWeakSince-conv*: $(P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{B}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
by (*simp add*: *iWeakSince-iSince-conv iT-Plus-neg-iSince-conv iT-Plus-neg-iAll-conv*)
lemma *iT-Minus-iWeakSince-conv*:
$(P\ t1.\ t1\ \mathcal{B}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{W}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
thm *iT-Minus-iWeakUntil-conv*[*symmetric*]
by (*simp add*: *iWeakSince-iSince-conv iT-Minus-iSince-conv iT-Minus-iAll-conv iWeakUntil-iUntil-conv*)
lemma *iT-Div-iWeakSince-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{B}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{B}\ t2\ I.\ Q\ (t2\ div\ k))$
by (*simp add*: *iWeakSince-iSince-conv iT-Div-iSince-conv iT-Div-iAll-conv*)

Release and Trigger operators

lemma *iT-Plus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Plus-iWeakUntil-conv*)
lemma *iT-Mult-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Mult-iWeakUntil-conv*)
lemma *iT-Plus-neg-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{R}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Plus-neg-iWeakUntil-conv*)
lemma *iT-Minus-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{T}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Minus-iWeakUntil-conv iTrigger-iSince-conv iWeakSince-iSince-conv disj-commute*)
lemma *iT-Div-iRelease-conv*: $(P\ t1.\ t1\ \mathcal{R}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{R}\ t2\ I.\ Q\ (t2\ div\ k))$
by (*simp add*: *iRelease-iWeakUntil-conv iT-Div-iWeakUntil-conv*)

lemma *iT-Plus-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus k).\ Q\ t2) = (P\ (t1 + k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 + k))$
by (*simp add*: *iTrigger-iWeakSince-conv iT-Plus-iWeakSince-conv*)
lemma *iT-Mult-iTrigger-conv*: $0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{T}\ t2\ (I \otimes k).\ Q\ t2) = (P\ (t1 * k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2 * k))$
by (*simp add*: *iTrigger-iWeakSince-conv iT-Mult-iWeakSince-conv*)
lemma *iT-Plus-neg-iTrigger-conv*: $(P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oplus\!- k).\ Q\ t2) = (P\ (t1 - k).\ t1\ \mathcal{T}\ t2\ (I \downarrow\geq k).\ Q\ (t2 - k))$
by (*simp add*: *iTrigger-iWeakSince-conv iT-Plus-neg-iWeakSince-conv*)
lemma *iT-Minus-iTrigger-conv*:
$(P\ t1.\ t1\ \mathcal{T}\ t2\ (k \ominus I).\ Q\ t2) = (P\ (k - t1).\ t1\ \mathcal{R}\ t2\ (I \downarrow\leq k).\ Q\ (k - t2))$
by (*fastsimp simp add*: *iTrigger-iWeakSince-conv iT-Minus-iWeakSince-conv iRelease-iUntil-conv iWeakUntil-iUntil-conv*)
lemma *iT-Div-iTrigger-conv*:
$0 < k \Longrightarrow (P\ t1.\ t1\ \mathcal{T}\ t2\ (I \oslash k).\ Q\ t2) = (P\ (t1\ div\ k).\ t1\ \mathcal{T}\ t2\ I.\ Q\ (t2\ div$

k))
by (*simp add*: *iTrigger-iWeakSince-conv iT-Div-iWeakSince-conv*)

end